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Well-posedness study for a time-domain spherical cloaking model

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ABSTRACT

In this paper, we study a time-domain spherical cloaking model recently introduced by Zhao and Hao (2009). This model is quite complicated and is composed of four coupled differential equations. Here we first prove the existence and uniqueness of a solution for this model. Then we obtain a stability result, which analysis is quite involved due to the coupling between the four variables. To our best knowledge, this is the first well-posedness study carried out for the time-domain cloaking model with metamaterials.

1. Introduction

The idea of invisibility cloaking using metamaterials started in 2006 when Pendry et al. [1] and Leonhardt [2] laid out the blueprints for making objects invisible to electromagnetic waves. In late 2006, a 2-D reduced cloak was successfully fabricated and demonstrated to work at 8.5 GHz and relied upon local resonances of split ring resonators [3]. This is the first practical realization of such a cloak, and the result matches well with the computer simulation [4] performed using the commercial package COMSOL. The cloaking technique of [1,2] is to establish a correspondence between physical material parameters (the material's permittivity and permeability) and coordinate transformations. The conceptual device constructed with these material parameters is able to guide waves to propagate around the cloaked region (usually the central region of the cloaking structure), and render the objects placed inside invisible to external electromagnetic radiations. It turns out that essentially the same idea was discussed earlier in 2003 by Greenleaf, Lassas, and Uhlmann [5,6] for electrical impedance tomography. Now this transformation technique is widely used in various cloak designs, and has earned a variety of names such as Transformation Electromagnetics, Transformation Optics, Transformation Acoustics, and Transformation Elastodynamics (see recent review papers [7–10] and the book [11]).

In addition to the transformation optics (and acoustics) technique, there are many different avenues towards electromagnetic and acoustic cloaking. Another kind of cloaking [12,13] requires a negative refractive index shell, which allows for the cloaking of a discrete set of dipoles when they are located within a given distance outside the shell. A third kind of cloaking uses complementary media to cloak objects at a distance outside the cloaking shell [14]. A fourth kind of cloaking is obtained via active scattering cancellation devices not completely surrounding the cloaked region (exterior cloaking) [15,16]. A very recent cloaking technique is to use zero index metamaterials loaded with normal dielectric defects [17,18].

Since 2006, study of using metamaterials to construct invisibility cloaks has been a very hot research topic. A search on "metamaterials and cloaking" over scholar.google.com (conducted on April 28, 2014) shows 1880 publications since

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2013. Most of them focus on engineering and physics. Compared to the huge amount of papers published in engineering and physics, there are not much mathematical analyses done for metamaterials and cloaking, even though numerical simulation in metamaterials [19,20] plays a very important role in cloaking structure design and validation of the theoretical predictions.

In recent years, mathematicians have started investigating this fascinating subject, but most works are still limited to frequency-domain or the quasi-static regime by mainly solving the Helmholtz equation [21–27], and the time-harmonic Maxwell's equations [28,29]. The advancement of broadband cloaks [30,31] makes time-domain cloaking simulation more appealing and necessary. Generally speaking, electromagnetic wave cloaking simulation boils down to solving metamaterial Maxwell's equations in either frequency-domain or time-domain. In 2012, we developed the first time-domain finite element method to simulate a cylindrical cloak [32], and completed the well-posedness study of this model in [33]. In this work, we carry out a rigorous analysis of the well-posedness for a time-domain spherical cloaking model recently developed and simulated by using the FDTD method [34]. Though there exist a few publications on well-posedness for metamaterial Maxwell's equations in frequency-domain (e.g. [35–37]) and time-domain [38], to the best of the author's knowledge, we are unaware of other works on the well-posedness study of time-domain cloaking models. The major challenge for the analysis is that this model is quite complicated, and is formed by four mixed order differential equations with four vector unknowns.

The rest of the paper is organized as follows. In Section 2, we provide a detailed derivation of the time-domain spherical cloak modeling equations, since the original paper does not even present the complete set of governing equations. Then in Section 3, we first prove the existence and uniqueness of our model problem, then we prove the stability of the model. We conclude the paper in Section 4.

2. The modeling equations

The permittivity and permeability of the ideal spherical cloak are given by [1]:

$$\epsilon_r = \mu_r = \frac{R_2}{R_2 - R_1}\left(\frac{r - R_1}{r}\right)^2, \quad R_1 \le r \le R_2, \tag{1}$$

$$\epsilon_\theta = \mu_\theta = \frac{R_2}{R_2 - R_1}, \qquad \epsilon_\phi = \mu_\phi = \frac{R_2}{R_2 - R_1}, \tag{2}$$

where R_1 and R_2 are the inner and outer radii of the cloak, and r denotes the radial distance from the center of the cloak.

Due to the inconvenience of cloaking simulation in spherical coordinate (e.g., COMSOL, the popular simulation software in this area, is only for Cartesian coordinate), the permittivity and permeability parameters given above have to be changed to Cartesian coordinate via the following transformation [34]:

$$\begin{aligned}\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{bmatrix} &= \begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\epsilon_r & 0 & 0\\ 0 & \epsilon_\theta & 0\\ 0 & 0 & \epsilon_\phi\end{bmatrix}\\ &\quad\times\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}.\end{aligned} \tag{3}$$

Substituting (3) into the constitutive equation

$$\epsilon_0\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix} = \begin{bmatrix}D_x\\ D_y\\ D_z\end{bmatrix},$$

we have (details see [34]):

$$\begin{aligned}\epsilon_0 E_x \;=\; &\left(\frac{1}{\epsilon_r}\sin^2\theta\cos^2\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\cos^2\phi + \frac{1}{\epsilon_\phi}\sin^2\phi\right)D_x\\ &+\left(\frac{1}{\epsilon_r}\sin^2\theta\sin\phi\cos\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\sin\phi\cos\phi - \frac{1}{\epsilon_\phi}\sin\phi\cos\phi\right)D_y\\ &+\left(\frac{1}{\epsilon_r}\sin\theta\cos\theta\cos\phi - \frac{1}{\epsilon_\theta}\sin\theta\cos\theta\cos\phi\right)D_z,\end{aligned} \tag{4}$$

$$\begin{aligned}\epsilon_0 E_y \;=\; &\left(\frac{1}{\epsilon_r}\sin^2\theta\sin\phi\cos\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\sin\phi\cos\phi - \frac{1}{\epsilon_\phi}\sin\phi\cos\phi\right)D_x\\ &+\left(\frac{1}{\epsilon_r}\sin^2\theta\sin^2\phi + \frac{1}{\epsilon_\theta}\cos^2\theta\sin^2\phi + \frac{1}{\epsilon_\phi}\cos^2\phi\right)D_y\\ &+\left(\frac{1}{\epsilon_r}\sin\theta\cos\theta\sin\phi - \frac{1}{\epsilon_\theta}\sin\theta\cos\theta\sin\phi\right)D_z,\end{aligned} \tag{5}$$

$$\epsilon_0 E_z = \left(\frac{1}{\epsilon_r}\sin\theta\cos\theta\cos\phi - \frac{1}{\epsilon_\theta}\sin\theta\cos\theta\cos\phi\right) D_x + \left(\frac{1}{\epsilon_r}\sin\theta\cos\theta\sin\phi - \frac{1}{\epsilon_\theta}\sin\theta\cos\theta\sin\phi\right) D_y + \left(\frac{1}{\epsilon_r}\cos^2\theta + \frac{1}{\epsilon_\theta}\sin^2\theta\right) D_z, \tag{6}$$

where ϵ_0 is the permittivity in air, $\boldsymbol{E} = (E_x, E_y, E_z)'$ and $\boldsymbol{D} = (D_x, D_y, D_z)'$ are the electric field and electric flux density, respectively.

