ON INCOMPRESSIBLE NAVIER-STOKES DYNAMICS: A NEW APPROACH FOR ANALYSIS AND COMPUTATION

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ABSTRACT. We show that in bounded domains with no-slip boundary conditions, the Navier-Stokes pressure can be determined in a such way that it is strictly dominated by viscosity. As a consequence, in a general domain we can treat the Navier-Stokes equations as a perturbed vector diffusion equation, instead of as a perturbed Stokes system. To illustrate the advantages of this view, we provide a simple proof of the unconditional stability of a difference scheme that is implicit only in viscosity and explicit in both pressure and convection terms, requiring no solution of stationary Stokes systems or inf-sup conditions.

1. Introduction

Consider the Navier-Stokes equations for incompressible fluid flow in Ω with no-slip boundary conditions on $\Gamma := \partial\Omega$:

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \nu\Delta\vec{u} + \vec{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\vec{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\vec{u} = 0 \quad \text{on } \Gamma. \tag{3}$$

Here $\vec{u}$ is the fluid velocity, p the pressure, and ν is the kinematic viscosity coefficient, assumed to be a fixed positive constant. Pressure plays a role like a Lagrange multiplier to enforce the incompressibility constraint, and this has been a main source of difficulties. In this paper we will describe some of the main results of [LLP], which indicate that the pressure can be obtained in a way that leads to considerable simplifications in both computation and analysis.

A standard way to determine p is via the Helmholtz-Hodge decomposition. We can rewrite (1) as

$$\partial_t \vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) = 0. \tag{4}$$

where $\mathcal{P}$ is the standard Helmholtz projection operator onto divergence-free fields.

In this formulation, solutions formally satisfy $\partial_t(\nabla\cdot\vec{u}) = 0$. The dissipation in (4) appears degenerate due to the fact that $\mathcal{P}$ annihilates gradients, so the analysis of (4) is usually restricted to spaces of divergence-free fields. But alternatives are possible if the pressure is determined differently when the velocity field has non-zero divergence. Instead of (4), we will consider the unconstrained formulation

$$\partial_t \vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) = \nu\nabla(\nabla\cdot\vec{u}). \tag{5}$$

There is no difference as long as $\nabla\cdot\vec{u} = 0$, of course. But we argue that (5) enjoys superior stability properties, for two reasons. The first is heuristic. The incompressibility constraint is enforced in a more robust way, because the divergence of velocity satisfies a diffusion equation with no-flux boundary conditions. Naturally, if $\nabla \cdot \vec{u} = 0$ initially, this remains true for all later time, and one has a solution of the standard Navier-Stokes equations (1)–(3).

The second reason is more profound. To explain, we recast (5) in the form (1) while explicitly identifying the separate contributions to the pressure term made by the convection and viscosity terms. Using the Helmholtz projection operator $\mathcal{P}$, we introduce the *Euler pressure* $p_{\rm E}$ and *Stokes pressure* $p_{\rm S}$ via the relations

$$\mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f}) = \vec{u}\cdot\nabla\vec{u} - \vec{f} + \nabla p_{\rm E}, \tag{6}$$

$$\mathcal{P}(-\Delta\vec{u}) = -\Delta\vec{u} + \nabla(\nabla\cdot\vec{u}) + \nabla p_{\rm S}. \tag{7}$$

This puts (5) into the form (1) with $p = p_{\rm E} + \nu p_{\rm S}$:

$$\partial_t\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p_{\rm E} + \nu\nabla p_{\rm S} = \nu\Delta\vec{u} + \vec{f}. \tag{8}$$

Since the Helmholtz projection is L^2-orthogonal, naturally the Stokes pressure from (7) satisfies

$$\int_\Omega |\nabla p_{\rm S}|^2 \le \int_\Omega |\Delta\vec{u}|^2 \quad \text{if } \nabla\cdot\vec{u} = 0. \tag{9}$$

In fact, it turns out that the Stokes pressure term is *strictly* dominated by the viscosity term, regardless of the divergence constraint. The estimate contained in the following theorem from [LLP] is key to a new treatment of Navier-Stokes dynamics in bounded domains.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) be a connected bounded domain with C^3 boundary. Then for any $\varepsilon > 0$, there exists $C \ge 0$ such that for all vector fields $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, the Stokes pressure p_S determined by (7) satisfies*

$$\int_\Omega |\nabla p_S|^2 \le \beta \int_\Omega |\Delta\vec{u}|^2 + C\int_\Omega |\nabla\vec{u}|^2. \tag{10}$$

where $\beta = \frac{2}{3} + \varepsilon$.

In this paper we will prove a slightly weaker version of this result that has a simpler proof. Namely, we will show that the estimate (10) holds for *some* $\beta < 1$. The full result that (10) holds with any $\beta > 2/3$, independent of the domain or the space dimension, is proved in [LLP] using differential geometry to establish sharp integrated Neumann-to-Dirichlet estimates for functions harmonic near the boundary of Ω.

Due to Theorem 1, we can treat the Navier-Stokes equations in bounded domains simply as a perturbation of the vector diffusion equation $\partial_t\vec{u} = \nu\Delta\vec{u}$, regarding both the pressure and convection terms as dominated by the viscosity term. To begin to see why, recall that the Laplace operator $\Delta\colon H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is an isomorphism, and note that $\nabla p_{\rm S}$ is determined by $\Delta\vec{u}$ via

$$\nabla p_{\rm S} = (I - \mathcal{P} - \mathcal{Q})\Delta\vec{u}, \qquad \mathcal{Q} := \nabla\nabla\cdot\Delta^{-1}. \tag{11}$$

Equation (5) can then be written

$$\begin{aligned} \partial_t\vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f}) &= \nu(\mathcal{P} + \mathcal{Q})\Delta\vec{u} \\ &= \nu\Delta\vec{u} - \nu(I - \mathcal{P} - \mathcal{Q})\Delta\vec{u}. \end{aligned} \tag{12}$$

Theorem 1 will allow us to regard the last term as a controlled perturbation.

This approach should be contrasted with the usual one that regards the Navier-Stokes equations as a perturbation of the Stokes system. To show the advantage of this point of view, we will sketch the proof from [LLP] of *unconditional stability* of a simple time-discretization scheme with explicit time-stepping for the pressure and nonlinear convection terms and that is implicit only in the viscosity term.

The discretization that we use is related to a class of extremely efficient numerical methods for incompressible flow [Ti, Pe, JL, GuS]. Thanks to the explicit treatment of the convection and pressure terms, the computation of the momentum equation is completely decoupled from the computation of the kinematic pressure Poisson equation used to enforce incompressibility. No stationary Stokes solver is necessary to handle implicitly differenced pressure terms. For three-dimensional flow in a general domain, the computation of incompressible Navier-Stokes dynamics is basically reduced to solving a heat equation and a Poisson equation at each time step. This class of methods is very flexible and can be used with all kinds of spatial discretization methods [JL], including finite difference, spectral, and finite element methods. The stability properties we establish here should be helpful in analyzing these methods.

