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Mathematical Simulation of Cloaking Metamaterial Structures

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Abstract. In this paper we present a rigorous derivation of the material parameters for both the cylinder and rectangle cloaking structures. Numerical results using these material parameters are presented to demonstrate the cloaking effect.

AMS subject classifications: 78M10, 65N30, 35L15

Key words: Maxwell's equations, metamaterial, finite element method, cloaking structures.

1 Introduction

In recent years, inspired by the pioneering work of Pendry et al. [15] and Leonhardt [10], there are lots of work devoted to the study of using metamaterials (e.g., [5,7,11]) to construct invisibility cloaks of different shapes (e.g., [1,9,14,18–20]). More details and references on cloaking can be found in recent reviews [2,6]. The basic principal behind this is the so-called transformation optics [10,15], which uses the coordinate transformation to design the material parameters to steer the light around some regions. Unfortunately, very few papers provided a clear derivation of the material parameters so that many researchers wasted a great deal of time on guessing those parameters and still could not obtain nice cloak results.

The main goal of this paper is to present a rigorous derivation of the material parameters for both the cylinder and rectangle cloaking structures. Detailed numerical results are provided to demonstrate our correct derivation and the cloaking effect achieved using these material parameters. Hopefully these will serve as benchmark problems so that other researchers can easily reproduce these models and inspire further advance in this area.

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2 The mathematical formulation

Modeling of electromagnetic phenomena at a fixed frequency ω is governed by the full Maxwell's equations (assuming a time harmonic variation of $\exp(i\omega t)$):

$$\nabla \times \boldsymbol{E} + i\omega\mu \boldsymbol{H} = 0, \qquad \nabla \times \boldsymbol{H} - i\omega\epsilon \boldsymbol{E} = 0, \tag{2.1}$$

where $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$ are the electric and magnetic fields, ϵ and μ are the permittivity and permeability of the material.

A very important property for Maxwell's equations is that Maxwell's equations are form invariant under coordinate transformations [16]. More specifically, under a coordinate transformation $\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x})$, the Eq. (2.1) keeps the same form in the transformed coordinate system [14]:

$$\nabla' \times \boldsymbol{E}' + i\omega\mu' \boldsymbol{H}' = 0, \qquad \nabla' \times \boldsymbol{H}' - i\omega\epsilon' \boldsymbol{E}' = 0, \tag{2.2}$$

where all new variables are given by

$$\boldsymbol{E}'(\boldsymbol{x}') = A^{-T}\boldsymbol{E}(\boldsymbol{x}), \quad \boldsymbol{H}'(\boldsymbol{x}') = A^{-T}\boldsymbol{H}(\boldsymbol{x}), \quad A = (A_{ij}), \quad A_{ij} = \frac{\partial x'_i}{\partial x_j}, \tag{2.3}$$

and

$$\mu'(\boldsymbol{x}') = \frac{A\mu(\boldsymbol{x})A^T}{\det(A)}, \qquad \epsilon'(\boldsymbol{x}') = \frac{A\epsilon(\boldsymbol{x})A^T}{\det(A)}. \tag{2.4}$$

2.1 Cylindrical cloak

Following [15], cloaking a central cylindrical region R_1 by a concentric cylindrical region of radius R_2 can be done using the following coordinate transformation:

$$r'(r,\theta) = \frac{R_2 - R_1}{R_2} r + R_1, \tag{2.5a}$$

$$\theta'(r,\theta) = \theta. \tag{2.5b}$$

Since the COMSOL solver is based on Cartesian coordinates, we have to transform the material parameters given in polar coordinates to Cartesian coordinates. In polar coordinates, we have

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \tan^{-1}\frac{x_2}{x_1}, \tag{2.6}$$

where we use the traditional notation: a point (x_1, x_2) in Cartesian coordinate system corresponds to a point (r, θ) in polar coordinate system.

