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On the L_p dual Minkowski problem

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ABSTRACT

The L_p dual curvature measure was introduced by Lutwak, Yang & Zhang in an attempt to unify the L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory. The characterization problem for L_p dual curvature measure, called the L_p dual Minkowski problem, is a fundamental problem in this unifying theory. The L_p dual Minkowski problem contains the L_p Minkowski problem and the dual Minkowski problem, two major problems in modern convex geometry that remain open in general. In this paper, existence results on the L_p dual Minkowski problem in the weak sense will be provided. Moreover, existence and uniqueness of the solution in the smooth category will also be demonstrated.

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1. Introduction

The classical Brunn–Minkowski theory focused on studying geometric invariants such as quermassintegrals (which include volume, surface area, and mean width) and geometric measures such as area measures and Federer's curvature measures. Among these measures are surface area measure and (Aleksandrov's) integral curvature, two most studied measures in the Brunn–Minkowski theory. The Minkowski problem and the Aleksandrov problem characterizing these two measures are influential problems not only in geometric analysis, but also in the theory of fully nonlinear partial differential equations.

The L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory are two theories that fundamentally extended the classical Brunn–Minkowski theory. The last two decades saw the rapid development of these two theories and they are now becoming the center focus of modern convex geometry.

The L_p Brunn–Minkowski theory came to life when Lutwak [49,50] introduced L_p surface area measure. When $p = 1$, the L_p surface area measure is the classical surface area measure. Since its introduction, the family of L_p surface area measure has quickly become the topic of many influential works. The L_p Minkowski problem is the problem of prescribing L_p surface area measure, which greatly generalizes the classical Minkowski problem. The family of L_p Minkowski problems contains important singular *unsolved* cases such as the logarithmic Minkowski problem (see Böröczky, Lutwak, Yang & Zhang [10]) and the centro-affine Minkowski problem.

The dual Brunn–Minkowski theory was introduced by Lutwak (see Schneider [58]) in the 1970s. The dual Brunn–Minkowski theory has been most effective in answering questions related to intersections. One major triumph of the dual Brunn–Minkowski theory is tackling the famous Busemann–Petty problem, see Gardner [21], Gardner, Koldobsky & Schlumprecht [23], Koldobsky [38–40], Lutwak [48], and Zhang [67]. Over the years, the dual theory has produced numerous profound concepts and results. See Gardner [22] and Schneider [58] for an overview of the theory. The dual theory makes extensive use of techniques from harmonic analysis. Recently, the dual Brunn–Minkowski theory took a huge step forward when Huang, Lutwak, Yang & Zhang [33] discovered the family of fundamental geometric measures—called dual curvature measures—in the dual theory. These measures are dual to Federer's curvature measures and are expected to play the same important role as area measures and curvature measures in the Brunn–Minkowski theory. The dual Minkowski problem is the problem of prescribing dual curvature measures. The dual Minkowski problem not only contains critical problems such as the logarithmic Minkowski problems and the Aleksandrov problem (prescribing curvature measure) as special cases, but also introduces intrinsic PDEs—something long missing—to the dual Brunn–Minkowski theory. The dual Minkowski problem, while still largely open, has been studied in [8,12,30,33,68,69].

A recent surprising discovery by Lutwak, Yang & Zhang [53] revealed that there exists a unifying theory that includes the classical Brunn–Minkowski theory, the L_p Brunn–Minkowski theory, and the dual Brunn–Minkowski theory. The latter two were never thought to be connected. In particular, they introduced the L_p dual curvature measure, which unifies both the aforementioned L_p surface area measures and dual curvature measures. The problem of prescribing this unifying family of measures is called the L_p dual Minkowski problem.

Problem 1.1 *(The L_p dual Minkowski problems).* Given a nonzero finite Borel measure μ on the unit sphere S^{n-1} and real numbers p, q, what are the necessary and sufficient conditions so that μ is exactly equal to $\widetilde{C}_{p,q}(K, \cdot)$ for some convex body $K \in \mathcal{K}_o^n$?

The L_p dual Minkowski problems will be thoroughly investigated in the current work.

When the given measure μ has a density f, the L_p dual Minkowski problems becomes the following Monge–Ampère type equation on S^{n-1}:

$$\det(h_{ij}(v) + h(v)\delta_{ij}) = f(v)h^{p-1}(v)(h^2(v) + |\nabla h(v)|^2)^{\frac{n-q}{2}}, \tag{1.1}$$

where f is a given positive smooth function on S^{n-1}, h is the unknown, δ_{ij} is the Kronecker delta, and ∇h and (h_{ij}) are the gradient and the Hessian of h on the unit sphere with respect to an orthonormal basis respectively.

It is worth noting that the L_p dual Minkowski problem contains the classical Minkowski problem and the Aleksandrov problem, as well as the unsolved logarithmic Minkowski problem [10,70] and the unsolved centro-affine Minkowski problem [19,71]. It not only unifies the L_p Minkowski problem [19,35,49] and the dual Minkowski problem posed in [33], but also includes the L_p Aleksandrov problem [34] and many new problems.

As mentioned previously, the family of L_p dual curvature measures, or sometimes simply referred to as *the* (p, q)*-th dual curvature measure* and denoted by $\widetilde{C}_{p,q}(K, \cdot)$ for $q \in \mathbb{R}$, were recently introduced by Lutwak, Yang & Zhang [53] in an effort to continue their groundbreaking works [33,34] with the first named author and to unify concepts never thought to be connected in the L_p Brunn–Minkowski theory and the dual Minkowski theory.

When $q = n$, the (p, n)-th dual curvature measure is up to a factor of n equal to the L_p surface area measure which is the core concept in the L_p Brunn–Minkowski theory. In this case, the L_p dual Minkowski problem becomes the L_p Minkowski problem, which includes the classical Minkowski problem solved by Minkowski, Fenchel & Jessen, Aleksandrov, etc. Regularity results on the classical Minkowski problem include the influential paper [17] by Cheng & Yau. When $p > 1$, the L_p Minkowski problem was solved by Lutwak [49] and Lutwak & Oliker [51] whenever the given data is even and by Chou & Wang [19] in the general case. The L_p Minkowski problem when $p < 1$ is much more complicated and contains *long open* problems such as the logarithmic Minkowski problem (solved in the symmetric case by Böröczky, Lutwak, Yang & Zhang [10] with recent major progress in the non-symmetric case made by Chen, Li & Zhu [16]) and the centro-affine Minkowski problem.

The logarithmic Minkowski problem characterizes cone volume measure which has been the central topic in a number of recent works. When the given data is even, the existence of solutions to the logarithmic Minkowski problem was completely solved in Böröczky, Lutwak, Yang, & Zhang [10]. In the general case (non-even case), important contributions were made by Zhu [70], later by Böröczky, Hegedűs & Zhu [6], and especially by Chen, Li & Zhu[16] more recently.

The centro-affine Minkowski problem characterizes the centro-affine surface area measure whose density in the smooth case is the centro-affine Gauss curvature. The characterization problem, in this case, is the centro-affine Minkowski problem posed in Chou & Wang [19]. See also Jian, Lu & Zhu [37], Lu & Wang [41], Zhu [71], etc., on this problem.

We emphasize again that the L_p dual Minkowski problem, considered in the current work, contain both the unsolved logarithmic Minkowski problem and the unsolved centro-affine Minkowski problem.

When $q = 0$, the L_p dual Minkowski problems becomes the L_p Aleksandrov problem posed by Huang, Lutawak, Yang, & Zhang [34], which is the L_p version of the Aleksandrov problem. Solutions in some special cases were given in [34].

When $p = 0$, the L_0 dual Minkowski problem becomes the *unsolved* dual Minkowski problem posed by Huang, Lutawak, Yang, & Zhang [33]. When $q < 0$, a complete solution to the dual Minkowski problem—including the existence and the uniqueness of the solution—was presented in [68]. When $q = 0$, the dual Minkowski problem becomes the Alesksandrov problem solved by Aleksandrov himself using a topological argument. The dual Minkowski problem gets much more challenging when $q > 0$ and only solutions in the case when the given data is even exist. When $0 < q < n$, a mass subspace inequality was given in [33] and proven to be sufficient. Although the condition is apparently necessary when $0 < q \leq 1$, examples of convex bodies that violates the given mass subspace inequality when $1 < q < n$ were independently discovered by Börřoczky, Henk, & Pollehn [8] and Zhao [69]. A better subspace mass inequality was proposed in [8,69], which was shown to be sufficient for the dual Minkowski problem when $q \in (1, n)$ is an integer in [69] and necessary when $q \in (1, n)$ (integers or not) in [8]. Very recently, Böröczky, Lutwak, Yang, Zhang & Zhao [12] showed the sufficiency of the subspace mass inequality when $q \in (1, n)$ is a non-integer, thus settling the existence part of the dual Minkowski problem when $q \in (0, n)$ and the given data is even. When $q = n$, the dual Minkowski problem becomes the logarithmic Minkowski problem mentioned above. When $q \geq n+1$, Henk & Pollehn [30] recently proposed a new subspace mass inequality and proved it to be necessary for the existence part of the dual Minkowski problem when the given data is even. Note again that, except for $q = n$ and $q \leq 0$, all existing results on the dual Minkowski problem are restricted to the case when the given measure is even.

Even more challenging than finding the correct necessary and sufficient conditions for the existence of solution to those newly posed Minkowski-type problems is establishing the uniqueness of the solution. So far, uniqueness of the solution has only been established for very few cases including the L_p Minkowski problem when $p \geq 1$, the Aleksandrov problem, the dual Minkowski problem when $q < 0$, and the logarithmic Minkowski problem when the dimension is 2 and the given data is even (see [9]).

