Boundary-Layer Behavior in the Fluid-Dynamic Limit for a Nonlinear Model Boltzmann Equation

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Abstract

In this paper, we study the fluid-dynamic limit for the one-dimensional Broadwell model of the nonlinear Boltzmann equation in the presence of boundaries. We consider an analogue of Maxwell's diffusive and reflective boundary conditions. The boundary layers can be classified as either compressive or expansive in terms of the associated characteristic fields. We show that both expansive and compressive boundary layers (before detachment) are nonlinearly stable and that the layer effects are localized so that the fluid dynamic approximation is valid away from the boundary. We also show that the same conclusion holds for short time without the structural conditions on the boundary layers. A rigorous estimate for the distance between the kinetic solution and the fluid-dynamic solution in terms of the mean-free path in the L^∞-norm is obtained provided that the interior fluid flow is smooth. The rate of convergence is optimal.

§1. Introduction

We study the boundary-layer behavior of the solutions to the one-dimensional Broadwell model of the nonlinear Boltzmann equation with an analogue of Maxwell's diffusive and diffusive-reflective boundary conditions at small mean-free path. This is one of the three connection problems in the fluid-dynamic approximation for a model Boltzmann equation proposed by BROADWELL [3].

The general Boltzmann equation of kinetic theory gives a statistical description of a gas of interacting particles. An important property of this equation is its asymptotic equivalence to the Euler or Navier-Stokes equations of compressible fluid dynamics, in the limit of small mean-free path. One expects that, away from initial layers, shock layers, and boundary layers, the Boltzmann solution should relax to its equilibrium state (local Maxwellian state) in the limit of small mean-free path, and that the gas should be governed by the macroscopic equations — the fluid equations. This is predicted by the method of normal solutions (on normal regions) based on the Hilbert expansion and the Chapman-Enskog expansions. Thus, to validate the fluid-dynamical approximation, it is necessary to complete the Hilbert expansion (or Chapman-Enskog expansion) with suitable initial data, boundary conditions, or matching conditions across shocks even at the formal level. Thus, one has to solve three connection problems across the layers within which the Hilbert expansion fails: to relate a given initial distribution function to the Hilbert (or Chapman-Enskog) solution which takes over after an initial transient (initial layer problem), to find the correct matching conditions for the two Hilbert solutions prevailing on each side for shock layers (shock-layer problem), and to relate a given boundary condition on the distribution function (for the kinetic theory) to the Hilbert solution which holds outside the boundary layers (boundary-layer problems). The rigorous mathematical justification of the fluid-dynamic approximation of Boltzmann solutions poses a challenging open problem in most important cases, in particular, in the case that there are shock layers and boundary layers in the fluid flow. This has been extensively studied in the literature. However, most of the previous works concentrate either on linearized Boltzmann equations [10, 9], or on initial layers for some models of the nonlinear Boltzmann equation [10, 7, 5, 6, 12, 15, 16] with notable exceptions [2, 17, 4, 18]. As for the boundary-layer problem, a qualitative theory exists for some models of steady Boltzmann equations [1], but very little is known for the unsteady problems. Since boundary layers are important because they describe the interactions of the gas molecules with the molecules of the solid body, i.e., the interaction between the body and the gas, to which one can trace the origin of the drag exerted by the gas on the body and the heat transfer between the gas and the solid boundaries, it is very important to understand the fluid-dynamical approximation when there are interactions of the gas with solid boundaries. It is expected that the fluid approximation is still valid away from the boundaries. The main difficulties in analyzing this problem are due to the complexity of the nonlocal collision operator in the Boltzmann equation, which makes it difficult to study the structures of the layer problems associated with the formal matched asymptotic analysis. Even when the structures of these layers are relatively easy to study as for the Broadwell model, the convergence cannot be obtained easily because the fluid-dynamical limits are highly singular, and the dissipative mechanisms are much weaker than those for the Navier-Stokes equations.

In this paper, we address the boundary-layer problem for the much simpler one-dimensional Broadwell model of the nonlinear Boltzmann equation with an analogue of Maxwell's diffusive and diffusive-reflective boundary conditions. The boundary layers can be classified as either compressive or expansive in terms of the associated characteristic fields. It turns out that this classification plays an important role on our stability analysis. We prove that both expansive and compressive boundary layers are nonlinearly stable (before detachment [14]) and the layer effects are localized. Thus the fluid approximation is justified for this model and a rigorous estimate of the convergence in the L^∞-norm in terms of the mean-free path is obtained provided the interior gas flow is smooth. The rate of convergence is optimal. We emphasize that the classification of layers is needed for long-time stability. In the case of short time, the convergence can be obtained quite easily without using the structure of the boundary layer. See Theorem 3.2 and §5.3.

The outline of our approach is as follows. We consider the initial-boundary-value problem for the Broadwell model with either diffusive or diffusive-reflective boundary conditions. The appropriate boundary conditions for the corresponding model Euler equations is formulated so that the initial-boundary-value problem for the Euler equation is well-posed and its solution can be realized as the limit of the corresponding Broadwell solution as the mean-free path goes to zero. This is achieved by matching the fluid solution with the boundary-layer solutions though conservation laws. This matched asymptotic analysis produces an approximate solution for the Broadwell equation with detailed layer structures near the boundary. Then the existence of the exact Broadwell solution and its convergence to the fluid solution away from the boundary are reduced to a nonlinear stability analysis. The main difficulty of the stiffness in the stability analysis is overcome by using energy estimates which depend crucially on the structures of the underlying boundary layers. In the case of compressive layers, this approach works before the detachment of the boundary layers.

This paper is organized as follows. In §2, the Broadwell model and its corresponding model Euler equations are introduced. Then we study the dynamic systems associated with the leading-order boundary layers. It turns out that this system can be integrated explicitly so that we can classify the layers as either compressive or expansive in terms of the rate of change of the associated characteristic speeds. As a consequence, we obtain suitable boundary conditions for the corresponding Euler equation and the well-posedness is verified. In §3, we state our main convergence theorems. The rest of the paper is devoted to the proof of the convergence theorems by using the approach outlined in the previous paragraph.

§2. Broadwell Model and Its Boundary Layers

§2.1. The Broadwell Model and the Corresponding Fluid Equations

The Broadwell model describes a gas as composed of particles of only six speeds with a binary collision law and spatial variation in only one direction. In one space dimension, the model takes the form [3]

$$\begin{aligned} \partial_t f^+ + \partial_x f^+ &= \tfrac{1}{\varepsilon}(f^0 f^0 - f^+ f^-), \\ \partial_t f^0 &= \tfrac{1}{2\varepsilon}(f^+ f^- - f^0 f^0), \\ \partial_t f^- - \partial_x f^- &= \tfrac{1}{\varepsilon}(f^0 f^0 - f^+ f^-), \end{aligned} \tag{2.1}$$

where ε is the mean-free path, f^+, f^0, and f^- denote the mass densities of gas particles with speed $1, 0$ and -1, respectively. In what follows, we use the vector notation $\boldsymbol{f} = (f^+, f^0, f^-)$. The fluid moments are defined as

$$\rho = f^+ + 4f^0 + f^-, \quad m = f^+ - f^-, \quad u = \frac{m}{\rho}, \tag{2.2a}$$

which are hydrodynamical quantities: the mass density, momentum, and fluid velocity respectively. We introduce another quantity z by

$$z = f^+ + f^-. \tag{2.2b}$$

Then the system (2.1) can be rewritten in terms of $\theta = (\rho, m, z)$ as

$$\begin{aligned} \partial_t \rho + \partial_x m &= 0, \\ \partial_t m + \partial_x z &= 0, \\ \partial_t z + \partial_x m &= \tfrac{1}{\varepsilon} q(\theta, \theta), \end{aligned} \tag{2.3a}$$

where

$$q(\theta_1, \theta_2) = \tfrac{1}{8}(\rho_1 - z_1)(\rho_2 - z_2) + \tfrac{1}{2}(m_1 m_2 - z_1 z_2). \tag{2.3b}$$

The state $\theta = (\rho, m, z)$ is said to be a local Maxwellian [3] if

$$\rho > 0, \quad |u| < c, \quad z = \rho\sigma(u), \tag{2.4a}$$

where

$$\sigma(u) = \tfrac{2}{3}\sqrt{1 + 3u^2} - \tfrac{1}{3}. \tag{2.4b}$$

By assuming the state to be in equilibrium, one can derive the following closed 2×2 system of conservation laws [3]:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho\sigma(u)) &= 0, \end{aligned} \tag{2.5}$$

which is called the model Euler equation which shares many properties of isentropic gas dynamics when the macroscopic speed of the gas is relatively small compared with the microscopic speed of the gas particles [3]. It has been shown by CAFLISCH [4] that the system (2.5) is strictly hyperbolic and genuinely nonlinear with characteristic speeds

$$\lambda_1 = 2\frac{u - \sqrt{\sigma(u)}}{3\sigma(u) + 1}, \quad \lambda_2 = 2\frac{u + \sqrt{\sigma(u)}}{3\sigma(u) + 1} \tag{2.6}$$

satisfying

$$-1 < \lambda_1(u) < 0 < \lambda_2(u) < 1 \quad \text{if } |u| < 1, \tag{2.7}$$

$$\frac{d\lambda_i(u)}{du} > 0, \quad i = 1, 2. \tag{2.8}$$

We study the initial-boundary-value problem for the Broadwell equations and the boundary-layer behavior of its solutions for small mean-free path. To isolate the effects of boundary layers, we assume that the initial state $\boldsymbol{f}_0 = (f_0^+, f_0^0, f_0^-)$ is a local Maxwellian and satisfies

$$0 < \gamma \leqq f_0^+(x), \quad f_0^0(x), \quad f_0^-(x) \leqq C \tag{2.9}$$

for some given positive constants γ and C.

§2.2. *Boundary Conditions and Well-Posedness of the Fluid Equations*

Let the boundary be given by

$$x = -\alpha t = s(t). \tag{2.10a}$$

To simplify the presentation, we assume that $0 < \alpha < 1$. We remark here that the cases $\alpha = 0$ and $\alpha = 1$ correspond to the uniform characteristic boundary conditions for the Broadwell equations, in which there are no strong boundary layers, so that the fluid-dynamic approximation can be easily justified by adapting our following analysis.

We consider the Broadwell equations on the region

$$\Omega_T = \{(x,t), s(t) \leqq x < +\infty, 0 \leqq t \leqq T\}. \tag{2.10b}$$

with initial data

$$(f^+, f^0, f^-)(x, t=0) = (f_0^+, f_0^0, f_0^-)(x) \tag{2.11}$$

satisfying (2.9), and two types of boundary conditions which are analogous to Maxwell's diffusive boundary conditions. One is the purely diffusive boundary condition

$$f^+(s(t),t) = f_b^+(t), \quad f^0(s(t),t) = f_b^0(t). \tag{2.12}$$

Another type is the diffusive-reflective boundary condition

$$f^+(s(t),t) = a(t)f^-(s(t),t), \quad 4f^0(s(t),t) = b(t)f^-(s(t),t) \tag{2.13}$$

where a and b are positive functions.

The gas near the boundary in general is not in an equilibrium state. In order to understand the leading-order behavior of the kinetic boundary layer, one can use the stretched variable $\xi = (x+\alpha t)/\varepsilon$ and look for the solution to (2.1) of the form $\boldsymbol{f}(\xi,t) = \boldsymbol{f}((x+\alpha t)/\varepsilon, t)$. Simple calculations show that up to the leading order, the solution is governed by the following system of ordinary differential equations in which t is regarded as a parameter:

$$\begin{aligned}(\alpha+1)\frac{df^+}{d\xi} &= f^0f^0 - f^+f^-,\\ -2\alpha\frac{df^0}{d\xi} &= f^0f^0 - f^+f^-,\\ (\alpha-1)\frac{df^-}{d\xi} &= f^0f^0 - f^+f^-.\end{aligned} \tag{2.14}$$

Corresponding to (2.12), the boundary data for (2.14) at $\xi = 0$ are given by

$$f^+(0) = f_b^+, \quad f^0(0) = f_b^0, \tag{2.15}$$

while for (2.13), the boundary condition for (2.14) takes the form

$$f^+(0) - af^-(0) = 0, \quad 4f^0(0) - bf^-(0) = 0. \tag{2.16}$$

The state at $\xi = +\infty$ is in the fluid region, hence taken to be a local Maxwellian in both cases

$$\boldsymbol{f}_\infty = (f_\infty^+, f_\infty^0, f_\infty^-), \quad f_\infty^+ f_\infty^- = (f_\infty^0)^2. \tag{2.17}$$

One can solve (2.14) explicitly and obtain the appropriate boundary condition for the model Euler equation (2.5) as follows.

We start with the case associated with the diffusive boundary condition (2.12). It follows from (2.14) that there exist two functions $c_1(t)$ and $c_2(t)$ independent of ξ such that

$$(\alpha + 1)f^+ + 2\alpha f^0 = c_1(t), \quad (\alpha - 1)f^- + 2\alpha f^0 = c_2(t). \tag{2.18}$$

Using the boundary condition (2.15) leads to

$$c_1(t) = (\alpha + 1)f_b^+ + 2\alpha f_b^0. \tag{2.19}$$

On the other hand, the boundary condition (2.17) yields

$$c_1(t) = \frac{\rho_b}{2}((\alpha + 1)(u_b + \sigma(u_b)) + \alpha(1 - \sigma(u_b))), \tag{2.20}$$

where we have rewritten (2.17) in terms of the fluid moments. Setting

$$\mathscr{B}(\rho, u)(t) \equiv \tfrac{1}{2}\rho(\sigma(u) + (\alpha + 1)u + \alpha)|_{(s(t),t)}, \tag{2.21}$$

we find the desired boundary condition for the Euler equations (2.5) to be

$$\mathscr{B}(\rho, u)(t) = (\alpha + 1)f_b^+(t) + 2\alpha f_b^0(t). \tag{2.22}$$

Next we derive the boundary condition for (2.5) corresponding to the diffusive-reflective boundary condition (2.13). Instead of (2.19), one gets from (2.16) and (2.18) that

$$\frac{2(\alpha + 1)a + \alpha b}{2(\alpha - 1) + \alpha b} = \frac{c_1}{c_2}. \tag{2.23}$$

Evaluating (2.18) at $\xi = \infty$ and using (2.17) in the macroscopic form, one can get

$$\begin{aligned} c_1 &= (\alpha + 1)f_\infty^+ + 2\alpha f_\infty^0 = \frac{\rho_b}{2}(\alpha + u_b + \sigma(u_b) + \alpha u_b), \\ c_2 &= (\alpha - 1)f_\infty^- + 2\alpha f_\infty^0 = \frac{\rho_b}{2}(\alpha + u_b - \sigma(u_b) - \alpha u_b). \end{aligned} \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$(\alpha + u_b)(1 + a - \alpha(1 - a)) = (\sigma(u_b) + \alpha u_b)(1 - a - \alpha(1 + a + b)). \tag{2.25}$$

In particular, (2.25) yields

$$u_b = -\alpha \tag{2.26a}$$

when

$$1 = a + \alpha(1 + a + b), \tag{2.26b}$$

which corresponds to the purely reflective boundary condition. In this case, the mass flux is conserved on the boundary, i.e.,

$$(1-\alpha)f^- = (1+\alpha)f^+ + 4\alpha f^0. \tag{2.26c}$$

Let u_b be the solution of (2.25). Then we have found the desired boundary condition for the model Euler equations (2.5) to be

$$u(s(t),t) = u_b(t). \tag{2.27}$$

We now show that the problem (2.5), (2.22), or (2.5), (2.27), is well-posed at least locally (in time). It is assumed that the boundary $x = -\alpha t$ is non-characteristic for (2.5), (2.22), or (2.5), (2.27). To show the local well-posedness, it suffices to check that the boundary condition accounts for the inflow on the boundary. To this end, we first rewrite the fluid equations (2.5) in the characteristic form [13]:

$$\partial_t \phi_+ + \lambda_+ \partial_x \phi_+ = 0,$$

$$\partial_t \phi_- + \lambda_- \partial_x \phi_- = 0,$$

in which the functions $\phi_\pm$ are the Riemann invariants of the form

$$\phi_\pm(\rho,u) = \rho^2(\sigma(u)-u^2)\exp\left\{ \pm 2\int_0^u \left(\frac{\sigma(w)}{1+3w^2}\right)^{1/2} \frac{dw}{\sigma(w)-w^2}\right\}.$$

Setting $\phi_\pm(x,t) = \phi_\pm(\rho(x,t),u(x,t))$, we obtain from direct computation that

$$\frac{\partial \mathcal{B}}{\partial \phi_+} = \frac{\rho}{12\phi_+\sqrt{\sigma(u)}}\Big(\big(u(\sqrt{1+3u^2}-1)+\sqrt{\sigma(u)}\big)\big(2\sqrt{1+3u^2}+(1+\alpha)u+\alpha-1\big)$$

$$+\big(2u+(1+\alpha)\sqrt{1+3u^2}\big)\big(2\sqrt{1+3u^2}-u^2-1\big)\Big) > 0.$$

Thus the implicit-function theorem implies that the inflow ϕ_+ can be represented in terms of a smooth function of the outflow ϕ_- and the given boundary values. Consequently, the initial-value problem (2.5), (2.22) is well-posed. The well-posedness of the initial-boundary-value problem (2.5), (2.27) follows from the following lemma whose proof is very tedious and is given in the Appendix A.

Lemma 2.1. *Assume that* $a \geqq 1/3$, $b \leqq 2/3$ *and* $\alpha \leqq 1/\sqrt{3}$. *Then there is a unique solution* u_b *to* (2.25) *which satisfies*

$$|u_b| < 1, \quad \lambda_1(u_b) < -\alpha. \tag{2.28}$$

We remark here that the specific bounds for a, b, and α are chosen just for the convenience of presentation of the proof, and can be relaxed somewhat. However, it can be shown that (2.28) fails when either α or b is close to 1.

