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Global stability for solutions to the exponential PDE describing epitaxial growth

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In this paper we prove the global existence, uniqueness, optimal large time decay rates, and uniform gain of analyticity for the exponential PDE $h_t = \Delta e^{-\Delta h}$ in the whole space $\mathbb{R}^d_x$. We assume the initial data is of medium size in the Wiener algebra $A(\mathbb{R}^d)$; we use the initial condition $\Delta h_0 \in A(\mathbb{R}^d)$ which is scale-invariant with respect to the invariant scaling of the exponential PDE. This exponential PDE was derived in [18] and more recently in [22].

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1. Introduction and main results

Epitaxial growth is an important physical process for forming solid films or other nano-structures. Indeed it is the only affordable method of high quality crystal growth for many semiconductor materials. It is also an important tool to produce some single-layer films to perform experimental research, highlighted by the recent breakthrough experiments on the quantum anomalous Hall effect and superconductivity above 100 K lead by Qikun Xue [3, 12].

This subject has been the focus of research from both physics and mathematics since the classic description of step dynamics in the work of Burton, Cabrera, and Frank in 1951 [2], Weeks [30] in the 1970's, the KPZ stochastic partial differential equation description beyond roughness transition in 1986 [16], and the mathematical analysis of Spohn in 1993 [29]. We refer to the books [26, 33] for a physical explanation of epitaxial growth. For more recent modeling and analysis, we refer to in particular to [6, 8, 13, 14, 21] and the references therein.

Epitaxy growth occurs as atoms, deposited from above, adsorb and diffuse on a crystal surface. Modeling the rates that the atoms hop and break bonds leads in the continuum limit to the degenerate 4th-order PDE $h_t = \Delta e^{-\Delta_p h}$, which involves the exponential nonlinearity and the p-Laplacian Δ_p, with $p = 1$, for example. This equation was first derived in [18] and more recently in [22]. In this paper, we will focus on this class of exponential PDE for the case $p = 2$ and we give a short derivation of the model below.

Let $h(x,t)$ be the height of a thin film. We consider the dynamics of atom deposition, evaporation, detachment and diffusion on a crystal surface in the epitaxial growth process. In absence of atom deposition and evaporation[1] and in the continuum limit, the above process can be well described by Fick's law:

$$h_t + \nabla \cdot J = 0, \quad J = -D_s \nabla \rho_s.$$

Here D_s is the surface diffusion constant and ρ_s is the equilibrium density of adatoms on a substrate of the thin film. ρ_s is described by the grand canonical ensemble $e^{-(E_s-\mu_s)/k_B T}$ up to a normalization constant, where E_s is the energy of pre adatom, μ_s is the chemical potential pre adatom, k_B is the Boltzmann constant and T is the temperature. We lump $e^{-E_s/k_B T}$ and the normalization constant into a reference density ρ^0 and then we arrive at the Gibbs–Thomson relation $\rho_s = \rho^0 e^{\mu_s/k_B T}$ which is connected to the theory of molecular capillarity [27].

In the continuum limit, the chemical potential μ_s is computed by the variation of free energy of the thin film. A simple broken-bond model for crystals consists of height columns described by $h = (h_i)_{i=1,\ldots N}$ with screw-periodic boundary conditions in the form

$$h_{i+N} = h_i + \alpha a N \quad \forall i \ ,$$

where α is the average slope and a is the side length. The column h_i is derived into h_i/a square boxes where an atom is placed to the center of each box. The atoms then connect to the nearest neighbor atoms with a bond from up, down, left and right. These bonds contain almost all the energy of the system. Hence we set the total energy of the system equal to

$$E(h) = -\gamma \cdot (\#\text{ of bonds}),$$

where γ is the energy per bond. The negative sign represents that the atoms prefer to stay together. It requests an amount of γ energy to brake the bond and separate two atoms. With the identity $x + |x| = 2x_+$ and some elementary computations, we can decompose the total energy $E(h)$ into the bulk contribution E_b and the surface contribution E_s. The bulk contribution is given by

$$E_b = -\frac{2\gamma}{a} \sum_{j=1}^{N} h_i + \frac{\gamma\alpha}{2} N.$$

Due to the conservation of mass $\sum_{j=1}^{N} h_i$, we know that E_b is independent of time and we can drop it from the energy computation. The surface contribution E_s is given by

$$E_s = \frac{\gamma}{2a} \sum_{i=1}^{N} |h_i - h_{i-1}| \ .$$

This free energy agrees with the computation in [30]. In general the free energy takes the form

$$E(h) = \frac{1}{p} \int |\nabla h|^p dx,$$

[1] In the case of atom deposition and evaporation with a constant rate a, we need to normalize the height by subtracting out $h(x,t) - at$. After doing that however we obtain the same equation.

or some linear combinations of those [23].

Now we can compute the chemical potential: $\mu_s = \frac{\delta E}{\delta h} = -\Delta_p h$ and the PDE becomes

$$h_t = \Delta e^{-\Delta_p h} \tag{1.1}$$

where, for simplicity, we have taken the constant coefficients $D_s \rho^0 = 1, k_B T = 1$. We refer to [22] and [18, 19] for a more physical derivation.

A linearized Gibbs–Thomson relation $\rho_s = \rho^0 e^{\mu_s / k_B T} \approx \rho^0 (1 + \mu_s / k_B T)$ is usually used in the physical modeling and it results the following PDE

$$h_t = \frac{D_s \rho^0}{k_B T} \Delta \Delta_p h. \tag{1.2}$$

Giga–Kohn [14] proved that there is a finite time extinction for (1.2). For the difficult case of $p = 1$, Giga–Giga [13] developed a H^{-1} total variation gradient flow to analyze this equation and they showed that the solution may instantaneously develop a jump discontinuity in the explicit example of crystal facet dynamics. This explicit construction of the jump discontinuity solution for facet dynamics was extended to the exponential PDE (1.1) in [19].

The exponential PDE (1.1) exhibits many distinguished behaviors in both the physical and the mathematical senses. The most important one is the asymmetry in the diffusivity for the convex and concave parts of height surface profiles. This can be seen directly if we recast (1.1) into the following weighted H^{-1} gradient flow with curvature-dependent mobility [14, 20]:

$$h_t = \nabla \cdot \mathcal{M} \nabla \frac{\delta E}{\delta h}, \quad \mathcal{M} = e^{-\Delta_p h}.$$

The exponential nonlinearity drastically distinguishes the diffusivity for the convex and concave surface and leads to the singular behavior of the solution.

In [20], a steady solution where Δh contains a delta function was constructed and the global existence of weak solutions with Δh as a Radon measure was proved for the case $p = 2$. In [11], a gradient flow method in a metric space was studied together with global existence and a free energy-dissipation inequality was obtained.

In the present paper, we will study the case $p = 2$ in the exponential PDE (1.1):

$$h_t = \Delta e^{-\Delta h} \quad \text{in } \mathbb{R}^d_x. \tag{1.3}$$

We will consider initial data $h_0(x)$. We will take advantage of the Wiener algebra $A(\mathbb{R}^d)$; $A(\mathbb{R}^d)$ is the space $\dot{\mathcal{F}}^{0,1}$ as defined in (1.5) in the next section. In particular in Section 1.2 our main results show that if $\Delta h_0 \in A(\mathbb{R}^d)$ with explicit norm size less than $\frac{52}{500}$, assuming additional conditions, then we can prove the global existence, uniqueness, uniform gain of analyticity, and the optimal large time decay rates (in the sense of Remark 1.5). We note that for $\lambda > 0$ the invariant scaling of (1.3) is

$$h^\lambda(t, x) = \lambda^{-2} h(\lambda^4 t, \lambda x), \tag{1.4}$$

and the condition $\Delta h_0 \in A(\mathbb{R}^d)$ is scale invariant (the exact condition we use is $h_0 \in \dot{\mathcal{F}}^{2,1}$ which is scale invariant).

In the next section we will introduce the necessary notation.

1.1 *Notation*

We introduce the following useful norms:

$$\|f\|^p_{\dot{\mathcal{F}}^{s,p}}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{sp} |\hat{f}(\xi,t)|^p d\xi, \quad s > -d/p, \quad 1 \leqslant p \leqslant 2. \tag{1.5}$$

We note that the Wiener algebra $A(\mathbb{R}^d)$ is $\dot{\mathcal{F}}^{0,1}$, and the condition $\Delta h_0 \in A(\mathbb{R}^d)$ is given by $h_0 \in \dot{\mathcal{F}}^{2,1}$. Here $\hat{f}$ is the standard Fourier transform of f:

$$\hat{f}(\xi) \stackrel{\text{def}}{=} \mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx. \tag{1.6}$$

When $p = 1$ we denote the norm by

$$\|f\|_s \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi. \tag{1.7}$$

We will use this norm generally for $s > -d$ and we refer to it as the *s-norm*. To further study the case $s = -d$, then for $s \geqslant -d$ we define the *Besov-type s-norm*:

$$\|f\|_{s,\infty} \stackrel{\text{def}}{=} \left\| \int_{C_k} |\xi|^s |\hat{f}(\xi)| \, d\xi \right\|_{\ell^\infty_k} = \sup_{k \in \mathbb{Z}} \int_{C_k} |\xi|^s |\hat{f}(\xi)| \, d\xi, \tag{1.8}$$

where for $k \in \mathbb{Z}$ we have

$$C_k = \{\xi \in \mathbb{R}^d : 2^{k-1} \leqslant |\xi| < 2^k\}. \tag{1.9}$$

Note that we have the inequality

$$\|f\|_{s,\infty} \leqslant \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi = \|f\|_s. \tag{1.10}$$

We note that

$$\|f\|_{-d/p,\infty} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for $p \in [1, 2]$ as is shown in [25, Lemma 5].

Further, when $p = 2$ we denote the norm (for $s > -d/2$) by

$$\|f\|^2_{\dot{\mathcal{F}}^{s,2}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|^2_{L^2(\mathbb{R}^d)}. \tag{1.11}$$

We also introduce following norms with analytic weights:

$$\|f\|^p_{\dot{\mathcal{F}}^{s,p}_\nu}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{sp} e^{p\nu(t)|\xi|} |\hat{f}(\xi,t)|^p d\xi, \quad s \geqslant 0, \quad p \in [1, 2], \tag{1.12}$$

for a positive function $\nu(t)$.

We also introduce the following notation for an iterated convolution

$$f^{*2}(x) = (f * f)(x) = \int_{\mathbb{R}^d} f(x-y) f(y) dy,$$

where $*$ denotes the standard convolution in $\mathbb{R}^d$. Furthermore in general

$$f^{*j}(x) = (f * \cdots * f)(x),$$

where the above contains $j-1$ convolutions of j copies of f. Then by convention when $j = 1$ we have $f^{*1} = f$, and further we use the convention $f^{*0} = 1$.

