INVERSE PROBLEMS, NON-ROUNDNESS AND FLAT PIECES OF THE EFFECTIVE BURNING VELOCITY FROM AN INVISCID QUADRATIC HAMILTON-JACOBI MODEL

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Abstract. The main goal of this paper is to understand finer properties of the effective burning velocity from a combustion model introduced by Majda and Souganidis [20]. Motivated by results in [4] and applications in turbulent combustion, we show that when the dimension is two and the flow of the ambient fluid is either weak or very strong, the level set of the effective burning velocity has flat pieces. Due to the lack of an applicable Hopf-type rigidity result, we need to identify the exact location of at least one flat piece. Implications on the effective flame front and other related inverse type problems are also discussed.

1. Introduction

We consider a flame propagation model proposed by Majda and Souganidis [20] described as follows. Suppose that $V : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth, mean zero, $\mathbb{Z}^n$-periodic and incompressible vector field. Let $T = T(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be the solution of the reaction-diffusion-convection equation

$$T_t + V \cdot DT = \kappa\, \Delta T + \frac{1}{\tau_r} f(T) \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

with given compactly supported initial data $T(x,0)$. Here κ and τ_r are positive constants proportional to the flame thickness, which has a small length scale denoted by $\varepsilon > 0$. The nonlinear function $f(T)$ is of KPP type, i.e.,

$$\begin{aligned} &f > 0 \text{ in } (0,1), \quad f < 0 \text{ in } (-\infty,0) \cup (1,\infty), \\ &f'(0) = \inf_{T>0} \frac{f(T)}{T} > 0. \end{aligned}$$

In turbulent combustions, the velocity field usually varies on small scales as well. We write $V = V\left(\frac{x}{\varepsilon^\gamma}\right)$ and, since the flame thickness is in general much smaller than the turbulence scale, as in [20], we set $\gamma \in (0,1)$ and write $\kappa = d\varepsilon$ and $\tau_r = \varepsilon$ for some given $d > 0$. To simplify notations, throughout this paper, we set $f'(0) = d = 1$. The corresponding equation becomes

$$T_t^\varepsilon + V\left(\frac{x}{\varepsilon^\gamma}\right) \cdot DT^\varepsilon = \varepsilon \Delta T^\varepsilon + \frac{1}{\varepsilon} f(T^\varepsilon) \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

2010 *Mathematics Subject Classification.* 35B27, 35B40, 35D40, 35R30, 37J50, 49L25 .

Key words and phrases. Aubry-Mather theory, effective Hamiltonian, effective burning velocity, flame propagation, homogenization, inverse problems, magnetic Hamiltonian, reaction-diffusion-convection equation, viscosity solutions, weak KAM theory.

The work of WJ is partially supported by NSF grant DMS-1515150, the work of HT is partially supported by NSF grant DMS-1615944, the work of YY is partially supported by NSF CAREER award #1151919.

which has a unique solution T^ε. It was proven in [20] that $T^\varepsilon \to 0$ locally uniformly in $\{(x,t) : Z < 0\}$, as $\varepsilon \to 0$, and $T^\varepsilon \to 1$ locally uniformly in the interior of $\{(x,t) : Z = 0\}$. Here, $Z \in C(\mathbb{R}^n \times [0,+\infty))$ is the unique viscosity solution of a variational inequality. Moreover, the set $\Gamma_t = \partial\{x \in \mathbb{R}^n : Z(x,t) < 0\}$ can be viewed as a moving front and it is shown to move with normal velocity:

$$v_{\vec{n}} = \alpha(\vec{n}),$$

where α, the so-called effective burning velocity, is defined as follows: for $p \in \mathbb{R}^n$,

$$\alpha(p) := \inf_{\lambda>0} \frac{1+\overline{H}(\lambda p)}{\lambda}. \tag{1.1}$$

