Superconvergence Analysis for Time-Dependent Maxwell's Equations in Metamaterials

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Received 31 July 2010; accepted 22 June 2011
Published online 1 September 2011 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/num.20703

In this article, we consider the time-dependent Maxwell's equations modeling wave propagation in metamaterials. One-order higher global superclose results in the L^2 norm are proved for several semidiscrete and fully discrete schemes developed for solving this model using nonuniform cubic and rectangular edge elements. Furthermore, L^∞ superconvergence at element centers is proved for the lowest order rectangular edge element. To our best knowledge, such pointwise superconvergence result and its proof are original, and we are unaware of any other publications on this issue. Numer Methods Partial Differential Eq 28: 1794–1816, 2012

Keywords: Maxwell's equations; metamaterial; mixed finite element method; superconvergence

I. INTRODUCTION

The metamaterials we are interested in are those artificially constructed electromagnetic nanomaterials with negative refraction index. The successful demonstration of such metamaterials in 2000 triggered a big wave in further study of such metamaterials and exploration of their applications in diverse areas such as subwavelength imaging and design of invisibility cloak. More details can be found in monographs such as [1–4] and references cited therein.

Because of the tremendous cost of metamaterials, numerical simulation plays a very important role in the investigation of wave propagation involving metamaterials. However, simulations are almost exclusively based on either the classic finite-difference time-domain (FDTD) method or

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Contract grant sponsor: NSFC; contract grant number: 11031006
Contract grant sponsor: Hunan Provicial NSF; contract grant number: 10JJ7001
Contract grant sponsor: National Science Foundation; contract grant number: DMS-0810896

commercial packages such as HFSS and COMSOL [5, p. 12]. Because of their limited capabilities (e.g., FDTD is not well suited for problems involving complex geometries, and COMSOL runs out memory so quickly for 3D simulations), there is an urgent need for developing more efficient and reliable software for metamaterial simulations [5, p. 28]. To our best knowledge, although there are many excellent work for the Maxwell's equations in vacuum (see, e.g., articles [6–11], books [12–14], and references cited therein), there is not much work devoted to the development and analysis of finite element methods (FEMs) for the Maxwell's equations involving metamaterials. Fernandes and Raffetto [15, 16] initiated the study of well posedness and finite element analysis for time-harmonic Maxwell's equations involving metamaterials. In recent years, we made some initial effort in developing and analyzing some FEM for time-domain Maxwell's equations involving metamaterials [17–19].

In our recent numerical experiments [17,18], we found that superconvergence phenomena exist for metamaterial simulations using FEMs. It is well known [20–22] that superconvergence results often happen when the underlying differential equations have smooth solutions and are solved on very structured meshes such as rectangular grids or strongly regular triangular grids. Many excellent superconvergence results have been obtained for elliptic and parabolic problems (e.g., [23–26]). However, there exist not many superconvergence results for Maxwell's equations. In 1994, Monk [27] obtained the first superconvergence result for Maxwell's equations in vacuum. Later, Brandts [28] presented another superconvergence analysis for 2D Maxwell's equations in vacuum. Also, Lin and Coworkers [29–31] systematically developed some global superconvergence results using the so-called Lin's integral identity technique [21, 32–34] developed in early 1990s. In 2008, Lin and Li [35] extended the superconvergence result for the vacuum case to three popular dispersive media models. However, all existing results are limited to semi-discrete schemes and obtained only in the L^2-norm. Here, we extend our previous work [35] to the metamaterial case. Superconvergence results are obtained for both semidiscrete and fully discrete schemes based on nonuniform cubic meshes. Furthermore, the present results are improved in that the convergence constant depends on time linearly, instead of exponentially because of the use of Gronwall's inequality.

In this article, $C > 0$ denotes a generic constant, which is independent of the finite element mesh size h and time step size τ. We also introduce some common notation [14]

$$H(\text{div};\Omega) = \left\{ \boldsymbol{v} \in (L^2(\Omega))^3; \quad \nabla \cdot \boldsymbol{v} \in (L^2(\Omega))^3 \right\},$$
$$H(\text{curl};\Omega) = \left\{ \boldsymbol{v} \in (L^2(\Omega))^3; \quad \nabla \times \boldsymbol{v} \in (L^2(\Omega))^3 \right\},$$
$$H_0(\text{curl};\Omega) = \{ \boldsymbol{v} \in H(\text{curl};\Omega); \quad \mathbf{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega \},$$

where $\alpha \geq 0$ is a real number, and Ω is a rectangular-type domain in $\mathcal{R}^3$ with boundary $\partial\Omega$. Let $(H^\alpha(\Omega))^3$ be the standard Sobolev space equipped with the norm $\|\cdot\|_\alpha$ and seminorm $|\cdot|_\alpha$. Specifically, $\|\cdot\|_0$ will mean the $(L^2(\Omega))^3$-norm. We equip $H(\text{curl};\Omega)$ with the norm $\|\boldsymbol{v}\|_{0,\text{curl}} = (\|\boldsymbol{v}\|_0^2 + \|\text{curl } \boldsymbol{v}\|_0^2)^{1/2}$. For clarity, in the rest of this article we introduce the vector notation

$$\boldsymbol{L}^2(\Omega) = \left(L^2(\Omega)\right)^3, \quad \boldsymbol{H}^\alpha(\Omega) = (H^\alpha(\Omega))^3.$$

The rest of this article is organized as follows. In Section II, we present the governing equations for metamaterials. In Section III, we develop two different semidiscrete schemes and prove the superclose results for both schemes. Then, in Section IV, we present a fully discrete scheme and its superclose analysis. Because of some major differences between the 2D and 3D cases, in Section V, we extend the 3D results to 2D. Especially, we prove the L^∞ superconvergence at element centers for the lowest order rectangular edge element. Finally, we conclude the article in Section VI.

II. THE GOVERNING EQUATIONS

The governing equations for modeling wave propagation in metamaterials are [19]:

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J}, \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E} - \boldsymbol{K}, \quad \text{in } \Omega \times (0, T), \tag{2}$$

$$\frac{\partial \boldsymbol{J}}{\partial t} + \Gamma_e \boldsymbol{J} = \epsilon_0 \omega_{pe}^2 \boldsymbol{E}, \quad \text{in } \Omega \times (0, T), \tag{3}$$

$$\frac{\partial \boldsymbol{K}}{\partial t} + \Gamma_m \boldsymbol{K} = \mu_0 \omega_{pm}^2 \boldsymbol{H}, \quad \text{in } \Omega \times (0, T), \tag{4}$$

where ϵ_0 and μ_0 are the permittivity and permeability in vacuum respectively, ω_{pe} and ω_{pm} are the electric and magnetic plasma frequencies, respectively, Γ_e and Γ_m are the electric and magnetic damping frequencies, respectively, $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{H}(\boldsymbol{x}, t)$ are the electric and magnetic fields, respectively, and $\boldsymbol{J}(\boldsymbol{x}, t)$ and $\boldsymbol{K}(\boldsymbol{x}, t)$ are the induced electric and magnetic currents, respectively. For simplicity, we assume that the boundary of Ω is perfect conducting so that

$$\mathbf{n} \times \boldsymbol{E} = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Furthermore, we assume that the initial conditions are

$$\boldsymbol{E}(\mathbf{x}, 0) = \boldsymbol{E}_0(\mathbf{x}), \quad \boldsymbol{H}(\mathbf{x}, 0) = \boldsymbol{H}_0(\mathbf{x}), \tag{6}$$

$$\boldsymbol{J}(\mathbf{x}, 0) = \boldsymbol{J}_0(\mathbf{x}), \quad \boldsymbol{K}(\mathbf{x}, 0) = \boldsymbol{K}_0(\mathbf{x}), \tag{7}$$

where $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{J}_0$, and $\boldsymbol{K}_0$ are some given functions.

To simplify the presentation and numerical simulation, we first rescale the governing equations (1)–(4). By introducing the notation

$$\tilde{t} = \frac{t}{\sqrt{\epsilon_0 \mu_0}}, \tilde{\Gamma}_m = \Gamma_m \sqrt{\epsilon_0 \mu_0}, \tilde{\Gamma}_e = \Gamma_e \sqrt{\epsilon_0 \mu_0}, \tilde{\omega}_m^2 = \epsilon_0 \mu_0 \omega_{pm}^2, \tilde{\omega}_e^2 = \epsilon_0 \mu_0 \omega_{pe}^2,$$

$$\tilde{\boldsymbol{E}} = \sqrt{\epsilon_0 \mu_0} \boldsymbol{E}, \quad \tilde{\boldsymbol{H}} = \mu_0 \boldsymbol{H}, \quad \tilde{\boldsymbol{J}} = \mu_0 \boldsymbol{J}, \quad \tilde{\boldsymbol{K}} = \sqrt{\epsilon_0 \mu_0} \boldsymbol{K},$$

it is not difficult to check that the Eqs. (1)–(4) can be written as

$$\frac{\partial \tilde{\boldsymbol{E}}}{\partial \tilde{t}} = \nabla \times \tilde{\boldsymbol{H}} - \tilde{\boldsymbol{J}}, \tag{8}$$

$$\frac{\partial \tilde{\boldsymbol{H}}}{\partial \tilde{t}} = -\nabla \times \tilde{\boldsymbol{E}} - \tilde{\boldsymbol{K}}, \tag{9}$$

$$\frac{\partial \tilde{\boldsymbol{J}}}{\partial \tilde{t}} + \tilde{\Gamma}_{\rm e}\tilde{\boldsymbol{J}} = \tilde{\omega}_{\rm e}^2\tilde{\boldsymbol{E}}, \tag{10}$$

$$\frac{\partial \tilde{\boldsymbol{K}}}{\partial \tilde{t}} + \tilde{\Gamma}_{\rm m}\tilde{\boldsymbol{K}} = \tilde{\omega}_{\rm m}^2\tilde{\boldsymbol{H}}, \tag{11}$$

which have the same form as the original governing equations (1)–(4) if we set $\epsilon_0 = \mu_0 = 1$ in (1)–(4). In the rest of this article, our discussion is based on the nondimensionalized form (8)–(11) by dropping all those tildes.

We like to remark that solving the metamaterial model (8)–(11) is quite challenging, because the governing equations cannot be reduced to a simple vector wave equation as in vacuum. Solving (10) yields

$$\boldsymbol{J}(\boldsymbol{x},t) = e^{-\Gamma_{\rm e}t}\boldsymbol{J}_0 + \omega_{\rm e}^2\int_0^t e^{-\Gamma_{\rm e}(t-s)}\boldsymbol{E}(\boldsymbol{x},s)ds \equiv -f + \tilde{\tilde{\boldsymbol{J}}}(\boldsymbol{E}). \tag{12}$$

Similarly, from (11) we have

$$\boldsymbol{K}(\boldsymbol{x},t) = e^{-\Gamma_{\rm m}t}\boldsymbol{K}_0 + \omega_{\rm m}^2\int_0^t e^{-\Gamma_{\rm m}(t-s)}\boldsymbol{H}(\boldsymbol{x},s)ds \equiv -g + \tilde{\tilde{\boldsymbol{K}}}(\boldsymbol{H}). \tag{13}$$

Differentiating (8) with respect to t and substituting (9) and (12) into it, we obtain

$$\begin{aligned} 0 &= \boldsymbol{E}_{tt} - \nabla\times\boldsymbol{H}_t + \boldsymbol{J}_t \\ &= \boldsymbol{E}_{tt} + \nabla\times\nabla\times\boldsymbol{E} + \nabla\times\boldsymbol{K} + \omega_{\rm e}^2\boldsymbol{E} - \Gamma_{\rm e}\left(e^{-\Gamma_{\rm e}t}\boldsymbol{J}_0 + \omega_{\rm e}^2\int_0^t e^{-\Gamma_{\rm e}(t-s)}\boldsymbol{E}(\boldsymbol{x},s)ds\right). \end{aligned} \tag{14}$$

Taking curl of (13) and substituting (8) into it, we have

$$\nabla\times\boldsymbol{K} = e^{-\Gamma_{\rm m}t}\nabla\times\boldsymbol{K}_0 + \omega_{\rm m}^2\int_0^t e^{-\Gamma_{\rm m}(t-s)}(\boldsymbol{E}_t + \boldsymbol{J})(\boldsymbol{x},s)ds. \tag{15}$$

Then replacing $\boldsymbol{J}$ of (15) by (12) and substituting the result into (14), we can obtain an equation involving one variable $\boldsymbol{E}$. As the resulting equation is so complicated, it is not a good idea of solving just one variable for the Maxwell's equations when metamaterials are involved. Hence, we better resort to the mixed FEM instead of the standard FEM.

