

MATHEMATICAL ANALYSIS AND TIME-DOMAIN FINITE ELEMENT SIMULATION OF CARPET CLOAK*

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Abstract. In this paper, a mathematical analysis of the popular carpet cloak model is carried out. The well-posedness of the model is first proved, and then a finite element time-domain method is developed for solving this model. Three carpet cloak simulations are demonstrated to show the effectiveness of our model and the developed algorithm. To the best of our knowledge, this is the first mathematical analysis carried out for the carpet cloak model. The numerical simulation using edge elements for the carpet cloak model is also original.

Key words. Maxwell's equations, metamaterials, carpet cloak, edge element

AMS subject classifications. 65N30, 35L15, 78-08

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1. Introduction. Since invisibility cloaks were first suggested by transformation optics theory in 2006 independently by Leonhardt [15] and Pendry, Schurig, and Smith [26] and by anomalous resonance by Milton and Nicorovici [24], much research has been done in the theoretical analysis, design, and fabrication of various types (cf. [9, 28] and references therein). We note that the first experimental verification of the invisibility cloak in the microwave regime was made by Schurig et al. [27], but it suffers some shortcomings. One problem is that it is difficult to manufacture such a metamaterial since the physical parameters are highly anisotropic; another problem is that such a cloak normally requires resonant structures, which lead to narrow band operation. One breakthrough work is due to Li and Pendry [20], who proposed the so-called carpet cloak by quasi-conformal mapping. This strategy circumvents both issues by transforming a bulging reflecting surface into a flat one, rendering anything within the bulging surface invisible to outside observers. Experimental realization of carpet cloaks was successfully demonstrated first at the microwave band by Liu et al [22]; it was then extended to terahertz and optical frequencies (see [8] and references therein). We comment that the transformation idea was discussed earlier by Greenleaf, Lassas, and Uhlmann [10] for electrical impedance tomography.

In recent years, mathematicians have started investigating the invisibility cloaking phenomenon, but most of their works are limited to the frequency-domain by studying the Helmholtz equation [13, 14, 4] and the time-harmonic Maxwell's equations [5, 7, 6, 21]. Very recently, the cloaking by anomalous resonance is mathematically justified for the first time in two and three dimensions [1, 2, 3]. The advent of broadband cloaks [11, 16, 22] inspired us to pursue the time-domain cloaking simulation and analysis. In [18], we developed a finite element time-domain (FETD) method to

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simulate the cylindrical cloak proposed by Pendry, Schurig, and Smith [26], and we carried out some mathematical analysis for that model. Later, we extended this work to an elliptical cloak and carried out the well-posedness study of both models [19]. A Green’s function approach for cylindrical cloaks is proposed in [23]. In this paper, we investigate the two-dimensional (2D) carpet cloak model from the mathematical point of view. Specifically, we derive the modeling equations and prove the well-posedness. We also develop an FETD method to simulate the cloaking phenomenon. To the best of our knowledge, this is the first mathematical treatment for the carpet cloak model. The development and analysis of this FETD method for this model is original.

The rest of the paper is organized as follows. In section 2, we first derive the governing equations for the carpet cloak. Then we prove the existence and uniqueness of the solution for the cloak modeling equations. A stability result is also obtained. In section 3, we propose an FETD scheme using edge elements to solve the carpet cloak model. A discrete stability result similar to the continuous case is proved. Then in section 4, three interesting carpet cloaking simulations are provided. We conclude the paper in section 5.

2. The modeling equations. Following [8], a triangular carpet cloak shown in Figure 1 (left) can be achieved with spatially homogeneous anisotropic dielectric materials. The cloaked region is the bottom triangle with vertices $(0, H_1)$, $(-d, 0)$, and $(d, 0)$. The cloaking region is the quadrilateral region formed by vertices $(-d, 0)$, $(0, H_1)$, $(d, 0)$, and $(0, H_2)$, where $H_1, d > 0$. In order to make the hidden objects inside the cloaked region invisible to outside observers, the permittivity and permeability in the cloaking region need to be specially designed.

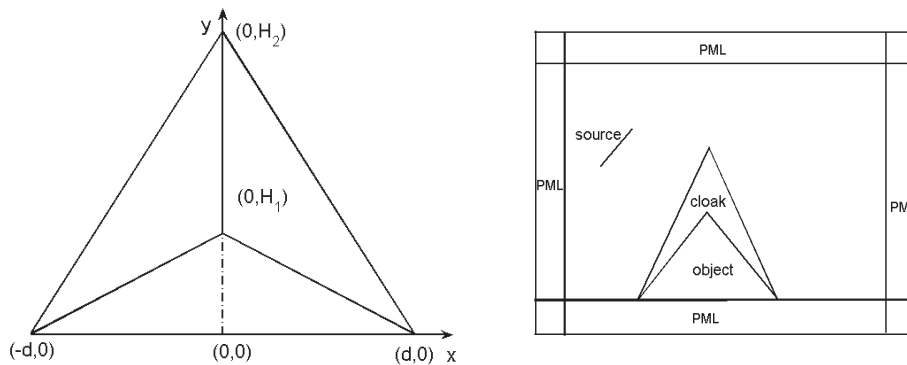


FIG. 1. The physical space of the carpet cloak.

Using the transformation optics theory (e.g., [17, sec. 9.2]), the relative permittivity and permeability in the cloaking region are given by [8]

$$\varepsilon = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} \frac{H_2}{H_2 - H_1} & -\frac{H_1 H_2}{(H_2 - H_1)d} \operatorname{sgn}(x) \\ -\frac{H_1 H_2}{(H_2 - H_1)d} \operatorname{sgn}(x) & \frac{H_2 - H_1}{H_2} + \frac{H_2}{H_2 - H_1} \left(\frac{H_1}{d}\right)^2 \end{bmatrix},$$

$$\mu = \frac{H_2}{H_2 - H_1},$$

where $\operatorname{sgn}(x)$ denotes the standard sign function. By the construction, it is assumed that $H_2 > H_1 > 0$.

It is easy to see that the characteristic equation of matrix ε is $\lambda^2 - (a+c)\lambda + 1 = 0$, which leads to the eigenvalues of matrix ε as

$$\lambda_1 = \frac{a+c - \sqrt{(a-c)^2 + 4b^2}}{2}, \quad \lambda_2 = \frac{a+c + \sqrt{(a-c)^2 + 4b^2}}{2}.$$

Moreover, the symmetric matrix ε can be diagonalized as

$$(2.1) \quad \varepsilon = P\Sigma P^T,$$

where matrices Σ and P are

$$\Sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix},$$

and elements $p_i, 1 \leq i \leq 4$, are given as

$$p_1 = \sqrt{\frac{\lambda_2 - a}{\lambda_2 - \lambda_1}}, \quad p_2 = \sqrt{\frac{a - \lambda_1}{\lambda_2 - \lambda_1}} \cdot \text{sgn}(x),$$

$$p_3 = -\sqrt{\frac{\lambda_2 - c}{\lambda_2 - \lambda_1}} \cdot \text{sgn}(x), \quad p_4 = \sqrt{\frac{c - \lambda_1}{\lambda_2 - \lambda_1}}.$$

It is not difficult to see that $\lambda_2 \geq \frac{a+c+|a-c|}{2} \geq a > 1$ holds true for any parameters H_1, H_2 , and d satisfying the constraint $H_2 > H_1 > 0$. Therefore, the relation $\lambda_1 \cdot \lambda_2 = 1$ implies that the eigenvalue $\lambda_1 \in (0, 1)$, which cannot be used directly in the time domain cloaking simulation. Similar to our previous work [18], we map λ_1 by the lossless Drude dispersion model,

$$\lambda_1(\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

where ω_p is the plasma frequency and ω is the general wave frequency.

