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Variational characterization for the planar dual Minkowski problem

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In this paper, we give a variational analysis to the planar dual Minkowski problem in the Sobolev space. With the new variational characterization, we can deal with existence results for prescribed not necessarily positive data. Meanwhile, functional inequalities and multiple solutions are also obtained.

1. Introduction

In convex geometric analysis, the Brunn–Minkowski theory and its generalization were established after the contribution of Brunn [11], Minkowski [45,46], Hilbert [25], Alekesandrov [2–6], Fenchel & Jessen [20] and so on. The central tasks are to solve Minkowski type problems and to establish related geometric inequalities, see, for example

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[42,51]. Analogy to the classical Brunn–Minkowski theory, the authors in [28] ask the following characterization problem.

Dual Minkowski problem. For any $q \in \mathbb{R}$, if μ is a finite Borel measure on $\mathbb{S}^{N-1}$, find necessary and sufficient conditions on μ so that it is the q-th dual curvature measure $\widetilde{C}_q(K, \cdot)$ of a convex body K in $\mathbb{R}^N$.

We plan to give a pure variational analysis to the dual Minkowski problem. To establish mathematical models for these celebrated problems, the support function u_K and radical function ρ_K of a convex body K in $\mathbb{R}^N$ with origin in its interior are introduced by

$$u_K(\theta) = \max\{\theta \cdot y : y \in K\},\ \theta \in \mathbb{R}^N; \quad \rho_K(\xi) = \max\{\lambda : \lambda\xi \in K\},\ \xi \in \mathbb{R}^N \setminus \{0\}.$$

With these notations, the first author with Lutwak, Yang and Zhang in [28] introduced the new geometric measure, the q-th dual curvature measure

$$\widetilde{C}_q(K, \eta) = \frac{1}{n} \int\limits_{\mathbb{S}^{N-1} \bigcap \alpha_K^*(\eta)} \rho_K^q(\xi) d\xi,$$

where α_K^* denotes reverse radial Gauss image of K in sphere $\mathbb{S}^{N-1}$. Suppose that K has a C^2 boundary with everywhere positive curvature, ∇ denotes the gradient operator with respect to a frame on $\mathbb{S}^{N-1}$ and E is the standard Riemannian metric matrix on $\mathbb{S}^{N-1}$. The density of dual curvature measure $\widetilde{C}_q(K, \cdot)$ equals to

$$(u_K^2 + |\nabla u_K|^2)^{\frac{q-N}{2}} u_K \det(\nabla^2 u_K + u_K E),$$

which is the integrand of dual quermassintegral, a fundamental geometric invariant in the dual Brunn–Minkowski theory [28,41,51]. When the measure μ has a density function g, the dual Minkowski problem is equivalent to the solvability of following Monge–Ampère equation

$$\det(\nabla^2 u_K + u_K E) = g(\theta) \frac{(u_K^2 + |\nabla u_K|^2)^{\frac{N-q}{2}}}{u_K},\ \theta \in \mathbb{S}^{N-1}. \tag{1.1}$$

Analytically, it is nontrivial to give a variational functional for the general nonlinear differential equation with gradient terms, even for the standard model

$$-\Delta u = c(x) + |\nabla u|^p. \tag{1.2}$$

Only when $p = 2$, the change of variable that $v = e^u - 1$ is valid for transforming (1.2) to a linear equation [31,37] as

$$-\Delta v = c(x)(v+1),$$

which has a variational structure. Besides the fully nonlinear Monge–Ampère operator, (1.1) with $q \neq N$ owns more complicated nonlinear gradient term than the standard model (1.2). The method involving change of variable seems invalid in the process of doing variational analysis to (1.1), even in two dimensions. One could also refer to other applications of variational functional to nonlinear differential equations with gradient terms, such as the famous Perelman functional in Section 1.1 of [48].

The first author with Lutwak, Yang and Zhang [28] developed a perturbation process and geometric variational formula after the work of Aleksandrov [4]. Let $\Omega \subset \mathbb{S}^{N-1}$ be a closed set which is not contained in any closed hemisphere of $\mathbb{S}^{N-1}$, $\rho_0 : \Omega \to (0, \infty)$ and $p : \Omega \to \mathbb{R}$ be two continuous functions. For $\delta > 0$, let $\rho_t : \Omega \to (0, \infty)$ be defined for $\xi \in \Omega$ and $t \in (-\delta, \delta)$ by

$$\log \rho_t(\xi) = \log \rho_0(\xi) + tp(\xi) + o(t, \xi).$$

The logarithmic family of convex hulls $\langle \rho_t \rangle = \mathrm{conv}\{\rho_t(\xi)\xi : \xi \in \mathbb{S}^{N-1}\}$, then its support function is differentiable with respect to the variational variable,

$$\left.\frac{d}{dt}\right|_{t=0} \log u_{\langle \rho_t \rangle}(\theta) = p(\alpha^*_{\langle \rho_0 \rangle}(\theta)). \tag{1.3}$$

It was proved in [28] that (1.1) is the Euler–Lagrange equation with respect to the geometric functional

$$\mathbb{F}(f) = \begin{cases} \int_{\mathbb{S}^{N-1}} g(\theta) \log f d\theta - \int_{\mathbb{S}^{N-1}} \log u_{\langle f \rangle}(\xi) d\xi, \\ \int_{\mathbb{S}^{N-1}} g(\theta) \log f d\theta + \frac{1}{q} \log \int_{\mathbb{S}^{N-1}} u^{-q}_{\langle f \rangle}(\xi) d\xi, \ q \neq 0, \end{cases} f \in \{f > 0 : f \in C(\mathbb{S}^{N-1})\},$$

in the sense of the Aleksandrov solution [2]. The functional $\mathbb{F}$ plays a key role in the study of the solvability of (1.1) [28]. It is also useful in the process of proving the regularities of solution to (1.1) via a Gauss curvature flow method by Li–Sheng–Wang [39]. Under the assumption that g is strictly positive, some other important contributions to the dual Minkowski problem are also obtained in [7,10,13,24,29,30,36,57,58], and so on.

The dual Minkowski problem (1.1) with $q = N$ is equivalent to the logarithmic Minkowski problem [9,52,59], which is a special case of the L_p Minkowski problems

$$\det(\nabla^2 u_K + u_K E) = g(\theta) u_K^{p-1}, \quad \theta \in \mathbb{S}^{N-1}. \tag{1.4}$$

Important contributions and applications of the L_p Minkowski problem are included in [16,17,19,21,22,26,27,32–36,40,42,47,49,52–55,60,61] and their references. Moreover, problem (1.4) relates to the L_p affine isoperimetric inequalities and Blaschke–Santaló inequality [8,19,38,43,44,54,56]. In the special case $N = 2$, several variational functionals in the Sobolev space were introduced to study the existence of (1.4) in [1,14,21,55] and functional inequalities were given in [15,19,54]. However, for the dual Minkowski problem (1.1), we know less about the related functional inequality and variational formula in the Sobolev space than the case of L_p Minkowski problem, even in two dimensions.

In this paper, we develop a new variational formula of the planar dual Minkowski problem in the Sobolev space to give the existence, non-uniqueness, and its related functional inequalities of problem (1.1) in two dimensions, namely,

$$u'' + u = g(\theta)\frac{(u^2 + |u'|^2)^{\frac{2-q}{2}}}{u}, \quad \theta \in \mathbb{S}^1. \tag{1.5}$$

Before we go any further, some notations should be introduced. Let $k > 1$, and $H^{1,k}(\mathbb{S}^1)$ be the usual Sobolev space, which is the completion of the smooth function space $C^\infty(\mathbb{S}^1)$ with respect to the Sobolev norm. We denote by $W^{1,k}(\mathbb{S}^1) = \{u \in H^{1,k}(\mathbb{S}^1) : u(\theta) = u(\theta + \pi), \theta \in \mathbb{S}^1\}$.

The main results of this paper are in the following.

Theorem 1.1. *Assume $q \geq 2$ be an even number, and $g \in W^{1,k}(\mathbb{S}^1)$ for some $k > 1$. If $\int_{\mathbb{S}^1} g(\theta)d\theta > 0$, then equation (1.5) has a positive solution $u \in C^2(\mathbb{S}^1)$. That is to say there is a solution for the planar dual Minkowski problem with the prescribed measure $gd\theta$ on $\mathbb{S}^1$.*

The assumption $\int_{\mathbb{S}^1} g(\theta)d\theta > 0$ implies that $g(\theta)$ may equal to zero on a potential interval. Hence, Theorem 1.1 is quite different from the previous existence results of the dual Minkowski problem [10,12,13,28,36,57], in which positive data g is supposed. In fact, the weaker assumption of g has been attempted by the first author with Lutwak, Yang and Zhang, it is known as the L_p-Aleksandrov integral curvature problem in all dimensions, see [29]. To solve the case $q > 2$ and get rid of the condition that g is strictly positive, we apply the variational method in the Sobolev space, instead of Aleksandrov's geometric variational method above-mentioned (see [4,28] for details). Indeed, our variational formula is direct and does not rely on the geometric dual variational formula (1.3). We prove in Section 2 below that (1.5) with any even $q \geq 2$ is an Euler–Lagrange equation of the functional

$$F_q(u) = \left(\int\limits_{\mathbb{S}^1} u^q d\theta - q\sum_{i=1}^{q/2} \tau_i \int\limits_{\mathbb{S}^1} u^{q-2i} u'^{\,2i} d\theta \right) \cdot \exp\left(-q \frac{\int_{\mathbb{S}^1} g(\theta) \ln u d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta} \right), \tag{1.6}$$

where

$$\tau_i = \frac{(q/2-1)!}{2(2i-1)i!(q/2-i)!}, \text{ for } i = 1, 2, \ldots, q/2. \tag{1.7}$$

For the special case $q = 2$, we see that $\tau_1 = 1/2$ and the functional

$$F_2(u) = \int_{\mathbb{S}^1} (u^2 - u'^2) d\theta \cdot \exp\left(-2 \frac{\int_{\mathbb{S}^1} g(\theta) \ln u d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta} \right)$$

is related with the logarithmic Minkowski problem

$$u'' + u = \frac{g(\theta)}{u}, \ \theta \in \mathbb{S}^1. \tag{1.8}$$

Chen made a breakthrough in [14] via firstly proving an existence result of (1.8) for not necessarily positive data g. In [14], a general affine isoperimetric inequality from [15] plays a key role in studying the geometric structure and the maximum of

$$T_{\alpha,p}(u) = \int_{\mathbb{S}^1} (\alpha u^2 - u'^2) d\theta \cdot \left(\int_{\mathbb{S}^1} g(\theta) u^p d\theta \right)^{-\frac{2}{p}}, \ -2 \leq p < 0,$$

where α is a parameter; and an approximating argument is applied to get the solution u_α of the following approximate equation of (1.8).

$$u'' + \alpha u = \frac{g(\theta)}{u}, \ \theta \in \mathbb{S}^1.$$

Then, the solution of (1.8) was obtained in [14] by the limitation of a subsequence of $\{u_\alpha\}$ as $\alpha \to 1$.

