

The Superconvergence Phenomenon and Proof of the MAC Scheme for the Stokes Equations on Non-uniform Rectangular Meshes

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Abstract For decades, the widely used finite difference method on staggered grids, also known as the marker and cell (MAC) method, has been one of the simplest and most effective numerical schemes for solving the Stokes equations and Navier–Stokes equations. Its superconvergence on uniform meshes has been observed by Nicolaidis (SIAM J Numer Anal 29(6):1579–1591, 1992), but the rigorous proof is never given. Its behavior on non-uniform grids is not well studied, since most publications only consider uniform grids. In this work, we develop the MAC scheme on non-uniform rectangular meshes, and for the first time we theoretically prove that the superconvergence phenomenon (i.e., second order convergence in the L^2 norm for both velocity and pressure) holds true for the MAC method on non-uniform rectangular meshes. With a careful and accurate analysis of various sources of errors, we observe that even though the local truncation errors are only first order in terms of mesh size, the global errors after summation are second order due to the amazing cancellation of local errors. This observation leads to the elegant superconvergence analysis even with non-uniform meshes. Numerical results are given to verify our theoretical analysis.

Keywords Stokes equations · Staggered grids · MAC scheme · Non-uniform rectangular grids

Mathematics Subject Classification 65M60 · 65N30 · 76D07

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1 Introduction

Due to the importance of incompressible flows, many numerical methods (such as finite difference methods, finite element methods, and finite volume methods) have been developed for solving the Stokes and Navier–Stokes equations over the past five decades (e.g., [5, 14] and references therein). However, it is arguably that “Among the many numerical methods that are available for solving the Navier–Stokes equations, one of the best and simplest is the marker-and-cell method.” (Girault and Lopez [4, p. 347]). The same point view has been shared by Han and Wu [6, p. 560]: “It is well known that the marker and cell (MAC) is one of the simplest and most effective numerical schemes for solving Stokes equations and Navier–Stokes equations.” The MAC method was introduced by Lebedev [12] and Daly et al. [3] in 1960s, and has been widely used in engineering applications as evidenced being the basis of many flow packages [17, p. 1579]. The MAC method is a finite difference method on rectangular cells with pressure approximated at the cell center, the x -component of velocity approximated at the midpoint of vertical edges of the cell, and the y -component of velocity approximated at the midpoint of horizontal edges of the cell (cf. Fig. 1).

Though the MAC method has been used successfully since 1965, its theoretical numerical analysis was not carried out until 1992 by Nicolaides and his collaborators [17, 18] by transforming the MAC method into a finite volume method. The difficulty is due to the fact that only a few mathematical tools are available for finite difference or finite volume methods [4, p. 348]. In 1996, Girault and Lopez [4] performed further analysis of the MAC method by interpreting it as a mixed finite element method for the vorticity–velocity–pressure formulation with special numerical quadratures. Then in 1998, Han and Wu [6] showed that

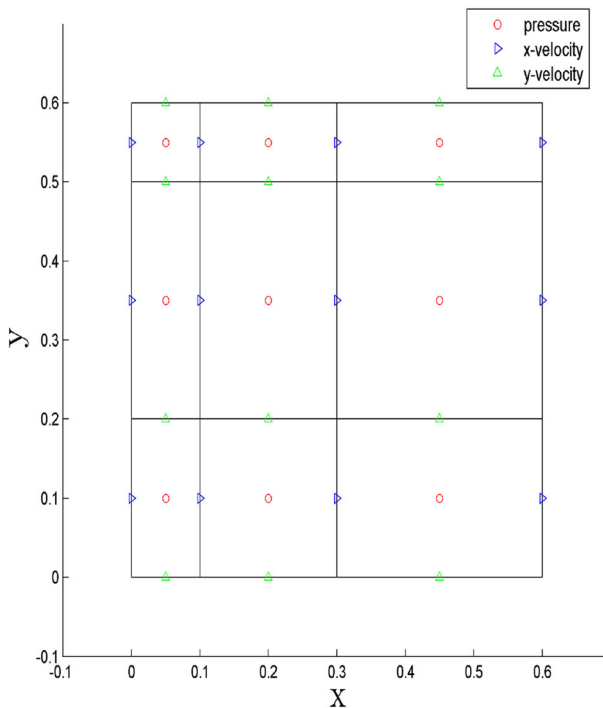


Fig. 1 An exemplary staggered grid

the MAC method can be obtained from a new mixed finite element method with the two components of velocity and pressure defined on three staggered rectangular grids. In 2008, Kanschat [11] showed that the MAC scheme is algebraically equivalent to the first-order local discontinuous Galerkin method of Cockburn et al. [2] with a proper quadrature. Their method is based on the $H(\text{div})$ -conforming Raviart–Thomas RT_0 element. Inspired by the work of Kanschat [11], very soon Mineev [15] demonstrated that the MAC scheme can be obtained by using the first-order Nédélec edge element on rectangular cells.

We would like to remark that all these papers [4, 6, 11, 17] proved that the MAC method has first order convergence $O(h)$ for both the velocity (in H^1 norm) and the pressure (in L^2 norm) on uniform rectangular meshes. But Nicolaidis [17, p. 1591] pointed out that numerical results suggest that the velocity is $O(h^2)$ without any proof. To our best knowledge, there is no any existing paper proving the $O(h^2)$ convergence for the velocity.

In this paper, inspired by the analysis technique developed by Weiser and Wheeler [24] for elliptic problems, and Süli and Monk [16, 23] for both elliptic and Maxwell’s equations solved by cell-centered non-uniform rectangular grids, we first derive the MAC scheme from a variational form, and then we show that the $O(h^2)$ superconvergence is indeed true for both the velocity and pressure on non-uniform rectangular grids. Superconvergence for other equations are well studied (cf. [7–9, 19, 25, 26], [13, Ch. 5] and references therein). Finally, we like to mention that (as one anonymous referee pointed out) there are some convergence analysis obtained for the MAC scheme developed for primitive equations (PEs), and planetary geostrophic equation (PGEs). For example, for the PEs the $O(h^2)$ convergence for the velocity in the L^2 norm is proved on uniform rectangular meshes by Samelson et al. [21], and for PGEs the $O(h^4)$ convergence in the L^2 norm is proved for the temperature on uniform rectangular meshes by Samelson et al. [22]. It would be interesting to explore in the future if our analysis can be extended to these problems on non-uniform rectangular grids.

The rest of the paper is organized as follows. In Sect. 2, we demonstrate that the MAC scheme can be derived from a variational form, which will be used late in the error analysis. In Sect. 3, we prove a discrete stability, which enjoys the same form in the continuous case. Section 4 is devoted to the $O(h^2)$ superconvergence analysis for the MAC scheme. Numerical examples are presented in Sect. 5 to support our theoretical analysis. We conclude the paper in Sect. 6.

2 Derivation of the MAC Scheme

Let us consider the Stokes problem in two dimensional space: find the velocity $\mathbf{u} = (u^x, u^y)^T$ and the pressure p such that

$$-\mu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega := (0, L_x) \times (0, L_y), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = 0, \quad \text{on } \partial\Omega, \tag{3}$$

where $\mathbf{f} = (f^x, f^y)^T$ denotes a given force, $\int_{\Omega} p \, dx \, dy = 0$, and $\mu > 0$ is the viscosity. For simplicity, we assume $\mu = 1$ in the rest paper. In the above, L_x and L_y denote the length of the rectangular domain Ω in the x - and y -directions, respectively.

To construct a cell-centered difference scheme for the Stokes problem, we introduce a non-uniform grid of Ω :

$$0 = x_0 < x_1 < \dots < x_M = L_x, \quad 0 = y_0 < y_1 < \dots < y_N = L_y,$$

and denote the cell center $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$. Furthermore, we approximate the pressure at each cell center $p(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ by $p_{i+\frac{1}{2},j+\frac{1}{2}}$, the x -velocity u^x at each x -edge center $u^x(x_i, y_{j+\frac{1}{2}})$ by $u^x_{i,j+\frac{1}{2}}$, and the y -velocity u^y at each y -edge center $u^y(x_{i-\frac{1}{2}}, y_j)$ by $u^y_{i-\frac{1}{2},j}$. To measure the convergence rate, we denote $h_x = \max_{1 \leq i \leq M}(x_i - x_{i-1})$, $h_y = \max_{1 \leq j \leq N}(y_j - y_{j-1})$, and $h = \max(h_x, h_y)$.