Let \mathbf{D} be the 2D electric displacement, and let \mathbf{E} be the 2D electric field. Substituting $\varepsilon = P\Sigma P^T$ into the constitutive equation $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$, and using the property $PP^T = P^T P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain

$$\varepsilon_0 \mathbf{E} = P\Sigma^{-1} P^T \mathbf{D},$$

which can be written in time-domain (assuming $e^{i\omega t}$ time dependence) as follows:

$$(2.2) \quad \varepsilon_0 \lambda_2 (\mathbf{E}_{t^2} + \omega_p^2 \mathbf{E}) = M_A \mathbf{D}_{t^2} + M_B \mathbf{D},$$

where matrices M_A and M_B are

$$M_A = \begin{pmatrix} p_1^2 \lambda_2 + p_2^2 & p_2 p_4 + p_1 p_3 \lambda_2 \\ p_2 p_4 + p_1 p_3 \lambda_2 & p_3^2 \lambda_2 + p_4^2 \end{pmatrix}, \quad M_B = \begin{pmatrix} p_2^2 & p_2 p_4 \\ p_2 p_4 & p_4^2 \end{pmatrix} \omega_p^2.$$

Here and below, for simplicity we denote \mathbf{E}_{t^k} as the k th derivative $\partial^k \mathbf{E} / \partial t^k, k \geq 1$. The same notation is used for other variables.

Coupling (2.2) with Faraday's law and Ampere's law, we obtain the governing equations for the carpet cloak:

$$(2.3) \quad \mathbf{D}_t = \nabla \times \mathbf{H},$$

$$(2.4) \quad \varepsilon_0 \lambda_2 (\mathbf{E}_{t^2} + \omega_p^2 \mathbf{E}) = M_A \mathbf{D}_{t^2} + M_B \mathbf{D},$$

$$(2.5) \quad \mu_0 \mu \mathbf{H}_t = -\nabla \times \mathbf{E},$$

where H denotes the magnetic field. Here and in the rest of paper, we use the 2D vector and scalar curl operators:

$$\nabla \times H = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x} \right)', \quad \nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \quad \forall \mathbf{E} = (E_x, E_y)',$$

where E_x and E_y denote the x and y components of the electric field \mathbf{E} , respectively. To make the carpet cloak model (2.3)–(2.5) complete, we assume that (2.3)–(2.5) satisfy the initial conditions

$$(2.6) \quad \mathbf{D}(\mathbf{x}, 0) = \mathbf{D}_0(\mathbf{x}), \quad \mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \quad H(\mathbf{x}, 0) = H_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega,$$

and the perfect conducting boundary condition (PEC)

$$(2.7) \quad \mathbf{n} \times \mathbf{E} = \mathbf{0} \quad \text{on } \partial\Omega,$$

where \mathbf{D}_0 , \mathbf{E}_0 , and H_0 are some properly given functions, \mathbf{n} is the unit outward normal vector to $\partial\Omega$, and Ω denotes the cloaking region.

LEMMA 2.1. *The matrix M_B is symmetric and nonnegative definite, and the matrix M_A is symmetric positive definite.*

Proof. First, it is easy to see that for any vector $(u, v)'$, we have

$$(u, v)M_B \begin{pmatrix} u \\ v \end{pmatrix} = \omega_p^2(p_2^2u^2 + 2p_2p_4uv + p_4^2v^2) = \omega_p^2(p_2u + p_4v)^2 \geq 0,$$

which proves the nonnegativeness of M_B .

Similarly, for any nonzero vector $(u, v)'$, we have

$$(2.8) \quad \begin{aligned} (u, v)M_A \begin{pmatrix} u \\ v \end{pmatrix} &= (p_1^2\lambda_2 + p_2^2)u^2 + 2(p_2p_4 + p_1p_3\lambda_2)uv + (p_3^2\lambda_2 + p_4^2)v^2 \\ &= \lambda_2(p_1u + p_3v)^2 + (p_2u + p_4v)^2 > 0, \end{aligned}$$

since the expression $\lambda_2(p_1u + p_3v)^2 + (p_2u + p_4v)^2$ equals zero if and only if $u = v = 0$. This completes the proof for the positive definiteness of M_A . \square

With Lemma 2.1, we can prove the existence and uniqueness of the solution for our carpet cloaking model.

THEOREM 2.2. *For any $t \in [0, T]$, there exists a unique solution $(\mathbf{E}(\cdot, t), \mathbf{D}(\cdot, t), H(\cdot, t)) \in (H_0(\text{curl}; \Omega))^2 \times H(\text{curl}; \Omega)$ of (2.3)–(2.7).*

Proof. For any function $u(t)$ defined for $t \geq 0$, let us denote its Laplace transform by $\hat{u}(s) = \mathcal{L}(u) = \int_0^\infty e^{-st}u(t)dt$. Taking the Laplace transforms of (2.3)–(2.5), respectively, we obtain

$$(2.9) \quad s\hat{\mathbf{D}} - \mathbf{D}_0 = \nabla \times \hat{H},$$

$$(2.10) \quad \varepsilon_0\lambda_2 \left(s^2\hat{\mathbf{E}} - s\mathbf{E}_0 - \partial_t\mathbf{E}(0) + \omega_p^2\hat{\mathbf{E}} \right) = M_A \left(s^2\hat{\mathbf{D}} - s\mathbf{D}_0 - \partial_t\mathbf{D}(0) \right) + M_B\hat{\mathbf{D}},$$

$$(2.11) \quad \mu_0\mu(s\hat{H} - H_0) = -\nabla \times \hat{\mathbf{E}}.$$

Multiplying (2.10) by s and then replacing $\hat{\mathbf{D}}$ by using (2.9), we have

$$(2.12) \quad \varepsilon_0\lambda_2(s^3 + s\omega_p^2)\hat{\mathbf{E}} = (s^2M_A + M_B)\nabla \times \hat{H} + \mathbf{f}_0(s),$$

where $\mathbf{f}_0(s) = \varepsilon_0\lambda_2(s^2\mathbf{E}_0 + s\partial_t\mathbf{E}(0)) + (s^2M_A + M_B)\mathbf{D}_0 - sM_A(s\mathbf{D}_0 - \partial_t\mathbf{D}(0))$.

Multiplying (2.12) by $\mu_0\mu s$ and then using (2.11) to eliminate \hat{H} , we obtain

$$(2.13) \quad \varepsilon_0\mu_0\mu\lambda_2(s^4 + s^2\omega_p^2)\hat{\mathbf{E}} = (s^2M_A + M_B)(\mu_0\mu\nabla \times \hat{H}_0 - \nabla \times \nabla \times \hat{\mathbf{E}}) + \mu_0\mu s\mathbf{f}_0(s),$$

which has a weak formulation as follows: Find $\hat{\mathbf{E}} \in H_0(\text{curl}; \Omega)$ such that

$$(2.14) \quad \varepsilon_0\mu_0\mu\lambda_2(s^4 + s^2\omega_p^2)(\hat{\mathbf{E}}, \mathbf{u}) + (s^2M_A + M_B)(\nabla \times \hat{\mathbf{E}}, \nabla \times \mathbf{u}) = (\mathbf{F}_0(s), \mathbf{u})$$

holds true for any $\mathbf{u} \in H_0(\text{curl}; \Omega)$. Here $\mathbf{F}_0(s) = \mu_0\mu(s^2M_A + M_B)\nabla \times \hat{H}_0 + \mu_0\mu s\mathbf{f}_0(s)$.