In the general case $q > 2$, we know less about the related inequalities for studying the geometric structure of F_q, such as the bound of F_q from above or below, which is the key point in the application of critical point theory. So, we have to establish some pertinent inequalities for estimating the bound of F_q. In fact, we shall show the following interesting functional inequality.

Theorem 1.2. *Assume $q \geq 2$ be an even number and $g \in W^{1,k}(\mathbb{S}^1)$ for some $k > 1$. Let τ_i be defined by (1.7). If $\int_{\mathbb{S}^1} g(\theta) d\theta > 0$, then there exists constant $C_q \geq 2\pi$ such that*

$$\left(\int_{\mathbb{S}^1} u^q d\theta - q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u^{q-2i} u'^{2i} d\theta \right) \cdot \exp\left(-q \frac{\int_{\mathbb{S}^1} g(\theta) \ln u d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta} \right) \leq C_q, \tag{1.9}$$

holds for all positive function $u \in W^{1,q}(\mathbb{S}^1)$. The equality in (1.9) holds if and only if $u = lw$, where the constant $l > 0$, and w is a solution of (1.5).

To obtain the inequality (1.9), a regularity theory for the weak solution of quasilinear differential equation is developed; then we prove the following new Poincaré-type inequality with its equality conditions.

$$\int_{\mathbb{S}^1} u^q d\theta \le q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u^{q-2i} u'^{\,2i} d\theta,\ \ u \in W^{1,q}(\mathbb{S}^1) \text{ with } u(\theta_0)=0 \text{ for some } \theta_0 \in \mathbb{S}^1.$$

Based on this analysis, we employ an approximating argument to deduce (1.9). By using the extremal functions of (1.9), we get the solvability of (1.5). Furthermore, there are other applications of (1.9). For the special case that $q = 2$ and $\min_{\theta\in\mathbb{S}^1} g(\theta) > 0$, (1.9) is equivalent to the log-Minkowski inequality [8] for origin-symmetric convex bodies in the plane. The inequality (1.9) is sharp in the sense that it may be invalid for functions with minimal period 2π as the following Theorem 1.3. In this sense, the assumption that the convex bodies are origin-symmetric is “a necessary condition ”for the log-Minkowski inequality in [8].

Theorem 1.3. *Let $q \ge 2$ be even and τ_i be defined by (1.7). There exist a series of 2π-periodic positive functions $\{u_n\} \subset H^{1,q}(\mathbb{S}^1)$ such that*

$$\lim_{n\to+\infty} \left(\int_{\mathbb{S}^1} u_n^q d\theta - q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u_n^{q-2i} u_n'^{\,2i} d\theta \right) \cdot \exp\left(-q \frac{\int_{\mathbb{S}^1} g(\theta) \ln u_n d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta} \right) = +\infty \tag{1.10}$$

holds for any given continuous positive function g.

In the last part of this paper, we focus on the uniqueness of solution to (1.5). For the special case $q = 2$, (1.5) is the logarithmic Minkowski problem in two dimensions, which has a unique π-periodic solution for positive g, see, for example [8,18,21]. If $g \equiv 1$, then $u = 1$ is obviously a solution of (1.5). Via giving an estimate to the value of C_q in (1.9), we show that (1.5) has the second solution for any even number $q \ge 6$ as follows.

Theorem 1.4. *Assume $g \equiv 1$. Let $q \ge 6$ be even.* (1.5) *has a non-constant π-periodic solution.*

The paper is organized as follows. In Section 2, we give a variational framework in the Sobolev space for studying the planar dual Minkowski problem (1.5). In Section 3, we show the regularity of weak solution and a new Poincaré-type inequality. In Section 4, we prove the main conclusions of this paper. We use C, c, C_i, c_i for $i \in \mathbb{N}$ to denote the constants whose values may change from line to line. We use $o(1)$, $O(\epsilon)$ to describe the asymptotic behavior of various quantities.

2. A variational framework in the Sobolev space

In this section, we establish a variational framework for the planar dual Minkowski problem with any even exponent $q \ge 2$. For $r \in \mathbb{N}$ and $a, b \in \mathbb{R}$, we will use the binomial formula

$$(a+b)^r = \sum_{i=0}^{r} C_r^i a^{r-i} b^i, \text{ where } C_r^i = r!/[i!(r-i)!]. \tag{2.1}$$

As in [23], we denote by $C^0(\mathbb{S}^1)$ the collection of continuous function on $\mathbb{S}^1$; for $m \in \mathbb{N}$ and $\gamma \in (0,1]$, let

$$C^m(\mathbb{S}^1) = \left\{u \in C^0(\mathbb{S}^1) : \partial^l u \in C^0(\mathbb{S}^1) \text{ for all } l = 0, 2, \dots, m\right\},$$

$$C^{m,\gamma}(\mathbb{S}^1) = \left\{u \in C^m(\mathbb{S}^1) : \sup_{\theta_1,\theta_2\in\mathbb{S}^1} \frac{|\partial^l u(\theta_1) - \partial^l u(\theta_2)|}{|\theta_1-\theta_2|^\gamma} < +\infty, \text{ for all } l = 0, 1, 2, \dots, m\right\}$$

be the usual differential function spaces and Hölder spaces, respectively. For $k > 1$, let $H^{1,k}(\mathbb{S}^1)$ and $W^{1,k}(\mathbb{S}^1)$ be two Sobolev spaces introduced above. We denote by

$$\|u\| = \left\{\int_{\mathbb{S}^1} |u|^k + |u'|^k d\theta\right\}^{1/k}$$

the usual norm of $H^{1,k}(\mathbb{S}^1)$ and $W^{1,k}(\mathbb{S}^1)$. For any even $q \geq 2$, the dual space of $W^{1,q}(\mathbb{S}^1)$ is equivalent to $W^{1,q/(q-1)}(\mathbb{S}^1)$. There exists a constant $C_0 > 0$ such that

$$\sup_{\theta\in\mathbb{S}^1} |u(\theta)| \leq C_0\|u\| \tag{2.2}$$

holds for all $u \in W^{1,q}(\mathbb{S}^1)$. Moreover, the embedding that $W^{1,q}(\mathbb{S}^1) \hookrightarrow C^{0,\gamma}(\mathbb{S}^1)$ is compact for $\gamma \in [0, (q-1)/q)$.

For even $q \geq 2$, we define a positive cone of $W^{1,q}(\mathbb{S}^1)$ as

$$M = \{u \in W^{1,q}(\mathbb{S}^1) : u(\theta) > 0, \text{ for } \theta \in \mathbb{S}^1\}.$$

We aim to get the solvability of planar dual problem (1.5) via the critical point in M of $F_q(u)$ defined by (1.6). To study the upper bound and critical points of $F_q(u)$ on M, we introduce a parameter t in $F_q(u)$ to get the following approximate functional

$$I_q := I_q(t,u) = \left(\int_{\mathbb{S}^1} t^q u^q d\theta - q\sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} t^{q-2i} u^{q-2i} u'^{\,2i} d\theta\right) \cdot \exp\left(-q\frac{\int_{\mathbb{S}^1} g(\theta)\ln u d\theta}{\int_{\mathbb{S}^1} g(\theta)d\theta}\right), \tag{2.3}$$

where the coefficients $\{\tau_i\}_{i=1}^{q/2}$ are given by (1.7). For the clearness of the following calculation, we rewrite these coefficients as $\tau_{q/2} = 1/[q(q-1)]$ and

$$\tau_i = \frac{(q/2-1)!}{2(2i-1)i!(q/2-i)!} = \frac{C_{q/2-1}^{i-1}}{2i(2i-1)} = \frac{C_{q/2-1}^{i}}{(q-2i)(2i-1)} \quad \forall\, i = 1, 2, \dots, q/2-1. \tag{2.4}$$

When the parameter $t = 1$, we see that $I_q(1, u)$ equals to $F_q(u)$. The introducing of parameter t here is useful for applying approximating argument as $t \to 1$. For $t > 0$, we see that the critical point of I_q in M is a positive solution of

$$(t^2u^2 + u'^2)^{\frac{q-2}{2}}(u'' + t^2u) = \lambda\frac{g(\theta)}{u}, \tag{2.5}$$

where λ is a multiplier. Once we obtain a positive solution w of (2.5) with $t = 1$ and $\lambda > 0$, then $\lambda^{-1/q}w$ is obviously the solution of (1.5). If $\int_{\mathbb{S}^1} g(\theta)d\theta > 0$, a standard argument in [50] can be applied to show that I_q is C^1 on M with a distance induced by the norm of $W^{1,q}(\mathbb{S}^1)$. For any $t > 0$, the critical point u of I_q in M is a weak solution of (2.5) with some given $\lambda \in \mathbb{R}$ in the sense of

$$\begin{aligned} t^q\int_{\mathbb{S}^1} u^{q-1}\varphi d\theta - \sum_{i=1}^{q/2-1}\tau_i t^{q-2i}\int_{\mathbb{S}^1}(q-2i)u^{q-2i-1}u'^{2i}\varphi + 2iu^{q-2i}u'^{2i-1}\varphi' d\theta \\ - \frac{1}{q-1}\int_{\mathbb{S}^1} u^{q-1}\varphi' d\theta = \lambda\int_{\mathbb{S}^1}\frac{g(\theta)}{u}\varphi d\theta, \quad \forall\varphi \in W^{1,q}(\mathbb{S}^1). \end{aligned} \tag{2.6}$$

In fact, we have the following theorem.