§2.3. Classifications of the Boundary Layers

To determine the structure of the boundary layer, we now solve (2.14) with boundary conditions (2.15), (2.17) or (2.16), (2.17). Substitute (2.18) into the second equation in (2.14) to get

$$\frac{df^0}{d\xi} = \frac{-(3\alpha^2+1)}{2\alpha(1-\alpha^2)}\left((f^0)^2 - \frac{2\alpha(c_1+c_2)}{3\alpha^2+1}f^0 + \frac{c_1c_2}{3\alpha^2+1}\right). \tag{2.29}$$

Since the Euler equation (2.5) with boundary data (2.22) or (2.27) and appropriate initial data has a smooth solution, it follows from the matching condition that

$$f^0_\infty = \frac{\rho}{4}(1-\sigma(u))(s(t),t), \tag{2.30}$$

where f^0_∞ is a root of the quadratic polynomial on the right side of (2.29). Define

$$f^0_{-\infty} = -f^0_\infty + \frac{2\alpha}{3\alpha^2+1}(u_{\rm b}+\alpha)\rho_{\rm b}. \tag{2.31}$$

Here and in what follows, we use the notation $\rho_{\rm b} = \rho(s(t),t)$, etc. It follows from the definitions of $c_1(t)$ and $c_2(t)$ that

$$c_1 + c_2 = f^+_\infty - f^-_\infty + \alpha(f^+_\infty + 4f^0_\infty + f^-_\infty) = (u_{\rm b}+\alpha)\rho_{\rm b}, \tag{2.32a}$$

$$c_1c_2 = (3\alpha^2+1)f^0_\infty f^0_{-\infty}. \tag{2.32b}$$

Consequently,

$$f^0_\infty + f^0_{-\infty} = \frac{2\alpha(c_1+c_2)}{3\alpha^2+1}, \quad f^0_\infty f^0_{-\infty} = \frac{c_1c_2}{3\alpha^2+1}, \tag{2.32c}$$

and so (2.29) becomes

$$\frac{df^0}{d\xi} = -c_\alpha(f^0 - f^0_\infty)(f^0 - f^0_{-\infty}) \tag{2.33a}$$

where

$$c_\alpha = \frac{3\alpha^2+1}{2\alpha(1-\alpha^2)}. \tag{2.33b}$$

Solving (2.33a), we obtain that

$$f^0(\xi) - f^0_\infty = \frac{(f^0_{\rm b} - f^0_\infty)(f^0_\infty - f^0_{-\infty})\exp(-C_\alpha(f^0_\infty - f^0_{-\infty})\xi)}{(f^0_{\rm b} - f^0_{-\infty})(f^0_{\rm b} - f^0_\infty)\exp(-C_\alpha(f^0_\infty - f^0_{-\infty})\xi)}. \tag{2.34}$$

Equations (2.34) and (2.18) give the corresponding formulas for f^+ and f^-. Our next lemma shows that the boundary layers approach the Maxwellian states exponentially fast as the fast variable goes to infinity.

Lemma 2.2. *If* $\lambda_-(u) < -\alpha$, *then* $f^0_{-\infty} < f^0_\infty$. *Furthermore, if* $f^0_b > f^0_{-\infty}$, *then*

$$|\boldsymbol{f}(\xi) - \boldsymbol{f}_\infty| \leqq C|f^0_b - f^0_\infty| \exp(-C(f^0_\infty - f^0_{-\infty})\xi). \tag{2.35}$$

The proof is given in Appendix A.

We remark here that for given boundary data, the condition that $f^0_b > f^0_{-\infty}$ is automatically satisfied if α is suitably small.

We now turn to the classification of boundary layers. Even though the gas near the boundary is not generally in equilibrium, it is appropriate to use the monotonicity of $\lambda_1(u)$ to describe the kinetic boundary layers. We say that a boundary layer is *compressive* if $d\lambda_1/d\xi < 0$, and *expansive* if $d\lambda_1/d\xi \geqq 0$.

Since the characteristic speeds are monotone functions of the macroscopic velocity u (cf. (2.8)), it is clear that the classification of the boundary layer depends on the monotonicity of u along the boundary-layer profile. Direct calculation using (2.14) shows that

$$\frac{du}{d\xi} = -\frac{4\rho_b(\alpha + u_b)}{(1-\alpha^2)\rho^2}\frac{df^0}{d\xi}. \tag{2.36}$$

It follows from this that there are four different cases depending on the speeds of the wall and the fluid:

	$\frac{df^0}{d\xi} < 0$	$\frac{df^0}{d\xi} > 0$
compressive layer $\left(\frac{d\lambda_1(u)}{d\xi} < 0\right)$	$u_b < -\alpha$	$u_b > -\alpha$
expansive layer $\left(\frac{d\lambda_1(u)}{d\xi} \geqq 0\right)$	$u_b \geqq -\alpha$	$u_b \leqq -\alpha$

Remarks. (i) One can similarly study the boundary layers for the model Navier-Stokes equations derived from (2.1) by the Chapman-Enskog expansion [4]. Viscous boundary layers can also be classified as either compressive or expansive. However, one can prove that viscous boundary layers exist only when $u_b > -\alpha$ [14]. Thus the boundary layers corresponding to $u_b \leqq -\alpha$ are due purely to the kinetic effects, which cannot be detected by the Chapman-Enskog expansions. This phenomena was observed previously in the steady problems for the GBK model (cf. [8]).

(ii) The compressible layers are not always stable and may detach from the boundary and become shocks. This is shown numerically in our forthcoming paper [14].

§3. The Fluid-Dynamic Limit

In this section we state our convergence results, which demonstrate, roughly speaking, that the boundary layers are nonlinearly stable before their detachment from the boundary, so that the fluid-dynamic approximation is still valid away from the moving boundary, provided that underlying fluid flow is smooth. Define

$$\Omega_T^\delta = \{(x,t), s(t) + \delta \leqq x < +\infty, 0 \leqq t \leqq T\}.$$

Here T is any finite positive number such that the initial-boundary-value problem for the model Euler equations (2.5) and either (2.22) or (2.27) has a sufficiently smooth solution $(\rho, m)(x,t)$ on the region Ω_T. In the case $df^0/d\xi > 0$, we assume further that the solution to the initial-boundary-value problem for the Euler equations, (2.5) and (2.22) lies in a δ_0-neighborhood of a global Maxwellian state (ρ^*, m^*, z^*), as do the boundary data f_b^+ and f_b^0. In the compressible layers, we also assume that the macroscopic speed is much slower than the microscopic speed. Then our convergence theorem can be stated as follows.

Theorem 3.1. *Assume that the boundary layer is either compressive or expansive for all* $t \in [0, T]$. *Let* $\boldsymbol{g}(x,t)$ *be the microscopic density distribution associated with the local Maxwellian* $(\rho, m)(x,t)$. *Then there exists an* $\varepsilon_0 > 0$ *such that for each* $0 < \varepsilon \leqq \varepsilon_0$, *the initial-boundary-value problem* (2.1), (2.12) *or* (2.1), (2.13) *has a unique smooth solution* $\boldsymbol{f}_\varepsilon(x,t)$ *such that*

$$\begin{aligned} &\boldsymbol{f}_\varepsilon(x+\alpha t, t) - \boldsymbol{g}(x+\alpha t, t) \in L^\infty([0,T], H^1(R_+)) \cap C([0,T]: L^2(R_+)),\\ &\frac{d\boldsymbol{f}_\varepsilon}{dt}(x+\alpha t, t) \in L^\infty([0,T]: L^2(R_+)). \end{aligned} \tag{3.1}$$

Furthermore, for any integer $n > 0$, *one can construct a bounded function* $\tilde{\boldsymbol{g}}(x,t,\varepsilon,n)$ *such that*

$$\sup_{0\leqq t \leqq T} \|\boldsymbol{f}_\varepsilon(\cdot + \alpha t, t) - \boldsymbol{g}(\cdot + \alpha t, t) - \varepsilon \tilde{\boldsymbol{g}}(\cdot + \alpha t, t, \varepsilon, n)\|_{L^\infty(R_+)} \leqq C_n \varepsilon^{n+1}. \tag{3.2}$$

In particular, for any $\delta > 0$, *there exists a* $C_\delta > 0$ *such that*

$$\sup_{\Omega_T^\delta} \|\boldsymbol{f}_\varepsilon(x+\alpha t, t) - \boldsymbol{g}(x+\alpha t, t)\| \leqq C_\delta \varepsilon. \tag{3.3}$$

For short time, then, the strength of the boundary layer is weak. We can obtain the following convergence result without taking into account the structure of the underlying boundary layer.

Theorem 3.2. *There exist suitable small positive constants* T_0 *and* ε_0 *such that the conclusions in* (3.1)–(3.3) *are true with* T *replaced by* T_0.

Remarks. 1. The rate of convergence in the theorem is optimal.

2. As indicated in (3.2), the principal asymptotic structure of the solution $\boldsymbol{f}_\varepsilon$ is explicitly described by the function $\boldsymbol{g} + \varepsilon\tilde{\boldsymbol{g}}$ which is constructed in detail by matching a boundary-layer expansion with the Hilbert expansion away from the boundary. This will be made clear in the proof of the theorem.

3. In the theorems, we assumed that the initial state is in equilibrium; thus the interesting problem of interaction of initial layers with boundary layers is completely ignored here. We also avoid the problem of interactions of boundary layers with shock layers by assuming that the solution of the model Euler equations is smooth. We are currently investigating these issues.

4. We present only the proof of the case $n = 0$; other cases can be treated similarly.

The rest of the paper is devoted to proving the theorems. As outlined in the introduction, the proofs use ideas similar to those of Strang and consists of two major parts. First we construct an accurate approximate solution of the Broadwell equations by matched asymptotic analysis. The constructions of the approximation solutions vary according to the properties of the boundary layers. In particular, a linear hyperbolic wave is needed in the case of the compressive layers to preserve the conservation of mass and momentum for the approximate solutions, which turns out to be crucial in our subsequent stability analysis. The dynamic systems associated with the boundary-layer expansions and the initial-boundary-value problems for the system of hyperbolic partial differential equations associated with the interior Hilbert expansions have to be solved simultaneously order by order. Higher-order expansions must be obtained in order to justify the validity of the lower-order expansions. The next main part is to prove that the approximate solution constructed here is nonlinearly stable, which implies the desired convergence results. We present the analysis for the case of the diffusive boundary condition in great detail in the next two sections. For the case of the diffusive-reflective boundary conditions, we sketch the main steps and point out only the major differences with the previous case. In the next section, we present the Hilbert and boundary-layer expansions and their matching for both cases. The stability analysis for the diffusive boundary condition is given in §5. Finally we deal with the case of diffusive-reflective boundary condition in §6.

§4. Matched Asymptotic Analysis

We now carry out the construction of approximation solutions by matched asymptotic analysis. We first introduce some necessary notations for the Broadwell equations in §4.1. The outer solutions away from the boundary are obtained by the Hilbert expansions for both types of boundary conditions in §4.2. We remark here that the boundary conditions needed for the outer solutions must come from matching with the boundary-layer solutions; thus one has to obtain the outer solutions and the boundary-layer solutions order by order simultaneously. However, for simplicity of presentation, we carry out the expansions separately. The boundary-layer solutions in the case of diffusive boundary conditions are carried out in §4.3, and those in the case of diffusive-reflective boundary condition are constructed in §4.4. This yields the desired approximate solutions.

§4.1. Preliminaries

We use the following notations introduced in [7]. Set

$$\boldsymbol{f} = \begin{pmatrix} f^+ \\ f^0 \\ f^- \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.1}$$

$$q(\boldsymbol{g}, \boldsymbol{h}) = g^0 h^0 - \tfrac{1}{2}(g^+ h^- + g^- h^+), \quad Q(\boldsymbol{g}, \boldsymbol{h}) = q(\boldsymbol{g}, \boldsymbol{h}) \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix}. \tag{4.2}$$

Then the Broadwell equations (2.1) can be written as

$$(\partial_t + V\partial_x)\boldsymbol{f}_\varepsilon = \tfrac{1}{\varepsilon} Q(\boldsymbol{f}_\varepsilon, \boldsymbol{f}_\varepsilon). \tag{4.3}$$

The linearized collision operator at $\boldsymbol{f}$ is given by

$$L_f \equiv 2Q(\boldsymbol{f}, \cdot) = -\begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix} (f^-, -2f^0, f^+). \tag{4.4}$$

The left and right eigenvectors of L_f may be chosen respectively as

$$\begin{aligned} \psi_1^f &= (1, 4, 1), \\ \psi_2^f &= (1, 0, -1), \\ \psi_3^f &= \frac{1}{f^+ + f^0 + f^-}(f^-, -2f^0, f^+), \end{aligned} \tag{4.5}$$

and $(\phi_1^f, \phi_2^f, \phi_3^f)$ so that

$$\langle \psi_i^f, \phi_j^f \rangle = \delta_{ij}, \quad i, j = 1, 2, 3. \tag{4.6}$$

The projection operator into the null-space of L_f is denoted by P_f and has the form

$$P_f \boldsymbol{h} = \rho^h \phi_1^f + m^h \phi_2^f. \tag{4.7}$$

Denote the inverse of L_f in the range of $I - P_f$ by K_f, i.e.,

$$K_f \boldsymbol{h} = L_f^{-1}(I - P_f)\boldsymbol{h} = \frac{-\langle \psi_3^f, \boldsymbol{h} \rangle}{f^+ + f^0 + f^-} \phi_3. \tag{4.8}$$

§4.2. The Hilbert Expansions

Away from the boundary, it is expected that the Broadwell solution can be well approximated by the regular expansion in the mean-free path:

$$\boldsymbol{f}_\varepsilon \sim \boldsymbol{g} + \varepsilon \boldsymbol{g}_1 + \varepsilon^2 \boldsymbol{g}_2 + \varepsilon^3 \boldsymbol{g}_3 + \cdots. \tag{4.9}$$

Substituting this expansion into (2.1) and comparing the coefficients of equal powers of ε, one easily gets

$$Q(\boldsymbol{g},\boldsymbol{g}) = 0, \tag{4.10a}$$

$$2Q(\boldsymbol{g},\boldsymbol{g}_1) = (\partial_t + V\partial_x)\boldsymbol{g}, \tag{4.10b}$$

$$2Q(\boldsymbol{g},\boldsymbol{g}_2) + Q(\boldsymbol{g}_1,\boldsymbol{g}_1) = (\partial_t + V\partial_x)\boldsymbol{g}_1, \tag{4.10c}$$

$$2Q(\boldsymbol{g},\boldsymbol{g}_3) + 2Q(\boldsymbol{g}_1,\boldsymbol{g}_2) = (\partial_t + V\partial_x)\boldsymbol{g}_2. \tag{4.10d}$$

We now discuss the solvability of each equation in (4.10). We first observe from (4.10a) that $\boldsymbol{g} = \boldsymbol{g}(x,t)$ is a local Maxwellian state. The solvability condition for (4.10b) yields

$$P_g(\partial_t + V\partial_x)\boldsymbol{g} = 0. \tag{4.11}$$

Setting

$$\rho = \langle\psi_1,\boldsymbol{g}\rangle, \quad m = \langle\psi_2,\boldsymbol{g}\rangle, \tag{4.12}$$

we have from (4.11) that

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho u) = 0,\\ &\partial_t(\rho u) + \partial_x(\rho\sigma(u)) = 0. \end{aligned} \tag{4.13a}$$

These are exactly the model Euler equations (2.5). We solve system (4.13a) with initial data

$$\rho(x,0) = \rho_0(x), \quad m(x,0) = m_0(x) \tag{4.13b}$$

subject to the boundary condition

$$\mathscr{B}(\rho,u) = (\alpha+1)f_{\mathrm{b}}^{+}(t) + 2\alpha f_{\mathrm{b}}^{0}(t), \tag{4.13c}$$

or

$$u(s(t),t) = u_{\mathrm{b}}(t), \tag{4.13c$'$}$$

corresponding to the diffusive and diffusive-reflective boundary conditions for the Broadwell equations, respectively (see (2.22) and (2.27)). So the solution is taken to be the given fluid solution. With $\boldsymbol{g}$ thus determined, the solution to (4.10b) can be rewritten explicitly as

$$\boldsymbol{g}_1 = \tilde{\boldsymbol{g}}_1 + K_g(I - P_g)(\partial_t + V\partial_x)\boldsymbol{g}, \quad (I - P_g)\tilde{\boldsymbol{g}}_1 = 0. \tag{4.14}$$

To derive the differential equations governing $\boldsymbol{g}_1$, we set

$$\rho_1 = \langle\psi_1,\tilde{\boldsymbol{g}}_1\rangle, \quad m_1 = \langle\psi_2,\tilde{\boldsymbol{g}}_1\rangle \tag{4.15}$$

and obtain from the solvability condition for (4.10c) that

$$P_g(\partial_t + V\partial_x)\tilde{\boldsymbol{g}}_1 + P_gDK_g(I - P_g)(\partial_t + V\partial_x)\boldsymbol{g} = 0, \tag{4.16}$$

which can be rewritten in terms of ρ_1 and m_1 in (4.16) as

$$\begin{aligned}\partial_t\rho_1 + \partial_x m_1 &= 0,\\ \partial_t m_1 + \partial_x(\rho_1(\sigma(u) - u\sigma'(u)) + m_1\sigma'(u)) &= \partial_x(\mu(u)\partial_x u).\end{aligned} \tag{4.17a}$$

Note that the principal part of (4.17a) is the linearized part of the model Euler equation at the Maxwellian $\boldsymbol{g}$, and so (4.17) is a strictly hyperbolic system. We now solve this system with initial data

$$\rho_1(x,0) = 0, \quad m_1(x,0) = 0 \tag{4.17b}$$

and the boundary condition

$$\mathscr{B}_{\mathrm{d}}(\rho_1, m_1) = \tfrac{1}{2}\mu(u)\,\partial_x u - \partial_t \int_0^\infty (h^+ + 2h^0)\,d\xi \tag{4.17c}$$

corresponding to (2.12). Here $\mu(u)$ is given by (2.8b), h^+ and h^0 are the components of the leading-order function in the boundary-layer expansion in the next section, and the boundary operator is defined as

$$\mathscr{B}_{\mathrm{d}}(\rho_1, m_1) \equiv \frac{\rho_1}{2}\left(\alpha + \frac{1-\sigma(u)}{1+3\sigma(u)}\right) + \frac{m_1}{2}\left(1 + \alpha + \frac{4u}{1+3\sigma(u)}\right). \tag{4.18}$$

In the case of (2.13), the boundary condition takes the form

$$\mathscr{B}_{\mathrm{r}}(\rho_1, m_1) = \gamma_2\mu(u)\partial_x u + \partial_t \int_0^\infty (h^+ + 2(1+\gamma_1)h^0 + \gamma_1 h^-)\,d\xi, \tag{4.17c$'$}$$

$$\mathscr{B}_{\mathrm{r}}(\rho_1, m_1) \equiv \frac{\gamma_3\sigma(u) + \gamma_4}{3\sigma(u)+1}\rho_1 + \frac{\gamma_5\sigma(u) + \gamma_6 u + \gamma_7}{3\sigma(u)+1}m_1, \tag{4.19}$$

where γ_i $(1 \leqq i \leqq 7)$ are some constants given explicitly in terms of a, b, and α. The derivations of (4.17c), (4.18), (4.17c$'$) and (4.19) will be given in the next two sections as consequences of matching with boundary-layer expansions. Assuming this, we show at the end of this section that the initial-boundary-value problems, (4.17)–(4.19), are well-posed.