As can be seen from the above discussion, even special cases of the L_p dual Minkowski problems can be extremely challenging. In the current work, we will study the L_p dual Minkowski problems in the case when $p \neq q$.

In Section 3, we will consider the L_p dual Minkowski problems (when $p \neq q$) in the weak sense using variational method. Because the behavior of the functional to be optimized changes considerably based on the values of p and q with respect to 0, results obtained here have to be split into three parts.

When $p > 0$ and $q < 0$, a complete characterization to existence part of the L_p dual Minkowski problem will be given. This is an extension to the existence results obtained in [68].

Theorem 1.2. *Let $p > 0$, $q < 0$, and μ be a non-zero finite Borel measure on S^{n-1}. There exists a convex body $K \in \mathcal{K}_o^n$ such that $\mu = \widetilde{C}_{p,q}(K, \cdot)$ if and only if μ is not concentrated on any closed hemisphere.*

When $p, q > 0$ and $p \neq q$, a complete solution to the existence part of the L_p dual Minkowski problem will be presented when the given data is even. Setting $q = n$, the following theorem contains the solution to the even L_p Minkowski problem when $p > 0$ and $p \neq n$ which was obtained in Lutwak [49] and Haberl, Lutwak, Yang & Zhang [27].

Theorem 1.3. *Let $p, q > 0$, $p \neq q$, and μ be a non-zero even Borel measure on S^{n-1}. There exists an origin symmetric convex body $K \in \mathcal{K}_e^n$ such that $\mu = \widetilde{C}_{p,q}(K, \cdot)$ if and only if μ is not concentrated in any great subsphere.*

The L_p Minkowski problem when $p \geq 1$ and $p \neq n$ was solved in the general case (not assuming evenness of the measure) in Chou & Wang [19] and later in Hug, Lutwak, Yang & Zhang [35]. Unfortunately, their solutions do not extend to this case in full generality. The key obstacle is the lack of a Minkowski type inequality in this more general setting. In particular, note that the L_p Minkowski problem when $0 < p < 1$ was studied by Zhu [72] and was essentially solved by Chen, Li & Zhu [15] recently.

When $p, q < 0$ and $p \neq q$, the following result will now be established.

Theorem 1.4. *Let $p, q < 0$, $p \neq q$, and μ be a non-zero finite even Borel measure on S^{n-1}. If μ vanishes on all great subsphere, then there exists an origin symmetric convex body $K \in \mathcal{K}_e^n$ such that $\mu = \widetilde{C}_{p,q}(K, \cdot)$.*

Note that the optimization problem to be introduced in Section 3 fails when the given measure μ has positive concentration in any great subsphere.

To complement and enrich the results obtained via variational method, solutions via continuity methods will be provided. In Section 4, both *the existence* and *the uniqueness* of the solution to the L_p dual Minkowski problem in the smooth category will be presented. Note that when $p = 0$, the following theorem provides regularity results to the dual Minkowski problem for negative indices.

Theorem 1.5. *Suppose $p > q$ and $0 < \alpha \leq 1$. For any given positive function $f \in C^{\alpha}(S^{n-1})$, there exists a unique solution $h \in C^{2,\alpha}(S^{n-1})$ to (1.1). If f is smooth, then the solution is also smooth.*

Since when $q = n$, the dual Minkowski problem becomes the L_p Minkowski problem, Theorem 1.5 also solves the L_p Minkowski problem for $p > n$ which was originally solved by Chou & Wang [19].

The uniqueness part of the L_p dual Minkowski problem when $p > q$ and the given measure is discrete (in the polytopal case) was proved by Lutwak, Yang & Zhang [53]. The uniqueness established in Theorem 1.5 settles the case when the given measure possesses a sufficiently smooth density and the solution is assumed to possess sufficient regularity. It remains unknown whether the uniqueness result still holds if the regularity assumption of the convex body is removed (see, for example [57]).

Note again that the uniqueness of the solution to the L_p dual Minkowski problem remains one of the biggest challenges in Minkowski type problems. While the uniqueness has been established for $p \geq 1$ by using the L_p Minkowski inequality [35], very little progress has been made for $p < 1$. Important recent results in this direction include Andrews [2], Böröczky, Lutwak, Yang & Zhang [9], Choi & Daskalopoulos [18], Huang, Liu & Xu [31], and Jian, Lu & Wang [36].

A major accomplishment in [53] is showing that the L_p dual curvature measures are valuations. See, for example, Alesker [1], Haberl [26], Haberl & Parapatits [28], Ludwig [42,43,45,46], Ludwig & Reitzner [47], Schuster [59,60], Schuster & Wannerer [61] and the references therein for important valuations in the theory of convex bodies.

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2. Preliminaries

This section is divided into three subsections. In the first subsection, basics in the theory of convex bodies will be covered. In the second subsection, the notion of L_p dual curvature measure, or the (p, q)-th dual curvature measure, will be introduced. Last but not least, in the third subsection, we will present the L_p dual Minkowski problems—the characterization problem for (p, q)-th dual curvature measure.

2.1. Basics in the theory of convex bodies

The book [58] by Schneider offers a comprehensive overview of the theory of convex bodies.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. We will write $C(S^{n-1})$ for the space of continuous functions on S^{n-1}. The subset $C^+(S^{n-1})$ of $C(S^{n-1})$ contain only positive functions whereas the subset $C_e(S^{n-1})$ contain only even functions. We will also write $C_e^+(S^{n-1})$ for $C^+(S^{n-1}) \cap C_e(S^{n-1})$.

A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The boundary of K is written as ∂K. Denote by $\mathcal{K}_0^n$ the class of convex bodies that contain the origin in their interiors in $\mathbb{R}^n$ and by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$.

Let K be a compact convex subset of $\mathbb{R}^n$. The support function h_K of K is defined by

$$h_K(y) = \max\{x \cdot y : x \in K\}, \quad y \in \mathbb{R}^n.$$

The support function h_K is a continuous function homogeneous of degree 1. Suppose K contains the origin in its interior. The radial function ρ_K is defined by

$$\rho_K(x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The radial function ρ_K is a continuous function homogeneous of degree -1. It is not hard to see that $\rho_K(u)u \in \partial K$ for all $u \in S^{n-1}$.

For each $f \in C^+(S^{n-1})$, the Wulff shape $[f]$ generated by f is the convex body defined by

$$[f] = \{x \in \mathbb{R}^n : x \cdot v \le f(v), \text{ for all } v \in S^{n-1}\}.$$

It is apparent that $h_{[f]} \le f$ and $[h_K] = K$ for each $K \in \mathcal{K}_0^n$.

The L_p combination of two convex bodies $K, L \in \mathcal{K}_0^n$ was first studied by Firey and was the starting point of the now rich L_p Brunn–Minkowski theory developed by Lutwak [49,50]. For $t > 0$, the L_p combination of K and L, denoted by $K +_p t \cdot L$, is defined to be the Wulff shape generated by h_t where

$$h_t = \begin{cases} (h_K^p + t h_L^p)^{\frac{1}{p}}, & \text{if } p \neq 0, \\ h_K h_L^t, & \text{if } p = 0. \end{cases}$$

When $p \ge 1$, by the convexity of ℓ_p norm, we get that

$$h_{K+_p t\cdot L}^p = h_K^p + t h_L^p.$$

Suppose K_i is a sequence of convex bodies in $\mathbb{R}^n$. We say K_i converges to a compact convex subset $K \subset \mathbb{R}^n$ if

$$\max\{|h_{K_i}(v) - h_K(v)| : v \in S^{n-1}\} \to 0, \tag{2.1}$$

as $i \to \infty$. If K contains the origin in its interior, equation (2.1) implies

$$\max\{|\rho_{K_i}(u) - \rho_K(u)| : u \in S^{n-1}\} \to 0,$$

as $i \to \infty$.

For a convex body $K \in \mathcal{K}_0^n$, the polar body of K is given by

$$K^* = \{y \in \mathbb{R}^n : y \cdot x \leq 1, \text{ for all } x \in K\}.$$

It is simple to check that $K^* \in \mathcal{K}_0^n$ and that

$$\begin{aligned} h_{K^*}(x) &= 1/\rho_K(x), \\ \rho_{K^*}(x) &= 1/h_K(x), \end{aligned} \tag{2.2}$$

for $x \in \mathbb{R}^n \setminus \{o\}$. Moreover, we have $(K^*)^* = K$.

By the definition of polar body and the relations (2.2), for $K, K_i \in \mathcal{K}_0^n$, we have K_i converges to K if and only if K_i^* converges to K^*.

For a compact convex subset K in $\mathbb{R}^n$ and $v \in S^{n-1}$, the supporting hyperplane $H(K, v)$ of K at v is given by

$$H(K, v) = \{x \in K : x \cdot v = h_K(v)\}.$$

By its definition, the supporting hyperplane $H(K, v)$ is non-empty and contains only boundary points of K. For $x \in H(K, v)$, we say v is an outer unit normal of K at $x \in \partial K$.