The existence of a unique solution $\hat{\mathbf{E}} \in H_0(\text{curl}; \Omega)$ of (2.14) is guaranteed by the Lax–Milgram lemma, since by Lemma 2.1 the matrix $s^2M_A + M_B$ is symmetric positive definite. The existence and uniqueness of $\hat{\mathbf{D}} \in H_0(\text{curl}; \Omega)$ and $\hat{H} \in H(\text{curl}; \Omega)$ are from (2.9)–(2.11). \square

Now we prove the following stability result for the solution of (2.3)–(2.5).

THEOREM 2.3. *For the solution (\mathbf{D}, \mathbf{E}) of (2.3)–(2.5) and any $t \in [0, T]$, the following stability holds true:*

$$\begin{aligned} & \left(\|\sqrt{M_A}\mathbf{D}_t\|^2 + \|\sqrt{M_B}\mathbf{D}\|^2 + \|\mathbf{E}_{t^2}\|^2 + \|\mathbf{E}_t\|^2 + \|\mathbf{E}\|^2 + \|\sqrt{M_A}\nabla \times \mathbf{E}_t\|^2 \right) (t) \\ & \leq C \left(\|\sqrt{M_A}\mathbf{D}_t\|^2 + \|\sqrt{M_B}\mathbf{D}\|^2 + \|\mathbf{E}_{t^2}\|^2 + \|\mathbf{E}_t\|^2 + \|\mathbf{E}\|^2 + \|\sqrt{M_A}\nabla \times \mathbf{E}_t\|^2 \right) (0), \end{aligned}$$

where the constant $C > 0$ depends on the physical parameters ε_0 , μ_0 , d , H_1 , H_2 , and ω_p .

Proof. The proof is composed of three major parts.

Part I. Multiplying (2.4) by \mathbf{D}_t and integrating the resultant over domain Ω , we have

$$\frac{1}{2} \frac{d}{dt} [(M_A\mathbf{D}_t, \mathbf{D}_t) + (M_B\mathbf{D}, \mathbf{D})] = \varepsilon_0\lambda_2(\mathbf{E}_{t^2}, \mathbf{D}_t) + \varepsilon_0\lambda_2\omega_p^2(\mathbf{E}, \mathbf{D}_t);$$

integrating this from 0 to t yields

$$(2.15) \quad \begin{aligned} & \frac{1}{2} [(M_A\mathbf{D}_t, \mathbf{D}_t) + (M_B\mathbf{D}, \mathbf{D})](t) - \frac{1}{2} [(M_A\mathbf{D}_t, \mathbf{D}_t) + (M_B\mathbf{D}, \mathbf{D})](0) \\ & = \int_0^t \varepsilon_0\lambda_2(\mathbf{E}_{t^2}, \mathbf{D}_t) ds + \int_0^t \varepsilon_0\lambda_2\omega_p^2(\mathbf{E}, \mathbf{D}_t) ds. \end{aligned}$$

Applying the Cauchy–Schwarz inequality and absorbing all terms involving initial conditions into $I_1(0)$, we can rewrite (2.15) as

$$(2.16) \quad \begin{aligned} & \frac{1}{2} [\|\sqrt{M_A}\mathbf{D}_t\|^2 + \|\sqrt{M_B}\mathbf{D}\|^2](t) \\ & \leq I_1(0) + \frac{\varepsilon_0\lambda_2}{2} \int_0^t (\|\mathbf{E}_{t^2}\|^2 + \|\mathbf{D}_t\|^2) ds + \frac{\varepsilon_0\lambda_2\omega_p^2}{2} \int_0^t (\|\mathbf{E}\|^2 + \|\mathbf{D}_t\|^2) ds. \end{aligned}$$

Part II. Similarly, multiplying (2.4) by \mathbf{E}_t and integrating the result over domain Ω , we have

$$\frac{1}{2} \varepsilon_0\lambda_2 \frac{d}{dt} [(\mathbf{E}_t, \mathbf{E}_t) + \omega_p^2(\mathbf{E}, \mathbf{E})] = (M_A\mathbf{D}_{t^2}, \mathbf{E}_t) + (M_B\mathbf{D}, \mathbf{E}_t);$$

integrating this from 0 to t , we obtain

$$(2.17) \quad \frac{1}{2}\varepsilon_0\lambda_2[(\mathbf{E}_t, \mathbf{E}_t) + \omega_p^2(\mathbf{E}, \mathbf{E})](t) - \frac{1}{2}\varepsilon_0\lambda_2[(\mathbf{E}_t, \mathbf{E}_t) + \omega_p^2(\mathbf{E}, \mathbf{E})](0) \\ = (M_A \mathbf{D}_t, \mathbf{E}_t)(t) - (M_A \mathbf{D}_t, \mathbf{E}_t)(0) - \int_0^t (M_A \mathbf{D}_s, \mathbf{E}_{t^2}) ds + \int_0^t (M_B \mathbf{D}_s, \mathbf{E}_t) ds.$$

Applying the Cauchy–Schwarz inequality and absorbing all terms involving initial conditions into $I_2(0)$, we can rewrite (2.17) as

$$(2.18) \quad \frac{\varepsilon_0\lambda_2}{2}[\|\mathbf{E}_t\|^2 + \omega_p^2\|\mathbf{E}\|^2](t) \leq I_2(0) + \frac{1}{4}\|\sqrt{M_A}\mathbf{D}_t\|^2(t) + \|\mathbf{E}_t\|^2(t) \\ + \frac{1}{2}\int_0^t (\|\sqrt{M_A}\mathbf{D}_s\|^2 + \|\mathbf{E}_{t^2}\|^2 + \|\sqrt{M_B}\mathbf{D}_s\|^2 + \|\mathbf{E}_t\|^2) ds.$$

Part III. Differentiating (2.4) with respect to t and using (2.3), we have

$$(2.19) \quad \varepsilon_0\lambda_2(\mathbf{E}_{t^3} + \omega_p^2\mathbf{E}_t) = M_A \nabla \times H_{t^2} + M_B \mathbf{D}_t.$$

Multiplying (2.19) by $\mu_0\mu$, and then replacing H by (2.5), we obtain

$$(2.20) \quad \varepsilon_0\mu_0\mu\lambda_2(\mathbf{E}_{t^3} + \omega_p^2\mathbf{E}_t) = -M_A \nabla \times \nabla \times \mathbf{E}_t + \mu_0\mu M_B \mathbf{D}_t.$$

Multiplying (2.20) by \mathbf{E}_{t^2} and integrating the result over domain Ω , we further have

$$(2.21) \quad \frac{1}{2}\varepsilon_0\mu_0\mu\lambda_2 \frac{d}{dt}[(\mathbf{E}_{t^2}, \mathbf{E}_{t^2}) + \omega_p^2(\mathbf{E}_t, \mathbf{E}_t)] + \frac{1}{2} \frac{d}{dt}(M_A \nabla \times \mathbf{E}_t, \nabla \times \mathbf{E}_t) = \mu_0\mu(M_B \mathbf{D}_t, \mathbf{E}_{t^2});$$

integrating this from $t = 0$ to t , we obtain

$$(2.22) \quad \frac{1}{2}\varepsilon_0\mu_0\mu\lambda_2[(\mathbf{E}_{t^2}, \mathbf{E}_{t^2}) + \omega_p^2(\mathbf{E}_t, \mathbf{E}_t)](t) + \frac{1}{2}(M_A \nabla \times \mathbf{E}_t, \nabla \times \mathbf{E}_t)(t) \\ - \frac{1}{2}(M_A \nabla \times \mathbf{E}_t, \nabla \times \mathbf{E}_t)(0) - \frac{1}{2}\varepsilon_0\mu_0\mu\lambda_2[(\mathbf{E}_{t^2}, \mathbf{E}_{t^2}) + \omega_p^2(\mathbf{E}_t, \mathbf{E}_t)](0) \\ = \int_0^t \mu_0\mu(M_B \mathbf{D}_s, \mathbf{E}_{t^2}) ds.$$