Following the same strategy, one can derive similar initial-boundary-value problems for $\boldsymbol{g}_2$ and $\boldsymbol{g}_3$. Details are omitted. Finally, setting

$$\boldsymbol{g}_4 = K_g(I - P_g)((\partial_t + V\partial_x)\boldsymbol{g}_3 - 2Q(\boldsymbol{g}_1, \boldsymbol{g}_3) - Q(\boldsymbol{g}_2, \boldsymbol{g}_2)), \tag{4.20}$$

we obtain a solution to (4.10e).

We now show that the initial-boundary-value problem (4.17) is indeed well-posed provided that h in (4.17c) is given. Rewrite the system in (4.17) as

$$\partial_t\begin{pmatrix}\rho_1\\ m_1\end{pmatrix} + \begin{pmatrix}0 & 1\\ \sigma(u) - u\sigma'(u) & \sigma'(u)\end{pmatrix}\partial_x\begin{pmatrix}\rho_1\\ m_1\end{pmatrix} + B(u)\begin{pmatrix}\rho_1\\ m_1\end{pmatrix} = f(u). \tag{4.21}$$

The eigenvalues $\lambda_\pm$ of this system are the same as those for the nonlinear Euler equation (2.5), and so

$$-1 < \lambda_1 < 0 < \lambda_2 < 1.$$

Let (ϕ_+, ϕ_-) be the characteristic variables for (4.21) so that

$$\phi_+ + \phi_- = \rho_1, \quad \lambda_2\phi_+ + \lambda_1\phi_- = m_1.$$

Equation (4.21) then becomes

$$\partial_t\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix} + \begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_1\end{pmatrix}\partial_x\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix} + B(u)\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix} = f(u).$$

It follows from the boundary condition and direct calculations that

$$\frac{\partial\mathscr{B}_\mathrm{d}}{\partial\phi_+} = \frac{2}{\alpha}(1 + g^+) + \frac{g^0 + \lambda_2(2g^+ + g^0)}{2(g^+ + g^0 + g^-)} > 0.$$

Consequently, by the implicit-function theorem one can represent the inflow ϕ_+ as a smooth function of the outflow ϕ_- and the boundary value $\mathscr{B}_\mathrm{d}$. Thus the initial-boundary-value problem (4.17) is well-posed.

§*4.3. Boundary-Layer Expansion I*

Next, we derive the boundary-layer solutions in the case of diffusive boundary conditions (2.12) for the Broadwell system (2.1). Near the boundary, the deviation of the Broadwell solution from the Euler solution is approximated by the singular expansion

$$\boldsymbol{h}(\xi,t) + \varepsilon\boldsymbol{h}_1(\xi,t) + \varepsilon^2\boldsymbol{h}_2(\xi,t) + \varepsilon^3\boldsymbol{h}_3(\xi,t) + \varepsilon^4\boldsymbol{h}_4(\xi,t) + \cdots, \tag{4.22}$$

where

$$\xi(x,t,\varepsilon) = \frac{x - s(t)}{\varepsilon}. \tag{4.23}$$

One can derive the governing equations for the boundary-layer solutions by requiring that the expression

$$\begin{aligned}\boldsymbol{f}_\varepsilon(x,t) \sim\; & \boldsymbol{g}(x,t) + \varepsilon\boldsymbol{g}_1(x,t) + \varepsilon^2\boldsymbol{g}_2(x,t) + \varepsilon^3\boldsymbol{g}_3(x,t) + \varepsilon^4\boldsymbol{g}_4(x,t) + \cdots\\ & + \boldsymbol{h}(\xi,t) + \varepsilon\boldsymbol{h}_1(\xi,t) + \varepsilon^2\boldsymbol{h}_2(\xi,t) + \varepsilon^3\boldsymbol{h}_3(\xi,t) + \varepsilon^4\boldsymbol{h}_4(\xi,t) + \cdots\end{aligned} \tag{4.24}$$

be a uniformly valid asymptotic solution for the initial-boundary-value problem of the Broadwell system. Substituting (4.24) into (2.1) yields the equations for each order of boundary-layer solutions:

$$(V + \alpha)\,\partial_\xi\boldsymbol{h} = Q(\boldsymbol{h},\boldsymbol{h}) + 2Q(\boldsymbol{g},\boldsymbol{h}), \tag{4.25a}$$

$$(V + \alpha)\,\partial_\xi\boldsymbol{h}_1 = 2Q(\boldsymbol{h} + \boldsymbol{g},\boldsymbol{h}_1) + 2Q(\boldsymbol{h}_1 + \xi\partial_x\boldsymbol{g},\boldsymbol{h}) - \partial_t\boldsymbol{h}, \tag{4.25b}$$

$$\begin{aligned}(V + \alpha)\,\partial_\xi\boldsymbol{h}_2 =\; & 2Q(\boldsymbol{h} + \boldsymbol{g},\boldsymbol{h}_2) + Q(\boldsymbol{h}_1,\boldsymbol{h}_1) + 2Q(\boldsymbol{g}_1 + \xi\partial_x\boldsymbol{g},\boldsymbol{h}_1)\\ & + 2Q(\boldsymbol{g}_2 + \xi\partial_x\boldsymbol{g}_1 + \tfrac{1}{2}\xi^2\partial_x^2\boldsymbol{g},\boldsymbol{h}) - \partial_t\boldsymbol{h}_1,\end{aligned} \tag{4.25c}$$

with similar equations for $\boldsymbol{h}_3$ and $\boldsymbol{h}_4$. The corresponding boundary conditions are

$$\begin{aligned} g^+(s(t),t) + h^+(0,t) &= f_b^+(t), \\ g^0(s(t),t) + h^0(0,t) &= f_b^0(t), \\ g_k^+(s(t),t) + h_k^+(0,t) &= 0, \quad k = 1,\dots,4, \\ g_k^0(s(t),t) + h_k^0(0,t) &= 0, \quad k = 1,\dots,4, \end{aligned} \tag{4.26}$$

$$\boldsymbol{h}_k(\xi,t) \to 0 \quad \text{uniformly as } \xi \to \infty, \quad k = 0,1,\dots,4. \tag{4.27}$$

The Hilbert and boundary-layer solutions can be obtained order by order separately. We start with the leading order term $\boldsymbol{h}$, which in §2 was shown to be

$$\begin{aligned} h^0(\xi) &= \frac{(f_b^0 - g^0)(g^0 - \tilde{g}^0)}{f_b^0 - \tilde{g}^0 - (f_b^0 - g^0)\exp(-C_\alpha(g^0 - \tilde{g}^0)\xi)} \exp(-C_\alpha(g^0 - \tilde{g}^0)\xi), \\ h^+(\xi) &= \frac{-2\alpha}{\alpha+1} h^0(\xi), \\ h^-(\xi) &= \frac{-2\alpha}{\alpha-1} h^0(\xi). \end{aligned} \tag{4.28}$$

where

$$\tilde{g}^0 \equiv -g^0 + \frac{2\alpha}{3\alpha^2+1}\rho(u+\alpha), \quad c_\alpha \equiv \frac{3\alpha^2+1}{2\alpha(1-\alpha^2)}. \tag{4.29}$$

Note that to simplify the notations, we use g^0, ρ, u to represent their corresponding values at the boundary $(s(t),t)$ in (4.29).

With $\boldsymbol{h}$ so determined, one can derive the boundary condition (4.17c), (4.18) for the first-order Hilbert solution and prove that this condition is a consequence of the matching conditions. Indeed, the solvability condition for (4.25b) with (4.27) yields

$$(\alpha+1)h_1^+ + 2\alpha h_1^0 = \partial_t \int_\xi^\infty (h^+ + 2h^0)\, d\xi. \tag{4.30}$$

It follows from this and boundary condition (4.26) that

$$(\alpha+1)g_1^+ + 2\alpha g_1^0 = -\partial_t \int_0^\infty (h^+ + 2h^0)\, d\xi. \tag{4.31}$$

On the other hand, one can write the first equation in (4.14) explicitly as

$$\boldsymbol{g}_1 = \rho_1 \phi_1^g + m_1 \phi_2^g - \tfrac{1}{2}\mu(u) u_x \phi_3. \tag{4.32}$$

This and direct calculations show that

$$(\alpha+1)g_1^+ + 2\alpha g_1^0 = \frac{\rho_1}{2}\left(\alpha + \frac{1-\sigma(u)}{1+3\sigma(u)}\right) + \frac{m_1}{2}\left(1 + \alpha + \frac{4u}{1+3\sigma(u)}\right) - \frac{1}{2}\mu(u)\,\partial_x u. \tag{4.33}$$

It should be noted that the first two terms on the right-hand side of (4.33) were defined to be the boundary operator in (4.18). Collecting (4.31) and (4.33) gives the desired boundary condition (4.17c). We note that the boundary condition (4.17c) only involves the leading-order Hilbert and boundary-layer solutions which have been completely determined, so that (4.17c) is well-defined.

Next we solve for the first-order boundary-layer solutions. Since $\boldsymbol{h}$ and $\boldsymbol{g}_1$ are given, integrating (4.25b) gives

$$\begin{aligned} h_1^0(\xi) &= -g_1^0 \exp\left(-C_\alpha \int_0^\xi (g^0 - \tilde{g}^0 + 2h^0)\,d\xi\right) \\ &\quad + \int_0^\xi \exp\left(-C_\alpha \int_{\xi'}^\xi (g^0 - \tilde{g}^0 + 2h^0)\,d\xi\right)\delta_0(\xi')\,d\xi', \\ h_1^+(\xi) &= \frac{-2\alpha}{\alpha+1}h_1^0(\xi) + \frac{1}{\alpha+1}\partial_t \int_\xi^\infty (h^+ + 2h^0)\,d\xi, \\ h_1^-(\xi) &= \frac{-2\alpha}{\alpha-1}h_1^0(\xi) + \frac{1}{\alpha-1}\partial_t \int_\xi^\infty (h^- + 2h^0)\,d\xi, \end{aligned} \tag{4.34}$$

where

$$\beta_0(\xi) = \frac{1}{1-\alpha^2}(h^0 + g^0)\,\partial_t \int_\xi^\infty (h^+ + 4h^0 + h^-)\,d\xi - \frac{1}{\alpha}q(\boldsymbol{g}_1 + \xi\partial_x \boldsymbol{g}, \boldsymbol{h}) - \frac{1}{\alpha}\partial_t h^0. \tag{4.35}$$

From the simple fact that

$$\int_\xi^\infty h^0\,d\xi = \frac{1}{c_\alpha}\ln\frac{f_b^0 - \tilde{g}^0}{f_b^0 - \tilde{g}^0 - (f_b^0 - g^0)\exp(-C_\alpha(g^0 - \tilde{g}^0)\,\xi)}$$

it follows that the boundary-layer solutions exponentially decay as $\xi \to \infty$.

Similarly, the solvability condition for (4.25c) and the matching conditions yield the desired boundary condition (4.23c), which in turn determines $\boldsymbol{g}_2$, and so the second-order boundary-layer solutions are given by

$$\begin{aligned} h_2^0(\xi) &= -g_2^0 \exp\left(-C_\alpha \int_0^\xi (g^0 - \tilde{g}^0 + 2h^0)\,d\xi\right) \\ &\quad + \int_0^\xi \exp\left(-C_\alpha \int_{\xi'}^\xi (g^0 - \tilde{g}^0 + 2h^0)\,d\xi\right)\beta_1(\xi')\,d\xi', \\ h_2^+(\xi) &= \frac{-2\alpha}{\alpha+1}h_2^0(\xi) + \frac{1}{\alpha+1}\partial_t \int_\xi^\infty (h_1^+ + 2h_1^0)\,d\xi, \\ h_2^-(\xi) &= \frac{-2\alpha}{\alpha-1}h_2^0(\xi) + \frac{1}{\alpha-1}\partial_t \int_\xi^\infty (h_1^- + 2h_1^0)\,d\xi, \end{aligned} \tag{4.36}$$

where

$$\beta_1(\xi) = \frac{1}{1-\alpha^2}(h^0 + g^0)\,\partial_t \int_\xi^\infty (h_1^+ + 4h_1^0 + h_1^-)\,d\xi - \frac{1}{2\alpha}q(\boldsymbol{h}_1, \boldsymbol{h}_1)$$

$$- \frac{1}{\alpha}q(\boldsymbol{g}_1 + \xi\partial_x \boldsymbol{g}, \boldsymbol{h}) - \frac{1}{\alpha}\,q(\boldsymbol{g}_2 + \xi\partial_x\boldsymbol{g}_1 + \frac{1}{2}\xi^2\partial_x^2\boldsymbol{g}, \boldsymbol{h}) - \frac{1}{\alpha}\partial_t\boldsymbol{h}_1. \quad (4.37)$$

It should be clear now that the higher-order solutions can be obtained in exactly the same way. In particular, one can compute $\boldsymbol{h}_3$ and $\boldsymbol{h}_4$ in detail. We note that all the higher-order boundary-layer solutions decay exponentially fast away from the boundary.

§*4.4. Boundary Layer Expansion II*

In the case of the diffusive-reflective boundary condition (2.13), the governing equations for the boundary-layer solutions are the same as in (4.25). The corresponding boundary conditions become

$$\begin{aligned} g_k^+(s(t),t) + h_k^+(0,t) &= a(t)(g_k^-(s(t),t) + h_k^-(0,t)), \quad k = 0,1,\ldots,4, \\ 4g_k^0(s(t),t) + 4h_k^0(0,t) &= b(t)(g_k^-(s(t),t) + h_k^-(0,t)), \quad k = 0,1,\ldots,4, \end{aligned} \quad (4.38)$$

$$\boldsymbol{h}_k(\xi,t) \to 0, \quad \text{uniformly as } \xi \to \infty,\ k = 0,1,\ldots,4. \quad (4.39)$$

The boundary-layer solutions of different orders can be obtained in a similar way as in the previous subsection. For example, the leading-order boundary-layer solutions are given by (4.28) and (4.29) with f_b^0 defined in our case as

$$f_b^0 = \frac{b}{2}\,\frac{(\alpha+1)g^+ + 2\alpha g^0}{\alpha + 2(1+\alpha)a}, \quad (4.40)$$

and the first-order boundary-layer solutions have the same forms as in (4.34) and (4.35) provided that one can justify the boundary conditions (4.17c′) and (4.19). This can be derived as follows. Note that the solvability condition for (4.25b) with (4.27) yields

$$\begin{aligned} (\alpha+1)h_1^+ + 2\alpha h_1^0 &= -\,\partial_t \int_\xi^\infty (h^+ + 2h^0)\,d\xi, \\ (\alpha-1)h_1^- + 2\alpha h_1^0 &= -\,\partial_t \int_\xi^\infty (h^+ + 2h^0)\,d\xi. \end{aligned} \quad (4.41)$$

Set

$$\gamma_1 = \frac{2a + 2\alpha a + \alpha b}{2 - 2\alpha - \alpha b}.$$

It follows from (4.41) that

$$(\alpha + 1)h_1^+ + 2(1+\gamma_1)\alpha h_1^0 + \gamma_1(\alpha - 1)h_1^- = -\partial_t \int_\xi^\infty (h^+ + 2(1+\gamma_1)h^0 + \gamma_1 h^-)\,d\xi$$

or

$$(\alpha + 1)(h_1^+ - ah_1^-) + \frac{\alpha}{2}(1+\gamma_1)(4h_1^0 - bh_1^-) = -\partial_t \int_\xi^\infty (h^+ + 2(1+\gamma_1)h^0 + \gamma_1 h^-)\,d\xi.$$

Combining this with the boundary condition (4.38) yields

$$(\alpha + 1)(g_1^+ - ag^-) + \frac{\alpha}{2}(1+\gamma_1)(4g_1^0 - bg_1^-) = -\partial_t \int_\xi^\infty (h^+ + 2(1+\gamma_1)h^0 + \gamma_1 h^-)\,d\xi. \tag{4.42}$$

Denoting the left-hand side of (4.42) as LHS, and using the expansion

$$\boldsymbol{g}_1 = \rho_1 \phi_1^g + m_1 \phi_2^g - \tfrac{1}{2}\mu(u)\,u_x \phi_3, \tag{4.43}$$

we compute that

$$\begin{aligned} 4(g^+ + g^0 + g^-)\,\mathrm{LHS} = {} & \rho_1(2(\alpha+1)(1-a)g^0 + \alpha(1+\gamma_1)(2g^+ - bg^0 + 2g^-)) \\ & + m_1(2(\alpha+1)(2g^+ + (1+a)g^0 + 2ag^-) \\ & + \alpha(1+\gamma_1)(-2g^+ + bg^0 + 2(1+b)g^-)) \\ & - \mu(u)u_x(2(\alpha+1)(1-a) - \alpha(2+b)(1+\gamma_1)) \\ & \times (g^+ + g^0 + g^-). \end{aligned}$$

In terms of macroscopic moments, this formula becomes

$$\mathrm{LHS} = \frac{\gamma_3\sigma(u) + \gamma_4}{3\sigma(u)+1}\rho_1 + \frac{\gamma_5\sigma(u) + \gamma_6 u + \gamma_7}{3\sigma(u)+1} m_1 - \gamma_2\,\mu(u)\,u_x, \tag{4.44}$$

where $\gamma_2, \ldots, \gamma_7$ are appropriate constants involving only a, b, and α. Combining (4.42) with (4.44) gives the desired boundary conditions (4.17c′) and (4.19). It can be verified very easily that all the boundary-layer solutions constructed above decay exponentially provided that $f_b^0 > \tilde{g}^0$, which holds true trivially for suitably small α.

§5. Stability Analysis I

In this section we prove the validity of the fluid-dynamic limit as stated in Theorems 3.1 and 3.2 in the case of diffusive boundary conditions for the Broadwell equations. The convergence analysis is done according to the structures of the boundary layers. The easier case, corresponding to the boundary layers satisfying $df^0/d\xi < 0$, is treated in §5.1. The main difficulty of stiffness across the boundary layer is overcome by making use of the nonnegative-definiteness of the normalized collision operator linearized around a carefully constructed approximate solution. The complementary case, corresponding to the boundary layer satisfying

$df^0/d\xi \geqq 0$, is more subtle and difficult. The method we are going to use is similar to that used in [11] and [18]. However, since a boundary does not satisfy an entropy condition as the shock layers do and, furthermore, since no smallness of the strength of boundary layers is assumed in our case, a more refined version of this method has to be used. This is carried out in §5.2. Finally, we study the fluid-dynamic limit for short time in §5.3.