Let $\omega \subset S^{n-1}$ be a Borel set. The radial Gauss image of K at ω, denoted by $\boldsymbol{\alpha}_K(\omega)$, is defined to be the set of all unit vectors v such that v is an outer unit normal of K at some boundary point $u\rho_K(u)$ where $u \in \omega$, i.e.,

$$\boldsymbol{\alpha}_K(\omega) = \{v \in S^{n-1} : v \cdot u\rho_K(u) = h_K(v) \text{ for some } u \in \omega\}.$$

When $\omega = \{u\}$ is a singleton, we usually write $\boldsymbol{\alpha}_K(u)$ instead of the more cumbersome notation $\boldsymbol{\alpha}_K(\{u\})$. Let ω_K be the subset of S^{n-1} such that $\boldsymbol{\alpha}_K(u)$ contains more than one element for each $u \in \omega_K$. By Theorem 2.2.5 in [58], the set ω_K has spherical Lebesgue measure 0. The radial Gauss map of K, denoted by α_K, is the map defined on $S^{n-1} \setminus \omega_K$ that takes each point u in its domain to the unique vector in $\boldsymbol{\alpha}_K(u)$. Hence α_K is defined almost everywhere on S^{n-1} with respect to the spherical Lebesgue measure.

Let $\eta \subset S^{n-1}$ be a Borel set. The reverse radial Gauss image of K, denoted by $\boldsymbol{\alpha}_K^*(\eta)$, is defined to be the set of all radial directions such that the corresponding boundary points have at least one outer unit normal in η, i.e.,

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : v \cdot u\rho_K(u) = h_K(v) \text{ for some } v \in \eta\}.$$

When $\eta = \{v\}$ is a singleton, we usually write $\boldsymbol{\alpha}_K^*(v)$ instead of the more cumbersome notation $\boldsymbol{\alpha}_K^*(\{v\})$. Let η_K be the subset of S^{n-1} such that $\boldsymbol{\alpha}_K^*(v)$ contains more than one element for each $v \in \eta_K$. By Theorem 2.2.11 in [58], the set η_K has spherical Lebesgue measure 0. The reverse radial Gauss map of K, denoted by α_K^*, is the map defined on $S^{n-1} \setminus \eta_K$ that takes each point v in its domain to the unique vector in $\boldsymbol{\alpha}_K^*(v)$. Hence α_K^* is defined almost everywhere on S^{n-1} with respect to the spherical Lebesgue measure.

2.2. (p, q)-th dual curvature measures

Dual quermassintegrals are the fundamental geometric invariants in the dual Brunn–Minkowski theory. Suppose $1 \le q \le n$ is an integer. The $(n-q)$-th dual quermassintegral of $K \in \mathcal{K}_0^n$, denoted by $\widetilde{W}_{n-q}(K)$, can be viewed as the average of q-dimensional intersection areas. That is,

$$\widetilde{W}_{n-q}(K) = \frac{\omega_n}{\omega_q} \int\limits_{G(n,q)} \mathcal{H}^q(K \cap \xi_q) d\xi_q,$$

where $\mathcal{H}^q$ is the q-dimensional Hausdorff measure and the integration is with respect to the Haar measure on the Grassmannian $G(n, q)$ containing all q-dimensional subspaces ξ_q in $\mathbb{R}^n$.

The dual quermassintegrals have the following integral representation (see Lutwak [48]):

$$\widetilde{W}_{n-q}(K) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_K^q(u) du,$$

which immediately allows us to extend the definition to all $q \in \mathbb{R}$.

The exact geometric measures that can be viewed as the differentials of dual quermassintegrals remained hidden until the ground-breaking work by Huang, Lutwak, Yang & Zhang [33]. For $q \neq 0$, the q-th dual curvature measure of $K \in \mathcal{K}_0^n$, denoted by $\widetilde{C}_q(K, \cdot)$, can be defined as the unique Borel measure on S^{n-1} that satisfies the following equation for each $L \in \mathcal{K}_0^n$:

$$\left.\frac{d}{dt}\right|_{t=0} \widetilde{W}_{n-q}(K +_0 t \cdot L) = q \int\limits_{S^{n-1}} \log h_L(v) d\widetilde{C}_q(K, v). \tag{2.3}$$

Note that there is a corresponding version of the formula that allows us to define $\widetilde{C}_0(K, \cdot)$, see [33]. Dual curvature measures are the notions in the dual Brunn–Minkowski theory that are dual to Federer's curvature measures.

The (p, q)-th dual curvature measure was very recently introduced by Lutwak, Yang & Zhang [53]. For $p, q \neq 0$, the (p, q)-th dual curvature measure of $K \in \mathcal{K}_0^n$, denoted by $\widetilde{C}_{p,q}(K, \cdot)$, is defined to be the unique Borel measure on S^{n-1} that satisfies the following equation for each $L \in \mathcal{K}_0^n$:

$$\left.\frac{d}{dt}\right|_{t=0} \widetilde{W}_{n-q}(K +_p t \cdot L) = q \int\limits_{S^{n-1}} h_L^p(v) d\widetilde{C}_{p,q}(K, v). \tag{2.4}$$

Note that when $p = 0$, the $(0, q)$-th dual curvature measure is defined as in (2.3) so that they are exactly the q-th dual curvature measure. There is a corresponding version of

(2.4) that allows us to define $\widetilde{C}_{p,q}(K, \cdot)$ for $q = 0$. It should also be pointed out that the (p, q)-th dual curvature measure defined in [53] is slightly more general than what we are using in the current paper.

Key properties of Lp dual curvature measures were shown in [53]. When viewed as a function from the set $\mathcal{K}_0^n$ (equipped with the Hausdorff metric) to the space of Borel measures on S^{n-1} (equipped with the weak topology), the (p, q)-th dual curvature measure is continuous. That is, for each $f \in C(S^{n-1})$,

$$\lim_{i\to\infty} \int_{S^{n-1}} f(v) d\widetilde{C}_{p,q}(K_i, v) = \int_{S^{n-1}} f(v) d\widetilde{C}_{p,q}(K_0, v),$$

given that $K_0, K_i \in \mathcal{K}_0^n$ and K_i converges to K_0. Moreover, it is also a valuation. That is,

$$\widetilde{C}_{p,q}(K \cup L, \cdot) + \widetilde{C}_{p,q}(K\cap, \cdot) = \widetilde{C}_{p,q}(K, \cdot) + \widetilde{C}_{p,q}(L, \cdot),$$

given that $K, L \in \mathcal{K}_0^n$ are such that $K \cup L \in \mathcal{K}_0^n$.

It was shown in [53] that the (p, q)-th dual curvature measure is absolutely continuous with respect to the q-th dual curvature measure and can be written as

$$d\widetilde{C}_{p,q}(K, \cdot) = h_K^{-p} d\widetilde{C}_q(K, \cdot).$$

The above equation works even when p or q is 0. By definition, it is immediate that the $(0, q)$-th dual curvature measure is exactly the q-th dual curvature measure. When $q = n$, $\widetilde{C}_n(K, \cdot)$ is the cone-volume measure,

$$d\widetilde{C}_n(K, \cdot) = \frac{1}{n} h_K dS_K(\cdot),$$

and thus $\widetilde{C}_{p,n}(K, \cdot)$ gives the L_p surface area measure,

$$d\widetilde{C}_{p,n}(K, \cdot) = \frac{1}{n} h_K^{1-p} dS(K, \cdot) = \frac{1}{n} dS_p(K, \cdot).$$

If K is a polytope that contains the origin in its interior with outer unit normals v_i, $i = 1, \ldots, m$, then the (p, q)-th curvature measure $\tilde{C}_{p,q}(K, \cdot)$ is discrete and is concentrated on $\{v_1, \ldots, v_m\}$. Namely,

$$\widetilde{C}_{p,q}(K, \cdot) = \sum_{i=1}^{m} c_i \delta_{v_i}(\cdot),$$

where

$$c_i = \frac{1}{n} h_K(v_i)^{-p} \int_{S^{n-1}\cap\Delta_i} \rho_K(u)^q \, du,$$

and Δ_i is the cone formed by the origin and the face of K with normal v_i.

If K is a convex body that has C^2 boundary with positive curvature and contains the origin in its interior, then $\widetilde{C}_{p,q}(K,\cdot)$ is absolutely continuous with respect to the Lebesgue measure,

$$\frac{d\widetilde{C}_{p,q}(K,v)}{dv} = \frac{1}{n} h_K^{1-p} (h_K^2 + |\nabla h_K|^2)^{\frac{q-n}{2}} \det((h_K)_{ij} + h_K \delta_{ij}),$$

where ∇h_K and $((h_K)_{ij})$ are the gradient and the Hessian of h_K on the unit sphere S^{n-1} with respect to an orthonormal basis respectively.

2.3. The L_p dual Minkowski problems

It is natural to consider the characterization problem for (p,q)-th dual curvature measure.

Problem 2.1 *(The L_p dual Minkowski problems).* Given a nonzero finite Borel measure μ on the unit sphere S^{n-1} and real numbers p, q, what are the necessary and sufficient conditions so that there exists a convex body $K \in \mathcal{K}_0^n$ satisfying

$$\widetilde{C}_{p,q}(K,\cdot) = \mu\,?$$

When $p = 0$, the L_p dual Minkowski problems becomes the dual Minkowski problem, which since its introduction in [33] has quickly established its fundamental role in the dual Brunn–Minkowski theory already generating works such as [8,12,30,33,68,69]. In particular, the case of $p = 0, q = n$ is the logarithmic Minkowski problem while $p = 0$, $q = 0$, it is the Aleksandrov problem.

When $q = n$, the L_p dual Minkowski problems becomes the L_p Minkowski problem which is fundamental in the L_p Brunn–Minkowski theory. The L_p Minkowski problem when $p > 1$ was solved in the even case by Lutwak [49] and in the general case by Chou–Wang [19]. A different approach was provided by Hug, Lutwak, Yang & Zhang [35]. A smooth solution was obtained by Huang & Lu [32] for $2 < p < n$. The solution of the L_p Minkowski problem plays a vital role in establishing various powerful sharp affine isoperimetric inequalities, see, e.g., Cianchi, Lutwak, Yang & Zhang [20], Lutwak, Yang & Zhang [52], and Zhang [66]. It contains important and unsolved singular cases such as the logarithmic Minkowski problem ($p = 0$) and the centroaffine Minkowski problem ($p = -n$).