Similarly, applying the Cauchy–Schwarz inequality to (2.22), and then multiplying both sides by $\frac{4}{\varepsilon_0\mu_0\mu\lambda_2\omega_p^2}$ and absorbing all terms involving initial conditions into $I_3(0)$, we can rewrite (2.22) as

$$(2.23) \quad \left(\frac{2}{\omega_p^2}\|\mathbf{E}_{t^2}\|^2 + 2\|\mathbf{E}_t\|^2 + \frac{2}{\varepsilon_0\mu_0\mu\lambda_2}\|\sqrt{M_A}\nabla \times \mathbf{E}_t\|^2 \right) (t) \\ \leq I_3(0) + \frac{2}{\varepsilon_0\lambda_2} \int_0^t (\|\sqrt{M_B}\mathbf{D}_s\|^2 + \|\mathbf{E}_{t^2}\|^2) ds.$$

Denote $LHS(t) = (\|\sqrt{M_A}\mathbf{D}_t\|^2 + \|\sqrt{M_B}\mathbf{D}_t\|^2 + \|\mathbf{E}_{t^2}\|^2 + \|\mathbf{E}_t\|^2 + \|\mathbf{E}\|^2 + \|\sqrt{M_A}\nabla \times \mathbf{E}_t\|^2)(t)$.

Summing up (2.16), (2.18), and (2.23), we can easily see that

$$LHS(t) \leq C(LHS(0) + \int_0^t LHS(s) ds),$$

which completes the proof by using the Gronwall inequality. \square

3. The FETD scheme. To design a finite element method to solve our carpet model, we assume that the physical domain Ω is partitioned by a family of regular rectangular or triangular mesh T^h with maximum mesh size h . Considering the usual low regularity of Maxwell's equations, here we just consider the lowest order Raviart–Thomas–Nédélec (RTN) mixed finite element spaces U_h and \mathbf{V}_h [25, 17]: For rectangular elements $K \in T^h$,

$$(3.1) \quad \begin{aligned} U_h &= \{\psi_h \in L^2(\Omega) : \psi_h|_K \in Q_{0,0}, \forall K \in T^h\}, \\ \mathbf{V}_h &= \{\phi_h \in H(\text{curl}; \Omega) : \phi_h|_K \in Q_{0,1} \times Q_{1,0}, \forall K \in T^h\}, \end{aligned}$$

where $Q_{i,j}$ denotes the space of polynomials whose degrees are less than or equal to i and j in variables x and y , respectively. Meanwhile, for triangular elements, we choose

$$(3.2) \quad \begin{aligned} U_h &= \{\psi_h \in L^2(\Omega) : \psi_h|_K \text{ is a piecewise constant}, \forall K \in T^h\}, \\ \mathbf{V}_h &= \{\phi_h \in H(\text{curl}; \Omega) : \phi_h|_K = \text{span}\{L_i \nabla L_j - L_j \nabla L_i\}, i, j = 1, 2, 3, \forall K \in T^h\}, \end{aligned}$$

where L_i denotes the standard linear basis function at vertex i of element K . The space

$$\mathbf{V}_h^0 = \{\phi_h \in \mathbf{V}_h, \mathbf{n} \times \phi_h = \mathbf{0} \text{ on } \partial\Omega\}$$

is introduced to impose the perfect conducting boundary condition $\mathbf{n} \times \mathbf{E} = \mathbf{0}$. We remark that the analysis carried out below holds true for arbitrary order RTN spaces.

For any discrete time solution u^n , we denote the following difference and average operators:

$$\begin{aligned} \delta_\tau u^n &= \frac{u^n - u^{n-1}}{\tau}, \quad \delta_\tau^2 u^n = \frac{u^n - 2u^{n-1} + u^{n-2}}{\tau^2}, \\ \hat{u}^{n+\frac{1}{2}} &= \frac{u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}}{2}, \quad \check{u}^{n+\frac{1}{2}} = \frac{u^{n+\frac{1}{2}} + 2u^{n-\frac{1}{2}} + u^{n-\frac{3}{2}}}{4} = \frac{\hat{u}^{n+\frac{1}{2}} + \hat{u}^{n-\frac{1}{2}}}{2}. \end{aligned}$$

Now we can construct a leap-frog scheme for the model equations (2.3)–(2.5): Given proper initial approximations $H_h^0, \mathbf{D}_h^{-\frac{1}{2}}, \mathbf{D}_h^{-\frac{3}{2}}, \mathbf{E}_h^{-\frac{1}{2}}, \mathbf{E}_h^{-\frac{3}{2}}$, for $n \geq 0$ find $\mathbf{D}_h^{n+\frac{1}{2}}, \mathbf{E}_h^{n+\frac{1}{2}} \in \mathbf{V}_h^0, H_h^{n+1} \in U_h$ such that

$$(3.3) \quad (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}, \phi_h) = (H_h^n, \nabla \times \phi_h),$$

$$(3.4) \quad \begin{aligned} \varepsilon_0 \lambda_2 (\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}, \varphi_h) + \varepsilon_0 \lambda_2 \omega_p^2 (\check{\mathbf{E}}_h^{n+\frac{1}{2}}, \varphi_h) \\ = (M_A \delta_\tau^2 \mathbf{D}_h^{n+\frac{1}{2}}, \varphi_h) + (M_B \check{\mathbf{D}}_h^{n+\frac{1}{2}}, \varphi_h), \end{aligned}$$

$$(3.5) \quad \mu_0 \mu (\delta_\tau H_h^{n+1}, \psi_h) = -(\nabla \times \mathbf{E}_h^{n+\frac{1}{2}}, \psi_h)$$

hold true for any $\phi_h, \varphi_h \in \mathbf{V}_h^0, \psi_h \in U_h$.

Note that the scheme (3.3)–(3.5) is explicit in that at each time step we can solve for $\mathbf{D}_h^{n+\frac{1}{2}}, \mathbf{E}_h^{n+\frac{1}{2}}$, and H_h^{n+1} from (3.3)–(3.5) sequentially.