From (2.6) and the relation $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we can obtain

$$\frac{\partial r}{\partial x_1} = \frac{x_1}{r} = \cos\theta, \qquad \frac{\partial r}{\partial x_2} = \frac{x_2}{r} = \sin\theta, \tag{2.7a}$$

$$\frac{\partial \theta}{\partial x_1} = -\frac{x_2}{r^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial x_2} = \frac{x_1}{r^2} = \frac{\cos\theta}{r}. \tag{2.7b}$$

For the transformation (2.5), by chain rule we can obtain

$$\begin{aligned}\frac{\partial x_1'}{\partial x_1} =&\frac{\partial x_1'}{\partial r'}\frac{\partial r'}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial x_1'}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\frac{\partial \theta}{\partial x_1}=\cos\theta\cdot\frac{R_2-R_1}{R_2}\cdot\cos\theta-r'\sin\theta\cdot\Big(-\frac{\sin\theta}{r}\Big)\\ =&\frac{R_2-R_1}{R_2}\cos^2\theta+\frac{r'}{r}\sin^2\theta=\frac{R_2-R_1}{R_2}+\frac{R_1}{r}\sin^2\theta.\end{aligned}$$

Using the same technique, we have

$$\begin{aligned}\frac{\partial x_1'}{\partial x_2} =&\frac{\partial x_1'}{\partial r'}\frac{\partial r'}{\partial r}\frac{\partial r}{\partial x_2}+\frac{\partial x_1'}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\frac{\partial \theta}{\partial x_2}=\cos\theta\cdot\frac{R_2-R_1}{R_2}\cdot\sin\theta-r'\sin\theta\cdot\Big(\frac{\cos\theta}{r}\Big)\\ =&\frac{R_2-R_1}{R_2}\sin\theta\cos\theta-\frac{r'}{r}\sin\theta\cos\theta=-\frac{R_1}{r}\sin\theta\cos\theta.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\frac{\partial x_2'}{\partial x_1} =&\frac{\partial x_2'}{\partial r'}\frac{\partial r'}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial x_2'}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\frac{\partial \theta}{\partial x_1}=\sin\theta\cdot\frac{R_2-R_1}{R_2}\cdot\cos\theta+r'\cos\theta\cdot\Big(-\frac{\sin\theta}{r}\Big)\\ =&\frac{R_2-R_1}{R_2}\sin\theta\cos\theta-\Big(\frac{R_2-R_1}{R_2}r+R_1\Big)r^{-1}\sin\theta\cos\theta=-\frac{R_1}{r}\sin\theta\cos\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial x_2'}{\partial x_2} =&\frac{\partial x_2'}{\partial r'}\frac{\partial r'}{\partial r}\frac{\partial r}{\partial x_2}+\frac{\partial x_2'}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\frac{\partial \theta}{\partial x_2}=\sin\theta\cdot\frac{R_2-R_1}{R_2}\cdot\sin\theta+r'\cos\theta\cdot\Big(\frac{\cos\theta}{r}\Big)\\ =&\frac{R_2-R_1}{R_2}\sin^2\theta+\Big(\frac{R_2-R_1}{R_2}r+R_1\Big)r^{-1}\cos^2\theta=\frac{R_2-R_1}{R_2}+\frac{R_1}{r}\cos^2\theta.\end{aligned}$$

From the derived $\partial x_i'/\partial x_j$, we obtain the transformation matrix

$$A=\left(\begin{array}{cc}\dfrac{R_2-R_1}{R_2}+\dfrac{R_1}{r}\sin^2\theta & -\dfrac{R_1}{r}\sin\theta\cos\theta\\ symmetric & \dfrac{R_2-R_1}{R_2}+\dfrac{R_1}{r}\cos^2\theta\end{array}\right), \tag{2.8}$$

which has determinant

$$\det(A)=\frac{R_2-R_1}{R_2}\Big(\frac{R_2-R_1}{R_2}+\frac{R_1}{r}\Big)=\Big(\frac{R_2-R_1}{R_2}\Big)^2\cdot\frac{r'}{r'-R_1}. \tag{2.9}$$