We additionally use the notation $A \lesssim B$ to mean that there exists a positive inessential constant $C > 0$ such that $A \leqslant CB$. The notation $\approx$ used as $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

1.2 *Main results*

In this section we present our main results. Our Theorem 1.1 below shows the global existence of solutions under a medium sized condition on the initial data as in Remark 1.2.

Theorem 1.1 *Consider initial data $h_0 \in \dot{\mathcal{F}}^{0,2} \cap \dot{\mathcal{F}}^{2,1}$ further satisfying $\|h_0\|_2 < y_*$ where $y_* > 0$ is given explicitly in Remark* 1.2. *Then there exists a global in time unique solution to* (1.3) *given by $h(t) \in C^0_t(\dot{\mathcal{F}}^{0,2} \cap \dot{\mathcal{F}}^{2,1})$ and we have that*

$$\|h\|_2(t) + \sigma_{2,1} \int_0^t \|h\|_6(\tau) d\tau \leqslant \|h_0\|_2 \tag{1.13}$$

with $\sigma_{2,1} > 0$ defined by (2.11).

In the next remark we explain the size of the constant.

REMARK 1.2 We can compute precisely the size of the constant y_* from Theorem 1.1. In particular the condition that it should satisfy is that

$$f_2(y_*) = (y_*^3 + 6y_*^2 + 7y_* + 1)e^{y_*} - 1 = \sum_{j=1}^{\infty} \frac{(j+1)^3}{j!} y_*^j < 1$$

Such a y_* can be taken to be $y_* \in (0, 52/500]$. For this reason we call the initial data "medium size". However $y_* \geqslant 105/1000$ is too big in our framework.

Now in the next theorem we prove the large time decay rates, and the propagation of additional regularity, for the solutions above.

Theorem 1.3 *We assume all the conditions in Theorem* 1.1. *We also assume that $\|h_0\|_{-d,\infty} < \infty$ but not necessarily small.*

In particular for any $\max\{-2, -d\} < s \leqslant 2$ *we have that*

$$\|h\|_s(t) \lesssim \|h_0\|_s, \tag{1.14}$$

assuming additionally that $\|h_0\|_s < \infty$ but not necessarily small.

In particular if $h_0 \in \dot{\mathcal{F}}^{s,1}$ and $h_0 \in \dot{\mathcal{F}}^{2,2}$ (these norms are not assumed to be small) then we conclude the large time decay rate

$$\|h(t)\|_s \lesssim (1+t)^{-(s+d)/4}, \tag{1.15}$$

where d is the spatial dimension in (1.3).

Then in the next theorem we explain the instant gain of uniform analyticity, at the optimal linear temporal growth rate of $t^{1/4}$, and the uniform large time decay rate of the analytic norms.

Theorem 1.4 *We assume all the conditions in Theorem* 1.1. *Additionally suppose that* $\|h_0\|_{-d,\infty} < \infty$, $h_0 \in \dot{\mathcal{F}}^{s,1}$ *for some fixed* $0 \leqslant s \leqslant 2$ *and* $h_0 \in \dot{\mathcal{F}}^{2,2}$ *(these norms are not assumed to be small).*

Then there exists a positive increasing function $\nu(t) > 0$ *such that* $\nu(t) \approx t^{1/4}$ *for large* $t \gtrsim 1$. *For this* $\nu(t)$*, the solution* $h(t,x)$ *from Theorem* 1.1 *further gains instant analyticity:* $h(t) \in C_t^0 \dot{\mathcal{F}}_\nu^{s,1}$. *And the analytic norm decays at the same rate:*

$$\|h(t)\|_{\dot{\mathcal{F}}_\nu^{s,1}} \lesssim (1+t)^{-(s+d)/4}. \tag{1.16}$$

The function $\nu(t)$ *is defined precisely in* (8.7).

In the remarks below we further explain the optimal linear uniform time decay rates and the optimal linear gain of analyticity with radius $\nu(t) \approx t^{1/4}$.

REMARK 1.5 We point out that the large time decay rates which we obtain in (1.15) and (1.16) (and also in (3.1) below) are the same as the optimal large time decay rates for the linearization of (1.3), which is given by

$$g_t + \Delta^2 g = 0, \tag{1.17}$$

obtained by removing the non-linear terms in the expansion of the nonlinearity as in (2.1) below.

In particular it can be shown by standard methods that if $g_0(x)$ is a tempered distribution vanishing at infinity and satisfying $\|g_0\|_{\rho,\infty} < \infty$, then one further has

$$\|g_0\|_{\rho,\infty} \approx \left\| t^{(s-\rho)/\gamma} \left\| e^{-t(-\Delta)^{\gamma/2}} g_0 \right\|_s \right\|_{L_t^\infty((0,\infty))}, \quad \text{for any } s \geqslant \rho,\, \gamma > 0.$$

It then follows from this equivalence that the optimal large time decay rate for the norm $\left\| e^{-t(-\Delta)^{\gamma/2}} g_0 \right\|_s$ is $t^{-(s-\rho)/\gamma}$. In particular the optimal decay rates for solutions of the linear equation (1.17) in the norm (1.7) are $t^{-(s-\rho)/4}$. These optimal linear large time decay rates are the same as the non-linear time decay rates in (1.15) and (3.1). These large time decay rates even hold for solutions to the equation (1.3) in the analytic norm as in (1.16).

When we say in this paper that the large time decay rates are optimal, we mean that we obtain the optimal linear decay rate as just described in Remark 1.5.

REMARK 1.6 Regarding the rate of growth of the radius of analyticity, we look at solutions to the following linear equation:

$$g_t + (-\Delta)^{\gamma/2} g = 0, \quad \gamma > 0. \tag{1.18}$$

If this equation has initial initial data g_0 then it's solution is $g = e^{-t(-\Delta)^{\gamma/2}} g_0$. We now take a look at the following quantity

$$\left\| e^{t^\alpha (-\Delta)^{1/2}} e^{-t(-\Delta)^{\gamma/2}} g_0 \right\|_s = \int_{\mathbb{R}^d} |\xi|^s e^{t^\alpha |\xi| - t|\xi|^\gamma} |\hat{g}_0(\xi)| \, d\xi.$$

One may try to take $\alpha > 0$ as large as possible to increase the temporal rate of growth of the radius of analyticity of solutions to (1.18). A larger $\alpha > 0$ gives a stronger estimate if the norm above is finite. In the whole space $\mathbb{R}^d$, it can be shown for $\alpha = \frac{1}{\gamma}$ that $t^\alpha |\xi| - t|\xi|^\gamma \leqslant 1$ holds uniformly for $0 \leqslant |\xi| \leqslant \infty$ and $t \geqslant 0$. If $\alpha > \frac{1}{\gamma}$ then it can be shown that $t^\alpha |\xi| - t|\xi|^\gamma$ is unbounded and goes to infinity as $t \to \infty$ along a large range of $|\xi|$ directions. By this reasoning, in the whole space $\mathbb{R}^d$, the optimal rate of growth of the temporal radius of analyticity for the linear equation (1.18) is $\alpha = \frac{1}{\gamma}$ in the analytic weight term $e^{t^\alpha|\xi|}$ in the above norm.

In this paper the relevant linearized equation is (1.17) (linearized from (2.1) below) for which the optimal linear temporal growth rate of analyticity as above is $e^{t^{1/4}|\xi|}$ with $\alpha = \frac{1}{4}$. We prove in Theorem 1.4 that the non-linear equation (1.3) also enjoys the uniform global in time gain of analyticity as in (1.16) with rate $\nu(t) \approx t^{1/4}$, and in addition the analytic norm satisfies the global in time uniform optimal linear large time decay rates.

Alternatively in the torus $\mathbb{T}^d$, by a similar analysis one can see that the optimal rate of growth of the radius of analyticity for the linear equation (1.18) when $\gamma \geqslant 1$ is $\alpha = 1$ in the analytic weight term $e^{t^\alpha|\xi|}$. This improvement can be shown directly because the Fourier modes are discrete and one does not have to handle the situation where the modes are becoming arbitrarily small as $|\xi| \to 0$.

1.3 *Related results, and methods used in the proof*

A key point in our paper is to do a Taylor expansion of the exponential non-linearity as in (2.1) below. Then one can take advantage of the fact that after taking the Fourier transform, then the products in the expansion are transformed into convolutions. Therefore one can use the structure of spaces such as $\dot{\mathcal{F}}^{2,1}$ to get useful global in time estimates like (2.10) without experiencing significant loss. Here we mention previous work such as [4, 5] where a related strategy was employed for the Muskat problem. Then we can obtain the optimal large time decay rates in the whole space using the global in time bounds that we obtain such as in (2.10) in combination with Fourier splitting techniques. The techniques to obtain the decay rates in the whole space have a long history, and we just briefly refer to the methods in [25, 28] and the discussion therein. To prove the uniform gain of analyticity, we perform a different splitting involving derivatives of the radius of analyticity $\nu'(t)$ from (8.7), and we acknowledge the methods from [7, 24] and [9] that are used for different equations.

We also mention that, after the mathematics in this paper had been completed, the paper [15] was posted showing the global existence of at least one weak solution to the exponential PDE (1.3) and the exponential large time decay, working on the torus $\mathbb{T}^d_x$. The paper [15] also uses the Taylor expansion of the exponential nonlinearity, and the condition $\Delta h_0 \in A(\mathbb{T}^d)$ with an equivalent size condition.

We further mention the recent subsequent paper of Ambrose [1] who works on the same equation (1.3) again in the torus $\mathbb{T}^d_x$ in the related scale invariant norm

$$\|h\|_{\mathcal{B}^s_\alpha} \overset{\text{def}}{=} \sum_{k\in\mathbb{Z}^n} |k|^s \sup_{t\in[0,\infty)} e^{\alpha t|k|}|\hat{h}(t,k)|.$$

Compared to the norms in this paper and in [15], then [1] moves the $\sup_t$ inside. Then, using the notation in this paper, if initially $\|h_0\|_{\mathcal{B}^2_0} = \|h_0\|_{\dot{\mathcal{F}}^{2,1}(\mathbb{T}^d)} < 1/4$, then [1] obtains global existence and gain of analyticity in $\mathcal{B}^2_\alpha(\mathbb{T}^d)$. In the whole space case, in the analogous space $\mathcal{B}^2_\alpha(\mathbb{R}^d)$, [1] obtains a local in time existence theorem of analytic solutions for initial data that is small in $\dot{\mathcal{F}}^{2,1}(\mathbb{R}^d)$. The smallness constant on the initial data for the local existence theorem is not carefully tracked in the whole space case.