Indeed, we will show here that our stability analysis easily adapts to proving unconditional stability for corresponding fully discrete finite-element methods with C^1 elements for velocity and C^0 elements for pressure. (For additional details and convergence results see [LLP].) It is important to note that we impose *no inf-sup compatibility condition* between the finite-element spaces for velocity and pressure.

Below, we will also describe how the unconstrained formulation above can be extended to handle nonhomogeneous boundary conditions. In [LLP] this formulation was exploited to establish a new result regarding existence and uniqueness for strong solutions with minimal compatibility conditions for total flux through the boundary. For further results and more complete references and proofs, we refer to [LLP].

2. Properties of the Stokes pressure

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^3 boundary Γ. For any $\vec{x} \in \Omega$ we let $\Phi(\vec{x}) = \operatorname{dist}(x, \Gamma)$ denote the distance from x to Γ. For any $s > 0$ we denote the set of points in Ω within distance s from Γ by

$$\Omega_s = \{\vec{x} \in \Omega \mid \Phi(\vec{x}) \leq s\}, \tag{13}$$

and set $\Omega_s^c = \Omega \backslash \Omega_s$ and $\Gamma_s = \{\vec{x} \in \Omega \mid \Phi(\vec{x}) = s\}$. Since Γ is C^3 and compact, there exists $s_0 > 0$ such that Φ is C^3 in Ω_{s_0} and its gradient is a unit vector, with $|\nabla\Phi(\vec{x})| = 1$ for every $\vec{x} \in \Omega_{s_0}$. We let

$$\vec{n}(\vec{x}) = -\nabla\Phi(\vec{x}), \tag{14}$$

then $\vec{n}(\vec{x})$ is the outward unit normal to $\Gamma_s = \partial\Omega_s^c$ for $s = \Phi(\vec{x})$, and $\vec{n} \in C^2(\bar{\Omega}_{s_0}, \mathbb{R}^N)$.

We let $\langle f, g \rangle_\Omega = \int_\Omega fg$ denote the L^2 inner product of functions f and g in Ω, and let $\|\cdot\|_\Omega$ denote the corresponding norm in $L^2(\Omega)$. We drop the subscript on the inner product and norm when the domain of integration is understood in context.

2.1. **Estimates related to the Neumann-to-Dirichlet map.**
Taking the divergence and normal component of both side of (7), we see that Stokes pressure

is harmonic in Ω, and is determined as the zero-mean solution of the Neumann boundary-value problem

$$\Delta p_{\mathrm{S}} = 0 \quad \text{in } \Omega, \qquad \vec{n} \cdot \nabla p_{\mathrm{S}} = \vec{n} \cdot (\Delta - \nabla\nabla\cdot)\vec{u} \quad \text{on } \Gamma. \tag{15}$$

A useful ingredient in our proof of (10) is an integrated version of a standard estimate that controls tangential gradients at the boundary (these are determined from Dirichlet data) in terms of the normal gradient (Neumann data).

Lemma 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. Then there exist positive constants r_0, C_1 and C_0 such that for any p that satisfies $\Delta p = 0$ in Ω, for any $s \in (0, r_0]$ we have*

$$\int_{\Gamma_s} |\nabla p|^2 \le C_1 \int_{\Gamma_s} |\vec{n} \cdot \nabla p|^2, \tag{16}$$

and furthermore,

$$\int_{\Omega_s} |\nabla p|^2 \le C_1 \int_{\Omega_s} |\vec{n} \cdot \nabla p|^2. \tag{17}$$

Proof: If Ω has smooth boundary and $s \in (0, s_0]$ is fixed, with s_0 taken as above, the estimate (16) is a consequence of classical elliptic theory as developed in the book of Lions and Magenes [LM], applied to the Neumann problem

$$\Delta p = 0 \quad \text{in } \Omega_s^c, \qquad \vec{n} \cdot \nabla p = g \quad \text{on } \Gamma_s. \tag{18}$$

Under the solvability condition $\int_{\Gamma_s} g = 0$, this theory yields a map $g \mapsto p \mapsto p|_{\Gamma_s}$ from $L^2(\Gamma_s) \to H^{3/2}(\Omega_s) \to H^1(\Gamma_s)$ that is bounded. This bounds the tangential part of ∇p in $L^2(\Gamma_s)$, which is sufficient to establish (16). We need to verify that these bounds are uniform in s for domains with C^2 boundary. We defer details to an appendix, where we sketch a streamlined version of the arguments of Lions and Magenes [LM] that yield the desired bounds, based on using trace theorems and the Lax-Milgram lemma to solve (18) with $g \in H^{-1/2}(\Gamma_s)$, and standard elliptic regularity theory to treat $g \in H^{1/2}(\Gamma_s)$, then interpolating to handle $g \in L^2(\Gamma_s)$. The key to obtaining uniform estimates is to work with a fixed system of local boundary-flattening maps that work simultaneously for all Γ_s — these are constructed using the distance function $\Phi(x)$.

Once (16) is established, one obtains (17) by simply integrating (16) in s.

2.2. **Identities at the boundary.** A key part of the proof of Theorem 1 involves boundary values of two quantities that involve the decomposition of $\vec{u} = (I - \vec{n}\vec{n}^t)\vec{u} + \vec{n}\vec{n}^t\vec{u}$ into parts parallel and normal to the boundary, for which we have the following Lemma. The proof involves only straightforward computations and a density argument; please see [LLP] for the details.

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class C^3. Then for any $\vec{u} \in H^2(\Omega, \mathbb{R}^N)$ with $\vec{u}|_\Gamma = 0$, the following is valid on Γ:*

(i) $\nabla \cdot \big((I - \vec{n}\vec{n}^t)\vec{u}\big) = 0$ *in* $H^{1/2}(\Gamma)$.
(ii) $\vec{n} \cdot (\Delta - \nabla\nabla\cdot)\big(\vec{n}\vec{n}^t\vec{u}\big) = 0$ *in* $H^{-1/2}(\Gamma)$.

2.3. **Identities for the Stokes pressure.** Given $\vec{u} \in H^2(\Omega,\mathbb{R}^N)\cap H^1_0(\Omega,\mathbb{R}^N)$, using the fact that $\mathcal{P}\nabla(\nabla\cdot\vec{u}) = 0$, from (7) we get

$$\nabla p_{\mathrm{S}} = (I-\mathcal{P})\Delta\vec{u} - \nabla\nabla\cdot\vec{u} = (I-\mathcal{P})(\Delta-\nabla\nabla\cdot)\vec{u} \tag{19}$$

Note that from the trace theorem for $H(\mathrm{div};\Omega)$ [GR, Theorem 2.5], whenever $\vec{a} \in L^2(\Omega,\mathbb{R}^N)$ and $\nabla\cdot\vec{a} = 0$ and $\vec{n}\cdot\vec{a}|_\Gamma = 0$, then $(I-\mathcal{P})\vec{a} = 0$. Thus, the Stokes pressure is not affected by any part of the velocity field that contributes nothing to $\vec{n}\cdot\vec{a}|_\Gamma$ where $\vec{a} = (\Delta-\nabla\nabla\cdot)\vec{u}$. Indeed, this means that the Stokes pressure is not affected by the part of the velocity field in the interior of Ω away from the boundary, and it is not affected by the normal component of velocity near the boundary, since $\vec{n}\cdot(\Delta-\nabla\nabla\cdot)(\vec{n}\vec{n}^t\vec{u})|_\Gamma = 0$ by Lemma 2.