LEMMA 3.1. *For the solution $(\mathbf{D}_h^{n+\frac{1}{2}}, \mathbf{E}_h^{n+\frac{1}{2}})$ of (3.3)–(3.5), we have*

$$(3.6) \quad \begin{aligned} \mu_0 \mu (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}, M_A \delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}}) + \frac{1}{2} (\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}\|^2) \\ \leq -\frac{1}{4} (\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{3}{2}}\|^2) \\ + \frac{\tau}{2} (M_A \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}, \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}). \end{aligned}$$

Proof. Using (3.3) with $\phi_h = M_A \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}$, (3.5) with $\psi_h = \nabla \times M_A \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}$, and the fact that M_A is piecewise constant, we obtain

$$\begin{aligned} & \mu_0 \mu (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}, M_A \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}) = \mu_0 \mu \tau (\delta_\tau H_h^n, \nabla \times M_A \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}) \\ & = -\tau (\nabla \times \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times M_A \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}) = -\tau (M_A \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}) \\ & = (M_A \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times (\mathbf{E}_h^{n-\frac{1}{2}} - \mathbf{E}_h^{n+\frac{1}{2}})) \\ & = \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}\|^2 - (M_A \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}). \end{aligned}$$

Adding $\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}\|^2$ to both sides of the above identity, we obtain

$$(3.7) \quad \begin{aligned} & \mu_0 \mu (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}, M_A \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}) + \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}\|^2 \\ & = \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}\|^2 + \tau (M_A \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}, \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}). \end{aligned}$$

Similarly, we have

$$(3.8) \quad \begin{aligned} & \mu_0 \mu (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}, M_A \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}) = -\tau (M_A \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}) \\ & = (M_A \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times (\mathbf{E}_h^{n-\frac{3}{2}} - \mathbf{E}_h^{n-\frac{1}{2}})) \leq \frac{1}{2} [\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{3}{2}}\|^2 \\ & - \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n-\frac{1}{2}}\|^2]. \end{aligned}$$

Taking the average of estimates of (3.7) and (3.8), we conclude the proof. \square

With Lemma 3.1, we can prove the following stability for the scheme (3.3)–(3.5).

THEOREM 3.2. *Denote the discrete solution $(\mathbf{D}_h^{n+\frac{1}{2}}, \mathbf{E}_h^{n+\frac{1}{2}})$ of (3.3)–(3.5), and the discrete energy*

$$(3.9) \quad \begin{aligned} ENG_n &= \frac{\varepsilon_0 \mu_0 \mu \lambda_2 (2 + \omega_p^2)}{4} \|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \frac{\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2}{2} \|\hat{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 + \frac{1}{2} \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}\|^2 \\ &+ \frac{1}{2} \|\sqrt{M_A} \delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \frac{1}{2} \|\sqrt{M_B} \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 \\ &+ \frac{\varepsilon_0 \mu_0 \mu \lambda_2}{2} \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \frac{1}{2} \|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2. \end{aligned}$$

Then for any $m \geq 1$ and sufficiently small constant $C_{cfl} > 0$, under the constraint

$$(3.10) \quad \tau \leq C_{cfl} h^2,$$

we have

$$ENG_m \leq C \left(ENG_0 + \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{-\frac{1}{2}}\|^2 + \|\sqrt{M_B} \delta_\tau \mathbf{D}_h^{-\frac{1}{2}}\|^2 \right),$$

where the constant $C > 0$ depends on the physical parameters $\varepsilon_0, \mu_0, d, H_1, H_2$, and ω_p but is independent of h and τ .

Proof. The proof follows similarly to that of the continuous case (i.e., Theorem 2.3) and is also composed of three major parts.

Part I. Choosing $\varphi_h = \tau \delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}}$ in (3.4) and using the Cauchy–Schwarz inequality, we have

$$\frac{\varepsilon_0 \lambda_2}{2} (\|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2) + \frac{\varepsilon_0 \lambda_2 \omega_p^2}{2} (\|\hat{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 - \|\hat{\mathbf{E}}_h^{n-\frac{1}{2}}\|^2)$$

$$\begin{aligned}
 &= (M_A \delta_\tau^2 \mathbf{D}_h^{n+\frac{1}{2}}, \tau \delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}}) + \left(M_B \frac{\hat{\mathbf{D}}_h^{n+\frac{1}{2}} + \hat{\mathbf{D}}_h^{n-\frac{1}{2}}}{2}, \tau \delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}} \right) \\
 &\leq (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}, M_A \delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}}) + \frac{\tau}{2} \left(\left\| M_B \frac{\hat{\mathbf{D}}_h^{n+\frac{1}{2}} + \hat{\mathbf{D}}_h^{n-\frac{1}{2}}}{2} \right\|^2 + \|\delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 \right) \\
 &\leq (\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}, M_A \delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}}) \\
 (3.11) \quad &+ \frac{\tau}{4} (\|M_B \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 + \|M_B \hat{\mathbf{D}}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2).
 \end{aligned}$$

Multiplying (3.11) by $\mu_0 \mu$, using Lemma 3.1, and summing up the resultant from $n = 1$ to m , we have

$$\begin{aligned}
 &\frac{\varepsilon_0 \mu_0 \mu \lambda_2}{2} (\|\delta_\tau \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\delta_\tau \mathbf{E}_h^{\frac{1}{2}}\|^2) + \frac{\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2}{2} (\|\hat{\mathbf{E}}_h^{m+\frac{1}{2}}\|^2 - \|\hat{\mathbf{E}}_h^{\frac{1}{2}}\|^2) \\
 &\quad + \frac{1}{2} (\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{\frac{1}{2}}\|^2) \\
 &\leq -\frac{1}{4} (\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{m-\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{-\frac{1}{2}}\|^2) \\
 &\quad + \frac{\tau}{2} \sum_{n=1}^m (M_A \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}, \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}) \\
 (3.12) \quad &+ \frac{\mu_0 \mu \tau}{4} \sum_{n=1}^m (\|M_B \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 + \|M_B \hat{\mathbf{D}}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2).
 \end{aligned}$$

Applying the Cauchy–Schwarz inequality for the second right-hand side term of (3.12), we have

$$\begin{aligned}
 &\frac{\varepsilon_0 \mu_0 \mu \lambda_2}{2} (\|\delta_\tau \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\delta_\tau \mathbf{E}_h^{\frac{1}{2}}\|^2) + \frac{\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2}{2} (\|\hat{\mathbf{E}}_h^{m+\frac{1}{2}}\|^2 - \|\hat{\mathbf{E}}_h^{\frac{1}{2}}\|^2) \\
 &\quad + \frac{1}{2} (\|\sqrt{M_A} \nabla \times \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{\frac{1}{2}}\|^2) \\
 &\leq \frac{1}{4} \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{-\frac{1}{2}}\|^2 + \frac{\tau}{4} \sum_{n=1}^m (\|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}\|^2) \\
 (3.13) \quad &+ \frac{\mu_0 \mu \tau}{4} \sum_{n=1}^m (\|M_B \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 + \|M_B \hat{\mathbf{D}}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2).
 \end{aligned}$$

Part II. Choosing $\varphi_h = \tau \delta_\tau \hat{\mathbf{D}}_h^{n+\frac{1}{2}}$ in (3.4) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
 &\frac{1}{2} (\|\sqrt{M_A} \delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 - \|\sqrt{M_A} \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}\|^2) + \frac{1}{2} (\|\sqrt{M_B} \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 - \|\sqrt{M_B} \hat{\mathbf{D}}_h^{n-\frac{1}{2}}\|^2) \\
 &\quad \leq \varepsilon_0 \lambda_2 (\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}, \tau \delta_\tau \hat{\mathbf{D}}_h^{n+\frac{1}{2}}) + \varepsilon_0 \lambda_2 \omega_p^2 \left(\frac{\hat{\mathbf{E}}_h^{n+\frac{1}{2}} + \hat{\mathbf{E}}_h^{n-\frac{1}{2}}}{2}, \tau \delta_\tau \hat{\mathbf{D}}_h^{n+\frac{1}{2}} \right) \\
 &\quad \leq \frac{\tau \varepsilon_0 \lambda_2}{2} (\|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}\|^2) \\
 (3.14) \quad &\quad + \frac{\tau \varepsilon_0 \lambda_2 \omega_p^2}{4} (\|\hat{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 + \|\hat{\mathbf{E}}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}\|^2).
 \end{aligned}$$