We would also like to mention [17] which studies the existence and uniqueness of fourth order equations using different methods.

1.4 *Self-similar solutions*

In this section we briefly mention the self-similar form of solutions to (1.3). We recall the scale invariance (1.4). We suppose that h is of the self-similar form

$$h(x,t) = t^{1/2} H(x/t^{1/4}). \tag{1.19}$$

Then h is invariant with respect to the scaling (1.4), and satisfies $||h(\cdot,t)||_2 = ||H||_2$ in the norm (1.7). Here we let $y = x/t^{1/4}$ be the self-similar variable. Further if the profile $H(y)$ satisfies the following equation

$$\frac{1}{2}H(y) - \frac{1}{4}y \cdot \nabla_y H(y) = e^{-\Delta_y H}\left(|\Delta_y \nabla_y H(y)|^2 - \Delta_y^2 H(y)\right),$$

then $h(x,t) = t^{1/2}H(x/t^{1/4})$ is a self-similar solution to (1.3). Here ∇_y and Δ_y are the gradient and the Laplacian in the variable y.

Note that the equation above can be equivalently written as

$$\frac{1}{2}H(y) - \frac{1}{4}y \cdot \nabla_y H(y) = \Delta_y e^{-\Delta_y H},$$

or alternatively after a Taylor expansion of the exponential it can be written as

$$\frac{2+d}{4}H(y) + \Delta_y^2 H = \frac{1}{4}\nabla_y \cdot (yH(y)) + \Delta_y \sum_{j=2}^{\infty} \frac{(-\Delta_y H)^j}{j!}. \tag{1.20}$$

Theorem 1.7 *Any self-similar solution of the form* (1.19) *to the equation* (1.3) *for which* $\|H\|_2 < y_*$, *for the constant* y_* *as in Remark* 1.2, *must satisfy* $\|H\|_2 = 0$.

Theorem 1.7 will be proven in Section 2.4.

1.5 *Outline of the paper*

The rest of the paper is organized as follows. In Section 2 we prove the a priori estimates for the exponential PDE (1.3) in the spaces $\dot{\mathcal{F}}^{s,p}$. Then in Section 3 we prove the large time decay rates in the whole space for a solution. After that in Section 4 we prove the uniform bounds in the Besov-type s-norms with negative indices including the critical index $\|h\|_{-d,\infty}$ where d is the dimension of $\mathbb{R}^d_x$. In Section 5 we prove the uniqueness of solutions. Then in Section 6 we sketch a proof of local existence and local gain of analyticity using an approximate regularized equation. And in Section 7 we explain how the results from the previous sections grant directly the proofs of Theorem 1.1 and Theorem 1.3. Lastly in Section 8 we explain how to obtain Theorem 1.4. This in particular uses the previous decay results (1.15) as well as previous results such as [7, 24]. In the Appendix A we present some plots of a few numerical simulations that were carried out for the exponential PDE (1.3) by Tom Witelski [31, 32].

2. A priori estimates

In this section we prove the apriori estimates for the exponential PDE in (1.3) and (2.1) in the spaces $\dot{\mathcal{F}}^{s,p}$ for $p \in [1,2]$. The key point is that we can prove a global in time Lyapunov inequality such as (2.10) below under an $O(1)$ medium size smallness condition on the initial data.

2.1 *A priori estimate in* $\dot{\mathcal{F}}^{2,1}$

We first establish the case of $\dot{\mathcal{F}}^{2,1}$ in order to explain the main idea in the simplest way. The equation (1.3) can be recast by Taylor expansion as

$$h_t + \Delta^2 h = \Delta \sum_{j=2}^{\infty} \frac{(-\Delta h)^j}{j!} \tag{2.1}$$

We look at this equation (2.1) using the Fourier transform (1.6) so that equation (1.3) is expressed as

$$\partial_t \hat{h}(\xi, t) + |\xi|^4 \hat{h}(\xi, t) = -|\xi|^2 \sum_{j=2}^{\infty} \frac{1}{j!} (|\cdot|^2 \hat{h})^{*j}(\xi, t)\,. \tag{2.2}$$

We multiply the above by $|\xi|^2$ to obtain

$$\partial_t |\xi|^2 \hat{h}(\xi, t) + |\xi|^6 \hat{h}(\xi, t) = -|\xi|^4 \sum_{j=2}^{\infty} \frac{1}{j!} (|\cdot|^2 \hat{h})^{*j}(\xi, t) \tag{2.3}$$

We will estimate this equation on the Fourier side in the following.

Our first step will be to estimate the infinite sum in (2.3). To this end notice that for any real number $s \geqslant 0$ the following triangle inequality holds:

$$|\xi|^s \leqslant j^{(s-1)^+} (|\xi - \xi_1|^s + \cdots + |\xi_{j-2} - \xi_{j-1}|^s + |\xi_{j-1}|^s), \tag{2.4}$$

where $(s-1)^+ = s-1$ if $s \geqslant 1$ and $(s-1)^+ = 0$ if $0 \leqslant s \leqslant 1$. We have further using the inequality (2.4) when $s \geqslant 1$ that

$$\int_{\mathbb{R}^d} |\xi|^s |(|\cdot|^2 \hat{h})^{*j}(\xi)|\, d\xi \leqslant j^s \int_{\mathbb{R}^d} |(|\cdot|^{s+2} \hat{h}) * (|\cdot|^2 \hat{h})^{*(j-1)}|\, d\xi \leqslant j^s \|h\|_{s+2} \|h\|_2^{j-1}. \tag{2.5}$$

Above we used Young's inequality repeatedly with $1 + 1 = 1 + 1$.

Observe that generally $\partial_t |\hat{h}| = \frac{1}{2}\big(\partial_t \hat{h} \overline{\hat{h}} + \hat{h} \overline{\partial_t \hat{h}}\big)|\hat{h}|^{-1}$. Now we multiply (2.3) by $\overline{\hat{h}} |\hat{h}|^{-1}(\xi, t)$, add the complex conjugate of the result, then integrate, and use (2.5) for $s = 4$ to obtain the following differential inequality

$$\frac{d}{dt} \|h\|_2 + \|h\|_6 \leqslant \|h\|_6 \sum_{j=2}^{\infty} \frac{j^4}{j!} \|h\|_2^{j-1}. \tag{2.6}$$

Now we denote the function

$$f_2(y) = \sum_{j=2}^{\infty} \frac{j^4}{j!} y^{j-1} = \sum_{j=1}^{\infty} \frac{(j+1)^3}{j!} y^j \tag{2.7}$$

Then (2.7) defines an entire function which is strictly increasing for $y \geqslant 0$ with $f_2(0) = 0$. In particular we choose the value y_* such that $f_2(y_*) = 1$.

Then (2.6) can be recast as

$$\frac{d}{dt}\|h\|_2 + \|h\|_6 \leqslant \|h\|_6 f_2\big(\|h\|_2\big) \tag{2.8}$$

If the initial data satisfies

$$\|h_0\|_2 < y_*, \tag{2.9}$$

then we can show that $\|h\|_2(t)$ is a decreasing function of t. Note that $y_* = y_{2*}$ in the notation from (2.18) below. In particular

$$f_2\big(\|h\|_2(t)\big) \leqslant f_2\big(\|h_0\|_2\big) < 1.$$

Using this calculation then (2.8) becomes

$$\frac{d}{dt}\|h\|_2 + \sigma_{2,1}\|h\|_6 \leqslant 0, \tag{2.10}$$

where

$$\sigma_{2,1} \overset{\text{def}}{=} 1 - f_2(\|h_0\|_2) > 0. \tag{2.11}$$

In particular if (2.9) holds, then $\|h\|_2(t) < y_*$ will continue to hold for a short time, which allows us to establish (2.10). The inequality (2.10) then defines a free energy and shows the dissipation production.

At the end of this section we look closer at the function $f_2(y)$:

$$f_2(y) = \sum_{j=1}^{\infty} \frac{(j+1)^3 y^j}{j!} = \sum_{j=1}^{\infty} \frac{(j(j-1)(j-2) + 6j(j-1) + 7j + 1)y^j}{j!},$$

which gives

$$f_2(y) = (y^3 + 6y^2 + 7y + 1)e^y - 1\,. \tag{2.12}$$

We know that $f_2(0) = 0$ and $f_2(y)$ is strictly increasing. Let y_* satisfy

$$(y_*^3 + 6y_*^2 + 7y_* + 1)e^{y_*} - 1 = 1 \tag{2.13}$$

Then $f_2(y_*) = 1$ as above.

To extend this analysis to the case where $s \neq 2$ we consider infinite series:

$$f_s(y) = \sum_{j=2}^{\infty} \frac{j^{s+2}}{j!} y^{j-1} = \sum_{j=1}^{\infty} \frac{(j+1)^{s+1}}{j!} y^j \tag{2.14}$$

Again $f_s(0) = 0$ and $f_s(y)$ is a strictly increasing entire function for any real s. We further remark that for $r \geqslant s$ we have the inequality

$$f_s(y) \leqslant f_r(y), \quad s \leqslant r, \quad \forall y \geqslant 0. \tag{2.15}$$

We further have a simple recursive relation

$$f_s(y) = \frac{d}{dy}\big(y f_{s-1}(y)\big), \quad f_{-1}(y) = e^y - 1.$$

This allows us to compute $f_s(y)$ for any s a non-negative integer as in (2.12).

2.2 *A priori estimate in the high order s-norm*

In this section we prove a high order estimate for any real number $s > -1$:

$$\partial_t |\xi|^s \hat{h}(\xi, t) + |\xi|^{s+4} \hat{h}(\xi, t) = -|\xi|^{s+2} \sum_{j=2}^{\infty} \frac{1}{j!} (|\xi|^2 \hat{h})^{*j}(\xi, t) \tag{2.16}$$

Using (2.5) and (2.16), one has

$$\frac{d}{dt}\|h\|_s + \|h\|_{s+4} \leqslant \|h\|_{s+4} \sum_{j=2}^{\infty} \frac{j^{s+2}}{j!} \|h\|_2^{j-1} \tag{2.17}$$

Now we recast (2.17) as

$$\frac{d}{dt}\|h\|_s + \|h\|_{s+4} \leqslant \|h\|_{s+4} f_s\big(\|h\|_2\big)$$

Let y_{s*} satisfy $f_s(y_{s*}) = 1$. If

$$\|h_0\|_2 < \min(y_{s*}, y_*). \tag{2.18}$$

Note that by (2.15) we have that $y_{s*} \leqslant y_{r*}$ for $s \leqslant r$. In particular we are using $y_{2*} = y_*$ in (2.9) and therefore $y_{s*} \leqslant y_*$ whenever $s \leqslant 2$.