This motivates us to focus on the part of velocity near and parallel to the boundary. We make the following decomposition. Fix $s>0$ so $2s<\min(s_0,r_0)$ and fix any smooth non-negative cutoff function $\xi : \Omega\to[0,1]$ with $\xi = 1$ on $\Omega_{s/2} = \{x\in\Omega \mid \mathrm{dist}(x,\Gamma)\le \frac{1}{2}s\}$ and $\xi = 0$ on $\Omega\backslash\Omega_s$. Then we can write

$$\vec{u} = \vec{u}_\perp + \vec{u}_\parallel \tag{20}$$

where

$$\vec{u}_\perp = \xi\vec{n}\vec{n}^t\vec{u} + (1-\xi)\vec{u}, \qquad \vec{u}_\parallel = \xi(I-\vec{n}\vec{n}^t)\vec{u}. \tag{21}$$

Now $\vec{u}_\perp = (\vec{n}\vec{n}^t)\vec{u}$ in Ω_s, and with $\vec{a}_\perp = (\Delta-\nabla\nabla\cdot)\vec{u}_\perp$ we have

$$\vec{a}_\perp \in L^2(\Omega,\mathbb{R}^N), \quad \nabla\cdot\vec{a}_\perp = 0, \quad \text{and} \quad \vec{n}\cdot\vec{a}_\perp|_\Gamma = 0 \tag{22}$$

by Lemma 2(ii). Hence $\langle\vec{a}_\perp,\nabla\phi\rangle = 0$ for all $\phi\in H^1(\Omega)$, i.e.,

$$(I-\mathcal{P})(\Delta-\nabla\nabla\cdot)\vec{u}_\perp = 0. \tag{23}$$

Combining this with (20) and (19) proves part (i) of the following.

Lemma 3. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^3 boundary, and let $\vec{u}\in H^2(\Omega,\mathbb{R}^N)\cap H^1_0(\Omega,\mathbb{R}^N)$. Let p_S and $\vec{u}_\parallel$ be defined as in (19) and (21) respectively. Then*

(*i*) *The Stokes pressure is determined by $\vec{u}_\parallel$ according to the formula*

$$\nabla p_S = (I-\mathcal{P})(\Delta-\nabla\nabla\cdot)\vec{u}_\parallel. \tag{24}$$

(*ii*) *For any $q\in H^1(\Omega)$ that satisfies $\Delta q = 0$ in the sense of distributions,*

$$\langle\Delta\vec{u}_\parallel - \nabla p_S, \nabla q\rangle = 0. \tag{25}$$

(*iii*) *In particular we can let $q = p_S$ in (ii), so $\langle\Delta\vec{u}_\parallel-\nabla p_S,\nabla p_S\rangle = 0$ and*

$$\|\Delta\vec{u}_\parallel\|^2 = \|\Delta\vec{u}_\parallel - \nabla p_S\|^2 + \|\nabla p_S\|^2. \tag{26}$$

Proof: We already proved (i). For (ii), note by Lemma 2(i) we have

$$\nabla\cdot\vec{u}_\parallel|_\Gamma = 0, \tag{27}$$

so $\nabla\cdot\vec{u}_\parallel\in H^1_0(\Omega)$. Using part (i), whenever $q\in H^1(\Omega)$ and $\Delta q = 0$ we get

$$\langle\nabla p_{\mathrm{S}},\nabla q\rangle = \langle\Delta\vec{u}_\parallel - \nabla\nabla\cdot\vec{u}_\parallel,\nabla q\rangle = \langle\Delta\vec{u}_\parallel,\nabla q\rangle. \tag{28}$$

This proves (ii), and then (iii) follows by the L^2 orthogonality. $\square$

2.4. **Proof of (10) for some $\beta < 1$.** Let $\vec{u} \in H^2(\Omega, \mathbb{R}^N) \cap H_0^1(\Omega, \mathbb{R}^N)$ and define the Stokes pressure ∇p_S by (7) and the decomposition $\vec{u} = \vec{u}_\perp + \vec{u}_\parallel$ as in the previous subsection. Then by part (iii) of Lemma 3 we have

$$\|\Delta\vec{u}\|^2 = \|\Delta\vec{u}_\perp\|^2 + 2\big\langle \Delta\vec{u}_\perp, \Delta\vec{u}_\parallel \big\rangle + \|\Delta\vec{u}_\parallel - \nabla p_S\|^2 + \|\nabla p_S\|^2. \tag{29}$$

We will establish the Theorem with the help of two further estimates.
Claim 1: For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ independent of $\vec{u}$ such that

$$\big\langle \Delta\vec{u}_\perp, \Delta\vec{u}_\parallel \big\rangle \ge -\varepsilon\|\Delta\vec{u}\|^2 - C_\varepsilon\|\nabla\vec{u}\|^2. \tag{30}$$

Claim 2: There exist positive constants β_1 and C_2 independent of $\vec{u}$ such that

$$\|\Delta\vec{u}_\parallel - \nabla p_S\|^2 \ge \beta_1\|\nabla p_S\|^2 - C_2\|\nabla\vec{u}\|^2. \tag{31}$$

Proof of claim 1: From the definitions in (21), we have

$$\Delta\vec{u}_\perp = \xi\vec{n}\vec{n}^t\Delta\vec{u} + (1-\xi)\Delta\vec{u} + R_1, \qquad \Delta\vec{u}_\parallel = \xi(I - \vec{n}\vec{n}^t)\Delta\vec{u} + R_2, \tag{32}$$

where $\|R_1\| + \|R_2\| \le C\|\nabla\vec{u}\|$ with C independent of $\vec{u}$. Since $I - \vec{n}\vec{n}^t = (I - \vec{n}\vec{n}^t)^2$,

$$\big(\xi\vec{n}\vec{n}^t\Delta\vec{u} + (1-\xi)\Delta\vec{u}\big) \cdot \big(\xi(I - \vec{n}\vec{n}^t)\Delta\vec{u}\big) = 0 + \xi(1-\xi)|(I - \vec{n}\vec{n}^t)\Delta\vec{u}|^2 \ge 0.$$