III. 3D SUPERCLOSE ANALYSIS FOR SEMIDISCRETE SCHEMES

In this section, we will discuss two different ways of solving the Maxwell's equations involving metamaterials. One way is to solve (8)–(11) directly. Another way is to reduce the system of (8)–(11) to two unknowns by using (12) and (13), i.e., in the integral-differential equation form:

$$\frac{\partial \boldsymbol{E}}{\partial t} = \nabla\times\boldsymbol{H} - \tilde{\tilde{\boldsymbol{J}}}(\boldsymbol{E}) + f, \tag{16}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\nabla\times\boldsymbol{E} - \tilde{\tilde{\boldsymbol{K}}}(\boldsymbol{H}) + g. \tag{17}$$

To design our mixed FEM, we partition Ω by a family of regular cubic meshes T^h with maximum mesh size h. Consider the Raviart-Thomas-Nédélec cubic elements (e.g., [36] and [14]):

$$\begin{aligned} \boldsymbol{U}_h &= \left\{ \boldsymbol{\psi}_h \in H(\text{div};\Omega) : \quad \boldsymbol{\psi}_h|_K \in Q_{k,k-1,k-1} \times Q_{k-1,k,k-1} \times Q_{k-1,k-1,k}, \quad \forall K \in T^h \right\}, \\ \boldsymbol{V}_h &= \left\{ \boldsymbol{\phi}_h \in H(\text{curl};\Omega) : \quad \boldsymbol{\phi}_h|_K \in Q_{k-1,k,k} \times Q_{k,k-1,k} \times Q_{k,k,k-1}, \quad \forall K \in T^h \right\}. \end{aligned}$$

Here, $Q_{i,j,k}$ denotes the space of polynomials whose degrees are less than or equal to i, j, k in variables x, y, z, respectively. It is easy to see that

$$\nabla \times \boldsymbol{V}_h \subset \boldsymbol{U}_h. \tag{18}$$

For superclose analysis, we need to define two operators. The first one is the standard $\boldsymbol{L}^2(\Omega)$-projection operator: For any $\boldsymbol{H} \in \boldsymbol{L}^2(\Omega)$, $P_h\boldsymbol{H} \in \mathbf{U}_h$ satisfies

$$(P_h\boldsymbol{H} - \boldsymbol{H}, \boldsymbol{\psi}_h) = 0, \quad \forall \psi_h \in \mathbf{U}_h.$$

Another one is the Nédélec interpolation operator Π_h, which is defined as: For any $\boldsymbol{E} \in H(\text{curl};\Omega)$, $\Pi_h\boldsymbol{E} \in \mathbf{V}_h$ satisfies [36, p. 331]:

$$\begin{aligned} &\int_{l_i} (\boldsymbol{E} - \Pi_h\boldsymbol{E}) \cdot \mathbf{t} q\, dl = 0, \quad \forall q \in P_{k-1}(l_i), i = 1, \ldots, 12, \\ &\int_{\sigma_i} ((\boldsymbol{E} - \Pi_h\boldsymbol{E}) \times \mathbf{n} \cdot \boldsymbol{q}\, d\sigma = 0, \quad \forall \boldsymbol{q} \in Q_{k-2,k-1}(\sigma_i) \times Q_{k-1,k-2}(\sigma_i), i = 1, \ldots, 6, \\ &\int_K (\boldsymbol{E} - \Pi_h\boldsymbol{E}) \cdot \boldsymbol{q}\, dK = 0, \quad \forall \boldsymbol{q} \in Q_{k-1,k-2,k-2} \times Q_{k-2,k-1,k-2} \times Q_{k-2,k-2,k-1}, \end{aligned}$$

where l_i and σ_i are the edges and faces of an element K, $\mathbf{t}$ is the unit tangent vector along the edge l_i, and $\mathbf{n}$ is the unit normal vector on face σ_i.

Furthermore, our superclose analysis depends on the following two fundamental results.

Lemma 3.1. [30, Lemma 3.1]. *On any cubic element K, for any $\boldsymbol{E} \in \boldsymbol{H}^{k+2}(K)$, we have*

$$\int_K \nabla \times (\boldsymbol{E} - \Pi_h\boldsymbol{E}) \cdot \boldsymbol{\psi}\, dxdydz = O(h^{k+1})\|\boldsymbol{E}\|_{k+2,K}\|\boldsymbol{\psi}\|_{0,K}, \quad \forall \boldsymbol{\psi}|_K \in \boldsymbol{U}_h(K).$$

Lemma 3.2. [30, Lemma 3.2]. *On any cubic element K, for any $\boldsymbol{E} \in \boldsymbol{H}^{k+1}(K)$, we have*

$$\int_K (\boldsymbol{E} - \Pi_h\boldsymbol{E}) \cdot \boldsymbol{\phi}\, dxdydz = O(h^{k+1})\|\boldsymbol{E}\|_{k+1,K}\|\boldsymbol{\phi}\|_{0,K}, \quad \forall \boldsymbol{\phi}|_K \in \boldsymbol{V}_h(K)$$

Note that in [30], the results of Lemmas 3.1 and 3.2 were stated for the whole domain Ω. But, the proofs of [30] actually show that the results hold true element-wisely. Readers can consult the original proofs of [30] and our detailed proof for rectangular elements presented in Section V. Below, we will discuss two different ways of solving the Maxwell's equations (8)–(11).

A. The Semidiscrete Scheme for an Integral-Differential Formulation

The first way is to solve the system (8)–(11) in the reduced integral-differential equation form:

$$\frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \tilde{\tilde{\boldsymbol{J}}}(\boldsymbol{E}) + f, \tag{19}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E} - \tilde{\tilde{\boldsymbol{K}}}(\boldsymbol{H}) + g. \tag{20}$$

Assuming existence of smooth solutions to (19) and (20), we can obtain its corresponding weak formulation: For any $t \in (0, T]$, find the solution $(\boldsymbol{E}, \boldsymbol{H}) \in H_0(\text{curl}; \Omega) \times \boldsymbol{L}^2(\Omega)$ such that

$$(\boldsymbol{E}_t, \boldsymbol{\phi}) - (\boldsymbol{H}, \nabla \times \boldsymbol{\phi}) + (\tilde{\tilde{\boldsymbol{J}}}(\boldsymbol{E}), \boldsymbol{\phi}) = (f, \boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} \in H_0(\text{curl}; \Omega), \tag{21}$$

$$(\boldsymbol{H}_t, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{E}, \boldsymbol{\psi}) + (\tilde{\tilde{\boldsymbol{K}}}(\boldsymbol{H}), \boldsymbol{\psi}) = (g, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega), \tag{22}$$

with the initial conditions

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega. \tag{23}$$

Note that the above derivation used the Stokes' formula

$$\int_\Omega \nabla \times \boldsymbol{\phi} \cdot \boldsymbol{H} d\boldsymbol{x} = \int_\Omega \boldsymbol{\phi} \cdot \nabla \times \boldsymbol{H} d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{n} \times \boldsymbol{\phi} \cdot \boldsymbol{H} ds. \tag{24}$$

Let $\boldsymbol{V}_h^0 = \{\boldsymbol{v} \in \boldsymbol{V}_h : \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$. Now we can construct a semidiscrete mixed FEM for solving (21) and (22): For any $t \in (0, T]$, find the solution $(\boldsymbol{E}^h, \boldsymbol{H}^h) \in \boldsymbol{V}_h^0 \times \boldsymbol{U}_h$ such that

$$\left(\boldsymbol{E}_t^h, \boldsymbol{\phi}_h\right) - (\boldsymbol{H}^h, \nabla \times \boldsymbol{\phi}_h) + (\tilde{\tilde{\boldsymbol{J}}}(\boldsymbol{E}^h), \boldsymbol{\phi}_h) = (f, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \tag{25}$$

$$\left(\boldsymbol{H}_t^h, \boldsymbol{\psi}_h\right) + (\nabla \times \boldsymbol{E}^h, \boldsymbol{\psi}_h) + (\tilde{\tilde{\boldsymbol{K}}}(\boldsymbol{H}^h), \boldsymbol{\psi}_h) = (g, \boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in \boldsymbol{U}_h, \tag{26}$$

with the initial conditions

$$\boldsymbol{E}^h(\boldsymbol{x}, 0) = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}^h(\boldsymbol{x}, 0) = P_h \boldsymbol{H}_0(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega. \tag{27}$$

For this scheme, we have the following superclose result:

Theorem 3.1. *Let $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}^h, \boldsymbol{H}^h)$ be the analytic and finite element solutions of (21) and (22) and (25) and (26) at time $t \in (0, T]$, respectively. Under the regularity assumptions*

$$\boldsymbol{E}_t \in L^\infty(0, T; \boldsymbol{H}^{k+1}(\Omega)), \quad \boldsymbol{E} \in L^\infty(0, T; \boldsymbol{H}^{k+2}(\Omega)),$$

where $k \geq 1$ is the order of the basis functions in spaces $\boldsymbol{U}_h$ and $\boldsymbol{V}_h$. Then, there exists a constant $C > 0$ independent of the mesh size h but linearly dependent on T, such that

$$\|\Pi_h \boldsymbol{E} - \boldsymbol{E}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|P_h \boldsymbol{H} - \boldsymbol{H}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}$$
$$\leq Ch^{k+1} \left(\|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))} \right).$$

Proof. Denote $\xi = \Pi_h \boldsymbol{E} - \boldsymbol{E}^h$, $\eta = P_h \boldsymbol{H} - \boldsymbol{H}^h$. Choosing $\boldsymbol{\phi} = \boldsymbol{\phi}_h = \xi$ and $\boldsymbol{\psi} = \boldsymbol{\psi}_h = \eta$ in (21) and (25), (22), and (26), respectively, and rearranging the resultants, we obtain the following error equations

$$\begin{aligned}(\xi_t, \xi) - (\eta, \nabla \times \xi) + (\tilde{\tilde{\boldsymbol{J}}}(\xi), \xi) &= ((\Pi_h \boldsymbol{E} - \boldsymbol{E})_t, \xi) - (P_h \boldsymbol{H} - \boldsymbol{H}, \nabla \times \xi) \\ &\quad + (\tilde{\tilde{\boldsymbol{J}}}(\Pi_h \boldsymbol{E} - \boldsymbol{E}), \xi), \\ (\eta_t, \eta) + (\nabla \times \xi, \eta) + (\tilde{\tilde{\boldsymbol{K}}}(\eta), \eta) &= ((P_h \boldsymbol{H} - \boldsymbol{H})_t, \eta) \\ &\quad + (\nabla \times (\Pi_h \boldsymbol{E} - \boldsymbol{E}), \eta) + (\tilde{\tilde{\boldsymbol{K}}}(P_h \boldsymbol{H} - \boldsymbol{H}), \eta).\end{aligned}$$