Then by (2.10) we have

$$f_s\big(\|h(\cdot, t)\|_2\big) \leqslant f_s\big(\|h_0\|_2\big) < 1\,.$$

Hence we conclude the energy-dissipation relation

$$\frac{d}{dt}\|h(\cdot, t)\|_s + \sigma_{s,1}\|h(\cdot, t)\|_{s+4} \leqslant 0,\ . \tag{2.19}$$

when (2.18) holds. Here we define $\sigma_{s,1} \stackrel{\text{def}}{=} \big(1 - f_s(\|h_0\|_2\big) > 0$.

Alternatively if $s > -d$ and $-2 < s \leqslant -1$ then we by a similar procedure we use (2.4) to obtain that

$$\frac{d}{dt}\|h\|_s + \|h\|_{s+4} \leqslant \|h\|_{s+4} f_{-1}\big(\|h\|_2\big).$$

And for $\|h_0\|_2 < y_*$ we similarly obtain $\frac{d}{dt}\|h(\cdot, t)\|_s + \sigma_{-1,1}\|h(\cdot, t)\|_{s+4} \leqslant 0$.

2.3 *A priori estimate in $\dot{\mathcal{F}}^{s,p}$*

In this section we prove a general $\dot{\mathcal{F}}^{s,p}$ for $1 < p \leqslant 2$. We multiply the equation (2.2) by $p|\xi|^{sp}\overline{\hat{h}}|\hat{h}|^{p-2}(\xi, t)$ and add the complex conjugate equation to obtain

$$\begin{aligned}
\partial_t \Big(|\xi|^{sp}|\hat{h}|^p(\xi, t)\Big) &+ p|\xi|^{sp+4}|\hat{h}|^p(\xi, t) \\
&= -\frac{p}{2}\sum_{j=2}^{\infty}\frac{1}{j!}|\xi|^{sp+2}\overline{\hat{h}}|\hat{h}|^{p-2}(\xi, t)(|\xi|^2\hat{h})^{*j}(\xi, t) \\
&\quad - \frac{p}{2}\sum_{j=2}^{\infty}\frac{1}{j!}|\xi|^{sp+2}\hat{h}|\hat{h}|^{p-2}(\xi, t)(|\xi|^2\overline{\hat{h}})^{*j}(\xi, t).
\end{aligned} \tag{2.20}$$

We will estimate this equation when $p \in (1,2]$. To this end, we split $ps = (p-1)s + s$, and we split $2 = \frac{(p-1)}{p}4 + \left(\frac{4}{p} - 2\right)$. We will do a Hölder inequality with $\frac{p-1}{p} + \frac{1}{p} = 1$, then use (2.4), and then Young's inequality repeatedly with $1 + \frac{1}{p} = \frac{1}{p} + 1$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} & |\xi|^{ps+2} |\bar{\hat{h}}(\xi)| |\hat{h}|^{p-2}(\xi) (|\cdot|^2 \hat{h}(\cdot))^{*j}(\xi)|\, d\xi \\
&\leqslant \| |\xi|^{s+4/p} \hat{h} \|_{L^p}^{p-1} \| |\xi|^{s+\frac{4}{p}-2} (|\xi|^2 \hat{h})^{*j} \|_{L^p} \\
&\leqslant \| |\xi|^{s+4/p} \hat{h} \|_{L^p}^{p-1} j^{s+\frac{4}{p}-2} \| (|\xi|^{s+4/p} \hat{h}) * (|\xi|^2 \hat{h})^{*(j-1)} \|_{L^p} \\
&\leqslant j^{s+\frac{4}{p}-2} \|h\|_{\dot{\mathcal{F}}^{s+4/p,p}}^p \|h\|_2^{j-1}.
\end{aligned} \tag{2.21}$$

Above we have used that $s + \frac{4}{p} - 2 \geqslant 1$ to use the first inequality in (2.5), or $s + \frac{4}{p} = 2$.

We now use (2.20) and (2.21) to obtain

$$\frac{d}{dt}\|h\|_{\dot{\mathcal{F}}^{s,p}}^p + p\|h\|_{\dot{\mathcal{F}}^{s+4/p,p}}^p \leqslant p\|h\|_{\dot{\mathcal{F}}^{s+4/p,p}}^p \sum_{j=2}^{\infty} \frac{j^{s+4/p-2}}{j!} \|h\|_2^{j-1} \tag{2.22}$$

The sum in the upper bound is $f_{s+4/p-4}(\|h\|_2)$ from (2.14). Similar to the previous discussions, we choose the positive real number y_{sp*} to satisfy $f_{s+4/p-4}(y_{sp*}) = 1$.

Then if

$$\|h_0\|_2 < \min(y_{sp*}, y_*), \tag{2.23}$$

it further holds that $f_{s+4/p-4}\big(\|h(\cdot,t)\|_2\big) \leqslant f_{s+4/p-4}\big(\|h_0\|_2\big) < 1$. Hence we again have the energy-dissipation relation

$$\frac{d}{dt}\|h(\cdot,t)\|_{\dot{\mathcal{F}}^{s,p}}^p + p\sigma_{s,p}\|h(\cdot,t)\|_{\dot{\mathcal{F}}^{s+4/p,p}}^p \leqslant 0\,. \tag{2.24}$$

when (2.23) holds. Here $\sigma_{s,p} \overset{\text{def}}{=} \big(1 - f_{s+4/p-4}(\|h_0\|_2)\big) > 0$.

Thus we have proven the general $\dot{\mathcal{F}}^{s,p}$ estimate for $1 < p \leqslant 2$ and $s + \frac{4}{p} - 2 \geqslant 1$ or $s + \frac{4}{p} = 2$. For instance for $p = 2$ then we have shown (2.24) for $s \geqslant 1$ and $s = 0$. The estimate (2.24) for the remaining range of $s \geqslant 0$ and $1 < p \leqslant 2$ can be handled by an analogous procedure using (2.4) and a slight modification of (2.21).

In particular (2.24) holds for $s = 0$ and $p = 2$ under only the assumption (2.9) because by (2.15) we have that $y_{02*} \leqslant y_*$. Similarly $y_{22*} \leqslant y_*$ and (2.24) holds for $s = 2$ and $p = 2$ under only the assumption (2.9). These are the main two additional estimates that we will use in this paper.

2.4 *Proof of Theorem* 1.7 *regarding self-similar solutions*

In this section we will prove Theorem 1.7 using the estimates from the previous sub-sections.

We recall the dynamic equation (2.1) and apply the the Fourier transform (1.6) as in (2.2), then the self-similar equation (1.20) after Fourier transform is

$$\frac{2+d}{4}\hat{H}(\xi) + |\xi|^4\hat{H}(\xi) = \frac{1}{4}\xi \cdot \nabla_\xi \hat{H} - |\xi|^2 \sum_{j=2}^{\infty} \frac{1}{j!} (|\cdot|^2 \hat{H})^{*j}(\xi,t)\,. \tag{2.25}$$

We multiply the above by $|\xi|^2 \overline{\hat{H}} |\hat{H}|^{-1}(\xi)$ and perform the estimate exactly the same as in (2.6) to obtain that

$$\frac{2+d}{2} \|H\|_2 + \|H\|_6 \leqslant \|H\|_6 f_2(\|H\|_2).$$

We therefore conclude as in (2.8)–(2.13) that if $\|H\|_2 \neq 0$ then we must have that $\|H\|_2 > y*$ since $f_2(y^*) = 1$. This completes the proof of Theorem 1.7.

3. Large time decay

In this section we prove the following large time decay rates in the whole space.

Proposition 3.1 *Given the solution to* (1.3) *from Theorem* 1.1. *Suppose additionally that* $\|h_0\|_s < \infty$ *for some* $\max\{-2, -d\} < s \leqslant 2$. *Further suppose* $\|h_0\|_{-d,\infty} < \infty$. *Assume that* $\|h_0\|^2_{\dot{\mathcal{F}}^{2,2}}$ *and* $\|h_0\|^2_{\dot{\mathcal{F}}^{0,2}}$ *are both initially finite. Then we have the following uniform decay estimate for* $t \geqslant 0$*:*

$$\|h\|_s \lesssim (1+t)^{-(s+d)/4}. \tag{3.1}$$

The implicit constant in the inequality above depends on $\|h_0\|_2$, $\|h_0\|_s$, $\|h_0\|_{-d,\infty}$, $\|h_0\|^2_{\dot{\mathcal{F}}^{2,2}}$, *and* $\|h_0\|^2_{\dot{\mathcal{F}}^{0,2}}$.

Notice that this decay only depends on the smallness of the $\|h_0\|_2$ norm. No other norm is required to be small. Further notice that Proposition 3.1 directly implies (1.15) in Theorem 1.3

A key step in proving (3.1) is to prove the following uniform estimate:

Proposition 3.2 *Given the solution from Theorem* 1.1. *Suppose additionally that* $\|h_0\|_{-d,\infty} < \infty$, $\|h_0\|^2_{\dot{\mathcal{F}}^{2,2}} < \infty$, *and* $\|h_0\|^2_{\dot{\mathcal{F}}^{0,2}} < \infty$. *Then we have*

$$\|h\|_{-d,\infty} \lesssim 1. \tag{3.2}$$

The proof of Proposition 3.2 will be given in Section 4. The goal of this section is to establish (3.1) by assuming (3.2).

We will use the following decay lemma from Patel–Strain [25]:

Lemma 3.3 *Suppose* $g = g(t,x)$ *is a smooth function with* $g(0,x) = g_0(x)$ *and assume that for some* $\mu \in \mathbb{R}$, $\|g_0\|_\mu < \infty$ *and*

$$\|g(t)\|_{\rho,\infty} \leqslant C_0$$

for some $\rho \geqslant -d$ *satisfying* $\rho < \mu$. *Let the following differential inequality hold for* $\gamma > 0$ *and for some* $C > 0$*:*

$$\frac{d}{dt}\|g\|_\mu \leqslant -C\|g\|_{\mu+\gamma}.$$

Then we have the uniform in time estimate

$$\|g\|_\mu(t) \lesssim \left(\|g_0\|_\mu + C_0\right)(1+t)^{-(\mu-\rho)/\gamma}.$$

This lemma is stated in the paper [25] with $\gamma = 1$, however the similar proof below assumes only that $\gamma > 0$. We include the proof for completeness.