Since $\epsilon_r = \mu_r \in [0, \frac{R_2 - R_1}{R_2}] < 1$, the cloak cannot be directly simulated [34] with (1). In this situation, ϵ_r and μ_r are often mapped by using some dispersive material models. In [34], the Drude model

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - j\omega\gamma}, \tag{7}$$

is used, where ω is the general wave frequency, and ω_p and γ are the electric plasma and collision frequencies of the material, respectively. By varying the plasma frequency with space, the radial dependent material parameters in (1) can be achieved. For example, in cloak simulation [39,34,32], ω_p is calculated using $\omega_p = \omega\sqrt{1-\epsilon_r}$ with ϵ_r calculated from (1) for the ideal lossless case.

The time-domain governing equation for the component E_x is given by [34, Eq. (16)]:

$$\begin{aligned}
&\epsilon_0\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right) E_x \\
&= \left[\sin^2\theta\cos^2\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) + \left(\frac{\cos^2\theta\sin^2\phi}{\epsilon_\theta} + \frac{\sin^2\phi}{\epsilon_\phi}\right)\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_x \\
&\quad + \left[\sin^2\theta\sin\phi\cos\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) + \left(\frac{\cos^2\theta\sin\phi\cos\phi}{\epsilon_\theta} - \frac{\sin\phi\cos\phi}{\epsilon_\phi}\right)\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_y \\
&\quad + \left[\sin\theta\cos\theta\cos\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) - \frac{\sin\theta\cos\theta\cos\phi}{\epsilon_\theta}\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_z.
\end{aligned} \tag{8}$$

Following the same idea as [34] by substituting (7) into (5), we obtain

$$\begin{aligned}
&\epsilon_0(\omega^2 - j\omega\gamma - \omega_p^2) E_y \\
&= \left[(\omega^2 - j\omega\gamma)\sin^2\theta\sin\phi\cos\phi + \left(\frac{\cos^2\theta\sin\phi\cos\phi}{\epsilon_\theta} - \frac{\sin\phi\cos\phi}{\epsilon_\phi}\right)(\omega^2 - j\omega\gamma - \omega_p^2)\right] D_x \\
&\quad + \left[(\omega^2 - j\omega\gamma)\sin^2\theta\sin^2\phi + \left(\frac{\cos^2\theta\sin^2\phi}{\epsilon_\theta} + \frac{\cos^2\phi}{\epsilon_\phi}\right)(\omega^2 - j\omega\gamma - \omega_p^2)\right] D_y \\
&\quad + \left[(\omega^2 - j\omega\gamma)\sin\theta\cos\theta\sin\phi - \frac{\sin\theta\cos\theta\sin\phi}{\epsilon_\theta}(\omega^2 - j\omega\gamma - \omega_p^2)\right] D_z,
\end{aligned}$$

which can be written in time domain (assuming time-harmonic variation of $e^{j\omega t}$) as

$$\begin{aligned}
&\epsilon_0\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right) E_y \\
&= \left[\sin^2\theta\sin\phi\cos\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) + \left(\frac{\cos^2\theta\sin\phi\cos\phi}{\epsilon_\theta} - \frac{\sin\phi\cos\phi}{\epsilon_\phi}\right)\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_x \\
&\quad + \left[\sin^2\theta\sin^2\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) + \left(\frac{\cos^2\theta\sin^2\phi}{\epsilon_\theta} + \frac{\cos^2\phi}{\epsilon_\phi}\right)\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_y \\
&\quad + \left[\sin\theta\cos\theta\sin\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) - \frac{\sin\theta\cos\theta\sin\phi}{\epsilon_\theta}\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_z.
\end{aligned} \tag{9}$$

Similarly, substituting (7) into (6), we have

$$\begin{aligned}
\epsilon_0\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right) E_z &= \left[\sin\theta\cos\theta\cos\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) - \frac{\sin\theta\cos\theta\cos\phi}{\epsilon_\theta}\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_x \\
&\quad + \left[\sin\theta\cos\theta\sin\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) - \frac{\sin\theta\cos\theta\sin\phi}{\epsilon_\theta}\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_y \\
&\quad + \left[\cos^2\theta\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right) + \frac{\sin^2\theta}{\epsilon_\theta}\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\right] D_z.
\end{aligned} \tag{10}$$

Denote the matrices M_A and M_B as:

$$M_A = \begin{bmatrix} \cos^2\theta\cos^2\phi + \sin^2\phi & \cos^2\theta\sin\phi\cos\phi - \sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ \cos^2\theta\sin\phi\cos\phi - \sin\phi\cos\phi & \cos^2\theta\sin^2\phi + \cos^2\phi & -\sin\theta\cos\theta\sin\phi \\ -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \end{bmatrix}$$

and

$$M_B = \begin{bmatrix} \sin^2\theta\cos^2\phi & \sin^2\theta\sin\phi\cos\phi & \sin\theta\cos\theta\cos\phi \\ \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi & \sin\theta\cos\theta\sin\phi \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\sin\phi & \cos^2\theta \end{bmatrix}.$$

Using the fact $\epsilon_\theta = \epsilon_\phi$, we can write (8)–(10) as a vector equation

$$\epsilon_0\epsilon_\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)\boldsymbol{E} = \left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_p^2\right)M_A\boldsymbol{D} + \epsilon_\phi\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right)M_B\boldsymbol{D}. \tag{11}$$

By the same technique, applying the Drude model

$$\mu_r(\omega) = 1 - \frac{\omega_{pm}^2}{\omega^2 - j\omega\gamma_m}, \tag{12}$$

to the magnetic constitutive equation

$$\mu_0\begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix},$$

we obtain

$$\mu_0\mu_\phi\left(\frac{\partial^2}{\partial t^2} + \gamma_m\frac{\partial}{\partial t} + \omega_{pm}^2\right)\boldsymbol{H} = \left(\frac{\partial^2}{\partial t^2} + \gamma_m\frac{\partial}{\partial t} + \omega_{pm}^2\right)M_A\boldsymbol{B} + \mu_\phi\left(\frac{\partial^2}{\partial t^2} + \gamma_m\frac{\partial}{\partial t}\right)M_B\boldsymbol{B}, \tag{13}$$

where μ_0 is the permeability in air, ω_{pm} and γ_m are the magnetic plasma and collision frequencies of the material, respectively, and $\boldsymbol{H} = (H_x, H_y, H_z)'$ and $\boldsymbol{B} = (B_x, B_y, B_z)'$ denote the magnetic field and magnetic flux density, respectively.