Using the L^2-projection property, and the fact that $\nabla \times \boldsymbol{V}_h \subset \boldsymbol{U}_h$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\xi\|_0^2 + \|\eta\|_0^2\right) + (\tilde{\tilde{\boldsymbol{J}}}(\xi), \xi) + (\tilde{\tilde{\boldsymbol{K}}}(\eta), \eta) \\ &\quad \le ((\Pi_h \boldsymbol{E} - \boldsymbol{E})_t, \xi) + (\tilde{\tilde{\boldsymbol{J}}}(\Pi_h \boldsymbol{E} - \boldsymbol{E}), \xi) + (\nabla \times (\Pi_h \boldsymbol{E} - \boldsymbol{E}), \eta). \quad (28)\end{aligned}$$

By Lemma 3.2 and the Cauchy-Schwarz inequality, we have

$$((\Pi_h \boldsymbol{E} - \boldsymbol{E})_t, \xi) \le C h^{k+1} \|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} \|\xi\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},$$

and

$$\begin{aligned}(\tilde{\tilde{\boldsymbol{J}}}(\Pi_h \boldsymbol{E} - \boldsymbol{E}), \xi) &= \omega_e^2 \int_0^t e^{-\Gamma_e(t-s)} (\Pi_h \boldsymbol{E} - \boldsymbol{E}, \xi) ds \\ &\le \omega_e^2 \int_0^t e^{-\Gamma_e(t-s)} \cdot C h^{k+1} \|\boldsymbol{E}\|_{k+1} \|\xi\|_0 ds \\ &\le C h^{k+1} \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} \|\xi\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},\end{aligned}$$

where in the last step we used the estimate

$$\int_0^t e^{-\Gamma_e(t-s)} ds = \frac{1}{\Gamma_e}(1 - e^{-\Gamma_e t}) \le \frac{1}{\Gamma_e}$$

and absorbed the physical parameters ω_e and Γ_e into the generic constant C.

Similarly, by Lemma 3.1 we have

$$(\nabla \times (\Pi_h \boldsymbol{E} - \boldsymbol{E}), \eta) \le C h^{k+1} \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))} \|\eta\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}.$$

Integrating (28) from 0 to t, using the fact that $\xi(0) = \eta(0) = 0$ and substituting the above estimates into the resultant, we obtain

$$\begin{aligned}\frac{1}{2}\left(\|\xi(t)\|_0^2 + \|\eta(t)\|_0^2\right) &\le t \cdot C h^{k+1} \Big(\|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} \|\xi\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\ &\quad + \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} \|\xi\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))} \|\eta\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \Big),\end{aligned}$$

where we used the inequality

$$\int_0^t (\tilde{\tilde{\boldsymbol{J}}}(\xi), \xi)dt \geq 0, \quad \int_0^t (\tilde{\tilde{\boldsymbol{K}}}(\eta), \eta)dt \geq 0, \tag{29}$$

which results from Remark 3.1.

Finally, using the arithmetic-geometry mean inequality ($ab \leq \frac{1}{4\delta}a^2 + \delta b^2$), then taking the maximum with respect to t, we have

$$\|\xi\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\eta\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq Ch^{2(k+1)} \left(\|\boldsymbol{E}_t\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{E}\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))} \right),$$

where C is linearly dependent on T^2. The proof completes by using the triangle inequality and the standard interpolation and projection error estimates. ∎

Remark 3.1. According to [37, (1.2)], a real-valued kernel $\beta(t) \in L_1(0, T)$ is called postive-definite if for each $T > 0$, β satisfies

$$\int_0^T \phi(t) \int_0^t \beta(t-s)\phi(s)dsdt \geq 0, \quad \forall \phi \in C[0, T]. \tag{30}$$

From Plancherel's theorem, for any kernel $\beta \in L_1(0, \infty)$, (30) holds if and only if

$$\int_0^\infty \beta(t)\cos(\alpha t)dt \geq 0, \quad \forall \alpha > 0. \tag{31}$$

It is easy to check that the kernel $\beta(t) = e^{-\Gamma_e t}$ satisfies (31), hence the β in $\tilde{\tilde{\boldsymbol{J}}}$ is a positive-definite kernel.

Remark 3.2. Applying the interpolation estimates and inverse inequality to the superclose result, we can easily obtain the following optimal L^∞ error estimate

$$\|\boldsymbol{E} - \boldsymbol{E}^h\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))} + \|\boldsymbol{H} - \boldsymbol{H}^h\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))} \leq Ch^k.$$

Using the postprocessing operators $\hat{\Pi}_{2h}$ and Π_{2h} defined in [30, 35], we can achieve the following global superconvergence result:

$$\|\hat{\Pi}_{2h}\boldsymbol{E} - \boldsymbol{E}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\Pi_{2h}\boldsymbol{H} - \boldsymbol{H}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq Ch^{k+1}.$$

B. The Semidiscrete Scheme for a Pure Differential Formulation

Here, we consider a semidiscrete scheme for solving the system (8)–(11) directly. Its corresponding weak formulation is as follows: For any $t \in (0, T]$, find the solutions $\boldsymbol{E} \in H_0(\text{curl}; \Omega)$, $\boldsymbol{J} \in H(\text{curl}; \Omega)$, $\boldsymbol{H}$, and $\boldsymbol{K} \in \boldsymbol{L}^2(\Omega)$ such that

$$(\boldsymbol{E}_t, \boldsymbol{\phi}) - (\boldsymbol{H}, \nabla \times \boldsymbol{\phi}) + (\boldsymbol{J}, \boldsymbol{\phi}) = 0, \quad \forall \boldsymbol{\phi} \in H_0(\text{curl}; \Omega), \tag{32}$$

$$(\boldsymbol{H}_t, \boldsymbol{\psi}) + (\nabla \times \boldsymbol{E}, \boldsymbol{\psi}) + (\boldsymbol{K}, \boldsymbol{\psi}) = 0, \quad \forall \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega), \tag{33}$$

$$(\boldsymbol{J}_t, \tilde{\boldsymbol{\phi}}) + \Gamma_e(\boldsymbol{J}, \tilde{\boldsymbol{\phi}}) - \omega_e^2(\boldsymbol{E}, \tilde{\boldsymbol{\phi}}) = 0, \quad \forall \tilde{\boldsymbol{\phi}} \in H(\text{curl}; \Omega), \tag{34}$$

$$(\boldsymbol{K}_t, \tilde{\boldsymbol{\psi}}) + \Gamma_m(\boldsymbol{K}, \tilde{\boldsymbol{\psi}}) - \omega_m^2(\boldsymbol{H}, \tilde{\boldsymbol{\psi}}) = 0, \quad \forall \tilde{\boldsymbol{\psi}} \in \boldsymbol{L}^2(\Omega), \tag{35}$$

with the initial conditions (6) and (7).

We can construct a semidiscrete mixed method for solving (32)–(35): For any $t \in (0, T]$, find the solutions $\boldsymbol{E}^h \in \boldsymbol{V}_h^0$, $\boldsymbol{J}^h \in \boldsymbol{V}_h$, $\boldsymbol{H}^h, \boldsymbol{K}^h \in \boldsymbol{U}_h$ such that

$$\left(\boldsymbol{E}_t^h, \boldsymbol{\phi}_h\right) - (\boldsymbol{H}^h, \nabla \times \boldsymbol{\phi}_h) + (\boldsymbol{J}^h, \boldsymbol{\phi}_h) = 0, \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \tag{36}$$

$$\left(\boldsymbol{H}_t^h, \boldsymbol{\psi}_h\right) + (\nabla \times \boldsymbol{E}^h, \boldsymbol{\psi}_h) + (\boldsymbol{K}^h, \boldsymbol{\psi}_h) = 0, \quad \forall \boldsymbol{\psi}_h \in \boldsymbol{U}_h, \tag{37}$$

$$\left(\boldsymbol{J}_t^h, \tilde{\boldsymbol{\phi}}_h\right) + \Gamma_{\mathrm{e}}(\boldsymbol{J}^h, \tilde{\boldsymbol{\phi}}_h) - \omega_{\mathrm{e}}^2(\boldsymbol{E}^h, \tilde{\boldsymbol{\phi}}_h) = 0, \quad \forall \tilde{\boldsymbol{\phi}}_h \in \boldsymbol{V}_h, \tag{38}$$

$$\left(\boldsymbol{K}_t^h, \tilde{\boldsymbol{\psi}}_h\right) + \Gamma_{\mathrm{m}}(\boldsymbol{K}^h, \tilde{\boldsymbol{\psi}}_h) - \omega_{\mathrm{m}}^2(\boldsymbol{H}^h, \tilde{\boldsymbol{\psi}}_h) = 0, \quad \forall \tilde{\boldsymbol{\psi}}_h \in \boldsymbol{U}_h, \tag{39}$$

with the initial approximations

$$\boldsymbol{E}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}_h^0(\boldsymbol{x}) = P_h \boldsymbol{H}_0(\boldsymbol{x}),$$
$$\boldsymbol{J}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{J}_0(\boldsymbol{x}), \quad \boldsymbol{K}_h^0(\boldsymbol{x}) = P_h \boldsymbol{K}_0(\boldsymbol{x}).$$

For this scheme, we have the following superclose result:

Theorem 3.2. *Let $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K})$ and $(\boldsymbol{E}^h, \boldsymbol{H}^h, \boldsymbol{J}^h, \boldsymbol{K}^h)$ be the analytic and finite element solutions of (32)–(35) and (36)–(39) at time $t \in (0, T]$, respectively. Under the regularity assumptions*

$$\boldsymbol{E}_t, \boldsymbol{J}_t, \boldsymbol{J} \in L^\infty(0, T; \boldsymbol{H}^{k+1}(\Omega)), \quad \boldsymbol{E} \in L^\infty(0, T; \boldsymbol{H}^{k+2}(\Omega)),$$

where $k \geq 1$ is the order of the basis functions in spaces $\boldsymbol{U}_h$ and $\boldsymbol{V}_h$. Then, there exists a constant $C > 0$ independent of the mesh size h but linearly dependent on T, such that

$$\begin{aligned}
&\|\Pi_h \boldsymbol{E} - \boldsymbol{E}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|P_h \boldsymbol{H} - \boldsymbol{H}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad + \frac{1}{\omega_{\mathrm{e}}} \|\Pi_h \boldsymbol{J} - \boldsymbol{J}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \frac{1}{\omega_{\mathrm{m}}} \|P_h \boldsymbol{K} - \boldsymbol{K}^h\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
&\leq Ch^{k+1}\Big(\|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{J}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))} + \|\boldsymbol{J}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\Big).
\end{aligned}$$