Substituting (2.8)-(2.9) into (2.4), we obtain the relative permittivity in the transformed space

$$\begin{aligned}\epsilon'=&\left(\begin{array}{cc}\epsilon'_{xx} & \epsilon'_{xy}\\ \epsilon'_{yx} & \epsilon'_{yy}\end{array}\right)=\frac{AA^T}{\det(A)}\\ =&\frac{1}{\det(A)}\left(\begin{array}{cc}\big(\frac{R_2-R_1}{R_2}\big)^2+\frac{R_1}{r}\big(2\frac{R_2-R_1}{R_2}+\frac{R_1}{r}\big)\sin^2\theta & -\frac{R_1}{r}\big(2\frac{R_2-R_1}{R_2}+\frac{R_1}{r}\big)\sin\theta\cos\theta\\ symmetric & \big(\frac{R_2-R_1}{R_2}\big)^2+\frac{R_1}{r}\big(2\frac{R_2-R_1}{R_2}+\frac{R_1}{r}\big)\cos^2\theta\end{array}\right).\end{aligned}$$

In summary, the material parameters in Cartesian coordinates become as follows:

$$\epsilon'_{xx} = \Big[\Big(\frac{R_2-R_1}{R_2}\Big)^2 + \frac{R_1}{r}\Big(2\frac{R_2-R_1}{R_2} + \frac{R_1}{r}\Big)\sin^2\theta\Big][\det(A)]^{-1},$$

$$\epsilon'_{xy} = \epsilon'_{yx} = \Big[-\frac{R_1}{r}\Big(2\frac{R_2-R_1}{R_2} + \frac{R_1}{r}\Big)\sin\theta\cos\theta\Big][\det(A)]^{-1},$$

$$\epsilon'_{yy} = \Big[\Big(\frac{R_2-R_1}{R_2}\Big)^2 + \frac{R_1}{r}\Big(2\frac{R_2-R_1}{R_2} + \frac{R_1}{r}\Big)\cos^2\theta\Big][\det(A)]^{-1},$$

and $\epsilon'_z = 1/\det(A)$. The permeability μ' has the same formula as permittivity ϵ.

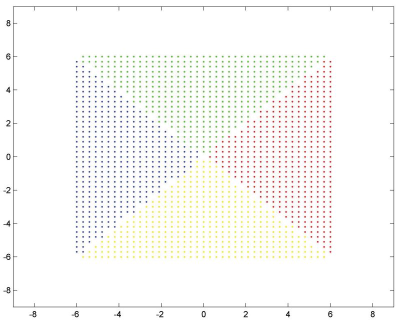
2.2 Square cloak

The same idea as the circle cloak can be used for design of a square-shaped cloak with inner square width $2S_1$ and outer square width $2S_2$. It can be seen that the coordinate transformation [18]

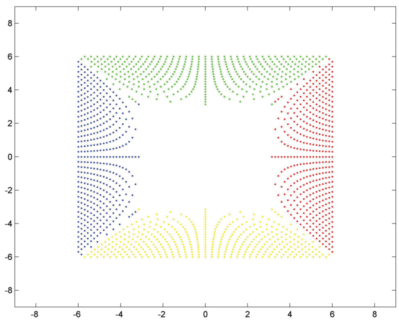
$$x'_1(x_1,x_2) = x_1\frac{S_2-S_1}{S_2} + S_1, \tag{2.10a}$$

$$x'_2(x_1,x_2) = x_2\Big(\frac{S_2-S_1}{S_2} + \frac{S_1}{x_1}\Big), \tag{2.10b}$$

mapped the right triangle in the original space into the right-subdomain in the transformed space (see Fig. 1).



(a)



(b)

Figure 1: (a) The original space; (b) The transformed space.