Summing up (3.14) from $n = 1$ to m , we obtain

$$\begin{aligned}
 & \frac{1}{2}(\|\sqrt{M_A}\delta_\tau \mathbf{D}_h^{m+\frac{1}{2}}\|^2 - \|\sqrt{M_A}\delta_\tau \mathbf{D}_h^{\frac{1}{2}}\|^2) + \frac{1}{2}(\|\sqrt{M_B}\hat{\mathbf{D}}_h^{m+\frac{1}{2}}\|^2 - \|\sqrt{M_B}\hat{\mathbf{D}}_h^{\frac{1}{2}}\|^2) \\
 & \leq \frac{\tau\varepsilon_0\lambda_2}{2} \sum_{n=1}^m (\|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}\|^2) \\
 (3.15) \quad & + \frac{\tau\varepsilon_0\lambda_2\omega_p^2}{4} \sum_{n=1}^m (\|\hat{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 + \|\hat{\mathbf{E}}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}\|^2).
 \end{aligned}$$

Part III. Subtracting (3.4) itself with n reduced by 1, then multiplying the result by $\mu_0\mu$, we obtain

$$\begin{aligned}
 & \varepsilon_0\mu_0\mu\lambda_2(\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}} - \delta_\tau^2 \mathbf{E}_h^{n-\frac{1}{2}}, \boldsymbol{\varphi}_h) + \varepsilon_0\mu_0\mu\lambda_2\omega_p^2\tau \left(\frac{\delta_\tau \hat{\mathbf{E}}_h^{n+\frac{1}{2}} + \delta_\tau \hat{\mathbf{E}}_h^{n-\frac{1}{2}}}{2}, \boldsymbol{\varphi}_h \right) \\
 & = \mu_0\mu(M_A(\delta_\tau^2 \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau^2 \mathbf{D}_h^{n-\frac{1}{2}}), \boldsymbol{\varphi}_h) \\
 (3.16) \quad & + \mu_0\mu\tau \left(M_B \frac{\delta_\tau \hat{\mathbf{D}}_h^{n+\frac{1}{2}} + \delta_\tau \hat{\mathbf{D}}_h^{n-\frac{1}{2}}}{2}, \boldsymbol{\varphi}_h \right).
 \end{aligned}$$

Using (3.3) and (3.5), we have

$$(3.17) \quad \mu_0\mu(M_A(\delta_\tau^2 \mathbf{D}_h^{n+\frac{1}{2}} - \delta_\tau^2 \mathbf{D}_h^{n-\frac{1}{2}}), \boldsymbol{\varphi}_h) = -\tau(M_A \nabla \times \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times \boldsymbol{\varphi}_h).$$

Choosing $\boldsymbol{\varphi}_h = \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}$ in both (3.16) and (3.17), we get

$$\begin{aligned}
 & \frac{\varepsilon_0\mu_0\mu\lambda_2}{2}(\|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\delta_\tau^2 \mathbf{E}_h^{n-\frac{1}{2}}\|^2) + \frac{\varepsilon_0\mu_0\mu\lambda_2\omega_p^2}{4}(\|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2) \\
 & \leq -(M_A \nabla \times \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times (\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}})) \\
 & \quad - \varepsilon_0\mu_0\mu\lambda_2\omega_p^2\tau \left(\frac{\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}} + \delta_\tau \mathbf{E}_h^{n-\frac{3}{2}}}{4}, \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}} \right) \\
 (3.18) \quad & \quad + \mu_0\mu\tau \left(M_B \frac{\delta_\tau \hat{\mathbf{D}}_h^{n+\frac{1}{2}} + \delta_\tau \hat{\mathbf{D}}_h^{n-\frac{1}{2}}}{2}, \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}} \right).
 \end{aligned}$$

Note that the first term on the right-hand side of (3.18) can be bounded as follows:

$$\begin{aligned}
 & -(M_A \nabla \times \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}, \nabla \times (\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}})) \\
 & = (M_A \nabla \times (\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}), \nabla \times (\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}})) \\
 & \quad - (M_A \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}, \nabla \times (\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}} - \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}})) \\
 & \leq \tau^2 \|\sqrt{M_A} \nabla \times \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \frac{1}{2}(\|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 \\
 (3.19) \quad & - \|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2).
 \end{aligned}$$

Substituting the estimate (3.19) into (3.18) and applying the Cauchy–Schwarz inequality to the last two terms of (3.18), we have

$$\frac{\varepsilon_0\mu_0\mu\lambda_2}{2}(\|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\delta_\tau^2 \mathbf{E}_h^{n-\frac{1}{2}}\|^2) + \frac{\varepsilon_0\mu_0\mu\lambda_2\omega_p^2}{4}(\|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2)$$

$$\begin{aligned}
& + \frac{1}{2} (\|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2) \\
& \leq \tau^2 \|\sqrt{M_A} \nabla \times \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 \\
& + \frac{\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2 \tau}{2} (\|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{n-\frac{3}{2}}\|^2 + \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2) \\
(3.20) \quad & + \frac{\mu_0 \mu \tau}{2} (\|\sqrt{M_B} \delta_\tau \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 + \|\sqrt{M_B} \delta_\tau \hat{\mathbf{D}}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2).
\end{aligned}$$

Summing up (3.20) from $n = 1$ to m , we obtain

$$\begin{aligned}
& \frac{\varepsilon_0 \mu_0 \mu \lambda_2}{2} (\|\delta_\tau^2 \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\delta_\tau^2 \mathbf{E}_h^{\frac{1}{2}}\|^2) + \frac{\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2}{4} (\|\delta_\tau \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\delta_\tau \mathbf{E}_h^{\frac{1}{2}}\|^2) \\
& + \frac{1}{2} (\|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{m+\frac{1}{2}}\|^2 - \|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{\frac{1}{2}}\|^2) \\
& \leq \tau^2 \sum_{n=1}^m \|\sqrt{M_A} \nabla \times \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \frac{\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2 \tau}{2} \sum_{n=1}^m (\|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2 \\
& + \|\delta_\tau \mathbf{E}_h^{n-\frac{3}{2}}\|^2 + \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2) \\
& + \frac{\mu_0 \mu \tau}{2} \sum_{n=1}^m (\|\sqrt{M_B} \delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \|\sqrt{M_B} \delta_\tau \mathbf{D}_h^{n-\frac{1}{2}}\|^2 + \|\sqrt{M_B} \delta_\tau \mathbf{D}_h^{n-\frac{3}{2}}\|^2) \\
(3.21) \quad & + \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2).
\end{aligned}$$

Adding up (3.13), (3.15), and (3.21), and using the definition (3.9), we have

$$\begin{aligned}
ENG_m & \leq C[ENG_0 + \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{-\frac{1}{2}}\|^2 + \|\delta_\tau \mathbf{E}_h^{-\frac{1}{2}}\|^2 + \|\sqrt{M_B} \delta_\tau \mathbf{D}_h^{-\frac{1}{2}}\|^2] \\
& + \frac{\tau}{4} \sum_{n=1}^m \{ \|\sqrt{M_A} \nabla \times \delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + \|\sqrt{M_A} \nabla \times \mathbf{E}_h^{n+\frac{1}{2}}\|^2 \\
& + \mu_0 \mu (2\|M_B \hat{\mathbf{D}}_h^{n+\frac{1}{2}}\|^2 + 2\|\delta_\tau \mathbf{E}_h^{n+\frac{1}{2}}\|^2) \\
& + 2\varepsilon_0 \lambda_2 (\|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + 2\|\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2) + \varepsilon_0 \lambda_2 \omega_p^2 (2\|\hat{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 + 2\|\delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2) \\
& + 4\tau \|\sqrt{M_A} \nabla \times \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 + 2\varepsilon_0 \mu_0 \mu \lambda_2 \omega_p^2 (2\|\delta_\tau \mathbf{E}_h^{n-\frac{1}{2}}\|^2 + \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2) \\
(3.22) \quad & + 2\mu_0 \mu (3\|\sqrt{M_B} \delta_\tau \mathbf{D}_h^{n+\frac{1}{2}}\|^2 + \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2) \}.
\end{aligned}$$

It is not difficult to see that when the time step τ is small enough, all terms on the right-hand side of (3.22) can be controlled by the corresponding left-hand side terms except $4\tau \|\sqrt{M_A} \nabla \times \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2$, which can be bounded as

$$(3.23) \quad 4\tau \|\sqrt{M_A} \nabla \times \delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2 \leq \tau \cdot C_{inv} h^{-2} \|\delta_\tau^2 \mathbf{E}_h^{n+\frac{1}{2}}\|^2,$$

where we used the standard inverse estimate.