Proof. Denote $\xi = \Pi_h \boldsymbol{E} - \boldsymbol{E}^h$, $\eta = P_h \boldsymbol{H} - \boldsymbol{H}^h$, $\tilde{\xi} = \Pi_h \boldsymbol{J} - \boldsymbol{J}^h$, and $\tilde{\eta} = P_h \boldsymbol{K} - \boldsymbol{K}^h$. Choosing $\boldsymbol{\phi} = \boldsymbol{\phi}_h = \xi$ in (32) and (36), $\boldsymbol{\psi} = \boldsymbol{\psi}_h = \eta$ in (33) and (37), $\tilde{\boldsymbol{\phi}} = \tilde{\boldsymbol{\phi}}_h = \tilde{\xi}$ in (34) and (38), and $\tilde{\boldsymbol{\psi}} = \tilde{\boldsymbol{\psi}}_h = \tilde{\eta}$ in (35) and (39), respectively, and rearranging the resultants, we obtain the error equations

$$(\xi_t, \xi) - (\eta, \nabla \times \xi) + (\tilde{\xi}, \xi) = ((\Pi_h \boldsymbol{E} - \boldsymbol{E})_t, \xi) - (P_h \boldsymbol{H} - \boldsymbol{H}, \nabla \times \xi) + (\Pi_h \boldsymbol{J} - \boldsymbol{J}, \xi),$$

$$(\eta_t, \eta) + (\nabla \times \xi, \eta) + (\tilde{\eta}, \eta) = ((P_h \boldsymbol{H} - \boldsymbol{H})_t, \eta) + (\nabla \times (\Pi_h \boldsymbol{E} - \boldsymbol{E}), \eta) + (P_h \boldsymbol{K} - \boldsymbol{K}, \eta),$$

$$(\tilde{\xi}_t, \tilde{\xi}) + \Gamma_{\mathrm{e}}(\tilde{\xi}, \tilde{\xi}) - \omega_{\mathrm{e}}^2(\xi, \tilde{\xi}) = ((\Pi_h \boldsymbol{J} - \boldsymbol{J})_t, \tilde{\xi}) + \Gamma_{\mathrm{e}}(\Pi_h \boldsymbol{J} - \boldsymbol{J}, \tilde{\xi}) - \omega_{\mathrm{e}}^2(\Pi_h \boldsymbol{E} - \boldsymbol{E}, \tilde{\xi}),$$

$$(\tilde{\eta}_t, \tilde{\eta}) + \Gamma_{\mathrm{m}}(\tilde{\eta}, \tilde{\eta}) - \omega_{\mathrm{m}}^2(\eta, \tilde{\eta}) = ((P_h \boldsymbol{K} - \boldsymbol{K})_t, \tilde{\eta}) + \Gamma_{\mathrm{m}}(P_h \boldsymbol{K} - \boldsymbol{K}, \tilde{\eta}) - \omega_{\mathrm{m}}^2(P_h \boldsymbol{H} - \boldsymbol{H}, \tilde{\eta}).$$

Dividing the last two equations by ω_e^2 and ω_m^2, respectively, then adding the above four equations together, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\xi\|_0^2+\|\eta\|_0^2+\frac{1}{\omega_e^2}\|\tilde{\xi}\|_0^2+\frac{1}{\omega_m^2}\|\tilde{\eta}\|_0^2\right)+\frac{\Gamma_e}{\omega_e^2}\|\tilde{\xi}\|_0^2+\frac{\Gamma_m}{\omega_m^2}\|\tilde{\eta}\|_0^2$$
$$=((\Pi_h\boldsymbol{E}-\boldsymbol{E})_t,\xi)+(\Pi_h\boldsymbol{J}-\boldsymbol{J},\xi)+(\nabla\times(\Pi_h\boldsymbol{E}-\boldsymbol{E}),\eta)$$
$$+\frac{1}{\omega_e^2}((\Pi_h\boldsymbol{J}-\boldsymbol{J})_t,\tilde{\xi})+\frac{\Gamma_e}{\omega_e^2}(\Pi_h\boldsymbol{J}-\boldsymbol{J},\tilde{\xi})-(\Pi_h\boldsymbol{E}-\boldsymbol{E},\tilde{\xi}), \quad (40)$$

where we used the L^2-projection property in the above derivation.

Using Lemmas 3.1 and 3.2 and the Cauchy-Schwarz inequality, we have

$$((\Pi_h\boldsymbol{E}-\boldsymbol{E})_t,\xi)\le Ch^{k+1}\|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\xi\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},$$
$$(\Pi_h\boldsymbol{J}-\boldsymbol{J},\xi)\le Ch^{k+1}\|\boldsymbol{J}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\xi\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},$$
$$(\nabla\times(\Pi_h\boldsymbol{E}-\boldsymbol{E}),\eta)\le Ch^{k+1}\|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))}\|\eta\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},$$
$$\frac{1}{\omega_e^2}((\Pi_h\boldsymbol{J}-\boldsymbol{J})_t,\tilde{\xi})\le\frac{1}{\omega_e^2}\cdot Ch^{k+1}\|\boldsymbol{J}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\tilde{\xi}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},$$
$$\frac{\Gamma_e}{\omega_e^2}(\Pi_h\boldsymbol{J}-\boldsymbol{J},\tilde{\xi})\le\frac{\Gamma_e}{\omega_e^2}\cdot Ch^{k+1}\|\boldsymbol{J}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\tilde{\xi}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))},$$
$$(\Pi_h\boldsymbol{E}-\boldsymbol{E},\tilde{\xi})\le Ch^{k+1}\|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\tilde{\xi}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}.$$

Substituting the above estimates into (40) and following the same technique as we used in Theorem 3.1, we obtain

$$\|\xi\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\eta\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\frac{1}{\omega_e^2}\|\tilde{\xi}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\frac{1}{\omega_m^2}\|\tilde{\eta}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}$$
$$\le Ch^{2(k+1)}\Big(\|\boldsymbol{E}_t\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}+\|\boldsymbol{J}\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}$$
$$+\|\boldsymbol{E}\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))}+\|\boldsymbol{J}_t\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\Big),$$

where the constant C is linearly dependent on T^2. ■

Remark 3.3. The same remark as Remark 3.2 holds true for this case.

IV. 3D SUPERCLOSE ANALYSIS FOR FULLY DISCRETE SCHEMES

To define our fully discrete scheme, we divide the time interval $(0,T)$ into M uniform subintervals by points $0=t_0<t_1<\cdots<t_M=T$, where $t_m=m\tau$, and denote the m-th subinterval by $I_m=[t_{m-1},t_m]$. Moreover, we define $\boldsymbol{u}^m=\boldsymbol{u}(\cdot,m\tau)$ for $0\le m\le M$, and the notation:

$$\delta_\tau\boldsymbol{u}^m=\frac{\boldsymbol{u}^m-\boldsymbol{u}^{m-1}}{\tau},\quad \bar{\boldsymbol{u}}^m=\frac{1}{2}(\boldsymbol{u}^m+\boldsymbol{u}^{m-1}).$$

Now we can formulate the Crank-Nicolson mixed finite element scheme for solving (32)–(35): For $m = 1, 2, \ldots, M$, find $\boldsymbol{E}_h^m \in \boldsymbol{V}_h^0$, $\boldsymbol{J}_h^m \in \boldsymbol{V}_h$, $\boldsymbol{H}_h^m, \boldsymbol{K}_h^m \in \boldsymbol{U}_h$ such that

$$\left(\delta_\tau \boldsymbol{E}_h^m, \boldsymbol{\phi}_h\right) - \left(\bar{\boldsymbol{H}}_h^m, \nabla \times \boldsymbol{\phi}_h\right) + \left(\bar{\boldsymbol{J}}_h^m, \boldsymbol{\phi}_h\right) = 0, \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \tag{41}$$

$$\left(\delta_\tau \boldsymbol{H}_h^m, \boldsymbol{\psi}_h\right) + \left(\nabla \times \bar{\boldsymbol{E}}_h^m, \boldsymbol{\psi}_h\right) + \left(\bar{\boldsymbol{K}}_h^m, \boldsymbol{\psi}_h\right) = 0, \quad \forall \boldsymbol{\psi}_h \in \boldsymbol{U}_h, \tag{42}$$

$$\left(\delta_\tau \boldsymbol{J}_h^m, \tilde{\boldsymbol{\phi}}_h\right) + \Gamma_{\mathrm{e}}\left(\bar{\boldsymbol{J}}_h^m, \tilde{\boldsymbol{\phi}}_h\right) - \omega_{\mathrm{e}}^2\left(\bar{\boldsymbol{E}}_h^m, \tilde{\boldsymbol{\phi}}_h\right) = 0, \quad \forall \tilde{\boldsymbol{\phi}}_h \in \boldsymbol{V}_h, \tag{43}$$

$$\left(\delta_\tau \boldsymbol{K}_h^m, \tilde{\boldsymbol{\psi}}_h\right) + \Gamma_{\mathrm{m}}\left(\bar{\boldsymbol{K}}_h^m, \tilde{\boldsymbol{\psi}}_h\right) - \omega_{\mathrm{m}}^2\left(\bar{\boldsymbol{H}}_h^m, \tilde{\boldsymbol{\psi}}_h\right) = 0, \quad \forall \tilde{\boldsymbol{\psi}}_h \in \boldsymbol{U}_h, \tag{44}$$

subject to the initial approximations

$$\boldsymbol{E}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}_h^0(\boldsymbol{x}) = P_h \boldsymbol{H}_0(\boldsymbol{x}), \tag{45}$$

$$\boldsymbol{J}_h^0(\boldsymbol{x}) = \Pi_h \boldsymbol{J}_0(\boldsymbol{x}), \quad \boldsymbol{K}_h^0(\boldsymbol{x}) = P_h \boldsymbol{K}_0(\boldsymbol{x}). \tag{46}$$

For this fully discrete scheme, we have the following superclose result:

Theorem 4.1. *Let ($\boldsymbol{E}^m, \boldsymbol{H}^m, \boldsymbol{J}^m$, and $\boldsymbol{K}^m$) and ($\boldsymbol{E}_h^m, \boldsymbol{H}_h^m, \boldsymbol{J}_h^m$, and $\boldsymbol{K}_h^m$) be the analytic and finite element solutions of (32)–(35) and (41)–(44) at time t_{m}, respectively. Under the regularity assumptions*

$$\boldsymbol{E}_t, \boldsymbol{J}_t, \boldsymbol{J} \in L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega)), \quad \boldsymbol{E} \in L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega)),$$
$$\boldsymbol{E}_{tt}, \boldsymbol{H}_{tt}, \boldsymbol{J}_{tt}, \boldsymbol{K}_{tt}, \nabla \times \boldsymbol{E}_{tt}, \nabla \times \boldsymbol{H}_{tt} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega)),$$

where $k \geq 1$ is the order of the basis functions in spaces $\boldsymbol{U}_h$ and $\boldsymbol{V}_h$. Then, there exists a constant $C > 0$ independent of the mesh size h but linearly dependent on T, such that

$$\begin{aligned}
&\max_{1\le m\le M}\left(\left\|\Pi_h \boldsymbol{E}^m - \boldsymbol{E}_h^m\right\|_0 + \left\|P_h \boldsymbol{H}^m - \boldsymbol{H}_h^m\right\|_0 + \left\|\Pi_h \boldsymbol{J}^m - \boldsymbol{J}_h^m\right\|_0 + \left\|P_h \boldsymbol{K}^m - \boldsymbol{K}_h^m\right\|_0\right)\\
&\le Ch^{k+1}\Big(\|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{J}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))} + \|\boldsymbol{J}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\Big)\\
&\quad + C\tau^2\Big(\|\nabla \times \boldsymbol{H}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{J}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\nabla \times \boldsymbol{E}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\
&\qquad + \|\boldsymbol{K}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{E}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{H}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\Big).
\end{aligned}$$