This means the leading term of $\big\langle \Delta\vec{u}_\perp, \Delta\vec{u}_\parallel \big\rangle$ is non-negative. Using the inequality $|\big\langle a, b \big\rangle| \le (\varepsilon/C)\|a\|^2 + (4C/\varepsilon)\|b\|^2$ and the bounds on R_1 and R_2 to estimate the remaining terms, it is easy to obtain (30).
Proof of claim 2: Recall that $\vec{u}_\parallel$ is supported in Ω_s, with

$$\Delta\vec{u}_\parallel = \xi(I - \vec{n}\vec{n}^t)\Delta\vec{u} + R_3 \tag{33}$$

where $\|R_3\| \le C\|\nabla\vec{u}\|$. Since $\vec{n} \cdot (I - \vec{n}\vec{n}^t)\Delta\vec{u} = 0$ we find

$$\|\vec{n} \cdot \Delta\vec{u}_\parallel\|_{\Omega_s} \le C_2\|\nabla\vec{u}\| \tag{34}$$

with $C_2 > 0$ independent of $\vec{u}$. Using $\|a + b\|^2 + \|a\|^2 \ge \frac{1}{2}\|b\|^2$ and Lemma 1, we get

$$\begin{aligned} \|\Delta\vec{u}_\parallel - \nabla p_S\|^2 &\ge \int_{\Omega_s^c} |\nabla p_S|^2 + \int_{\Omega_s} |\vec{n} \cdot (\Delta\vec{u}_\parallel - \nabla p_S)|^2 \\ &\ge \int_{\Omega_s^c} |\nabla p_S|^2 + \frac{1}{2}\int_{\Omega_s} |\vec{n} \cdot \nabla p_S|^2 - \int_{\Omega_s} |\vec{n} \cdot \Delta\vec{u}_\parallel|^2 \\ &\ge \beta_1\|\nabla p_S\|^2 - C_2\|\nabla\vec{u}\|^2. \end{aligned} \tag{35}$$

with $\beta_1 = \min(1, 1/(2C_1))$. This establishes Claim 2.

Now we conclude the proof of the theorem. Combining the claims with (29), we get

$$(1 + 2\varepsilon)\|\Delta\vec{u}\|^2 \ge (1 + \beta_1)\|\nabla p_S\|^2 - (C_2 + 2C_\varepsilon)\|\nabla\vec{u}\|^2. \tag{36}$$

Taking $\varepsilon > 0$ so that $2\varepsilon < \beta_1$ yields (10) with $\beta = (1 + 2\varepsilon)/(1 + \beta_1) < 1$. □

3. Unconditional stability of time discretization with pressure explicit

In this section we exploit Theorem 1 to establish the unconditional stability of a simple time discretization scheme for the initial-boundary-value problem for (5), our unconstrained formulation of the Navier-Stokes equations. We focus here on the case of two and three dimensions.

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with boundary Γ of class C^3. We consider the initial-boundary-value problem

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p_{\mathrm{E}} + \nu\nabla p_{\mathrm{S}} = \nu\Delta\vec{u} + \vec{f} \qquad (t > 0,\ x \in \Omega), \tag{37}$$

$$\vec{u} = 0 \qquad (t \geq 0,\ x \in \Gamma), \tag{38}$$

$$\vec{u} = \vec{u}_{\mathrm{in}} \qquad (t = 0,\ x \in \Omega). \tag{39}$$

We assume $\vec{u}_{\mathrm{in}} \in H^1_0(\Omega, \mathbb{R}^N)$ and $\vec{f} \in L^2(0, T; L^2(\Omega, \mathbb{R}^N))$ for some given $T > 0$.

Theorem 1 tells us that the Stokes pressure can be strictly controlled by the viscosity term. This allows us to treat the pressure term explicitly, so that the update of pressure is decoupled from that of velocity. This can make corresponding fully discrete numerical schemes very efficient (see [JL], also [Ti, Pe, GuS]). Here, through Theorem 1, we will prove that the following spatially continuous time discretization scheme has surprisingly good stability properties:

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} - \nu\Delta\vec{u}^{n+1} = \vec{f}^n - \vec{u}^n \cdot \nabla\vec{u}^n - \nabla p^n_{\mathrm{E}} - \nu\nabla p^n_{\mathrm{S}}, \tag{40}$$

$$\nabla p^n_{\mathrm{E}} = (I - \mathcal{P})(\vec{f}^n - \vec{u}^n \cdot \nabla\vec{u}^n), \tag{41}$$

$$\nabla p^n_{\mathrm{S}} = (I - \mathcal{P})\Delta\vec{u}^n - \nabla(\nabla \cdot \vec{u}^n), \tag{42}$$

$$\vec{u}^n\big|_{\Gamma} = 0. \tag{43}$$

We set

$$\vec{f}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \vec{f}(t)\, dt, \tag{44}$$

and take $\vec{u}^0 \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$ to approximate $\vec{u}_{\mathrm{in}}$ in $H^1_0(\Omega, \mathbb{R}^N)$. It is evident that for all $n = 0, 1, 2, \ldots$, given $\vec{u}^n \in H^2 \cap H^1_0$ one can determine $\nabla p^n_{\mathrm{E}} \in L^2$ and $\nabla p^n_{\mathrm{S}} \in L^2$ from (41) and (42) and advance to time step $n+1$ by solving (40) as an elliptic boundary-value problem with Dirchlet boundary values to obtain $\vec{u}^{n+1}$.

Let us begin making estimates — our main result is stated as Theorem 2 below. Dot (40) with $-\Delta u^{n+1}$ and use (41) and $\|I - \mathcal{P}\| \leq 1$ to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2 + \|\nabla\vec{u}^{n+1} - \nabla\vec{u}^n\|^2\Big) + \nu\|\Delta\vec{u}^{n+1}\|^2 \\
&\qquad \leq \|\Delta\vec{u}^{n+1}\|\Big(2\|\vec{f}^n - \vec{u}^n \cdot \nabla\vec{u}^n\| + \nu\|\nabla p^n_{\mathrm{S}}\|\Big) \\
&\qquad \leq \frac{\varepsilon_1}{2}\|\Delta\vec{u}^{n+1}\|^2 + \frac{2}{\varepsilon_1}\|\vec{f}^n - \vec{u}^n \cdot \nabla\vec{u}^n\|^2 + \frac{\nu}{2}\big(\|\Delta\vec{u}^{n+1}\|^2 + \|\nabla p^n_{\mathrm{S}}\|^2\big)
\end{aligned} \tag{45}$$

for any $\varepsilon_1 > 0$. This gives

$$\begin{aligned}
&\frac{1}{\Delta t}\Big(\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2\Big) + (\nu - \varepsilon_1)\|\Delta\vec{u}^{n+1}\|^2 \\
&\qquad \leq \frac{8}{\varepsilon_1}\Big(\|\vec{f}^n\|^2 + \|\vec{u}^n \cdot \nabla\vec{u}^n\|^2\Big) + \nu\|\nabla p^n_{\mathrm{S}}\|^2.
\end{aligned} \tag{46}$$

By Theorem 1, for some $\beta < 1$ one has

$$\nu\|\nabla p_S^n\|^2 \le \nu\beta\|\Delta\vec{u}^n\|^2 + \nu C_\beta\|\nabla\vec{u}^n\|^2. \tag{47}$$