By integrating the x -component of Eq. (1) over cell $\Omega_{i,j+\frac{1}{2}} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_j, y_{j+1})$, we have

$$\begin{aligned} & - \left[\int_{y_j}^{y_{j+1}} \left(\frac{\partial u^x}{\partial x} \left(x_{i+\frac{1}{2}}, y \right) - \frac{\partial u^x}{\partial x} \left(x_{i-\frac{1}{2}}, y \right) \right) dy \right. \\ & \quad \left. + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{\partial u^x}{\partial y} (x, y_{j+1}) - \frac{\partial u^x}{\partial y} (x, y_j) \right) dx \right] \\ & \quad + \int_{y_j}^{y_{j+1}} \left(p \left(x_{i+\frac{1}{2}}, y \right) - p \left(x_{i-\frac{1}{2}}, y \right) \right) dy \\ & = \int_{\Omega_{i,j+\frac{1}{2}}} f^x(x, y) dx dy, \quad 1 \leq i \leq M - 1, 0 \leq j \leq N - 1. \end{aligned} \tag{4}$$

Then approximating those edge integrals by midpoint values, the volume integral by its center value, and dividing both sides by $|\Omega_{i,j+\frac{1}{2}}| = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})(y_{j+1} - y_j)$, we obtain

$$\begin{aligned} & - \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \left(\frac{u^x_{i+1,j+\frac{1}{2}} - u^x_{i,j+\frac{1}{2}}}{x_{i+1} - x_i} - \frac{u^x_{i,j+\frac{1}{2}} - u^x_{i-1,j+\frac{1}{2}}}{x_i - x_{i-1}} \right) \\ & - \frac{1}{y_{j+1} - y_j} \left(\frac{u^x_{i,j+\frac{3}{2}} - u^x_{i,j+\frac{1}{2}}}{y_{j+\frac{3}{2}} - y_{j+\frac{1}{2}}} - \frac{u^x_{i,j+\frac{1}{2}} - u^x_{i,j-\frac{1}{2}}}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} \right) \\ & + \frac{p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = f^x_{i,j+\frac{1}{2}}, \end{aligned} \tag{5}$$

which can be rewritten as follows:

$$- \frac{\Delta_x^+ u^x_{i,j+\frac{1}{2}} - \Delta_x^- u^x_{i,j+\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} - \frac{\Delta_y^+ u^x_{i,j+\frac{1}{2}} - \Delta_y^- u^x_{i,j+\frac{1}{2}}}{y_{j+1} - y_j} + \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}} = f^x_{i,j+\frac{1}{2}}, \tag{6}$$

where we used the following standard forward and backward difference operators:

$$\Delta_x^+ u^x_{i,j+\frac{1}{2}} = \frac{u^x_{i+1,j+\frac{1}{2}} - u^x_{i,j+\frac{1}{2}}}{x_{i+1} - x_i}, \quad \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}.$$

Similar notations are used for operations in the y -direction. Here we denote $f^x_{i,j+\frac{1}{2}} = f^x(x_i, y_{j+\frac{1}{2}})$.

Similarly, by integrating the y -component of Eq. (1) over cell $\Omega_{i-\frac{1}{2},j} = (x_{i-1}, x_i) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, we have

$$\begin{aligned}
 & - \left[\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\frac{\partial u^y}{\partial x}(x_i, y) - \frac{\partial u^y}{\partial x}(x_{i-1}, y) \right) dy \right. \\
 & \quad \left. + \int_{x_{i-1}}^{x_i} \left(\frac{\partial u^y}{\partial y} \left(x, y_{j+\frac{1}{2}} \right) - \frac{\partial u^y}{\partial y} \left(x, y_{j-\frac{1}{2}} \right) \right) dx \right] \\
 & \quad + \int_{x_{i-1}}^{x_i} \left(p \left(x, y_{j+\frac{1}{2}} \right) - p \left(x, y_{j-\frac{1}{2}} \right) \right) dx \\
 & = \int_{\Omega_{i-\frac{1}{2},j}} f^y(x, y) dx dy, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N - 1. \tag{7}
 \end{aligned}$$

Then approximating those edge integrals by midpoint values, the volume integral by its center value, and using the difference operators introduced above, we can obtain the cell-centered difference scheme for the y -velocity:

$$- \frac{\Delta_x^+ u_{i-\frac{1}{2},j}^y - \Delta_x^- u_{i-\frac{1}{2},j}^y}{x_i - x_{i-1}} - \frac{\Delta_y^+ u_{i-\frac{1}{2},j}^y - \Delta_y^- u_{i-\frac{1}{2},j}^y}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} + \Delta_y^- p_{i-\frac{1}{2},j+\frac{1}{2}} = f_{i-\frac{1}{2},j}^y. \tag{8}$$

By integrating Eq. (3) over cell $\Omega_{i-\frac{1}{2},j+\frac{1}{2}} = (x_i - x_{i-1}) \times (y_j, y_{j+1})$, and carrying similar techniques as above, we can obtain the cell-centered difference scheme for Eq. (3):

$$\Delta_x^- u_{i,j+\frac{1}{2}}^x + \Delta_y^+ u_{i-\frac{1}{2},j}^y = 0. \tag{9}$$

In summary, we recover the famous MAC scheme for the Stokes problem (1)–(3):

$$- \frac{\Delta_x^+ u_{i,j+\frac{1}{2}}^x - \Delta_x^- u_{i,j+\frac{1}{2}}^x}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} - \frac{\Delta_y^+ u_{i,j+\frac{1}{2}}^x - \Delta_y^- u_{i,j+\frac{1}{2}}^x}{y_{j+1} - y_j} + \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}} = f_{i,j+\frac{1}{2}}^x, \tag{10}$$

$$- \frac{\Delta_x^+ u_{i-\frac{1}{2},j}^y - \Delta_x^- u_{i-\frac{1}{2},j}^y}{x_i - x_{i-1}} - \frac{\Delta_y^+ u_{i-\frac{1}{2},j}^y - \Delta_y^- u_{i-\frac{1}{2},j}^y}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} + \Delta_y^- p_{i-\frac{1}{2},j+\frac{1}{2}} = f_{i-\frac{1}{2},j}^y, \tag{11}$$

$$\Delta_x^- u_{i,j+\frac{1}{2}}^x + \Delta_y^+ u_{i-\frac{1}{2},j}^y = 0. \tag{12}$$

3 Stability Analysis

In this section, we will carry out the stability analysis for the scheme (10)–(12).

First, let us introduce various discrete norms to be used in the following analysis:

$$\|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2 = \sum_{i=1}^M \sum_{j=0}^{N-1} (x_i - x_{i-1})(y_{j+1} - y_j) \Delta_x^- u_{i,j+\frac{1}{2}}^x \cdot \Delta_x^- u_{i,j+\frac{1}{2}}^x, \tag{13}$$

$$\|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2 = \sum_{i=0}^M \sum_{j=0}^N (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) \Delta_y^- u_{i,j+\frac{1}{2}}^x \cdot \Delta_y^- u_{i,j+\frac{1}{2}}^x, \tag{14}$$

$$\|\Delta_x^- u_{i-\frac{1}{2},j}^y\|^2 = \sum_{i=1}^{M+1} \sum_{j=0}^N (x_{i-\frac{1}{2}} - x_{i-\frac{3}{2}}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) \Delta_x^- u_{i-\frac{1}{2},j}^y \cdot \Delta_x^- u_{i-\frac{1}{2},j}^y, \tag{15}$$

$$\|\Delta_y^- u_{i-\frac{1}{2},j}^y\|^2 = \sum_{i=1}^M \sum_{j=1}^N (x_i - x_{i-1}) (y_j - y_{j-1}) \Delta_y^- u_{i-\frac{1}{2},j}^y \cdot \Delta_y^- u_{i-\frac{1}{2},j}^y, \tag{16}$$

where functions out of the index bounds ($0 \leq i \leq M, 0 \leq j \leq N$) are treated as zeros, and any coordinates out of the physical domain are treated as their neighboring values. For example, in (14) we treat

$$x_{-\frac{1}{2}} = y_{-\frac{1}{2}} = 0, \quad x_{M+\frac{1}{2}} = L_x, \quad y_{N+\frac{1}{2}} = L_y.$$

Note that to easily distinguish between the various norms, we leave the subindices inside the norms. For functions without operators Δ_x^\pm or Δ_y^\pm , the discrete norm is defined as the function value at the cell center multiplying the cell area, e.g.,

$$\|u_{i,j+\frac{1}{2}}^x\|^2 = \sum_{i=0}^M \sum_{j=0}^{N-1} (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+1} - y_j) |u_{i,j+\frac{1}{2}}^x|^2. \tag{17}$$

To prove the stability result for our scheme (10)–(12), we need the following lemmas.

Lemma 3.1

$$\begin{aligned} & \sum_{i=0}^M \sum_{j=0}^N (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}} \\ & + \sum_{i=1}^M \sum_{j=1}^N (x_i - x_{i-1}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) u_{i-\frac{1}{2},j}^y \Delta_y^- p_{i-\frac{1}{2},j+\frac{1}{2}} = 0. \end{aligned} \tag{18}$$

Proof Note that

$$\begin{aligned} & \sum_{i=0}^M \sum_{j=0}^N (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \cdot \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}} \\ & = \sum_i \sum_j (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \cdot (p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}}) \\ & = \sum_j (y_{j+1} - y_j) \left(\sum_i u_{i,j+\frac{1}{2}}^x \cdot p_{i+\frac{1}{2},j+\frac{1}{2}} - \sum_i u_{i,j+\frac{1}{2}}^x \cdot p_{i-\frac{1}{2},j+\frac{1}{2}} \right) \\ & = - \sum_j (y_{j+1} - y_j) \sum_i \left(u_{i,j+\frac{1}{2}}^x - u_{i-1,j+\frac{1}{2}}^x \right) p_{i-\frac{1}{2},j+\frac{1}{2}} \\ & = - \sum_i \sum_j (x_i - x_{i-1}) (y_{j+1} - y_j) \cdot \left(\Delta_x^- u_{i,j+\frac{1}{2}}^x \right) \cdot p_{i-\frac{1}{2},j+\frac{1}{2}}, \end{aligned} \tag{19}$$

where in the third step we used the identity $\sum_i u_{i,j+\frac{1}{2}}^x \cdot p_{i+\frac{1}{2},j+\frac{1}{2}} = \sum_i u_{i-1,j+\frac{1}{2}}^x \cdot p_{i-\frac{1}{2},j+\frac{1}{2}}$. Here and below, without confusion we may not write out the specific bounds of i and j .

By the same technique, we have

$$\begin{aligned}
 & \sum_i \sum_j (x_i - x_{i-1}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) u_{i-\frac{1}{2},j}^y \cdot \Delta_y^- p_{i-\frac{1}{2},j+\frac{1}{2}} \\
 &= \sum_i (x_i - x_{i-1}) \cdot \left(\sum_j u_{i-\frac{1}{2},j}^y p_{i-\frac{1}{2},j+\frac{1}{2}} - \sum_j u_{i-\frac{1}{2},j}^y p_{i-\frac{1}{2},j-\frac{1}{2}} \right) \\
 &= - \sum_i (x_i - x_{i-1}) \sum_j \left(u_{i-\frac{1}{2},j+1}^y - u_{i-\frac{1}{2},j}^y \right) \cdot p_{i-\frac{1}{2},j+\frac{1}{2}} \\
 &= - \sum_i \sum_j (x_i - x_{i-1}) (y_{j+1} - y_j) \cdot \left(\Delta_y^+ u_{i-\frac{1}{2},j}^y \right) \cdot p_{i-\frac{1}{2},j+\frac{1}{2}}. \tag{20}
 \end{aligned}$$

The proof is completed by adding (19) and (20), and using (12). □

We also need the following discrete Poincaré inequality.