§5.1. *Convergence Analysis for* $df^0/d\xi < 0$

To carry out the program outlined after Theorem 3.1, we use a truncation of the series (4.24) as our approximate solution to the initial-boundary-value problem (2.1), (2.11), and (2.12), so that we can decompose the solution $\boldsymbol{f}_\varepsilon(x,t)$ into the sum of the approximate solution with an error term $\varepsilon^2\tilde{\boldsymbol{e}}_\varepsilon(x,t)$, i.e.,

$$\begin{aligned}\boldsymbol{f}_\varepsilon(x,t) &= \boldsymbol{g}(x,t) + \varepsilon\boldsymbol{g}_1(x,t) + \varepsilon^2\boldsymbol{g}_2(x,t) + \varepsilon^3\boldsymbol{g}_3(x,t) + \varepsilon^4\boldsymbol{g}_4(x,t)\\ &\quad + \boldsymbol{h}(\xi,t) + \varepsilon\boldsymbol{h}_1(\xi,t) + \varepsilon^2\boldsymbol{h}_2(\xi,t) + \varepsilon^3\boldsymbol{h}_3(\xi,t) + \varepsilon^4\boldsymbol{h}_4(\xi,t) + \varepsilon^2\tilde{\boldsymbol{e}}_\varepsilon(x,t),\end{aligned} \tag{5.1}$$

where ξ is defined in (4.23). By construction, we have

$$\begin{gathered}(\partial_t + V\partial_x)\tilde{\boldsymbol{e}}_\varepsilon = \tfrac{1}{\varepsilon}L\tilde{\boldsymbol{e}}_\varepsilon + \varepsilon Q(\tilde{\boldsymbol{e}}_\varepsilon,\tilde{\boldsymbol{e}}_\varepsilon) + \tilde{L}_1\tilde{\boldsymbol{e}}_\varepsilon + \varepsilon^2\tilde{\boldsymbol{r}}_\varepsilon,\\ \tilde{\boldsymbol{e}}_\varepsilon(x,0) = 0,\\ \tilde{e}_\varepsilon^+(-\alpha t,t) = \tilde{e}_\varepsilon^0(-\alpha t,t) = 0,\end{gathered} \tag{5.2}$$

where

$$\begin{aligned}L &= L_{\boldsymbol{g}+\boldsymbol{h}} = 2Q(\boldsymbol{g}+\boldsymbol{h},\cdot),\\ \tilde{L}_1 &= 2Q(\boldsymbol{g}_1 + \boldsymbol{h}_1 + \varepsilon(\boldsymbol{g}_2+\boldsymbol{h}_2) + \varepsilon^2(\boldsymbol{g}_3+\boldsymbol{h}_3) + \varepsilon^3(\boldsymbol{g}_4+\boldsymbol{h}_4),\cdot),\end{aligned} \tag{5.3}$$

$$\begin{aligned}\tilde{\boldsymbol{r}}_\varepsilon &= 2Q(\xi\partial_x\tilde{\boldsymbol{g}}_4 + \tfrac{1}{2}\xi^2\partial_x^2\tilde{\boldsymbol{g}}_3 + \tfrac{1}{3!}\xi^3\partial_x^3\tilde{\boldsymbol{g}}_2 + \tfrac{1}{4!}\xi^4\partial_x^4\tilde{\boldsymbol{g}}_1 + \tfrac{1}{5!}\xi^5\partial_x^5\tilde{\boldsymbol{g}},\boldsymbol{h})\\ &\quad + 2Q(\xi\partial_x\tilde{\boldsymbol{g}}_3 + \tfrac{1}{2}\xi^2\partial_x^2\tilde{\boldsymbol{g}}_2 + \tfrac{1}{3!}\xi^3\partial_x^3\tilde{\boldsymbol{g}}_1 + \tfrac{1}{4!}\xi^4\partial_x^4\tilde{\boldsymbol{g}},\boldsymbol{h}_1)\\ &\quad + 2Q(\xi\partial_x\tilde{\boldsymbol{g}}_2 + \tfrac{1}{2}\xi^2\partial_x^2\tilde{\boldsymbol{g}}_1 + \tfrac{1}{3}\xi^3\partial_x^3\tilde{\boldsymbol{g}},\boldsymbol{h}_2)\\ &\quad + 2Q(\xi\partial_x\tilde{\boldsymbol{g}}_1 + \tfrac{1}{2}\xi^2\partial_x^2\tilde{\boldsymbol{g}},\boldsymbol{h}_3) + 2Q(\xi\partial_x\tilde{\boldsymbol{g}},\boldsymbol{h}_4)\\ &\quad + Q(2(\boldsymbol{g}_1+\boldsymbol{h}_1) + 2\varepsilon(\boldsymbol{g}_2+\boldsymbol{h}_2) + 2\varepsilon^2(\boldsymbol{g}_3+\boldsymbol{h}_3) + \varepsilon^3(\boldsymbol{g}_4+\boldsymbol{h}_4),\boldsymbol{g}_4+\boldsymbol{h}_4)\\ &\quad + Q(2(\boldsymbol{g}_2+\boldsymbol{h}_2) + \varepsilon(\boldsymbol{g}_3+\boldsymbol{h}_3),\boldsymbol{g}_3+\boldsymbol{h}_3).\end{aligned} \tag{5.4}$$

It follows from the structures of the Hilbert and boundary-layer solutions that $\tilde{\boldsymbol{r}}_\varepsilon$ are bounded in $W^{1,\infty}$.

Let P be the diagonal matrix defined by

$$P = \mathrm{diag}(\sqrt{g^- + h^-}, 2\sqrt{g^0+h^0}, \sqrt{g^+ + h^+}). \tag{5.5}$$

One can symmetrize the system in (5.2) by reformulating the problem in terms of the new variables

$$e_\varepsilon(x,t) = (P\tilde{e}_\varepsilon)(x+\alpha t, t), \tag{5.6}$$

as

$$\begin{gathered} \partial_t e_\varepsilon + (\alpha + V)\partial_x e_\varepsilon = \tfrac{1}{\varepsilon}\tilde{L}e_\varepsilon + \tfrac{\alpha}{\varepsilon}h^0_\xi B_\varepsilon e_\varepsilon + \varepsilon\Gamma(e_\varepsilon, e_\varepsilon) + L_1 e_\varepsilon + \varepsilon r_\varepsilon, \\ e_\varepsilon(x,0) = 0, \\ e^+_\varepsilon(0,t) = e^0_\varepsilon(0,t) = 0, \end{gathered} \tag{5.7}$$

where

$$\Gamma(f,h) = PQ(P^{-1}f,\, P^{-1}h), \tag{5.8a}$$

$$r_\varepsilon = P\tilde{r}_\varepsilon, \tag{5.8b}$$

$$\tilde{L} = PLP^{-1}, \tag{5.8c}$$

$$\begin{aligned} L_1 &= P\tilde{L}_1P^{-1} \\ &\quad + \frac{1}{2}\operatorname{diag}\left(\frac{(\partial_t + V\partial_x)g^- + \partial_t h^-}{g^- + h^-}, \frac{(\partial_t + V\partial_x)g^0 + \partial_t h^0}{g^0 + h^0}, \frac{(\partial_t + V\partial_x)g^+ + \partial_t h^+}{g^+ + h^+}\right) \end{aligned} \tag{5.8d}$$

$$B_\varepsilon = \operatorname{diag}\left(\frac{1+\alpha}{1-\alpha}\frac{1}{g^- + h^-}, \frac{1}{2(g^0 + h^0)}, \frac{1-\alpha}{1+\alpha}\frac{1}{g^+ + h^+}\right). \tag{5.8e}$$

It should be noted that the main advantage of the new error equations (5.7) over (5.2) is that

$$\tilde{L} \leqq 0,\ B_\varepsilon > 0. \tag{5.9}$$

Since (5.7) is a hyperbolic system, it is straightforward to prove the local existence and uniqueness for the initial-boundary-value problem (5.7) and (5.8) for fixed $\varepsilon > 0$ in the space

$$L^\infty([0,\tau], H^1(R_+)) \cap C([0,\tau], L^2(R_+)). \tag{5.10}$$

To obtain global existence and the desired convergence estimate, one needs only to derive an appropriate a priori estimate on the solutions to (5.7). Let e_ε be such a solution in the space defined in (5.10), and let e_ε satisfy

$$\sup_{0 \leqq t \leqq t_0} \|e_\varepsilon\|_{L^\infty} \leqq C, \tag{5.11}$$

where $t_0 \leqq T$, and the positive constant C is independent of ε. We now proceed to derive an a priori estimate. Taking the inner product of both sides of (5.7) with e_ε, integrating by parts, noting that the boundary layer satisfies $\partial_\xi h^0 < 0$, and assuming (5.9), the boundary conditions, and estimate on r_ε, and the a priori bounds (5.11) on e_ε, we arrive at the basic energy inequality

$$\partial_t \|e_\varepsilon\|^2_{L^2} + (1-\alpha)(e^-(0,t))^2 \leqq C\|e_\varepsilon\|^2_{L^2} + O(\varepsilon)\|e_\varepsilon\|_{L^2}. \tag{5.12}$$

Consequently, by the Gronwall inequality, we get

$$\sup_{0 \leqq t \leqq t_0} \|\boldsymbol{e}_\varepsilon\|_{L^2} \leqq O(\varepsilon). \tag{5.13}$$

To justify the a priori bounds (5.11) and obtain the rate of convergence, we need to estimate the higher-order derivatives. It turns out to be convenient to estimate the time derivative first. Set

$$\bar{\boldsymbol{e}}_\varepsilon = \partial_t \boldsymbol{e}_\varepsilon. \tag{5.14}$$

Differentiating (5.7) with respect to t gives

$$\begin{aligned} \partial_t \bar{\boldsymbol{e}}_\varepsilon + (\alpha + V)\partial_x \bar{\boldsymbol{e}}_\varepsilon = \tfrac{1}{\varepsilon}\tilde{L}\bar{\boldsymbol{e}}_\varepsilon + \tfrac{\alpha}{\varepsilon}h^0_\xi B_\varepsilon \bar{\boldsymbol{e}}_\varepsilon + 2\varepsilon\Gamma(\bar{\boldsymbol{e}}_\varepsilon, \boldsymbol{e}_\varepsilon) + L_1\bar{\boldsymbol{e}}_\varepsilon + \varepsilon\partial_t \boldsymbol{r}_\varepsilon \\ + \tfrac{1}{\varepsilon}(\tilde{L} + \alpha h^0_\xi B_\varepsilon + \varepsilon L_1)_t \boldsymbol{e}_\varepsilon + \varepsilon P_t Q(P^{-1}\boldsymbol{e}_\varepsilon, P^{-1}\boldsymbol{e}_\varepsilon) \\ + 2\varepsilon P Q(P_t^{-1}\boldsymbol{e}_\varepsilon, P^{-1}\boldsymbol{e}_\varepsilon), \end{aligned} \tag{5.15}$$

$$\bar{\boldsymbol{e}}_\varepsilon(x,0) = \varepsilon \boldsymbol{r}_\varepsilon,$$

$$\bar{e}^+_\varepsilon(0,t) = \bar{e}^0_\varepsilon(0,t) = 0,$$

where the initial data for $\bar{\boldsymbol{e}}_\varepsilon$ are obtained by using the equations in (5.7). Noting that time is a slow variable in (5.15) and taking into account the estimate (5.13), we obtain that

$$\partial_t \|\bar{\boldsymbol{e}}_\varepsilon\|^2_{L^2} \leqq O(\varepsilon) + C\|\bar{\boldsymbol{e}}_\varepsilon\|^2_{L^2} + C\|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2}. \tag{5.16}$$

It follows that

$$\sup_{0 \leqq t \leqq t_0} \|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2} \leqq C. \tag{5.17}$$

This and the equations in (5.7) yield

$$\sup_{0 \leqq t \leqq t_0} \|\partial_x \boldsymbol{e}_\varepsilon\|_{L^2} \leqq C \sup_{0 \leqq t \leqq t_0} (\|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2} + \tfrac{1}{\varepsilon}\|\boldsymbol{e}_\varepsilon\|_{L^2} + \varepsilon\|\boldsymbol{r}_\varepsilon\|_{L^2}) \leqq C. \tag{5.18}$$

Therefore, by the Sobolev inequality, we get the desired super-norm estimate, i.e.,

$$\sup_{0 \leqq t \leqq t_0} \|\boldsymbol{e}_\varepsilon\|_{L^\infty} \leqq \sup_{0 \leqq t \leqq t_0} \sqrt{\|\boldsymbol{e}_\varepsilon\|_{L^2}\|\partial_x \boldsymbol{e}_\varepsilon\|_{L^2}} \leqq O(\sqrt{\varepsilon}), \tag{5.19}$$

which not only justifies the a priori assumption (5.11), but also gives the desired convergence result. Hence the theorem is proved in this case.

§5.2. *Stability Analysis for $df^0/d\xi \geqq 0$*

In this case, a simple modification of the previous analysis does not suffice. We need a careful modification of the analysis used for the shock layer in [18, 11]. We assume in the rest of this section that the solution to the initial-boundary-value problem for the Euler equations (2.5), (2.22) lies in a δ_0-neighborhood of a global Maxwellian state (ρ^*, m^*, z^*), as do the boundary data f_b^+ and f_b^0.

As mentioned before, in order to exploit the property that $df^0/d\xi \geqq 0$ in the boundary layer, it is desirable to conserve the mass and momentum for the error terms. To this end, as in the case of the shock layer [18], we construct a linear hyperbolic wave in addition to the Hilbert and boundary-layer solutions in the approximate solution. The hyperbolic wave is defined to be the solution of

$$\begin{aligned}(\partial_t + V\partial_x)\boldsymbol{d} &= -(\partial_t + V\partial_x)\boldsymbol{g}_4 - \partial_t \boldsymbol{h}_4,\\ \boldsymbol{d}(x,0) &= 0, \\ d^+(s(t),t) &= d^0(s(t),t) = 0,\end{aligned} \tag{5.20}$$

where $\boldsymbol{g}_4$ and $\boldsymbol{h}_4$ are defined in (4.10) and (4.25), respectively. Since this system is linear, the solution can be obtained explicitly by integration along characteristic lines.

The approximate Broadwell solution and the decomposition of the exact solution to the initial-boundary-value problem (2.1), (2.11) and (2.12) can be defined in terms of the microscopic distributions in exactly the same way as in §5.1 with the term $\varepsilon^4\boldsymbol{d}(x,t)$ added. However, it turns out to be more convenient to work with the macroscopic variables. Thus, we define the approximate solution as

$$\begin{aligned}\underline{\theta}_\varepsilon(x,t) &= \theta(x,t) + \varepsilon\theta_1(x,t) + \varepsilon^2\theta_2(x,t) + \varepsilon^3\theta_3(x,t) + \varepsilon^4(\theta_4(x,t) + \boldsymbol{d})\\ &\quad + \Theta(\xi,t) + \varepsilon\Theta_1(\xi,t) + \varepsilon^2\Theta_2(\xi,t) + \varepsilon^3\Theta_3(\xi,t) + \varepsilon^4\Theta_4(\xi,t),\end{aligned} \tag{5.21}$$

where $\theta_k = (\rho_k, m_k, z_k)$ and $\Theta_k = (P_k, M_k, Z_k)$ are the Hilbert and boundary-layer solutions, respectively, which are constructed in §4, but written in terms of macroscopic variables. Substitute $\underline{\theta} = (\underline{\rho}, \underline{m}, \underline{z})$ into (2.3) to get

$$\begin{aligned}\partial_t\underline{\rho} + \partial_x\underline{m} &= 0,\\ \partial_t\underline{m} + \partial_x\underline{z} &= 0,\\ \partial_t\underline{z} + \partial_x\underline{m} &= \tfrac{1}{\varepsilon}q(\underline{\theta},\underline{\theta}) + \varepsilon^4\bar{r}_\varepsilon,\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}\bar{r}_\varepsilon &= 2q(\theta\partial_x\theta_{4b} + \tfrac{1}{2}\theta^2\partial_x^2\theta_{3b} + \tfrac{1}{3!}\theta^3\partial_x^3\theta_{2b} + \tfrac{1}{4!}\theta^4\partial_x^4\theta_{1b} + \tfrac{1}{5!}\theta^5\partial_x^5\theta_b, \Theta)\\ &\quad + 2q(\theta\partial_x\theta_{3b} + \tfrac{1}{2}\theta^2\partial_x^2\theta_{2b} + \tfrac{1}{3!}\theta^3\partial_x^3\theta_{1b} + \tfrac{1}{4!}\theta^4\partial_x^4\theta_b, \Theta_1)\\ &\quad + 2q(\theta\partial_x\tilde{\theta}_{2b} + \tfrac{1}{2}\theta^2\partial_x^2\theta_{1b} + \tfrac{1}{3}\theta^3\partial_x^3\theta_b, \Theta_2)\\ &\quad + 2q(\theta\partial_x\theta_{1b} + \tfrac{1}{2}\theta^2\partial_x^2\theta_b, \Theta_3) + 2Q(\theta\partial_x\theta_b, \Theta_4)\\ &\quad + q(2(\theta_1 + \Theta_1) + 2\varepsilon(\theta_2 + \Theta_2) + 2\varepsilon^2(\theta_3 + \Theta_3) + \varepsilon^3(\theta_4 + \Theta_4 + d_\varepsilon), \theta_4 + \Theta_4 + d_\varepsilon)\\ &\quad + q(2(\theta_2 + \Theta_2) + \varepsilon(\theta_3 + \Theta_3), \theta_3 + \Theta_3).\end{aligned} \tag{5.23}$$

Let $\theta_\varepsilon = (\rho_\varepsilon, m_\varepsilon, z_\varepsilon)$ be the solution of (2.1), (2.11), and (2.12). We write

$$\theta_\varepsilon(x,t) = \underline{\theta}_\varepsilon(x,t) + \varepsilon^2\tilde{\eta}_\varepsilon(x,t), \tag{5.24}$$

where $\tilde{\eta} = (\tilde{\phi}, \tilde{\psi}, \tilde{\omega})$. The error equation is then reduced to

$$
\begin{aligned}
&\partial_t \tilde{\phi}_\varepsilon + \partial_x \tilde{\psi}_\varepsilon = 0,\\
&\partial_t \tilde{\psi}_\varepsilon + \partial_x \tilde{\omega}_\varepsilon = 0,\\
&\partial_t \tilde{\omega}_\varepsilon + \partial_x \tilde{\psi}_\varepsilon = \tfrac{2}{\varepsilon} q(\underline{\theta}, \tilde{\eta}_\varepsilon) + \varepsilon q(\tilde{\eta}_\varepsilon, \tilde{\eta}_\varepsilon) + \varepsilon^2 r_\varepsilon,\\
&\tilde{\eta}_\varepsilon(x,0) = 0,\\
&\tilde{\psi}_\varepsilon(-\alpha t, t) + \tilde{\omega}_\varepsilon(-\alpha t, t) = \tilde{\phi}_\varepsilon(-\alpha t, t) - \tilde{\omega}_\varepsilon(-\alpha t, t) = 0.
\end{aligned}
\tag{5.25}
$$