It is easy to prove that the transformation matrix in this case is

$$A_r = \begin{pmatrix} \frac{S_2-S_1}{S_2} & 0 \\ -\frac{x_2S_1}{x_1^2} & \frac{S_2-S_1}{S_2} + \frac{S_1}{x_1} \end{pmatrix}, \tag{2.11}$$

which has determinant

$$\det(A_r) = \frac{S_2-S_1}{S_2}\Big(\frac{S_2-S_1}{S_2} + \frac{S_1}{x_1}\Big). \tag{2.12}$$

Mapping the unit permittivity tensor $\epsilon = I$ by (2.4), we obtain

$$\epsilon'_r = \frac{A_r A_r^T}{\det(A_r)} = \begin{pmatrix} \left(\frac{S_2-S_1}{S_2}\right)^2 & -\frac{x_2S_1}{x_1^2}\cdot\frac{S_2-S_1}{S_2} \\ symmetric & \left(\frac{x_2S_1}{x_1^2}\right)^2 + \left(\frac{S_2-S_1}{S_2}+\frac{S_1}{x_1}\right)^2 \end{pmatrix} [\det(A_r)]^{-1}. \tag{2.13}$$

Corresponding formulas for the upper, left and bottom sub-domain of the cloak can be similarly obtained by applying rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

to the right sub-domain with rotation angles $\theta = \pi/2$, π and $3\pi/2$, respectively.

For the upper sub-domain, we have

$$\begin{pmatrix} \tilde{x_1} \\ \tilde{x_2} \end{pmatrix} = R\Big(\frac{\pi}{2}\Big)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

applying which to (2.10a)-(2.10b), we have the coordinate transformation for the upper sub-domain as follows:

$$\begin{aligned} \tilde{x}'_1 &= -x'_2 = -x_2\Big(\frac{S_2-S_1}{S_2}+\frac{S_1}{x_1}\Big) = \tilde{x}_1\Big(\frac{S_2-S_1}{S_2}+\frac{S_1}{\tilde{x}_2}\Big), \\ \tilde{x}'_2 &= x'_1 = x_1\frac{S_2-S_1}{S_2}+S_1 = \tilde{x}_2\frac{S_2-S_1}{S_2}+S_1, \end{aligned}$$

which leads to

$$A_u = \begin{pmatrix} \frac{S_2-S_1}{S_2}+\frac{S_1}{\tilde{x}_2} & -\frac{\tilde{x}_1S_1}{\tilde{x}_2^2} \\ 0 & \frac{S_2-S_1}{S_2} \end{pmatrix}, \qquad \det(A_u) = \Big(\frac{S_2-S_1}{S_2}+\frac{S_1}{\tilde{x}_2}\Big)\cdot\frac{S_2-S_1}{S_2}, \tag{2.14}$$

and

$$\epsilon'_u = \begin{pmatrix} \left(\frac{S_2-S_1}{S_2}+\frac{S_1}{\tilde{x}_2}\right)^2 + \left(\frac{\tilde{x}_1S_1}{\tilde{x}_2^2}\right)^2 & -\frac{S_2-S_1}{S_2}\cdot\frac{\tilde{x}_1S_1}{\tilde{x}_2^2} \\ symmetric & \left(\frac{S_2-S_1}{S_2}\right)^2 \end{pmatrix} [\det(A_u)]^{-1}.$$

For the left sub-domain, we have the coordinate transformation:

$$\begin{aligned} \tilde{x}'_1 &= -x'_1 = -\Big(x_1\frac{S_2-S_1}{S_2}+S_1\Big) = \tilde{x}_1\frac{S_2-S_1}{S_2}-S_1, \\ \tilde{x}'_2 &= -x'_2 = -x_2\Big(\frac{S_2-S_1}{S_2}+\frac{S_1}{x_1}\Big) = \tilde{x}_2\Big(\frac{S_2-S_1}{S_2}-\frac{S_1}{\tilde{x}_1}\Big), \end{aligned}$$