In summary, coupling the constitutive equations (11) and (13) with the Faraday's Law, Ampere's Law, we obtain the governing equations for modeling the ideal spherical cloak:

$$\boldsymbol{B}_t = -\nabla\times\boldsymbol{E}, \tag{14}$$

$$\boldsymbol{D}_t = \nabla\times\boldsymbol{H}, \tag{15}$$

$$\epsilon_0\epsilon_\phi(\boldsymbol{E}_{t^2} + \gamma\boldsymbol{E}_t + \omega_p^2\boldsymbol{E}) = M_A(\boldsymbol{D}_{t^2} + \gamma\boldsymbol{D}_t + \omega_p^2\boldsymbol{D}) + \epsilon_\phi M_B(\boldsymbol{D}_{t^2} + \gamma\boldsymbol{D}_t), \tag{16}$$

$$\mu_0\mu_\phi(\boldsymbol{H}_{t^2} + \gamma_m\boldsymbol{H}_t + \omega_{pm}^2\boldsymbol{H}) = M_A(\boldsymbol{B}_{t^2} + \gamma_m\boldsymbol{B}_t + \omega_{pm}^2\boldsymbol{B}) + \mu_\phi M_B(\boldsymbol{B}_{t^2} + \gamma_m\boldsymbol{B}_t), \tag{17}$$

where for simplicity we denote u_{t^k} for the kth derivative of a function u with respect to t. To complete the problem, we supplement (14)–(17) with the perfectly conduct (PEC) boundary condition

$$\boldsymbol{n}\times\boldsymbol{E} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{18}$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$, and the initial conditions

$$\boldsymbol{B}(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x}), \qquad \boldsymbol{D}(\boldsymbol{x},0) = \boldsymbol{D}_0(\boldsymbol{x}), \qquad \boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \qquad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \forall\,\boldsymbol{x}\in\Omega, \tag{19}$$

where $\boldsymbol{B}_0(\boldsymbol{x})$, $\boldsymbol{D}_0(\boldsymbol{x})$, $\boldsymbol{E}_0(\boldsymbol{x})$ and $\boldsymbol{H}_0(\boldsymbol{x})$ are some given functions, and Ω denotes the spherical shell $R_1 \le r \le R_2$.

We like to remark that the PEC boundary condition is imposed on the inner boundary so that any object can be cloaked inside, since no wave can be penetrated into the inner sphere (the so-called cloaked region). The spherical shell is often called as the cloaking region. Outside the cloaking region is the free space, which is governed by the standard Maxwell's equations in air. Note that with the choice

$$\omega_p = \omega_{pm} = 0, \qquad \gamma = \gamma_m = 0, \qquad \epsilon_\phi = \mu_\phi = 1,$$

and usage of the identity $M_A + M_B = I$, (16)–(17) are reduced to the simple constitutive identities $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$ and $\boldsymbol{B} = \mu_0\boldsymbol{H}$, i.e., the standard Maxwell's equations in air are a special case of the spherical cloak modeling equations (14)–(17). This fact is used in the numerical simulation implementation (see [39,34,32]).

3. The well-posedness of the cloaking model

Before we prove the well-posedness of the model (14)–(19), we need the following property.

Lemma 3.1. (i) *The matrix M_A is symmetric and non-negative definite.*
(ii) *Under the constraint*

$$1 < \epsilon_\phi \le 2, \tag{20}$$

the matrix $M = M_A + \epsilon_\phi M_B$ is symmetric positive definite.

Proof. (i) By the definition of M_A, it is easy to see that for any vector $(u, v, w)'$, we have

$$\begin{aligned}(u, v, w)M_A \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= (\cos^2\theta\cos^2\phi + \sin^2\phi)u^2 - 2uv\sin^2\theta\sin\phi\cos\phi - 2uw\sin\theta\cos\theta\cos\phi \\ &\quad + (\cos^2\theta\sin^2\phi + \cos^2\phi)v^2 - 2vw\sin\theta\cos\theta\sin\phi + w^2\sin^2\theta \\ &= (w\sin\theta\cos\phi - u\cos\theta)^2 + (w\sin\theta\sin\phi - v\cos\theta)^2 + (u\sin\phi\sin\theta - v\cos\phi\sin\theta)^2 \ge 0,\end{aligned} \tag{21}$$

which proves the non-negativeness of M_A.

(ii) Similarly, by the definitions of M_B, we have

$$\begin{aligned}(u, v, w)M_B \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= u^2\sin^2\theta\cos^2\phi + 2uv\sin^2\theta\sin\phi\cos\phi + 2uw\sin\theta\cos\theta\cos\phi \\ &\quad + 2vw\sin\theta\cos\theta\sin\phi + v^2\sin^2\theta\sin^2\phi + w^2\cos^2\theta \\ &= (u\sin\theta\cos\phi + v\sin\theta\sin\phi)^2 + (u\sin\theta\cos\phi + w\cos\theta)^2 + (v\sin\theta\sin\phi + w\cos\theta)^2 \\ &\quad - u^2\sin^2\theta\cos^2\phi - v^2\sin^2\theta\sin^2\phi - w^2\cos^2\theta.\end{aligned} \tag{22}$$

Using the identities (21)–(22), and condition $1 < \epsilon_\phi \le 2$, we obtain

$$\begin{aligned}&(u, v, w)(M_A + \epsilon_\phi M_B) \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &> (w\sin\theta\cos\phi - u\cos\theta)^2 + (w\sin\theta\sin\phi - v\cos\theta)^2 + (u\sin\phi\sin\theta - v\cos\phi\sin\theta)^2 \\ &\quad + (u\sin\theta\cos\phi + v\sin\theta\sin\phi)^2 + (u\sin\theta\cos\phi + w\cos\theta)^2 + (v\sin\theta\sin\phi + w\cos\theta)^2 \\ &\quad - \epsilon_\phi[u^2\sin^2\theta\cos^2\phi + v^2\sin^2\theta\sin^2\phi + w^2\cos^2\theta] \\ &= u^2 + v^2 + w^2 - (\epsilon_\phi - 1)[u^2\sin^2\theta\cos^2\phi + v^2\sin^2\theta\sin^2\phi + w^2\cos^2\theta] \\ &\ge u^2(1 - \sin^2\theta\cos^2\phi) + v^2(1 - \sin^2\theta\sin^2\phi) + w^2(1 - \cos^2\theta) \ge 0,\end{aligned}$$

which shows the positive definiteness of M. Note that we used (20) in the last step. □

By the definition of ϵ_ϕ, we see that the constraint (20) is equivalent to $R_2 \ge 2R_1$, which means that the cloaking region needs large enough. We are unsure if this constraint is necessary for the practical cloak, since the simulation of [34] is only carried out for $R_2 = 2R_1$. We like to remark that a similar restriction $b > 2a$ is imposed in [40] for the cylindrical cloak obtained with the virtual space to physical space mapping $r = [\frac{a}{b}(\frac{r'}{b} - 2) + 1]r' + a$, where $r' \in [0, b]$. Here a and b are the radii of the inner and outer circles.

