Relaxation and diffusion enhanced dispersive waves†

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The development of shocks in nonlinear hyperbolic conservation laws may be regularized through either diffusion or relaxation. However, we have observed surprisingly that for some physical problems, when both of the smoothing factors – diffusion and relaxation – coexist, under appropriate asymptotic assumptions, the dispersive waves are enhanced. This phenomenon is studied asymptotically in the sense of the Chapman–Enskog expansion and demonstrated numerically.

1. Introduction

Relaxation occurs when the underlying material is in non-equilibrium, and usually takes the form of source terms in hyperbolic conservation laws. The relaxation is often stiff when the relaxation time is much shorter than the scales of other physical quantities. The effect of stiff relaxation is important in a wide range of problems of physical significance. In water waves, relaxation is associated with the balance of gravitational force and the friction with the riverbed (Stoker 1958). Sharp slope and rough riverbed give a stiff relaxation that causes the flooding. In thermo-non-equilibrium gases, the internal state variable satisfies a stiff rate equation that measures the departure of the relaxation from the local equilibrium (de Groot & Mazur 1984). In rarefied gas dynamics, the stiff relaxation describes the interaction of particles with small mean free path, and the small mean free path limit recovers the Euler and Navier–Stokes equations (Chapman & Cowling 1970). Other relaxation phenomena occur in traffic flow, viscoelasticity with memory and magnetohydrodynamics, etc.

In the spirit of the Chapman–Enskog expansion for kinetic equations, the asymptotic analysis on these stiff relaxation problems shows that stiff relaxation plays the role of diffusion to the leading order approximation with respect to the small relaxation time, provided that a suitable characteristic condition is met (Liu 1987; Chen *et al.* 1994). Thus, similar to the diffusion itself, the relaxation terms smooth the singularities – the shock waves – of the hyperbolic conservation laws.

Although diffusion or relaxation each plays the role of smoothing out the shock

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waves in hyperbolic conservation laws, the coexistence of both relaxation and diffusion in such systems has not been well understood. Whereas the real physical problems mentioned above usually involve both relaxation and diffusion, for simplicity the diffusion term is often ignored. Here we stress the problem of the coexistence of relaxation and diffusion. Surprisingly, we have found that, under some appropriate asymptotic assumptions, the dispersive waves are enhanced by the interaction of the relaxation and the diffusion. In a viscous water wave equation that describes flooding we have found the solitary waves of the Korteweg–de Vries (KdV) type (Zabusky & Kruskal 1965). In a viscous isentropic thermo-non-equilibrium gas with a single relaxation process we have derived a dispersive equations of the Boussinesq type (Newell 1985). Similar behaviour also exists in a viscous Broadwell model of the nonlinear Boltzmann equation (Broadwell 1964).

2. A relaxation model

We begin with a simple model of the relaxation system proposed in Jin & Xin (1994). Although this is an artificial model, it does possess the key physical properties that we shall address, thus serving to illustrate our idea. Consider the relaxation system of Jin & Xin (1994) with a viscous term:

$$\left.\begin{aligned} \partial_t v + \partial_x u &= 0, \\ \partial_t u + a\,\partial_x v &= -\frac{1}{\varepsilon}(u - f(v)) + \nu\,\partial_x(g(v)\partial_x u). \end{aligned}\right\} \tag{2.1}$$

Here v is some conserved physical quantity, u is some rate variable that measures the departure of the relaxation from the local equilibrium. $\varepsilon \ll 1$ is the relaxation time, ν is the constant viscosity coefficient, and a is a positive constant such that

$$a - f'(v)^2 > 0. \tag{2.2}$$

The viscosity term in (2.1) is the Navier–Stokes viscosity term if $g(v) = v^{-1}$. Let $\delta = \varepsilon^{1/4}$; we look for the long wave asymptotic solution under a new scaling $t \mapsto \delta t$ and $x \mapsto \delta x$. Although this scaling does not fully resolve the small relaxation rate of $O(\varepsilon)$, it does give a more detailed wave structure that becomes significant only in the transition regions such as shock layers or other layers of sharp gradient. Under this scaling, equation (2.1) becomes

$$\partial_t v + \partial_x u = 0, \quad \partial_t u + a\,\partial_x v = -(1/\delta^3)\,(u - f(v)) + (\nu/\delta)\,\partial_x(g(v)\partial_x u). \tag{2.3}$$