Proof. For some $\delta, \kappa > 0$ to be chosen, and $s \in \mathbb{R}$, we initially observe that

$$\begin{aligned}
\|g\|_\kappa &= \int_{\mathbb{R}^d} |\xi|^\kappa |\hat{g}(\xi)| d\xi \\
&\geqslant \int_{|\xi|>(1+\delta t)^s} |\xi|^\kappa |\hat{g}(\xi)| d\xi \\
&\geqslant (1+\delta t)^{s\beta} \int_{|\xi|>(1+\delta t)^s} |\xi|^{\kappa-\beta} |\hat{g}(\xi)| d\xi \\
&= (1+\delta t)^{s\beta} \Big(\|g\|_{\kappa-\beta} - \int_{|\xi|\leqslant(1+\delta t)^s} |\xi|^{\kappa-\beta} |\hat{g}(\xi)| d\xi \Big).
\end{aligned}$$

Using this inequality with $\kappa = \mu + \gamma$ and $\beta = \gamma$, we obtain that

$$\begin{aligned}
\frac{d}{dt}\|g\|_\mu + C(1+\delta t)^{s\gamma}\|g\|_\mu &\leqslant -C\|g\|_{\mu+\gamma} + C(1+\delta t)^{s\gamma}\|g\|_\mu \\
&\leqslant C(1+\delta t)^{s\gamma} \int_{|\xi|\leqslant(1+\delta t)^s} |\xi|^\mu |\hat{g}(\xi)| d\xi.
\end{aligned}$$

Then, using the sets C_k as in (1.9) and defining χ_S to be the characteristic function on a set S, the upper bound in the last inequality can be bounded as follows

$$\begin{aligned}
\int_{|\xi|\leqslant(1+\delta t)^s} |\xi|^\mu |\hat{g}(\xi)| d\xi &= \sum_{k\in\mathbb{Z}} \int_{C_k} \chi_{\{|\xi|\leqslant(1+\delta t)^s\}} |\xi|^\mu |\hat{g}| \, d\xi \\
&\approx \sum_{2^k\leqslant(1+\delta t)^s} \int_{C_k} |\xi|^\mu |\hat{g}| \, d\xi \\
&\lesssim \|g\|_{\rho,\infty} \sum_{2^k\leqslant(1+\delta t)^s} 2^{k(\mu-\rho)} \\
&\lesssim \|g\|_{\rho,\infty}(1+\delta t)^{s(\mu-\rho)} \sum_{2^k(1+\delta t)^{-s}\leqslant 1} 2^{k(\mu-\rho)}(1+\delta t)^{-s(\mu-\rho)} \\
&\lesssim \|g\|_{\rho,\infty}(1+\delta t)^{s(\mu-\rho)} \\
&\lesssim C_0(1+\delta t)^{s(\mu-\rho)},
\end{aligned}$$

where the implicit constant in the inequalities does not depend on t. In particular we have used that the following uniform in time estimate holds

$$\sum_{2^k(1+\delta t)^{-s}\leqslant 1} 2^{k(\mu-\rho)}(1+\delta t)^{-s(\mu-\rho)} \lesssim 1.$$

Combining the above inequalities, we obtain that

$$\frac{d}{dt}\|g\|_\mu + C(1+\delta t)^{s\gamma}\|g\|_\mu \lesssim C_0(1+\delta t)^{s\gamma}(1+\delta t)^{s(\mu-\rho)}. \tag{3.3}$$

In the following estimate will use (3.3) with $s = -1/\gamma$, we suppose $a > (\mu - \rho)/\gamma > 0$, and we choose $\delta > 0$ such that $a\delta = C$. We then obtain that

$$\begin{aligned}\frac{d}{dt}((1+\delta t)^a \|g\|_\mu) &= (1+\delta t)^a \frac{d}{dt}\|g\|_\mu + a\delta \|g\|_\mu (1+\delta t)^{a-1} \\ &= (1+\delta t)^a \frac{d}{dt}\|g\|_\mu + C\|g\|_\mu (1+\delta t)^{a-1} \\ &= (1+\delta t)^a \Big(\frac{d}{dt}\|g\|_\mu + C(1+\delta t)^{-1}\|g\|_\mu\Big) \\ &\lesssim C_0 (1+\delta t)^{a-1-(\mu-\rho)/\gamma}.\end{aligned}$$

Since $a > (\mu - \rho)/\gamma$, we integrate in time to obtain that

$$(1+\delta t)^a \|g(t)\|_\mu \lesssim \|g_0\|_\mu + \frac{C_0}{\delta}(1+\delta t)^{a-(\mu-\rho)/\gamma}.$$

We conclude our proof by dividing both sides of the inequality by $(1+\delta t)^a$. □

We have established the differential energy inequalities (2.10) and (2.19) for the equation (1.3). Thus to prove the time decay in (3.1) it remains only to establish (3.2).

4. Proof of the uniform bound $\|h\|_{-d,\infty}(t) \lesssim 1$

In this section we will prove the uniform bound in (3.2).

Proof of Proposition 3.2. We recall (2.2), and we uniformly bound the integral over C_k for each $j \in \mathbb{Z}$ as in (1.8) and (1.9). We obtain the following differential inequality

$$\frac{d}{dt}\int_{C_k} |\xi|^{-d} |\hat{h}(\xi,t)| d\xi + \int_{C_k} d\xi\, |\xi|^{-d+4} |\hat{h}(\xi,t)| \leqslant \sum_{j=2}^{\infty} \frac{1}{j!} \int_{C_k} d\xi\, |\xi|^{-d+2} (|\xi|^2 |\hat{h}|)^{*j}(\xi,t). \tag{4.1}$$

We will estimate the upper bound. We can estimate the integral as

$$\begin{aligned}\int_{C_k} d\xi\, |\xi|^{-d+2} (|\xi|^2 \hat{h})^{*j}(\xi,t) &\lesssim 2^{-d(k-1)} \int_{C_k} d\xi\, |\xi|^2 (|\xi|^2 \hat{h})^{*j}(\xi,t) \\ &\lesssim \||\cdot|^2 (|\cdot|^2 \hat{h})^{*j}(\xi,t)\|_{L^\infty_\xi}\end{aligned} \tag{4.2}$$

The last inequality holds because the integral over C_k is of size 2^{dk}.

We will use (2.4). We also use Young's inequality, first with $1 + \frac{1}{\infty} = \frac{1}{2} + \frac{1}{2}$, and again with $1 + \frac{1}{2} = 1 + \frac{1}{2}$ repeatedly to obtain:

$$\begin{aligned}\||\cdot|^2 (|\cdot|^2 \hat{h})^{*j}(\cdot)\|_{L^\infty_\xi} &\lesssim j^2 \||\cdot|^4 \hat{h}(\cdot)\|_{L^2_\xi} \|(|\cdot|^2 \hat{h})^{*(j-1)}(\cdot)\|_{L^2_\xi} \\ &\lesssim j^2 \|h\|_{\dot{\mathcal{F}}^{4,2}} \||\cdot|^2 \hat{h}(\cdot)\|_{L^2_\xi} \||\cdot|^2 \hat{h}(\cdot)\|_{L^1_\xi}^{j-2} \\ &\lesssim j^2 \|h\|_{\dot{\mathcal{F}}^{4,2}} \|h\|_{\dot{\mathcal{F}}^{2,2}} \|h\|_2^{j-2}.\end{aligned} \tag{4.3}$$

Now we plug (4.2) into (4.1) to obtain

$$\frac{d}{dt}\int_{C_k} |\xi|^{-d}|\hat{h}(\xi,t)|d\xi + \int_{C_k} d\xi\, |\xi|^{-d+4}|\hat{h}(\xi,t)| \leqslant \sum_{j=2}^{\infty}\frac{1}{j!}\||\cdot|^2(|\cdot|^2\hat{h})^{*j}(\cdot)\|_{L^\infty_\xi}. \tag{4.4}$$

We further estimate the upper bound using (4.3) to obtain

$$\sum_{j=2}^{\infty}\frac{1}{j!}\||\cdot|^2(|\cdot|^2\hat{h})^{*j}(\cdot,t)\|_{L^\infty_\xi} \lesssim \|h(t)\|_{\dot{\mathcal{F}}^{4,2}}\|h(t)\|_{\dot{\mathcal{F}}^{2,2}}\sum_{j=2}^{\infty}\frac{j^2}{j!}\|h(t)\|_2^{j-2}$$
$$\lesssim \|h(t)\|_{\dot{\mathcal{F}}^{4,2}}\|h(t)\|_{\dot{\mathcal{F}}^{2,2}}.$$

In the above we have used that

$$\sum_{j=2}^{\infty}\frac{j^2}{j!}\|h(t)\|_2^{j-2} \lesssim 1.$$

The above holds because the sum initially $\sum_{j=2}^{\infty}\frac{j^2}{j!}\|h_0\|_2^{j-2} \lesssim 1$ converges generally. Then we further use the estimate (2.10) to see that $\|h(t)\|_2 \leqslant \|h_0\|_2$.

We conclude from integrating (4.4) and using the above estimates that

$$\int_{C_k} |\xi|^{-d}|\hat{h}(\xi,t)|d\xi + \int_0^t ds\int_{C_k} d\xi\, |\xi|^{-d+4}|\hat{h}(\xi,s)|$$
$$\leqslant \int_{C_k} |\xi|^{-d}|\hat{h}_0(\xi)|d\xi + \int_0^t ds\|h(t)\|_{\dot{\mathcal{F}}^{4,2}}\|h(t)\|_{\dot{\mathcal{F}}^{2,2}}. \tag{4.5}$$

Thus as long as we make the proper assumptions to bound

$$\int_0^t ds\|h(t)\|_{\dot{\mathcal{F}}^{4,2}}\|h(t)\|_{\dot{\mathcal{F}}^{2,2}} \leqslant \sqrt{\int_0^t ds\|h(t)\|^2_{\dot{\mathcal{F}}^{4,2}}\,dt}\sqrt{\int_0^t \|h(t)\|^2_{\dot{\mathcal{F}}^{2,2}}\,dt} \lesssim 1, \tag{4.6}$$

then the bound (4.5) with (4.6) implies Proposition 3.2.

However when $\|h_0\|_{\dot{\mathcal{F}}^{2,2}}$ and $\|h_0\|_{\dot{\mathcal{F}}^{0,2}}$ are both initially finite then (2.24) implies that (4.6) holds. Here we have used (2.24) with $p = 2$, $s = 2$ and $s = 0$, respectively. Then we obtain the bound (3.2). □

5. Uniqueness

In this section we prove the uniqueness of solutions to (1.3) which satisfy (2.9).