Theorem 2.1. *Assume $q \geq 2$ be an even number, $t > 0$ be a parameter and $\int_{\mathbb{S}^1} g(\theta)d\theta > 0$. Let τ_i be defined by (1.7). If positive function $u \in W^{1,q}(\mathbb{S}^1)$ is a critical point of I_q in (2.3), then u is a weak solution of (2.5) in the sense of (2.6) with*

$$\lambda = \left(\int_{\mathbb{S}^1} t^q u^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} t^{q-2i}u^{q-2i}u'^{2i}d\theta\right) \Big/ \int_{\mathbb{S}^1} g(\theta)d\theta. \tag{2.7}$$

Furthermore, if $u \in W^{2,2}(\mathbb{S}^1)$, then

$$\begin{aligned} t^q\int_{\mathbb{S}^1} u^{q-1}\varphi d\theta - \sum_{i=1}^{q/2-1}\tau_i t^{q-2i}\int_{\mathbb{S}^1}(q-2i)u^{q-2i-1}u'^{2i}\varphi + 2iu^{q-2i}u'^{2i-1}\varphi' d\theta \\ - \frac{1}{q-1}\int_{\mathbb{S}^1} u'^{q-1}\varphi' d\theta = \int_{\mathbb{S}^1}(t^2u^2 + u'^2)^{q/2-1}(u'' + t^2u)\varphi d\theta, \quad \forall\varphi \in W^{1,q}(\mathbb{S}^1). \end{aligned} \tag{2.8}$$

Therefore $u(\theta)$ is a solution of (2.5) for almost everywhere $\theta \in \mathbb{S}^1$, i.e.

$$\int_{\mathbb{S}^1}(t^2u^2 + u'^2)^{q/2-1}(u'' + t^2u)\varphi d\theta = \lambda\int_{\mathbb{S}^1}\frac{g(\theta)}{u}\varphi d\theta, \quad \forall\varphi \in W^{1,q}(\mathbb{S}^1). \tag{2.9}$$

Proof. Let $u \in M$ be a critical point of I_q, then, the Fréchet derivative $I'_q(t,u) = 0$ in $W^{1,q/(q-1)}(\mathbb{S}^1)$. This implies that (2.6) holds with λ given by (2.7). In the following, we show that u is a weak solution of (2.5) via proving that (2.8) and (2.9) hold under additional assumption $u \in W^{2,2}(\mathbb{S}^1)$. For any $\varphi \in W^{1,q}(\mathbb{S}^1)$ and $i = 1, 2, \ldots, q/2$, by applying the Newton–Leibniz formula and an approximating argument, we obtain that

$$\int_{\mathbb{S}^1} u^{q-2i} u'^{\,2i-1} \varphi' d\theta = -(q-2i) \int_{\mathbb{S}^1} u^{q-2i-1} u'^{\,2i} \varphi d\theta - (2i-1) \int_{\mathbb{S}^1} u^{q-2i} u'^{\,2i-2} u'' \varphi d\theta.$$

Via applying this formula and rewriting the right-hand side of equation (2.8), we deduce that

$$\begin{aligned}
&\text{the right-hand side of (2.8)} = t^q \int_{\mathbb{S}^1} u^{q-1} \varphi d\theta + \int_{\mathbb{S}^1} u'^{\,q-2} u'' \varphi d\theta \\
&\quad + \sum_{i=1}^{q/2-1} t^{q-2i} \tau_i \int_{\mathbb{S}^1} 2i(2i-1) u^{q-2i} u'^{\,2i-2} u'' \varphi - (q-2i)(1-2i) u^{q-2i-1} u'^{\,2i} \varphi d\theta, \\
&= \underbrace{t^q \int_{\mathbb{S}^1} u^{q-1} \varphi d\theta + \sum_{i=1}^{q/2-1} t^{q-2i} \tau_i \int_{\mathbb{S}^1} (q-2i)(2i-1) u^{q-2i-1} u'^{\,2i} \varphi d\theta}_{(I)} \\
&\quad + \underbrace{\int_{\mathbb{S}^1} u'^{\,q-2} u'' \varphi d\theta + \sum_{i=1}^{q/2-1} t^{q-2i} \tau_i \int_{\mathbb{S}^1} 2i(2i-1) u^{q-2i} u'^{\,2i-2} u'' \varphi d\theta}_{(II)} \\
&:= (I) + (II).
\end{aligned} \tag{2.10}$$

By (2.4), $\tau_i = C^i_{q/2-1}/[(q-2i)(2i-1)]$ for $i = 1, 2, \ldots, q/2-1$. Then we simplify the formula (I) as

$$\begin{aligned}
(I) &= \int_{\mathbb{S}^1} \left(t^q u^{q-1} + \sum_{i=1}^{q/2-1} t^{q-2i} \tau_i (q-2i)(2i-1) u^{q-2i-1} u'^{\,2i} \right) \varphi d\theta \\
&= \int_{\mathbb{S}^1} \sum_{i=0}^{q/2-1} C^i_{q/2-1} t^{q-2i} u^{q-2i-1} u'^{\,2i} \varphi d\theta \\
&= \int_{\mathbb{S}^1} (t^2 u^2 + u'^{\,2})^{\frac{q-2}{2}} t^2 u \varphi d\theta.
\end{aligned} \tag{2.11}$$

For $i = 1, 2, \ldots, q/2-1$, we have $\tau_i = C^{i-1}_{q/2-1}/[2i(2i-1)]$ by (2.4), it follows that

$$
\begin{aligned}
(II) &= \int_{\mathbb{S}^1} u'^{\,q-2} u'' \varphi d\theta - q \sum_{i=1}^{q/2-1} t^{q-2i} \tau_i \int_{\mathbb{S}^1} 2i(2i-1) u^{q-2i} u'^{\,2i-2} u'' \varphi d\theta. \\
&= \int_{\mathbb{S}^1} \left(u'^{\,q-2} + \sum_{i=1}^{q/2-1} C_{q/2-1}^{i-1} t^{q-2i} u^{q-2i} u'^{\,2i-2} \right) u'' \varphi d\theta. \\
&= \int_{\mathbb{S}^1} \left(u'^{\,q-2} + \sum_{j=0}^{q/2-2} C_{q/2-1}^{j} t^{q-2j-2} u^{q-2j-2} u'^{\,2j} \right) u'' \varphi d\theta. \qquad (2.12) \\
&= \int_{\mathbb{S}^1} \left(\sum_{j=0}^{q/2-1} C_{q/2-1}^{j} t^{q-2j-2} u^{q-2j-2} u'^{\,2j} \right) u'' \varphi d\theta. \\
&= \int_{\mathbb{S}^1} (t^2 u^2 + u'^{\,2})^{\frac{q-2}{2}} u'' \varphi d\theta.
\end{aligned}
$$

Via (2.10)–(2.12) we obtain (2.8). And (2.9) follows from (2.6) and (2.8). □

For any $\alpha < \beta$, we define some Sobolev spaces as

$$
W^{1,q}(\alpha,\beta) = \left\{ u \in L^1(\alpha,\beta) : \int_\alpha^\beta |u|^q + |u'|^q d\theta < +\infty \right\},
$$

$$
W_0^{1,q}(\alpha,\beta) = \left\{ u \in W^{1,q}(\alpha,\beta) : u(\alpha) = u(\beta) = 0 \right\},
$$

$$
W^{2,2}(\alpha,\beta) = \left\{ u \in L^1(\alpha,\beta) : \int_\alpha^\beta |u|^2 + |u''|^2 d\theta < +\infty \right\}.
$$

In the next section, we also need a local version of (2.8).

Corollary 2.2. *Let $q \geq 2$ be even number, τ_i be defined by (1.7). Assume $u \in W^{2,2}(\alpha,\beta)$ and $\varphi \in W_0^{1,q}(\alpha,\beta)$. Then,*

$$
\begin{aligned}
&t^q \int_\alpha^\beta u^{q-1} \varphi d\theta - \sum_{i=1}^{q/2-1} \tau_i t^{q-2i} \int_\alpha^\beta (q-2i) u^{q-2i-1} u'^{\,2i} \varphi + 2i u^{q-2i} u'^{\,2i-1} \varphi' d\theta \\
&\quad - \frac{1}{q-1} \int_\alpha^\beta u'^{\,q-1} \varphi' d\theta = \int_\alpha^\beta (t^2 u^2 + u'^{\,2})^{q/2-1} (u'' + t^2 u) \varphi d\theta.
\end{aligned} \qquad (2.13)
$$

Proof. (2.13) follows by a similar calculation of (2.10)–(2.12). We omit the details here. □

3. Regularity of weak solution and a Poincaré-type inequality

In the rest of this paper, the regularity of weak solutions to a quasilinear elliptic equation will be used repeatedly. However, to the best of the authors' knowledge, there is not a suitable theorem in reference for being applied directly. For the sake of the completeness of our paper, we give the proof of the local differentiability to the weak solution as follows.

Lemma 3.1. *Let $q \geq 2$ be an even number, and $t > 0$ be a parameter. Assume that $f(\theta) \in C^{0,\gamma}(\mathbb{S}^1)$ for some $\gamma \in (0,1)$, and $u \in W^{1,q}(\mathbb{S}^1)$ be a weak solution of*

$$(t^2u^2 + u'^2)^{\frac{q-2}{2}}(u'' + t^2u) = f(\theta), \tag{3.1}$$

in the sense of

$$\begin{aligned} t^q \int_{\mathbb{S}^1} u^{q-1}\varphi d\theta - \sum_{i=1}^{q/2-1} \tau_i t^{q-2i} \int_{\mathbb{S}^1} (q-2i)u^{q-2i-1}u'^{2i}\varphi + 2iu^{q-2i}u'^{2i-1}\varphi' d\theta \\ - \frac{1}{q-1}\int_{\mathbb{S}^1} u'^{q-1}\varphi' d\theta = \int_{\mathbb{S}^1} f(\theta)\varphi d\theta, \quad \forall \varphi \in W^{1,q}(\mathbb{S}^1), \end{aligned} \tag{3.2}$$

where τ_i is defined by (1.7). If $u(\theta_0) > 0$ for given $\theta_0 \in \mathbb{S}^1$, then $u''(\theta)$ is continuous at $\theta = \theta_0$.