Here, $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is a convex function called the effective Hamiltonian. For each $p \in \mathbb{R}^n$, $\overline{H}(p)$ is defined to be the unique constant (ergodic constant) such that the following cell problem

$$H(p+Du,x) = |p+Du|^2 + V(x)\cdot(p+Du) = \overline{H}(p) \quad \text{in } \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \tag{1.2}$$

admits a periodic viscosity solution $u \in C^{0,1}(\mathbb{T}^n)$. See [18] for the general statement. There is no viscous term in (1.2) because $\gamma < 1$ (see [20, Proposition 1.1]). Note that α, by definition, has positive homogeneity of degree 1. By the level-set approach, the effective flame front Γ_t can be described as the zero level set of $F = F(x,t)$, which satisfies

$$F_t + \alpha(DF) = 0$$

with $\Gamma_0 = \{F(x,0) = 0\}$. Thus, $\alpha(p)$ can be viewed as one way to model turbulent flame speed, a quantity of significant importance in turbulent combustion. See [11, 25] for comparisons between $\alpha(p)$ and the turbulent flame speed modeled by the G-equation (a popular level-set approach model in combustion community).

The original Hamiltonian $H(p,x) = |p|^2 + V(x)\cdot p$ is similar to the so called Mañé Hamiltonian (or magnetic Lagrangian) in the dynamical system community. Throughout this paper, we assume that V is smooth, $\mathbb{Z}^n$-periodic, incompressible and has mean zero, i.e.,

$$\operatorname{div}(V) = 0 \quad \text{and} \quad \int_{\mathbb{T}^n} V\,dx = 0. \tag{1.3}$$

Under these assumptions, it is easy to check that

$$\overline{H}(0) = 0, \quad \overline{H}(p) \geq |p|^2 \quad \text{and} \quad \alpha(p) \geq 2|p|.$$

Moreover, $\alpha(p)$ is convex. See Lemma 2.1.

Practically speaking, it is always desirable to get more information of the turbulent flame speed (effective burning velocity). In combustion literature, the turbulent flame speed is often considered to be isotropic and various explicit formulas have been introduced to quantify it. See [1, 2] and the references therein. So it is natural to ask whether there exist some non-trivial V such that the corresponding $\alpha(p)$ is isotropic. In addition, from the mathematical perspective, it is a very interesting and challenging problem to rigorously identify the shape of the effective Hamiltonian or other effective quantities. In this paper, we are interested in understanding some refined properties of the effective burning velocity $\alpha(p)$. In particular,

Question 1. If the flow is not at rest, that is, the velocity field V is not constantly zero, can the convex level set $\{p \in \mathbb{R}^n : \alpha(p) = 1\}$ be strictly convex? A simpler inverse type question is whether $\alpha(p) = c|p|$ for some $c > 0$ (i.e., isotropic) implies that $V \equiv 0$.

Remark 1.1. *When $n = 2$, $\alpha(p)$ is actually C^1 away from the origin (Lemma 2.1). If the initial flame front is the circle $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, then, owing to Theorem 2.4, the effective front at $t > 0$ is $\Gamma_t = \{x + tD\alpha(x) : x \in S^1\}$ which is a C^1 and strictly convex curve. Obviously, if α is Euclidean, that is $\alpha(p) = |p|$, then Γ_t is round for all $t > 0$. If the level curve $\{\alpha(p) = 1\}$ contains a flat piece, then there exist x_0, $x_1 \in S^1$ such that*

$$D\alpha(x_s) \equiv D\alpha(x_0) \quad \text{for } x_s = (1-s)x_0 + sx_1 \text{ and } s \in [0,1] \ .$$

In view of the positive homogeneity of α, $D\alpha(p) = D\alpha(x_0) = D\alpha(x_1)$ for all $p \in S^1$ between x_0 and x_1, i.e. $(p - x_0) \cdot (p - x_1) < 0$. Owing to the representation of Γ_t, the arc between x_0 and x_1 of S^1 is translated in time and is contained in the front Γ_t. This somehow implies that along the direction $D\alpha(x_0)$, the linear transport overwhelms the nonlinear reaction term and dominates the spread of flame, i.e., the propagation behaves like $F_t + D\alpha(x_0) \cdot DF = 0$.