Case 1. The stiff relaxation. First we ignore the viscosity term by setting $\nu = 0$. When $\varepsilon \ll 1$ (or equivalently $\delta \ll 1$), the leading term approximation of the relaxation system (2.3) gives the local equilibrium $u = f(v)$ and the nonlinear hyperbolic conservation law

$$\partial_t v + \partial_x f(v) = 0. \tag{2.4}$$

The solution of (2.4) may develop shock waves in finite time even if the initial condition is smooth. One can use the Hilbert expansion or the Chapman–Enskog expansion to derive the higher order approximation. Adopting the strategy of the Chapman–Enskog expansion, which allows contributions of different orders in ε to the time derivative from the space derivative, one obtains the convection-diffusion

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equation

$$\partial_t v + \partial_x f(v) = \delta^3 \, \partial_x([a - f'(v)^2]\partial_x v). \tag{2.5}$$

The stability condition for this approximation is (2.2), usually referred to as the subcharacteristic condition (Whitham 1974; Liu 1987). This equation explains the smoothing effect of the stiff relaxation.

One can obtain a closed equation for v from (2.3) by eliminating u:

$$\partial_t v + \partial_x f(v) = \delta^3 \, (a\partial_{xx} v - \partial_{tt} v). \tag{2.6}$$

This can be viewed as a first-order wave equation perturbed by a second-order linear wave. The linear stability of this kind of wave hierarchy was studied in Whitham (1974). One can recover the convection-diffusion equation (2.5) by applying the approximation $\partial_t v = -\partial_x f(v) + O(\delta^3)$ in (2.6).

Case 2. The diffusion. If the relaxation term is not present in (2.1), then the solution will be dissipative. For example, if $g(v) = 1$, a simple linear analysis shows that the normal modes solution is of the form $e^{ikx-\beta t}$ with the decay rate $\beta \sim a\delta/\nu$. The decay rate not only is independent of the frequency k but decays slower for larger viscosity coefficient ν.

Case 3. The coexistence of stiff relaxation and diffusion. When both stiff relaxation and diffusion terms appear in equation (2.3), then contrary to physical intuition, we can obtain solitary waves. To see this, eliminating u from the relaxation system (2.1) gives

$$\partial_t v + \partial_x f(v) = \delta^3 \, (a\partial_{xx} v - \partial_{tt} v) - \nu\delta^2 \, \partial_{xx}(g(v)\partial_t v). \tag{2.7}$$

Applying the approximation

$$\partial_t v = -\partial_x f(v) + O(\nu\delta^2 + \delta^3) \tag{2.8}$$

to (2.7), one arrives at

$$\partial_t v + \partial_x f(v) = \delta^3 \, \partial_x([a - f'(v)^2]\partial_x v) + \nu\delta^2 \, \partial_{xx}(g(v)f'(v)\partial_x v) + O(\delta^4). \tag{2.9}$$

When the diffusion coefficient dominates in the sense that $\nu \gg \delta = \varepsilon^{1/4}$, the dispersion term in (2.9) dominates the dissipation term, and solitary waves will develop after the formation of shock waves, a well-known phenomenon for dispersive waves. In fact, by taking

$$f(v) = 3v^2, \quad g(v) = 1/f'(v) \tag{2.10}$$

in (2.9) we recover the KdV equation

$$\partial_t v + 6\, v\, \partial_x v = \nu\delta^2 \, \partial_{xxx} v, \tag{2.11}$$

after ignoring the $O(\delta^3)$ term. This term can be ignored only if the subcharacteristic condition (2.2) is satisfied. Otherwise this term becomes the ill-posed backward heat equation and the result will be quite different.