Using this in (46), one obtains

$$\begin{aligned}\frac{1}{\Delta t}\Big(\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2\Big) &+ (\nu-\varepsilon_1)\big(\|\Delta\vec{u}^{n+1}\|^2 - \|\Delta\vec{u}^n\|^2\big)\\ &+ (\nu-\varepsilon_1-\nu\beta)\|\Delta\vec{u}^n\|^2\\ &\le \frac{8}{\varepsilon_1}\big(\|\vec{f}^n\|^2 + \|\vec{u}^n\cdot\nabla\vec{u}^n\|^2\big) + \nu C_\beta\|\nabla\vec{u}^n\|^2.\end{aligned} \tag{48}$$

At this point the pressure has been dealt with. Then, it is rather straightforward and standard to derive from Ladyzhenskaya's inequalities (see [LLP]) that

$$\|\vec{u}^n\cdot\nabla\vec{u}^n\|^2 \le \varepsilon_2\|\Delta\vec{u}^n\|^2 + C\|\nabla\vec{u}^n\|^6, \tag{49}$$

for any $\varepsilon_2 > 0$. Plug this into (48) and take ε_1, $\varepsilon_2 > 0$ satisfying $\nu - \varepsilon_1 > 0$ and $\varepsilon := \nu - \varepsilon_1 - \nu\beta - 8\varepsilon_2/\varepsilon_1 > 0$. We get

$$\begin{aligned}&\frac{1}{\Delta t}\big(\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2\big) + (\nu-\varepsilon_1)\big(\|\Delta\vec{u}^{n+1}\|^2 - \|\Delta\vec{u}^n\|^2\big) + \varepsilon\|\Delta\vec{u}^n\|^2\\ &\quad\le \frac{8}{\varepsilon_1}\|\vec{f}^n\|^2 + C\|\nabla\vec{u}^n\|^6 + \nu C_\beta\|\nabla\vec{u}^n\|^2.\end{aligned} \tag{50}$$

A Gronwall-type argument now leads to a simplification of the stability result in [LLP]:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with C^3 boundary, and assume $\vec{f} \in L^2(0,T;L^2(\Omega,\mathbb{R}^N))$ for some given $T > 0$ and $\vec{u}^0 \in H_0^1(\Omega,\mathbb{R}^N) \cap H^2(\Omega,\mathbb{R}^N)$. Consider the time-discrete scheme (40)-(44). Then there exist positive constants T^* and C_3, such that whenever $n\Delta t \le T^*$, we have*

$$\sup_{0\le k\le n}\|\nabla\vec{u}^k\|^2 + \sum_{k=0}^{n}\|\Delta\vec{u}^k\|^2\Delta t \le C_3. \tag{51}$$

The constants T^* and C_3 depend only upon Ω, ν and

$$M_0 := \|\nabla\vec{u}^0\|^2 + \nu\Delta t\|\Delta\vec{u}^0\|^2 + \int_0^T \|\vec{f}\|^2.$$

Proof: Put

$$z_n = \|\nabla\vec{u}^n\|^2 + (\nu-\varepsilon_1)\Delta t\|\Delta\vec{u}^n\|^2, \quad w_n = \varepsilon\|\Delta\vec{u}^n\|^2, \quad b_n = \|\vec{f}^n\|^2, \tag{52}$$

and note that from (44) we have that $\sum_{k=0}^{n-1} b_k^2\Delta t \le \int_0^T |\vec{f}(t)|^2\,dt$ as long as $n\Delta t \le T$. Then by (50),

$$z_{n+1} + w_n\Delta t \le z_n + C\Delta t(b_n + z_n + z_n^3), \tag{53}$$

where we have replaced $\max\{8/\varepsilon_1, C, \nu C_\beta\}$ by C. Summing from 0 to $n-1$ yields

$$z_n + \sum_{k=0}^{n-1} w_k\Delta t \le CM_0 + C\Delta t\sum_{k=0}^{n-1}(z_k + z_k^3) =: y_n. \tag{54}$$

The quantities y_n so defined increase with n and satisfy

$$y_{n+1} - y_n = C\Delta t(z_n + z_n^3) \le C\Delta t(y_n + y_n^3). \tag{55}$$

Now set $F(y) = \ln(\sqrt{1+y^2}/y)$ so that $F'(y) = -(y+y^3)^{-1}$. Then on $(0,\infty)$, F is positive, decreasing and convex, and we have

$$F(y_{n+1}) - F(y_n) = F'(\xi_n)(y_{n+1} - y_n) \geq -\frac{y_{n+1} - y_n}{y_n + y_n^3} \geq -C\Delta t, \tag{56}$$

whence

$$F(y_n) \geq F(y_0) - Cn\Delta t = F(CM_0) - Cn\Delta t. \tag{57}$$

Choosing any $T^* > 0$ so that $C_* := F(CM_0) - CT^* > 0$, we infer that as long as $n\Delta t \leq T^*$ we have $y_n \leq F^{-1}(C_*)$, and this together with (54) yields the stability estimate (51). □

4. Unconditional stability for C^1/C^0 finite element methods without inf-sup conditions

The simplicity of the stability proof for the time-discrete scheme above allows us to easily establish the unconditional stability of corresponding fully discrete finite-element methods that use C^1 elements for the velocity field and C^0 elements for pressure.

We suppose that for some sequence of positive values of h approaching zero, $X_h \subset H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$ is a finite-dimensional space containing the approximate velocity field, and suppose $Y_h \subset H^1(\Omega)/\mathbb{R}$ is a finite-dimensional space containing approximate pressures. We discretize (37) in a straightforward way, implicitly only in the viscosity term and explicitly in the pressure and nonlinear terms. The resulting scheme was also derived in [JL] and is equivalent to a space discretization of the scheme in (40)–(44). Given the approximate velocity $\vec{u}_n^h$ at the n-th time step, we determine $p_h^n \in Y_h$ and $\vec{u}_h^{n+1} \in X_h$ by requiring

$$\big\langle \nabla p_h^n + \nu\nabla\nabla\cdot\vec{u}_h^n - \nu\Delta\vec{u}_h^n + \vec{u}_h^n\cdot\nabla\vec{u}_h^n - \vec{f}^n, \nabla\phi_h \big\rangle = 0 \quad \forall \phi_h \in Y_h, \tag{58}$$

$$\Big\langle \frac{\nabla\vec{u}_h^{n+1} - \nabla\vec{u}_h^n}{\Delta t}, \nabla\vec{v}_h \Big\rangle + \big\langle \nu\Delta\vec{u}_h^{n+1}, \Delta\vec{v}_h \big\rangle = \big\langle \nabla p_h^n + \vec{u}_h^n\cdot\nabla\vec{u}_h^n - \vec{f}^n, \Delta\vec{v}_h \big\rangle \quad \forall \vec{v}_h \in X_h. \tag{59}$$