Proof. For $q = 2$, (3.1) is a linear equation. The conclusion is obvious. In the following part, we assume $q > 2$. Since $2\epsilon := u(\theta_0) > 0$ and $u \in W^{1,q}(\mathbb{S}^1) \subset C^{0,(q-1)/q}(\mathbb{S}^1)$, there exists $\delta > 0$ such that

$$|u(\theta) - 2\epsilon| < \epsilon \text{ for all } \theta \in (\theta_0 - \delta, \theta_0 + \delta). \tag{3.3}$$

For a small $h \in (0, \delta/8)$, we denote by $\Delta^h v := \Delta^h v(\theta) = (v(\theta+h) - v(\theta))/h$ the difference quotient of v. We define a cut-off function $\xi \in C_0^\infty(\mathbb{S}^1)$ such that $|\xi'(\theta)| < 8/\delta$ for all $\theta \in \mathbb{S}^1$, $\xi(\theta) = 1$ for $|\theta - \theta_0| < \delta/4$ and $\xi(\theta) = 0$ for $|\theta - \theta_0| > \delta/2$. Let $\varphi = \Delta^{-h}(\xi^2\Delta^h u)$ in (3.2), we deduce that

$$\int_{\mathbb{S}^1} A(u,u')[\Delta^{-h}(\xi^2\Delta^h u)]' d\theta + \int_{\mathbb{S}^1} B(u,u')\Delta^{-h}(\xi^2\Delta^h u) d\theta = \int_{\mathbb{S}^1} f\Delta^{-h}(\xi^2\Delta^h u) d\theta, \tag{3.4}$$

where $A(x,y)$ and $B(x,y)$ are binary polynomials defined by

$$A(x,y) = -2\sum_{i=1}^{q/2} i\tau_i t^{q-2i}x^{q-2i}y^{2i-1} \text{ and}$$

$$B(x,y)=t^q x^{q-1}-\sum_{i=1}^{q/2-1}(q-2i)\tau_i t^{q-2i}x^{q-2i-1}y^{2i}. \tag{3.5}$$

By the definition of difference quotient, we have $[\Delta^{-h}(\xi^2\Delta^h u)]'=\Delta^{-h}(\xi^2\Delta^h u)'=\Delta^{-h}(\xi^2\Delta^h u'+2\xi\xi'\Delta^h u)$ and $\int_{\mathbb{S}^1} w\Delta^{-h}v d\theta=-\int_{\mathbb{S}^1} v\Delta^h w d\theta$ for any $v,w\in L^2(\mathbb{S}^1)$. Applying these properties in (3.4) we deduce that

$$\int_{\mathbb{S}^1}\Delta^h A(u,u')(\xi^2\Delta^h u'+2\xi\xi'\Delta^h u)d\theta+\int_{\mathbb{S}^1}\xi^2\Delta^h B(u,u')\Delta^h u d\theta=-\int_{\mathbb{S}^1} f\Delta^{-h}(\xi^2\Delta^h u)d\theta. \tag{3.6}$$

And a direct calculation induces that

$$\begin{aligned}\Delta^h A(u(\theta),u'(\theta))&=A_1\Delta^h u(\theta)+A_2\Delta^h u'(\theta),\\ \Delta^h B(u(\theta),u'(\theta))&=B_1\Delta^h u(\theta)+B_2\Delta^h u'(\theta),\end{aligned} \tag{3.7}$$

where

$$\begin{pmatrix} A_1:=A_1(\theta,h)=\int_0^1\frac{\partial}{\partial x}A\left((1-s)u(\theta)+su(\theta+h),u'(\theta+h)\right)ds,\\ A_2:=A_2(\theta,h)=\int_0^1\frac{\partial}{\partial y}A\left(u(\theta),(1-s)u'(\theta)+su'(\theta+h)\right)ds,\\ B_1:=B_1(\theta,h)=\int_0^1\frac{\partial}{\partial x}B\left((1-s)u(\theta)+su(\theta+h),u'(\theta+h)\right)ds,\\ B_2:=B_2(\theta,h)=\int_0^1\frac{\partial}{\partial y}B\left(u(\theta),(1-s)u'(\theta)+su'(\theta+h)\right)ds.\end{pmatrix} \tag{3.8}$$

By applying formula (3.7) in equation (3.6), we obtain that

$$\begin{aligned}\int_{\mathbb{S}^1}A_2\xi^2(\Delta^h u')^2 d\theta=&-\int_{\mathbb{S}^1}A_1\xi^2\Delta^h u\Delta^h u'+2A_1\xi\xi'(\Delta^h u)^2+2A_2\xi\xi'\Delta^h u\Delta^h u' d\theta\\ &-\int_{\mathbb{S}^1}B_1\xi^2(\Delta^h u)^2+B_2\xi^2\Delta^h u\Delta^h u'+f\Delta^{-h}(\xi^2\Delta^h u)d\theta.\end{aligned} \tag{3.9}$$

From (3.5), we see that

$$\frac{\partial A(x,y)}{\partial x}=-2\sum_{i=1}^{q/2-1}i(q-2i)\tau_i t^{q-2i}x^{q-2i-1}y^{2i-1},$$

$$\frac{\partial A(x,y)}{\partial y} = -2\tau_1 t^{q-2} x^{q-2} - 2\sum_{i=2}^{q/2} i(2i-1)\tau_i t^{q-2i} x^{q-2i-1} y^{2i-2},$$

$$\frac{\partial B(x,y)}{\partial x} = (q-1)t^q x^{q-2} - \sum_{i=1}^{q/2-1} (q-2i)(q-2i-1)\tau_i t^{q-2i} x^{q-2i-2} y^{2i},$$

$$\frac{\partial B(x,y)}{\partial y} = -2\sum_{i=1}^{q/2-1} i(q-2i)\tau_i t^{q-2i} x^{q-2i-1} y^{2i-1}.$$

By applying these formulas and (3.3) in (3.8), we obtain a constant $c > 1$ depending only on t, q, δ, ϵ, and that the following estimates

$$|A_1| + |B_1| + |B_2| \le c(1 + |u'(\theta)| + |u'(\theta+h)|)^{q-2}, \tag{3.10}$$

$$\frac{1}{c}(1 + |u'(\theta)| + |u'(\theta+h)|)^{q-2} \le -A_2 \le c(1 + |u'(\theta)| + |u'(\theta+h)|)^{q-2} \tag{3.11}$$

hold for all $\theta \in (\theta_0 - \delta/2, \theta_0 + \delta/2)$. By applying (3.10), (3.11) and the Young's inequality in (3.9), we deduce that

$$\begin{aligned} &\frac{1}{2c}\int_{\mathbb{S}^1} \xi^2 (1 + |u'(\theta)| + |u'(\theta+h)|)^{q-2} (\Delta^h u')^2 d\theta \\ &\le C\int_{\mathbb{S}^1} (\xi^2 + |\xi\xi'|)(1 + |u'(\theta)| + |u'(\theta+h)|)^{q-2} (\Delta^h u)^2 d\theta + \left|\int_{\mathbb{S}^1} f\Delta^{-h}(\xi^2\Delta^h u) d\theta\right|, \end{aligned} \tag{3.12}$$

where $C > 1$ depends only on t, q, δ and ϵ. Let $h \to 0^+$, we have

$$\left|\int_{\mathbb{S}^1} f\Delta^{-h}(\xi^2\Delta^h u) d\theta\right| \le \max_{\theta\in\mathbb{S}^1} |f(\theta)| \int_{\mathbb{S}^1} 2|\xi\xi'\Delta^h u| + |\xi^2\Delta^h u'| d\theta + o(1);$$

it follows from the Young's inequality that there exists $C_\nu > 0$, depending only on $\nu > 0$, δ and q, such that

$$\left|\int_{\mathbb{S}^1} f\Delta^{-h}(\xi^2\Delta^h u) d\theta\right| \le \max_{\theta\in\mathbb{S}^1} |f(\theta)| \left(\int_{\mathbb{S}^1} 2\xi|\xi'||\Delta^h u|^q d\theta + \nu\int_{\mathbb{S}^1} \xi^2 |\Delta^h u'|^2 d\theta + C_\nu + o(1)\right). \tag{3.13}$$

Since $u \in W^{1,q}(\mathbb{S}^1)$, let ν be small enough and $h \to 0^+$, from (3.12) and (3.13) we deduce that

$$\int_{\theta_0-\delta/4}^{\theta_0+\delta/4} (u'')^2 d\theta \leq C_1 \int_{\theta_0-\delta}^{\theta_0+\delta} |u'|^q d\theta + C_2, \tag{3.14}$$

where C_1 and C_2 depend only on t, ϵ, δ and q. That is, $u''(\theta)$ exists for almost everywhere $\theta \in (\theta_0 - \delta/4, \theta_0 + \delta/4)$ and

$$u \in W^{2,2}(\theta_0 - \delta/4, \theta_0 + \delta/4) \subset C^{1,1}(\theta_0 - \delta/4, \theta_0 + \delta/4). \tag{3.15}$$

Let $\zeta \in W^{1,q}(\mathbb{S}^1)$ with compact support in $(\theta_0 - \delta/4, \theta_0 + \delta/4)$, we define $\phi(\theta) = (t^2u^2 + u'^{\,2})^{\frac{2-q}{2}}\zeta$ for $\theta \in (\theta_0 - \delta/4, \theta_0 + \delta/4)$ and $\phi(\theta) = 0$ for $\theta \in \mathbb{S}^1 \setminus (\theta_0 - \delta/4, \theta_0 + \delta/4)$. Then it is easy to check that $\phi \in W^{1,q}(\mathbb{S}^1)$ by combining (3.3) and (3.15). Let $\varphi = \phi$ in (3.2), by applying (2.13) with $\alpha = \theta_0 - \delta/4$ and $\beta = \theta_0 + \delta/4$ we obtain that