In figure 1 we display and compare the numerical computations on the relaxation system (2.3) for the above three cases with $a = 4$, $f(v) = 0.5v^2$ and $g(v) = 1$. We start with the initial data $v(0, x) = 0.5(1 + \cos(2x))$ and $u(0, x) = 0.5v(0, x)^2$. We solve (2.3) with the relaxation time $\epsilon = 10^{-6}$, and plot all solutions at $t = 3$. For case 1, we take $\nu = 0$ so there is no viscosity and the solution is plotted by the dashed line. We see that a shock wave corresponding to the leading term

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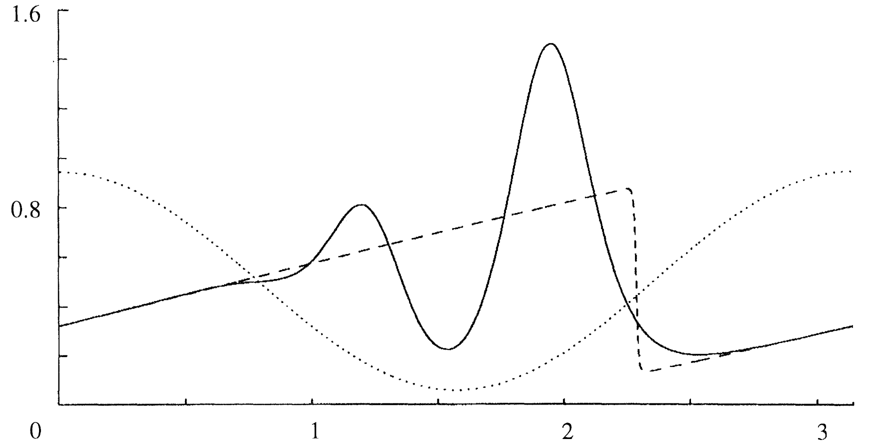

Figure 1. Relaxation (dashed line), diffusion (dotted line), and the enhanced solitary waves (solid line) of the relaxation system (2.3). $t = 3$.

approximation (2.5) develops, moves to the right and decays. For case 2, assume there is no relaxation terms and the viscosity coefficient in (2.3) equals 100 (corresponding to small δ). We plot the solution with the dotted line. We see a diffusive behaviour that forces the solution to decay exponentially with a rate of $e^{-a\nu t/\delta}$. For case 3, both the relaxation and the diffusion terms are present, and we plot the solution by the solid line. It can be seen that after the formation of the shock wave the solution develops oscillations behind the shock, which eventually become solitary waves. This is a typical dispersive behaviour that is solely enhanced by the interaction of the relaxation and the diffusion.

3. Water wave

To apply this analysis to physical problems, we first consider a viscous water wave equation that describes the flooding down a sharply inclined open channel with large riverbed friction. Using dimensionless variables, this flooding flow can be described by the following equations for mass and momentum in the lagrangian coordinates (Stoker 1958):

$$\left.\begin{aligned} \partial_t v - \partial_x u &= 0, \\ \partial_t u - v^{-3}\partial_x v &= (1/\varepsilon)(v^{-1} - F^{-2}u^2) + \nu\partial_x(v^{-1}\partial_x u). \end{aligned}\right\} \tag{3.1}$$

Here v, the specific volume, is the reciprocal of the height of the river, u is the mean velocity of the flow, ε is the dimensionless small parameter that measures the sharpness of the riverbed slope and the friction of the riverbed, and ν is the viscosity coefficient. The Froude number F is the dimensionless speed of undisturbed flow of unit height with friction and gravitational forces in perfect balance. Let $\delta = \varepsilon^{1/4}$. Under the long wave asymptotic scaling $t \mapsto \delta t$ and $x \mapsto \delta x$, equation (3.1) becomes

$$\left.\begin{aligned} \partial_t v - \partial_x u &= 0, \\ \partial_t u - v^{-3}\partial_x v &= (1/\delta^3)(v^{-1} - F^{-2}u^2) + (\nu/\delta)\partial_x(v^{-1}\partial_x u). \end{aligned}\right\} \tag{3.2}$$