Lemma 3.2

$$\begin{aligned}
 (i) \quad & \|u_{i,j+\frac{1}{2}}^x\|^2 \leq \frac{L_x^2}{2} \|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2 + \frac{L_y^2}{2} \|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2, \\
 (ii) \quad & \|u_{i-\frac{1}{2},j}^y\|^2 \leq \frac{L_x^2}{2} \|\Delta_x^- u_{i-\frac{1}{2},j}^y\|^2 + \frac{L_y^2}{2} \|\Delta_y^- u_{i-\frac{1}{2},j}^y\|^2.
 \end{aligned}$$

Proof (i) By definition of the discrete norm (17), we have

$$\begin{aligned}
 \|u_{i,j+\frac{1}{2}}^x\|^2 &= \sum_{i=0}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) (y_{j+1} - y_j) \cdot \left(u_{i,j+\frac{1}{2}}^x \right)^2 \\
 &= \frac{1}{2} \left\{ \sum_{i=0}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) (y_{j+1} - y_j) \left| \sum_{m=1}^i (x_m - x_{m-1}) \Delta_x^- u_{m,j+\frac{1}{2}}^x \right|^2 \right. \\
 &\quad \left. + \sum_{i=0}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) (y_{j+1} - y_j) \left| \sum_{n=0}^j (y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}) \Delta_y^- u_{i,n+\frac{1}{2}}^x \right|^2 \right\} \\
 &\leq \frac{1}{2} \left\{ \sum_{i=0}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) (y_{j+1} - y_j) \right. \\
 &\quad \left[\sum_{m=1}^i (x_m - x_{m-1}) \cdot \sum_{m=1}^i (x_m - x_{m-1}) |\Delta_x^- u_{m,j+\frac{1}{2}}^x|^2 \right] \\
 &\quad + \sum_{i=0}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) (y_{j+1} - y_j) \\
 &\quad \left. \left[\sum_{n=1}^j (y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}) \cdot \sum_{n=1}^j (y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}) |\Delta_y^- u_{i,n+\frac{1}{2}}^x|^2 \right] \right\} \\
 &\leq \frac{1}{2} \left(\sum_{m=1}^M \sum_{j=0}^{N-1} (x_m - x_{m-1}) (y_{j+1} - y_j) |\Delta_x^- u_{m,j+\frac{1}{2}}^x|^2 \right)
 \end{aligned}$$

$$\begin{aligned} & \left(\sum_{i=0}^M (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) \cdot \sum_{m=1}^i (x_m - x_{m-1}) \right) \\ & + \frac{1}{2} \left(\sum_{i=0}^M \sum_{n=1}^{N-1} (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}) |\Delta_y^- u_{i,n+\frac{1}{2}}^y|^2 \right) \\ & \left(\sum_{j=0}^{N-1} (y_{j+1} - y_j) \cdot \sum_{n=1}^j (y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}) \right) \\ & \leq \frac{L_x^2}{2} \|\Delta_x^- u_{m,j+\frac{1}{2}}^x\|^2 + \frac{L_y^2}{2} \|\Delta_y^- u_{i,n+\frac{1}{2}}^y\|^2, \end{aligned}$$

which completes the proof. (ii) The proof is similar to (i). □

Remark 3.1 If a function v only vanishes on one edge of Ω , the discrete Poincaré inequality can be modified as:

$$\begin{aligned} \|v_{i,j+\frac{1}{2}}\|^2 & \leq L_x^2 \|\Delta_x^- v_{i,j+\frac{1}{2}}\|^2 \quad \forall v|_{x=0} = 0, \\ \|v_{i-\frac{1}{2},j}\|^2 & \leq L_y^2 \|\Delta_y^- v_{i-\frac{1}{2},j}\|^2 \quad \forall v|_{y=0} = 0. \end{aligned}$$

To make the stability analysis clearer, we first prove the following identities.

Lemma 3.3

- (i) $-\sum_{i=1}^M \sum_{j=0}^{N-1} (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \cdot \left(\Delta_x^+ u_{i,j+\frac{1}{2}}^x - \Delta_x^- u_{i,j+\frac{1}{2}}^x \right) = \|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2,$
- (ii) $-\sum_{i=0}^M \sum_{j=0}^{N-1} (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) u_{i,j+\frac{1}{2}}^x \cdot \left(\Delta_y^+ u_{i,j+\frac{1}{2}}^x - \Delta_y^- u_{i,j+\frac{1}{2}}^x \right) = \|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2,$
- (iii) $-\sum_{i=1}^M \sum_{j=0}^N (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) u_{i-\frac{1}{2},j}^y \cdot \left(\Delta_x^+ u_{i-\frac{1}{2},j}^y - \Delta_x^- u_{i-\frac{1}{2},j}^y \right) = \|\Delta_x^- u_{i-\frac{1}{2},j}^y\|^2,$
- (iv) $-\sum_{i=1}^M \sum_{j=0}^N (x_i - x_{i-1}) u_{i-\frac{1}{2},j}^y \cdot \left(\Delta_y^+ u_{i-\frac{1}{2},j}^y - \Delta_y^- u_{i-\frac{1}{2},j}^y \right) = \|\Delta_y^- u_{i-\frac{1}{2},j}^y\|^2.$

Proof (i) It is easy to see that

$$\begin{aligned} & -\sum_{i=1}^M \sum_{j=0}^{N-1} (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \cdot \left(\Delta_x^+ u_{i,j+\frac{1}{2}}^x - \Delta_x^- u_{i,j+\frac{1}{2}}^x \right) \\ & = -\sum_{j=0}^{N-1} (y_{j+1} - y_j) \left[\sum_{i=1}^M u_{i,j+\frac{1}{2}}^x \cdot \Delta_x^+ u_{i,j+\frac{1}{2}}^x - \sum_{i=1}^M u_{i,j+\frac{1}{2}}^x \cdot \Delta_x^- u_{i,j+\frac{1}{2}}^x \right] \\ & = \sum_{j=0}^{N-1} (y_{j+1} - y_j) \sum_{i=1}^M \left(u_{i,j+\frac{1}{2}}^x - u_{i-1,j+\frac{1}{2}}^x \right) \Delta_x^- u_{i,j+\frac{1}{2}}^x \\ & = \sum_{i=1}^M \sum_{j=0}^{N-1} (x_i - x_{i-1}) (y_{j+1} - y_j) \Delta_x^- u_{i,j+\frac{1}{2}}^x \cdot \Delta_x^- u_{i,j+\frac{1}{2}}^x = \|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2, \end{aligned}$$

where we used the definition (13) in the last step.

(ii) Similarly, we have

$$\begin{aligned}
 & - \sum_{i=0}^M \sum_{j=0}^{N-1} (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) u_{i,j+\frac{1}{2}}^x \cdot \left(\Delta_y^+ u_{i,j+\frac{1}{2}}^x - \Delta_y^- u_{i,j+\frac{1}{2}}^x \right) \\
 & = \sum_{i=0}^M (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) \left[\sum_{j=0}^{N-1} u_{i,j+\frac{1}{2}}^x \cdot \Delta_y^- u_{i,j+\frac{1}{2}}^x - \sum_{j=0}^{N-1} u_{i,j+\frac{1}{2}}^x \cdot \Delta_y^+ u_{i,j+\frac{1}{2}}^x \right] \\
 & = \sum_{i=0}^M \sum_{j=0}^{N-1} (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) \Delta_y^- u_{i,j+\frac{1}{2}}^x \cdot \Delta_y^- u_{i,j+\frac{1}{2}}^x = \|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2,
 \end{aligned}$$

where we used the definition (14) in the last step.

The proofs of (iii) and (iv) can be carried out similarly. □

With the above preparations, we can obtain the following stability for our scheme.

Theorem 3.1

$$\begin{aligned}
 & \|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2 + \|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2 + \|\Delta_x^- u_{i-\frac{1}{2},j}^y\|^2 + \|\Delta_y^- u_{i-\frac{1}{2},j}^y\|^2 \\
 & \leq C \left(\|f_{i,j+\frac{1}{2}}^x\|^2 + \|f_{i-\frac{1}{2},j}^y\|^2 \right).
 \end{aligned}$$

Proof Multiplying (10) by $(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})(y_{j+1} - y_j)u_{i,j+\frac{1}{2}}^x$, and summing up all i and j , we obtain

$$\begin{aligned}
 & - \sum_i \sum_j (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \left(\Delta_x^+ u_{i,j+\frac{1}{2}}^x - \Delta_x^- u_{i,j+\frac{1}{2}}^x \right) \\
 & - \sum_i \sum_j (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) u_{i,j+\frac{1}{2}}^x \left(\Delta_y^+ u_{i,j+\frac{1}{2}}^x - \Delta_y^- u_{i,j+\frac{1}{2}}^x \right) \\
 & + \sum_i \sum_j (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}} \\
 & = \sum_i \sum_j (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+1} - y_j) u_{i,j+\frac{1}{2}}^x f_{i,j+\frac{1}{2}}^x, \tag{21}
 \end{aligned}$$

where we skipped the detailed lower and upper indices for both i and j .