$$\begin{aligned}
&\int_{\theta_0-\delta/4}^{\theta_0+\delta/4} (u'' + t^2u)\zeta d\theta = \int_{\theta_0-\delta/4}^{\theta_0+\delta/4} (t^2u^2 + u'^{\,2})^{q/2-1}(u'' + t^2u)\phi d\theta \\
&= t^q \int_{\theta_0-\delta/4}^{\theta_0+\delta/4} u^{q-1}\phi d\theta - \frac{1}{q-1} \int_{\theta_0-\delta/4}^{\theta_0+\delta/4} u'^{\,q-1}\phi' d\theta \\
&\quad - \sum_{i=1}^{q/2-1} \tau_i t^{q-2i} \int_{\theta_0-\delta/4}^{\theta_0+\delta/4} (q-2i)u^{q-2i-1}u'^{\,2i}\phi + 2iu^{q-2i}u'^{\,2i-1}\phi' d\theta \\
&= t^q \int_{\mathbb{S}^1} u^{q-1}\phi d\theta - \frac{1}{q-1} \int_{\mathbb{S}^1} u'^{\,q-1}\phi' d\theta \\
&\quad - \sum_{i=1}^{q/2-1} \tau_i t^{q-2i} \int_{\mathbb{S}^1} (q-2i)u^{q-2i-1}u'^{\,2i}\phi + 2iu^{q-2i}u'^{\,2i-1}\phi' d\theta \\
&= \int_{\mathbb{S}^1} f(\theta)\phi d\theta = \int_{\theta_0-\delta/4}^{\theta_0+\delta/4} f(\theta)(t^2u^2 + u'^{\,2})^{\frac{2-q}{2}}\zeta d\theta.
\end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that $u(\theta)$ satisfies equation $u''(\theta) + t^2u(\theta) = f(\theta)(t^2u^2 + u'^{\,2})^{\frac{2-q}{2}}$ almost everywhere in $(\theta_0 - \delta/4, \theta_0 + \delta/4)$. By (3.3), (3.15) and the assumption $f \in C^{0,\gamma}(\mathbb{S}^1)$, we deduce that $f(\theta)(t^2u^2 + u'^{\,2})^{\frac{2-q}{2}}$ is smooth enough over $(\theta_0 - \delta/4, \theta_0 + \delta/4)$, that a standard regularity theorem can be applied for deducing $u \in C^2(\theta_0 - \delta/4, \theta_0 + \delta/4)$. $\square$

To estimate the bound of functional I_q, we need the following inequalities.

Lemma 3.2. *Let $q > 0$ be an even number, $u \in W^{1,q}(\mathbb{S}^1)$ with $u(\theta_0) = 0$ for some $\theta_0 \in \mathbb{S}^1$. For each $i = 1, 2, \ldots, q/2$, there exists $c_i > 0$ such that*

$$\int_{\mathbb{S}^1} u^q(\theta)d\theta \leq c_i \int_{\mathbb{S}^1} u^{q-2i}u'^{\,2i}d\theta. \tag{3.17}$$

Proof. For each $i = 1, 2, \ldots, q/2$, by using the Newton–Leibniz formula and Hölder inequality we derive that

$$\begin{aligned}
\int_{\mathbb{S}^1} u^q(\theta)d\theta &= \int_{\mathbb{S}^1} \left(\int_{\theta_0}^{\theta} \frac{du^{\frac{q}{2i}}(t)}{dt}dt \right)^{2i} d\theta = \left(\frac{q}{2i}\right)^{2i} \int_{\mathbb{S}^1} \left(\int_{\theta_0}^{\theta} u^{\frac{q}{2i}-1}(t)u'(t)dt \right)^{2i} d\theta \\
&\leq \left(\frac{q}{2i}\right)^{2i} \int_{\mathbb{S}^1} \left((\theta-\theta_0)^{2i-1} \int_{\theta_0}^{\theta} u^{q-2i}u'^{\,2i}dt \right) d\theta \\
&\leq \left(\frac{q}{2i}\right)^{2i} \frac{(2\pi)^{2i}}{2i} \int_{\mathbb{S}^1} u^{q-2i}u'^{\,2i}d\theta.
\end{aligned}$$

Let $c_i = (q/2i)^{2i}(2\pi)^{2i}/2i$, we obtain (3.17). □

The inequality (3.17) is not enough for a direct application to estimate the bound of functional I_q. However, (3.17) is useful to prove the following new Poincaré-type inequality (3.18) with equality conditions, which will play an essential role in the process of studying the bound of I_q.

Lemma 3.3. *Let $\theta_0 \in \mathbb{S}^1$ be a given point, we denote by $W^{1,q}_{\theta_0} = \{u \in W^{1,q}(\mathbb{S}^1) : u(\theta_0) = 0\}$ a subspace of $W^{1,q}(\mathbb{S}^1)$. Assume $q > 0$ be an even number, τ_i be defined by (1.7) and $u \in W^{1,q}_{\theta_0}$. Then,*

$$\int_{\mathbb{S}^1} u^q d\theta \leq q\sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u^{q-2i}u'^{\,2i}d\theta. \tag{3.18}$$

And the equality in (3.18) holds if and only if $u(\theta) = l|\sin(\theta-\theta_0)|$ with $l \in \mathbb{R}$.

Proof. It is clear that (3.18) holds for $u = 0$. For the case $u \neq 0$, we consider the following eigenvalue problem

$$\mu := \inf_{u\in W^{1,q}_{\theta_0}, u\neq 0} \frac{q\sum\limits_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u^{q-2i}u'^{\,2i}d\theta - \int_{\mathbb{S}^1} u^q d\theta}{\int_{\mathbb{S}^1} u^q d\theta}.$$

If $\mu \geq 0$, we obtain (3.18). Otherwise $\mu < 0$, a contradiction will be deduced in the following. By (3.17), we see that $\mu > -\infty$. There exists a sequence $\{u_n\} \subset W^{1,q}_{\theta_0}$ such that $|u_n|_q^q := \int_{\mathbb{S}^1} u_n^q d\theta = 1$ and

$$\lim_{n\to+\infty} q\sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u_n^{q-2i} u_n'^{2i} d\theta - \int_{\mathbb{S}^1} u_n^q d\theta = \mu \in (-\infty, 0).$$

It follows that $\{u_n\}$ is bounded in $W^{1,q}(\mathbb{S}^1)$. By the compactness of embedding $W^{1,q}_{\theta_0} \hookrightarrow C^{0,\gamma}(\mathbb{S}^1)$ with $\gamma \in [0, (q-1)/q)$, we obtain a function $u_0 \in W^{1,q}_{\theta_0}$ such that $|u_0|_q = 1$, $u_0(\theta_0) = 0$ and up to a subsequence,

$$\lim_{n\to+\infty} \int_{\mathbb{S}^1} u_n^q d\theta = \int_{\mathbb{S}^1} u_0^q d\theta \quad \text{and} \quad \lim_{n\to+\infty} \int_{\mathbb{S}^1} u_n^{q-2i} u_n'^{2i} d\theta \geq \int_{\mathbb{S}^1} u_0^{q-2i} u_0'^{2i} d\theta$$

for $i = 1, 2, \ldots, q/2$. Hence, $q\sum\limits_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u_0^{q-2i} u_0'^{2i} d\theta - \int_{\mathbb{S}^1} u_0^q d\theta = \mu$. The variational principle implies that

$$\begin{aligned} &\sum_{i=1}^{q/2-1} \tau_i \int_{\mathbb{S}^1} (q-2i) u_0^{q-2i-1} u_0'^{2i} \varphi + 2i u_0^{q-2i} u_0'^{2i-1} \varphi' d\theta \\ &\quad - \int_{\mathbb{S}^1} u_0^{q-1} \varphi d\theta + \frac{1}{q-1} \int_{\mathbb{S}^1} u_0'^{q-1} \varphi' d\theta = \mu \int_{\mathbb{S}^1} u_0^{q-1} \varphi d\theta, \quad \forall \varphi \in W^{1,q}_{\theta_0}. \end{aligned} \tag{3.19}$$

Without loss of generality, we assume $u_0 \geq 0$. Since u_0 is $\pi-$periodic and $|u_0|_q = 1$, there exists an interval $[\theta_1, \theta_2] \subset [0, \pi]$ such that $u_0(\theta) > 0$ for $\theta \in (\theta_1, \theta_2)$ and $u_0(\theta_1) = u_0(\theta_2) = 0$. By applying Lemma 3.1 with $f(\theta) = \mu u_0^{q-1}(\theta)$, we see that u_0 is C^2 smooth over interval (θ_1, θ_2). In (3.19), let φ be a smooth function with compact support in (θ_1, θ_2); it follows from Corollary 2.2 that

$$(u_0^2 + u_0'^2)^{\frac{q-2}{2}} [u_0''(\theta) + u_0(\theta)] + \mu u_0^{q-1}(\theta) = 0, \quad \theta \in (\theta_1, \theta_2). \tag{3.20}$$

Let $\sigma_0^2(\theta) = u_0^2(\theta) + u_0'^2(\theta)$. Via multiplying both sides of (3.20) by $u_0'(\theta)$, we deduce that

$$(\sigma_0^q(\theta))' = -\mu (u_0^q(\theta))', \quad \theta \in (\theta_1, \theta_2). \tag{3.21}$$

Let $|u_0(\alpha)| = \max\limits_{\theta\in(\theta_1,\theta_2)} |u_0(\theta)| > 0$, then $u_0'(\alpha) = 0$. By integrating both sides of (3.21) over interval (θ_1, θ), we obtain $(1+\mu)u_0^q(\alpha) = \lim\limits_{\beta\to\theta_1^+} u_0'^q(\beta) \geq 0$ for $\theta = \alpha$, and $\sigma_0^q(\theta) + \mu u_0^q(\theta) = C$ for $\theta \in (\theta_1, \theta_2)$, where C is a constant. It follows that $\mu \geq -1$. If $\mu = -1$, we have $C = 0$ by the obvious fact that $\sigma_0^q(\alpha) = u_0^q(\alpha)$. Then we deduce that $u_0'(\theta) = 0$, therefore

$u_0(\theta) \equiv 0$ for all $\theta \in (\theta_1, \theta_2)$, which contradicts with $u_0(\theta) > 0$ over interval (θ_1, θ_2). Hence, we obtain that

$$-1 < \mu < 0 \text{ and } \lim_{\theta \to \theta_1^+} {u_0'}^q(\theta) = (1+\mu) u_0^q(\alpha) > 0. \tag{3.22}$$

By a similar argument, we obtain that $\lim_{\theta \to \theta_2^-} {u_0'}^q(\theta) = (1+\mu) u_0^q(\alpha) > 0$. So,

$$(u_0^2(\theta) + {u_0'}^2(\theta))^{\frac{2-q}{2}} \in L^\infty(\theta_1, \theta_2) \text{ for } q > 2.$$

Via multiplying by $(u_0^2 + {u_0'}^2)^{\frac{2-q}{2}} u_0$ on both sides of (3.20) and then integrating them over interval (θ_1, θ_2) we get

$$\int_{\theta_1}^{\theta_2} u_0^2 - {u_0'}^2 d\theta = -\mu \int_{\theta_1}^{\theta_2} u_0^2 (u_0^2 + {u_0'}^2)^{\frac{2-q}{2}} d\theta > 0, \tag{3.23}$$

where we have used the first part of (3.22). Since $[\theta_1, \theta_2] \subset [0, \pi]$, we see that (3.23) contradicts to the Wirtinger's inequality $\int_{\theta_1}^{\theta_2} u_0^2 d\theta \leq (\theta_2 - \theta_1)^2/\pi^2 \int_{\theta_1}^{\theta_2} {u_0'}^2 d\theta$.