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The second equation in (3.2) gives

$$v^{-1} - F^{-2}u^2 + \nu\delta^2\partial_x(v^{-1}\partial_x u) = \delta^3(\partial_t u - v^{-3}\partial_x u) \equiv \delta^3 u_1 \tag{3.3}$$

where u_1 will be further evaluated later. From (3.3),

$$u = Fv^{-1/2} + \tfrac{1}{2}\nu\delta^2 F v^{1/2}\partial_x(v^{-1}\partial_x u) - \tfrac{1}{2}\delta^3 F v^{1/2} u_1 + O(\delta^4). \tag{3.4}$$

Applying $u = Fv^{-1/2} + O(\delta^2)$ to the right-hand side of (3.4) yields

$$u = F\,v^{-1/2} + \tfrac{1}{6}\,\nu\delta^2\,F^2\,v^{1/2}\,\partial_{xx}(v^{-3/2}) - \tfrac{1}{2}\,\delta^3\,F\,v^{1/2}u_1 + O(\delta^4)\,. \tag{3.5}$$

By using (3.3) and (3.5),

$$\begin{aligned} u_1 &= \partial_t u - v^{-3}\partial_x v \\ &= -\tfrac{1}{2}Fv^{-3/2}\partial_t v - v^{-3}\partial_x v + O(\delta^2) \\ &= -\tfrac{1}{2}Fv^{-3/2}\partial_x u - v^{-3}\partial_x v + O(\delta^2) \\ &= -\tfrac{1}{2}Fv^{-3/2}\partial_x(Fv^{-1/2}) - v^{-3}\partial_x v + O(\delta^2) \\ &= (\tfrac{1}{4}F^2 - 1)v^{-3}\partial_x v + O(\delta^2). \end{aligned} \tag{3.6}$$

Now plugging u_1 back into (3.5) and using the first equation of (3.2) we have

$$\partial_t v - F\partial_x(v^{-1/2}) = \tfrac{1}{6}\,\nu\delta^2\,F^2\,\partial_x(v^{1/2}\partial_{xx}(v^{-3/2})) + \tfrac{1}{2}\delta^3\,F\,\partial_x((1 - F^2/4)v^{-5/2}\partial_x v) + O(\delta^4). \tag{3.7}$$

This is a nonlinear convection–diffusion–dispersion equation. Clearly the stability requirement imposes the well-known stability condition $F < 2$ that corresponds to the subcharacteristic condition (2.2) for the relaxation system (2.1). In the stable case, we can ignore the $O(\delta^3)$ term in (3.7) to get the following dispersive equation,

$$\partial_t v - F\partial_x(v^{-1/2}) = \tfrac{1}{6}\,\nu\delta^2\,F^2\,\partial_x(v^{1/2}\partial_{xx}(v^{-3/2})). \tag{3.8}$$

This results the development of solitary waves after the formation of the shock waves, a dispersive behaviour similar to that of the KdV equation.

In (3.8) v is a conserved quantity. Let h be the height of the flooding river. By definition $h = v^{-1}$. Some algebraic manipulation on (3.8) gives the following conservation for h:

$$\partial_t h + \tfrac{1}{5}\,F\partial_x(h^{5/2}) = -\tfrac{1}{6}\,\nu\delta^2\,F^2\,\partial_x(h^{3/2}\partial_{xx}(h^{3/2})) + \tfrac{1}{9}\,\nu\delta^2\,F^2\partial_x([\partial_x(h^{3/2})]^2)\,. \tag{3.9}$$

We can also determine the travelling wave of (3.8) that moves with a constant speed σ. Set $v(x,t) = 1/\Phi(\xi)$ and $\xi = x - \sigma t$, then (3.8) becomes

$$\frac{\partial\Phi}{\partial\xi} = \tfrac{8}{3}\,F^{-1}\delta^{-1}\sqrt{-3\sigma - 2F\Phi^{1/2}}. \tag{3.10}$$

The solution of the ordinary differential equation (3.10) gives the desirable travelling wave. It exists when σ is negative. Because we use the lagrangian coordinates, this indicates that the solitary waves always move slower than the flow.