The logarithmic Minkowski problem studies cone volume measures that appeared in a growing number of works. See, for example, Barthe, Guédon, Mendelson & Naor [4], Böröczky & Henk [7], Böröczky, Lutwak, Yang & Zhang [9–11], Henk & Linke [29], Ludwig [44], Ludwig & Reitzner [47], Naor [54], Naor & Romik [55], Paouris & Werner [56], Stancu [62,63], Xiong [65], Zhu [70], and Zou & Xiong [73]. The logarithmic Minkowski problem has strong connections with isotropic measures (Böröczky, Lutwak, Yang & Zhang [11]), curvature flows (Andrews [2,3]), and the log-Brunn–Minkowski inequality (e.g., Böröczky, Lutwak, Yang & Zhang [9], Xi & Leng [64]), an inequality stronger than the classical Brunn–Minkowski inequality.

When the given measure μ has a density f, the L_p dual Minkowski problems is equivalent to the following Monge–Ampère equation on S^{n-1},

$$\det(h_{ij}(v) + h(v)\delta_{ij}) = f(v)h^{p-1}(v)(h^2(v) + |\nabla h(v)|^2)^{\frac{n-q}{2}},\ v \in S^{n-1}. \tag{2.5}$$

Since the unit balls of finite dimensional Banach spaces are origin-symmetric convex bodies and the dual curvature measure of an origin-symmetric convex body is even, it is of great interest to study the following even L_p dual Minkowski problems.

Problem 2.2 *(The even L_p dual Minkowski problems).* Given a nonzero even finite Borel measure μ on the unit sphere S^{n-1} and real numbers p, q, what are the necessary and sufficient conditions so that there exists a convex body $K \in \mathcal{K}_e^n$ satisfying

$$\widetilde{C}_{p,q}(K, \cdot) = \mu\,?$$

In the current work, we will study the L_p dual Minkowski problems when $p \neq q$ from two different angles. One is from convex geometric analysis point of view where we will solve the L_p dual Minkowski problems using variational method. The solution will include cases when the given measure possesses no density. This will be done in Section 3. In addition to that, we will solve the Monge–Ampère equation (2.5) by using continuity methods and Caffarelli's regularity results of the Minkowski problem [13, 14]. Both existence and uniqueness of the solution (in the smooth category) will be demonstrated. This will be done in Section 4.

3. A variational method to weak solutions

In this section, we will use variational method to study weak solutions to the L_p dual Minkowski problems when $p \neq q$ and $p, q \neq 0$. Note that when $p = 0$, this problem is known as the dual Minkowski problem, and when $q = 0$, this problem is known as the L_p Aleksandrov problem. Both these special cases have already been considered.

3.1. An associated optimization problem

To obtain a weak solution using variational method, the first step is to convert the existence problem into an optimization problem whose optimizer is exactly the solution to the original problem.

For any nonzero finite Borel measure μ on S^{n-1} and $p, q \neq 0$, define $\Phi_{p,q} : \mathcal{K}_o^n \to \mathbb{R}$ by

$$\Phi_{p,q}(Q) = -\frac{1}{p} \log \int_{S^{n-1}} h_Q(v)^p \, d\mu(v) + \frac{1}{q} \log \int_{S^{n-1}} \rho_Q(u)^q \, du,$$

for each $Q \in \mathcal{K}_o^n$.

We consider the maximization problem

$$\sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_o^n\}.$$

Lemma 3.1. *Let $p, q \neq 0$ and μ be a nonzero finite Borel measure on S^{n-1}. If $K \in \mathcal{K}_o^n$ satisfies*

$$\int_{S^{n-1}} h_K^p(v) d\mu(v) = \widetilde{W}_{n-q}(K), \tag{3.1}$$

and

$$\Phi_{p,q}(K) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_o^n\}, \tag{3.2}$$

then

$$\mu = \widetilde{C}_{p,q}(K, \cdot).$$

Proof. For each $f \in C^+(S^{n-1})$, let $[f]$ be the Wulff shape generated by f, i.e.,

$$[f] = \{x \in \mathbb{R}^n : x \cdot v \leq f(v) \text{ for all } v \in S^{n-1}\} \in \mathcal{K}_o^n.$$

Define the functional $\Psi_{p,q} : C^+(S^{n-1}) \to \mathbb{R}$ by

$$\Psi_{p,q}(f) = -\frac{1}{p} \log \int_{S^{n-1}} f(v)^p \, d\mu(v) + \frac{1}{q} \log \int_{S^{n-1}} \rho_{[f]}(u)^q \, du.$$

Note that $\Psi_{p,q}(f)$ is homogeneous of degree 0, i.e. for all $\lambda > 0$ and $f \in C^+(S^{n-1})$,

$$\Psi_{p,q}(\lambda f) = \Psi_{p,q}(f).$$

We claim that

$$\Psi_{p,q}(f) \leq \Psi_{p,q}(h_K), \tag{3.3}$$

for each $f \in C^+(S^{n-1})$. Indeed, since $h_{[f]} \leq f$ and $[h_K] = K$, we have

$$\Psi_{p,q}(f) \le \Psi_{p,q}(h_{[f]}) = \Phi_{p,q}([f]) \le \Phi_{p,q}(K) = \Psi_{p,q}(h_K),$$

where the second inequality sign follows from (3.2).

For any $g \in C(S^{n-1})$ and $t \in (-\delta, \delta)$ where $\delta > 0$ is sufficiently small, let

$$h_t(v) = h_K(v)e^{tg(v)}. \tag{3.4}$$

By Theorem 4.5 of [33],

$$\frac{d}{dt}\widetilde{W}_{n-q}([h_t])\Big|_{t=0} = q \int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K,v). \tag{3.5}$$

By (3.3), the definition of $\Psi_{p,q}$, and (3.5), we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Psi_{p,q}(h_t)\Big|_{t=0} \\ &= \frac{d}{dt}\left(-\frac{1}{p}\log \int_{S^{n-1}} h_t(v)^p\, d\mu(v) + \frac{1}{q}\log \widetilde{W}_{n-q}([h_t])\right)\Bigg|_{t=0} \\ &= -\left(\int_{S^{n-1}} h_K^p(v) d\mu(v)\right)^{-1} \int_{S^{n-1}} h_K(v)^p g(v)\, d\mu(v) + \frac{1}{\widetilde{W}_{n-q}(K)} \int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K,v). \end{aligned}$$

By (3.1), we have

$$h_K(v)^p\, d\mu(v) = d\widetilde{C}_q(K,v).$$

We conclude that $\mu = \widetilde{C}_{p,q}(K,\cdot)$. □

Whenever $p \neq q$, we can take advantage of the different degrees of homogeneity of the two sides in (3.1) and get the following lemma.

Lemma 3.2. *Let $p, q \neq 0$, $p \neq q$, and μ be a nonzero finite Borel measure on S^{n-1}. If $K \in \mathcal{K}_o^n$ satisfies*

$$\Phi_{p,q}(K) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_o^n\},$$

then there exists a constant $c > 0$ such that

$$\mu = \widetilde{C}_{p,q}(cK, \cdot).$$

Proof. Note that since $p \neq q$, we may find a constant $c > 0$ such that

$$\int_{S^{n-1}} h_{cK}^p(v)d\mu(v) = \widetilde{W}_{n-q}(cK).$$

Since $\Phi_{p,q}$ is homogeneous of degree 0, we have

$$\Phi_{p,q}(cK) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_o^n\}.$$

By Lemma 3.1, we have

$$\mu = \widetilde{C}_{p,q}(cK, \cdot). \qquad \square$$

We remark that there is an obvious way to adapt Lemmas 3.1 and 3.2 and their proofs so that they can be used to treat the case when the given measure μ is even and the solution convex body K is origin-symmetric. In particular, we have the following lemma.

Lemma 3.3. *Let $p, q \neq 0$, $p \neq q$, and μ be a nonzero even finite Borel measure on S^{n-1}. If $K \in \mathcal{K}_e^n$ satisfies*

$$\Phi_{p,q}(K) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_e^n\},$$

then there exists a constant $c > 0$ such that

$$\mu = \widetilde{C}_{p,q}(cK, \cdot).$$

Lemmas 3.2 and 3.3 convert the existence part of the L_p dual Minkowski problem into an optimization problem. The rest of this section is aimed to prove an optimizer exists.

3.2. Existence of an optimizer

Since the behavior of $\Phi_{p,q}$ changes based on the values of p and q, we shall divide our results into three cases: i) $p > 0$ and $q < 0$; ii) $p, q > 0$ and $p \neq q$; iii) $p, q < 0$ and $p \neq q$. The case $p < 0$, $q > 0$ remains unsolved. Progress along that direction is of great interest.

Let us first deal with the case $p > 0$, $q < 0$.

Lemma 3.4. *Let $p > 0$, $q < 0$, and μ be a non-zero finite Borel measure on S^{n-1}. There exists a convex body $K \in \mathcal{K}_o^n$ such that $\Phi_{p,q}(K) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_o^n\}$ if μ is not concentrated on any closed hemisphere.*

Proof. Suppose Q_l is a maximization sequence; i.e.,

$$\lim_{l\to\infty} \Phi_{p,q}(Q_l) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_o^n\}.$$

Note that $\Phi_{p,q}$ is homogeneous of degree 0. So we may (after rescaling) assume that $\int_{S^{n-1}} \rho_{Q_l}^q(u)du = 1$.