which leads to

$$A_l = \begin{pmatrix} \frac{S_2-S_1}{S_2} & 0 \\ \frac{\tilde{x}_2S_1}{\tilde{x}_1^2} & \frac{S_2-S_1}{S_2}-\frac{S_1}{\tilde{x}_1} \end{pmatrix}, \qquad \det(A_l) = \frac{S_2-S_1}{S_2}\cdot\Big(\frac{S_2-S_1}{S_2}-\frac{S_1}{\tilde{x}_1}\Big), \tag{2.15}$$

and

$$\epsilon_l' = \begin{pmatrix} \big(\frac{S_2-S_1}{S_2}\big)^2 & \frac{S_2-S_1}{S_2}\cdot\frac{\tilde{x}_2 S_1}{\tilde{x}_1^2} \\ symmetric & \big(\frac{\tilde{x}_2 S_1}{\tilde{x}_1^2}\big)^2 + \big(\frac{S_2-S_1}{S_2} - \frac{S_1}{\tilde{x}_1}\big)^2 \end{pmatrix} [\det(A_l)]^{-1}.$$

Similarly, we can obtain the coordinate transformation for the bottom sub-domain:

$$\begin{aligned} \tilde{x}_1' &= x_2' = x_2\Big(\frac{S_2-S_1}{S_2} + \frac{S_1}{x_1}\Big) = \tilde{x}_1\Big(\frac{S_2-S_1}{S_2} - \frac{S_1}{\tilde{x}_2}\Big), \\ \tilde{x}_2' &= -x_1' = -x_1\frac{S_2-S_1}{S_2} + S_1 = \tilde{x}_2\frac{S_2-S_1}{S_2} - S_1, \end{aligned}$$

which leads to

$$\begin{aligned} A_b &= \begin{pmatrix} \frac{S_2-S_1}{S_2} - \frac{S_1}{\tilde{x}_2} & \frac{\tilde{x}_1 S_1}{\tilde{x}_2^2} \\ 0 & \frac{S_2-S_1}{S_2} \end{pmatrix}, \qquad \det(A_b) = \Big(\frac{S_2-S_1}{S_2} - \frac{S_1}{\tilde{x}_2}\Big)\cdot\frac{S_2-S_1}{S_2}, \\ \epsilon_b' &= \begin{pmatrix} \big(\frac{S_2-S_1}{S_2} - \frac{S_1}{\tilde{x}_2}\big)^2 + \big(\frac{\tilde{x}_1 S_1}{\tilde{x}_2^2}\big)^2 & \frac{S_2-S_1}{S_2}\cdot\frac{\tilde{x}_1 S_1}{\tilde{x}_2^2} \\ symmetric & \big(\frac{S_2-S_1}{S_2}\big)^2 \end{pmatrix} [\det(A_b)]^{-1}. \end{aligned}$$

3 Numerical results

In this section, we present some results obtained by our derived material formulas in both cases. Our implementation is based on COMSOL Multiphysics package (`www.comsol.com`) for 2-D transverse electric modeling, i.e., by combining (2.2) into an equation just involving one variable $\boldsymbol{E} = E_z$:

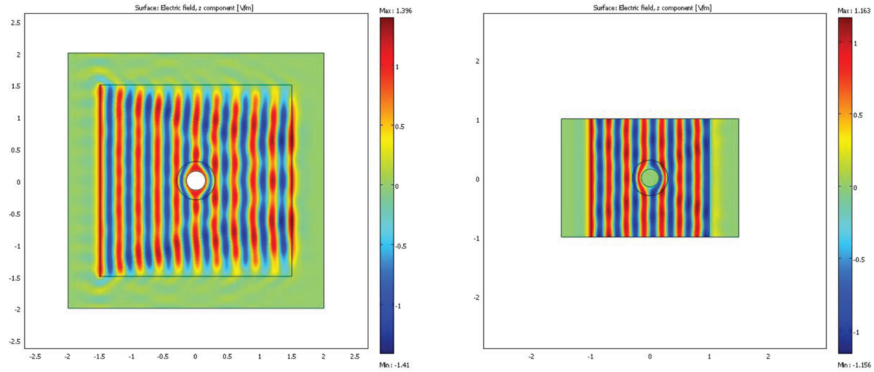
$$\nabla \times (\mu_r^{-1}\nabla \times \boldsymbol{E}) - k_0^2\epsilon_r\boldsymbol{E} = 0, \tag{3.1}$$

where μ_r and ϵ_r are the relative permeability and permittivity, and k_0 denotes the wave number of free space $k_0 = \omega\sqrt{\epsilon_0\mu_0} = \omega/c_0$, here c_0 is the speed of light in free space.