The numerical calculation in which these phenomena were observed were made by solving the flooding water equation (3.2) with $\delta = 10^{-2}$ (thus $\varepsilon = 10^{-8}$), $\nu = 1$, $F = 1$ and the periodic initial condition $v(0,x) = (1 + 0.5\cos(2x))^{-2/3}$ and $u(0,x) = F/\sqrt{v(0,x)}$. We depict the solution of $h = v^{-1}$ in Figure 2 at $t = 0$ (the

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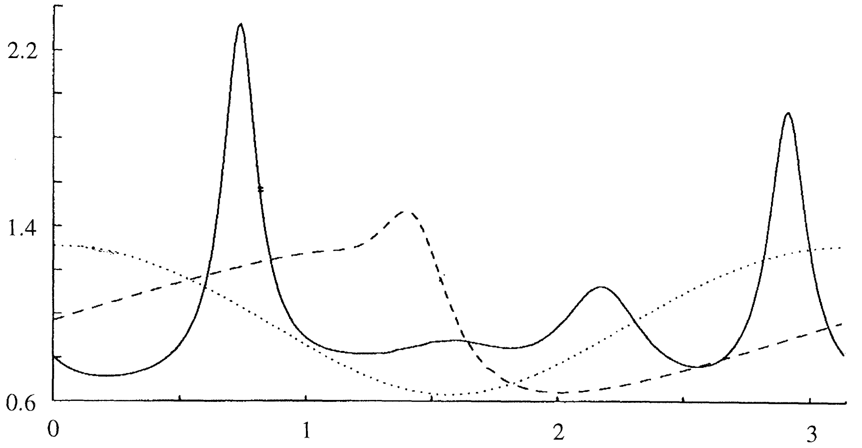

Figure 2. Solitary waves of the viscous flooding river equation (3.2). ——, $t = 6.5$; – – –, $t = 1.75$; ······, $t = 0$.

dotted line), 1.75 (the dashed line) and 6.5 (the solid line). Although starting with smooth initial data, the solution will develop shock waves in the breakdown time. Due to the dispersion effect, some oscillations form behind the shock. At later time these oscillations become the solitary waves, each of them preserves the original shape and velocity after passing through the others. These results are quite similar to those of the KdV equation, and agree with the experiment carried out directly on the reduced equation (3.8).

4. Thermo-non-equilibrium gases

Such dispersive phenomenon may also exist in thermo-non-equilibrium gases. Consider an isentropic flow of a gas in lagrangian coordinates with a single relaxation process (de Groot & Mazur 1984):

$$\left.\begin{aligned} \partial_t v - \partial_x u = 0, \quad \partial_t u + \partial_x p(v,\xi) = 0, \\ \partial_t \xi = (1/\varepsilon)(\xi^*(v) - \xi) + \nu \partial_x (g(v) \partial_x \xi). \end{aligned}\right\} \tag{4.1}$$

Here v is the specific volume, u is the velocity, $p = p(v,\xi)$ is the pressure such that $\partial_v p < 0$. The internal variable ξ satisfies a rate equation that measures the departure of the relaxation process from the local thermal equilibrium. $\xi^*(v)$ is the equilibrium state, ε is the relaxation time, and ν is the viscosity coefficient. Let $\delta = \varepsilon^{1/4}$. Under the long wave asymptotic scaling $t \mapsto \delta t$ and $x \mapsto \delta x$, equation (4.1) becomes

$$\left.\begin{aligned} \partial_t v - \partial_x u = 0, \quad \partial_t u + \partial_x p(v,\xi) = 0, \\ \partial_t \xi = (1/\delta^3)(\xi^*(v) - \xi) + (\nu/\delta) \partial_x (g(v) \partial_x \xi). \end{aligned}\right\} \tag{4.2}$$