We claim that Q_l^* is uniformly bounded. If not, after taking subsequence, we may find $v_l \in S^{n-1}$ such that $\rho_{Q_l^*}(v_l) \to \infty$ as $l \to \infty$. By the definition of polar body and the support function,

$$\begin{aligned} 1 &= \int_{S^{n-1}} \rho_{Q_l}^q(u)du \\ &= \int_{S^{n-1}} h_{Q_l^*}^{-q}(u)du \\ &\geq \int_{S^{n-1}} \rho_{Q_l^*}(v_l)^{-q}(v_l \cdot u)_+^{-q}du \\ &= \rho_{Q_l^*}(v_l)^{-q} \int_{S^{n-1}} (v_1 \cdot u)_+^{-q}du. \end{aligned}$$

Here $(t)_+ = \max\{t, 0\}$ for any $t \in \mathbb{R}$. Since $\int_{S^{n-1}}(v_1 \cdot u)_+^{-q}du$ is positive and $q < 0$, we have $\rho_{Q_l^*}(v_l)$ is bounded. This is a contradiction to the choice of v_l. Hence Q_l^* is uniformly bounded.

By Blaschke's selection theorem, (after taking a subsequence) we may assume that Q_l^* converges to a compact convex subset $K_0 \subset \mathbb{R}^n$.

We will show that K_0 has the origin in its interior. If not, then the origin is on the boundary of K_0 and therefore, there exists $u_0 \in S^{n-1}$ such that $h_{K_0}(u_0) = 0$. Since Q_l^* converges to K_0, we have

$$\lim_{l\to\infty} h_{Q_l^*}(u_0) = 0.$$

For each $\delta > 0$, define

$$\omega_\delta = \{v \in S^{n-1} : v \cdot u_0 > \delta\}.$$

Since μ is not concentrated in any closed hemisphere, there exists $\delta_0 > 0$ such that $\mu(\omega_{\delta_0}) > 0$. Since $\rho_{Q_l^*}(v)v \cdot u_0 \leq h_{Q_l^*}(u_0)$, we have $\rho_{Q_l^*}(v)$ goes to 0 uniformly on ω_{δ_0}.

This, together with $\int_{S^{n-1}} \rho_{Q_l}^q(u)du = 1$ and $p > 0$, implies

$$\begin{aligned} \Phi_{p,q}(Q_l) &= -\frac{1}{p} \log \int_{S^{n-1}} h_{Q_l}^p(v)d\mu(v) \\ &\leq -\frac{1}{p} \log \int_{\omega_{\delta_0}} h_{Q_l}^p(v)d\mu(v) \end{aligned}$$

$$= -\frac{1}{p}\log\int_{\omega_{\delta_0}}\rho_{Q_l^*}^{-p}(v)d\mu(v)$$

$$\to -\infty,$$

which is a contradiction to Q_l being a maximization sequence. Hence K_0 contains the origin in its interior. Let $K = K_0^*$. Since Q_l^* converges to K_0, Q_l converges to K. That K is a maximizer now follows directly from the continuity of $\Phi_{p,q}$ and the fact that Q_l is a maximizing sequence. □

The "if" part of the following theorem follows directly from Lemmas 3.2 and 3.4 whereas the "only if" part is obvious.

Theorem 3.5. *Let $p > 0$, $q < 0$ and μ be a non-zero finite Borel measure on S^{n-1}. There exists a convex body $K \in \mathcal{K}_o^n$ such that $\mu = \tilde{C}_{p,q}(K,\cdot)$ if and only if μ is not concentrated on any closed hemisphere.*

Let us now consider the case $p, q > 0$ and $p \neq q$.

The following two lemmas provide important estimates for establishing the existence of an optimizer.

Lemma 3.6. *Let $p, \varepsilon_0 > 0$ and μ be a non-zero even finite Borel measure on S^{n-1}. Suppose $e_{1l}, \cdots, e_{nl}$ is a sequence of orthonormal basis in $\mathbb{R}^n$ and $\{a_l\}$ is a sequence of positive real numbers. Assume $e_{1l}, \cdots, e_{nl}$ converges to an orthonormal basis $e_1, \cdots, e_n$ in $\mathbb{R}^n$. Define*

$$G_l = \{x \in \mathbb{R}^n : |x\cdot e_{1l}|^2 + \cdots + |x\cdot e_{n-1,l}|^2 \le a_l^2 \text{ and } |x\cdot e_{n,l}|^2 \le \varepsilon_0\}.$$

If μ is not concentrated in any great subsphere, then there exists $c, L > 0$ (independent of l) such that

$$\int_{S^{n-1}} h_{G_l}^p(v)d\mu(v) \ge c,$$

for each $l > L$.

Proof. Note that $\pm\varepsilon_0 e_{n,l} \in G_l$. Hence, by the definition of support function,

$$h_{G_l}(v) \ge \varepsilon_0|v\cdot e_{n,l}|, \tag{3.6}$$

for each $v \in S^{n-1}$. For each $\delta > 0$, define

$$\omega_\delta = \{v \in S^{n-1} : |v\cdot e_n| > \delta\}.$$

By monotone convergence theorem, and the fact that μ is not concentrated in any great subsphere,

$$\lim_{\delta\to 0^+} \mu(\omega_\delta) = \mu(S^{n-1} \setminus \operatorname{span}\{e_1, \cdots, e_{n-1}\}) > 0$$

Hence, there exists $\delta_0 > 0$ such that $\mu(\omega_{\delta_0}) > 0$.

Since e_{nl} converges to e_n, there exists $L > 0$ such that for $l > L$, we have

$$|e_{nl} - e_n| < \delta_0/2.$$

This, together with (3.6), implies that for $v \in \omega_{\delta_0}$ and $l > L$,

$$h_{G_l}(v) \geq \varepsilon_0(|v \cdot e_n| - |v \cdot (e_n - e_{nl})|) \geq \varepsilon_0\delta_0/2.$$

Hence,

$$\int_{S^{n-1}} h_{G_l}^p(v)d\mu(v) \geq \int_{\omega_{\delta_0}} h_{G_l}^p(v)d\mu(v) \geq \left(\frac{1}{2}\varepsilon_0\delta_0\right)^p \mu(\omega_{\delta_0}) =: c. \qquad \square$$

Let $0 < a < 1$ and $e_1, \cdots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Define

$$T_a = \{x \in \mathbb{R}^n : |x \cdot e_1| \leq a \text{ and } |x \cdot e_2|^2 + \cdots + |x \cdot e_n|^2 \leq 1\}.$$

The following lemma estimates the dual quermassintegral of the generalized ellipsoid T_a.

Lemma 3.7. *Let $0 < a < 1$ and $e_1, \cdots e_n$ be an orthonormal basis in $\mathbb{R}^n$. If $q > 0$, then*

$$\lim_{a\to 0^+} \int_{S^{n-1}} \rho_{T_a}^q(u)du = 0.$$

Proof. Since $q > 0$, we have $\rho_{T_a}^q$ is bounded. Thus, by dominated convergence theorem,

$$\lim_{a\to 0^+} \int_{S^{n-1}} \rho_{T_a}^q(u)du = \int_{S^{n-1}} \lim_{a\to 0^+} \rho_{T_a}^q(u)du.$$

It is simple to see that

$$\lim_{a\to 0^+} \rho_{T_a}(u) = \begin{cases} 1, & \text{if } u \in \operatorname{span}\{e_2, \cdots, e_n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we get the desired result. $\square$

We are ready to show the existence of an optimizer.

Lemma 3.8. *Let $p, q > 0$, $p \neq q$, and μ be a non-zero even finite Borel measure on S^{n-1}. If μ is not concentrated in any great subsphere, then there exists $K \in \mathcal{K}_e^n$ such that*

$$\Phi_{p,q}(K) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_e^n\}. \tag{3.7}$$

Proof. Let $\{Q_l\} \subset \mathcal{K}_e^n$ be a maximizing sequence; i.e.,

$$\lim_{l\to\infty} \Phi_{p,q}(Q_l) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_e^n\}.$$

Since $\Phi_{p,q}$ is homogeneous of degree 0, we may assume the diameter of Q_l is 1. By Blaschke's selection theorem, we may assume (take subsequence is necessary) that Q_l converges to an origin-symmetric compact subset K of $\mathbb{R}^n$. Note that, by the continuity of $\Phi_{p,q}$, if K contains the origin in its interior, then K satisfies (3.7).

To prove K contains the origin in its interior, we argue by contradiction and assume K is contained in some $(n-1)$-dimensional subspace.

Since Q_l is origin-symmetric, there exists an ellipsoid E_l (namely, the John ellipsoid, or the ellipsoid of maximal volume contained in Q_l) such that

$$E_l \subset Q_l \subset \sqrt{n}E_l.$$

Assume

$$E_l = \left\{x \in \mathbb{R}^n : \frac{|x\cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x\cdot e_{nl}|^2}{a_{nl}^2} \leq 1\right\},$$

for some orthonormal basis $e_{1l}, \cdots, e_{nl}$ in $\mathbb{R}^n$ and $0 < a_{1l} \leq \cdots \leq a_{nl} \leq 1$. By taking subsequences, we may assume

$$\lim_{l\to\infty} a_{il} = a_i,$$

for $i = 1, \cdots, n$ and $e_{1l}, \cdots, e_{nl}$ converges to an orthonormal basis $e_1, \cdots, e_n$ in $\mathbb{R}^n$. Obviously, $0 \leq a_1 \leq \cdots \leq a_n \leq 1$. Since K is contained in some lower dimensional subspace and $E_l \subset Q_l$, we have $a_1 = 0$. On the other hand, since the diameter of Q_l is 1 and that $Q_l \subset \sqrt{n}E_l$, we have $a_n > 0$. Hence there exists $\varepsilon_0 > 0$ such that $a_{nl} > \varepsilon_0$ for each $l > 0$.