Proof. Integrating (32)–(35) in time over I_{m} and dividing all by τ, we have

$$(\delta_\tau \boldsymbol{E}^m, \boldsymbol{\phi}) - \left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{H}(s)ds, \nabla \times \boldsymbol{\phi}\right) + \left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{J}(s)ds, \boldsymbol{\phi}\right) = 0, \tag{47}$$

$$(\delta_\tau \boldsymbol{H}^m, \boldsymbol{\psi}) + \left(\nabla \times \frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{E}(s)ds, \boldsymbol{\psi}\right) + \left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{K}(s)ds, \boldsymbol{\psi}\right) = 0, \tag{48}$$

$$(\delta_\tau \boldsymbol{J}^m, \tilde{\boldsymbol{\phi}}) + \Gamma_{\mathrm{e}}\left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{J}(s)ds, \tilde{\boldsymbol{\phi}}\right) - \omega_{\mathrm{e}}^2\left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{E}(s)ds, \tilde{\boldsymbol{\phi}}\right) = 0, \tag{49}$$

$$(\delta_\tau \boldsymbol{K}^m, \tilde{\boldsymbol{\psi}}) + \Gamma_{\mathrm{m}}\left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{K}(s)ds, \tilde{\boldsymbol{\psi}}\right) - \omega_{\mathrm{m}}^2\left(\frac{1}{\tau}\int_{I_{\mathrm{m}}} \boldsymbol{H}(s)ds, \tilde{\boldsymbol{\psi}}\right) = 0. \tag{50}$$

Denote $\xi_h^m = \Pi_h \boldsymbol{E}^m - \boldsymbol{E}_h^m, \eta_h^m = P_h \boldsymbol{H}^m - \boldsymbol{H}_h^m, \tilde{\xi}_h^m = \Pi_h \boldsymbol{J}^m - \boldsymbol{J}_h^m$, and $\tilde{\eta}_h^m = P_h \boldsymbol{K}^m - \boldsymbol{K}_h^m$. Subtracting (41)–(44) from (47)–(50) with $\boldsymbol{\phi} = \boldsymbol{\phi}_h$, $\boldsymbol{\psi} = \boldsymbol{\psi}_h$, $\tilde{\boldsymbol{\phi}} = \tilde{\boldsymbol{\phi}}_h$, and $\tilde{\boldsymbol{\psi}} = \tilde{\boldsymbol{\psi}}_h$, we can obtain the error equations

$$
\begin{aligned}
i.\quad & \left(\delta_\tau \xi_h^m, \boldsymbol{\phi}_h\right) - \left(\bar{\eta}_h^m, \nabla \times \boldsymbol{\phi}_h\right) + \left(\bar{\tilde{\xi}}_h^m, \boldsymbol{\phi}_h\right) \\
& = (\delta_\tau(\Pi_h \boldsymbol{E}^m - \boldsymbol{E}^m), \boldsymbol{\phi}_h) - \left(P_h \bar{\boldsymbol{H}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{H}(s)ds, \nabla \times \boldsymbol{\phi}_h\right) \\
& + \left(\Pi_h \bar{\boldsymbol{J}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{J}(s)ds, \boldsymbol{\phi}_h\right), \\
ii.\quad & \left(\delta_\tau \eta_h^m, \boldsymbol{\psi}_h\right) + \left(\nabla \times \bar{\xi}_h^m, \boldsymbol{\psi}_h\right) + \left(\bar{\tilde{\eta}}_h^m, \boldsymbol{\psi}_h\right) \\
& = (\delta_\tau(P_h \boldsymbol{H}^m - \boldsymbol{H}^m), \boldsymbol{\psi}_h) + \left(\nabla \times \left(\Pi_h \bar{\boldsymbol{E}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{E}(s)ds\right), \boldsymbol{\psi}_h\right) \\
& + \left(P_h \bar{\boldsymbol{K}}_h^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{K}(s)ds, \boldsymbol{\psi}_h\right), \\
iii.\quad & \left(\delta_\tau \tilde{\xi}_h^m, \tilde{\boldsymbol{\phi}}_h\right) + \Gamma_\mathrm{e}\left(\bar{\tilde{\xi}}_h^m, \tilde{\boldsymbol{\phi}}_h\right) - \omega_\mathrm{e}^2\left(\bar{\xi}_h^m, \tilde{\boldsymbol{\phi}}_h\right) \\
& = \left(\delta_\tau(\Pi_h \boldsymbol{J}^m - \boldsymbol{J}^m), \tilde{\boldsymbol{\phi}}_h\right) + \Gamma_\mathrm{e}\left(\Pi_h \bar{\boldsymbol{J}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{J}(s)ds, \tilde{\boldsymbol{\phi}}_h\right) \\
& - \omega_\mathrm{e}^2\left(\Pi_h \bar{\boldsymbol{E}}_h^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{E}(s)ds, \tilde{\boldsymbol{\phi}}_h\right), \\
iv.\quad & \left(\delta_\tau \tilde{\eta}_h^m, \tilde{\boldsymbol{\psi}}_h\right) + \Gamma_\mathrm{m}\left(\bar{\tilde{\eta}}_h^m, \tilde{\boldsymbol{\psi}}_h\right) - \omega_\mathrm{m}^2\left(\bar{\eta}_h^m, \tilde{\boldsymbol{\psi}}_h\right) \\
& = \left(\delta_\tau(P_h \boldsymbol{K}^m - \boldsymbol{K}^m), \tilde{\boldsymbol{\psi}}_h\right) + \Gamma_\mathrm{m}\left(P_h \bar{\boldsymbol{K}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{K}(s)ds, \tilde{\boldsymbol{\psi}}_h\right) \\
& - \omega_\mathrm{m}^2\left(P_h \bar{\boldsymbol{H}}_h^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{H}(s)ds, \tilde{\boldsymbol{\psi}}_h\right).
\end{aligned}
$$

Choosing $\phi_h = \tau\bar{\xi}_h^m, \psi_h = \tau\bar{\eta}_h^m, \tilde{\phi}_h = \tau\bar{\tilde{\xi}}_h^m$, and $\tilde{\psi}_h = \tau\bar{\tilde{\eta}}_h^m$ in the above error equations, dividing the last two equations by ω_e^2 and ω_m^2, adding the resultants together, and using the property of operator P_h, we obtain

$$
\begin{aligned}
& \frac{1}{2}\left[\|\xi_h^m\|_0^2 - \|\xi_h^{m-1}\|_0^2 + \|\eta_h^m\|_0^2 - \|\eta_h^{m-1}\|_0^2 + \frac{1}{\omega_\mathrm{e}^2}\left(\|\tilde{\xi}_h^m\|_0^2 - \|\tilde{\xi}_h^{m-1}\|_0^2\right) + \frac{1}{\omega_\mathrm{m}^2}\left(\|\tilde{\eta}_h^m\|_0^2 - \|\tilde{\eta}_h^{m-1}\|_0^2\right)\right] \\
& \quad \le \tau\left(\delta_\tau(\Pi_h \boldsymbol{E}^m - \boldsymbol{E}^m), \bar{\xi}_h^m\right) - \tau\left(\bar{\boldsymbol{H}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{H}(s)ds, \nabla \times \bar{\xi}_h^m\right) \\
& \quad + \tau\left(\Pi_h \bar{\boldsymbol{J}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{J}(s)ds, \bar{\tilde{\xi}}_h^m\right) + \tau\left(\nabla \times \left(\Pi_h \bar{\boldsymbol{E}}^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{E}(s)ds\right), \bar{\eta}_h^m\right) \\
& \quad + \tau\left(\bar{\boldsymbol{K}}_h^m - \frac{1}{\tau}\int_{I_\mathrm{m}} \boldsymbol{K}(s)ds, \bar{\tilde{\eta}}_h^m\right) + \frac{\tau}{\omega_\mathrm{e}^2}\left(\delta_\tau(\Pi_h \boldsymbol{J}^m - \boldsymbol{J}^m), \bar{\tilde{\xi}}_h^m\right)
\end{aligned}
$$

$$+\frac{\tau\Gamma_{\rm e}}{\omega_{\rm e}^2}\left(\Pi_h\bar{\boldsymbol{J}}^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{J}(s)ds,\bar{\bar{\xi}}_h^m\right)-\tau\left(\Pi_h\bar{\boldsymbol{E}}_h^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{E}(s)ds,\bar{\bar{\xi}}_h^m\right)$$
$$+\frac{\tau\Gamma_{\rm m}}{\omega_{\rm m}^2}\left(\bar{\boldsymbol{K}}^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{K}(s)ds,\bar{\bar{\eta}}_h^m\right)-\tau\left(\bar{\boldsymbol{H}}_h^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{H}(s)ds,\bar{\bar{\eta}}_h^m\right)$$
$$=\sum_{i=1}^{10}{\rm Err}_i. \tag{51}$$

Now we just need to estimate all $\mathrm{Err}_i, i = 1,\dots,10$. By the Mean-Value Theorem and Lemma 3.2, we have

$$\begin{aligned}{\rm Err}_1&=\tau\big(\delta_\tau(\Pi_h\boldsymbol{E}^m-\boldsymbol{E}^m),\bar{\xi}_h^m\big)=\tau\big((\Pi_h\boldsymbol{E}-\boldsymbol{E})_t(\boldsymbol{x},t_{\rm m}-\theta\tau),\bar{\xi}_h^m\big)\\&\le\tau\cdot Ch^{k+1}\|\boldsymbol{E}_t(\cdot,t_{\rm m}-\theta\tau)\|_{k+1}\left\|\bar{\xi}_h^m\right\|_0\le\tau\cdot Ch^{k+1}\|\boldsymbol{E}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\xi_h\|_{l^\infty(L^2)},\end{aligned}$$

where we introduced the notation

$$\|\xi_h\|_{l^\infty(L^2)}=\max_{1\le m\le M}\left\|\xi_h^m\right\|_0.$$

Using integration by parts and the formula [18, Lemma 5.1]

$$\left\|\frac{1}{2}(\boldsymbol{u}(\cdot,t_{m-1})+\boldsymbol{u}(\cdot,t_{\rm m}))-\frac{1}{\tau}\int_{t_{m-1}}^{t_{\rm m}}\boldsymbol{u}(t)dt\right\|_0^2\le\frac{\tau^3}{4}\int_{t_{m-1}}^{t_{\rm m}}\|\boldsymbol{u}_{tt}(t)\|_0^2dt\quad\forall\boldsymbol{u}\in H^2(0,T;\boldsymbol{L}^2(\Omega)), \tag{52}$$

we obtain

$$\begin{aligned}{\rm Err}_2&=\tau\left(\bar{\boldsymbol{H}}^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{H}(s)ds,\nabla\times\bar{\xi}_h^m\right)=\tau\left(\nabla\times\bar{\boldsymbol{H}}^m-\frac{1}{\tau}\int_{I_{\rm m}}\nabla\times\boldsymbol{H}(s)ds,\bar{\xi}_h^m\right)\\&\le\tau\left(\frac{\tau^3}{4}\int_{I_{\rm m}}\|\nabla\times\boldsymbol{H}_{tt}(s)\|_0^2ds\right)^{1/2}\left\|\bar{\xi}_h^m\right\|_0\\&\le\frac{\tau^3}{2}\|\nabla\times\boldsymbol{H}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\xi_h\|_{l^\infty(L^2)}.\end{aligned}$$

Using the inequality (52) and Lemma 3.2, we have

$$\begin{aligned}{\rm Err}_3&=\tau\left(\Pi_h\bar{\boldsymbol{J}}^m-\bar{\boldsymbol{J}}^m+\bar{\boldsymbol{J}}^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{J}(s)ds,\bar{\xi}_h^m\right)\\&\le\left(\tau\cdot Ch^{k+1}\|\boldsymbol{J}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}+\frac{\tau^3}{2}\|\boldsymbol{J}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\right)\|\xi_h\|_{l^\infty(L^2)}.\end{aligned}$$