To study the well-posedness, we need to introduce the functional spaces

$$H(\mathrm{curl}; \Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^3;\ \nabla \times \boldsymbol{v} \in (L^2(\Omega))^3\}, \tag{23}$$

$$H_0(\mathrm{curl}; \Omega) = \{\boldsymbol{v} \in H(\mathrm{curl}; \Omega);\ \boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega\}. \tag{24}$$

Now we can prove the existence and uniqueness of the solution for our cloaking model. Let $T > 0$ be the final simulation time for our model.

Theorem 3.1. *Under the constraint (20) and the following assumptions:*

$$\boldsymbol{E}(\boldsymbol{x}, 0), \boldsymbol{E}_t(\boldsymbol{x}, 0), \boldsymbol{E}_{t^2}(\boldsymbol{x}, 0), \nabla \times \boldsymbol{E}(\boldsymbol{x}, 0), \nabla \times \boldsymbol{E}_t(\boldsymbol{x}, 0) \in (L^2(\Omega))^3,$$

$$\boldsymbol{H}(\boldsymbol{x}, 0), \boldsymbol{H}_t(\boldsymbol{x}, 0), \boldsymbol{H}_{t^2}(\boldsymbol{x}, 0), \nabla \times \boldsymbol{H}(\boldsymbol{x}, 0), \nabla \times \boldsymbol{H}_t(\boldsymbol{x}, 0) \in (L^2(\Omega))^3,$$

there exists a unique solution $(\boldsymbol{E}(\boldsymbol{x}, t), H(\boldsymbol{x}, t)) \in C([0, T]; H_0(\mathrm{curl}; \Omega)) \times C([0, T]; H(\mathrm{curl}; \Omega))$.

Proof. Differentiating (11) with respect to t and using (15), we obtain

$$\begin{aligned}\epsilon_0\epsilon_\phi(\boldsymbol{E}_{t^3}+\gamma\boldsymbol{E}_{t^2}+\omega_p^2\boldsymbol{E}_t) &= (M_A+\epsilon_\phi M_B)(\boldsymbol{D}_{t^3}+\gamma\boldsymbol{D}_{t^2})+\omega_p^2 M_A\boldsymbol{D}_t\\ &= M(\nabla\times\boldsymbol{H}_{t^2}+\gamma\nabla\times\boldsymbol{H}_t)+\omega_p^2M_A\nabla\times\boldsymbol{H}.\end{aligned} \tag{25}$$

Similarly, differentiating (13) with respect to t and using (14), we obtain

$$\begin{aligned}\mu_0\mu_\phi(\boldsymbol{H}_{t^3}+\gamma_m\boldsymbol{H}_{t^2}+\omega_{pm}^2\boldsymbol{H}_t) &= (M_A+\epsilon_\phi M_B)(\boldsymbol{B}_{t^3}+\gamma_m\boldsymbol{B}_{t^2})+\omega_{pm}^2 M_A\boldsymbol{B}_t\\ &= -M(\nabla\times\boldsymbol{E}_{t^2}+\gamma_m\nabla\times\boldsymbol{E}_t)-\omega_{pm}^2M_A\nabla\times\boldsymbol{E}.\end{aligned} \tag{26}$$

For any function $u(t)$ defined for $t\ge 0$, let us denote its Laplace transform by $\hat{u}(s)=\mathcal{L}(u)=\int_0^\infty e^{-st}u(t)dt$. Taking the Laplace transform of (25) and (26), respectively, we obtain

$$\epsilon_0\epsilon_\phi(s^3+\gamma s^2+\omega_p^2 s)\hat{\boldsymbol{E}} = M(s^2+\gamma s)\nabla\times\hat{\boldsymbol{H}}+\omega_p^2M_A\nabla\times\hat{\boldsymbol{H}}+\boldsymbol{f}_0(s), \tag{27}$$

$$\mu_0\mu_\phi(s^3+\gamma_m s^2+\omega_{pm}^2 s)\hat{\boldsymbol{H}} = -M(s^2+\gamma_m s)\nabla\times\hat{\boldsymbol{E}}-\omega_{pm}^2M_A\nabla\times\hat{\boldsymbol{E}}+\boldsymbol{g}_0(s), \tag{28}$$

where $\boldsymbol{f}_0(s)$ and $\boldsymbol{g}_0(s)$ contain all the terms related to initial conditions and are given as:

$$\boldsymbol{f}_0(s) = \epsilon_0\epsilon_\phi[(s^2+\gamma s+\omega_p^2)\boldsymbol{E}(0)+(s+\gamma)\boldsymbol{E}_t(0)+\boldsymbol{E}_{t^2}(0)]-M[(s+\gamma)\nabla\times\boldsymbol{H}(0)+\nabla\times\boldsymbol{H}_t(0)],$$

$$\boldsymbol{g}_0(s) = \mu_0\mu_\phi[(s^2+\gamma_m s+\omega_{pm}^2)\boldsymbol{H}(0)+(s+\gamma_m)\boldsymbol{H}_t(0)+\boldsymbol{H}_{t^2}(0)]+M[(s+\gamma_m)\nabla\times\boldsymbol{E}(0)+\nabla\times\boldsymbol{E}_t(0)].$$

With the notation $p_e(s) = s^3+\gamma s^2+\omega_p^2 s$, $p_m(s) = s^3+\gamma_m s^2+\omega_{pm}^2 s$, $M_e = (s^2+\gamma s)M+\omega_p^2M_A$, and $M_m = (s^2+\gamma_m s)M+\omega_{pm}^2M_A$, we can rewrite (27) and (28) as

$$\epsilon_0\epsilon_\phi p_e(s)\hat{\boldsymbol{E}} = M_e\nabla\times\hat{\boldsymbol{H}}+\boldsymbol{f}_0(s), \tag{29}$$

$$\mu_0\mu_\phi p_m(s)\hat{\boldsymbol{H}} = -M_m\nabla\times\hat{\boldsymbol{E}}+\boldsymbol{g}_0(s). \tag{30}$$

Substituting $\hat{\boldsymbol{H}}$ of (30) into (29), we obtain

$$\epsilon_0\mu_0\epsilon_\phi\mu_\phi p_e(s)\hat{\boldsymbol{E}} = M_e\nabla\times\left[-\frac{M_m}{p_m(s)}\nabla\times\hat{\boldsymbol{E}}+\frac{\boldsymbol{g}_0(s)}{p_m(s)}\right]+\mu_0\mu_\phi\boldsymbol{f}_0(s),$$

which can be rewritten as

$$\epsilon_0\mu_0\epsilon_\phi\mu_\phi p_e(s)M_e^{-1}\hat{\boldsymbol{E}}+\nabla\times\left(\frac{M_m}{p_m(s)}\nabla\times\hat{\boldsymbol{E}}\right) = \boldsymbol{f}_0^*(s), \tag{31}$$

where we denote $\boldsymbol{f}_0^*(s)=\nabla\times\frac{\boldsymbol{g}_0(s)}{p_m(s)}+\mu_0\mu_\phi M_e^{-1}\boldsymbol{f}_0(s)$. Note that the invertibility of M_e is guaranteed by Lemma 3.1.