The results presented below were carried out using COMSOL installed on a Dell Latitude D630 laptop with 2GB of RAM and 2.50GHz CPU. For comparison purpose, all tests are done with 1GHz incident plane wave, quadratic elements, and an efficient direct solver SPOOLES provided by COMSOL.

3.1 Cylinder cloak

For this test, the cylinder cloak shell is located at $r \in (0.15, 0.3)$, i.e., $R_1 = 0.15$m, $R_2 = 0.3$m, a PML with 0.5m thickness is used outside the box $[-1.5, 1.5]^2$. We first tested a big structure with 81280 elements, and 163041 degrees of freedom (DOFs), whose solution time is 11.032s. To reduce the computational cost, we then put the cloak shell inside a smaller rectangle $[-1.5, 1.5] \times [-1, 1]$, and impose a PML of 0.5m



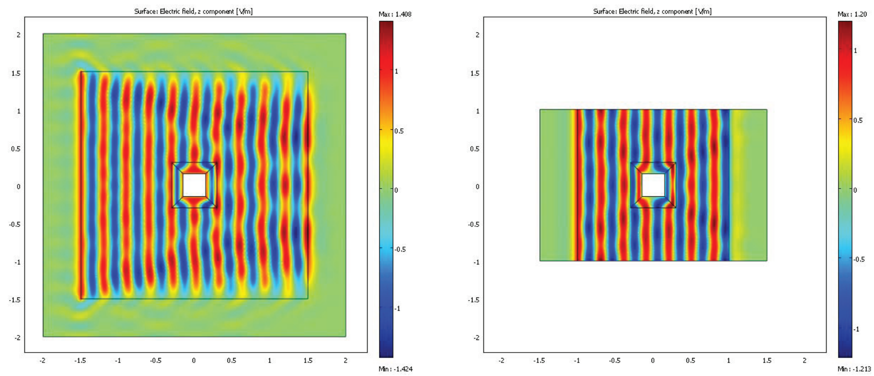
(a) (b)

Figure 2: The real part of the electric-field phasor obtained for the cylinder cloak: (a) A big structure; (b) A small structure.

on each end. In this case, the total number of elements is 58496, DOFs are 117409 and the solution time is 6.781s. The obtained electric field distributions for both cases are shown in Fig. 2, which shows that the phase fronts are completely restored after the wave moves out the cloaked area, i.e., this structure clearly demonstrates the cloaking effect.

3.2 Square cloak

The square cloak has the same geometry as the cylindrical case, except that we replace the circular shell by a rectangular shell with the same size. The big structure setting uses 58760 elements, 118128 DOFs, and solution time of 7.172s. The small structure has 28800 elements, 44976 DOFs, and solution time of 2.453s. The obtained electric field distribution is shown in Fig. 3, which also shows that this structure clearly demonstrates the cloaking effect.



(a) (b)

Figure 3: The real part of the electric-field phasor obtained for the square cloak: (a) A big structure; (b) A small structure.

4 Concluding remarks

In this paper, we presented a rigorous derivation of the material parameters for both the cylinder and rectangle cloaking structures. Detailed numerical results obtained using COMSOL are presented to demonstrate the cloaking effect. Hopefully these two will serve as benchmark problems for computational scientists who are interested in this exciting area. More advanced computational algorithms (e.g., [4, 8, 12, 13, 17] and references therein) developed for solving Maxwell's equations in both free space and dispersive media will be investigated for cloaking modeling in the future.

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