By similar arguments, we have

$$\begin{aligned}{\rm Err}_4&=\tau\left(\nabla\times(\Pi_h\bar{\boldsymbol{E}}^m-\bar{\boldsymbol{E}}^m+\bar{\boldsymbol{E}}^m-\frac{1}{\tau}\int_{I_{\rm m}}\boldsymbol{E}(s)ds),\bar{\eta}_h^m\right)\\&\le\left(\tau\cdot Ch^{k+1}\|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))}+\frac{\tau^3}{2}\|\nabla\times\boldsymbol{E}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\right)\|\eta_h\|_{l^\infty(L^2)},\end{aligned}$$

and

$$\mathrm{Err}_5 = \tau\left(\bar{\boldsymbol{K}}_h^m - \frac{1}{\tau}\int_{I_m}\boldsymbol{K}(s)ds, \bar{\eta}_h^m\right) \le \frac{\tau^3}{2}\|\boldsymbol{K}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\eta_h\|_{l^\infty(L^2)}.$$

Similar to Err_1, we have

$$\mathrm{Err}_6 = \frac{\tau}{\omega_e^2}\left(\delta_\tau(\Pi_h\boldsymbol{J}^m - \boldsymbol{J}^m), \bar{\tilde{\xi}}_h^m\right) \le \frac{\tau}{\omega_e^2}\cdot Ch^{k+1}\|\boldsymbol{J}_t\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\|\xi_h\|_{l^\infty(L^2)}.$$

Similar to Err_3, we have

$$\begin{aligned}\mathrm{Err}_7 &= \frac{\tau\Gamma_e}{\omega_e^2}\left(\Pi_h\bar{\boldsymbol{J}}^m - \bar{\boldsymbol{J}}^m + \bar{\boldsymbol{J}}^m - \frac{1}{\tau}\int_{I_m}\boldsymbol{J}(s)ds, \bar{\tilde{\xi}}_h^m\right)\\ &\le \frac{\Gamma_e}{\omega_e^2}\left(\tau\cdot Ch^{k+1}\|\boldsymbol{J}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \frac{\tau^3}{2}\|\boldsymbol{J}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\right)\|\tilde{\xi}_h\|_{l^\infty(L^2)},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Err}_8 &= \tau\left(\Pi_h\bar{\boldsymbol{E}}_h^m - \bar{\boldsymbol{E}}_h^m + \bar{\boldsymbol{E}}_h^m - \frac{1}{\tau}\int_{I_m}\boldsymbol{E}(s)ds, \bar{\tilde{\xi}}_h^m\right)\\ &\le \left(\tau\cdot Ch^{k+1}\|\boldsymbol{E}\|_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \frac{\tau^3}{2}\|\boldsymbol{E}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\right)\|\tilde{\xi}_h\|_{l^\infty(L^2)}.\end{aligned}$$

Similar to Err_5, we have

$$\mathrm{Err}_9 = \frac{\tau\Gamma_m}{\omega_m^2}\left(\bar{\boldsymbol{K}}^m - \frac{1}{\tau}\int_{I_m}\boldsymbol{K}(s)ds, \bar{\tilde{\eta}}_h^m\right) \le \frac{\Gamma_m}{\omega_m^2}\cdot\frac{\tau^3}{2}\|\boldsymbol{K}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\tilde{\eta}_h\|_{l^\infty(L^2)},$$

and

$$\mathrm{Err}_{10} = \tau\left(\bar{\boldsymbol{H}}_h^m - \frac{1}{\tau}\int_{I_m}\boldsymbol{H}(s)ds, \bar{\tilde{\eta}}_h^m\right) \le \frac{\tau^3}{2}\|\boldsymbol{H}_{tt}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\tilde{\eta}_h\|_{l^\infty(L^2)}.$$

Substituting the above estimates into (51), summing up the result from $m = 1$ to any $n \le M$ with the fact that

$$\xi_h^0 = \eta_h^0 = \tilde{\xi}_h^0 = \tilde{\eta}_h^0 = 0,$$

then using the arithmetic-geometric mean inequality and taking the maximum with respect to n, we obtain

$$\begin{aligned}&\|\xi_h\|^2_{l^\infty(L^2)} + \|\eta_h\|^2_{l^\infty(L^2)} + \|\tilde{\xi}_h\|^2_{l^\infty(L^2)} + \|\tilde{\eta}_h\|^2_{l^\infty(L^2)}\\ &\quad\le Ch^{2(k+1)}\Big(\|\boldsymbol{E}_t\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{J}\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))} + \|\boldsymbol{E}\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+2}(\Omega))}\\ &\qquad + \|\boldsymbol{J}_t\|^2_{L^\infty(0,T;\boldsymbol{H}^{k+1}(\Omega))}\Big) + C\tau^4\Big(\|\nabla\times\boldsymbol{H}_{tt}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{J}_{tt}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\ &\qquad + \|\nabla\times\boldsymbol{E}_{tt}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{K}_{tt}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{E}_{tt}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{H}_{tt}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\Big),\end{aligned}$$

which concludes the proof. Note that C linearly depends on T^2. ■

Remark 4.1. Similarly, we can formulate the leap-frog mixed finite element scheme for solving (32)–(35): Given initial approximations $\boldsymbol{E}_h^0, \boldsymbol{K}_h^0, \boldsymbol{H}_h^{\frac{1}{2}}$, and $\boldsymbol{J}_h^{\frac{1}{2}}$, for $m = 1, 2, \ldots$, find $\boldsymbol{E}_h^m \in \boldsymbol{V}_h^0$, $\boldsymbol{J}_h^{m+\frac{1}{2}} \in \boldsymbol{V}_h$, $\boldsymbol{H}_h^{m+\frac{1}{2}}$, and $\boldsymbol{K}_h^m \in \boldsymbol{U}_h$ such that

$$\left(\frac{\boldsymbol{E}_h^m - \boldsymbol{E}_h^{m-1}}{\tau}, \boldsymbol{\phi}_h\right) - \left(\boldsymbol{H}_h^{m-\frac{1}{2}}, \nabla \times \boldsymbol{\phi}_h\right) + \left(\boldsymbol{J}_h^{m-\frac{1}{2}}, \boldsymbol{\phi}_h\right) = 0, \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h^0, \tag{53}$$

$$\left(\frac{\boldsymbol{H}_h^{m+\frac{1}{2}} - \boldsymbol{H}_h^{m-\frac{1}{2}}}{\tau}, \boldsymbol{\psi}_h\right) + \left(\nabla \times \boldsymbol{E}_h^m, \boldsymbol{\psi}_h\right) + \left(\boldsymbol{K}_h^m, \boldsymbol{\psi}_h\right) = 0, \quad \forall \boldsymbol{\psi}_h \in \boldsymbol{U}_h, \tag{54}$$

$$\left(\frac{\boldsymbol{J}_h^{m+\frac{1}{2}} - \boldsymbol{J}_h^{m-\frac{1}{2}}}{\tau}, \tilde{\boldsymbol{\phi}}_h\right) + \Gamma_{\mathrm{e}}\left(\frac{1}{2}\left(\boldsymbol{J}_h^{m+\frac{1}{2}} + \boldsymbol{J}_h^{m-\frac{1}{2}}\right), \tilde{\boldsymbol{\phi}}_h\right) - \omega_{\mathrm{e}}^2\left(\boldsymbol{E}_h^m, \tilde{\boldsymbol{\phi}}_h\right) = 0, \quad \forall \tilde{\boldsymbol{\phi}}_h \in \boldsymbol{V}_h, \tag{55}$$

$$\left(\frac{\boldsymbol{K}_h^m - \boldsymbol{K}_h^{m-1}}{\tau}, \tilde{\boldsymbol{\psi}}_h\right) + \Gamma_{\mathrm{m}}\left(\frac{1}{2}\left(\boldsymbol{K}_h^m + \boldsymbol{K}_h^{m-1}\right), \tilde{\boldsymbol{\psi}}_h\right) - \omega_{\mathrm{m}}^2\left(\boldsymbol{H}_h^{m-\frac{1}{2}}, \tilde{\boldsymbol{\psi}}_h\right) = 0, \quad \forall \tilde{\boldsymbol{\psi}}_h \in \boldsymbol{U}_h, \tag{56}$$

subject to the initial approximations (45) and (46). Combining the above proof techniques with that developed for the leap-frog scheme [18], we can obtain the following superclose result:

$$\max_{1\le m}\left(\left\|\Pi_h \boldsymbol{E}^m - \boldsymbol{E}_h^m\right\|_0 + \left\|P_h \boldsymbol{H}^{m+\frac{1}{2}} - \boldsymbol{H}_h^{m+\frac{1}{2}}\right\|_0 + \left\|\Pi_h \boldsymbol{J}^{m+\frac{1}{2}} - \boldsymbol{J}_h^{m+\frac{1}{2}}\right\|_0 + \left\|P_h \boldsymbol{K}^m - \boldsymbol{K}_h^m\right\|_0\right)$$
$$\le C(\tau^2 + h^{k+1}).$$

V. POINTWISE SUPERCONVERGENCE FOR 2D RECTANGULAR EDGE ELEMENTS

In this section, we first show that the superclose results obtained for cubic elements can be extended to rectangular elements in a two-dimensional domain. Then, we prove the pointwise superconvergence at the element centers for the lowest order edge element. Note that in the 2D Maxwell's equations, one field is a vector, whereas the other one has to be a scalar. Without loss of generality, we assume that the electrical field $\boldsymbol{E}$ is a vector, whereas the magnetic field H is a scalar. To make the extension clearly, we define the 2D curl operators:

$$\nabla \times H = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right)', \quad \nabla \times \boldsymbol{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}, \quad \forall \boldsymbol{E} \equiv (E_1, E_2). \tag{57}$$

For a 2D domain Ω, we partition it by a family of regular rectangular meshes T^h with maximum mesh size h. The corresponding Nédélec rectangular elements can be defined as: For any $k \ge 1$,

$$U_h = \{\psi_h \in L^2(\Omega): \quad \psi_h|_K \in Q_{k-1,k-1}, \quad \forall K \in T^h\},$$
$$\boldsymbol{V}_h = \{\boldsymbol{\phi}_h \in H(\mathrm{curl}; \Omega): \quad \boldsymbol{\phi}_h|_K \in Q_{k-1,k} \times Q_{k,k-1}, \quad \forall K \in T^h\}.$$

Here, $Q_{i,j}$ denotes the space of polynomials whose degrees are less than or equal to i, j in variables x, y, respectively. It is easy to see that $\nabla \times \boldsymbol{V}_h \subset U_h$ still holds.

For the 2D case, we need to modify the definition for the operator $\Pi_h \boldsymbol{E} \in \mathbf{V}_h$ as follows:

$$\int_{l_i} (\boldsymbol{E} - \Pi_h \boldsymbol{E}) \cdot \mathbf{t} q dl = 0, \quad \forall q \in P_{k-1}(l_i), i = 1, \ldots, 4, \tag{58}$$

$$\int_K (\boldsymbol{E} - \Pi_h \boldsymbol{E}) \cdot \boldsymbol{q} dxdy = 0, \quad \forall \boldsymbol{q} \in Q_{k-1,k-2} \times Q_{k-2,k-1}, \tag{59}$$

where l_i are the edges of an element K, $\mathbf{t}$ is the unit tangent vector along the edge l_i. When $k = 1$, $\Pi_h \boldsymbol{E}$ is defined by (58) only.