Consider the weak formulation of (31): Find $\hat{\boldsymbol{E}}\in H_0(\text{curl};\Omega)$ such that

$$\epsilon_0\mu_0\epsilon_\phi\mu_\phi p_e(s)(M_e^{-1}\hat{\boldsymbol{E}},\boldsymbol{\phi})+\left(\frac{M_m}{p_m(s)}\nabla\times\hat{\boldsymbol{E}},\nabla\times\boldsymbol{\phi}\right) = (\boldsymbol{f}_0^*(s),\boldsymbol{\phi}), \tag{32}$$

holds true for any $\boldsymbol{\phi}\in H_0(\text{curl};\Omega)$. The existence of a unique solution $\hat{\boldsymbol{E}}\in H_0(\text{curl};\Omega)$ of (32) is guaranteed by the Lax–Milgram lemma, since by Lemma 3.1 the matrices M_e and M_e are symmetric positive definite.

Using the fact $\hat{\boldsymbol{E}}\in H_0(\text{curl};\Omega)$, it is easy to see that (29) implies $\nabla\times\hat{\boldsymbol{H}}\in\mathbf{L}^2(\Omega)$, and (30) implies $\hat{\boldsymbol{H}}\in\mathbf{L}^2(\Omega)$. This proves the existence and uniqueness of $\hat{\boldsymbol{H}}\in H(\text{curl};\Omega)$. The inverse Laplace transforms of functions $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{H}}$ are solutions of the time-dependent problem (25)–(26). This completes the proof. □

The rest of this section is devoted to the proof of a stability result for the spherical cloak model (11)–(19). The proof is composed of the following three lemmas.

Lemma 3.2.

$$\begin{aligned}&\frac{\epsilon_0^2\epsilon_\phi^2}{4}(\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(t)+(\|\sqrt{M}\boldsymbol{D}_t\|^2+\|\omega_p\sqrt{M_A}\boldsymbol{D}\|^2)(t)\\ &\quad\le\frac{\epsilon_0^2\epsilon_\phi^2}{2}(2\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(0)+\frac{5}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2(0)+2\|\omega_p\sqrt{M_A}\boldsymbol{D}\|^2(0)\\ &\qquad+\frac{\epsilon_0^2\epsilon_\phi^2}{2}\int_0^t[5\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+(2+4\gamma^2)\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+4\|\omega_p^2\sqrt{M^{-1}}\boldsymbol{E}\|^2]dt\\ &\qquad+\int_0^t\left[\frac{13+\gamma^2}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2+\frac{1}{2}\|\omega_p^2\sqrt{M^{-1}}M_A\boldsymbol{D}\|^2\right]dt.\end{aligned} \tag{33}$$

Proof. I. Multiplying (16) by $\boldsymbol{D}_t$ and integrating the resultant over Ω, we have

$$\frac{1}{2}\frac{d}{dt}[(M\boldsymbol{D}_t,\boldsymbol{D}_t)+(\omega_p^2M_A\boldsymbol{D},\boldsymbol{D})]+\gamma(M\boldsymbol{D}_t,\boldsymbol{D}_t)=\epsilon_0\epsilon_\phi(\boldsymbol{E}_{t^2}+\gamma\boldsymbol{E}_t+\omega_p^2\boldsymbol{E},\boldsymbol{D}_t). \tag{34}$$

Integrating (34) from 0 to t, and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\frac{1}{2}(\|\sqrt{M}\boldsymbol{D}_t\|^2+\|\omega_p\sqrt{M_A}\boldsymbol{D}\|^2)(t)-\frac{1}{2}(\|\sqrt{M}\boldsymbol{D}_t\|^2+\|\omega_p\sqrt{M_A}\boldsymbol{D}\|^2)(0)\\&\quad\le\int_0^t\epsilon_0\epsilon_\phi(\boldsymbol{E}_{t^2}+\gamma\boldsymbol{E}_t+\omega_p^2\boldsymbol{E},\boldsymbol{D}_t)dt\\&\quad\le\int_0^t\left[\frac{\epsilon_0^2\epsilon_\phi^2}{2}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\|\gamma\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p^2\sqrt{M^{-1}}\boldsymbol{E}\|^2)+\frac{3}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2\right]dt,\end{aligned} \tag{35}$$

where we dropped the term $\gamma(M\boldsymbol{D}_t,\boldsymbol{D}_t)$ due to the positivity of matrix M proved by Lemma 3.1.

II. Multiplying both sides of (16) by $\epsilon_0\epsilon_\phi M^{-1}\boldsymbol{E}_t$ and integrating the resultant over Ω, we have

$$\frac{\epsilon_0^2\epsilon_\phi^2}{2}\frac{d}{dt}[\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2]+\epsilon_0^2\epsilon_\phi^2\gamma\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2=\epsilon_0\epsilon_\phi(\boldsymbol{D}_{t^2}+\gamma\boldsymbol{D}_t+\omega_p^2M^{-1}M_A\boldsymbol{D},\boldsymbol{E}_t), \tag{36}$$

integrating which from 0 to t, we have

$$\begin{aligned}&\frac{\epsilon_0^2\epsilon_\phi^2}{2}[(\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(t)-(\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(0)]\\&\quad\le\epsilon_0\epsilon_\phi[(\boldsymbol{D}_t,\boldsymbol{E}_t)(t)-(\boldsymbol{D}_t,\boldsymbol{E}_t)(0)]-\int_0^t\epsilon_0\epsilon_\phi(\boldsymbol{D}_t,\boldsymbol{E}_{t^2})dt+\int_0^t\epsilon_0\epsilon_\phi(\gamma\boldsymbol{D}_t+\omega_p^2M^{-1}M_A\boldsymbol{D},\boldsymbol{E}_t)dt\\&\quad\le\frac{\epsilon_0^2\epsilon_\phi^2}{4}\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\sqrt{M}\boldsymbol{D}_t\|^2+\frac{\epsilon_0^2\epsilon_\phi^2}{2}\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2(0)+\frac{1}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2(0)\\&\qquad+\int_0^t\left[\frac{\epsilon_0^2\epsilon_\phi^2}{2}\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\frac{1}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2\right]dt\\&\qquad+\int_0^t\left[\frac{1}{2}(\|\gamma\sqrt{M}\boldsymbol{D}_t\|^2+\|\omega_p^2\sqrt{M^{-1}}M_A\boldsymbol{D}\|^2)+\epsilon_0^2\epsilon_\phi^2\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2\right]dt,\end{aligned} \tag{37}$$