Setting

$$
A_\varepsilon = \tfrac{1}{4}(\underline{\rho} - \underline{z}), \quad B_\varepsilon = \tfrac{1}{4}(\underline{\rho} + 3\underline{z}), \tag{5.26}
$$

we can transform the third equation of (5.25) into

$$
A_\varepsilon \tilde{\phi} + \underline{m}\tilde{\psi} - B_\varepsilon \tilde{\omega} = \varepsilon \partial_t \tilde{\omega}_\varepsilon + \varepsilon \partial_x \tilde{\psi}_\varepsilon - \varepsilon^2 q(\tilde{\eta}_\varepsilon, \tilde{\eta}_\varepsilon) - \varepsilon^3 r_\varepsilon. \tag{5.27}
$$

We reformulate the problem by using the substitutions

$$
\tilde{\phi}_\varepsilon = \bar{\phi}_x, \quad \tilde{\psi}_\varepsilon = \bar{\psi}_x, \quad \tilde{\omega}_\varepsilon = \bar{\omega}. \tag{5.28}
$$

In terms of $\bar{\eta} = (\bar{\phi}, \bar{\psi}, \bar{\omega})$, we have

$$
\begin{aligned}
&\partial_t \bar{\phi} + \partial_x \bar{\psi} = 0,\\
&\partial_t \bar{\psi} + \bar{\omega} = 0,\\
&B_\varepsilon \partial_t \bar{\psi} + \underline{m}\partial_x \bar{\psi} + A_\varepsilon \partial_x \bar{\phi} = \varepsilon(\partial_{xx}\bar{\psi} - \partial_{tt}\bar{\psi}) - \varepsilon^2 q(\bar{\phi}_x, \bar{\psi}_x, -\bar{\psi}_t) - \varepsilon^3 r_\varepsilon,\\
&\bar{\eta}(x,0) = 0,\\
&\bar{\psi}_x(-\alpha t, t) + \bar{\omega}(-\alpha t, t) = \bar{\phi}_x(-\alpha t, t) - \bar{\omega}(-\alpha t, t) = 0.
\end{aligned}
\tag{5.29}
$$

Equation (5.29) can be further simplified by introducing the scalings

$$
\bar{\phi}(x,t) = \varepsilon\phi(y,\tau), \quad \bar{\psi}(x,t) = \varepsilon\psi(y,\tau), \quad y = \frac{x + \alpha t}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}. \tag{5.30}
$$

We obtain

$$
\begin{aligned}
L_1(\phi,\psi) &\equiv \phi_r + \alpha\phi_y + \psi_y = 0,\\
L_2(\phi,\psi) &\equiv (\psi_r + \alpha\psi_y)_r + \alpha(\psi_r + \alpha\psi_y)_y - \psi_{yy} + A\phi_y + (m + \alpha B)\psi_y + B\psi_r\\
&= -\varepsilon^2 q(\phi_y, \psi_y, -(\psi_r + \alpha\psi_y)) + \varepsilon^3 r,\\
&\phi(y,0) = \psi(y,0) = \psi_r(y,0) = 0,\\
&\phi_y(0,\tau) + \psi_y(0,\tau) = 0,\\
&\phi(0,\tau) + \psi(0,\tau) = 0,
\end{aligned}
\tag{5.31}
$$

Our remaining task is to estimate the solution of (5.31). In what follows, we use $H^l(R_+)(l \geqq 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$, and $\|\ \|$ denotes the usual L_2-norm. Also we use k's to denote any positive constants which are independent of ε, y and τ.

Define the solution space for (5.31) by

$$X(0,\tau) = \{(\phi,\psi): (\phi,\psi) \in C(0,\tau: H^2(R_+)), \psi_\tau \in C(0,\tau: H^1(R_+))\} \tag{5.32}$$

with $0 < \tau < T/\varepsilon$. Suppose that for some $0 \leqq \tau_0 < T/\varepsilon$, there exists a solution (ϕ,ψ) to (5.31), such that $(\phi,\psi) \in X(0,\tau_0)$. We assume a priori that

$$\sup_{0 \leqq s \leqq \tau_0} (\|(\phi,\psi)\|_2 + \|\psi_\tau\|_1) \leqq C. \tag{5.33}$$

Then the main result in this subsection is the following a priori estimate.

Proposition 5.1 (A priori estimate). *Let* $(\phi,\psi) \in X(0,\tau_0)$ *be a solution to* (5.31) *satisfying* (5.33). *Then, there is an* $\varepsilon_0 > 0$ *such that*

$$\sup_{0 \leqq \tau \leqq \tau_0} (\|(\phi,\psi)(\tau)\|_2^2 + \|\psi_\tau(\tau)\|_1^2 + \int_0^{\tau_0} (\|(\phi_y,\psi_y)(\cdot,\tau)\|_1^2 + \|\psi_r(\cdot,\tau)\|_1^2)\, d\tau \leqq K\varepsilon \tag{5.34}$$

for $0 < \varepsilon \leqq \varepsilon$

The proof of this proposition occupies the rest of this subsection. We first list some properties of the approximate Broadwell solution $\underline{\theta}_\varepsilon$, which play important roles in our energy analysis later on. For any vector $e = (e_1, e_2, e_3)$ satisfying $e_3^2 > e_2^2$, define

$$\kappa_1(e) = \tfrac{1}{4}(e_1 + e_3 - 2\sqrt{e_3^2 - e_2^2}), \quad \kappa_2(e) = \tfrac{1}{4}(e_1 + e_3 + 2\sqrt{e_3^2 - e_2^2}). \tag{5.35}$$

Also set $\kappa_1^\varepsilon = \kappa_1(\underline{\theta}_\varepsilon)$ and $\kappa_2^\varepsilon = \kappa_2(\underline{\theta}_\varepsilon)$.

Lemma 5.1. *There exists a positive constant* ε_1 *such that if* $\varepsilon \leqq \varepsilon_1$, *then*

(1)
$$k_1 < z^\varepsilon < K_1, \quad k_2 < A^\varepsilon < K_2, \quad k_3 < B^\varepsilon < K_3, \qquad k_4 < \rho^\varepsilon < K_4, |m^\varepsilon| < \rho^\varepsilon \tag{5.36}$$

for some positive constants k *and* K, *independent of* t, ε, *and* x,

(2)
$$\max_{\varepsilon, y, \tau} A^\varepsilon \leqq \max_{\varepsilon, y, \tau} \kappa_1^\varepsilon < \min_{\varepsilon, y, \tau} \kappa_2^\varepsilon \leqq \min_{\varepsilon, y, \tau} B^\varepsilon, \tag{5.37}$$

(3)
$$-\alpha A_y^{-1} = \frac{\alpha(1-\alpha^2)}{4(f^0)^2} P_{0y} + O(\varepsilon). \tag{5.38}$$

Proof. Note that

$$\partial_y f^0 = \frac{1-\alpha^2}{4} \partial_y P_0, \tag{5.39}$$

which follows from (2.14). The proof of the lemma is now similar to that in [18]. □

We now proceed to derive the main estimate (5.34). We carry out the detailed analysis for the expansive layers, i.e., $u_b + \alpha \leqq 0$; the analysis for the case of the compressive layer is similar to that of the shock layer in [18], which we sketch at the end of this section for completeness. We start with the basic L_2 estimate. Set

$$\begin{aligned} I_1 &\equiv -\psi_y L_1 + A^{-1}(\psi_\tau + \alpha\psi_y)L_2 = \varepsilon^2 A^{-1}(\psi_\tau + \alpha\psi_y)(\varepsilon r - q), \\ I_2 &\equiv \phi L_1 + A^{-1}\psi L_2 = \varepsilon^2 A^{-1}\psi(\varepsilon r - q). \end{aligned} \tag{5.40}$$

Let λ be an appropriate constant to be determined later. Compute the expression $I_1 + \lambda I_2$ to get

$$\partial_\tau(E_1 + E_2 + E_3) + E_4 + E_5 + E_6 + E_7 + \partial_y E_8 = \varepsilon^2 A^{-1}(\psi_r + \alpha\psi_y + \lambda\psi)(\varepsilon r - q), \tag{5.41}$$

where

$$\begin{aligned} E_1 &= \tfrac{1}{2}A^{-1}(\lambda A\phi^2 - 2A\phi\psi_y + \psi_y^2) \equiv E_1(\phi, \psi_y), \\ E_2 &= \tfrac{1}{2}A^{-1}(\lambda B\psi^2 + 2\lambda\psi(\psi_\tau + \alpha\psi_y) + (\psi_\tau + \alpha\psi_y)^2) \\ &\equiv E_2(\psi, \psi_\tau + \alpha\psi_y), \\ E_3 &= -\tfrac{\lambda}{2}(A_\tau^{-1} + \alpha A_y^{-1})\psi^2 \equiv E_3(\psi), \\ E_4 &= A^{-1}((B - \lambda)(\psi_\tau + \alpha\psi_y)^2 + m(\psi_\tau + \alpha\psi_y)\psi_y + (\lambda - A)\psi_y^2) \\ &\equiv E_4(\psi_\tau + \alpha\psi_y, \psi_y), \\ E_5 &= \tfrac{\lambda}{2}\psi^2((1 + \alpha^2)A_{yy}^{-1} - (A^{-1}(m + \alpha B))_y) \equiv E_5(\psi), \\ E_6 &= -A_y^{-1}(\alpha(\psi_\tau + \alpha\psi_y)^2 + 2(\psi_\tau + \alpha\psi_y)\psi_y - \alpha\psi_y^2) + 2\alpha A_y^{-1}\psi\psi_y \\ &\equiv E_6(\psi, \psi_y, \psi_\tau + \alpha\psi_y). \\ E_7 &= \tfrac{\lambda}{2}\psi^2(A_{\tau\tau}^{-1} + 2\alpha A_{yy}^{-1} - (BA^{-1})_\tau - \tfrac{1}{2}A_\tau^{-1}((\psi_r + \alpha\psi_y)^2 + (\psi_y^2) - 2A\phi\psi_y) \\ &\equiv E_7(\phi, \psi, \psi_y, \psi_\tau + \alpha\psi_y), \\ E_8 &= \tfrac{\alpha}{2}A^{-1}(\psi_\tau + \alpha\psi_y)^2 - A^{-1}\psi_\tau\psi_y - \tfrac{\alpha}{2}A^{-1}\psi_y^2 + \psi_\tau\phi \\ &\quad + \lambda\alpha A^{-1}\psi(\psi_\tau + \alpha\psi_y) - \lambda A^{-1}\psi\psi_y + \tfrac{\lambda}{2}\alpha\phi^2 + \lambda\phi\psi \\ &\quad + \tfrac{\lambda}{2}\psi^2(A^{-1}(m + \alpha B) - (1 + \alpha^2)A_y^{-1} - \alpha A_\tau^{-1}) \\ &\equiv E_8(\phi, \psi, \psi_\tau, \psi_y, \psi_\tau + \alpha\psi_y). \end{aligned} \tag{5.42}$$

Each term can be estimated by using Lemma 5.1 as follows. First, by (5.37), one can choose λ so that

$$\max_{\varepsilon, y, \tau} A \leqq \max_{\varepsilon, y, \tau} \kappa_1 < \lambda < \min_{\varepsilon, y, \tau} \kappa_2 \leqq \min_{\varepsilon, y, \tau} B. \tag{5.43}$$

Hence,

$$A^2 - \lambda A < 0, \quad \lambda^2 - B\lambda < 0, \quad m^2 - \psi(B - \lambda)(\lambda - A) < 0. \tag{5.44}$$

Consequently, there exist positive constants k, K such that

$$\begin{aligned} k(\phi^2 + \psi_y^2) &\leqq E_1 \leqq K(\phi^2 + \psi_y^2), \\ k[\psi^2 + (\psi_\tau + \alpha\psi_y)^2] &\leqq E_2 \leqq K[\psi^2 + (\psi_\tau + \alpha\psi_y)^2], \\ k[\psi_y^2 + (\psi_\tau + \alpha\psi_y)^2] &\leqq E_4 \leqq K[\psi_y^2 + (\psi_\tau + \alpha\psi_y)^2]. \end{aligned} \tag{5.45}$$

Next, it follows from (5.38) that

$$E_3 \geqq k_5 |P_{0y}| \psi^2 - O(1)\varepsilon\psi^2. \tag{5.46}$$

To compute E_5, one computes

$$\begin{aligned} &(1+\alpha^2)\left(\frac{1}{f^0}\right)_{yy} - (A^{-1}(m+\alpha B))_y \\ &\quad = (1+\alpha^2)\frac{2(f_y^0)^2 - f^0 f_{yy}^0}{f_0^3} - \left(\frac{f^+ - f^- + \alpha(f^+ + f^0 + f^-)}{f^0}\right)_y. \end{aligned} \tag{5.47}$$

First, direct calculation using (2.33) gives

$$2(f_y^0)^2 - f^0 f_{yy}^0 = c_\alpha((f_\infty^0 + f_{-\infty}^0) f^0 - 2 f_\infty^0 f_{-\infty}^0) f_y^0.$$

Next, using (2.18) yields

$$\left(\frac{f^+ - f^- + \alpha(f^+ + f^0 + f^-)}{f^0}\right)_y = \left(\frac{c_1 + c_2}{f^0} - 3\alpha\right)_y = -\frac{(c_1 + c_2) f^0 f_y^0}{(f^0)^3}.$$

Thus,

$$(1+\alpha^2)(2(f_y^0)^2 - f^0 f_{yy}^0) + (c_1 + c_2) f^0 = \frac{2(c_1 + c_2)}{1-\alpha^2}(f^0 - f_\infty^0) + 2c_\alpha (f_\infty^0)^2$$

where we have used

$$c_\alpha (f_\infty^0 + f_{-\infty}^0) = \frac{c_1 + c_2}{1 - \alpha^2},$$

$$2c_\alpha f_\infty^0 f_{-\infty}^0 = -2c_\alpha (1-\alpha^2)(f_\infty^0)^2 + \frac{c_1 + c_2}{1-\alpha^2} f_\infty^0,$$

which follow from (2.31) and (2.32). Consequently,

$$(1+\alpha^2)\left(\frac{1}{f^0}\right)_{yy} - (A^{-1}(m+\alpha B))_y = 2f_y^0 \frac{(c_1 + c_2)(f^0 - f_\infty^0) + c_\alpha(1-\alpha^2)(f_\infty^0)^2}{(1-\alpha^2)(f^0)^3}.$$

This and (5.38) lead to

$$E_5 \geqq k_6 |P_{0y}| \psi^2 - O(1)\varepsilon\psi^2. \tag{5.48}$$

Next, it follows from the construction of $\underline{\theta}_\varepsilon$, (2.33), and the assumption that the Euler solution lies in a small δ_0-neighborhood of the Maxwellian state (ρ^*, m^*, z^*) that

$$|E_6| \leqq \frac{k_6}{2} |P_{0y}| \psi^2 + K(\delta_0^2 + \varepsilon)((\psi_\tau + \alpha\psi_y)^2 + \psi_y^2) + O(\varepsilon)\psi^2 \tag{5.49}$$

where K is a positive constant independent of δ_0 and ε. Finally,

$$E_7 \geqq -O(\varepsilon)((\psi_\tau + \alpha\psi_y)^2 + \psi_y^2 + \phi^2 + \psi^2). \tag{5.50}$$

Now integrate (5.41) over $R_+ \times [0, \tau]$, and use estimates (5.45)–(5.50) to show that

$$\begin{aligned}
&\|\phi(\tau)\|^2 + \|\psi(\tau)\|_1^2 + \|\psi_\tau + \alpha\psi_y\|^2 + \int_0^\infty |P_{0y}|\psi^2\,dy \\
&\quad + \int_0^\tau \|(\psi_y, \psi_\tau + \alpha\psi_y)\|^2\,ds + \int_0^\tau\int_0^\infty |P_{0y}|\psi^2\,dy\,ds - E_8(0) \\
&\leqq O(\varepsilon)\int_0^\tau \|(\phi, \psi, \psi_y, \psi_\tau + \alpha\psi_y)\|^2\,ds \\
&\quad + K(\delta_0^2 + \varepsilon)\int_0^\tau \|(\psi_y, \psi_\tau + \alpha\psi_y)\|^2\,ds \\
&\quad + K\varepsilon^3 \int_0^\tau\int_0^\infty (|\psi_\tau + \alpha\psi_y| + |\psi|)|r|\,dy\,ds \\
&\quad + K\varepsilon^3 \int_0^t\int_0^\infty (|\psi_\tau + \alpha\psi_y| + |\psi|)|q|\,dy\,ds.
\end{aligned} \tag{5.51}$$

It remains to deal with the boundary terms. As a consequence of the boundary conditions in (5.31), we find that

$$\begin{aligned}
\phi_y(0,\tau) + \psi_y(0,\tau) &= 0, \\
\phi(0,\tau) + \psi(0,\tau) &= 0, \\
\phi_\tau(0,\tau) + \psi_\tau(0,\tau) &= 0, \\
\phi_\tau(0,\tau) + \alpha\phi_y(0,\tau) + \psi_y(0,\tau) &= 0.
\end{aligned} \tag{5.52}$$

It follows that

$$\begin{aligned}
-E_8(0) &= (1-\alpha)A^{-1}\phi_y(0)^2 + (1-\alpha)(1+\lambda A^{-1})\phi_y(0)\phi(0) \\
&\quad + \tfrac{\lambda}{2}(2 - \alpha - A^{-1}(m + \alpha B) + (1+\alpha^2)A_y^{-1} + \alpha A_\tau^{-1})\phi(0)^2 \\
&= (1-\alpha)A^{-1}\phi_y(0)^2 + (1-\alpha)(1+\lambda A^{-1})\phi_y(0)\phi(0) \\
&\quad + \tfrac{\lambda}{2}(2 + 2\alpha - A^{-1}\rho(u+\alpha) + (1+\alpha^2)A_y^{-1} + \alpha A_\tau^{-1})\phi(0)^2 \\
&\equiv -E_8(\phi(0), \phi_y(0)).
\end{aligned} \tag{5.53}$$

This is a positive quadratic form for $(\phi(0), \phi_y(0))$ provided that λA^{-1} is close to 1, which is the case under our assumption that the macroscopic speed is much smaller than the microscopic speed.