The 2D superclose analysis depends on the following fundamental results.

Lemma 5.1. *For any $\boldsymbol{u} \in H(curl; K)$ and $q \in Q_{k-1,k-1}(K), k \geq 1$, we have*

$$\int_K \nabla \times (\boldsymbol{u} - \Pi_h \boldsymbol{u}) \cdot q dxdy = 0.$$

Proof. The proof follows from the Stokes' formula

$$\int_K \nabla \times (\boldsymbol{u} - \Pi_h \boldsymbol{u}) \cdot q dxdy = \int_{\partial K} (\boldsymbol{u} - \Pi_h \boldsymbol{u}) \cdot \mathbf{t} q dl + \int_K (\boldsymbol{u} - \Pi_h \boldsymbol{u}) \cdot (\nabla \times q) dxdy$$

and the property (58) and (59) for the operator Π_h. ■

Let P_h be the L^2-projection operator onto the space U_h. Then, we have

Lemma 5.2. *For any $w \in L^2(K)$ and $\boldsymbol{\phi}|_K \in Q_{k-1,k} \times Q_{k,k-1}, k \geq 1$, we have*

$$\int_K (w - P_h w) \cdot \nabla \times \boldsymbol{\phi} dxdy = 0.$$

Lemma 5.3. *Let $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$ be an arbitrary rectangular element. Then for any $\boldsymbol{u} \in H(curl; K)$ and $\boldsymbol{\phi}|_K \in Q_{k-1,k} \times Q_{k,k-1}, k \geq 1$, we have*

$$\int_K (u_1 - (\Pi_h \boldsymbol{u})_1) \phi_1 dxdy = O\left(h_y^{k+1}\right) \left\| \partial_y^{k+1} u_1 \right\|_{0,K} \|\phi_1\|_{0,K}, \tag{60}$$

$$\int_K (u_2 - (\Pi_h \boldsymbol{u})_2) \phi_2 dxdy = O\left(h_x^{k+1}\right) \left\| \partial_x^{k+1} u_2 \right\|_{0,K} \|\phi_2\|_{0,K}, \tag{61}$$

where u_1, u_2 and ϕ_1, ϕ_2 are the two components of $\boldsymbol{u}$ and $\boldsymbol{\phi}$, respectively. Hence, we have

$$\int_K (\boldsymbol{u} - \Pi_h \boldsymbol{u}) \cdot \boldsymbol{\phi} dxdy = O(h^{k+1}) \|\boldsymbol{u}\|_{k+1,K} \|\boldsymbol{\phi}\|_{0,K}.$$

Proof. Note that

$$\int_K (\boldsymbol{u} - \Pi_h \boldsymbol{u}) \cdot \boldsymbol{\phi} dxdy = \int_K (u_1 - (\Pi_h \boldsymbol{u})_1) \phi_1 dxdy + \int_K (u_2 - (\Pi_h \boldsymbol{u})_2) \phi_2 dxdy.$$

Hence, we just need to consider the first inner product.

i. First, let us consider the $k = 1$ case. In this case, $\phi_1 \in Q_{0,1}$, by the Taylor expansion, we obtain

$$\int_K (u_1 - (\Pi_h \boldsymbol{u})_1)\phi_1 dxdy = \int_K (u_1 - (\Pi_h \boldsymbol{u})_1)[\phi_1(x_c, y_c) + (y - y_c)\partial_y \phi_1(x_c, y_c)]dxdy. \quad (62)$$

Denote the functions

$$A(x) = \frac{1}{2}\left[(x - x_c)^2 - h_x^2\right], \quad B(y) = \frac{1}{2}\left[(y - y_c)^2 - h_y^2\right]. \quad (63)$$

Note that in the proof below we will constantly use the facts that:

$$A(x) = 0 \quad \text{on } x = x_c \pm h_x, \quad B(y) = 0 \quad \text{on } y = y_c \pm h_y. \quad (64)$$

Using integration by parts and the identity $\partial_{yy}B(y) = 1$, (58) and (64), we have

$$\begin{aligned}
&\int_K (u_1 - (\Pi_h \boldsymbol{u})_1)dxdy = \int_K (u_1 - (\Pi_h \boldsymbol{u})_1)\partial_{yy}B(y)dxdy \\
&= \int_{x=x_c-h_x}^{x_c+h_x} (u_1 - (\Pi_h \boldsymbol{u})_1)\partial_y B(y)|_{y=y_c-h_y}^{y_c+h_y} dx - \int_K (u_1 - (\Pi_h \boldsymbol{u})_1)_y \partial_y B(y)dxdy \\
&= \int_K (u_1 - (\Pi_h \boldsymbol{u})_1)_{yy} B(y)dxdy = \int_K \partial_{yy}u_1 \cdot B(y)dxdy,
\end{aligned}$$

where in the last step we used the fact that $(\Pi_h \boldsymbol{u})_1 \in Q_{0,1}$.

Similarly, by the identity $y - y_c = \frac{1}{6}\partial_y^3(B^2(y))$ and integration by parts, we obtain

$$\begin{aligned}
&\int_K (u_1 - (\Pi_h \boldsymbol{u})_1)(y - y_c)dxdy = \int_K (u_1 - (\Pi_h \boldsymbol{u})_1) \cdot \frac{1}{6}\partial_y^3(B^2(y))dxdy \\
&= \int_{x=x_c-h_x}^{x_c+h_x} (u_1 - (\Pi_h \boldsymbol{u})_1) \cdot \frac{1}{6}\partial_y^2(B^2(y))\Big|_{y=y_c-h_y}^{y_c+h_y} dx - \int_K (u_1 - (\Pi_h \boldsymbol{u})_1)_y \cdot \frac{1}{6}\partial_y^2(B^2(y))dxdy \\
&= \int_K (u_1 - (\Pi_h \boldsymbol{u})_1)_{yy} \cdot \frac{1}{6}\partial_y(B^2(y))dxdy = \int_K \partial_{yy}u_1 \cdot \frac{1}{6}(B^2(y))_y dxdy.
\end{aligned}$$

Substituting the above integral identities into (62) and using the inverse estimate, we have

$$\begin{aligned}
&\int_K (u_1 - (\Pi_h \boldsymbol{u})_1)\phi_1 dxdy \\
&= \int_K \partial_{yy}u_1 \cdot B(y) \cdot \phi_1(x_c, y_c)dxdy + \int_K \partial_{yy}u_1 \cdot \frac{1}{6}(B^2(y))_y \cdot \partial_y \phi_1(x_c, y_c)dxdy \\
&= \int_K \partial_{yy}u_1 \cdot B(y) \cdot [\phi_1(x, y) - (y - y_c)\partial_y \phi_1(x, y)]dxdy \\
&\quad + \int_K \partial_{yy}u_1 \cdot \frac{1}{3}B(y) \cdot (y - y_c)\partial_y \phi_1(x, y)dxdy \\
&= O\left(h_y^2\right)\|\partial_{yy}u_1\|_{0,K}\|\phi_1\|_{0,K}.
\end{aligned}$$

By the same arguments, we can have

$$\int_K (u_2 - (\Pi_h \boldsymbol{u})_2)\phi_2 dxdy = O\big(h_x^2\big)\|\partial_{xx}u_2\|_{0,K}\|\phi_2\|_{0,K},$$

which completes the proof for the $k = 1$ case.

ii. Now we consider the proof for $k \geq 2$. In this case, $\phi_1 \in Q_{k-1,k}$ can be expanded as

$$\phi_1(x,y) = \sum_{i=0}^{k-1}\sum_{j=0}^{k-2} \frac{(x-x_c)^i}{i!}\frac{(y-y_c)^j}{j!}\partial_x^i\partial_y^j\phi_1(x_c,y_c)$$
$$+ \frac{(y-y_c)^{k-1}}{(k-1)!}\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c) + \frac{(y-y_c)^k}{k!}\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^k\phi_1(x_c,y_c).$$

By the interpolation property (59), we have

$$\int_K (u_1 - (\Pi_h\boldsymbol{u})_1)\sum_{i=0}^{k-1}\sum_{j=0}^{k-2}\frac{(x-x_c)^i}{i!}\frac{(y-y_c)^j}{j!}\partial_x^i\partial_y^j\phi_1(x_c,y_c)dxdy = 0.$$

Using integration by parts, the interpolation property (58) and (59), and the identity

$$\frac{(y-y_c)^{k-1}}{(k-1)!} = \frac{2^k}{(2k)!}\partial_y^{k+1}(B^k(y)) + P_{k-3}(y),$$

where $P_{k-3}(y)$ is a polynomial of degree $k-3$, we obtain

$$\int_K (u_1 - (\Pi_h\boldsymbol{u})_1)\cdot\frac{(y-y_c)^{k-1}}{(k-1)!}\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c)dxdy$$
$$= \int_{x=x_c-h_x}^{x_c+h_x}(u_1 - (\Pi_h\boldsymbol{u})_1)\cdot\frac{2^k}{(2k)!}\partial_y^k(B^k(y))\Big|_{y=y_c-h_y}^{y_c+h_y}\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c)dx$$
$$- \int_K \partial_y(u_1 - (\Pi_h\boldsymbol{u})_1)\cdot\frac{2^k}{(2k)!}\partial_y^k(B^k(y))\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c)dxdy$$
$$= -\int_{x=x_c-h_x}^{x_c+h_x}\partial_y(u_1 - (\Pi_h\boldsymbol{u})_1)\cdot\frac{2^k}{(2k)!}\partial_y^{k-1}(B^k(y))\Big|_{y=y_c-h_y}^{y_c+h_y}\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c)dx$$
$$+ \int_K \partial_{yy}(u_1 - (\Pi_h\boldsymbol{u})_1)\cdot\frac{2^k}{(2k)!}\partial_y^{k-1}(B^k(y))\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c)dxdy$$
$$= (-1)^{k+1}\int_K \partial_y^{k+1}(u_1 - (\Pi_h\boldsymbol{u})_1)\cdot\frac{2^k}{(2k)!}B^k(y)\sum_{i=0}^{k-1}\frac{(x-x_c)^i}{i!}\partial_x^i\partial_y^{k-1}\phi_1(x_c,y_c)dxdy$$
$$= O\big(h_y^{k+1}\big)\big\|\partial_y^{k+1}u_1\big\|_{0,K}\|\phi_1\|_{0,K},$$

where in the last step we used the fact that $(\Pi_h\boldsymbol{u})_1 \in Q_{k-1,k}$.