Substituting (3.23) into (3.22), choosing τ small enough, and using the discrete Gronwall inequality, we conclude the proof. \square

4. Numerical results. In this section, we present some numerical simulation results for the carpet cloak model solved by the FETD algorithm (3.3)–(3.5). To simulate the cloak phenomenon, we surround the physical domain by a perfectly

matched layer (PML); see Figure 1 (right). In this paper, we use the classical 2D Berenger PML, whose governing equations can be written as

$$(4.1) \quad \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_x \end{pmatrix} \mathbf{E} = \nabla \times H_z,$$

$$(4.2) \quad \mu_0 \frac{\partial H_{zx}}{\partial t} + \sigma_{mx} H_{zx} = -\frac{\partial E_y}{\partial x},$$

$$(4.3) \quad \mu_0 \frac{\partial H_{zy}}{\partial t} + \sigma_{my} H_{zy} = \frac{\partial E_x}{\partial y},$$

where $H_z = H_{zx} + H_{zy}$ denotes the magnetic field, and the parameters σ_i and $\sigma_{m,i}, i = x, y$, are the electric and magnetic conductivities in the x - and y - directions, respectively. In our simulation, we use a PML with 12 rectangular cells in thickness around the physical domain.

To couple the PML model with the scheme (3.3)–(3.5) developed for solving the blanket cloak model, we construct a similar leap-frog scheme for solving the PML model (4.1)–(4.3) in the PML region: For any $n \geq 0$, find $\mathbf{E}_h^{n+\frac{1}{2}} \in \mathbf{V}_h^0, H_{zx,h}^{n+1}, H_{zy,h}^{n+1} \in U_h$ such that

$$(4.4) \quad \varepsilon_0 \left(\frac{\mathbf{E}_h^{n+\frac{1}{2}} - \mathbf{E}_h^{n-\frac{1}{2}}}{\tau}, \phi_h \right) + \left(\begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_x \end{pmatrix} \frac{\mathbf{E}_h^{n+\frac{1}{2}} + \mathbf{E}_h^{n-\frac{1}{2}}}{2}, \phi_h \right) = (H_{zx,h}^n + H_{zy,h}^n, \nabla \times \phi_h),$$

$$(4.5) \quad \mu_0 \left(\frac{H_{zx,h}^{n+1} - H_{zx,h}^n}{\tau}, \psi_{1,h} \right) + \left(\sigma_{mx} \frac{H_{zx,h}^{n+1} + H_{zx,h}^n}{2}, \psi_{1,h} \right) = -\left(\frac{\partial}{\partial x} E_{y,h}^{n+\frac{1}{2}}, \psi_{1,h} \right),$$

$$(4.6) \quad \mu_0 \left(\frac{H_{zy,h}^{n+1} - H_{zy,h}^n}{\tau}, \psi_{2,h} \right) + \left(\sigma_{my} \frac{H_{zy,h}^{n+1} + H_{zy,h}^n}{2}, \psi_{2,h} \right) = \left(\frac{\partial}{\partial y} E_{x,h}^{n+\frac{1}{2}}, \psi_{2,h} \right)$$

hold true for any $\phi_h \in \mathbf{V}_h^0$ and any $\psi_{1,h}, \psi_{2,h} \in U_h$.

In our simulation, we choose $H_1 = 0.05m, H_2 = 0.2m, d = 0.2m$, and the physical domain $\Omega = [-0.3, 0.3] m \times [0, 0.3] m$, which is partitioned by a uniform triangular mesh with a mesh size $h = 0.00625$. The PML region surrounding Ω is partitioned by a uniform rectangular mesh. Our final mesh yields 53330 total edges, 26960 total triangular elements, and 6258 total rectangular elements. In the test, we choose the time step size $\tau = 2 * 10^{-13}$ s, and the total number of time steps 15000; i.e., the final simulation time $T = 3.0$ nanosecond (ns).

Example 1. The incident wave is generated by a plane wave source $H_z = 0.1 \sin(\omega t)$ imposed at line $x = -0.3$, where $\omega = 2\pi f$ with frequency $f = 3.0$ GHz. The numerical magnetic fields H_z and electric fields E_y at different time steps are shown in Figures 2 and 3, respectively. Both figures show clearly that the plane wave pattern is recovered very well after passing through the cloaking region, which makes any objects hidden inside the cloaked region invisible to observers at the far end.

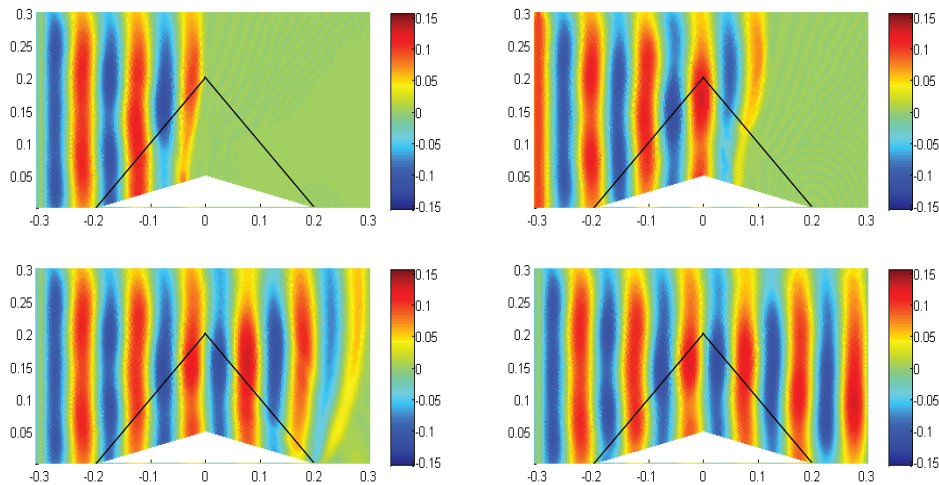


FIG. 2. *Example 1. The numerical magnetic fields H_z at 5000, 7000, 10000, and 15000 time steps (oriented counterclockwise).*