We are to show the scheme above is unconditionally stable. First, we take $\phi_h = p_h$ in (58). Due to the fact that $\big\langle \mathcal{P}(\Delta - \nabla\nabla\cdot)\vec{u}_h^n, \nabla p_h^n \big\rangle = 0$, we directly deduce from the Cauchy-Schwarz inequality that

$$\|\nabla p_h^n\| \leq \|\nu\nabla p_{\mathrm{S}}(u_h^n)\| + \|\vec{u}_h^n\cdot\nabla\vec{u}_h^n - \vec{f}^n\| \tag{60}$$

where

$$\nabla p_{\mathrm{S}}(u_h^n) = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}_h^n \tag{61}$$

is the Stokes pressure associated with $\vec{u}_h^n$. (Note $\nabla p_{\mathrm{S}}(u_h^n)$ need not lie in the space Y_h). Now, taking $\vec{v}_h = \vec{u}_h^{n+1}$ in (59) and arguing just as in (45), we obtain an exact analog of (46), namely

$$\begin{aligned} &\frac{1}{\Delta t}\Big(\|\nabla\vec{u}_h^{n+1}\|^2 - \|\nabla\vec{u}_h^n\|^2\Big) + (\nu - \varepsilon_1)\|\Delta\vec{u}_h^{n+1}\|^2 \\ &\quad \leq \frac{8}{\varepsilon_1}\Big(\|\vec{f}^n\|^2 + \|\vec{u}_h^n\cdot\nabla\vec{u}_h^n\|^2\Big) + \nu\|\nabla p_{\mathrm{S}}(\vec{u}_h^n)\|^2. \end{aligned} \tag{62}$$

Proceeding now exactly as in section 3 leads to the following result.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with C^3 boundary, and suppose $X_h \subset H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, $Y_h \subset H^1(\Omega)/\mathbb{R}$. Assume $\vec{f} \in L^2(0,T; L^2(\Omega,\mathbb{R}^N))$ for some given $T > 0$ and $\vec{u}^0_h \in X_h$. Consider the finite-element scheme (58)-(59) with (44). Then there exist positive constants T^* and C_4, such that whenever $n\Delta t \le T^*$, we have*

$$\sup_{0\le k\le n} \|\nabla \vec{u}^k_h\|^2 + \sum_{k=0}^{n} \|\Delta \vec{u}^k_h\|^2 \Delta t \le C_4. \tag{63}$$

The constants T^* and C_4 depend only upon Ω, ν and

$$M_{0h} := \|\nabla \vec{u}^0_h\|^2 + \nu\Delta t \|\Delta \vec{u}^0_h\|^2 + \int_0^T \|\vec{f}\|^2.$$

In [LLP] further arguments are given that establish the convergence of these approximation schemes up to the maximal time of existence for the strong solution of the unconstrained Navier-Stokes formulation (37)-(39).

5. Non-homogeneous side conditions

Consider the Navier-Stokes equations with non-homogeneous boundary conditions and divergence constraint:

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \nu\Delta\vec{u} + \vec{f} \qquad (t>0, x\in\Omega), \tag{64}$$

$$\nabla\cdot\vec{u} = h \qquad (t\ge 0, x\in\Omega), \tag{65}$$

$$\vec{u} = \vec{g} \qquad (t\ge 0, x\in\Gamma), \tag{66}$$

$$\vec{u} = \vec{u}_{\rm in} \qquad (t=0, x\in\Omega). \tag{67}$$

What we have done before can be viewed as replacing the divergence constraint (65) by decomposing the pressure via the formulae in (6) and (7) in such a way that the divergence constraint is enforced automatically. It turns out that in the non-homogeneous case a very similar procedure works. One can simply use the Helmholtz decomposition to identify Euler and Stokes pressure terms *exactly as before* via the formulae (6) and (7), but in addition another term is needed in the total pressure to deal with the inhomogeneities. Equation (5) is replaced by

$$\partial_t\vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) + \nabla p_{gh} = \nu\nabla(\nabla\cdot\vec{u}). \tag{68}$$

The equation that determines the inhomogeneous pressure p_{gh} can be found by dotting with $\nabla\phi$ for $\phi\in H^1(\Omega)$, formally integrating by parts and plugging in the side conditions: We require

$$\big\langle \nabla p_{gh}, \nabla\phi\big\rangle = -\big\langle \partial_t(\vec{n}\cdot\vec{g}), \phi\big\rangle_\Gamma + \big\langle \partial_t h, \phi\big\rangle + \big\langle \nu\nabla h, \nabla\phi\big\rangle \tag{69}$$

for all $\phi\in H^1(\Omega)$. With this definition, we see from (68) that

$$\big\langle \partial_t\vec{u}, \nabla\phi\big\rangle - \big\langle \partial_t(\vec{n}\cdot\vec{g}), \phi\big\rangle_\Gamma + \big\langle \partial_t h, \phi\big\rangle = \big\langle \nu\nabla(\nabla\cdot\vec{u} - h), \nabla\phi\big\rangle \tag{70}$$

for every $\phi\in H^1(\Omega)$. This will mean $w := \nabla\cdot\vec{u} - h$ is a weak solution of

$$\partial_t w = \nu\Delta w \ \text{ in } \Omega, \qquad \vec{n}\cdot\nabla w = 0 \ \text{ on } \Gamma, \tag{71}$$

with initial condition $w = \nabla\cdot\vec{u}_{\rm in} - h\big|_{t=0}$. So the divergence constraint will be enforced through exponential diffusive decay as before (see (84) below).

The total pressure in (64) now has the representation

$$p = p_{\rm E} + \nu p_{\rm S} + p_{gh}, \tag{72}$$

where the Euler pressure $p_{\rm E}$ and the Stokes pressure $p_{\rm S}$ are determined exactly by (6) and (7) as before, and p_{gh} is determined up to a constant by the forcing functions g and h through the weak-form pressure Poisson equation (69). Our unconstrained formulation of (64)-(67) then takes the form

$$\partial_t\vec u+\vec u\cdot\nabla\vec u+\nabla p_{\rm E}+\nu\nabla p_{\rm S}+\nabla p_{gh}=\nu\Delta\vec u+\vec f\qquad (t>0,\ x\in\Omega), \tag{73}$$

$$\vec u=\vec g\qquad (t\ge 0,\ x\in\Gamma), \tag{74}$$

$$\vec u=\vec u_{\rm in}\qquad (t=0,\ x\in\Omega). \tag{75}$$

We shall state an existence and uniqueness result for strong solutions of the unconstrained formulation (73)–(75). We refer to [LLP] for the proof, which is based on using a classical trace theorem of Lions and Magenes to reduce the problem to one with homogeneous boundary and initial conditions. Then one uses the stability of the time-differencing scheme together with a standard weak compactness method to get the existence.