which can be further reduced to

$$\begin{aligned}&\frac{\epsilon_0^2\epsilon_\phi^2}{4}(\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(t)\\&\quad\le\|\sqrt{M}\boldsymbol{D}_t\|^2+\frac{\epsilon_0^2\epsilon_\phi^2}{2}(2\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(0)+\frac{1}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2(0)\\&\qquad+\int_0^t\left[\frac{\epsilon_0^2\epsilon_\phi^2}{2}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+2\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)+\frac{1+\gamma^2}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2+\frac{1}{2}\|\omega_p^2\sqrt{M^{-1}}M_A\boldsymbol{D}\|^2\right]dt.\end{aligned} \tag{38}$$

Multiplying (35) by four and adding the resultant to (38) completes the proof. □

We like to remark that in the proof of Lemma 3.2, we cannot simply cancel out the term $\int_0^t\epsilon_0\epsilon_\phi(\boldsymbol{D}_t,\boldsymbol{E}_{t^2})dt$ appearing in both (35) and (37). Otherwise, the term $\epsilon_0\epsilon_\phi(\boldsymbol{D}_t,\boldsymbol{E}_t)(t)$ cannot be controlled. Furthermore, we have to keep ω_p, M_A, M and M^{-1} inside the L_2 norm, since they are spatial dependent. Actually, it is this fact that complicates the stability analysis.

Lemma 3.3.

$$\begin{aligned}&\frac{1}{4}(\|\sqrt{M}\boldsymbol{B}_t\|^2+\|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2)(t)+\mu_0^2\mu_\phi^2(\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+\|\omega_{pm}\sqrt{M^{-1}}\boldsymbol{H}\|^2)(t)\\&\quad\le\|\sqrt{M}\boldsymbol{B}_t\|^2(0)+\frac{1}{2}\|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2(0)+\frac{\mu_0^2\mu_\phi^2}{2}(5\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+4\|\omega_{pm}\sqrt{M^{-1}}\boldsymbol{H}\|^2)(0)\\&\qquad+\int_0^t\left\{\frac{\mu_0^2\mu_\phi^2}{2}[(13+\gamma_m^2)\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+\|\omega_{pm}^2\sqrt{M^{-1}}\boldsymbol{H}\|^2]+\frac{5}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2\right\}dt\\&\qquad+\int_0^t\left\{(1+2\gamma_m^2)\|\sqrt{M}\boldsymbol{B}_t\|^2+2\|\omega_{mp}^2\sqrt{M^{-1}}M_A\boldsymbol{B}\|^2\right\}dt.\end{aligned} \tag{39}$$

Proof. I. Multiplying (17) by $\boldsymbol{B}_t$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\sqrt{M}\boldsymbol{B}_t\|^2+\|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2)+\gamma_m(M\boldsymbol{B}_t,\boldsymbol{B}_t)=\mu_0\mu_\phi(\boldsymbol{H}_{t^2},\boldsymbol{B}_t)+\mu_0\mu_\phi(\gamma_m\boldsymbol{H}_t+\omega_{pm}^2\boldsymbol{H},\boldsymbol{B}_t),$$

integrating which from 0 to t, and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{2}[(\|\sqrt{M}\boldsymbol{B}_t\|^2+\|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2)(t)-(\|\sqrt{M}\boldsymbol{B}_t\|^2+\|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2)(0)]\\
&\quad\le\mu_0\mu_\phi\int_0^t(\boldsymbol{H}_{t^2},\boldsymbol{B}_t)dt+\int_0^t\mu_0\mu_\phi(\gamma_m\boldsymbol{H}_t+\omega_{pm}^2\boldsymbol{H},\boldsymbol{B}_t)dt\\
&\quad=\mu_0\mu_\phi[(\boldsymbol{H}_t,\boldsymbol{B}_t)(t)-(\boldsymbol{H}_t,\boldsymbol{B}_t)(0)]-\mu_0\mu_\phi\int_0^t(\boldsymbol{H}_t,\boldsymbol{B}_{t^2})dt+\int_0^t\mu_0\mu_\phi(\gamma_m\boldsymbol{H}_t+\omega_{pm}^2\boldsymbol{H},\boldsymbol{B}_t)dt\\
&\quad\le\frac{1}{4}\|\sqrt{M}\boldsymbol{B}_t\|^2(t)+\mu_0^2\mu_\phi^2\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2(t)+\frac{1}{2}\|\sqrt{M}\boldsymbol{B}_t\|^2(0)+\frac{\mu_0^2\mu_\phi^2}{2}\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2(0)\\
&\qquad+\int_0^t\left[\frac{\mu_0^2\mu_\phi^2}{2}\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+\frac{1}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2\right]dt\\
&\qquad+\int_0^t\left[\frac{\mu_0^2\mu_\phi^2}{2}(\|\gamma_m\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+\|\omega_{pm}^2\sqrt{M^{-1}}\boldsymbol{H}\|^2)+\|\sqrt{M}\boldsymbol{B}_t\|^2\right]dt,
\end{aligned}\tag{40}$$

where in the last step we used the identity $\boldsymbol{B}_{t^2}=-\nabla\times\boldsymbol{E}_t$.

II. Multiplying (17) by $\mu_0\mu_\phi M^{-1}\boldsymbol{H}_t$ and integrating over Ω, we have

$$\mu_0^2\mu_\phi^2(M^{-1}(\boldsymbol{H}_{t^2}+\gamma_m\boldsymbol{H}_t+\omega_{pm}^2\boldsymbol{H}),\boldsymbol{H}_t)=\mu_0\mu_\phi(\boldsymbol{B}_{t^2}+\gamma_m\boldsymbol{B}_t+\omega_{pm}^2M^{-1}M_A\boldsymbol{B},\boldsymbol{H}_t),$$

integrating which from 0 to t, and using the Cauchy–Schwarz inequality, we further have

$$\begin{aligned}
&\frac{\mu_0^2\mu_\phi^2}{2}[(\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+\|\omega_{pm}\sqrt{M^{-1}}\boldsymbol{H}\|^2)(t)-(\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2+\|\omega_{pm}\sqrt{M^{-1}}\boldsymbol{H}\|^2)(0)]\\
&\quad\le\int_0^t\left[\frac{1}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2+\frac{\mu_0^2\mu_\phi^2}{2}\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2\right]dt\\
&\qquad+\int_0^t\left[\frac{1}{2}\|\gamma_m\sqrt{M}\boldsymbol{B}_t\|^2+\frac{1}{2}\|\omega_{pm}^2\sqrt{M^{-1}}M_A\boldsymbol{B}\|^2+\mu_0^2\mu_\phi^2\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2\right]dt.
\end{aligned}\tag{41}$$

Multiplying (41) by four and adding the resultant to (40) conclude the proof. □

From Lemmas 3.2 and 3.3, we can see that all the right hand terms except $\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2$ and $\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2$ in both (33) and (39) can be controlled by those left hand side terms. In the following lemma, we will build up an estimate to bound these two terms.