By using the third equation of (4.2) we have

$$\xi = \xi^*(v) + \nu \delta^2 \, \partial_x (g(v) \partial_x \xi) + \delta^3 \xi_1, \tag{4.3}$$

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where ξ_1 is given by

$$\xi_1 = -\partial_t \xi = -\partial_v \xi^* \partial_t v + O(\delta^2) = -\partial_v \xi^* \, \partial_x u + O(\delta^2). \tag{4.4}$$

Now, by (4.3) and (4.4)

$$p(v,\xi) = p(v,\xi^*) + \nu\delta^2 \, \partial_\xi p(v,\xi^*) \, \partial_x (g(v)\partial_x \xi^*(v)) \\ - \delta^3 \, \partial_\xi p(v,\xi^*) \, \partial_v \xi^* \, \partial_x u + O(\delta^4). \tag{4.5}$$

This implies

$$\begin{aligned} \partial_t v - \partial_x u &= 0, \\ \partial_t u + \partial_x p(v,\xi^*) &= -\nu\delta^2 \, \partial_x (\partial_\xi p(v,\xi^*) \, \partial_x [g(v)\partial_x \xi^*(v)]) \\ &\quad + \delta^3 \, \partial_x (\partial_v \xi^* \, \partial_\xi p(v,\xi^*) \, \partial_x u) + O(\delta^4). \end{aligned} \tag{4.6}$$

The stability condition is

$$\partial_v \xi^*(v) \, \partial_\xi p(v,\xi^*) > 0. \tag{4.7}$$

With proper choice of the equilibrium state ξ^*, equation (4.6) is a dispersive-dissipative hyperbolic system. Clearly the dispersion term dominates the dissipation, thus solitary waves will emerge after the formation of shock waves. Eliminating u from (4.6) and ignoring the $O(\delta^3)$ term, we get the following Boussinesq type equation

$$\partial_{tt} v + \partial_{xx} p(v,\xi^*(v)) = -\nu\delta^2 \, \partial_{xx} (\partial_\xi p(v,\xi^*) \, \partial_x [g(v)\partial_x \xi^*(v)]). \tag{4.8}$$

A simple choice of p is $p(v,\xi) = q(v) - \alpha\xi$ for some positive constant α, which only takes into account the first order deviations from the thermodynamical equilibrium. Let $g(v) = 1/\partial_v \xi^*(v)$ and $q(v) = -v$, $\xi^*(v) = -3v^2/\alpha$. Then (4.8) becomes the exact Boussinesq equation

$$\partial_{tt} v - \partial_{xx} v + 3\partial_{xx}(v^2) = \nu\alpha\delta^2 \partial_{xxxx} v. \tag{4.9}$$

5. A viscous Broadwell model

We have also observed similar dispersive phenomenon in 'viscous' rarefied gas dynamics. Consider the Broadwell model of the nonlinear Boltzmann equation that describes a gas as composed of only four speeds with a binary collision law and spatial variation in only one direction (Broadwell 1964):

$$\left.\begin{aligned} \partial_t f_+ + \partial_x f_+ &= (1/\varepsilon)(f_0^2 - f_+ f_-), \\ \partial_t f_0 &= (1/\varepsilon)(f_+ f_- - f_0^2), \\ \partial_t f_- - \partial_x f_- &= (1/\varepsilon)(f_0^2 - f_+ f_-). \end{aligned}\right\} \tag{5.1}$$