In the following, we show that the equality in (3.18) holds if and only if $u = l|\sin(\theta - \theta_0)|$ for $l \in \mathbb{R}$. Firstly, we check the sufficient condition. Let $v = l|\sin(\theta - \theta_0)| \in W_{\theta_0}^{1,q}$ with $l \in \mathbb{R}$, we see that $v''(\theta) + v(\theta) = 0$ for $\theta \neq \theta_0 + m\pi$ with $m \in \mathbb{Z}$. And it is not difficult to check that v is a weak solution of

$$(u^2 + {u'}^2)^{\frac{q-2}{2}} [u'' + u] = 0, \quad \theta \in \mathbb{S}^1.$$

By a direct calculation or applying the Corollary 2.2 with $u = \varphi = v$, we deduce that

$$\int_{\mathbb{S}^1} v^q d\theta = q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} v^{q-2i} {v'}^{2i} d\theta.$$

Let u_0 be a minimum of $\mu = 0$ in $W_{\theta_0}^{1,q}$. Similar as the process of deducing for (3.20) we obtain that $u_0(\theta)$ is C^2 smooth over $\mathbb{S}^1 \setminus \{\theta \in \mathbb{S}^1 : u_0(\theta) = 0\}$, and

$$(u_0^2(\theta) + {u_0'}^2(\theta))^{\frac{q-2}{2}} [u_0''(\theta) + u_0(\theta)] = 0 \quad \text{for } \theta \in \mathbb{S}^1 \setminus \{\theta \in \mathbb{S}^1 : u_0(\theta) = 0\}. \tag{3.24}$$

Hence,

$$\begin{aligned} (\sigma_0^q(\theta))' &= q(u_0^2(\theta) + {u_0'}^2(\theta))^{\frac{q-2}{2}} [u_0''(\theta) + u_0(\theta)] u_0'(\theta) = 0 \\ &\text{for } \theta \in \mathbb{S}^1 \setminus \{\theta \in \mathbb{S}^1 : u_0(\theta) = 0\}. \end{aligned} \tag{3.25}$$

Since u_0 is a continuous function, the set $\mathbb{S}^1 \setminus \{\theta \in \mathbb{S}^1 : u_0(\theta) = 0\}$ may be consist of many subintervals (θ_i, η_i) with $|u_0(\theta)| > 0$ for $\theta \in (\theta_i, \eta_i)$ and $u_0(\theta_i) = u_0(\eta_i) = 0$, $(i = 1, 2, 3, \dots)$. Then, by integrating (3.25) over (θ_i, η_i), we deduce that

$$\lim_{\theta\to\theta_i^+} {u_0'}^2(\theta) = \lim_{\theta\to\eta_i^-} {u_0'}^2(\theta) = \max_{\theta\in(\theta_i,\eta_i)} u_0^2(\theta) > 0. \tag{3.26}$$

From (3.25) and (3.26) we see that $\sigma_0^2(\theta) = u_0^2(\theta) + {u_0'}^2(\theta) \equiv \max_{\theta\in(\theta_i,\eta_i)} u_0^2(\theta) > 0$ for $\theta \in (\theta_i, \eta_i)$. It follows from (3.24) that u_0 is a nontrivial solution of the boundary value problem

$$u''(\theta) + u(\theta) = 0, \quad \theta \in (\theta_i, \eta_i),\ u(\theta_i) = u(\eta_i) = 0.$$

So $\eta_i - \theta_i = \pi$ and $u_0(\theta) = l|\sin(\theta - \theta_i)|$ with $l \neq 0$. Since u_0 is $\pi-$periodic and $u_0(\theta_0) = 0$, we see that $\theta_i \equiv \theta_0$ and $\eta_i \equiv \theta_0 + \pi$, $(i = 1, 2, 3, \dots)$. Hence $u_0(\theta) = l|\sin(\theta - \theta_0)|$. And the equality in (3.18) also holds obviously when $u = 0$. So, we get the necessary condition. □

4. Proof of the main theorems

Before proving the main theorems of this paper, we need the following conclusions.

Lemma 4.1. *Let $q > 0$ be even number, τ_i be defined by (1.7) and $t \in (0, 1)$. Assume $\{u_n\} \subset W^{1,q}(\mathbb{S}^1)$ be a sequence of positive functions, and that*

$$\int_{\mathbb{S}^1} t^q u_n^q d\theta \geq q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} t^{q-2i} u_n^{q-2i} {u_n'}^{2i} d\theta. \tag{4.1}$$

If $\|u_n\| \equiv 1$, then there exists a positive function $v \in W^{1,q}(\mathbb{S}^1)$, and up to a subsequence,

$$u_n \rightharpoonup v \text{ weakly in } W^{1,q}(\mathbb{S}^1), \quad \text{and } u_n \to v \text{ uniformly on } \mathbb{S}^1, \text{ as } n \to +\infty. \tag{4.2}$$

Proof. Since $\{u_n\}$ is bounded in $W^{1,q}(\mathbb{S}^1)$, there exists a $v \in W^{1,q}(\mathbb{S}^1)$, up to a subsequence of $\{u_n\}$, $u_n \overset{n\to+\infty}{\rightharpoonup} v$ weakly in $W^{1,q}(\mathbb{S}^1)$. By the compactness of embedding from $W^{1,q}(\mathbb{S}^1)$ to $C^{0,\gamma}(\mathbb{S}^1)$ with $\gamma \in [0, (q-1)/q)$, we see that the second part of (4.2) holds. By (4.2) and $u_n > 0$, we deduce that $v \geq 0$. If $v \equiv 0$, it follows from (4.2) that, up to a subsequence,

$$\int_{\mathbb{S}^1} u_n^q d\theta = o(1) \quad \text{and} \quad \int_{\mathbb{S}^1} {u_n'}^q d\theta = 1 + o(1), \quad \text{as } n \to +\infty. \tag{4.3}$$

From (4.3) and (4.1) we obtain a contradiction. Hence $v \not\equiv 0$ and $|v|_q > 0$. By combining (4.1) and (4.2) we obtain that

$$\int_{\mathbb{S}^1} t^q v^q d\theta \geq q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} t^{q-2i} v^{q-2i} v'^{\,2i} d\theta.$$

Since $t \in (0,1)$ and $|v|_q > 0$, it follows that

$$\int_{\mathbb{S}^1} v^q d\theta > q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} v^{q-2i} v'^{\,2i} d\theta. \tag{4.4}$$

If $v(\theta_0) = 0$ for some $\theta_0 \in \mathbb{S}^1$, by Lemma 3.3 we deduce that

$$\int_{\mathbb{S}^1} v^q d\theta \leq q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} v^{q-2i} v'^{\,2i} d\theta,$$

which contradicts to (4.4). So, $v(\theta) > 0$ for all $\theta \in \mathbb{S}^1$. □

Theorem 4.2. *Let $q \geq 2$ be even, and τ_i be defined by (1.7). Assume $g(\theta) \in W^{1,k}(\mathbb{S}^1)$ for some $k > 1$. If $\int_{\mathbb{S}^1} g(\theta)d\theta > 0$, there exist $u_1 \in M$ and constant $C_q \geq 2\pi$ such that*

$$\left(\int_{\mathbb{S}^1} u^q d\theta - q \sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u^{q-2i} u'^{\,2i} d\theta\right) \cdot \exp\left(-q \frac{\int_{\mathbb{S}^1} g(\theta) \ln u d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta}\right) \leq I_q(1, u_1) =: C_q \tag{4.5}$$

holds for all positive functions $u \in W^{1,q}(\mathbb{S}^1)$.

Proof. Since $\int_{\mathbb{S}^1} g(\theta)d\theta > 0$, we see that $I_q(t,u)$ is C^1 on M. And I_q is also homogeneous on u with degree zero, that is, $I_q(t, lu) = I_q(t,u)$ for $l > 0$. We prove this theorem by the following six steps.

Step 1. For $t \in (0,1)$, there exists $u_t \in M$ such that $\max_{u \in M} I_q(t,u) = I_q(t,u_t) \geq 2\pi t^q$ and $\|u_t\| = 1$.

Let $\{u_n\} \subset M$ be a maximizing sequence of $\max_{u \in M} I_q(t,u) \geq I_q(t,1) = 2\pi t^q$. Without loss of generality, let $\|u_n\| \equiv 1$ by the homogeneity of I_q on u. Then, we apply Lemma 4.1 to get a positive function $u_t \in M$ and a subsequence of $\{u_n\}$ such that (4.2) holds with $v = u_t$. It follows that

$$\lim_{n \to +\infty} \int_{\mathbb{S}^1} g(\theta) \ln u_n d\theta = \int_{\mathbb{S}^1} g(\theta) \ln u_t d\theta \quad \text{and} \quad \int_{\mathbb{S}^1} u_t^{q-2i} u_t'^{\,2i} d\theta \leq \inf \lim_{n \to +\infty} \int_{\mathbb{S}^1} u_n^{q-2i} u_n'^{\,2i} d\theta$$

holds for all $i = 1, 2, \ldots, q/2$. Hence $I_q(t,u_t) = \lim_{n \to +\infty} I_q(t,u_n) = \max_{u \in M} I_q(t,u) \geq I_q(t,1) = 2\pi t^q$. By the property that I_q is homogeneous on u, we replaced u_t by $u_t/\|u_t\|$ so that $\|u_t\| = 1$.