Lemma 3.4. *Under the assumption $\gamma_m=\gamma$, we have*

$$\begin{aligned}
&\frac{\epsilon_0\epsilon_\phi\mu_0\mu_\phi}{2}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)(t)+\frac{1}{4}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(t)\\
&\quad\le\frac{\epsilon_0\epsilon_\phi\mu_0\mu_\phi}{2}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)(0)+\frac{3}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(0)\\
&\qquad+\frac{\gamma_m^2}{2}\|\sqrt{M}\boldsymbol{B}_t\|^2(0)+\frac{1}{2}\|\omega_{pm}^2M_A\sqrt{M^{-1}}\boldsymbol{B}\|^2(0)+2\gamma_m^2\|\sqrt{M}\boldsymbol{B}_t\|^2(t)+2\|\omega_{pm}^2\sqrt{M^{-1}}M_A\boldsymbol{B}\|^2(t)\\
&\qquad+\int_0^t\left[\left(\frac{1}{2}+\gamma_m\right)\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2+\frac{1}{2}\|\omega_{pm}^2\sqrt{M^{-1}}M_A\boldsymbol{B}_t\|^2\right]dt\\
&\qquad+\int_0^t\frac{\mu_0\mu_\phi}{2}(\|\omega_{pm}^2\sqrt{M}\boldsymbol{D}_t\|^2+\|\omega_p^2\sqrt{M^{-1}}M_A\boldsymbol{D}_t\|^2+2\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2)dt.
\end{aligned}\tag{42}$$

Proof. I. Multiplying (16) by $\mu_0\mu_\phi M^{-1}$, then differentiating both sides with respect to t and using (15), we have

$$\epsilon_0\epsilon_\phi\mu_0\mu_\phi M^{-1}(\boldsymbol{E}_{t^3}+\gamma\boldsymbol{E}_{t^2}+\omega_p^2\boldsymbol{E}_t)=\mu_0\mu_\phi(\nabla\times\boldsymbol{H}_{t^2}+\gamma\nabla\times\boldsymbol{H}_t+\omega_p^2M^{-1}M_A\boldsymbol{D}_t),$$

multiplying which by $\boldsymbol{E}_{t^2}$ and integrating over Ω, we obtain

$$\frac{\epsilon_0\epsilon_\phi\mu_0\mu_\phi}{2}\frac{d}{dt}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2) \le \mu_0\mu_\phi(\nabla\times\boldsymbol{H}_{t^2}+\gamma\nabla\times\boldsymbol{H}_t+\omega_p^2M^{-1}M_A\boldsymbol{D}_t,\boldsymbol{E}_{t^2})$$
$$= \mu_0\mu_\phi(\boldsymbol{H}_{t^2}+\gamma\boldsymbol{H}_t,\nabla\times\boldsymbol{E}_{t^2})+\mu_0\mu_\phi(\omega_p^2M^{-1}M_A\boldsymbol{D}_t,\boldsymbol{E}_{t^2}), \qquad (43)$$

where in the last step we used integration by parts and the PEC boundary condition (18).

Integrating (43) from 0 to t, we have

$$\frac{\epsilon_0\epsilon_\phi\mu_0\mu_\phi}{2}[(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)(t)-(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2+\|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)(0)]$$
$$\le \int_0^t \mu_0\mu_\phi(\boldsymbol{H}_{t^2}+\gamma\boldsymbol{H}_t,\nabla\times\boldsymbol{E}_{t^2})dt+\int_0^t \mu_0\mu_\phi(\omega_p^2M^{-1}M_A\boldsymbol{D}_t,\boldsymbol{E}_{t^2})dt. \qquad (44)$$

II. Multiplying (17) by $\nabla\times\boldsymbol{E}_{t^2}$, integrating over Ω, and using the identity $\boldsymbol{B}_t=-\nabla\times\boldsymbol{E}$, we have

$$\mu_0\mu_\phi(\boldsymbol{H}_{t^2}+\gamma_m\boldsymbol{H}_t,\nabla\times\boldsymbol{E}_{t^2})$$
$$=-(M(\nabla\times\boldsymbol{E}_t+\gamma_m\nabla\times\boldsymbol{E}),\nabla\times\boldsymbol{E}_{t^2})+(\omega_{pm}^2M_A\boldsymbol{B},\nabla\times\boldsymbol{E}_{t^2})-\mu_0\mu_\phi(\omega_{pm}^2\boldsymbol{H},\nabla\times\boldsymbol{E}_{t^2}),$$

integrating which from 0 to t, we obtain

$$\mu_0\mu_\phi\int_0^t(\boldsymbol{H}_{t^2}+\gamma_m\boldsymbol{H}_t,\nabla\times\boldsymbol{E}_{t^2})dt = -\frac{1}{2}[\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(t)-\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(0)]$$
$$-\int_0^t(\gamma_mM\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E}_{t^2})dt+\int_0^t(\omega_{pm}^2M_A\boldsymbol{B},\nabla\times\boldsymbol{E}_{t^2})dt$$
$$-\int_0^t\mu_0\mu_\phi(\omega_{pm}^2\boldsymbol{H},\nabla\times\boldsymbol{E}_{t^2})dt. \qquad (45)$$

The last three terms of (45) can be bounded as follows. First, using integration by parts, the identity $\nabla\times\boldsymbol{E}=-\boldsymbol{B}_t$ and the Cauchy–Schwarz inequality, we have

$$-\int_0^t(\gamma_mM\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E}_{t^2})dt = -\gamma_m(M\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E}_t)(t)+\gamma_m(M\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E}_t)(0)$$
$$+\int_0^t\gamma_m(M\nabla\times\boldsymbol{E}_t,\nabla\times\boldsymbol{E}_t)dt$$
$$\le \frac{1}{8}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(t)+2\|\gamma_m\sqrt{M}\boldsymbol{B}_t\|^2(t)$$
$$+\frac{1}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(0)+\frac{1}{2}\|\gamma_m\sqrt{M}\boldsymbol{B}_t\|^2(0)+\int_0^t\gamma_m\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2dt. \qquad (46)$$