Similarly, multiplying (11) by $(x_i - x_{i-1})(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})u_{i-\frac{1}{2},j}^y$, we obtain

$$\begin{aligned}
 & - \sum_i \sum_j (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) u_{i-\frac{1}{2},j}^y \left(\Delta_x^+ u_{i-\frac{1}{2},j}^y - \Delta_x^- u_{i-\frac{1}{2},j}^y \right) \\
 & - \sum_i \sum_j (x_i - x_{i-1}) u_{i-\frac{1}{2},j}^y \left(\Delta_y^+ u_{i-\frac{1}{2},j}^y - \Delta_y^- u_{i-\frac{1}{2},j}^y \right) \\
 & + \sum_i \sum_j (x_i - x_{i-1}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) u_{i-\frac{1}{2},j}^y \Delta_y^- p_{i-\frac{1}{2},j+\frac{1}{2}} \\
 & = \sum_i \sum_j (x_i - x_{i-1}) (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) u_{i-\frac{1}{2},j}^y f_{i-\frac{1}{2},j}^y. \tag{22}
 \end{aligned}$$

Adding (21) and (22), using the identities given in Lemmas 3.1 and 3.3, then using the Cauchy–Schwarz inequality to those two right-hand side terms and Lemma 3.2, we have

$$\begin{aligned} & \|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2 + \|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2 + \|\Delta_x^- u_{i-\frac{1}{2},j}^y\|^2 + \|\Delta_y^- u_{i-\frac{1}{2},j}^y\|^2 \\ & \leq \delta_1 \left(\frac{L_x^2}{2} \|\Delta_x^- u_{i,j+\frac{1}{2}}^x\|^2 + \frac{L_y^2}{2} \|\Delta_y^- u_{i,j+\frac{1}{2}}^x\|^2 \right) + \frac{1}{4\delta_1} \|f_{i,j+\frac{1}{2}}^x\|^2 \\ & \quad + \delta_2 \left(\frac{L_x^2}{2} \|\Delta_x^- u_{i-\frac{1}{2},j}^y\|^2 + \frac{L_y^2}{2} \|\Delta_y^- u_{i-\frac{1}{2},j}^y\|^2 \right) + \frac{1}{4\delta_2} \|f_{i-\frac{1}{2},j}^y\|^2, \end{aligned}$$

which concludes the proof by choosing δ_1 and δ_2 sufficiently small. □

Remark 3.2 Multiplying (1) by \mathbf{u} , integrating by parts and using (2), we easily have

$$\|\nabla \mathbf{u}\|_0^2 = (\mathbf{f}, \mathbf{u}) \leq \frac{1}{4\delta} \|\mathbf{f}\|_0^2 + \delta \|\mathbf{u}\|_0^2.$$

Using the Poincaré inequality [1, p. 135]

$$\|\mathbf{u}\|_0^2 \leq C \|\nabla \mathbf{u}\|_0^2, \quad \forall \mathbf{u} \in (H_0^1(\Omega))^2,$$

and choosing $\delta > 0$ small enough, we obtain the stability in continuous form

$$\|\nabla \mathbf{u}\|_0^2 \leq C \|\mathbf{f}\|_0^2, \tag{23}$$

whose discrete form is given in Theorem 3.1.

4 Error Analysis

In this section, we shall derive the error analysis for our scheme. First, let us denote the errors

$$\begin{aligned} e_{i,j+\frac{1}{2}}^x &= u^x \left(x_i, y_{j+\frac{1}{2}} \right) - u_{i,j+\frac{1}{2}}^x, \\ e_{i-\frac{1}{2},j}^y &= u^y \left(x_{i-\frac{1}{2}}, y_j \right) - u_{i-\frac{1}{2},j}^y, \\ e_{i+\frac{1}{2},j+\frac{1}{2}}^p &= p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) - p_{i+\frac{1}{2},j+\frac{1}{2}}, \end{aligned}$$

where $u^x(x_i, y_{j+\frac{1}{2}})$, $u^y(x_{i-\frac{1}{2}}, y_j)$ and $p(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ denote the exact solutions u^x , u^y and p of the Stokes problem (1)–(3) at those specific points, respectively.

From our scheme derivation process, we can obtain the following error equations:

$$-\frac{\Delta_x^+ e_{i,j+\frac{1}{2}}^x - \Delta_x^- e_{i,j+\frac{1}{2}}^x}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} - \frac{\Delta_y^+ e_{i,j+\frac{1}{2}}^x - \Delta_y^- e_{i,j+\frac{1}{2}}^x}{y_{j+1} - y_j} + \Delta_x^- e_{i+\frac{1}{2},j+\frac{1}{2}}^p = R_{i,j+\frac{1}{2}}^x, \tag{24}$$

$$-\frac{\Delta_x^+ e_{i-\frac{1}{2},j}^y - \Delta_x^- e_{i-\frac{1}{2},j}^y}{x_i - x_{i-1}} - \frac{\Delta_y^+ e_{i-\frac{1}{2},j}^y - \Delta_y^- e_{i-\frac{1}{2},j}^y}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} + \Delta_y^- e_{i-\frac{1}{2},j+\frac{1}{2}}^p = R_{i-\frac{1}{2},j}^y, \tag{25}$$

$$\Delta_x^- e_{i,j+\frac{1}{2}}^x + \Delta_y^+ e_{i-\frac{1}{2},j}^y = R_{i-\frac{1}{2},j+\frac{1}{2}}^d, \tag{26}$$

where detailed expressions for R^x , R^y and R^d will be given below.

From (4) and (10), we have

$$\begin{aligned}
 R_{i,j+\frac{1}{2}}^x &= -\frac{\Delta_x^+ u^x(x_i, y_{j+\frac{1}{2}}) - \Delta_x^- u^x(x_i, y_{j+\frac{1}{2}})}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} - \frac{\Delta_y^+ u^x(x_i, y_{j+\frac{1}{2}}) - \Delta_y^- u^x(x_i, y_{j+\frac{1}{2}})}{y_{j+1} - y_j} \\
 &\quad + \Delta_x^- p(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) - f_{i,j+\frac{1}{2}}^x \\
 &= \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{y_j}^{y_{j+1}} \left(\frac{\partial u^x}{\partial x}(x_{i+\frac{1}{2}}, y) - \frac{\partial u^x}{\partial x}(x_{i-\frac{1}{2}}, y) \right) dy \\
 &\quad - \frac{\Delta_x^+ u^x(x_i, y_{j+\frac{1}{2}}) - \Delta_x^- u^x(x_i, y_{j+\frac{1}{2}})}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \\
 &\quad + \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{\partial u^x}{\partial y}(x, y_{j+1}) - \frac{\partial u^x}{\partial y}(x, y_j) \right) dx \\
 &\quad - \frac{\Delta_y^+ u^x(x_i, y_{j+\frac{1}{2}}) - \Delta_y^- u^x(x_i, y_{j+\frac{1}{2}})}{y_{j+1} - y_j} \\
 &\quad - \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{y_j}^{y_{j+1}} \left(p(x_{i+\frac{1}{2}}, y) - p(x_{i-\frac{1}{2}}, y) \right) dy + \Delta_x^- p(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \\
 &\quad + \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} f^x(x, y) dx dy - f^x(x_i, y_{j+\frac{1}{2}}) \\
 &= R_1^x + R_2^x + R_3^x + R_4^x. \tag{27}
 \end{aligned}$$

Similarly, from (7) and (11), we have

$$\begin{aligned}
 R_{i-\frac{1}{2},j}^y &= -\frac{\Delta_x^+ u^y(x_{i-\frac{1}{2}}, y_j) - \Delta_x^- u^y(x_{i-\frac{1}{2}}, y_j)}{x_i - x_{i-1}} - \frac{\Delta_y^+ u^y(x_{i-\frac{1}{2}}, y_j) - \Delta_y^- u^y(x_{i-\frac{1}{2}}, y_j)}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} \\
 &\quad + \Delta_y^- p(x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}}) - f_{i-\frac{1}{2},j}^y \\
 &= \frac{1}{|\Omega_{i-\frac{1}{2},j}|} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\frac{\partial u^y}{\partial x}(x_i, y) - \frac{\partial u^y}{\partial x}(x_{i-1}, y) \right) dy \\
 &\quad - \frac{\Delta_x^+ u^y(x_{i-\frac{1}{2}}, y_j) - \Delta_x^- u^y(x_{i-\frac{1}{2}}, y_j)}{x_i - x_{i-1}} \\
 &\quad + \frac{1}{|\Omega_{i-\frac{1}{2},j}|} \int_{x_{i-1}}^{x_i} \left(\frac{\partial u^y}{\partial y}(x, y_{j+\frac{1}{2}}) - \frac{\partial u^y}{\partial y}(x, y_{j-\frac{1}{2}}) \right) dx \\
 &\quad - \frac{\Delta_y^+ u^y(x_{i-\frac{1}{2}}, y_j) - \Delta_y^- u^y(x_{i-\frac{1}{2}}, y_j)}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}} \\
 &\quad - \frac{1}{|\Omega_{i-\frac{1}{2},j}|} \int_{x_{i-1}}^{x_i} \left(p(x, y_{j+\frac{1}{2}}) - p(x, y_{j-\frac{1}{2}}) \right) dx + \Delta_y^- p(x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}})
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{|\Omega_{i-\frac{1}{2},j}|} \int_{\Omega_{i-\frac{1}{2},j}} f^y(x, y) dx dy - f^y\left(x_{i-\frac{1}{2}}, y_j\right) \\
 & = R_1^y + R_2^y + R_3^y + R_4^y.
 \end{aligned} \tag{28}$$

Finally, from (12), it is easy to see that

$$R_{i-\frac{1}{2},j+\frac{1}{2}}^d = \Delta_x^- u^x\left(x_i, y_{j+\frac{1}{2}}\right) + \Delta_y^+ u^y\left(x_{i-\frac{1}{2}}, y_j\right). \tag{29}$$

To prove the error analysis of our scheme, below we first estimate R_1^x . Denote the maximum norm for the r -th derivatives of any function w as $|D^r w|_\infty = \left| \frac{\partial^r w}{\partial x^m \partial y^n} \right|_{\infty, \bar{\Omega}}$, where $m + n = r$, and $m, n \geq 0$.