Let Ω be a bounded, connected domain in $\mathbb{R}^N$ ($N=2$ or 3) with boundary Γ of class C^3. We assume

$$\vec u_{\rm in}\in H^1(\Omega,\mathbb{R}^N), \tag{76}$$

$$\vec f\in L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \tag{77}$$

$$\begin{aligned}\vec g\in H^{3/4}(0,T;L^2(\Gamma,\mathbb{R}^N))\cap L^2(0,T;H^{3/2}(\Gamma,\mathbb{R}^N))\\ \cap\{\vec g\mid \partial_t(\vec n\cdot\vec g)\in L^2(0,T;H^{-1/2}(\Gamma))\},\end{aligned} \tag{78}$$

$$h\in H_h:=L^2(0,T;H^1(\Omega))\cap H^1(0,T;(H^1)'(\Omega)). \tag{79}$$

Here $(H^1)'$ is the space dual to H^1. We also make the compatibility assumptions

$$\vec g=\vec u_{\rm in}\quad \text{when } t=0,\ x\in\Gamma, \tag{80}$$

$$\big\langle\partial_t(\vec n\cdot\vec g),1\big\rangle_\Gamma=\big\langle\partial_t h,1\big\rangle_\Omega. \tag{81}$$

We define

$$V:=L^2(0,T;H^2(\Omega,\mathbb{R}^N))\cap H^1(0,T;L^2(\Omega,\mathbb{R}^N)), \tag{82}$$

and note we have the embeddings ([Ev, p. 288], [Te, p. 176])

$$V\hookrightarrow C([0,T],H^1(\Omega,\mathbb{R}^N)),\quad H_h\hookrightarrow C([0,T],L^2(\Omega)). \tag{83}$$

Theorem 4. *Let Ω be a bounded, connected domain in $\mathbb{R}^N$ ($N=2$ or 3) and assume (76)-(81). Then there exists $T^*>0$ so that a unique strong solution of (73)-(75) exists on $[0,T^*]$, with*

$$\begin{aligned}\vec u\in& L^2(0,T^*;H^2(\Omega,\mathbb{R}^N))\cap H^1(0,T^*;L^2(\Omega,\mathbb{R}^N)),\\ p=&\nu p_S+p_E+p_{gh}\in L^2(0,T^*;H^1(\Omega)/\mathbb{R}),\end{aligned}$$

where p_E and p_S are defined in (6) and (7), $p_{gh}\in L^2(H^1(\Omega)/\mathbb{R})$ satisfies (69).

Moreover, $\vec u\in C([0,T^*],H^1(\Omega,\mathbb{R}^N))$ and

$$\nabla\cdot\vec u-h\in L^2(0,T^*;H^1(\Omega))\cap H^1(0,T^*;(H^1)'(\Omega))$$

is a smooth solution of the heat equation for $t>0$ with no-flux boundary conditions. The map $t\mapsto\|\nabla\cdot\vec u-h\|^2$ is smooth for $t>0$ and we have the dissipation identity

$$\frac{d}{dt}\frac12\|\nabla\cdot\vec u-h\|^2+\nu\|\nabla(\nabla\cdot\vec u-h)\|^2=0. \tag{84}$$

If we further assume $h \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *and* $\nabla\cdot\vec{u}_{\rm in} \in H^1(\Omega)$, *then*

$$\nabla\cdot\vec{u} \in L^2(0,T^*;H^2(\Omega)) \cap H^1(0,T^*;L^2(\Omega)).$$

Acknowledgments

The fact (related to Lemma 1) that $\vec{n}\cdot\nabla p \in L^2(\Omega_s)$ implies $\nabla p \in L^2(\Omega)$ for harmonic p was proved some years ago by Oscar Gonzalez and RLP (unpublished) through a partitioning and flattening argument. RLP is grateful for this collaboration. This material is based upon work supported by the National Science Foundation under grant no. DMS 03-05985 (RLP) and DMS-0107218 (JGL).

Appendix A. Uniform bounds on the Neumann-to-Dirichlet map

Here we provide a more detailed sketch of the proof of inequality (16) in Lemma 1, based on summarizing the relevant arguments from [LM, vol. I] and taking into account the finite regularity of the boundary.

Without loss of generality we may assume Ω is connected, so the same holds for Ω_s for small s (see below). For solutions of the Neumann problem

$$\Delta p = 0 \quad \text{in } \Omega_s^c, \qquad \vec{n}\cdot\nabla p = g \quad \text{on } \Gamma_s, \tag{85}$$

we seek to bound the tangential gradient of p by g in $L^2(\Gamma_s)$, uniformly for small s. This is accomplished by interpolating between maps

$$H^{-1/2}(\Gamma_s) \xrightarrow{A_1} H^1(\Omega_s) \xrightarrow{T_1} H^{1/2}(\Gamma_s),$$
$$H^{1/2}(\Gamma_s) \xrightarrow{A_2} H^2(\Omega_s) \xrightarrow{T_2} H^{3/2}(\Gamma_s).$$

Here the maps T_1 and T_2 are the trace maps. The map A_1 gives the zero-average weak solution of the Neumann problem (18), by applying the Lax-Milgram lemma to the weak form of (85):

$$\int_{\Omega_s^c} \nabla p\cdot\nabla q = \int_{\Gamma_s} gq \qquad \text{for all } q\in H^1(\Omega_s^c) \text{ with } \int_{\Omega_s^c} q = 0. \tag{86}$$

For our given solution of (85) we have $p-\bar{p}_s = A_1 g$ where $\bar{p}_s = |\Omega_s^c|^{-1}\int_{\Omega_s^c} p$. Since

$$\left|\int_{\Omega_s^c} gq\right| \le \|g\|_{H^{-1/2}(\Gamma_s)}\|T_1 q\|_{H^{1/2}(\Gamma_s)} \le C_0\|g\|_{H^{-1/2}(\Gamma_s)}\|q\|_{H^1(\Omega_s^c)} \tag{87}$$

by a trace theorem [LM, vol. I, p. 41], the bound

$$\|p-\bar{p}_s\|_{H^1(\Omega_s^c)} \le C_1\|g\|_{H^{-1/2}(\Gamma_s)} \tag{88}$$

follows by taking $q = p-\bar{p}_s$ and using Poincaré's inequality.

The map A_2 is the restriction of A_1 and is bounded by elliptic regularity theory using only the C^2 regularity of the boundary. Using that the trace maps T_1 and T_2 are bounded, we obtain bounds

$$\|p-\bar{p}_s\|_{H^{1/2}(\Gamma_s)} \le C_2\|g\|_{H^{-1/2}(\Gamma_s)}, \qquad \|p-\bar{p}_s\|_{H^{3/2}(\Gamma_s)} \le C_3\|g\|_{H^{1/2}(\Gamma_s)}. \tag{89}$$

By an abstract interpolation theorem [LM, vol. I, p. 27], the map T_1A_1 restricts to a bounded map between interpolation spaces,

$$TA : [H^{-1/2}(\Gamma_s), H^{1/2}(\Gamma_s)]_{1/2} \to [H^{1/2}(\Gamma_s), H^{3/2}(\Gamma_s)]_{1/2} \tag{90}$$

with $\|TA\| \le C_4$ where C_4 depends only on C_2 and C_3. By [LM, vol. I, p. 36] we have the isomorphisms

$$L^2(\Gamma_s) \cong [H^{1/2}(\Gamma_s), H^{3/2}(\Gamma_s)]_{1/2}, \qquad H^1(\Gamma_s) \cong [H^{-1/2}(\Gamma_s), H^{1/2}(\Gamma_s)]_{1/2} \tag{91}$$

with equivalent norms, and so the map $g \mapsto TAg = (p - \bar{p}_s)|_{\Gamma_s}$ is bounded from $L^2(\Gamma_s)$ to $H^1(\Gamma_s)$. This leads to the estimate

$$\begin{aligned} \|\nabla p\|^2_{L^2(\Gamma_s)} &= \|\nabla(p - \bar{p}_s)\|^2_{L^2(\Gamma_s)} \\ &\le \|p - \bar{p}_s\|^2_{H^1(\Gamma_s)} + \|\vec{n} \cdot \nabla p\|^2_{L^2(\Gamma_s)} \le C_4 \|\vec{n} \cdot \nabla p\|^2_{L^2(\Gamma_s)}, \end{aligned} \tag{92}$$

which corresponds to (16).