Here f_+, f_0 and f_- denote the mass densities of gas particle with speed $1, 0$ and -1 respectively, ε is the mean free path. Introducing the fluid moment quantities of density ρ, momentum m and z as

$$\rho = f_+ + 2f_0 + f_-, \quad m = f_+ - f_-, \quad z = f_+ + f_-, \tag{5.2}$$

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the Broadwell equation (5.1) is then equivalent to

$$\left.\begin{aligned}\partial_t\rho+\partial_x m=0,\quad \partial_t m+\partial_x z=0,\\ \partial_t z+\partial_x m=-(1/2\varepsilon)\,(2\rho z-\rho^2-m^2)+\nu\rho\partial_{xx}z,\end{aligned}\right\}\tag{5.3}$$

after addition of an artificial viscosity term $\nu\rho\partial_{xx}z$ in the third equation, which is introduced here to study the interaction between the relaxation and the diffusion. Let $\delta=\varepsilon^{1/4}$. Using the long wave time and space variables $t\mapsto\delta t$ and $x\mapsto\delta x$, equation (5.3) becomes

$$\left.\begin{aligned}\partial_t\rho+\partial_x m=0,\quad \partial_t m+\partial_x z=0,\\ \partial_t z+\partial_x m=-(1/2\delta^3)\,(2\rho z-\rho^2-m^2)+(\nu/\delta)\rho\partial_{xx}z.\end{aligned}\right\}\tag{5.4}$$

First, from (5.4),

$$z=\tfrac{1}{2}(\rho+m^2/\rho)+\nu\delta^2\partial_{xx}z-\delta^3\frac{\partial_t z+\partial_x m}{\rho}.\tag{5.5}$$

Taking the derivative with respective to x and use the second equation of (5.4), one has

$$\partial_t m+\tfrac{1}{2}\partial_x(\rho+m^2/\rho)-\nu\delta^2\,\partial_{txx}m=\delta^3\partial_x\frac{\partial_t z+\partial_x m}{\rho}.\tag{5.6}$$

Let $u=m/\rho$. Taking t derivative on (5.5) leads to

$$\begin{aligned}\partial_t z&=\tfrac{1}{2}(\partial_t\rho-u^2\partial_t\rho+2u\partial_t m)+O(\delta^2)\\ &=-\tfrac{1}{2}(1-u^2)\partial_x m+u\partial_t m+O(\delta^2).\end{aligned}\tag{5.7}$$

Similarly, taking x derivative on (5.5) gives

$$\partial_t m=-\partial_x z=-\tfrac{1}{2}(1-u^2)\partial_x\rho-u\partial_x m+O(\delta^2).\tag{5.8}$$

Applying (5.8) in (5.7) gives

$$\frac{\partial_t z+\partial_x m}{\rho}=\tfrac{1}{2}(1-u^2)\partial_x u+O(\delta^2).\tag{5.9}$$

Now substituting (5.5) and (5.9) in (5.6) yields

$$\begin{aligned}\partial_t m+\tfrac{1}{2}\partial_x(\rho+m^2/\rho)&=-\tfrac{1}{2}\nu\delta^2\,\partial_{xxx}(\rho+m^2/\rho)\\ &\quad+\tfrac{1}{2}\delta^3\partial_x((1-u^2)\partial_x u)+O(\delta^4).\end{aligned}\tag{5.10}$$

Here the stability criterion implies the subcharacteristic condition $|u|<1$, indicating that the macroscopic fluid speed u should not exceed the microscopic speed of the particles. After ignoring $O(\delta^3)$ term, we arrive at the dispersive equations

$$\partial_t\rho+\partial_x m=0,\quad \partial_t m+\tfrac{1}{2}\,\partial_x(\rho+m^2/\rho)=-\tfrac{1}{2}\nu\delta\,\partial_{xxx}(\rho+m^2/\rho).\tag{5.11}$$

6. Conclusion

In conclusion we have found that the interaction of relaxation and diffusion enhances the dispersive waves under some appropriate asymptotic scalings. This important phenomenon cannot occur if either the relaxation or the diffusion is

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neglected. Our analysis provides an effective tool to study such behaviour, and can also be applied to other physical problems with similar structure of relaxation and diffusion.

For stiff relaxation problems without the viscosity term, Hunter has also derived a set of dispersive wave equations using the Chapman–Enskog expansion (Hunter 1993). His expansion was based on long wave and weak nonlinearity assumptions, thus differs from our phenomena here.

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