Proposition 5.1 *Given two solutions h_1 and h_2 to* (1.3) *with the same initial data h_0 satisfying* (2.9). *Then $\|h_1 - h_2\|_2 = 0$. If we further assume that the initial data satisfies $\|h_0\|_0 < \infty$, then $\|h_1 - h_2\|_0 = 0$. In particular $\|h_1 - h_2\|_{L^\infty_x} = 0$.*

Proof of Proposition 5.1. We consider the equation (2.1) satisfied by both h_1 and h_2. Then we have that

$$(h_1 - h_2)_t + \Delta^2(h_1 - h_2) = \Delta\sum_{j=2}^{\infty}\frac{(-\Delta h_1)^j - (-\Delta h_2)^j}{j!}.$$

We further have the algebraic identity

$$(-\Delta h_1)^j - (-\Delta h_2)^j = -(\Delta h_1 - \Delta h_2)\left(\sum_{m=0}^{j-1}(-\Delta h_1)^{j-1-m}(-\Delta h_2)^m\right).$$

We take the Fourier transform to obtain

$$\partial_t\left(\hat{h}_1(\xi,t) - \hat{h}_2(\xi,t)\right) + |\xi|^4\left(\hat{h}_1(\xi,t) - \hat{h}_2(\xi,t)\right)$$
$$= -|\xi|^2\left(|\cdot|^2\left(\hat{h}_1 - \hat{h}_2\right)\right) * \sum_{j=2}^{\infty}\frac{1}{j!}\left(\sum_{m=0}^{j-1}(|\cdot|^2\hat{h}_1)^{*(j-1-m)} * (|\cdot|^2\hat{h}_2)^{*m}\right). \quad (5.1)$$

Then we obtain that

$$\frac{d}{dt}\|h_1 - h_2\|_s + \|h_1 - h_2\|_{s+4} = \mathcal{U}_s. \quad (5.2)$$

Above $\mathcal{U}_s$ is the integral of the right side of (5.1) multiplied by $|\xi|^s$:

$$\mathcal{U}_s \overset{\text{def}}{=} -\int_{\mathbb{R}^d} d\xi\, |\xi|^{2+s}\sum_{j=2}^{\infty}\frac{1}{j!}\left(\sum_{m=0}^{j-1}\left(|\cdot|^2\left(\hat{h}_1 - \hat{h}_2\right)\right) * (|\cdot|^2\hat{h}_1)^{*(j-1-m)} * (|\cdot|^2\hat{h}_2)^{*m}\right).$$

We will consider the cases $s = 2$ and then $s = 0$.

When $s = 2$ above, we use (2.4) and Young's inequality to obtain

$$\mathcal{U}_2 \leqslant \|h_1 - h_2\|_6\sum_{j=2}^{\infty}\frac{j^4}{j!}\max\{\|h_1\|_2, \|h_2\|_2\}^{j-1}$$
$$+ \|h_1 - h_2\|_2\max\{\|h_1\|_6, \|h_2\|_6\}\sum_{j=2}^{\infty}\frac{j^5}{j!}\max\{\|h_1\|_2, \|h_2\|_2\}^{j-2}.$$

Above we use $\max\{\|h_1\|_2, \|h_2\|_2\}$ only to reduce the number of terms that we need to write down.

Now since the initial data satisfies (2.9) then we have

$$\sum_{j=2}^{\infty}\frac{j^4}{j!}\max\{\|h_1\|_2, \|h_2\|_2\}^{j-1} \leqslant \sum_{j=2}^{\infty}\frac{j^4}{j!}\|h_0\|_2^{j-1} < 1.$$

Also $\sum_{j=2}^{\infty}\frac{j^5}{j!}\max\{\|h_1\|_2, \|h_2\|_2\}^{j-2} \leqslant \sum_{j=2}^{\infty}\frac{j^5}{j!}\|h_0\|_2^{j-2} \lesssim 1$. Then after integrating (5.2) with $s = 2$ in time, we obtain that

$$\|h_1 - h_2\|_2(t) + \sigma\int_0^t \|h_1 - h_2\|_6(s)\, ds \lesssim \int_0^t \|h_1 - h_2\|_2(s)\max\{\|h_1\|_6, \|h_2\|_6\}(s)\, ds.$$

Here we use $\sigma = \sigma_{2,1} > 0$ from (2.11). Notice that $\int_0^t \max\{\|h_1\|_6, \|h_2\|_6\}(s)\, ds < \infty$ by (2.10). Now the Gronwall inequality implies that $\|h_1 - h_2\|_2 = 0$.

We turn to the case $s = 0$ in (5.2). We will obtain an upper bound for $\mathcal{U}_0$. We will use (2.4) (with $s = 2$ in (2.4)). Then with Young's inequality we obtain

$$\mathcal{U}_0 \leqslant \|h_1 - h_2\|_4 \sum_{j=2}^{\infty} \frac{j^2}{j!} \max\{\|h_1\|_2, \|h_2\|_2\}^{j-1} + \|h_1 - h_2\|_2 \max\{\|h_1\|_4, \|h_2\|_4\} \sum_{j=2}^{\infty} \frac{j^3}{j!} \max\{\|h_1\|_2, \|h_2\|_2\}^{j-2}.$$

However we know from the previous case that $\|h_1 - h_2\|_2 = 0$ therefore the second term above is zero. Then similar to the previous case, for a $\delta > 0$ we obtain

$$\|h_1 - h_2\|_0(t) + \delta \int_0^t \|h_1 - h_2\|_4(s)\, ds \leqslant 0.$$

We conclude that $\|h_1 - h_2\|_0 = 0$. □

We remark that the same methods can be used to prove that $\|h_1 - h_2\|_{\dot{\mathcal{F}}^{0,2}} = 0$.

6. Local existence and approximation

In this section we prove the local existence theorem using a suitable approximation scheme. Since the methods in this section are rather standard, therefore we provide a sketch of the key ideas.

Proposition 6.1 *Consider initial data $h_0 \in \dot{\mathcal{F}}^{0,2}$ further satisfying $\|h_0\|_2 < y_*$ where $y_* > 0$ is given explicitly in Remark 1.2.*

Then there exists a time $T > 0$ and an interval $[0, T]$ upon which we have a local in time unique solution to (1.3) given by $h(t) \in C^0([0, T]; \dot{\mathcal{F}}^{0,2} \cap \dot{\mathcal{F}}^{2,1})$. This solution also gains instant analyticity on $[0, T]$ as

$$\|h\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}(t) \leqslant \|h_0\|_2 e^{bT},$$

where $\nu(t) = bt$ for some fixed $b \in (0, 1)$ with $b = b(\|h_0\|_2)$.

To prove this we perform a regularization of (2.1) as follows. Let ζ_t be the heat kernel in $\mathbb{R}^d$ for $t > 0$. We will consider ζ_ϵ with $\epsilon > 0$ so that ζ_ϵ is an approximation to the identity as $\epsilon \to 0$. We define the regularized equation as:

$$\partial_t h^\epsilon + \Delta^2(\zeta_\epsilon * \zeta_\epsilon * h^\epsilon) = \Delta \sum_{j=2}^{\infty} \frac{(-\Delta(\zeta_\epsilon * \zeta_\epsilon * h^\epsilon))^j}{j!}, \quad h_0^\epsilon = \zeta_\epsilon * h_0. \tag{6.1}$$

This regularized system (6.1) can be directly estimated using all of the apriori estimates from the previous sections. In particular all the previous estimates for (1.3) in this paper continue to straightforwardly apply to the approximate problem (6.1).

These estimates allow us to prove a local existence theorem for the regularized system (6.1) using the Picard theorem on a Banach space $C^0([0, T_\epsilon]; \dot{\mathcal{F}}^{4,2} \cap \dot{\mathcal{F}}^{0,2})$. We find the abstract evolution system given by $\partial_t h^\epsilon = F(h^\epsilon)$ where F is Lipschitz on the open set $\{f(x) \in \dot{\mathcal{F}}^{4,2} \cap \dot{\mathcal{F}}^{0,2} : \|f\|_{\dot{\mathcal{F}}^{2,1}} < y_*\}$. Observe that $h_0^\epsilon \in \dot{\mathcal{F}}^{4,2}$ since $h_0 \in \dot{\mathcal{F}}^{0,2}$. Further, since the convolutions are taken with the heat kernel, we can prove analyticity for h^ϵ.

In particular directly following (2.19) we obtain for some $\delta_1 > 0$ that

$$\frac{d}{dt}\|h^\epsilon(t)\|_2 + \delta_1\|\zeta_\epsilon * \zeta_\epsilon * h^\epsilon\|_6 \leqslant 0.$$

Similarly following following (2.24) we obtain for some $\delta_2 > 0$ that

$$\frac{d}{dt}\|h^\epsilon(t)\|^2_{\dot{\mathcal{F}}^{0,2}} + \delta_2\|\zeta_\epsilon * \zeta_\epsilon * h^\epsilon\|^2_{\dot{\mathcal{F}}^{2,2}} \leqslant 0\,.$$

From these estimates, we can obtain the convergence needed to take a limit as $\epsilon \to 0$ in (6.1) and obtain the unique solution from Proposition 6.1 on the uniform time interval $[0, T]$ for some $T > 0$.

In the following we show how to to reach analyticity in short time as in Proposition 6.1. The approximation scheme in (6.1) is well designed to reach the analytic regime in short time, and maintain the analyticity in the limit as $\epsilon \to 0$. Below, we explain the gain of analyticity with the a priori estimate.

6.1 *Reach analyticity in a short time*

We use $\nu(t) = bt$ (for some $b > 0$ to be determined) in the analytic space (1.12) with $s = 2$ and $p = 1$. Note that $|\xi|^3 \leqslant |\xi|^6 + |\xi|^2$. From (2.3) we obtain the following differential inequality:

$$\frac{d}{dt}\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu} + (1-\nu'(t))\|h\|_{\dot{\mathcal{F}}^{6,1}_\nu} \leqslant \nu'(t)\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \|h\|_{\dot{\mathcal{F}}^{6,1}_\nu}\sum_{j=2}^{\infty}\frac{j^4}{j!}\|h\|^{j-1}_{\dot{\mathcal{F}}^{2,1}_\nu} \tag{6.2}$$

We recall f_2 in (2.7) and (2.14). We have the estimate

$$\frac{d}{dt}\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu} + (1-b)\|h\|_{\dot{\mathcal{F}}^{6,1}_\nu} \leqslant b\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \|h\|_{\dot{\mathcal{F}}^{6,1}_\nu} f_2(\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu}) \tag{6.3}$$

Recalling (2.9), we can choose a small $T > 0$ such that

$$\|h_0\|_2 e^{bT} < y_*.$$

Then by choosing T smaller if necessary, on $0 \leqslant t \leqslant T$ by continuity we have

$$f_2(\|h(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu}) \leqslant b_1 < 1,$$

where $b_1 = b_1(\|h_0\|_2, T)$. We choose $b > 0$ small enough so that $\delta = 1 - b - b_1 > 0$, and then we have

$$\frac{d}{dt}\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \delta\|h\|_{\dot{\mathcal{F}}^{6,1}_\nu} \leqslant b\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu}\,. \tag{6.4}$$

Then we apply the Grönwall inequality to (6.4) to obtain

$$\|h\|_{\dot{\mathcal{F}}^{2,1}_\nu}(t) \leqslant \|h_0\|_2 e^{bT} \leqslant y_*.$$

This completes the proof of the gain of analyticity, and Proposition 6.1.