The last two integrals on the right-hand side of (5.51) can be estimated as follows. First, by the Cauchy inequality and the structure of the approximate solution, we have

$$\varepsilon^3 \int_0^\tau \int (|\psi_\tau + \alpha\psi_y| + |\psi|)\,|r|\,dy\,ds$$
$$\leqq \varepsilon^3 \int_0^\tau \|(\psi, \psi_\tau + \alpha\psi_y)\|^2\,ds + \varepsilon^3 \int_0^\tau \|r\|^2\,ds$$
$$\leqq \varepsilon^3 \int_0^\tau \|(\psi, \psi_\tau + \alpha\psi_y)\|^2\,ds + O(\varepsilon)$$

for all $\tau \in [0, \tau_0]$. Next, Sobolev's inequality gives

$$\varepsilon^3 \int_0^\tau \int (|\psi| + |\psi_\tau + \alpha\psi_y|)\,|q|\,dy\,ds$$
$$\leqq O(\varepsilon^2) \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau + \alpha\psi_y)\|^2\,ds, \tag{5.54}$$

which yields the estimate on the nonlinear terms. We thus conclude from (5.51)–(5.54) that

$$\|\phi(\tau)\|^2 + \|\psi(\tau)\|_1^2 + \|(\psi_\tau + \alpha\psi_y)\|^2 + \int_0^\infty |P_{0y}|\psi^2\,dy$$
$$+ \int_0^\tau \|(\psi_y, \psi_\tau + \alpha\psi_y)\|^2\,ds + \int_0^\tau \int |P_{0y}|\psi^2\,dy\,ds$$
$$\leqq O(\varepsilon) \int_0^\tau (\|\phi\|^2 + \|\psi\|_1^2 + \|(\psi_\tau + \alpha\psi_y)\|^2)\,ds$$
$$+ O(\varepsilon^2) \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau + \alpha\psi_y)\|^2\,ds + O(\varepsilon) \tag{5.55}$$

for suitable small ε and δ_0.

The next step is to estimate ϕ_y. Calculate the identity

$$I_3 \equiv (\psi_\tau + \alpha\psi_y + \phi_y)\,\partial_y L_1 + \phi_y L_2 = \varepsilon^2 \phi_y(\varepsilon r(y,\tau) - q(\phi_y, \psi_y, -(\phi_\tau + \alpha\psi_y))) \tag{5.56}$$

to get

$$\partial_\tau(\tfrac{1}{2}\alpha^2\phi_y^2 + (\psi_\tau + \alpha\psi_y)\phi_y - \tfrac{1}{2}\psi_y^2) + A\phi_y^2 + \phi_y(B\psi_\tau + (m + \alpha B)\psi_y) + \partial_y E_9$$
$$= \varepsilon^3 \phi_y r(y,\tau) - \varepsilon^2 \phi_y q, \tag{5.57}$$

where

$$E_9 = \tfrac{1}{2}\alpha(\phi_y^2 + \psi_y^2) + \psi_\tau\psi_y + \alpha\phi_y(\psi_\tau + \alpha\psi_y) \equiv E_9(\phi_y, \psi_\tau, \psi_y). \tag{5.58}$$

Integrate (5.57) by parts over $R_+ \times [0,\tau]$ to show that

$$\|\phi_y(\tau)\|^2 + \int_0^\tau \|\phi_y\|^2\,ds - E_9(0)$$
$$\leqq O(1)\|(\psi_y, \psi_\tau + \alpha\psi_y)\|^2 + O(1)\int_0^\tau \|(\psi_y,\psi_\tau)\|^2\,ds$$
$$+ O(\varepsilon^2)\int_0^\tau \|(\phi_y,\psi_y,\psi_\tau+\alpha\psi_y)\|^2\,ds + O(\varepsilon^3)\int_0^\tau \|r\|^2\,ds. \tag{5.59}$$

It follows from the boundary relations (5.52) that

$$-E_9(0) = (1-\alpha)(\phi_y(0))^2 \equiv -E_9(\phi_y(0)) \geqq 0. \tag{5.60}$$

Combining (5.55) with (5.59) and using them successively, one finds that

$$\|\phi(\tau)\|_1^2 + \|\psi(\tau)\|_1^2 + \|(\psi_\tau(\tau)\|^2 + \int_0^\infty |P_{0y}|\psi^2\,dy$$
$$+ \int_0^\tau \|(\phi_y,\psi_y,\psi_\tau)\|^2\,ds + \int_0^\tau\int |P_{0y}|\psi^2\,dy\,ds$$
$$\leqq O(\varepsilon)\int_0^\tau (\|\phi,\psi\|_1^2 + \|\psi_\tau\|^2)\,ds + O(\varepsilon) \tag{5.61}$$

for some positive constant K. Applying Gronwall's inequality yields

$$\|\phi(\tau)\|_1^2 + \|\psi(\tau)\|_1^2 + \|(\psi_\tau + (\tau)\|^2 + \int_0^\infty |P_{0y}|\psi^2\,dy$$
$$+ \int_0^\tau \|(\phi_y,\psi_y,\psi_\tau)\|^2\,ds + \int_0^\tau\int |P_{0y}|\psi^2\,dy\,ds \leqq O(\varepsilon) \tag{5.62}$$

for all $\tau \in [0,\tau_0]$, $\tau_0 \leqq T/\varepsilon$.

To justify the a priori assumption (5.33), it is necessary to estimate the higher-order derivatives of (ϕ,ψ). This can be done in a way similar to that for the basic L^2 estimate. For completeness, we outline it here. Set

$$\begin{aligned} I_4 &\equiv -\psi_{y\tau}\partial_\tau L_1 + A^{-1}\partial_\tau(\psi_\tau+\alpha\psi_y)\partial_\tau L_2 = \varepsilon^2 A^{-1}\partial_\tau(\psi_\tau+\alpha\psi_y)(\varepsilon r_\tau - q_\tau),\\ I_5 &\equiv \phi_\tau\partial_\tau L_1(\phi,\psi) + A^{-1}\psi_\tau L_2(\phi,\psi) = \varepsilon^2 A^{-1}\psi_\tau(\varepsilon r_\tau - q_\tau),\\ I_6 &\equiv \partial_\tau(\psi_\tau+\alpha\psi_y+\phi_y)\,\partial_{\tau y}L_1(\phi,\psi) + \phi_{\tau y}\partial_\tau L_2(\phi,\psi) = \varepsilon^2\phi_{\tau y}(\varepsilon r_\tau - q_\tau). \end{aligned} \tag{5.63}$$

As in (5.40), (5.41), lengthy calculations show that the expression $I_4 + \lambda I_5$ becomes

$$\begin{aligned} &\partial_\tau(E_1(\phi_\tau,\psi_{\tau y}) + E_2(\psi_\tau,\psi_{\tau y}) + E_3(\psi_\tau))\\ &\quad + E_4(\psi_{\tau\tau}+\alpha\psi_{\tau y},\psi_{\tau y}) + E_5(\psi_\tau) + E_6(\psi_\tau,\psi_{\tau y},\psi_{\tau\tau}+\alpha\psi_{\tau y})\\ &\quad + E_7(\phi_\tau,\psi_\tau,\psi_{\tau y},\psi_{\tau\tau}+\alpha\psi_{\tau y}) + \partial_y E_8(\phi_\tau,\psi_\tau,\psi_{\tau\tau},\psi_{\tau y},\psi_{\tau\tau}+\alpha\psi_{\tau y})\\ &= \varepsilon^2 A^{-1}(\psi_\tau+\alpha\psi_y+\lambda\psi)_\tau(\varepsilon r_\tau - q_\tau) - E_{10}(\phi_y,\psi_\tau,\psi_y,\psi_\tau+\alpha\psi_y), \end{aligned} \tag{5.64}$$

where

$$E_{10} = A^{-1}A_\tau(\psi_\tau + \alpha\psi_y)_\tau\phi_y + A^{-1}(m + \alpha B)_\tau(\psi_\tau + \alpha\psi_y)_\tau\psi_y + A^{-1}B_\tau(\psi_\tau + \alpha\psi_y)\psi_\tau$$
$$+ \lambda A^{-1}A_\tau\psi_\tau\phi_y + \lambda A^{-1}(m + \alpha B)_\tau\psi_\tau\psi_y + \lambda A^{-1}B_\tau\psi_\tau\psi_\tau$$
$$\equiv E_{10}(\phi_y, \psi_\tau, \psi_y, \psi_\tau + \alpha\psi_y), \tag{5.65}$$

and where the constant λ is chosen as before. Similarly, we compute I_6 to get

$$\partial_\tau(\tfrac{1}{2}\phi_{y\tau}^2 + (\psi_{\tau\tau} + \alpha\psi_{y\tau})\phi_{y\tau} - \tfrac{1}{2}\psi_{y\tau}^2) + A\phi_{y\tau}^2$$
$$+ \phi_{y\tau}(B\psi_{\tau\tau} + (m + \alpha B)\psi_{y\tau}) + \partial_y E_9(\phi_{y\tau}, \psi_{\tau\tau}, \psi_{y\tau})$$
$$= \varepsilon^2\phi_{y\tau}(\varepsilon r_\tau - q_\tau) - E_{11}(\phi_y, \phi_{y\tau}, \psi_\tau, \psi_y), \tag{5.66}$$

where

$$E_{11} = A_\tau\phi_y\phi_{y\tau} + (m + \alpha B)_\tau\phi_{y\tau}\psi_y + B_\tau\phi_{y\tau}\psi_\tau \equiv E_{11}(\phi_y, \phi_{y\tau}, \psi_\tau, \psi_y). \tag{5.67}$$

Now integrating (5.64) and (5.66) over $R_+ \times [0, \tau]$, one can derive as in (5.51) and (5.59) that

$$\|(\phi_\tau(\tau)\|^2 + \|\psi_\tau(\tau)\|_1^2 + \|\psi_{\tau\tau} + \alpha\psi_{y\tau}\| + \int_0^{+\infty} |P_{0y}|\psi_\tau^2\,dy$$
$$+ \int_0^\tau \|(\psi_{y\tau}, \psi_{\tau\tau} + \alpha\psi_{y\tau})\|^2\,ds + \int_0^\tau\!\!\int |P_{0y}|\psi_\tau^2\,dy - E_8(0)$$
$$\leqq K(\delta_0^2 + \varepsilon)\int_0^\tau \|(\psi_{y\tau}, \psi_{\tau\tau} + \alpha\psi_{y\tau})\|^2\,ds$$
$$+ O(\varepsilon)\int_0^\tau \|(\phi_\tau, \psi_\tau)\|^2\,ds$$
$$+ \varepsilon^3\int_0^\tau\!\!\int |\psi_{\tau\tau} + \alpha\psi_{y\tau} + \lambda\psi_\tau|\,|r_\tau|\,dy\,ds$$
$$+ \varepsilon^2\int_0^\tau\!\!\int |\psi_{\tau\tau} + \alpha\psi_{y\tau} + \lambda\psi_\tau|\,|q_\tau|\,dy\,ds + \int_0^\tau |E_{10}|\,d\tau, \tag{5.68}$$

$$\|\phi_{y\tau}(\tau)\|^2 + \int_0^\tau \|\phi_{y\tau}\|^2\,ds - E_9(0)$$
$$\leqq K\|(\psi_{\tau\tau}, \psi_{y\tau})(\tau)\|^2 + O(\varepsilon^2)\int_0^\tau\!\!\int_0^\infty |\phi_{y\tau}|(\varepsilon|r_\tau| + |q_\tau|)\,dy\,ds + \int_0^\tau |E_{11}|\,d\tau. \tag{5.69}$$

As before, the boundary terms satisfy

$$E_8(0) = E_8(\phi_\tau(0), \phi_{\tau y}(0)) \leqq 0, \tag{5.70}$$
$$E_9(0) = E_9(\phi_{\tau y}(0)) \leqq 0. \tag{5.71}$$

The integrals involving r_τ in (5.68) and (5.69) can be estimated as follows:

$$\varepsilon^3 \int_0^\tau \int |\psi_{\tau\tau} + \alpha\psi_{y\tau} + \lambda\psi_\tau| |r_\tau| \, dy \, ds$$

$$\leqq \varepsilon \int_0^\tau \|\psi_{\tau\tau} + \alpha\psi_{y\tau}\|^2 \, ds + O(\varepsilon) \int_0^\tau \|\psi_\tau\|^2 \, ds + O(\varepsilon^5) \int_0^\tau \|r_\tau\|^2 \, ds$$

$$\leqq \varepsilon \int_0^\tau \|\psi_{\tau\tau} + \alpha\psi_{y\tau}\|^2 \, ds + O(\varepsilon) \int_0^\tau \|\psi_\tau\|^2 \, ds + O(\varepsilon^2). \tag{5.72}$$

Similarly,

$$\varepsilon^2 \int_0^\tau \int |\phi_{y\tau}| |r_\tau| \, dy \, ds \leqq \varepsilon \int_0^\tau \|\phi_{y\tau}\|^2 \, ds + K\varepsilon^2. \tag{5.73}$$

Using the structure of q and Sobolev's inequality leads to

$$\varepsilon^2 \int_0^\tau \int |\psi_{\tau\tau} + \alpha\psi_{y\tau} + \lambda\psi_\tau| |q_\tau| \, dy \, ds + \int_0^\tau \int |\phi_{y\tau}| |q_\tau| \, dy \, ds$$

$$\leqq O(\varepsilon^2) \int_0^\tau (\|\phi, \phi_\tau, \psi, \psi_\tau)\|_1^2 + \|\psi_{\tau\tau}\|^2) \, ds. \tag{5.74}$$

Collecting (5.68)–(5.74), we have

$$\|\phi_\tau(\tau)\|_1^2 + \|\psi_\tau(\tau)\|_1^2 + \|\psi_{\tau\tau}(\tau)\|^2 + \int_0^\tau (\|\phi_\tau, \psi_\tau)\|_1^2 + \|\psi_{\tau\tau}\|^2) \, ds + \int_0^\tau \int |P_{0y}| \psi_\tau^2 \, dy \, ds$$

$$\leqq O(\varepsilon) \int_0^\tau (\|\phi_\tau, \psi_\tau)\|_1^2 + \|\psi_{\tau\tau}\|^2) \, ds + O(\varepsilon^2) + K \int_0^\tau (|E_{10}| + |E_{11}|) \, d\tau. \tag{5.75}$$

On the other hand, simple calculation shows that

$$K \int_0^\tau |E_{10}| \, d\tau \leqq \tfrac{1}{2} \int_0^\tau \|\psi_{\tau\tau} + \alpha\psi_{y\tau}\|^2 \, d\tau + O(1) \int_0^\tau \|\phi_y, \psi_y, \psi_\tau\|^2 \, d\tau,$$

$$K \int_0^\tau |E_{11}| \, d\tau \leqq \tfrac{1}{2} \int_0^\tau \|\phi_{\tau y}\|^2 \, d\tau + O(1) \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau)\|^2 \, d\tau.$$

We finally arrive at

$$\|\phi_\tau(\tau)\|_1^2 + \|\psi_\tau(\tau)\|_1^2 + \|\psi_{\tau\tau}(\tau)\|^2$$

$$+ \int_0^\tau (\|\phi_\tau, \psi_\tau)\|_1^2 + \|\psi_{\tau\tau}\|^2) \, ds + \int_0^\tau \int |P_{0y}| \psi_\tau^2 \, dy \, ds \leqq K\varepsilon \tag{5.76}$$

for all $\tau \in [0, \tau_0], \tau_0 \leqq T/\varepsilon$.

It follows from this and the equations in (5.31) that

$$\|(\phi, \psi)(\tau)\|_2^2 + \|\psi_\tau(\tau)\|_1^2 \leqq K\varepsilon \tag{5.77}$$

for all $\tau \in [0, \tau_0]$, $\tau \leqq T/\varepsilon$.

This proves Proposition 5.1 in the case of the expansive layers, i.e., $u_b + \alpha \leqq 0$. For the compressive layers, i.e., $u_b + \alpha \geqq 0$, the previous analysis has to be combined with that for the shock layers in [18]. The major modification is that the E_5 and E_8 in (5.42) must be redefined as

$$\begin{aligned} E_5 &= -\tfrac{\lambda}{2}\psi^2(A^{-1}(m+\alpha B))_y \equiv E_5(\psi), \\ E_8 &= \tfrac{\alpha}{2}A^{-1}(\psi_\tau + \alpha\psi_y)^2 - A^{-1}\psi_\tau\psi_y - \tfrac{\alpha}{2}A^{-1}\psi_y^2 + \psi_\tau\phi + \lambda\alpha A^{-1}\psi(\psi_\tau + \alpha\psi_y) \\ &\quad - \lambda A^{-1}\psi\psi_y + \tfrac{\lambda}{2}\alpha\phi^2 + \lambda\phi\psi + \tfrac{\lambda}{2}\psi^2(A^{-1}(m+\alpha B) - \alpha A_\tau^{-1}) \\ &\equiv E_8(\phi, \psi, \psi_\tau, \psi_y, \psi_\tau + \alpha\psi_y). \end{aligned} \tag{5.78}$$

with corresponding changes in $E_8(0)$. Then the desired estimate can be obtained for suitably small α by imitating the previous analysis and that in [18]. Details are omitted. Thus the proof of Proposition 5.1 is completed.

Since the system in (5.31) is hyperbolic, it is standard to prove the local (in time) existence and uniqueness of the solution to the initial-boundary-value problem (5.31) in the space X (see (5.32)) [18]. From this and the a priori estimate Proposition 5.1, one may conclude, by using the standard continuous induction argument for hyperbolic equations, that the unique solution to problem (5.31) exists up to T/ε, and, furthermore, that the estimate (5.34) holds for all $\tau_0 \leqq T/\varepsilon$. Consequently, Theorem 3.1 follows from this, from the structures of our approximate solutions, and from the reformulation (5.24), (5.28), and (5.30).