Lemma 4.1

$$\begin{aligned}
 R_1^x & = \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{y_j}^{y_{j+1}} \left(\frac{\partial u^x}{\partial x}\left(x_{i+\frac{1}{2}}, y\right) - \frac{\partial u^x}{\partial x}\left(x_{i-\frac{1}{2}}, y\right) \right) dy \\
 & \quad - \frac{\Delta_x^+ u^x\left(x_i, y_{j+\frac{1}{2}}\right) - \Delta_x^- u^x\left(x_i, y_{j+\frac{1}{2}}\right)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \\
 & = -\frac{1}{12} (x_{i+1} + x_{i-1} - 2x_i) \frac{\partial^3 u^x}{\partial x^3}\left(x_i, y_{j+\frac{1}{2}}\right) + O(h^2) \|D^4 u^x\|_\infty,
 \end{aligned} \tag{30}$$

Proof (i) Note that R_1^x can be rewritten as follows

$$R_1^x = \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} \left\{ \frac{\partial^2 u^x}{\partial x^2}(x, y) dx dy - \frac{\Delta_x^+ u^x\left(x_i, y_{j+\frac{1}{2}}\right) - \Delta_x^- u^x\left(x_i, y_{j+\frac{1}{2}}\right)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \right\} dx dy. \tag{31}$$

By the Taylor expansion, we have

$$\begin{aligned}
 \Delta_x^+ u^x\left(x_i, y_{j+\frac{1}{2}}\right) & = \frac{\partial u_*^x}{\partial x} + \frac{1}{2}(x_{i+1} - x_i) \frac{\partial^2 u_*^x}{\partial x^2} \\
 & \quad + \frac{1}{6}(x_{i+1} - x_i)^2 \frac{\partial^3 u_*^x}{\partial x^3} + \frac{1}{24}(x_{i+1} - x_i)^3 \frac{\partial^4 \hat{u}^x}{\partial x^4},
 \end{aligned} \tag{32}$$

where $\frac{\partial^k u_*^x}{\partial x^k}$, $k = 1, 2, 3$, denote the derivatives evaluated at point $(x_i, y_{j+\frac{1}{2}})$, and $\frac{\partial^4 \hat{u}^x}{\partial x^4}$ denotes the derivative at some point between $(x_i, y_{j+\frac{1}{2}})$ and $(x_{i+1}, y_{j+\frac{1}{2}})$.

Similarly, by the Taylor expansion, we can obtain

$$\begin{aligned}
 \Delta_x^- u^x\left(x_i, y_{j+\frac{1}{2}}\right) & = \frac{\partial u_*^x}{\partial x} + \frac{1}{2}(x_{i-1} - x_i) \frac{\partial^2 u_*^x}{\partial x^2} \\
 & \quad + \frac{1}{6}(x_{i-1} - x_i)^2 \frac{\partial^3 u_*^x}{\partial x^3} + \frac{1}{24}(x_{i-1} - x_i)^3 \frac{\partial^4 \check{u}^x}{\partial x^4},
 \end{aligned} \tag{33}$$

where $\frac{\partial^4 \check{u}^x}{\partial x^4}$ denotes the derivative at some point between $(x_{i-1}, y_{j+\frac{1}{2}})$ and $(x_i, y_{j+\frac{1}{2}})$.

Combining (32) and (33), we have

$$\begin{aligned} & \frac{\Delta_x^+ u^x \left(x_i, y_{j+\frac{1}{2}}\right) - \Delta_x^- u^x \left(x_i, y_{j+\frac{1}{2}}\right)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \\ &= \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \\ & \left[\frac{x_{i+1} - x_{i-1}}{2} \frac{\partial^2 u_*^x}{\partial x^2} + \frac{x_{i+1} - x_{i-1}}{6} (x_{i+1} + x_{i-1} - 2x_i) \frac{\partial^3 u_*^x}{\partial x^3} + O(h_x^3) \left\| \frac{\partial^4 u^x}{\partial x^4} \right\|_\infty \right] \\ &= \frac{\partial^2 u_*^x}{\partial x^2} + \frac{1}{3} (x_{i+1} + x_{i-1} - 2x_i) \frac{\partial^3 u_*^x}{\partial x^3} + O(h_x^2) \left\| \frac{\partial^4 u^x}{\partial x^4} \right\|_\infty, \end{aligned} \tag{34}$$

where in the last step we used the following property

$$x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} - \frac{x_{i-1} + x_i}{2} = \frac{x_{i+1} - x_{i-1}}{2}. \tag{35}$$

On the other hand, using the Taylor expansion:

$$\frac{\partial^2 u^x}{\partial x^2} = \frac{\partial^2 u_*^x}{\partial x^2} + (x - x_i) \frac{\partial^3 u_*^x}{\partial x^3} + \left(y - y_{j+\frac{1}{2}}\right) \frac{\partial^3 u_*^x}{\partial x^2 \partial y} + O(h^2) \|D^4 u^x\|_\infty.$$

Integrating this expansion over cell $\Omega_{i,j+\frac{1}{2}}$ and using the property

$$\int_{\Omega_{i,j+\frac{1}{2}}} (y - y_{j+\frac{1}{2}}) dx dy = 0,$$

we have

$$\frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} \frac{\partial^2 u^x}{\partial x^2} = \frac{\partial^2 u_*^x}{\partial x^2} + \frac{1}{4} (x_{i+1} + x_{i-1} - 2x_i) \frac{\partial^3 u_*^x}{\partial x^3} + O(h^2) \|D^4 u^x\|_\infty. \tag{36}$$

Subtracting (34) from (36), we have

$$R_1^x = -\frac{1}{12} (x_{i+1} + x_{i-1} - 2x_i) \frac{\partial^3 u_*^x}{\partial x^3} + O(h^2) \|D^4 u^x\|_\infty,$$

which concludes the proof. □

Remark 4.1 For simplicity, in Lemma 4.1 we used the maximum norm to bound the error. If we want to bound the error by the Sobolev norm, we have to use the Taylor expansion with error in integral form.

Furthermore, we like to remark that R_1^x is only $O(h)$ for a non-uniform grid, since $x_{i+1} + x_{i-1} - 2x_i \neq 0$ unless a uniform mesh is considered. Hence the traditional error analysis technique by using the stability result directly can lead to only $O(h)$ convergence. Such issue has been noticed before for elliptic problems (e.g., [23,24]) and Maxwell’s equations [16]. By following similar ideas, we can derive $O(h^2)$ error estimate by summing up all the contributions of local truncation errors as shown in the next lemma.

Lemma 4.2

$$\sum_{i=1}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right) (y_{j+1} - y_j) R_1^x \cdot e_{i,j+\frac{1}{2}}^x \leq Ch_x^2 \|\Delta_x^- e_{i,j+\frac{1}{2}}^x\|.$$

Proof Denote $b_{i,j+\frac{1}{2}} = \frac{\partial^3 u^x}{\partial x^3}(x_i, y_{j+\frac{1}{2}})$. For the dominant term $(x_{i+1} + x_{i-1} - 2x_i)b_{i,j+\frac{1}{2}}$ of R_1^x , using property (35), we have

$$\begin{aligned}
 & \sum_{i=1}^M \sum_{j=0}^{N-1} \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right) (y_{j+1} - y_j)(x_{i+1} + x_{i-1} - 2x_i) \cdot e^x_{i,j+\frac{1}{2}} b_{i,j+\frac{1}{2}} \\
 &= \sum_{j=0}^{N-1} (y_{j+1} - y_j) \sum_{i=1}^M \left(\frac{x_{i+1} - x_{i-1}}{2}\right) (x_{i+1} + x_{i-1} - 2x_i) e^x_{i,j+\frac{1}{2}} b_{i,j+\frac{1}{2}} \\
 &= \sum_{j=0}^{N-1} (y_{j+1} - y_j) \sum_{i=1}^M \frac{(x_{i+1} - x_i) + (x_i - x_{i-1})}{2} \\
 &\quad \cdot [(x_{i+1} - x_i) - (x_i - x_{i-1})] e^x_{i,j+\frac{1}{2}} b_{i,j+\frac{1}{2}} \\
 &= \frac{1}{2} \sum_{j=0}^{N-1} (y_{j+1} - y_j) \sum_{i=1}^M [(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2] e^x_{i,j+\frac{1}{2}} b_{i,j+\frac{1}{2}} \\
 &= \frac{1}{2} \sum_{j=0}^{N-1} (y_{j+1} - y_j) \sum_{i=1}^M (x_i - x_{i-1})^2 \left(e^x_{i-1,j+\frac{1}{2}} b_{i-1,j+\frac{1}{2}} - e^x_{i,j+\frac{1}{2}} b_{i,j+\frac{1}{2}} \right) \\
 &= -\frac{1}{2} \sum_{i=1}^M \sum_{j=0}^{N-1} (x_i - x_{i-1})(y_{j+1} - y_j) \cdot (x_i - x_{i-1})^2 \\
 &\quad \left[b_{i,j+\frac{1}{2}} \Delta_x^- e^x_{i,j+\frac{1}{2}} + e^x_{i-1,j+\frac{1}{2}} \Delta_x^- b_{i,j+\frac{1}{2}} \right] \\
 &\leq Ch_x^2 \|\Delta_x^- e^x_{i,j+\frac{1}{2}}\|, \tag{37}
 \end{aligned}$$

where in the last step we used the Cauchy–Schwarz inequality, the notation of norm (13), and the identity

$$\Delta_x^-(a_i b_i) = b_i \Delta_x^- a_i + a_{i-1} \Delta_x^- b_i.$$

Similar estimates can be obtained for R_2^x . Now we investigate the error caused by R_3^x .