Define

$$T_l = \{x \in \mathbb{R}^n : |x\cdot e_{1l}| \leq a_{1l} \text{ and } |x\cdot e_{2l}|^2 + \cdots + |x\cdot e_{nl}|^2 \leq 1\},$$

and

$$G_l = \{x \in \mathbb{R}^n : |x\cdot e_{1l}|^2 + \cdots + |x\cdot e_{n-1,l}|^2 \leq a_{1l}^2/2 \text{ and } |x\cdot e_{nl}| \leq \varepsilon_0/\sqrt{2}\}.$$

It is easy to see that

$$G_l \subset E_l \subset Q_l \subset \sqrt{n}E_l \subset \sqrt{n}T_l.$$

Hence, by the fact that $p, q > 0$,

$$\Phi_{p,q}(Q_l) \leq -\frac{1}{p}\log \int_{S^{n-1}} h_{G_l}^p d\mu(v) + \frac{1}{q}\log \int_{S^{n-1}} \rho_{\sqrt{n}T_l}(u)^q du.$$

By Lemma 3.6, there exists $L > 0$ and $c \in \mathbb{R}$ such that for each $l > L$,

$$-\frac{1}{p}\log \int_{S^{n-1}} h_{G_l}^p d\mu(v) \leq c.$$

This and Lemma 3.7 imply

$$\lim_{l\to\infty} \Phi_{p,q}(Q_l) \leq c + \log\sqrt{n} + \frac{1}{q}\log \lim_{l\to\infty} \int_{S^{n-1}} \rho_{T_l}(u)^q du = -\infty.$$

This is a contradiction to Q_l being a maximizing sequence. □

The “if” part of the following theorem follows immediately from Lemmas 3.3 and 3.8 whereas the “only if” part is obvious.

Theorem 3.9. *Let $p, q > 0$, $p \neq q$, and μ be a non-zero even finite Borel measure on S^{n-1}. There exists $K \in \mathcal{K}_e^n$ such that $\mu = \widetilde{C}_{p,q}(K, \cdot)$ if and only if μ is not concentrated in any great subsphere.*

Finally, let us consider the case $p, q < 0$ and $p \neq q$. The proof to the following lemma is in the spirit of the proof to Lemma 6.5 in [34].

Lemma 3.10. *Let $p, q < 0$, $p \neq q$, and μ be a non-zero finite even Borel measure on S^{n-1}. If μ vanishes on all great subsphere, then there exists $K \in \mathcal{K}_e^n$ such that*

$$\Phi_{p,q}(K) = \sup\{\Phi_{p,q}(Q) : Q \in \mathcal{K}_e^n\}.$$

Proof. Suppose $\{Q_l\} \subset \mathcal{K}_e^n$ is a maximizing sequence. Since $\Phi_{p,q}$ is homogeneous of degree 0, we may assume $\int_{S^{n-1}} \rho_{Q_l}^q(u)du = 1$. Arguing as in Lemma 3.4, we have Q_l^* is uniformly bounded. Assume the bound is M. That is $Q_l^* \subset MB_n$, where B_n is the n-dimensional unit ball.

By Blaschke's selection theorem, we may assume Q_l^* converges to an origin-symmetric compact convex subset $Q_0 \subset \mathbb{R}^n$. Note that if Q_0 contains the origin in its interior, then we may take K to be Q_0^* by the continuity of $\Phi_{p,q}$ and the fact that Q_l converges to K.

To show Q_0^* contains the origin in its interior, let us prove by contradiction. Assume there exists $u_0 \in S^{n-1}$ such that $h_{Q_0^*}(\pm u_0) = 0$.

For each $\delta > 0$, define

$$\omega_\delta = \{v \in S^{n-1} : |v \cdot u_0| > \delta\}.$$

Note that $\rho_{Q_l^*}$ goes to 0 uniformly on ω_δ.

Since μ is finite and μ vanishes on all great subsphere, we have

$$\lim_{\delta \to 0^+} \mu(S^{n-1} \setminus \omega_\delta) = 0.$$

For each $\varepsilon > 0$, choose $\delta_0 > 0$ so that $\mu(S^{n-1} \setminus \omega_{\delta_0}) < \varepsilon M^p/2$. Since $\rho_{Q_l^*}$ goes to 0 uniformly on ω_{δ_0} and $p < 0$, we may choose L so that for each $l > L$,

$$\rho_{Q_l^*}(v)^{-p} < \frac{1}{|\mu|}\varepsilon/2,$$

for each $v \in \omega_{\delta_0}$.

Thus, we have for each $l > L$,

$$\begin{aligned}\int_{S^{n-1}} h_{Q_l}^p(v)d\mu(v) &= \int_{\omega_{\delta_0}} \rho_{Q_l^*}^{-p}(v)d\mu(v) + \int_{S^{n-1}\setminus\omega_{\delta_0}} \rho_{Q_l^*}^{-p}(v)d\mu(v) \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon\end{aligned}$$

Hence

$$\lim_{l\to\infty} \int_{S^{n-1}} h_{Q_l}^p(v)d\mu(v) = 0.$$

This, and the fact that $\int_{S^{n-1}} \rho_{Q_l}^q(u)du = 1$, implies that

$$\lim_{l\to\infty} \Phi_{p,q}(Q_l) = -\infty,$$

which is a contradiction to Q_l being a maximizing sequence. □

The following theorem follows directly from Lemmas 3.3 and 3.10.

Theorem 3.11. *Let $p, q < 0$, $p \neq q$, and μ be a non-zero finite even Borel measure on S^{n-1}. If μ vanishes on all great subsphere, then there exists $K \in \mathcal{K}_e^n$ such that $\mu = \tilde{C}_{p,q}(K, \cdot)$.*

4. Regularity and existence of smooth solutions

In this section, we will consider the regularity of the L_p dual Minkowski problem when $p > q$ and the given measure has a density. In particular, the following solution to the Monge–Ampère equation (2.5) will be obtained.

Theorem 4.1. *Suppose $p > q$ and $0 < \alpha \leq 1$. For any given positive function $f \in C^\alpha(S^{n-1})$, there exists a unique solution $h \in C^{2,\alpha}(S^{n-1})$ to (2.5). If f is smooth, then the solution is also smooth.*

For notational simplicity, define

$$F(h_{ij}, h) = \det(h_{ij} + h\delta_{ij}),$$

and

$$J(\nabla h, h) = f(u)h^{p-1}(h^2 + |\nabla h|^2)^{\frac{n-q}{2}},$$

for each positive function $h \in C^{2,\alpha}(S^{n-1})$.

4.1. Uniqueness

In order to prove Theorem 4.1, we first show the uniqueness part as [25,57]. It remains unknown whether the uniqueness result still holds if the regularity assumption of the convex body is removed (see, for example [57]).

Lemma 4.2. *Suppose $h^1, h^2 \in C^{2,\alpha}(S^{n-1})$ are solutions to the Monge–Ampère equation with $p > q$*

$$\det(h_{ij} + h\delta_{ij}) = f(u)h^{p-1}(h^2 + |\nabla h|^2)^{\frac{n-q}{2}}, \ u \in S^{n-1}. \tag{4.1}$$

Then $h^1 \equiv h^2$.

Proof. We prove this by contradiction. Without loss of generality, we may assume $h^1 > h^2$ somewhere on S^{n-1}. Hence, there exists a constant $t \geq 1$ such that

$$th^2 - h^1 \geq 0 \text{ on } S^{n-1}, \qquad \text{and } th^2 - h^1 = 0 \text{ at some point } P \in S^{n-1}.$$

For $p > q$, by homogeneity of F and J, the fact that h^1, h^2 solve (4.1), and that $t \geq 1$, we have

$$F(h^1_{ij}, h^1) = J(\nabla h^1, h^1),$$
$$F(th^2_{ij}, th^2) = t^{q-p}J(\nabla th^2, th^2) \leq J(\nabla th^2, th^2).$$

Hence,

$$
\begin{aligned}
0 &\geq F(th^2_{ij}, th^2) - F(h^1_{ij}, h^1) + J(\nabla h^1, h^1) - J(\nabla th^2, th^2) \\
&= \int_0^1 \frac{dF(\varepsilon th^2_{ij} + (1-\varepsilon)h^1_{ij}, \varepsilon th^2 + (1-\varepsilon)h^1)}{d\varepsilon} d\varepsilon - \int_0^1 \frac{dJ(\varepsilon \nabla th^2 + (1-\varepsilon)\nabla h^1)}{d\varepsilon} d\varepsilon \\
&= \int_0^1 \frac{\partial F(\varepsilon th^2_{ij} + (1-\varepsilon)h^1_{ij}, \varepsilon th^2 + (1-\varepsilon)h^1)}{\partial h_{ij}} d\varepsilon [((th^2 - h^1)_{ij} + (th^2 - h^1)\delta_{ij})] \\
&\quad - \int_0^1 \frac{\partial J(\varepsilon \nabla th^2 + (1-\varepsilon)\nabla h^1)}{\partial h_i} d\varepsilon (th^2 - h^1)_i \\
&\quad - \int_0^1 \frac{\partial J(\varepsilon \nabla th^2 + (1-\varepsilon)\nabla h^1)}{\partial h} d\varepsilon (th^2 - h^1).
\end{aligned}
$$

That is, $th^2 - h^1$ satisfies an elliptic inequality. The strong maximum principle (Theorem 3.5 in [24]) yields $th^2 \equiv h^1$ on S^{n-1}. This implies that th^2 solves (4.1), i.e.,

$$
\begin{aligned}
F(th^2_{ij}, th^2) &= J(\nabla th^2, th^2) \\
&= t^{(p-1-q+n)} J(\nabla h^2, h^2) \\
&= t^{(p-1-q+n)} F(h^2_{ij}, h^2).
\end{aligned}
$$

But the homogeneity of F implies

$$
F(th^2_{ij}, th^2) = t^{n-1} F(h^2_{ij}, h^2).
$$

Thus, we have $t^{p-q} = 1$, which implies that $t = 1$. □

Note that the condition $p > q$ comes in critically in the above lemma.