Similarly, using the identity $\frac{(y-y_c)^k}{k!} = \frac{2^{k+1}}{(2k+2)!}\partial_y^{k+2}(B^{k+1}(y)) + P_{k-2}(y)$, where $P_{k-2}(y)$ is a polynomial of degree $k-2$, we obtain

$$\int_K (u_1 - (\Pi_h \boldsymbol{u})_1) \cdot \frac{(y-y_c)^k}{k!} \sum_{i=0}^{k-1} \frac{(x-x_c)^i}{i!} \partial_x^i \partial_y^k \phi_1(x_c, y_c) dx dy$$

$$= (-1)^{k+1} \int_K \partial_y^{k+1}(u_1 - (\Pi_h \boldsymbol{u})_1) \cdot \frac{2^{k+1}}{(2k+2)!}(B^{k+1}(y))_y \sum_{i=0}^{k-1} \frac{(x-x_c)^i}{i!} \partial_x^i \partial_y^k \phi_1(x_c, y_c) dx dy$$

$$= O\left(h_y^{k+1}\right) \left\| \partial_y^{k+1} u_1 \right\|_{0,K} \|\phi_1\|_{0,K}.$$

Combining the above estimates, we have

$$\int_K (u_1 - (\Pi_h \boldsymbol{u})_1)\phi_1 dx dy = O\left(h_y^{k+1}\right) \left\| \partial_y^{k+1} u_1 \right\|_{0,K} \|\phi_1\|_{0,K}.$$

By the same arguments, we can obtain

$$\int_K (u_2 - (\Pi_h \boldsymbol{u})_2)\phi_2 dx dy = O\left(h_x^{k+1}\right) \left\| \partial_x^{k+1} u_2 \right\|_{0,K} \|\phi_2\|_{0,K},$$

which concludes the proof for the $k \geq 2$ case. ■

With Lemmas 5.1–5.3, we can see that Theorems 3.1, 3.2, and 4.1 hold true for 2D rectangular elements. Below, we want to show that for the lowest order edge element (i.e., $k = 1$ in U_h and $\boldsymbol{V}_h$), we have one-order higher superconvergence in the L_∞-norm at rectangular element centers.

Lemma 5.4. *Let* $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$ *be an arbitrary rectangular element. Then for any* $\boldsymbol{u} \in H(\text{curl}; K)$ *and* $\Pi_h \boldsymbol{u}|_K \in Q_{0,1} \times Q_{1,0}$, *we have*

$$(\boldsymbol{u} - \Pi_h \boldsymbol{u})(x_c, y_c) = O(h^2). \tag{65}$$

Proof. For the lowest order edge element $Q_{0,1} \times Q_{1,0}$, the interpolation $\Pi_h \boldsymbol{u}$ of any $\boldsymbol{u} \in H(\text{curl}; K)$ can be written as

$$\Pi_h \boldsymbol{u}(x, y) = \sum_{j=1}^{4} \left(\frac{1}{|l_j|} \int_{l_j} \boldsymbol{u} \cdot \boldsymbol{t}_j dl \right) \boldsymbol{N}_j(x, y), \tag{66}$$

where we denote l_j the four edges of the element, which start from the bottom edge and orient counterclockwisely. Furthermore, $|l_j|$ and $\boldsymbol{t}_j$ represent the length of edge l_j and the unit tangent vector along l_j, respectively. The edge element basis functions $\boldsymbol{N}_j$ are as follows:

$$\boldsymbol{N}_1 = \begin{pmatrix} \frac{(y_c+h_y)-y}{2h_y} \\ 0 \end{pmatrix}, \quad \boldsymbol{N}_2 = \begin{pmatrix} 0 \\ \frac{x-(x_c-h_x)}{2h_x} \end{pmatrix}, \quad \boldsymbol{N}_3 = \begin{pmatrix} \frac{(y_c-h_y)-y}{2h_y} \\ 0 \end{pmatrix}, \quad \boldsymbol{N}_4 = \begin{pmatrix} 0 \\ \frac{x-(x_c+h_x)}{2h_x} \end{pmatrix}.$$

By (66) and the notation $\boldsymbol{u} = (u_1, u_2)'$, we have

$$
\begin{aligned}
&\Pi_h \boldsymbol{u}(x_c, y_c) \\
&= \frac{1}{2h_x}\int_{l_1} u_1(x, y_c - h_y)dx \cdot \begin{pmatrix} \frac{(y_c+h_y)-y_c}{2h_y} \\ 0 \end{pmatrix} + \frac{1}{2h_y}\int_{l_2} u_2(x_c + h_x, y)dy \cdot \begin{pmatrix} 0 \\ \frac{x_c-(x_c-h_x)}{2h_x} \end{pmatrix} \\
&\quad - \frac{1}{2h_x}\int_{l_3} u_1(x, y_c + h_y)dx \cdot \begin{pmatrix} \frac{(y_c-h_y)-y_c}{2h_y} \\ 0 \end{pmatrix} - \frac{1}{2h_y}\int_{l_4} u_2(x_c - h_x, y)dy \cdot \begin{pmatrix} 0 \\ \frac{x_c-(x_c+h_x)}{2h_x} \end{pmatrix},
\end{aligned}
$$

from which we obtain the first component

$$
\begin{aligned}
&\frac{1}{4h_x}\left(\int_{x_c-h_x}^{x_c+h_x} u_1(x, y_c - h_y)dx + \int_{x_c-h_x}^{x_c+h_x} u_1(x, y_c + h_y)dx\right) \\
&\quad = \frac{1}{4h_x}\left(\int_{x_c-h_x}^{x_c+h_x} \left[u_1(x_c, y_c - h_y) + (x - x_c)\partial_x u_1(x_c, y_c - h_y) + O\left(h_x^2\right)\right]dx\right. \\
&\quad\quad \left. + \int_{x_c-h_x}^{x_c+h_x} \left[u_1(x_c, y_c + h_y) + (x - x_c)\partial_x u_1(x_c, y_c + h_y) + O\left(h_x^2\right)\right]dx\right) \\
&\quad = \frac{1}{2}[u_1(x_c, y_c - h_y) + u_1(x_c, y_c + h_y)] + O\left(h_x^2\right),
\end{aligned}
$$

where we used the Taylor expansion and the fact that $\int_{x_c-h_x}^{x_c+h_x}(x - x_c)dx = 0$. Using the Taylor expansion one more time, we can easily see that

$$
\begin{aligned}
((\Pi_h \boldsymbol{u})_1 - u_1)(x_c, y_c) &= \frac{1}{2}[u_1(x_c, y_c - h_y) + u_1(x_c, y_c + h_y)] - u_1(x_c, y_c) + O\left(h_x^2\right) \\
&= O\left(h_x^2\right) + O\left(h_y^2\right).
\end{aligned}
$$

By the same arguments, we can obtain the same estimate for the second component:

$$
((\Pi_h \boldsymbol{u})_2 - u_2)(x_c, y_c) = O\left(h_x^2\right) + O\left(h_y^2\right),
$$

which completes the proof. ■

Theorem 5.1. *Let (x_c, y_c) be the center of a rectangular element $K = [x_c - h_x, x_c + h_x] \times [y_c - h_y, y_c + h_y]$, and $\boldsymbol{E}^h$ and $\boldsymbol{H}^h$ be the lowest order finite element solution of (36)–(39), i.e., $\boldsymbol{E}^h|_K \in Q_{0,1} \times Q_{1,0}$ and $\boldsymbol{H}^h|_K \in Q_{0,0}$. Under the assumption that the L^2 estimates of $\Pi_h \boldsymbol{E} - \boldsymbol{E}^h$ and $P_h \boldsymbol{H} - \boldsymbol{H}^h$ over element K is below the average over the whole domain Ω or the L^2 estimates are almost uniformly distributed, i.e.,*

$$
\begin{aligned}
\int_K |\Pi_h \boldsymbol{E} - \boldsymbol{E}^h|^2 dK &\le \frac{C}{N}\int_\Omega |\Pi_h \boldsymbol{E} - \boldsymbol{E}^h|^2 dK, \\
\int_K |P_h \boldsymbol{H} - \boldsymbol{H}^h|^2 dK &\le \frac{C}{N}\int_\Omega |P_h \boldsymbol{H} - \boldsymbol{H}^h|^2 dK, \qquad (67)
\end{aligned}
$$

where N denotes the total number of elements over Ω. Then on a quasi-uniform mesh we have the L^∞ superconvergence

$$|(\boldsymbol{E}-\boldsymbol{E}^h)(x_c,y_c)|+|(\boldsymbol{H}-\boldsymbol{H}^h)(x_c,y_c)|\le Ch^2. \tag{68}$$

Proof. Using the fact that the m-point Gaussian quadrature holds exactly for all polynomials up to degree $2m-1$, and the Cauchy-Schwarz inequality, for the first component of error $\Pi_h\boldsymbol{E}-\boldsymbol{E}^h$ we easily have

$$\begin{aligned}
|(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h)_1(x_c,y_c)| &= \left|\frac{1}{|K|}\int_K(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h)_1dxdy\right| \\
&\le \frac{1}{|K|}\left(\int_K|(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h)_1|^2dxdy\right)^{1/2}\left(\int_K 1^2dxdy\right)^{1/2} \\
&\le \frac{1}{|K|^{1/2}}\left(\frac{1}{N}\int_\Omega|(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h)_1|^2dxdy\right)^{1/2}\le\frac{1}{(N|K|)^{1/2}}\cdot Ch^2\le Ch^2,
\end{aligned}\tag{69}$$

where we used Theorem 3.1 and the fact that $N|K|\approx \text{meas}(\Omega)=O(1)$. Here, we denote $|K|$ for the area of element K. Similar estimate can be obtained for the second component, i.e.,

$$|(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h)_2(x_c,y_c)|=O(h^2),$$

from which and Lemma 5.1, we obtain

$$(\boldsymbol{E}-\boldsymbol{E}^h)(x_c,y_c)=(\boldsymbol{E}-\Pi_h\boldsymbol{E})(x_c,y_c)+(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h)(x_c,y_c)=O(h^2).$$

Note that for any function $f(x,y)$, by Taylor expansion, we have

$$\frac{1}{|K|}\int_K f(x,y)dxdy-f(x_c,y_c)=\frac{1}{|K|}\int_K(f(x,y)-f(x_c,y_c))dxdy \tag{70}$$

$$=\frac{1}{|K|}\int_K[(x-x_c)\partial_x f(x_c,y_c)+(y-y_c)\partial_y f(x_c,y_c)+O(h^2)]dxdy=O(h^2), \tag{71}$$

using which, the fact that $\int_K(P_h\boldsymbol{H}-\boldsymbol{H})dxdy=0$ and similar arguments used in (69), we have

$$\begin{aligned}
(\boldsymbol{H}-\boldsymbol{H}^h)(x_c,y_c) &\approx \frac{1}{|K|}\int_K(\boldsymbol{H}-\boldsymbol{H}^h)(x,y)dxdy+O(h^2) \\
&= \frac{1}{|K|}\int_K(P_h\boldsymbol{H}-\boldsymbol{H}^h)(x,y)dxdy+O(h^2) \\
&\le \frac{1}{|K|}\left(\int_K|P_h\boldsymbol{H}-\boldsymbol{H}^h|^2dxdy\right)^{1/2}\left(\int_K 1^2dxdy\right)^{1/2}+O(h^2)\le Ch^2,
\end{aligned}\tag{72}$$

which concludes the proof. ■

Remark 5.1. By similar arguments, for the fully discrete scheme (41)–(44) we have

$$\max_{1\le m\le M}\left(\left|\left(\boldsymbol{E}^m-\boldsymbol{E}_h^m\right)(x_c,y_c)\right|+\left|\left(\boldsymbol{H}^m-\boldsymbol{H}_h^m\right)(x_c,y_c)\right|\right)\le C(h^2+\tau^2).$$

VI. CONCLUSIONS

In this article, we consider the time-dependent Maxwell's equations modeling wave propagation in metamaterials. We presented detailed superconvergence analysis for this model solved by several semidiscrete and fully discrete schemes. We believe that similar 3D L^∞ superconvergence can happen at the element centers or face centers; hopefully, detailed analysis and numerical results will be presented in our future article.

The authors thank anonymous referees for their helpful comments that improved the article. They also thank Professor John Whiteman's kind help during the review process. This work is inspired by the IPAM workshop "Metamaterials: Applications, Analysis and Modeling" held at UCLA during January 25–29, 2010. As Co-chair of this workshop, Jichun Li highly appreciates IPAM's kind support for sponsoring this workshop.

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