Lemma 4.3

$$\begin{aligned}
 R_3^x &= \Delta_x^- p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}\right) - \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{y_j}^{y_{j+1}} \left(p \left(x_{i+\frac{1}{2}}, y\right) - p \left(x_{i-\frac{1}{2}}, y\right)\right) dy \\
 &= O(h^2) \|D^3 p\|_\infty. \tag{38}
 \end{aligned}$$

Proof Note that R_3^x can be rewritten as

$$R_3^x = \Delta_x^- p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}\right) - \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} \frac{\partial p}{\partial x}(x, y) dx dy. \tag{39}$$

Using the Taylor expansion at $(x_i, y_{j+\frac{1}{2}})$, we have

$$\Delta_x^- p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}\right) = \frac{\partial p_*}{\partial x} + \frac{1}{2} \left(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}} - 2x_i\right) \frac{\partial^2 p_*}{\partial x^2} + O(h_x^2) \|\frac{\partial^3 p}{\partial x^3}\|_\infty, \tag{40}$$

where $\frac{\partial^k p_*}{\partial x^k}$, $k = 1, 2$, denote the derivatives evaluated at point $(x_i, y_{j+\frac{1}{2}})$.

Taking the Taylor expansion of $\frac{\partial p}{\partial x}$ at $(x_i, y_{j+\frac{1}{2}})$, and then integrating over cell $\Omega_{i,j+\frac{1}{2}}$, we obtain

$$\frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} \frac{\partial p}{\partial x} = \frac{\partial p^*}{\partial x} + \frac{1}{2} (x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}} - 2x_i) \frac{\partial^2 p^*}{\partial x^2} + O(h^2) \|D^3 p\|_\infty. \tag{41}$$

The proof is completed by substituting (40) and (41) into (39). □

Lemma 4.4

$$\begin{aligned} (i) \quad R_4^x &= \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} f^x(x, y) dx dy - f^x \left(x_i, y_{j+\frac{1}{2}} \right) \\ &= \frac{1}{2} (x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}} - 2x_i) \frac{\partial f^x}{\partial x} \left(x_i, y_{j+\frac{1}{2}} \right) + O(h^2) \|D^2 f^x\|_\infty, \\ (ii) \quad \sum_{i=1}^M \sum_{j=0}^{N-1} (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) (y_{j+1} - y_j) R_4^x \cdot e^x_{i,j+\frac{1}{2}} &\leq Ch^2 \|\Delta_x^- e^x_{i,j+\frac{1}{2}}\|. \end{aligned} \tag{42}$$

Proof (i) Using the Taylor expansion of f^x at $(x_i, y_{j+\frac{1}{2}})$, and then integrating over cell $\Omega_{i,j+\frac{1}{2}}$, we obtain

$$\frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} f^x = f^x_* + \frac{1}{2} (x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}} - 2x_i) \frac{\partial f^x}{\partial x} + O(h^2) \|D^2 f^x\|_\infty, \tag{43}$$

which completes the proof of (i).

(ii) The proof of (ii) follows the same idea of Lemma 4.2. □

Lemma 4.5

$$\begin{aligned} R_{i-\frac{1}{2},j+\frac{1}{2}}^d &= \Delta_x^- u^x \left(x_i, y_{j+\frac{1}{2}} \right) + \Delta_y^+ u^y \left(x_{i-\frac{1}{2}}, y_j \right) \\ &= O(h_x^2) \left\| \frac{\partial^3 u^x}{\partial x^3} \right\|_\infty + O(h_y^2) \left\| \frac{\partial^3 u^y}{\partial y^3} \right\|_\infty. \end{aligned} \tag{44}$$

Proof Taking the Taylor expansion of all functions at $(x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}})$, we have

$$\begin{aligned} \Delta_x^- u^x \left(x_i, y_{j+\frac{1}{2}} \right) &= \frac{\partial u^x_*}{\partial x} + \frac{1}{2} (x_i + x_{i-1} - 2x_{i-\frac{1}{2}}) \frac{\partial^2 u^x_*}{\partial x^2} + O(h_x^2) \left\| \frac{\partial^3 u^x}{\partial x^3} \right\|_\infty, \\ \Delta_y^+ u^y \left(x_{i-\frac{1}{2}}, y_j \right) &= \frac{\partial u^y_*}{\partial y} + \frac{1}{2} (y_j + y_{j+1} - 2y_{j+\frac{1}{2}}) \frac{\partial^2 u^y_*}{\partial y^2} + O(h_y^2) \left\| \frac{\partial^3 u^y}{\partial y^3} \right\|_\infty. \end{aligned}$$

Summing up the above two estimates, and using (29) and the properties

$$x_{i-\frac{1}{2}} = \frac{x_i + x_{i-1}}{2}, \quad y_{j+\frac{1}{2}} = \frac{y_j + y_{j+1}}{2},$$

we conclude the proof. □

With the above preparations, now we can prove the following convergence result.

Theorem 4.1 Under the assumption that the pressure error is $O(h^2)$, i.e.,

$$\|e_{i-\frac{1}{2},j+\frac{1}{2}}^p\| \leq Ch^2, \tag{45}$$

we have the discrete H_1 error estimate

$$\|\Delta_x^- e_{i,j+\frac{1}{2}}^x\| + \|\Delta_y^- e_{i,j+\frac{1}{2}}^x\| + \|\Delta_x^- e_{i-\frac{1}{2},j}^y\| + \|\Delta_y^- e_{i-\frac{1}{2},j}^y\| \leq Ch^2, \tag{46}$$

and the discrete L_2 error estimate

$$\|e_{i,j+\frac{1}{2}}^x\| + \|e_{i-\frac{1}{2},j}^y\| \leq Ch^2. \tag{47}$$

Proof The proof follows a similar idea to the stability analysis developed in Theorem 3.1. One major difference is that the discrete divergence equation (26) is not zero anymore by comparison to (12).

Multiplying (24) by $|\Omega_{i,j+\frac{1}{2}}|e_{i,j+\frac{1}{2}}^x$, and (25) by $|\Omega_{i-\frac{1}{2},j}|e_{i-\frac{1}{2},j}^y$, summing up the results over all i and j , and using Lemma 3.3 (with u replaced by e), we obtain

$$\begin{aligned} & \|\Delta_x^- e_{i,j+\frac{1}{2}}^x\|^2 + \|\Delta_y^- e_{i,j+\frac{1}{2}}^x\|^2 + \|\Delta_x^- e_{i-\frac{1}{2},j}^y\|^2 + \|\Delta_y^- e_{i-\frac{1}{2},j}^y\|^2 \\ &= \sum_i \sum_j |\Omega_{i,j+\frac{1}{2}}| e_{i,j+\frac{1}{2}}^x R_{i,j+\frac{1}{2}}^x + \sum_i \sum_j |\Omega_{i-\frac{1}{2},j}| e_{i-\frac{1}{2},j}^y R_{i-\frac{1}{2},j}^y \\ & - \sum_i \sum_j |\Omega_{i,j+\frac{1}{2}}| e_{i,j+\frac{1}{2}}^x \cdot \Delta_x^- e_{i+\frac{1}{2},j+\frac{1}{2}}^p - \sum_i \sum_j |\Omega_{i-\frac{1}{2},j}| e_{i-\frac{1}{2},j}^y \cdot \Delta_y^- e_{i-\frac{1}{2},j+\frac{1}{2}}^p. \end{aligned} \tag{48}$$

Using Lemmas 4.2-4.4 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & \sum_i \sum_j |\Omega_{i,j+\frac{1}{2}}| e_{i,j+\frac{1}{2}}^x R_{i,j+\frac{1}{2}}^x + \sum_i \sum_j |\Omega_{i-\frac{1}{2},j}| e_{i-\frac{1}{2},j}^y R_{i-\frac{1}{2},j}^y \\ & \leq Ch^2 \left(\|\Delta_x^- e_{i,j+\frac{1}{2}}^x\| + \|\Delta_y^- e_{i,j+\frac{1}{2}}^x\| + \|\Delta_x^- e_{i-\frac{1}{2},j}^y\| + \|\Delta_y^- e_{i-\frac{1}{2},j}^y\| \right). \end{aligned} \tag{49}$$

Following the proof of Lemma 3.1, we obtain

$$\begin{aligned} & \sum_i \sum_j |\Omega_{i,j+\frac{1}{2}}| e_{i,j+\frac{1}{2}}^x \cdot \Delta_x^- e_{i+\frac{1}{2},j+\frac{1}{2}}^p + \sum_i \sum_j |\Omega_{i-\frac{1}{2},j}| e_{i-\frac{1}{2},j}^y \cdot \Delta_y^- e_{i-\frac{1}{2},j+\frac{1}{2}}^p \\ &= - \sum_i \sum_j (x_i - x_{i-1})(y_{j+1} - y_j) \cdot \left[\Delta_x^- e_{i,j+\frac{1}{2}}^x + \Delta_y^+ e_{i-\frac{1}{2},j}^y \right] \cdot e_{i-\frac{1}{2},j+\frac{1}{2}}^p \\ & \leq \|R_{i-\frac{1}{2},j+\frac{1}{2}}^d\| \cdot \|e_{i-\frac{1}{2},j+\frac{1}{2}}^p\| \leq Ch^2 \|e_{i-\frac{1}{2},j+\frac{1}{2}}^p\|, \end{aligned} \tag{50}$$

where in the last step we used the Cauchy–Schwarz inequality and Lemma 4.5.

Substituting (49), and (50) with the assumption (45) into (48), we conclude the proof of (46).

Using Lemma 3.2 (with u replaced by e) to (46), we have (47). □

Remark 4.2 Theorem 4.1 shows that we have the $O(h^2)$ superconvergence on the non-uniform rectangular grids, which is a much better result than those obtained in earlier works (e.g., [4,6,17]). Of course, this result is based on the assumption (45). Actually, from the above proof it is not difficult to see that if we replace the assumption (45) by the normal requirement

$$\|e_{i-\frac{1}{2},j+\frac{1}{2}}^p\| \leq C, \tag{51}$$

(i.e., the error of pressure is bounded, which is equivalent to that the numerical pressure is bounded), then we recover the usual $O(h)$ convergence for the velocity in both discrete H_1 and L_2 norms.