Step 2. For each $t \in (0,1)$, let u_t be given by step 1. Then $u_t \in C^2(\mathbb{S}^1)$ is a positive solution of

$$(t^2u^2 + u'^2)^{\frac{q-2}{2}}(u'' + t^2u) = a_t\frac{g(\theta)}{u},\ \theta \in \mathbb{S}^1, \tag{4.6}$$

where

$$a_t = \left(\int_{\mathbb{S}^1} t^q u_t^q d\theta - q\sum_{i=1}^{q/2}\tau_i \int_{\mathbb{S}^1} t^{q-2i}u_t^{q-2i}u_t'^{2i}d\theta\right) / \int_{\mathbb{S}^1} g(\theta)d\theta > 0. \tag{4.7}$$

By step 1, $u_t \in M$ is a critical point of I_q. It follows from Theorem 2.1 that u_t is a weak solution of (2.5) with $\lambda = a_t$, where a_t is given by (4.7). Since $u_t(\theta) > 0$ for $\theta \in \mathbb{S}^1$, we have that $g(\theta)/u \in C^{0,\gamma}(\mathbb{S}^1)$ for some $\gamma \in (0,1)$. It follows from Lemma 3.1 that $u_t \in C^2(\mathbb{S}^1)$. By applying (2.9) of Theorem 2.1, we deduce that u_t is a positive solution of (4.6).

Step 3. Let $\epsilon_t = \min_{\theta\in\mathbb{S}^1} u_t(\theta) > 0$. If ϵ_t converges to 0 as $t \to 1^-$, then $a_t = O(\epsilon_t)$ as $t \to 1^-$.

For $t \in (0,1)$, from (4.7) we deduce that

$$a_t\int_{\mathbb{S}^1} g(\theta)d\theta = \int_{\mathbb{S}^1} t^q u_t^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} t^{q-2i}u_t^{q-2i}u_t'^{2i}d\theta \le \int_{\mathbb{S}^1} u_t^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} u_t^{q-2i}u_t'^{2i}d\theta.$$

Let $v_t = u_t - \epsilon_t$. If $\epsilon_t \to 0$, then $\{v_t\}$ is also bounded in $W^{1,q}(\mathbb{S}^1)$. It follows that

$$\begin{aligned} a_t\int_{\mathbb{S}^1} g(\theta)d\theta &\le \int_{\mathbb{S}^1}(v_t+\epsilon_t)^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1}(v_t+\epsilon_t)^{q-2i}v_t'^{2i}d\theta \\ &= \int_{\mathbb{S}^1} v_t^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} v_t^{q-2i}v_t'^{2i}d\theta + \sum_{i=0}^{q-1}C_q^i\int_{\mathbb{S}^1} v_t^i\epsilon_t^{q-i}d\theta \\ &\quad - q\sum_{i=1}^{q/2}\tau_i\sum_{j=0}^{q-2i-1}C_{q-2i}^j\epsilon_t^{q-2i-j}\int_{\mathbb{S}^1} v_t'^{2i}v_t^j d\theta \\ &= \int_{\mathbb{S}^1} v_t^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} v_t^{q-2i}v_t'^{2i}d\theta + O(\epsilon_t). \end{aligned} \tag{4.8}$$

Since $\min_{\theta\in\mathbb{S}^1} v_t(\theta) = 0$, by applying Lemma 3.3 we deduce that

$$\int_{\mathbb{S}^1} v_t^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} v_t^{q-2i}v_t'^{2i}d\theta \le 0.$$

This and (4.8) show that $a_t = O(\epsilon_t)$ if $\epsilon_t \to 0$ in the process of $t \to 1^-$.

Step 4. There exists $u_1 \not\equiv 0$ such that, up to a subsequence, $u_t \rightharpoonup u_1$ weakly in $W^{1,q}(\mathbb{S}^1)$ and $u_t \to u_1$ strongly in $C^{0,\frac{q-1}{2q}}(\mathbb{S}^1)$ as $t \to 1^-$. And there exists a positive constant $\sigma > 0$ such that $\inf\limits_{t\in(0,1)} \max\limits_{\theta\in\mathbb{S}^1} u_t(\theta) \geq \sigma > 0$.

Since $\|u_t\| \equiv 1$, there exists $u_1(\theta) \in W^{1,q}(\mathbb{S}^1)$ and a subsequence of $\{u_t\}$ such that $u_t \rightharpoonup u_1$ weakly in $W^{1,q}(\mathbb{S}^1)$ as $t \to 1^-$. By the compactness of embedding from $W^{1,q}(\mathbb{S}^1)$ to $C^{0,\frac{q-1}{2q}}(\mathbb{S}^1)$, we have a subsequence of $\{u_t\}$ that $u_t \to u_1$ strongly in $C^{0,\frac{q-1}{2q}}(\mathbb{S}^1)$ as $t \to 1^-$. If $u_1(\theta) \equiv 0$, we see that $u_t(\theta)$ converges to 0 uniformly on $\theta \in \mathbb{S}^1$ as $t \to 1^-$. Let $t \to 1^-$, by using Hölder inequality we see that $\int_{\mathbb{S}^1} u_t^{q-2i} u_t'^{2i} d\theta = o(1)$ for all $i = 1, 2, \ldots, q/2 - 1$, and therefore $\int_{\mathbb{S}^1} |u_t'|^q d\theta = 1 + o(1)$ via $\|u_t\| \equiv 1$. These together with the definition of a_t in (4.7) imply that $a_t < 0$ as t being close to 1, which is a contradiction. Hence $u_1 \not\equiv 0$. And $\inf\limits_{t\in(0,1)} \max\limits_{\theta\in\mathbb{S}^1} u_t(\theta) \geq \sigma > 0$ is obvious by a similar argument.

Step 5. Let u_1 be given by step 4, then $u_1(\theta) > 0$ for all $\theta \in \mathbb{S}^1$. Therefore $u_1 \in M$.

From step 4, we see that u_1 is the limitation of a subsequence of $\{u_t\}$ in a Hölder space. If the sequence $\epsilon_t := \min\limits_{\theta\in\mathbb{S}^1} u_t(\theta)$ has a positive bound from below. Then, u_1 is a positive function; therefore, $u_1 \in M$. We prove these conclusions by the following contradiction. If a subsequence of $\{\epsilon_t\}$ converges to 0 as $t \to 1^-$. It follows from step 3 that $a_t = O(\epsilon_t)$ as $t \to 1^-$. Let $\alpha, \beta \in \mathbb{S}^1$. Since $u_t \in C^2(\mathbb{S}^1)$ and $g \in W^{1,k}(\mathbb{S}^1)$ for some $k > 1$, we have

$$
\begin{aligned}
\left| \int_\alpha^\beta a_t \frac{g(\theta) u_t'}{u_t} d\theta \right| &= \left| a_t \left[g(\beta) \ln u_t(\beta) - g(\alpha) \ln u_t(\beta) \right] - \int_\alpha^\beta a_t g'(\theta) \ln u_t d\theta \right| \\
&\leq C\epsilon_t \left[|g(\beta) \ln u_t(\beta)| + |g(\alpha) \ln u_t(\beta)| \right] + C\epsilon_t \max_{\theta\in\mathbb{S}^1} |\ln u_t(\theta)| \int_{\mathbb{S}^1} |g'(\theta)|^k d\theta \qquad (4.9) \\
&\leq C \left(\max_{\theta\in\mathbb{S}^1} |g(\theta)| + \int_{\mathbb{S}^1} |g'(\theta)|^k d\theta \right) \epsilon_t \max_{\theta\in\mathbb{S}^1} |\ln(u_t(\theta))|.
\end{aligned}
$$

On the other hand, the fact $\max\limits_{\theta\in\mathbb{S}^1} |u_t(\theta)| \leq C_0 \|u_t\| = C_0$ implies that

$$
\epsilon_t \max_{\theta\in\mathbb{S}^1} |\ln(u_t(\theta))| = \epsilon_t |\ln(\min_{\theta\in\mathbb{S}^1} u_t(\theta))| = -\epsilon_t \ln \epsilon_t \to 0 \text{ as } \epsilon_t \to 0^+.
$$

Since $g \in W^{1,k}(\mathbb{S}^1)$, from (4.9) we deduce that

$$
\left| \int_\alpha^\beta a_t \frac{g(\theta) u_t'}{u_t} d\theta \right| \to 0 \text{ as } t \to 1^-.
$$

Via multiplying both sides of (4.6) by u_t', then integrating them over interval (α, β) we obtain

$$\frac{1}{q}\int_{\alpha}^{\beta}\frac{d(t^2u_t^2+{u_t'}^2)^{\frac{q}{2}}}{d\theta}d\theta=\int_{\alpha}^{\beta}\frac{a_tg(\theta)u_t'}{u_t}d\theta\overset{t\to 1^-}{\to}0. \tag{4.10}$$

For each $t\in(0,1)$, let β be a global maximum of u_t, i.e., $U_t:=u_t(\beta)=\max_{\theta\in\mathbb{S}^1}u_t(\theta)$ and $u_t'(\beta)=0$. It follows from (4.10) that

$$\frac{1}{q}\left[U_t^q-(t^2u_t^2(\alpha)+{u_t'}^2(\alpha))^{\frac{q}{2}}\right]=\frac{1}{q}\int_{\alpha}^{\beta}\frac{d(t^2u_t^2+{u_t'}^2)^{\frac{q}{2}}}{d\theta}d\theta\overset{t\to 1^-}{\to}0. \tag{4.11}$$

By step 4, we see that $U_t\geq\sigma>0$ uniformly on $t\in(0,1)$. Let $1-t$ be small enough, (4.11) shows that

$$u_t^2(\alpha)+{u_t'}^2(\alpha)>\sigma^2/2>0,\ \text{for all }\alpha\in\mathbb{S}^1.$$

Furthermore, let α be a global minimum of u_t, i.e. $u_t(\alpha)=\min_{\theta\in\mathbb{S}^1}u_t(\theta)=\epsilon_t$ and $u_t'(\alpha)=0$. Then, we deduce the following contradiction.

$$\sigma^2/2<u_t^2(\alpha)+{u_t'}^2(\alpha)=\epsilon_t^2\to 0\text{ as }t\to 1^-.$$

Step 6. Let u_1 be given by step 4, then (4.5) holds for all positive function $u\in W^{1,q}(\mathbb{S}^1)$.