§5.3. Convergence Analysis for Short Time

Finally we turn to the proof of Theorem 3.2 in the case of diffusive boundary condition (2.12). Since for short time, the boundary layer has not fully developed yet, its structure is not important. We can prove Theorem 3.2 by modifying the analysis given in §5.2. Since the strategy is exactly the same as in §5.2, we only sketch the necessary different estimates here. The crucial step is to show that the a priori estimate (5.24) holds for only $\tau_0 < T_0/\varepsilon$ with T_0 suitably small without conditions on the structure of the boundary layer and δ_0. It follows easily by checking the proof of Proposition 5.1 that one has only to estimate $E_6(\psi, \psi_y, \psi_\tau + \alpha\psi_y)$, $E_6(\psi_\tau, \psi_{\tau y}, \psi_{\tau\tau} + \alpha\psi_{\tau y})$, and $\int |P_{0y}|(\psi^2 + \psi_y^2)\,dy$. It follows from (5.42) that

$$|E_6(\psi, \psi_y, \psi_\tau + \alpha\psi_y)| \leqq O(1)|P_{0y}|\psi^2 + O(1)(|P_{0y} + \varepsilon)((\psi_\tau + \alpha\psi_y)^2 + \psi_y^2), \tag{5.79}$$

$$\begin{aligned} |E_6(\psi_\tau, \psi_{\tau y}, \psi_{\tau\tau} + \alpha\psi_{\tau y})| &\leqq O(1)|P_{0y}||\psi_\tau|^2 + O(1)(|P_{0y} + \varepsilon)((\psi_{\tau\tau} + \alpha\psi_{\tau y})^2 + \psi_{\tau y}^2) \\ &\quad + O(1)\varepsilon\psi_\tau^2. \end{aligned} \tag{5.80}$$

Let $\delta(t)$ denote the strength of the boundary layer, i.e., $\delta(t) = |f_b^0 - f_\infty^0|$. Then by the structure of the boundary layer (2.35) and compatibility condition, we have

$$\delta(0) = 0, \; |P_{0y}(y,t)| \leqq O(1)\delta(t), \quad \int_0^\infty |P_{0y}(y,t)|\,y\,dy < O(1)\delta(t). \tag{5.81}$$

Consequently, $\max_{0 \leqq t \leqq T_0} \delta(t)$ is sufficiently small if T_0 is suitably small. Using (5.81) and the simple inequality

$$|\psi(y,\tau)| \leqq |y|^{1/2} \| \psi(\cdot,\tau)\|_{L^2(R_+)},$$

we have

$$\int_0^\infty |P_{0y}|\psi^2\,dy \leqq \int_0^\infty |P_{0y}(y,t)|\,|y|\,\|\psi_y(\cdot,\tau)\|_{L^2}^2\,dy$$
$$\leqq O(1)\delta(t)\|\psi_y(\tau)\|_{L^2}^2, \tag{5.82}$$

$$\int_0^\infty |P_{0y}|\psi_\tau^2\,dy \leqq O(1)\delta(t)\|\psi_{y\tau}(\tau)\|_{L^2}^2. \tag{5.83}$$

Using (5.79), (5.81), (5.82), and the derivation of (5.62), one concludes that there exists a constant $T_0 > 0$ (suitably small) such that

$$\|\phi,\psi,\psi_\tau)(\tau)\|_1^2 + \int_0^\tau \|(\phi_y,\psi_y,\psi_\tau)(\tau)\|_{L^2}^2\,ds \leqq O(\varepsilon) \tag{5.84}$$

for all $\tau \in [0,\tau_0]$, $\tau_0 \leqq T_0/\varepsilon$. Similarly, it follows from (5.80), (5.81), (5.83), and the derivation of (5.76) that

$$\|\phi_\tau(\tau)\|_1^2 + \|\psi_\tau(\tau)\|_1^2 + \|\psi_{\tau\tau}(\tau)\|^2 + \int_0^\tau (\|\phi_\tau,\psi_\tau)\|_1^2 + \|\psi_{\tau\tau}\|^2)\,ds \leqq K\varepsilon \tag{5.85}$$

for all $\tau \in [0,\tau_0]$, $\tau_0 \leqq T_0/\varepsilon$ with suitably small $T_0 > 0$. Combining (5.84) with (5.85) yields the desired a priori estimate — Proposition 5.1 for all $\tau_0 \in (0, T_0/\varepsilon)$. Theorem 3.2 now follows easily as in the previous subsection.

§6. Stability Analysis II

In this section we treat the case of the diffusive-reflective boundary conditions. Since the stability analysis follows the same basic line of reasoning as for the case of purely diffusive boundary conditions except for the treatment of boundary terms, we only point out the main differences in the analysis and provide the key estimates involving the boundary terms.

§6.1. *Stability Analysis for $df^0/d\xi \leqq 0$*

The strategy is same as in §5.1. We only sketch the slight change involving the boundary conditions. In our case, instead of (5.7), the difference between the approximate and the exact solution of the Broadwell equations becomes

$$\partial_t \boldsymbol{e}_\varepsilon + (\alpha + V)\,\partial_x \boldsymbol{e}_\varepsilon = \tfrac{1}{\varepsilon}\tilde{L}\boldsymbol{e}_\varepsilon + \tfrac{\alpha}{\varepsilon}h_\xi^0 B_\varepsilon \boldsymbol{e}_\varepsilon + \varepsilon\Gamma(\boldsymbol{e}_\varepsilon,\boldsymbol{e}_\varepsilon) + L_1\boldsymbol{e}_\varepsilon + \varepsilon\boldsymbol{r}_\varepsilon,$$
$$\boldsymbol{e}_\varepsilon(x,0) = 0, \tag{6.1}$$
$$e_\varepsilon^+(0,t) - a(t)e_\varepsilon^-(0,t) = 4e_\varepsilon^0(0,t) - b(t)e_\varepsilon^-(0,t) = 0.$$

In the same way as for (5.13), one can show that

$$\partial_t \|\boldsymbol{e}_\varepsilon\|_{L^2}^2 + ((1-\alpha)(e^-(0,t))^2 - (1+\alpha)(e^+(0,t))^2 - \alpha(e^0(0,t))^2)$$
$$\leqq C\|\boldsymbol{e}_\varepsilon\|_{L^2}^2 + O(\varepsilon)\|\boldsymbol{e}_\varepsilon\|_{L^2}. \tag{6.2}$$

It follows from the boundary conditions in (6.1) that the second term on the left-hand side of this inequality becomes

$$(1-\alpha)(e^-(0,t))^2 - (1+\alpha)(e^+(0,t))^2 - \alpha(e^0(0,t))^2$$
$$= (1 - a^2 - \alpha(1 + a^2 + \tfrac{1}{16}b^2))(e^-(0,t))^2, \tag{6.3}$$

which is positive provided that

$$1 \geqq a^2 + \alpha(1 + a^2 + \tfrac{1}{16}b^2). \tag{6.4}$$

We note that inequality (6.4) is a consequence of our assumption that the macroscopic speed of the gas is suitably small compared with the microscopic speed of the gas particles. Indeed, in the case of the compressive layer, for which $u_b + \alpha \leqq 0$, (6.4) holds since $\alpha \leqq |u_b|$ is suitably small in this case. For the expansive layers, the inequality $u_b + \alpha \geqq 0$ implies that

$$\alpha(1 + a + b) + a \leqq 1.$$

Consequently, (6.4) holds without any assumption. We thus obtain the desired basic energy estimate (5.14). Note also that (6.2) yields

$$\int_0^t (e^-(0,s))^2\, ds \leqq O(\varepsilon^2). \tag{6.5}$$

The equations for the time derivative of the error

$$\bar{\boldsymbol{e}}_\varepsilon = \partial_t \boldsymbol{e}_\varepsilon \tag{6.6}$$

are the same as in (5.16), while the boundary conditions change to

$$\begin{aligned} \bar{e}_\varepsilon^+(0,t) &= a(t)\bar{e}_\varepsilon^-(0,t) + a'(t)e_\varepsilon^-(0,t), \\ 4\bar{e}_\varepsilon^0(0,t) &= b(t)\bar{e}_\varepsilon^-(0,t) + b'(t)e_\varepsilon^-(0,t). \end{aligned} \tag{6.7}$$

Thus an estimate like that used before gives

$$\partial_t \|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2}^2 + I \leqq O(\varepsilon) + C\|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2}^2 + C\|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2}, \tag{6.8}$$

where I represents all the terms involving boundary conditions:

$$\begin{aligned} I &= (1-\alpha)(\bar{e}^-(0,t))^2 - (1+\alpha)(\bar{e}^+\ (0,t))^2 - \alpha(\bar{e}^0(0,t))^2 \\ &= (1 - a^2 - \alpha(1 + a^2 + \tfrac{1}{16}b^2))(\bar{e}^-(0,t))^2 \\ &\quad - ((1+\alpha)a^2 + \tfrac{1}{16}\alpha b^2)'\bar{e}^- e^- - ((1+\alpha)(a')^2 + \tfrac{1}{16}\alpha(b')^2(\bar{e})^2. \end{aligned}$$

It follows from (6.4) and some easy manipulations that

$$I \geqq -C(e^-)^2. \tag{6.9}$$

Collecting (6.5), (6.8), and (6.9) yields

$$\sup_{0 \leqq t \leqq t_0} \|\bar{\boldsymbol{e}}_\varepsilon\|_{L^2} \leqq C. \tag{6.10}$$

Now the theorem can be proved in the same way as in §5.1.

§6.2. *Stability Analysis for $df^0/d\xi > 0$.*

The analysis in §5.2 can be repeated with the following notable exceptions. First, instead of (5.21), the linear hyperbolic wave is defined to be the solution of

$$\begin{aligned} (\partial_t + V\partial_x)\boldsymbol{d} &= -(\partial_t + V\partial_x)\boldsymbol{g}_4 - \partial_t \boldsymbol{h}_4, \\ \boldsymbol{d}(x,0) &= 0, \\ d^+(s(t),t) &= a(t)d^-(s(t),t), \\ 4d^0(s(t),t) &= b(t)d^-(s(t),t). \end{aligned} \tag{6.11}$$

The approximate and exact solutions to the problem (2.1), (2.11), and (2.13) are constructed exactly as in §5.2 (see (5.25)–(5.29)), while problem (5.30) becomes

$$\begin{aligned} \partial_t\bar\phi + \partial_x\bar\psi &= 0, \\ \partial_t\bar\psi + \bar\omega &= 0, \\ B_\varepsilon\partial_t\bar\psi + \underline{m}\partial_x\bar\psi + A_\varepsilon\partial_x\bar\phi &= \varepsilon(\partial_{xx} - \partial_{tt}\bar\psi) - \varepsilon^2 q(\bar\phi_x, \bar\psi_x, -\bar\psi_t) - \varepsilon^3 r_\varepsilon, \\ \bar\eta(x,0) &= 0, \\ (1 + a(t))\bar\psi_x(-\alpha t, t) + (1 - a(t))\bar\omega(-\alpha t, t) &= 0, \\ 2\bar\phi_x(-\alpha t, t) + b(t)\bar\psi_x(-\alpha t, t) - (2 + b(t))\bar\omega(-\alpha t, t) &= 0. \end{aligned} \tag{6.12}$$

We reformulate problem (6.12) by rescaling the problem as in (5.31). One needs to derive the boundary conditions for (ϕ, ψ) corresponding to those in (5.32). It follows from the boundary conditions in (6.12) that at the boundary $y = 0$,

$$\begin{aligned} (1 + a)\bar\psi_x + (1 - a)\bar\omega &= 0, \\ 2\bar\phi_x + b\bar\psi_x - (2 + b)\bar\omega &= 0. \end{aligned} \tag{6.13}$$

Eliminating ω gives

$$(1 + a)(2 + b)\bar\psi_x + (1 - a)(2\bar\phi_x + b\bar\psi_x) = 0$$

or

$$(1 - a)\bar\phi_x + (1 + a + b)\bar\psi_x = 0.$$

In terms of ϕ and ψ, we have

$$(1 - a)\phi_y + (1 + a + b)\psi_y = 0. \tag{6.14}$$

Now, substituting $\bar{\omega} = -(\psi_\tau + \alpha\psi_y)$ into (6.13) yields

$$(1+a)\psi_y - (1-a)(\psi_\tau + \alpha\psi_y) = 0.$$

This, together with

$$\phi_\tau + \alpha\phi_y + \psi_y = 0$$

leads to

$$\begin{pmatrix} 0 & 1+a-\alpha(1-a) \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} \phi_y \\ \psi_y \end{pmatrix} + \begin{pmatrix} 0 & a-1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \phi_\tau \\ \psi_\tau \end{pmatrix} = 0.$$

Combining this with (6.14) implies that

$$(1+a-\alpha(1-a))\phi_\tau + (1-a-\alpha(1+a+b))\psi_\tau = 0. \tag{6.15}$$

For simplicity of presentation, we assume that a and b do not depend on the time τ. Then it follows from (6.15) and the initial data that the

$$(1+a-\alpha(1-a))\phi + (1-a-\alpha(1+a+b))\psi = 0 \tag{6.16}$$

holds at the boundary. In particular, for the purely reflective boundary, $u = -\alpha$, and so $\phi = 0$ at the boundary. Hence the counterpart of (5.32) now becomes

$$\begin{aligned} &L_1(\phi,\psi) \equiv \phi_\tau + \alpha\phi_y + \psi_y = 0, \\ &L_2(\phi,\psi) \equiv (\psi_\tau + \alpha\psi_y)_\tau + \alpha(\psi_\tau + \alpha\psi_y)_y - \psi_{yy} + A\phi_y + (m+\alpha B)\psi_y + B\psi_\tau \\ &\qquad = -\varepsilon^2 q(\phi_y, \psi_y, -(\psi_\tau + \alpha\psi_y)) + \varepsilon^3 r, \\ &\qquad \phi(y,0) = \psi(y,0) = \psi_\tau(y,0) = 0, \\ &(1+a-\alpha(1-a))\phi(0,\tau) + (1-a-\alpha(1+a+b))\psi(0,\tau) = 0, \\ &\qquad (1-a)\phi_y(0,\tau) + (1+a+b)\psi_y(0,\tau) = 0. \end{aligned} \tag{6.17}$$

The a priori estimate on the solution of (6.17), as in Proposition 5.1, can be derived as in §5.2, except for the treatment of the boundary terms, which are described by

$$\begin{aligned} E_8 = {} & \tfrac{\alpha}{2}A^{-1}(\psi_\tau + \alpha\psi_y)^2 - A^{-1}\psi_\tau\psi_y - \tfrac{\alpha}{2}A^{-1}\psi_y^2 + \psi_\tau\phi + \lambda\alpha A^{-1}\psi(\psi_\tau + \alpha\psi_y) \\ & - \lambda A^{-1}\psi\psi_y + \tfrac{\lambda}{2}\alpha\phi^2 + \lambda\phi\psi + \tfrac{\lambda}{2}\psi^2(A^{-1}(m+\alpha B) - (1+\alpha^2)A_y^{-1} - \alpha A_\tau^{-1}). \end{aligned}$$

To estimate E_8 at the boundary, we note that

$$\begin{aligned} \psi_\tau + \alpha\psi_y &= \frac{1+a}{1-a}\psi_y, \\ \psi_\tau &= \frac{1+a-\alpha(1-a)}{1-a}\psi_y, \\ \phi &= \frac{\alpha(1+a+b)+a-1}{1+a-\alpha(1-a)}\psi. \end{aligned} \tag{6.18}$$

Hence,

$$
\begin{aligned}
E_8(0) = \frac{\alpha}{2}A^{-1} &= \frac{(1+a)^2}{(1-a)^2}\psi_y^2 - \frac{1+a-\alpha(1-a)}{1-a}A^{-1}\psi_y^2 - \frac{\alpha}{2}A^{-1}\psi_y^2 \\
&+ \frac{1+a-\alpha(1-a)}{(1-a)}\frac{\alpha(1+a+b)+a-1}{1+a-\alpha(1-a)}\psi\psi_y + \lambda\alpha A^{-1}\frac{1+a}{1-a}\psi\psi_y \\
&- \lambda A^{-1}\psi\psi_y + \frac{\lambda}{2}\alpha\left(\frac{\alpha(1+a+b)+a-1}{1+a-\alpha(1-a)}\right)^2\psi^2 \\
&+ \lambda\frac{\alpha(1+a+b)+a-1}{1+a-\alpha(1-a)}\psi^2 \\
&+ \frac{\lambda}{2}\psi^2(A^{-1}(m+\alpha B) - (1+\alpha^2)A_y^{-1} - \alpha A_\tau^{-1}).
\end{aligned}
\tag{6.19}
$$

Set

$$
\begin{aligned}
\mu_0 &= a + \alpha(1+a+b) - 1, \\
\mu_1 &= \alpha(1+a) - 1 + a, \\
\mu_2 &= 1 + a - \alpha(1-a),
\end{aligned}
\tag{6.20}
$$

and also use the notations:

$$
\begin{aligned}
I_1 &= A^{-1}\frac{1-\alpha-a^2(1+\alpha)}{(1-a)^2}, \\
I_2 &= \frac{\lambda}{2}\bigg(\frac{-\mu_0}{\mu_2^2}(\alpha(3a-3+\alpha(1+a+b))+2(1+a)) \\
&\qquad - A^{-1}\rho(u+\alpha) + 3\alpha - (1+\alpha^2)A_y^{-1} - \alpha A_\tau^{-1}\bigg).
\end{aligned}
\tag{6.21}
$$

We now rewrite (6.19) as

$$
-E_8(0) = I_1\psi_y^2 + \frac{\mu_0 + \lambda A^{-1}\mu_1}{1-a}\psi\psi_y + I_2\psi^2,
\tag{6.22}
$$

which can be regarded as a quadratic form for ψ_y and ψ. To show that this is positive-definite, we need to check the positivity of I_1, I_2, and the determinant of the quadratic form defined as

$$
\begin{aligned}
I_3 &= 2\lambda A^{-1}(1-\alpha-a^2(1+\alpha)) \\
&\quad \times\bigg(3\alpha + \frac{-\mu_0}{\mu_2^2}(\alpha(3a-3+\alpha(1+a+b))+2(1+a)) \\
&\quad - A^{-1}\rho(u+\alpha) - (1+\alpha^2)A_y^{-1} - \alpha A_\tau^{-1}\bigg) - (\mu_0 + \lambda A^{-1}\mu_1)^2.
\end{aligned}
\tag{6.23}
$$

There are two cases. First we treat the expansive boundary layer. Then $\mu_0 \leqq 0$ (see (2.26) and (2.35)), so that

$$1 - \alpha - a^2(\alpha + 1) = -(1 + a)\mu_0 + \alpha(2a + ab + b) > 0, \tag{6.24}$$

$$\alpha(3a - 3 + \alpha(1 + a + b)) + 2(1 + a) = (1 - \alpha)(2 - \alpha) + a(1 + \alpha)(2 + \alpha) + b\alpha^2 > 0.$$

It follows that both I_1 and I_2 are positive since $u + \alpha \leqq 0$, and (5.39) and (5.40) hold. To show that $I_3 > 0$, one computes that at the boundary

$$A^{-1}\rho(u + \alpha) = \frac{4}{b}\mu_0, \tag{6.25}$$

and so

$$\begin{aligned} I_3 = {} & 2(-\mu_0(1 + a) + \alpha(2a + ab + b)) \\ & \times\left(3\alpha + \frac{4}{b}\mu_0 + \frac{-\mu_0}{\mu_2^2}((1 - \alpha)(2 - \alpha) + a(1 + \alpha)(2 + \alpha) + \alpha b \right. \\ & \left. - (1 + \alpha^2)A_y^{-1} - \alpha A_\tau^{-1})\right) - (2\mu_0 + \alpha b)^2. \end{aligned}$$