In the rest of this section, we want to show that the assumption (45) actually holds true when the pressure $p = 0$ on just one boundary edge of Ω . Hence $\mathbf{u} = 0$ is only imposed for either u^x or u^y on that edge, and this change of boundary conditions does not affect all previous results (cf. Remark 3.2).

Taking the divergence of (1) and using (2), we have

$$\Delta p = \nabla \cdot \mathbf{f} := g. \tag{52}$$

Integrating (52) over cell $\Omega_{i+\frac{1}{2},j+\frac{1}{2}}$ and following the same idea used for developing the MAC scheme for the Stokes problem, we can obtain the cell-center difference scheme for the pressure:

$$\frac{\Delta_x^+ p_{i+\frac{1}{2},j+\frac{1}{2}} - \Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}}}{x_{i+1} - x_i} + \frac{\Delta_y^+ p_{i+\frac{1}{2},j+\frac{1}{2}} - \Delta_y^- p_{i+\frac{1}{2},j+\frac{1}{2}}}{y_{j+1} - y_j} = g_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{53}$$

Multiplying (53) by $p_{i+\frac{1}{2},j+\frac{1}{2}} |\Omega_{i+\frac{1}{2},j+\frac{1}{2}}|$, and summing up all i and j , we have

$$\|\Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}}\|^2 + \|\Delta_y^- p_{i+\frac{1}{2},j+\frac{1}{2}}\|^2 \leq \|g_{i+\frac{1}{2},j+\frac{1}{2}}\| \cdot \|p_{i+\frac{1}{2},j+\frac{1}{2}}\|, \tag{54}$$

which, along with the discrete Poincaré inequality for p (cf. Remark 3.2 by assuming $p = 0$ on one edge of Ω), we have the stability:

$$\|\Delta_x^- p_{i+\frac{1}{2},j+\frac{1}{2}}\|^2 + \|\Delta_y^- p_{i+\frac{1}{2},j+\frac{1}{2}}\|^2 \leq C \|g_{i+\frac{1}{2},j+\frac{1}{2}}\|^2. \tag{55}$$

Following the steps used above for deriving the error estimate for the Stokes problem, we can easily obtain the error estimate:

$$\|\Delta_x^- \hat{e}_{i+\frac{1}{2},j+\frac{1}{2}}^p\| + \|\Delta_y^- \hat{e}_{i+\frac{1}{2},j+\frac{1}{2}}^p\| \leq Ch^2, \tag{56}$$

where the error of p is denoted by $\hat{e}_{i+\frac{1}{2},j+\frac{1}{2}}^p = p(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) - p_{i+\frac{1}{2},j+\frac{1}{2}}$.

From (56) and the discrete Poincaré inequality for $\hat{e}_{i+\frac{1}{2},j+\frac{1}{2}}^p$, we have $\|\hat{e}_{i+\frac{1}{2},j+\frac{1}{2}}^p\| \leq Ch^2$, which justifies the validity of our assumption (45).

5 Numerical Results

In this section, we present some numerical results to justify our theoretical analysis. For simplicity, we fix our domain $\Omega = (0, 1)^2$, and $\mu = 1$ in (1). We solve a test problem of [20], which has the exact solution of

$$p = 150(x - 0.5)(y - 0.5),$$

$$u^x(x, y) = -256x^2(x - 1)^2y(y - 1)(2y - 1), \quad u^y(x, y) = -u^x(y, x),$$

where the corresponding \mathbf{f} is given by (1). Using the MAC scheme for the problem on uniform and nonuniform meshes, we obtain the errors and the relative errors of pressure and velocity given in Tables 1 and 2, respectively. The non-uniform mesh used here is generated from the corresponding uniform mesh by refining the prime-numbered intervals in both x and y directions (cf. Fig. 2).

Table 1 Convergence results obtained with uniform rectangular grids

Meshes	$\ e_p(h)\ _0$	$\ e_u^x(h)\ _0$	$\ e_u(h)\ _0$	$\varepsilon_p(h)$	$\varepsilon_u(h)$	$\frac{\varepsilon_p(h)}{\varepsilon_p(2h)}$	$\frac{\varepsilon_u(h)}{\varepsilon_u(2h)}$
$h = \frac{1}{4}$	0.1454	0.096	0.1358	0.0124	0.1167	–	–
$h = \frac{1}{8}$	0.0483	0.0177	0.025	0.0039	0.0245	0.3323	0.1843
$h = \frac{1}{16}$	0.011	0.0036	0.0051	8.7952e-4	0.0051	0.2267	0.2026
$h = \frac{1}{32}$	0.0025	7.9287e-4	0.0011	1.9667e-4	0.0011	0.2243	0.2212
$h = \frac{1}{64}$	5.7207e-04	1.8581e-4	2.6277e-04	4.5777e-5	2.6393e-4	0.2329	0.2343
$h = \frac{1}{128}$	1.3747e-04	4.4949e-5	6.3568e-05	1.09e-5	6.3861e-5	0.2403	0.2419

Table 2 Convergence results obtained with nonuniform rectangular grids

Meshes	$\ e_p(h)\ _0$	$\ e_u^x(h)\ _0$	$\ e_u(h)\ _0$	$\ \varepsilon_p(h)\ _0$	$\ \varepsilon_u(h)\ _0$	$\frac{\ e_p(h)\ _0}{\ e_p(2h)\ _0}$	$\frac{\ e_u(h)\ _0}{\ e_u(2h)\ _0}$
$h = \frac{1}{4}$	0.0613	0.0589	0.0833	0.0051	0.0722	–	–
$h = \frac{1}{8}$	0.0945	0.0097	0.0138	0.0076	0.0138	1.5429	0.1653
$h = \frac{1}{16}$	0.0264	0.0025	0.0035	0.0021	0.0036	0.2793	0.2574
$h = \frac{1}{32}$	0.0079	6.5578e-4	9.2741e-4	6.3199e-4	9.3092e-4	0.299	0.2618
$h = \frac{1}{64}$	0.0019	1.5353e-4	2.1713e-04	1.5032e-4	2.1810e-4	0.238	0.2341
$h = \frac{1}{128}$	4.5341e-4	3.7202e-5	5.2611e-05	3.6274e-5	5.2855e-5	0.2413	0.2423

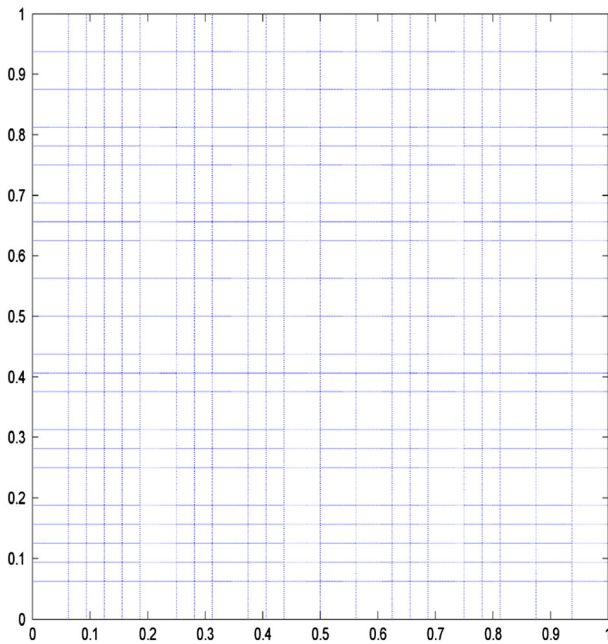


Fig. 2 The non-uniform mesh generated from the 16×16 uniform mesh

To evaluate the convergence rate, we define the following absolute errors for velocity and pressure:

$$\begin{aligned} \|e_u^x\|_0^2 &:= \sum_{i=0}^M \sum_{j=0}^{N-1} \left[u_{i,j+\frac{1}{2}}^x - u^x \left(x_i, y_{j+\frac{1}{2}} \right) \right]^2 \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left(y_{j+1} - y_j \right), \\ \|e_u^y\|_0^2 &:= \sum_{i=0}^{M-1} \sum_{j=0}^N \left[u_{i+\frac{1}{2},j}^y - u^y \left(x_{i+\frac{1}{2}}, y_j \right) \right]^2 \left(x_{i+1} - x_i \right) \left(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right), \\ \|e_u\|_0^2 &:= \|e_u^x\|_0^2 + \|e_u^y\|_0^2, \\ \|e_p\|_0^2 &:= \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \left[p_{i+\frac{1}{2},j+\frac{1}{2}} - p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) \right]^2 \left(x_{i+1} - x_i \right) \left(y_{j+1} - y_j \right), \end{aligned}$$

and the relative errors for velocity and pressure:

$$\begin{aligned} \varepsilon_u^x &:= \frac{\|e_u^x\|_0}{\|u^x\|_0} = \frac{\sqrt{\sum_{i=0}^M \sum_{j=0}^{N-1} \left[u_{i,j+\frac{1}{2}}^x - u^x \left(x_i, y_{j+\frac{1}{2}} \right) \right]^2 \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left(y_{j+1} - y_j \right)}}{\sqrt{\sum_{i=0}^M \sum_{j=0}^{N-1} \left[u^x \left(x_i, y_{j+\frac{1}{2}} \right) \right]^2 \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left(y_{j+1} - y_j \right)}}, \\ \varepsilon_u^y &:= \frac{\|e_u^y\|_0}{\|u^y\|_0} = \frac{\sqrt{\sum_{i=0}^{M-1} \sum_{j=0}^N \left[u_{i+\frac{1}{2},j}^y - u^y \left(x_{i+\frac{1}{2}}, y_j \right) \right]^2 \left(x_{i+1} - x_i \right) \left(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right)}}{\sqrt{\sum_{i=0}^{M-1} \sum_{j=0}^N \left[u^y \left(x_{i+\frac{1}{2}}, y_j \right) \right]^2 \left(x_{i+1} - x_i \right) \left(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right)}}, \\ \varepsilon_u &:= \frac{\|e_u\|_0}{\|u\|_0} = \frac{\sqrt{\|e_u^x\|_0^2 + \|e_u^y\|_0^2}}{\sqrt{\|u^x\|_0^2 + \|u^y\|_0^2}}, \\ \varepsilon_p &:= \frac{\|e_p\|_0}{\|p\|_0} = \frac{\sqrt{\sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \left[p_{i+\frac{1}{2},j+\frac{1}{2}} - p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) \right]^2 \left(x_{i+1} - x_i \right) \left(y_{j+1} - y_j \right)}}{\sqrt{\sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \left[p \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) \right]^2 \left(x_{i+1} - x_i \right) \left(y_{j+1} - y_j \right)}}. \end{aligned}$$

We also display the ratios of relative errors of adjacently refined meshes (i.e. $\frac{\varepsilon_p(h)}{\varepsilon_p(2h)}$ and $\frac{\varepsilon_u(h)}{\varepsilon_u(2h)}$) for uniform meshes and nonuniform meshes in Tables 1 and 2, respectively. It is clear that the ratios of relative errors converges to the expected value of 0.25 with enhanced mesh resolution.