4.2. Existence

The existence part of Theorem 4.1 will be demonstrated in this subsection.

According to the path (3.4) of variational formula, we define L_h to be the linearized operator of (4.1) at h, that is, for each $\zeta \in C^{2,\alpha}(S^{n-1})$,

$$
\begin{aligned}
L_h(\zeta) &= \left. \frac{d(F((h_\varepsilon \zeta)_{ij}, h_\varepsilon \zeta) - J(\nabla(h_\varepsilon \zeta), h_\varepsilon \zeta))}{d\varepsilon} \right|_{\varepsilon=0} \\
&= \frac{\partial F}{\partial h_{ij}} ((h\zeta)_{ij} + h\zeta\delta_{ij}) - \frac{\partial J}{\partial h_i} (h\zeta)_i - \frac{\partial J}{\partial h} h\zeta, \qquad (4.2)
\end{aligned}
$$

where $h_\varepsilon = he^{\varepsilon\zeta}$.

Lemma 4.3. *Suppose $p > q$ and $h > 0$ is a solution to (4.1). If $\zeta \in C^{2,\alpha}(S^{n-1})$ solves*

$$L_h(\zeta) = 0,$$

then $\zeta \equiv 0$ on S^{n-1}.

Proof. From $L_h(\zeta) = 0$ and (4.2), we have

$$\begin{aligned} 0 = L_h(\zeta) &= \frac{\partial F}{\partial h_{ij}}((h\zeta)_{ij} + h\zeta\delta_{ij}) - \frac{\partial J}{\partial h_i}(h\zeta)_i - \frac{\partial J}{\partial h}h\zeta \\ &= \zeta L_h(1) + h\frac{\partial F}{\partial h_{ij}}\zeta_{ij} + 2\frac{\partial F}{\partial h_{ij}}h_i\zeta_j - h\frac{\partial J}{\partial h_i}\zeta_i. \end{aligned}$$

Note that the matrix $\left(\frac{\partial F}{\partial h_{ij}}\right)_{(n-1)\times(n-1)}$ is positive definite. Therefore, at any (global) minimum point of z, one has

$$h\frac{\partial F}{\partial h_{ij}}\zeta_{ij} + +2\frac{\partial F}{\partial h_{ij}}h_i\zeta_j - h\frac{\partial J}{\partial h_i}\zeta_i \geq 0,$$

which implies

$$\zeta L_h(1) \leq 0.$$

By homogeneity of the Monge–Ampère equation (4.1), it is simple to see that if $p > q$,

$$L_h(1) = (q-p)J(h, \nabla h) < 0.$$

Thus, the minimum value of ζ must be nonnegative. Similarly, the maximum value of ζ must be nonpositive. Hence $\zeta \equiv 0$. □

Now we are ready to prove Theorem 4.1.

Proof. By Lemma 4.2, only the existence part needs a proof.

We will use the continuity methods to establish the existence. For each $t \in [0,1]$, consider the following family of equations

$$\det(h_{ij} + h\delta_{ij}) = [(1-t) + tf]h^{p-1}(h^2 + |\nabla h|^2)^{\frac{n-q}{2}}. \tag{4.3}$$

Let I be the set defined by

$$I = \{t \in [0,1] | (4.3) \text{ admits a solution } h_t \in C^{k+2}(S^{n-1})\}.$$

Clearly $0 \in I$ since $h \equiv 1$ is a solution to (4.3) when $t = 0$. We prove that $I = [0,1]$ by showing I is both open and closed.

The openness of I follows from Lemma 4.3.

It remains to prove the closeness of I, which is equivalent to making the following a priori estimates.

- Step I, C^0 estimates: Assume the solution h of (2.5) attains its maximal at $u_0 \in S^{n-1}$. Then

$$h(u_0)^{n-1} \geq f(u_0)h(u_0)^{p-1+n-q} \Rightarrow h(u_0) \leq \left(\frac{1}{\inf f}\right)^{\frac{1}{p-q}}$$

provided $p > q$. In the same way, we have for $p > q$,

$$\left(\frac{1}{\sup f}\right)^{\frac{1}{p-q}} \leq h \leq \left(\frac{1}{\inf f}\right)^{\frac{1}{p-q}}, \ x \in \mathbb{R}^{n-1}. \tag{4.4}$$

- Step II, C^1 estimates: Assume that $h \in C^{2,\alpha}(S^{n-1})$ is a solution of (2.5) for $q < p$. By (4.4),

$$|\nabla h| \leq \left(\frac{1}{\inf f}\right)^{\frac{1}{p-q}}. \tag{4.5}$$

Indeed, consider the maximum point u_0 of the function $h^2 + |\nabla h|^2$, we have

$$(h_{ij} + h\delta_{ij})h_j = 0.$$

If $\det(h_{ij} + h\delta_{ij}) \neq 0$, then $|\nabla h(u_0)| = 0$. Thus $\max(h^2 + |\nabla h|^2) \leq \max h^2$. If $\det(h_{ij} + h\delta_{ij}) = 0$, we consider $h_\varepsilon = h + \varepsilon$ for $\varepsilon > 0$. Repeat the above process, one has $\max(h_\varepsilon^2 + |\nabla h_\varepsilon|^2) \leq \max h_\varepsilon^2$. Let $\varepsilon \to 0$, we get (4.5).
- Step III, C^2 estimates: (4.4) and (4.5) say that the Gauss curvature G satisfies

$$\frac{1}{C_5} \leq \det(h_{ij} + h\delta_{ij}) \leq C_5.$$

With the above bound, Caffarelli's result [14] implies $h \in C^{1,\alpha}$. Then $C^{2,\alpha}$ follows from Caffarelli's Schauder estimate in Theorem 4 in [13].

To be more precise, we may first transfer (2.5) from S^{n-1} to $\mathbb{R}^{n-1}$ (see pp. 13–18 in Pogorelov [57]) and then apply Caffarelli [13,14]. For readers' convenience, we include a brief sketch for the transfer process.

We claim that for $e \in S^{n-1}$ and a convex solution h to (2.5), then (after extending h to $S^{n-1} \setminus \{o\}$ as a positively homogeneous function of degree 1) the function $v(x) = h(x + e)$ defined for $x \in e^\perp$ is a solution to

$$\det(v_{ij}) = \frac{f\left(\frac{x+e}{|x+e|}\right)}{|x+e|^{n+p}} v^{p-1}(|Dv|^2 + (v - x \cdot Dv)^2)^{\frac{n-q}{2}}, \qquad x \in e^{\perp}.$$

Here Dv is the regular Euclidean gradient of v in $e^{\perp}$.

For $x \in e^{\perp}$, define $\pi(x) = \frac{x+e}{|x+e|}$. Lemma 3.1 in Bianchi, Böröczky & Colesanti [5] showed the claim for $q = n$. Therefore, to prove our claim, we only need to show

$$\left(h^2 + |\nabla h|^2\right)\big|_{\pi(x)} = |Dv|^2 + (v - x \cdot Dv)^2, \qquad x \in e^{\perp}. \tag{4.6}$$

Towards this end, notice that since h is positively homogeneous of degree 1,

$$\left(h^2 + |\nabla h|^2\right)\big|_{\pi(x)} = |Dh(x+e)|^2. \tag{4.7}$$

Here Dh is the regular Euclidean gradient of h in $\mathbb{R}^n$. By the definition of v and the fact that $x \in e^{\perp}$, we have

$$D_w v(x) = D_w h(x+e), \qquad \forall w \in e^{\perp}. \tag{4.8}$$

On the other side, note that $h(x+te) = th(\frac{x}{t} + e) = tv(\frac{x}{t})$. Hence

$$\frac{\partial h(x+te)}{\partial t} = v\left(\frac{x}{t}\right) - \frac{1}{t} x \cdot Dv\left(\frac{x}{t}\right),$$

or

$$D_e h(x+e) = v(x) - x \cdot Dv. \tag{4.9}$$

Equation (4.6) now follows immediately from (4.7), (4.8), and (4.9). □

References

[1] S. Alesker, Continuous rotation invariant valuations on convex sets, Ann. of Math. (2) 149 (1999) 977–1005.

[2] B. Andrews, Gauss curvature flow: the fate of the rolling stones, Invent. Math. 138 (1999) 151–161.

[3] B. Andrews, Classification of limiting shapes for isotropic curve flows, J. Amer. Math. Soc. 16 (2003) 443–459 (electronic).

[4] F. Barthe, O. Guédon, S. Mendelson, A. Naor, A probabilistic approach to the geometry of the l_p^n-ball, Ann. Probab. 33 (2005) 480–513.