By step 5, we see that $u_1\in M$. If there exists $w\in M$ such that $I_q(1,w)>I_q(1,u_1)$, it follows from the definition of I_q and the conclusion of step 4 that

$$\lim_{t\to 1^-}I_q(t,w)=I_q(1,w)>I_q(1,u_1)\geq\lim_{t\to 1^-}I_q(t,u_t)$$

Let t_0 be close to 1 enough, then

$$I_q(t_0,w)>I_q(t_0,u_{t_0}),$$

which contradicts with the fact that $I_q(t,u_t)=\max_{u\in M}I_q(t,u)$ for $t\in(0,1)$ in step 1. Hence $C_q:=I_q(1,u_1)=\max_{u\in M}I_q(1,u)\geq I_q(1,1)=2\pi$. □

In the following, we give the proofs of our main theorems.

Proof of Theorem 1.1. Let $u_1\in M$ and constant $C_q\geq 2\pi$ be obtained in Theorem 4.2. Then u_1 is a critical point of $I_q(1,u)$ in M. By applying Theorem 2.1 we see that u_1 is also a weak solution of

$$(u^2+{u'}^2)^{\frac{q-2}{2}}(u''+u)=\lambda_1\frac{g(\theta)}{u},\ \theta\in\mathbb{S}^1, \tag{4.12}$$

where

$$\lambda_1 = \frac{\int_{\mathbb{S}^1} u_1^q d\theta - q\sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u_1^{q-2i} u_1'^{2i} d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta} = \frac{C_q}{\int_{\mathbb{S}^1} g(\theta) d\theta} \exp\left(\frac{q\int_{\mathbb{S}^1} g(\theta)\ln u_1 d\theta}{\int_{\mathbb{S}^1} g(\theta) d\theta}\right) > 0.$$

We see that $u_1 \in C^{0,(q-1)/q}(\mathbb{S}^1)$ by the embedding theorem. Then, by applying the Lemma 3.1 with $f(\theta) = g(\theta)/u_1$ we obtain that $u_1 \in C^2(\mathbb{S}^1)$. Therefore u_1 is a positive solution of (4.12). By apply the homogeneous property of (4.12) and (1.5), we see that $\lambda_1^{-1/q} u_1$ is a positive solution of (1.5). □

Proof of Theorem 1.2. The inequality (1.9) is a direct conclusion of (4.5) in Theorem 4.2. Let w be the solution of (1.5) obtained by the maximum of I_q in M (see the proof of Theorem 1.1). By using the homogeneous property of $I_q(1,u)$ on u, it is easy to check that the equality in (1.9) holds if and only if $u = lw$ for $l > 0$. □

Proof of Theorem 1.3. Let $v(\theta) = |\sin(\theta)|$, by Lemma 3.3 we see that

$$\int_0^\pi v^q d\theta - q\sum_{i=1}^{q/2} \tau_i \int_0^\pi v^{q-2i} v'^{\,2i} d\theta = 0. \tag{4.13}$$

Let $u_0 \in W^{1,q}(\mathbb{S}^1)$ be defined by $u_0(\theta) = |v(2\theta/3)|$ for $\theta \in [0, 3\pi/2]$ and $u_0(\theta) = 0$ for $\theta \in (3\pi/2, 2\pi)$. A simple calculation and (4.13) show that

$$\begin{aligned}
&\int_{\mathbb{S}^1} u_0^q d\theta - q\sum_{i=1}^{q/2} \tau_i \int_{\mathbb{S}^1} u_0^{q-2i} u_0'^{2i} d\theta \\
&= \int_0^{3\pi/2} u_0^q d\theta - q\sum_{i=1}^{q/2} \tau_i \int_0^{3\pi/2} u_0^{q-2i} u_0'^{2i} d\theta \\
&= \frac{3}{2}\left(\int_0^\pi v^q d\theta - q\sum_{i=1}^{q/2} \tau_i 2^{2i}/3^{2i} \int_0^\pi v^{q-2i} v'^{\,2i} d\theta\right) \\
&= \frac{3}{2} q \sum_{i=1}^{q/2} \tau_i (1 - 2^{2i}/3^{2i}) \int_0^\pi v^{q-2i} v'^{\,2i} d\theta := c_0 > 0.
\end{aligned}$$

Let $u_n = u_0 + 1/n \geq 1/n$, then $\{u_n\} \subset H^{1,q}(\mathbb{S}^1)$, $\|u_n - u_0\| = o(1)$ as $n \to +\infty$ and

$$\lim_{n\to+\infty}\int_{\mathbb{S}^1} g(\theta)\ln u_n d\theta$$

$$= \lim_{n\to+\infty}\left(\int_0^{2\pi/3} g(\theta)\ln(\sin(2\theta/3)+1/n)d\theta - \int_{2\pi/3}^{2\pi} g(\theta)\ln n d\theta\right) = -\infty.$$

Then we obtain (1.10) by

$$\lim_{n\to+\infty}\left(\int_{\mathbb{S}^1} u_n^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} u_n^{q-2i}{u_n'}^{2i}d\theta\right)\cdot\exp\left(-q\frac{\int_{\mathbb{S}^1} g(\theta)\ln u_n d\theta}{\int_{\mathbb{S}^1} g(\theta)d\theta}\right)$$

$$\geq \frac{c_0}{2}\lim_{n\to+\infty}\exp\left(-q\frac{\int_{\mathbb{S}^1} g(\theta)\ln u_n d\theta}{\int_{\mathbb{S}^1} g(\theta)d\theta}\right) = +\infty. \quad \square$$

To study the multiple solutions of (1.5), we need the following corollary of Theorem 1.2.

Corollary 4.3. *Let $q \geq 2$ be an even number, τ_i be defined by (1.7). Then, there exists $c_q \geq 2\pi$ such that*

$$\left(\int_{\mathbb{S}^1} u^q d\theta - q\sum_{i=1}^{q/2}\tau_i\int_{\mathbb{S}^1} u^{q-2i}{u'}^{2i}d\theta\right)\cdot\exp\left(\frac{-q}{2\pi}\int_{\mathbb{S}^1}\ln u d\theta\right) \leq c_q \tag{4.14}$$

holds for all positive function $u \in W^{1,q}(\mathbb{S}^1)$. The maximum value c_q can be attained by taking u as a solution of (1.5) with $g \equiv 1$. For any function $v \in W^{1,q}(\mathbb{S}^1)$, we have

$$\frac{1}{c_q}\int_{\mathbb{S}^1}\left(1 - q\sum_{i=1}^{q/2}\tau_i {v'}^{2i}\right)\cdot\exp(qv)d\theta \leq \exp\left(\int_{\mathbb{S}^1}\frac{qv}{2\pi}d\theta\right). \tag{4.15}$$

Moreover, $c_2 = 2\pi$, and $c_q > 2\pi$ for any even number $q \geq 6$.

Proof. By applying Theorem 1.2 with $g \equiv 1$, there exists $c_q \geq 2\pi$ such that (4.14) holds; and the equality in (4.14) holds when u is a solution of (1.5) with $g \equiv 1$. For $v \in W^{1,q}(\mathbb{S}^1)$, let $u = e^v$ in (4.14), we obtain that

$$J_q(v) := \frac{1}{2\pi}\int_{\mathbb{S}^1}\left(1 - q\sum_{i=1}^{q/2}\tau_i {v'}^{2i}\right)\exp(qv)d\theta\cdot\exp\left(-\int_{\mathbb{S}^1}\frac{qv}{2\pi}d\theta\right) \leq \frac{c_q}{2\pi}. \tag{4.16}$$

Then (4.15) follows directly from (4.16). For $q = 2$, we see that constant 1 is the unique $\pi-$ periodic solution of (1.5) when $g \equiv 1$, for example, see [8,18,21]. By Theorem 1.2, we obtain that $c_2 = I_2(1,1) = 2\pi$. In the following part, we estimate $c_q > 2\pi$ for $q \geq 6$ by setting some specific functions v in (4.16). Let $v_0 = 2(|\sin(\theta)| - 2/\pi)/q$, then $\int_{\mathbb{S}^1} v_0 d\theta = 0$. We have

$$\begin{aligned}\frac{c_q}{2\pi} \geq J_q(v_0) &= \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\left(1 - q\sum_{i=1}^{q/2}\tau_i {v_0'}^{2i}\right)\exp(qv_0)d\theta \\ &= \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\left(1 - q\sum_{i=1}^{q/2}\frac{\tau_i}{(q/2)^{2i}}\cos^{2i}(\theta)\right)\exp\left(2|\sin(\theta)| - \frac{4}{\pi}\right)d\theta.\end{aligned} \tag{4.17}$$

On the other hand, for $q > 4$ we have

$$\sum_{i=1}^{q/2}\frac{\tau_i}{(q/2)^{2i}} = \sum_{i=1}^{q/2}\frac{(q/2-1)!}{2(2i-1)i!(q/2-i)!(q/2)^{2i}} < \frac{1}{q}\sum_{i=1}^{q/2}\frac{1}{(q/2)^i} < \frac{3}{q^2}.$$

If $q \geq 20$, it follows from (4.17) that

$$\frac{c_q}{2\pi} \geq J_q(v_0) \geq \frac{q-3}{q}\frac{1}{\pi}\int\limits_0^{\pi}\exp\left(2\sin(\theta) - \frac{4}{\pi}\right)d\theta > \frac{1.18\times(q-3)}{q} > 1. \tag{4.18}$$

Let $w_0 = |\sin(\theta)| - \frac{2}{\pi}$, then $\int_{\mathbb{S}^1} w_0 d\theta = 0$. By using numerical method, we obtain the following table consisting of the approximate values of $J_q := J_q(w_0)$ for some even numbers $q < 20$.

q	6	8	10	12	14	16	18
J_q	1.42	2.34	4.08	7.36	13.55	25.35	47.99

This combining with (4.18) show that $c_q \geq 2\pi J_q > 2\pi$ for any even number $q \geq 6$. □

Remark 4.4. We note that the special case of (4.14) with $q = 2$ has been obtained in the Corollary 8 of [33].

Remark 4.5. We conjecture that $c_4 > 2\pi$ and $\lim\limits_{q\to+\infty} c_q = +\infty$. It is interesting to confirm them.

Proof of Theorem 1.4. We denote by w one of the maximums of I_q in M. If $w \equiv l > 0$ is a constant, then $\max\limits_{u\in M} I_q(1,u) = I_q(1,l) = 2\pi$. For the case that $q \geq 6$ is even and $g \equiv 1$, Corollary 4.3 shows that $I_q(1,w) = \max\limits_{u\in M} I_q(1,u) = c_q > 2\pi$; it follows that w is not a constant. □

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