The errors and convergence rates of velocity and pressure obtained on various uniform and non-uniform rectangular grids are displayed in Figs. 3 and 4, where we observe that the convergence rates are of second order in the L^2 norm, which confirm our theoretical analysis.

6 Conclusions

The MAC method as an effective numerical method for solving 2D Stokes problems has been widely used in practice. However, the second-order (super-)convergence of this method on non-uniform meshes has not been pointed out and theoretically analyzed. In this work,

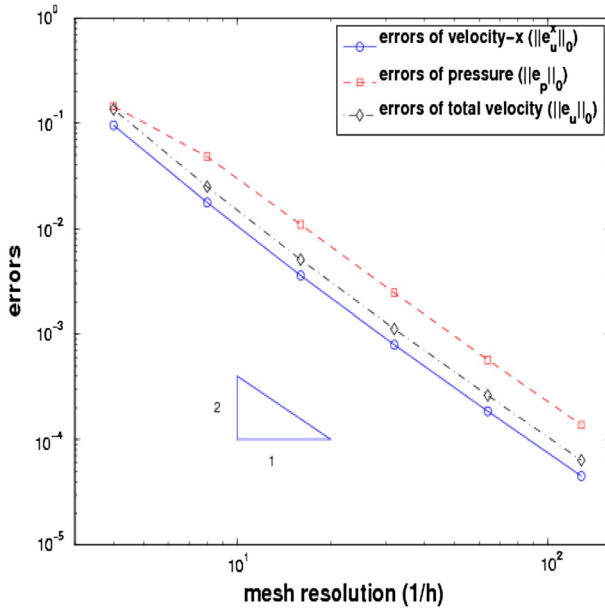


Fig. 3 Convergence results with uniform meshes

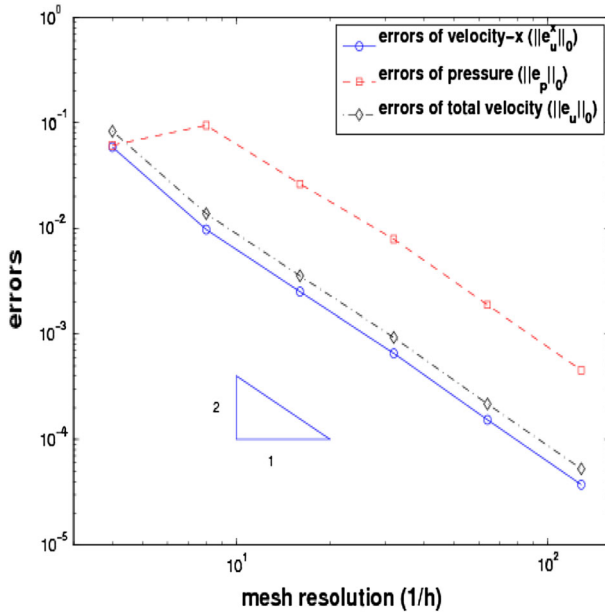


Fig. 4 Convergence results with nonuniform meshes

we prove rigorously in the first time that the L^2 errors for both the velocity and pressure converge in the second order in terms of mesh size. Numerical results are provided to verify the theoretical results. We like to mention that similar results hold true for 3D Stokes equations

with similar cubic cell-centered grids. Some possible future works are to investigate the convergence analysis for high order MAC scheme for the Stokes problem [10] and for the full Navier–Stokes equation on nonuniform meshes.

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References

1. Brenner, S., Scott, L.R.: *The Mathematical Theory of Finite Element Methods*, 3rd edn. Springer, Berlin (2008)
2. Cockburn, B., Kanschat, G., Schötzau, D.: A locally conservative LDG method for the incompressible Navier–Stokes equations. *Math. Comp.* **74**, 1067–1095 (2005)
3. Daly, B.J., Harlow, F.H., Shannon, J.P., Welch, J.E.: *The MAC Method*, Tech. report LA-3425, Los Alamos Scientific Laboratory, University of California (1965)
4. Girault, V., Lopez, H.: Finite-element error estimates for the MAC scheme. *IMA J. Numer. Anal.* **16**, 347–379 (1996)
5. Gustafsson, B., Nilsson, J.: Boundary conditions and estimates for the steady Stokes equations on staggered grids. *J. Sci. Comput.* **15**, 29–59 (2000)
6. Han, H., Wu, X.: A new mixed finite element formulation and the MAC method for the Stokes equations. *SIAM J. Numer. Anal.* **35**(2), 560–571 (1998)
7. Huang, Y., Li, J., Wu, C., Yang, W.: Superconvergence analysis for linear tetrahedral edge elements. *J. Sci. Comput.* doi:[10.1007/s10915-014-9848-7](https://doi.org/10.1007/s10915-014-9848-7)
8. Huang, Y., Li, J., Yang, W., Sun, S.: Superconvergence of mixed finite element approximations to 3-D Maxwell’s equations in metamaterials. *J. Comp. Phys.* **230**, 8275–8289 (2011)
9. Huang, Y., Zhang, S.: Supercloseness of the divergence-free finite element solutions on rectangular grids. *Commun. Math. Stat.* **1**, 143–162 (2013)
10. Ito, K., Qiao, Z.: A high order compact MAC finite difference scheme for the Stokes equations: augmented variable approach. *J. Comp. Phys.* **227**, 8177–8190 (2008)
11. Kanschat, G.: Divergence-free discontinuous Galerkin schemes for the Stokes equations and the MAC scheme. *Int. J. Numer. Meth. Fluids* **56**, 941–950 (2008)
12. Lebedev, V.L.: Difference analogues of orthogonal decompositions, fundamental differential operators and certain boundary-value problems of mathematical physics. *Z. Vycisl. Mat. i Mat. Fiz.* **4**, 449–465 (1964)
13. Li, J., Huang, Y.: *Time-Domain Finite Element Methods for Maxwell’s Equations in Metamaterials*, Springer Series in Computational Mathematics, vol. 43. Springer, Berlin (2013)
14. Li, M., Tang, T., Fornberg, B.: A compact fourth-order finite difference scheme for the steady incompressible Navier–Stokes equations. *Int. J. Numer. Meth. Fluids* **20**, 1137–1151 (1995)
15. Mineev, P.D.: Remarks on the links between low-order DG methods and some finite-difference schemes for the Stokes problem. *Int. J. Numer. Meth. Fluids* **58**, 307–317 (2008)
16. Monk, P., Sili, E.: A convergence analysis of Yee’s scheme on nonuniform grids. *SIAM J. Numer. Anal.* **32**(2), 393–412 (1994)
17. Nicolaides, R.A.: Analysis and convergence of the MAC scheme I. The linear problem. *SIAM J. Numer. Anal.* **29**(6), 1579–1591 (1992)
18. Nicolaides, R.A., Wu, X.: Analysis and convergence of the MAC scheme. II. Navier–Stokes equations. *Math. Comput.* **65**, 29–44 (1996)
19. Qiao, Z., Yao, C., Jia, S.: Superconvergence and extrapolation analysis of a nonconforming mixed finite element approximation for time-harmonic Maxwell’s equations. *J. Sci. Comput.* **46**, 1–19 (2011)
20. Rannacher, R., Turek, S.: Simple nonconforming quadrilateral Stokes element. *Numer. Methods Partial Differ. Equ.* **8**, 97–111 (1992)
21. Samelson, R., Temam, R., Wang, C., Wang, S.: Surface pressure Poisson equation formulation of the primitive equations: numerical schemes. *SIAM J. Numer. Anal.* **41**, 1163–1194 (2003)
22. Samelson, R., Temam, R., Wang, C., Wang, S.: A fourth-order numerical method for the planetary geostrophic equations with inviscid geostrophic balance. *Numer. Math.* **107**, 669–705 (2007)

23. Süli, E.: Convergence of finite volume schemes for Poisson's equation on nonuniform meshes. *SIAM J. Numer. Anal.* **28**(5), 1419–1430 (1991)
24. Weiser, A., Wheeler, M.F.: On convergence of block-centered finite differences for elliptic problems. *SIAM J. Numer. Anal.* **25**(2), 351–375 (1988)
25. Yang, Y., Shu, C.-W.: Analysis of optimal superconvergence of discontinuous Galerkin method for linear hyperbolic equations. *SIAM J. Numer. Anal.* **50**, 3110–3133 (2012)
26. Zhang, Z. (ed.): Special issues of “Superconvergence and a Posteriori Error Estimates in Finite Element Methods”. *Int. J. Numer. Anal. Model* **2**(1), 1–126 (2005) and **3**(3), 255–376 (2006)