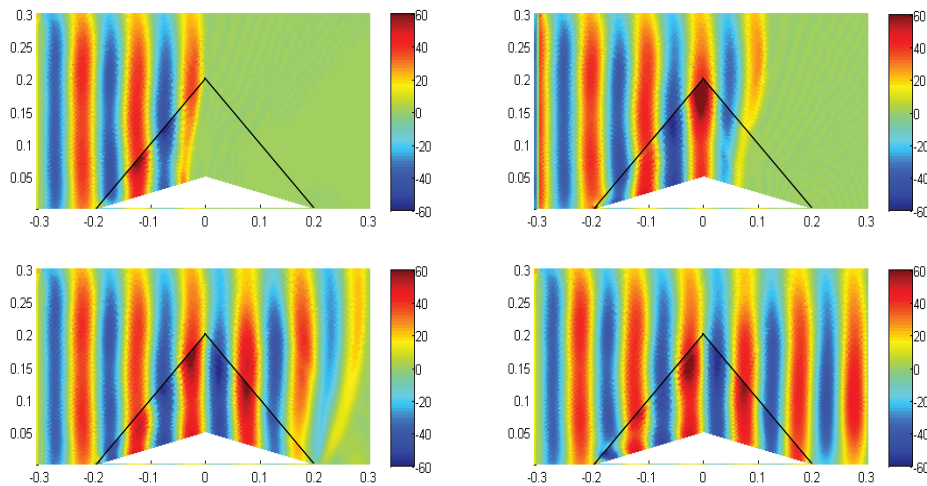


FIG. 3. *Example 1. The numerical electric fields E_y at 5000, 7000, 10000, and 15000 time steps (oriented counterclockwise).*

Example 2. The incident wave is generated by a Gaussian wave

$$H_z(x, y, t) = 0.1e^{-(y-0.15)^2/(60L)^2} \sin(\omega t)$$

imposed along a slanted line $y = x + 0.45$, where $L = 0.004\sqrt{2}$ and $\omega = 2\pi f$ with frequency $f = 6.0$ GHz. The numerical magnetic fields H_z at different time steps are presented in Figure 4. So that the reader may appreciate the cloak phenomenon, in Figure 5 we present the magnetic fields H_z obtained without the cloaking material. It is clear that the cloak phenomenon disappears if the cloaking material is removed.

Example 3. Since the ideal cloak requires that the permittivity and permeability be anisotropic, and this arrangement is quite difficult to construct, the following

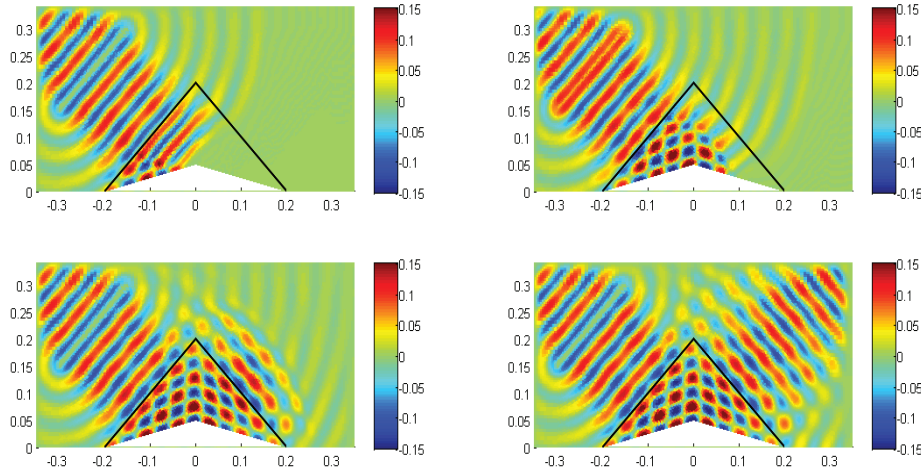


FIG. 4. Example 2. The numerical magnetic fields H_z at 5000, 7000, 10000, and 15000 time steps (oriented counterclockwise).

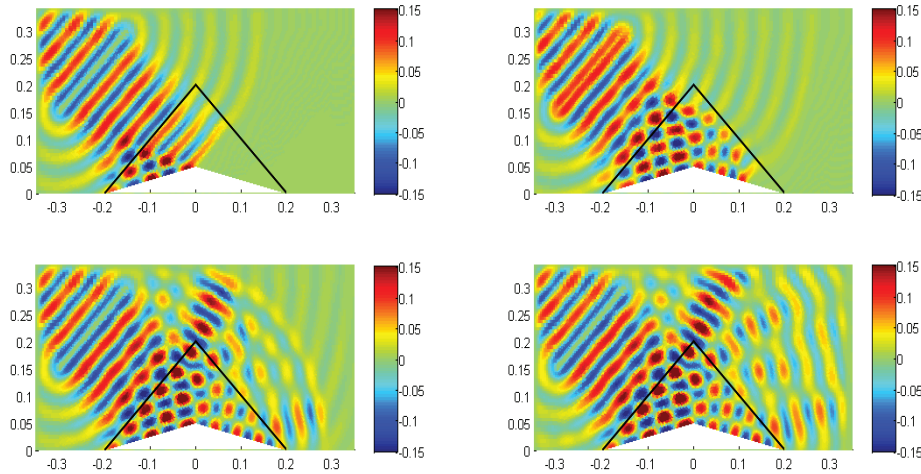


FIG. 5. Example 2. The numerical magnetic fields H_z at 5000, 7000, 10000, and 15000 time steps (oriented counterclockwise) obtained with the cloaking material removed.

reduced cloak material is suggested [8]:

$$\mu = 1, \quad \varepsilon = \frac{H_2}{H_2 - H_1} \begin{bmatrix} \frac{H_2}{H_2 - H_1} & -\frac{H_1 H_2}{(H_2 - H_1)d} \operatorname{sgn}(x) \\ -\frac{H_1 H_2}{(H_2 - H_1)d} \operatorname{sgn}(x) & \frac{H_2 - H_1}{H_2} + \frac{H_2}{H_2 - H_1} \left(\frac{H_1}{d}\right)^2 \end{bmatrix},$$

and the reduced cloak materials can be realized by natural anisotropic materials [8]. We solve Example 2 again by using this simplified permittivity and permeability. The calculated magnetic fields H_z at different time steps are presented in Figure 6, which shows that the cloak phenomenon is almost the same as Figure 4 for the ideal permittivity and permeability.

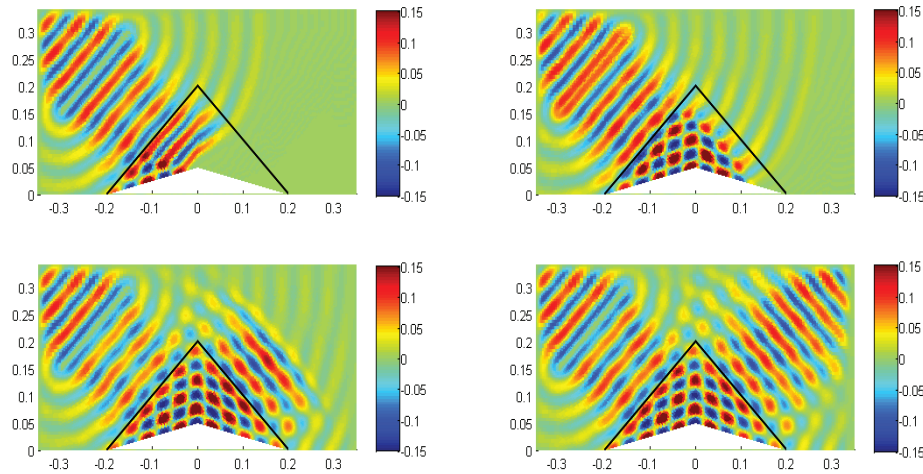


FIG. 6. Example 3. The calculated magnetic fields H_z at 5000, 7000, 10000, and 15000 time steps (oriented counterclockwise) with the simplified cloak material.

5. Conclusion. In this paper, we present a mathematical formulation of the popular carpet cloak model. The well-posedness of the model is established, and a finite element time-domain method is developed for solving this model. To the best of our knowledge, this is the first mathematical treatment for the carpet cloak model.

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