7. Global existence

In this section we briefly collect our previous estimates and explain the proofs of Theorem 1.1 and Theorem 1.3. Note that if (2.9) holds then (2.10) also holds. Also (2.24) holds for $s = 0$ and $p = 2$ under only the assumption (2.9) because by (2.15) we have that $y_{02*} \leqslant y_*$. These global in time bounds combined with Proposition 5.1 and Proposition 6.1 yield directly the proof of Theorem 1.1.

We now explain the proof of Theorem 1.3. Notice that the analysis in Section 2.2 directly yields (1.14). And the fast large time decay rates (1.15) follow from Proposition 3.1 and Proposition 3.2 under the assumptions used in the statement of (1.15).

8. Long time existence and decay in the analytic norms

In this section we will present finally the proof of Theorem 1.4. This will show the global in time uniform gain of analyticity with radius of analyticity that grows like $t^{1/4}$ for large $t \gtrsim 1$. This will also show the uniform large time decay rates of the analytic norms with the optimal linear decay rate as in Remark 1.5.

8.1 *Gevrey estimates and the radius of analyticity*

We consider $\nu(t) > 0$, and now we look at estimates for (2.1) in the $\dot{\mathcal{F}}^{s,1}_{\nu}$ space with $0 \leqslant s \leqslant 2$. We will show that the radius of analyticity grows like $\nu(t) \approx t^{1/4}$ which is the optimal linear growth rate as in Remark 1.6.

We multiply (2.2) by $|\xi|^s e^{\nu(t)|\xi|}\bar{\hat{h}}|\hat{h}|^{-1}(\xi,t)$ and then we obtain

$$
\begin{aligned}
\partial_t \left(|\xi|^s e^{\nu(t)|\xi|}|\hat{h}|(\xi,t)\right) - \nu'(t)|\xi|^{s+1}e^{\nu(t)|\xi|}|\hat{h}|(\xi,t) + |\xi|^{s+4}e^{\nu(t)|\xi|}|\hat{h}|(\xi,t)&\\
= -\frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j!}|\xi|^{s+2}e^{\nu(t)|\xi|}\bar{\hat{h}}|\hat{h}|^{-1}(\xi,t)(|\xi|^2\hat{h})^{*j}(\xi,t)&\\
-\frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j!}|\xi|^{s+2}e^{\nu(t)|\xi|}\hat{h}|\hat{h}|^{-1}(\xi,t)(|\xi|^2\bar{\hat{h}})^{*j}(\xi,t).& \quad (8.1)
\end{aligned}
$$

To estimate the nonlinear term on the right side we use Young's inequality as

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\xi|^{s+2}e^{\nu(t)|\xi|}|\bar{\hat{h}}(\xi)|\hat{h}|^{-1}(\xi)(|\cdot|^2\hat{h}(\cdot))^{*j}(\xi)|\,d\xi&\\
\leqslant \| |\xi|^{s+2}e^{\nu(t)|\xi|}(|\xi|^2\hat{h})^{*j}\|_{L^1_\xi}&\\
\leqslant j^{s+2}\|(|\xi|^{s+4}e^{\nu(t)|\xi|}\hat{h}) * (|\xi|^2 e^{\nu(t)|\xi|}\hat{h})^{*(j-1)}\|_{L^1_\xi}&\\
\leqslant j^{s+2}\|f\|_{\dot{\mathcal{F}}^{s+4,1}_{\nu}}(t)\|h\|^{j-1}_{\dot{\mathcal{F}}^{2,1}_{\nu}}.& \quad (8.2)
\end{aligned}
$$

Now we use (8.2) to obtain the following estimate

$$
\frac{d}{dt}\|h\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} \leqslant \nu'(t)\|h\|_{\dot{\mathcal{F}}^{s+1,1}_{\nu}} - \|h\|_{\dot{\mathcal{F}}^{s+4,1}_{\nu}} + \|h\|_{\dot{\mathcal{F}}^{s+4,1}_{\nu}}\sum_{j=2}^{\infty}\frac{j^{s+2}}{j!}\|h\|^{j-1}_{\dot{\mathcal{F}}^{2,1}_{\nu}}. \quad (8.3)
$$

We will use (8.3) to simultaneously prove a global bound and large time decay rates.

For now we will focus on the second two terms on the left side of (8.3). We use Hölder's inequality with $s+1 = \frac{3}{4}s + \frac{1}{4}(s+4)$, then Young's inequality with $\frac{3}{4}+\frac{1}{4} = 1$, and multiply and divide by $\nu(t)^{3/4}$ and then we have that

$$\|h\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} \leqslant \|h\|^{3/4}_{\dot{\mathcal{F}}^{s,1}_\nu} \|h\|^{1/4}_{\dot{\mathcal{F}}^{s+4,1}_\nu} \leqslant \frac{3}{4}\nu(t)^{-1}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} + \frac{1}{4}\nu(t)^3\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}. \tag{8.4}$$

Further, since $e^x \leqslant 1 + xe^x$ for $x \geqslant 0$, using also (8.4) we also have that

$$\begin{aligned}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} &= \int_{\mathbb{R}^d} |\xi|^s e^{\nu(t)|\xi|}|\hat{h}(\xi,t)|d\xi \\ &\leqslant \int_{\mathbb{R}^d} |\xi|^s|\hat{h}(\xi,t)|d\xi + \nu(t)\int_{\mathbb{R}^d} |\xi|^{s+1}e^{\nu(t)|\xi|}|\hat{h}(\xi,t)|d\xi \\ &\leqslant \|h\|_{\dot{\mathcal{F}}^{s,1}} + \frac{3}{4}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} + \frac{\nu(t)^4}{4}\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}.\end{aligned}$$

We conclude that

$$\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leqslant 4\|h\|_{\dot{\mathcal{F}}^{s,1}} + \nu(t)^4\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}. \tag{8.5}$$

This is one estimate that we will use just below.

Looking at (8.3), now we estimate the following difference using (8.4) and (8.5)

$$\begin{aligned}\nu'(t)\|h\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} &\leqslant \frac{3}{4}\nu'(t)\nu(t)^{-1}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} + \frac{1}{4}\nu'(t)\nu(t)^3\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} - \|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} \\ &\leqslant 3\nu'(t)\nu(t)^{-1}\|h\|_{\dot{\mathcal{F}}^{s,1}} + \frac{3}{4}\nu'(t)\nu(t)^{-1}\nu(t)^4\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} \\ &\quad + \frac{1}{4}\nu'(t)\nu(t)^3\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} - \|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}.\end{aligned} \tag{8.6}$$

We will also use this estimate momentarily.

First with $b > 0$ from Proposition 6.1, for $t \geqslant 0$, we choose

$$\nu(t) = ((bt_0)^4 + at)^{1/4}, \tag{8.7}$$

for some $t_0 > 0$ and $a > 0$ to be determined. Then $\nu'(t) = \frac{a}{4}((bt_0)^4 + at)^{-3/4}$ and $\nu'(t) = \frac{a}{4}\nu(t)^{-3}$. Further then

$$\nu'(t)\nu(t)^{-1}\nu(t)^4 = \frac{a}{4},$$

and $\nu'(t)\nu(t)^{-1} = \frac{a}{4}\nu(t)^{-4}$.

And then from (8.6) we have

$$\nu'(t)\|h\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} \leqslant \frac{3a}{4}\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}} - \left(1 - \frac{a}{4}\right)\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}.$$

Later we will choose $0 < a < 4$ small.

Now from (8.5) we have

$$-\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} \leqslant -\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} + 4\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}}.$$

We choose $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = 1$. Then we have

$$\nu'(t)\|h\|_{\dot{\mathcal{F}}^{s+1,1}_\nu} - \|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}$$
$$\leqslant \left(\frac{3a}{4} + 4\alpha\left(1-\frac{a}{4}\right)\right)\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}} - \alpha\left(1-\frac{a}{4}\right)\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} - \beta\left(1-\frac{a}{4}\right)\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}.$$

These are the main upper bounds that we will use next.

Now returning to (8.3), we obtain the following differential inequality

$$\frac{d}{dt}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} + \delta\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leqslant \lambda\nu(t)^{-4}\|h\|_{\dot{\mathcal{F}}^{s,1}} - \kappa\|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu} + \|h\|_{\dot{\mathcal{F}}^{s+4,1}_\nu}\sum_{j=2}^{\infty}\frac{j^{s+2}}{j!}\|h\|^{j-1}_{\dot{\mathcal{F}}^{2,1}_\nu}. \tag{8.8}$$

In the above $\delta = \alpha\big(1-\frac{a}{4}\big) > 0$ is a small constant and $0 < \kappa \overset{\text{def}}{=} \beta\big(1-\frac{a}{4}\big) < 1$ can be chosen arbitrarily close to 1. Further $\lambda \overset{\text{def}}{=} \big(\frac{3a}{4} + 4\alpha\big(1-\frac{a}{4}\big)\big) > 0$ can be chosen to be small.

We will use the estimate (8.8), combined with the following procedure to obtain the global decay of the analytic norm with radius (8.7). For now we restrict to the case $s = 2$. We start with the solution from Theorem 1.1 with initial data satisfying (2.9). Further as in Theorem 1.3 we assume that $h_0 \in \dot{\mathcal{F}}^{2,2}$, $h_0 \in \dot{\mathcal{F}}^{0,2}$, and $\|h_0\|_{-d,\infty} < \infty$. Then from (1.15) we can choose a large time $T_1 = T_1(\epsilon') > 0$ such that for $t \geqslant 0$ we have

$$\|h(T_1 + t)\|_2 \leqslant \epsilon'\nu(t)^{-(2+d)}, \tag{8.9}$$

where we will choose $\epsilon' > 0$ small in a moment.

From the local existence result in Proposition 6.1, we know that equation (2.1) has a gain of analyticity on a local time interval starting with the initial data described in the previous paragraph. We take initial data that for the gain of analyticity as $\|h(T_1)\|_2 < \epsilon$ where $\epsilon = \epsilon(\epsilon', t_0) > 0$ is small and $\epsilon \to 0$ as $\epsilon' \to 0$. Then for a short time interval $[T_1, T_1 + 2t_0]$ from Proposition 6.1 we still have

$$\|h(T_1 + t)\|_{\dot{\mathcal{F}}^{2,1}_\mu} < \epsilon$$

for all $t \in [T_1, T_1 + 2t_0]$ where as in Proposition 6.1 we use $\mu(t) = bt$.