Note that

$$(1 + a)\frac{8}{b}u_0^2 \geqq 8\mu_0^2,$$

$$2\alpha(2a + ab + b)(3\alpha) \geqq 6\alpha^2 b \geqq 2\alpha^2 b^2.$$

Consequently,

$$I_3 > 0. \tag{6.26}$$

This completes the boundary estimate in the case of the expansive boundary layer. Next we turn to the study of the compressive boundary layer, in which case we have $\mu_0 \geqq 0$. For technical reasons, we assume further that $\mu_0 \leqq \alpha b/4$ and $\alpha \leqq \frac{1}{4}$. Then

$$-\alpha < u \leqq -\alpha - \frac{\alpha b}{4(1 + a + b)}. \tag{6.27}$$

It is easy to see that the numerator of I_1 is estimated by

$$-(1 + a)\mu_0 + \alpha(2a + ab + b) \geqq (ab - \mu_0)(1 + a) \geqq \tfrac{3}{4}\alpha b(1 + a) > 0. \tag{6.28}$$

Since $\mu_0 \geqq 0$ and $\alpha \leqq \frac{1}{4}$, we have

$$a \geqq 1 - \alpha(1 + a + b) \geqq 1 - 3\alpha \geqq \frac{\alpha}{\alpha + 1}.$$

This implies that $\mu_2 \geqq 1$ in our case. The sum of the first three terms in I_2 is then bounded from below by

$$\frac{\lambda}{2}\left(3\alpha - \frac{4\mu_0}{b} - \mu_0(\alpha(3a - 3 + \alpha(1 + a + b)) + 2(1 + a))\right)$$

$$\geqq \frac{\lambda}{2}\left(2\alpha - \frac{1}{4}\alpha b(3\alpha^2 + 2 + 2a)\right) \geqq \frac{3}{8}\lambda\alpha > 0, \tag{6.29}$$

so that I_2 is positive. Arguing in a similar way, we obtain that

$$2\lambda A^{-1}(1 - \alpha - a^2(1 + \alpha)) \geqq 2\alpha b,$$

$$3\alpha + \frac{-\mu_0}{\mu_2^2}(\alpha(3a - 3 + \alpha(1 + a + b)) + 2(1 + a))$$
$$- A^{-1}\rho(u + \alpha) - (1 + \alpha^2)A_y^{-1} - \alpha A_\tau^{-1} \geqq \tfrac{3}{2}\alpha,$$

$$(\mu_0 + \lambda A^{-1}\mu_1)^2 = (\mu_0 + \lambda A^{-1}(\mu_0 - \alpha b))^2 \leqq \tfrac{1}{4}\alpha^2 b^2.$$

This immediately yields

$$I_3 \geqq 3\alpha^2 b - \tfrac{1}{4}\alpha^2 b^2 > 0. \tag{6.30}$$

In summary, we have shown that $E_8(0) \leqq 0$ in all the cases. Since the boundary estimates for the higher-order derivatives can be obtained in exactly the same way, the rest of the analysis for the derivation of the priori estimate for the solution of (6.17), as described in Proposition 5.1, can be carried out in the same way as in §5.2 with a few slight modifications, so that the proof of Theorem 3.1 in this case is similar to those in §5.2. Details are omitted.

Finally, one can prove Theorem 3.2 in the case of diffusive-reflexive boundary conditions as in §5.3 combined with the modification in this subsection. We note that all the conditions on the structure of the boundary layer and the technical assumption made in the previous analysis can be avoided by using the fact that the boundary layers are weak for short time. Thus the proof of Theorem 3.2 is complete.

Appendix A. Proof of Lemmas 2.1 and 2.2

Proof of Lemma 2.1. Assume that $a \geqq 1/3$, $b \leqq 2/3$, and $\alpha \leqq 1/\sqrt{3}$. We show that there is a unique solution u_b to

$$(\alpha + u_b)(1 + a - \alpha(1 - a)) = (\sigma(u_b) + \alpha u_b)(1 - a - \alpha(1 + a + b)) \tag{A.1}$$

such that

$$|u_b| < 1, \quad \lambda_1(u_b) < -\alpha. \tag{A.2}$$

Since $\lambda_1(u)$ is monotone in u, it is easy to check that (A.2) is equivalent to

$$-1 < u < u_\alpha, \tag{A.3a}$$

where

$$u_\alpha = \frac{-4\alpha + \sqrt{16\alpha^2 + (1-3\alpha^2)^3(1+\alpha^2)}}{(1-3\alpha^2)^2}, \tag{A.3b}$$

which is the solution of $\lambda_1(u_\alpha) = -\alpha$ satisfying $|u_\alpha| < 1$.

Setting

$$\begin{aligned} a_1 &= 1 - a - \alpha(1 + a + b), \\ a_2 &= \alpha(1 - a - \alpha(1 + a + b)) - (1 + a) + \alpha(1 - a), \\ a_3 &= \alpha(1 + a - \alpha(1 - a)), \end{aligned} \tag{A.4}$$

we can then rewrite (A.1) as

$$a_1\sigma(u_b) + a_2 u_b = a_3,$$

or equivalently,

$$2a_1\sqrt{1+3u^2} = a_1 + 3a_3 - 3a_2 u. \tag{A.5}$$

Squaring (A.5) leads to

$$(9a_2^2 - 12a_1^2)u^2 - 6(a_1 + 3a_3)a_2 u + (a_1 + 3a_3)^2 - 4a_1^2 = 0. \tag{A.6}$$

Equation (A.6) has real roots if and only if

$$12^2 a_1^2(a_2^2 + 2a_1a_3 + 3a_3^2 - a_1^2) \geqq 0, \tag{A.7}$$

and the two roots are

$$u_\pm = \frac{a_2(a_1 + 3a_3) \pm 2|a_1|\sqrt{a_2^2 + 2a_1a_3 + 3a_3^2 - a_1^2}}{3a_2^2 - 4a_1^2}. \tag{A.8}$$

We now show that u_+ satisfies both (A.3a) and (A.5). First, we check (A.7). Since

$$a_2 = \alpha a_1 - \frac{1}{\alpha}a_3, \tag{A.9}$$

we have

$$a_2^2 + 2a_1a_3 + 3a_3^2 - a_1^2 = \frac{1}{\alpha^2}(a_3^2 - \alpha^2\alpha_1^2 + 3\alpha^2a_3^2 + \alpha^4a_1^2).$$

On the other hand,

$$\begin{aligned} a_3^2 - \alpha^2a_1^2 &= 4\alpha^2(1-\alpha^2)a - \alpha^4b^2 + 2\alpha^3b(1 - \alpha - a(1+\alpha)) \\ &= 4\alpha^2(1-\alpha^2)a + b^2\alpha^4 + 2\alpha^3ba_1 \\ &\geqq 4\alpha^2(1-\alpha^2)a + b^2\alpha^4 - \alpha^2b^2 - \alpha^4a_1^2 \\ &= \alpha^2(1-\alpha^2)(4a - b^2) - \alpha^4a_1^2 \\ &\geqq -\alpha^4a_1^2, \end{aligned}$$

where we have assumed that $b \leqq \frac{2}{3}$ and $a \geqq \frac{1}{3}$. It follows that

$$a_2^2 + 2a_1a_3 + 3a_3^2 - a_1^2 \geqq 3a_3^2 \geqq 0, \tag{A.10}$$

which proves (A.7). In the following argument, we use the fact that

$$a_2 < 0. \tag{A.11}$$

In fact, due to (A.9) this is trivial in the case $a_1 < 0$. For $a_1 > 0$, one has

$$a_2 = \alpha a_1 - \frac{1}{\alpha} a_3 \leqq \frac{3\alpha^2 - 1}{\alpha} a_3 < 0$$

provided that $3a_3 - a_1 \geqq 0$. When $3a_3 - a_1 < 0$, it follows from

$$a_2^2 + 2a_1a_3 + 3a_3^2 - a_1^2 = a_2^2 + (a_1 + a_3)(3a_3 - a_1) \geqq 0 \tag{A.12}$$

that $a_2 \neq 0$. Moreover, at $\alpha = 0$, we have

$$a_2 = -1 - a < 0, \quad 3a_3 - a_1 = a - 1 < 0.$$

Note further that $3a_3 - a_1$ is an increasing function of α on the interval $(0, 1/\sqrt{3})$. Consequently (A.11) holds.

To show (A.3) for u_+, we rewrite u_+ as

$$u_+ = \frac{(a_1 + a_3)(3a_3 - a_1)}{a_2(a_1 + 3a_3) - 2|a_1|\sqrt{a_2^2 + (a_1 + a_3)(3a_3 - a_1)}}. \tag{A.13}$$

One checks easily that the denominator in (A.13) is always negative. Indeed, when either (i) $a_1 > 0$, or (ii) $a_1 < 0$ with $a_1 + 3a_3 \geqq 0$, this follows trivially since a_2 is negative. The case that $a_1 + 3a_3 < 0$ never occurs for $\alpha \in (0, 1/\sqrt{3})$, which follows by a simple calculation.

We now show that $-1 < u_+$. Since the denominator in (A.13) is negative, it suffices to show that

$$(a_1 + a_3)(3a_3 - a_1) < |a_2|(a_1 + 3a_3) + 2|a_1|\sqrt{a_2^2 + (a_1 + a_3)(3a_3 - a_1)}. \tag{A.14}$$

There are two cases:

Case 1: $a_1 > 0$. In this case, it is sufficient to prove (A.14) for $3a_3 - a_1 \geqq 0$. Assuming that $3\alpha^2 < 1$, we compute that

$$3a_3 - a_1 < \frac{1}{\alpha}(3a_3 - a_1) < \frac{1}{\alpha}(3a_3 - 3\alpha^2 a_1) = 3|a_2|. \tag{A.15}$$

Consequently, we obtain the estimate

$$(a_1 + a_3)(3a_3 - a_1) < 3|a_2|(a_1 + a_3) = |a_2|(a_1 + 3a_3) + 2a_1|a_2|, \tag{A.16}$$

which yields (A.14).

Case 2. $a_1 < 0$. When $a_1 + a_3 \leqq 0$, (A.14) holds trivially. To deal with the case that $a_1 + a_3 > 0$, we observe that

$$\begin{aligned} &|a_2|(a_1 + 3a_3) + 2|a_1|\sqrt{a_2^2 + (a_1 + a_3)(3a_3 - a_1)} \\ &\qquad \geqq |a_2|(a_1 + 3a_3) + 2|a_1|\,|a_2| = -a_2(3a_3 - a_1). \end{aligned} \tag{A.17}$$

Thus (A.14) follows provided that

$$a_1 + a_3 < -a_2. \tag{A.18}$$

However, inequality (A.18) is a direct consequence of $a_1 < 0$ and $\alpha < 1$.

Next we show that $u_+ < u_\alpha$. Consider first the simpler case: $a_1 < 0$. Observe that $u_\alpha > 0$ for $\alpha < 1/\sqrt{3}$. Furthermore, we can easily check that the numerator in (A.13) vanishes at $a_1 + a_3 = 0$ which occurs only for $\alpha > 1/\sqrt{3}$, and hence $u_+ < 0$. Thus the conclusion follows in this case. Next, we deal with the case $a_1 > 0$. It can be easily seen that we need only consider $0 < \alpha < \alpha^*$, where $\alpha^* \in (0, \frac{1}{6})$ is the only root of $3a_3 - a_1 = 0$ on $(0, 1)$. Note that $u_\alpha \geqq \frac{1}{2}$ at $\alpha = \frac{1}{6}$. It thus suffices to verify for $\alpha \in (0, \alpha^*)$ that

$$u_+ \leqq \tfrac{1}{2}, \tag{A.19}$$

which is equivalent to

$$\alpha(2 + \alpha)a_1^2 \leqq (1 + 4\alpha - 3\alpha^2)a_1 a_3 + (6\alpha + 3)a_3^2. \tag{A.20}$$

Observe that $2\alpha a_1 < a_3$ for $1 > a > \frac{1}{3}$. It follows that

$$-3\alpha^2 a_1 a_3 + (6\alpha + 3)a_3^2 > 0,$$

$$\alpha(2 + \alpha)a_1^2 < \tfrac{1}{2}(1 + \alpha)a_1 a_3 < (1 + 4\alpha)a_1 a_3.$$

This ensures (A.20), and hence (A.19) follows.

Finally, it remains to show that u_+ is a solution of (A.5). Thus we must verify that

$$a_1 + 3a_3 - 3a_2 u_+ \geqq 0. \tag{A.21}$$

Simple manipulation shows that (A.21) is equivalent to

$$\begin{aligned} &3a_2(a_1 + a_3)(3a_3 - a_1) \\ &\quad \geqq (a_1 + 3a_3)\left(a_2(a_1 + 3a_3) - 2|a_1|\sqrt{a_2^2 + (a_1 + a_3)(3a_3 - a_1)}\right). \end{aligned} \tag{A.22}$$

As before we separate the two cases according to $a_1 > 0$ or $a_1 < 0$. In the case $a_1 > 0$, it need to be clear that we need only prove (A.22) for $3a_3 - a_1 \geqq 0$. Then

$$\begin{aligned} 3a_2(a_1 + a_3)(3a_3 - a_1) &\geqq 3a_2(a_1 + 3a_3)(a_1 + 3a_3) \\ &= (a_1 + 3a_3)(a_2(a_1 + 3a_3) - 2a_1|a_2|) \\ &\geqq (a_1 + 3a_3)\left(a_2(a_1 + 3a_3) - 2a_1\sqrt{a_2^2 + (a_1 + a_3)(3a_3 - a_1)}\right), \end{aligned}$$

which ensures (A.22). Consider now the complementary case $a_1 < 0$. Our assumption then implies that when $a_1 + a_3 > 0$,

$$\begin{aligned} 3a_2(a_1 + a_3)(3a_3 - a_1) &\geqq a_2(a_1 + a_3)(3a_1 - a_1) \\ &= (a_1 + 3a_3)(a_2(a_1 + 3a_3) - 2|a_1||a_2|) \\ &\geqq (a_1 + 3a_3)\left(a_2(a_1 + 3a_3) - 2|a_1|\sqrt{a_2^2 + (a_1 + a_3)(3a_3 - a_1)}\right). \end{aligned}$$

If $a_1 + a_3 < 0$, then (A.13) shows that $u_+ > 0$. Hence, since $a_1 + 3a_3 > 0$ remains valid for $\alpha \in (0, 1/\sqrt{3})$, it follows that (A.21) holds. The proof of the lemma is completed.

Proof of Lemma 2.2. It follows from the structure of the boundary layers (2.33) that we only need to prove the first statement in the lemma. Equation (2.30) implies that $f^0_{-\infty} < f^0_\infty$ can be written in the form

$$\frac{2\alpha}{1+3\alpha^2}\rho(u+\alpha) < 2f^0_\infty = \frac{1}{2}\rho(1-\sigma(u)),$$

or equivalently,

$$(1+3\alpha^2)\sqrt{1+3u^2} < 2(1-3\alpha u). \tag{A.23}$$

On the other hand, the assumption that $\lambda_-(u) < -\alpha$ yields

$$u + \alpha\sqrt{1+3u^2} < \sqrt{\sigma(u)}. \tag{A.24}$$

In the case that

$$u + \alpha\sqrt{1+3u^2} \geqq 0,$$

we obtain from (A.24) that

$$(1+3\alpha^2)(1+3u^2) < 2(1-3u\alpha)\sqrt{1+3u^2},$$

which immediately gives (A.23). For the complementary case

$$u + \alpha\sqrt{1+3u^2} < 0,$$

we estimate directly that

$$\begin{aligned} 2(1-3\alpha u) &> 2 + 6\alpha^2\sqrt{1+3u^2} \\ &> (1-3\alpha^2)\sqrt{1+3u^2} + 6\alpha^2\sqrt{1+3u^2} \\ &= (1+3\alpha^2)\sqrt{1+3u^2}, \end{aligned}$$

where we have used the inequality

$$2 \geqq \sqrt{1+3u^2} \geqq (1-3\alpha^2)\sqrt{1+3u^2}$$

which follows from $|u| \leqq 1$. This completes the proof of the lemma.

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References

1. C. Bardos, R. Caflisch & B. Nicolaenko, *The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas*, Comm. Pure Appl. Math. **39** (1986), 323–352.
2. C. Bardos, F. Golse & C. D. Levermore, *Fluid dynamic limits of kinetic equations II: Convergence proofs for the Boltzmann equation*, Comm. Pure Appl. Math., to appear.
3. J. E. Broadwell, *Shock structure in a simple discrete velocity gas*, Phys. Fluids **7** (1964), 1243–1247.
4. R. E. Caflisch, *Navier-Stokes and Boltzmann shock profiles for a model of gas dynamics*, Comm. Pure Appl. Math. **32** (1979), 521–554.
5. R. E. Caflisch, *The fluid dynamic limit of the nonlinear Boltzmann equation*, Comm. Pure Appl. Math. **33** (1989), 651–656.
6. R. E. Caflisch & Tai-Ping Liu, *Stability of shock waves for the Broadwell equations*, Comm. Math. Phys. **114** (1980), 103–130.
7. R. E. Caflisch & G. C. Papanicolaou, *The fluid dynamical limit of a nonlinear model Boltzmann equation*, Comm. Pure Appl. Math. **32** (1979), 589–616.
8. C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York, 1988.
9. R. Ellis & M. Pinsky, *The first and second fluid approximation to the linearized Boltzmann equation*, J. Math. Pure Appl. **54** (1975), 125–156.
10. H. Grad, *Singular and nonuniform limits of solutions of the Boltzmann equations*, SIAM, Amer. Math. Soc. Proceedings I, Transport Theory (1969), 296–308.
11. S. Kawashima, *Global solutions to the initial-boundary value problems for the discrete Boltzmann equation*, preprint (1992).
12. S. Kawashima, A. Matsumura & T. Nishida, *On the fluid approximation to the Boltzmann equation at the level of the Navier-Stokes equation*, Comm. Math. Phys. **70** (1979), 97–124.
13. H.-O. Kreiss, *Initial boundary value problem for hyperbolic systems*, Comm. Pure Appl. Math **23** (1970), 277–295.
14. Jian-Guo Liu & Zhouping Xin, *Kinetic and viscous boundary layers for the Broadwell equations*, submitted to Transport Th. Stat. Phys.
15. T. Nishida, *Fluid dynamical limit of the nonlinear Boltzmann equation in the level of the compressible Euler equation*, Comm. Math. Phys. **61** (1978), 119–148.
16. T. Platkowski & R. Illner, *Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory*, SIAM Rev. **30** (1988), 213–255.
17. M. Slemrod & A. E. Tzavaras, *Self-similar fluid dynamic limits for the Broadwell system*, preprint (1992).
18. Zhouping Xin, *The fluid-dynamic limit of the Broadwell model of the nonlinear Boltzmann equation in the presence of shocks*, Comm. Pure. Math. **44** (1991), 679–713.

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