We now argue that these bounds are uniform in s for $0 < s \le s_0$ small. For this purpose one should consider how the fractional-order Sobolev spaces $H^r(\Gamma_s)$ and the norms in those spaces are defined, and how one obtains the bounds in trace theorems, in Poincaré's inequality, in the elliptic regularity estimate, and in comparing equivalent norms. All this can be done (except for Poincaré's inequality, which we discuss below) using a fixed partition of unity on Ω and a fixed family of maps that locally flatten the boundaries Γ_s simultaneously.

To describe how this can work, let $\Phi(x) = \pm\,\mathrm{dist}(x, \Gamma)$ be the signed distance function on $\mathbb{R}^N$, positive in Ω and negative outside. Then for some $s_0 > 0$, Φ is C^2 for $0 \le |\Phi(x)| \le s_0$. Let $x_0 \in \Gamma$ be arbitrary. Translate and rotate coordinates as necessary so $x_0 = 0$ and the normal $\vec{n}(x_0) = -\vec{e}_N$. For $x = (\hat{x}, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, let

$$\psi(x) = (\hat{x}, \Phi(\hat{x}, x_N)). \tag{93}$$

Evidently the map ψ locally flattens all Γ_s simultaneously — we have $\psi(x) = (\hat{x}, s)$ for all $x \in \Gamma_s$. By the proof of the inverse function theorem, ψ is a C^2 diffeomorphism of $B(x_0, r_0)$ onto its image, for some $r_0 \in (0, s_0]$ fixed and independent of x_0.

Let $r_k = 2^{-k} r_0$, $k = 1, 2, \ldots$. The compact set $\bar{\Omega}_{r_2}$ is covered by the union of balls $B(x_0, r_1)$ for $x_0 \in \Gamma$. Hence there is a finite subcover by balls $B(x^{(j)}, r_1)$, $j = 1, \ldots, m$. We denote by ψ_j the associated diffeomorphism on $B_j = B(x^{(j)}, r_0)$; we have $\vec{e}_N \cdot \psi_j(x) = \Phi(x)$ for $|x - x^{(j)}| < r_0$. We fix a partition of unity on $\bar{\Omega}_{r_2}$ subordinate to this finite cover by B_j. I.e., we fix $\alpha_j \in C_0^\infty(B_j)$, $j = 1, \ldots, m$, such that $\sum_{j=1}^m \alpha_j(x) = 1$ for all $x \in \bar{\Omega}_{r_2}$.

The norms in the fractional-order Sobolev spaces $H^r(\Gamma_s)$ can then be defined as follows for $|r| \le 2$. Given $s \in [0, r_2]$, the norm of any $u \in C^2(\Gamma_s)$ in $H^r(\Gamma_s)$ is given by the norm of the C^2 function $\hat{x} \mapsto \{(\alpha_j u) \circ \psi_j^{-1}(\hat{x}, s)\}_{j=1}^m$ in $H^r(\mathbb{R}^{N-1})^m$. With this fixed system of cutoffs and flattening maps, one can check that the trace theorem for T_1 and T_2, the elliptic regularity theory for A_2, and the norm equivalences that must be invoked to obtain the estimates in (89), all involve bounds that are valid uniformly for $s \in (0, r_1]$. It remains to check Poincaré's inequality.

Uniform bounds in Poincaré's inequality. In the procedure that we have outlined, to obtain a uniform bound in (88) we need that the constant in Poincaré's inequality

$$\int_{\Omega_s^c} |p - \bar{p}_s|^2 \le C \int_{\Omega_s^c} |\nabla p|^2, \qquad \bar{p}_s = \frac{1}{|\Omega_s^c|} \int_{\Omega_s^c} p, \tag{94}$$

can be taken independent of s for small s. Showing this is somewhat related to showing that Ω_s^c is connected for small s, which we can do as follows: Since Ω is connected, any two points in Ω_s^c can be connected by a path in Ω. On this path we can replace points in Ω_s by the projection along normals onto $\Gamma_{s'}$, where $s'-s$ is positive and so small that the new path is contained in Ω_s^c and still connects the same two points.

To prove the uniform Poincaré inequality (94), we reduce to a fixed s by integrating along normals as follows. Recall that the distance function $\Phi(x) = \text{dist}(x,\Gamma)$ is C^2 for $\Phi(x) \le s_0$. Fix $\varepsilon = \frac{1}{2}s_0$, and suppose $0 < s \le \varepsilon$. Let q be a smooth function on Ω_s^c. Then if $s < \Phi(x) < \varepsilon$ we integrate $\partial_t q(x - t\vec{n}(x))$ from $t=0$ to ε and apply the Cauchy-Schwarz inequality to get

$$|q(x)|^2 \le 2|q(x-\varepsilon\vec{n}(x))|^2 + 2\varepsilon \int_0^\varepsilon |\nabla q(x - t\vec{n}(x))|^2\,dt. \tag{95}$$

Integrating this in the domain where $s < \Phi(x) < \varepsilon$ and changing variables on the right hand side we get

$$\int_{s<\Phi(x)<\varepsilon} |q(x)|^2\,dx \le C\left(\int_{\varepsilon<\Phi(y)<2\varepsilon} |q(y)|^2\,dy + \int_{s<\Phi(y)<2\varepsilon} |\nabla q(y)|^2\,dy\right) \tag{96}$$

with C independent of q and s. Now, given a smooth function p on Ω_s^c, use this inequality for $q = p - \bar{p}_\varepsilon$, together with the standard Poincaré inequality for the domain Ω_ε^c, and the L^2-optimality of the average in Ω_s^c, to conclude that

$$\int_{\Omega_s^c} |p - \bar{p}_s|^2 \le \int_{\Omega_s^c} |p-\bar{p}_\varepsilon|^2 \le C\left(\int_{\Omega_\varepsilon^c} |p-\bar{p}_\varepsilon|^2 + \int_{\Omega_s^c} |\nabla p|^2\right) \le C\int_{\Omega_s^c} |\nabla p|^2 \tag{97}$$

for some C independent of p and s.

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