[5] G. Bianchi, K.J. Böröczky, A. Colesanti, Smoothness in the L_p Minkowski problem for $p < 1$, arXiv:1706.06310, 2017.

[6] K.J. Böröczky, P. Hegedűs, G. Zhu, On the discrete logarithmic Minkowski problem, Int. Math. Res. Not. IMRN 2016 (6) (2016) 1807–1838.

[7] K.J. Böröczky, M. Henk, Cone-volume measure of general centered convex bodies, Adv. Math. 286 (2016) 703–721.

[8] K.J. Böröczky, M. Henk, H. Pollehn, Subspace concentration of dual curvature measures of symmetric convex bodies, J. Differential Geom. (2018), in press.

[9] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn–Minkowski inequality, Adv. Math. 231 (2012) 1974–1997.

[10] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. 26 (2013) 831–852.
[11] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Affine images of isotropic measures, J. Differential Geom. 99 (2015) 407–442.
[12] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Y. Zhao, The dual Minkowski problem for symmetric convex bodies, preprint.
[13] L. Caffarelli, Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation, Ann. of Math. (2) 131 (1990) 135–150.
[14] L. Caffarelli, A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity, Ann. of Math. (2) 131 (1) (1990) 129–134.
[15] S. Chen, Q. Li, G. Zhu, On the L_p Monge–Ampère equation, J. Differential Equations 263 (8) (2017) 4997–5011.
[16] S. Chen, Q. Li, G. Zhu, The logarithmic Minkowski problem for non-symmetric measures, Trans. Amer. Math. Soc. (2018), https://doi.org/10.1090/tran/7499.
[17] S.Y. Cheng, S.T. Yau, On the regularity of the solution of the n-dimensional Minkowski problem, Comm. Pure Appl. Math. 29 (1976) 495–516.
[18] K. Choi, P. Daskalopoulos, Uniqueness of closed self-similar solutions to the gauss curvature flow, preprint.
[19] K.-S. Chou, X.-J. Wang, The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205 (2006) 33–83.
[20] A. Cianchi, E. Lutwak, D. Yang, G. Zhang, Affine Moser–Trudinger and Morrey–Sobolev inequalities, Calc. Var. Partial Differential Equations 36 (2009) 419–436.
[21] R.J. Gardner, A positive answer to the Busemann–Petty problem in three dimensions, Ann. of Math. (2) 140 (1994) 435–447.
[22] R.J. Gardner, Geometric Tomography, second edition, Encyclopedia of Mathematics and Its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.
[23] R.J. Gardner, A. Koldobsky, T. Schlumprecht, An analytic solution to the Busemann–Petty problem on sections of convex bodies, Ann. of Math. (2) 149 (1999) 691–703.
[24] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 224, Springer-Verlag, Berlin, 1983.
[25] P. Guan, C. Lin, X.-N. Ma, The existence of convex body with prescribed curvature measures, Int. Math. Res. Not. IMRN (2009) 1947–1975.
[26] C. Haberl, Minkowski valuations intertwining with the special linear group, J. Eur. Math. Soc. (JEMS) 14 (2012) 1565–1597.
[27] C. Haberl, E. Lutwak, D. Yang, G. Zhang, The even Orlicz Minkowski problem, Adv. Math. 224 (2010) 2485–2510.
[28] C. Haberl, L. Parapatits, The centro-affine Hadwiger theorem, J. Amer. Math. Soc. 27 (2014) 685–705.
[29] M. Henk, E. Linke, Cone-volume measures of polytopes, Adv. Math. 253 (2014) 50–62.
[30] M. Henk, H. Pollehn, Necessary subspace concentration conditions for the even dual Minkowski problem, Adv. Math. 323 (2018) 114–141.
[31] Y. Huang, J. Liu, L. Xu, On the uniqueness of L_p-Minkowski problems: the constant p-curvature case in $\mathbb{R}^3$, Adv. Math. 281 (2015) 906–927.
[32] Y. Huang, Q.P. Lu, On the regularity of the L_p Minkowski problem, Adv. in Appl. Math. 50 (2013) 268–280.
[33] Y. Huang, E. Lutwak, D. Yang, G. Zhang, Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems, Acta Math. 216 (2016) 325–388.
[34] Y. Huang, E. Lutwak, D. Yang, G. Zhang, The L_p-Aleksandrov problem for L_p-integral curvature, J. Differential Geom. (2018), in press.
[35] D. Hug, E. Lutwak, D. Yang, G. Zhang, On the L_p Minkowski problem for polytopes, Discrete Comput. Geom. 33 (2005) 699–715.
[36] H. Jian, J. Lu, X.-J. Wang, Nonuniqueness of solutions to the L_p-Minkowski problem, Adv. Math. 281 (2015) 845–856.
[37] H. Jian, J. Lu, G. Zhu, Mirror symmetric solutions to the centro-affine Minkowski problem, Calc. Var. Partial Differential Equations 55 (2016) 41.
[38] A. Koldobsky, Intersection bodies, positive definite distributions, and the Busemann–Petty problem, Amer. J. Math. 120 (1998) 827–840.

[39] A. Koldobsky, The Busemann–Petty problem via spherical harmonics, Adv. Math. 177 (2003) 105–114.
[40] A. Koldobsky, Stability in the Busemann–Petty and Shephard problems, Adv. Math. 228 (2011) 2145–2161.
[41] J. Lu, X.-J. Wang, Rotationally symmetric solutions to the L_p-Minkowski problem, J. Differential Equations 254 (2013) 983–1005.
[42] M. Ludwig, Ellipsoids and matrix-valued valuations, Duke Math. J. 119 (2003) 159–188.
[43] M. Ludwig, Intersection bodies and valuations, Amer. J. Math. 128 (2006) 1409–1428.
[44] M. Ludwig, General affine surface areas, Adv. Math. 224 (2010) 2346–2360.
[45] M. Ludwig, Minkowski areas and valuations, J. Differential Geom. 86 (2010) 133–161.
[46] M. Ludwig, Valuations on Sobolev spaces, Amer. J. Math. 134 (2012) 827–842.
[47] M. Ludwig, M. Reitzner, A classification of SL(n) invariant valuations, Ann. of Math. (2) 172 (2010) 1219–1267.
[48] E. Lutwak, Intersection bodies and dual mixed volumes, Adv. Math. 71 (1988) 232–261.
[49] E. Lutwak, The Brunn–Minkowski–Firey theory. I. Mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993) 131–150.
[50] E. Lutwak, The Brunn–Minkowski–Firey theory. II. Affine and geominimal surface areas, Adv. Math. 118 (1996) 244–294.
[51] E. Lutwak, V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem, J. Differential Geom. 41 (1995) 227–246.
[52] E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities, J. Differential Geom. 62 (2002) 17–38.
[53] E. Lutwak, D. Yang, G. Zhang, L_p dual curvature measures, Adv. Math. 329 (2018) 85–132.
[54] A. Naor, The surface measure and cone measure on the sphere of l_p^n, Trans. Amer. Math. Soc. 359 (2007) 1045–1079 (electronic).
[55] A. Naor, D. Romik, Projecting the surface measure of the sphere of l_p^n, Ann. Inst. Henri Poincaré Probab. Stat. 39 (2003) 241–261.
[56] G. Paouris, E. Werner, Relative entropy of cone measures and L_p centroid bodies, Proc. Lond. Math. Soc. 3 (104) (2012) 253–286.
[57] A.V. Pogorelov, The Minkowski Multidimensional Problem, V.H. Winston, Washington, 1978.
[58] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, expanded edition, Encyclopedia of Mathematics and Its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.
[59] F.E. Schuster, Valuations and Busemann–Petty type problems, Adv. Math. 219 (2008) 344–368.
[60] F.E. Schuster, Crofton measures and Minkowski valuations, Duke Math. J. 154 (2010) 1–30.
[61] F.E. Schuster, T. Wannerer, GL(n) contravariant Minkowski valuations, Trans. Amer. Math. Soc. 364 (2012) 815–826.
[62] A. Stancu, The discrete planar L_0-Minkowski problem, Adv. Math. 167 (2002) 160–174.
[63] A. Stancu, On the number of solutions to the discrete two-dimensional L_0-Minkowski problem, Adv. Math. 180 (2003) 290–323.
[64] D. Xi, G. Leng, Dar's conjecture and the log-Brunn–Minkowski inequality, J. Differential Geom. 103 (2016) 145–189.
[65] G. Xiong, Extremum problems for the cone volume functional of convex polytopes, Adv. Math. 225 (2010) 3214–3228.
[66] G. Zhang, The affine Sobolev inequality, J. Differential Geom. 53 (1999) 183–202.
[67] G. Zhang, A positive solution to the Busemann–Petty problem in $\mathbf{R}^4$, Ann. of Math. (2) 149 (1999) 535–543.
[68] Y. Zhao, The dual Minkowski problem for negative indices, Calc. Var. Partial Differential Equations 56 (2) (2017) 18.
[69] Y. Zhao, Existence of solutions to the even dual Minkowski problem, J. Differential Geom. (2018), in press.
[70] G. Zhu, The logarithmic Minkowski problem for polytopes, Adv. Math. 262 (2014) 909–931.
[71] G. Zhu, The centro-affine Minkowski problem for polytopes, J. Differential Geom. 101 (2015) 159–174.
[72] G. Zhu, The L_p Minkowski problem for polytopes for $0 < p < 1$, J. Funct. Anal. 269 (2015) 1070–1094.
[73] D. Zou, G. Xiong, Orlicz–John ellipsoids, Adv. Math. 265 (2014) 132–168.