This is how we choose $t_0 > 0$ small from (8.7) to guarantee the above based upon our choice of ϵ. We use estimate (8.8) with $s = 2$ starting at time $T_1 + t_0$ with $\nu(t) = ((bt_0)^4 + at)^{1/4}$ as in (8.7). To ease the notation, in the rest of this paragraph we write $\tilde{h}(t) = h(T_1 + t_0 + t)$ and $\tilde{h}_0 = h(T_1 + t_0)$. Now following arguments analogous to below (2.8) using (8.8) with $s = 2$ we have

$$\frac{d}{dt}\|\tilde{h}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \delta\nu(t)^{-4}\|\tilde{h}\|_{\dot{\mathcal{F}}^{2,1}_\nu} \leqslant \lambda\nu(t)^{-4}\|\tilde{h}\|_{\dot{\mathcal{F}}^{2,1}} - \kappa\|\tilde{h}\|_{\dot{\mathcal{F}}^{6,1}_\nu} + \|\tilde{h}\|_{\dot{\mathcal{F}}^{6,1}_\nu}\sum_{j=2}^{\infty}\frac{j^4}{j!}\|\tilde{h}\|^{j-1}_{\dot{\mathcal{F}}^{2,1}_\nu}$$
$$\leqslant \epsilon\lambda\nu(t)^{-(2+d)-4} - C_1\|\tilde{h}\|_{\dot{\mathcal{F}}^{6,1}_\nu} \leqslant C\lambda\nu(t)^{-(2+d)-4}. \tag{8.10}$$

Here we used that $\|\tilde{h}\|_{\dot{\mathcal{F}}^{2,1}} \lesssim \nu(t)^{-(2+d)}$. Above we can take $C > 0$ and $C_1 > 0$ since, as above, we can choose $\|\tilde{h}\|_{\dot{\mathcal{F}}^{2,1}_\nu}$ to be arbitrarily small. We multiply by $\nu(t)^{4\delta/a}$ to obtain

$$\frac{d}{dt}\left(\nu(t)^{4\delta/a}\|\tilde{h}(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\right) \leqslant C\lambda\nu(t)^{4\delta/a-(2+d)-4}.$$

Note that $\frac{d}{dt}\nu(t)^{4\delta/a} = \delta\nu(t)^{4\delta/a-4}$. Then we integrate to obtain

$$\|\tilde{h}(t)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \leqslant \frac{C\lambda}{\delta - a(2+d)/4}\frac{a}{4}\nu(t)^{-(2+d)} + \|\tilde{h}_0\|_{\dot{\mathcal{F}}^{2,1}_\nu}\nu(0)^{4\delta/a}\nu(t)^{-4\delta/a}. \tag{8.11}$$

This concludes the main estimates of this paragraph.

Now choosing $a > 0$ sufficiently small, depending upon the other parameters in (8.11), allows us to propagate the assumption that $\|h(T_1 + t_0 + t)\|_{\dot{\mathcal{F}}^{2,1}_\nu} < \epsilon$ for all $t \geqslant 0$. Therefore (8.10) and (8.11) hold for all times $t \geqslant 0$. We conclude that

$$\|h(T+t)\|_{\dot{\mathcal{F}}^{2,1}_\nu} = \int_{\mathbb{R}^d} |\xi|^2 e^{\nu(t)|\xi|}|\hat{h}(\xi, T+t)|d\xi \lesssim (1+t)^{-(2+d)/4}, \tag{8.12}$$

which holds uniformly for some fixed $T > 0$ and all $t \geqslant 0$.

We can also prove (8.12) for any $0 \leqslant s \leqslant 2$ by using the same technique, and obtain the decay rate in (1.16). These estimates now grant Theorem 1.4. Q.E.D.

We remark that one can also control the exponential PDE (1.3) globally in time in the norms of $\dot{\mathcal{F}}^{s,1}$ for $s \geqslant 2$ and in $\dot{\mathcal{F}}^{s,p}$ for general $s \geqslant 0$ and $p \in (1, 2]$ using only the smallness assumption from (2.9) that is used in Theorem 1.1. The idea is to use the large time decay in (1.15) for $s = 2$ to show that after a time $T > 0$ the norm $\|h(T)\|_2$ can be as small as we need in order to control the sums such as in the upper bounds of (2.17) and (2.22). In a similar way one can also obtain control of the analytic norms such as $\dot{\mathcal{F}}^{s,p}_\nu$.

Appendix

A. Numerical simulations

Here we present some numerical simulations of (3) to illustrate some aspects of the main results. The computations were contributed by Thomas Witelski [31, 32] and carried out specifically for this paper.

The equation (1.3) was simulated in one spatial dimension with periodic boundary conditions on $0 \leqslant x \leqslant 2\pi$. Computations were done using a fully-implicit backward Euler time-stepping scheme with a second-order accurate finite-difference discretization in space with $2^{14} = 16,384$ grid points (results were also validated against a Fourier pseudo-spectral code). Initial conditions for each simulation were taken to be $h_0(x) = A\sin(x)$ with $A > 0$, which gives $||h_0||_2 = A$ using the norm (1.7).

Results for three different values for A are shown to illustrate behaviors starting from initial data below or above the critical value $y_* \approx 0.104835667$. Figure 1 shows that below y_*, for $A = 0.1$, the $\dot{\mathcal{F}}^{2,1}$ norm is monotone decreasing in time, as expected from Theorem 1. In this particular case, the evolution of the $\dot{\mathcal{W}}^{2,\infty}$ norm (here, also starting from value A) is almost indistinguishable from the $\dot{\mathcal{F}}^{2,1}$, but it is always a lower bound for that norm. The decay of profiles of $h_{xx}(x,t)$ are also shown. Note that here $\dot{\mathcal{W}}^{2,\infty}$ denotes the standard homogeneous Sobolev norm with two derivatives in the L^∞ space.

Further analysis is needed to better understand the behaviors for $A > y_*$, but numerical simulations can be suggestive of the dynamics that can occur. Figure 2 shows results starting from $A = 0.3$. The value of the $\dot{\mathcal{W}}^{2,\infty}$ is still monotone decreasing, but now the evolution of the $\dot{\mathcal{F}}^{2,1}$

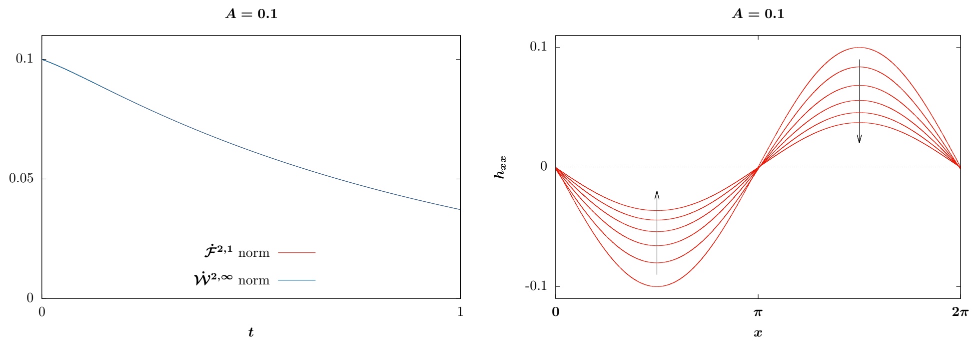

FIG. 1. Numerical simulation of evolution from initial data $h_0 = A\sin(x)$ with $\frac{1}{10} = A < y_*$ showing monotone decay of the $\dot{\mathcal{F}}^{2,1}$ norm

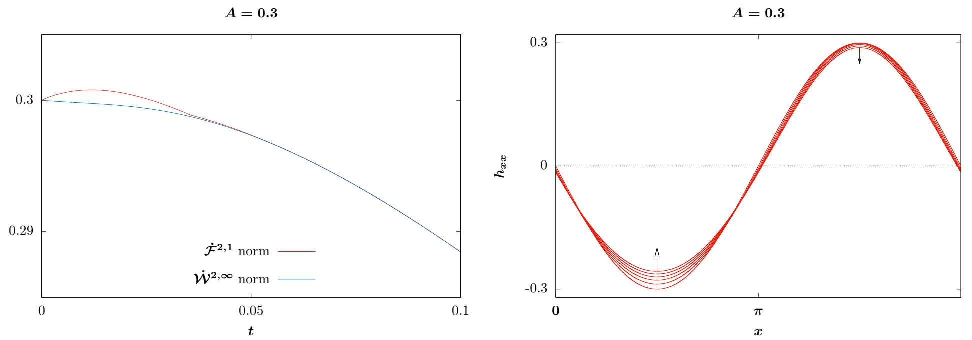

FIG. 2. Numerical simulation of evolution from initial data $h_0 = A\sin(x)$ with $\frac{3}{10} = A > y_*$ showing that $\dot{\mathcal{F}}^{2,1}$ norm is no longer monotone decreasing when the conditions of Theorem 1 are violated

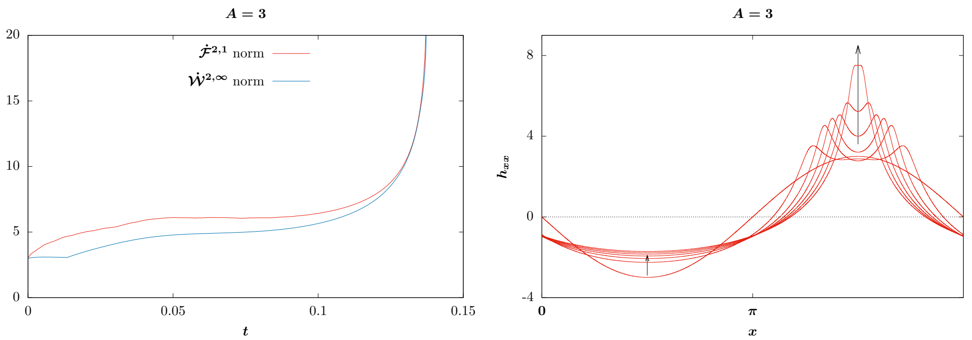

FIG. 3. Numerical simulation of evolution from significantly larger initial data $h_0 = A\sin(x)$ with $3 = A > y_*$ having both norms being non-monotone in time and suggesting finite-time blow-up

norm is not monotone. For larger initial data, Figure 3 shows results starting from $A = 3$. Here both norms are non-monotone in time and seem to suggest formation of a finite-time singularity as h_{xx}

blows-up. Numerical evidence for finite-time blow was given in an earlier 2013 paper by Marzuola and Weare [22].

The ebook version of this paper contains full color images of the plots in of Fig. 1–3.

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