Similarly, we can obtain

$$\int_0^t(\omega_{pm}^2M_A\boldsymbol{B},\nabla\times\boldsymbol{E}_{t^2})dt=(\omega_{pm}^2M_A\boldsymbol{B},\nabla\times\boldsymbol{E}_t)(t)-(\omega_{pm}^2M_A\boldsymbol{B},\nabla\times\boldsymbol{E}_t)(0)-\int_0^t(\omega_{pm}^2M_A\boldsymbol{B}_t,\nabla\times\boldsymbol{E}_t)dt$$
$$\le\frac{1}{8}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(t)+2\|\omega_{pm}^2\sqrt{M^{-1}}M_A\boldsymbol{B}\|^2(t)+\frac{1}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(0)$$
$$+\frac{1}{2}\|\omega_{pm}^2\sqrt{M^{-1}}M_A\boldsymbol{B}\|^2(0)+\int_0^t\left[\frac{1}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2+\frac{1}{2}\|\omega_{pm}^2\sqrt{M^{-1}}M_A\boldsymbol{B}_t\|^2\right]dt. \qquad (47)$$

Using the PEC boundary condition (18) and the identity $\nabla\times\boldsymbol{H}=\boldsymbol{D}_t$, we have

$$\int_0^t\mu_0\mu_\phi(\omega_{pm}^2\boldsymbol{H},\nabla\times\boldsymbol{E}_{t^2})dt = \int_0^t\mu_0\mu_\phi(\omega_{pm}^2\boldsymbol{D}_t,\boldsymbol{E}_{t^2})dt$$
$$\le\int_0^t\frac{\mu_0\mu_\phi}{2}(\|\omega_{pm}^2\sqrt{M}\boldsymbol{D}_t\|^2+\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2)dt. \qquad (48)$$

Substituting (46)–(48) into (45), then into (44), and bounding the last term of (44) as follows

$$\int_0^t\mu_0\mu_\phi(\omega_p^2M^{-1}M_A\boldsymbol{D}_t,\boldsymbol{E}_{t^2})dt.\le\int_0^t\frac{\mu_0\mu_\phi}{2}(\|\omega_p^2\sqrt{M^{-1}}M_A\boldsymbol{D}_t\|^2+\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2)dt,$$

we conclude the proof. □

We like to remark that the assumption $\gamma_m = \gamma$ always holds true in practical simulations with the Drude model [20]. Most often, researchers working on cloaking simulation (e.g., [34,32]) pay more attention to the lossless case $\gamma_m = \gamma = 0$. Here we prove the stability for the general case with a loss.

Theorem 3.2. *Denote a constant $\gamma_{\max}$, which satisfies the condition*

$$\gamma_{\max} > 8\max\{\gamma_m^2, \max_{\overline{\Omega}}(\omega_{pm}\sqrt{M_A M^{-1}})\}. \tag{49}$$

Then under the assumptions $\gamma_m = \gamma$, we have for any $t \in [0, T]$:

$$\begin{aligned}
&\frac{\epsilon_0^2\epsilon_\phi^2}{4}(\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2 + \|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(t) + (\|\sqrt{M}\boldsymbol{D}_t\|^2 + \|\omega_p\sqrt{M_A}\boldsymbol{D}\|^2)(t)\\
&\quad + \gamma_{\max}\left[\frac{1}{4}(\|\sqrt{M}\boldsymbol{B}_t\|^2 + \|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2)(t) + \mu_0^2\mu_\phi^2(\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2 + \|\omega_{pm}\sqrt{M^{-1}}\boldsymbol{H}\|^2)(t)\right]\\
&\quad + \frac{\epsilon_0\epsilon_\phi\mu_0\mu_\phi}{2}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2 + \|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)(t) + \frac{1}{4}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(t)\\
&\le CI_0(\boldsymbol{E}_{t^2}(0), \boldsymbol{E}_t(0), \boldsymbol{E}(0), \boldsymbol{D}_t(0), \boldsymbol{D}(0), \boldsymbol{B}_t(0), \boldsymbol{B}(0), \boldsymbol{H}_t(0), \boldsymbol{H}(0)),
\end{aligned}$$

where the constant $C > 0$ depends on physical parameters $\epsilon_0, \mu_0, R_1, R_2, \gamma, \gamma_m, \omega_p, \omega_{pm}$, and the function $I_0(\cdots)$ is given as:

$$\begin{aligned}
I_0 = {}& \frac{\epsilon_0^2\epsilon_\phi^2}{2}(2\|\sqrt{M^{-1}}\boldsymbol{E}_t\|^2 + \|\omega_p\sqrt{M^{-1}}\boldsymbol{E}\|^2)(0) + \frac{5}{2}\|\sqrt{M}\boldsymbol{D}_t\|^2(0) + 2\|\omega_p\sqrt{M_A}\boldsymbol{D}\|^2(0)\\
&+ \|\sqrt{M}\boldsymbol{B}_t\|^2(0) + \frac{1}{2}\|\omega_{pm}\sqrt{M_A}\boldsymbol{B}\|^2(0) + \frac{\mu_0^2\mu_\phi^2}{2}(5\|\sqrt{M^{-1}}\boldsymbol{H}_t\|^2 + 4\|\omega_{pm}\sqrt{M^{-1}}\boldsymbol{H}\|^2)(0)\\
&+ \frac{\epsilon_0\epsilon_\phi\mu_0\mu_\phi}{2}(\|\sqrt{M^{-1}}\boldsymbol{E}_{t^2}\|^2 + \|\omega_p\sqrt{M^{-1}}\boldsymbol{E}_t\|^2)(0) + \frac{3}{2}\|\sqrt{M}\nabla\times\boldsymbol{E}_t\|^2(0)\\
&+ \frac{\gamma_m^2}{2}\|\sqrt{M}\boldsymbol{B}_t\|^2(0) + \frac{1}{2}\|\omega_{pm}^2 M_A\sqrt{M^{-1}}\boldsymbol{B}\|^2(0).
\end{aligned}$$

Proof. From Lemmas 3.2–3.4 and the Gronwall inequality, we see that all the right hand side terms can be controlled by the left hand side terms, except the following two terms

$$2\gamma_m^2\|\sqrt{M}\boldsymbol{B}_t\|^2(t) + 2\|\omega_{pm}^2 M_A\sqrt{M^{-1}}\boldsymbol{B}\|^2(t), \tag{50}$$

which can be controlled by multiplying (39) by $\gamma_{\max}$.

The proof is completed by applying the Gronwall inequality to the summation of (33), the $\gamma_{\max}$ multiple of (39), and (42). □

4. Conclusions

In this paper, we carried out the well-posedness study of a time-domain spherical cloaking model introduced by Zhao and Hao [34]. Due to the extensive computing power needed for simulating this model as pointed out in [34]: "100 processors and 220 gigabyte (GB) memory were used to run the parallel dispersive FDTD simulations. Each simulation lasts around 45 h (13,000 time steps) before reaching the steady-state.", we did not pursue the numerical simulation here. In the future, we plan to develop some efficient time-domain finite element methods (such as hp method [41] or DG methods [42,43]) to simulate this spherical cloaking model.

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