Before stating the main results, we review some related works that partly motivate the study of the above questions from the mathematical perspective. Consider the metric Hamiltonian $H(p,x) = \sum_{1\le i,j\le n} a_{ij}(x)p_ip_j$ with smooth, periodic and positive definite coefficient (a_{ij}). It was proven in a very interesting paper of Bangert [4] that, for $n = 2$, if the convex level curve $\{p \in \mathbb{R}^2 : \overline{H}(p) \le 1\}$ is strictly convex, then (a_{ij}) must be a constant matrix. The argument consists of two main ingredients. First, through a delicate analysis using two dimensional topology, Bangert showed that if the level set is strictly convex at a point, then the corresponding Mather set of that point is the whole torus $\mathbb{T}^2$ and it is foliated by minimizing geodesics pointing to a specific direction. Secondly, a well-known theorem of Hopf [17], which says that a periodic Riemannian metric on $\mathbb{R}^2$ without conjugate points must be flat, was then applied for the conclusion. Part of Bangert's results (e.g. foliation of the 2d torus by minimizing orbits) was extended to Tonelli Hamiltonians in [21] for more general surfaces. Combining with the Hopf-type rigidity result in [6] for magnetic Hamiltonian, still for $n = 2$, it is easy to derive that the level set of the $\overline{H}$ associated with the Mañé-type Hamiltonian (1.2) must contain flat pieces unless $V \equiv 0$. We would like to point out that the non-strict convexity has not been established for general Tonelli Hamiltonian due to the lack of Hopf's rigidity result for Finsler metrics. See [26, 24] for instance.

The main difficulty in our situation is that the effective burning velocity α is related to the effective Hamiltonian $\overline{H}$ through a variational formulation; see (1.1). In particular, the level set of α is not the same as that of $\overline{H}$ and Hopf-type rigidity results are not applicable. In contrast to the proof in [4], we need to figure out the exact location of at least one flat piece in our proofs, which is of independent interest.

In this paper, we establish some results concerning Question 1 when the flow is either very weak or very strong. The first theorem is for any dimension.

Theorem 1.1. *Assume that V is not constantly zero. Then there exists $\varepsilon_0 > 0$ such that when $\varepsilon \in (0, \varepsilon_0)$, the level curve $S_\varepsilon = \{p \in \mathbb{R}^n : \alpha_\varepsilon(p) = 1\}$ is not round (or equivalently, the function α_ε is not Euclidean). Here, α_ε is the effective burning velocity associated with the flow velocity εV.*

In two dimensional space, thanks to Lemma 2.1, $\alpha(p) \in C^1(\mathbb{R}^2 \backslash \{0\})$. We prove further that the level curve of α is not strictly convex under small or strong advections by identifying the location of at least one flat piece. To state the result precisely, we recall that, for a set $S \subset \mathbb{R}^n$, a point p is said to be *a linear point of* S if there exists a unit vector q and a positive number $\mu_0 > 0$ such that the line segment $\{p + tq : t \in [0, \mu_0]\} \subseteq S$. In addition, $k \in \mathbb{Z}^n$ is called a *frequency* of V if $\int_{(0,1)^n} V e^{-i2\pi k \cdot x}\, dx \neq 0$. Write $k^\perp$ as the rotation of k by $\frac{\pi}{2}$ counterclockwise and

$$\mathcal{F}_V = \{\frac{k^\perp}{|k^\perp|} : \ k \text{ is a frequency of V}\}. \tag{1.4}$$

Theorem 1.2. *Assume that $n = 2$ and V is not constantly zero. Then*

1. (weak flow) *there exists $\varepsilon_0 > 0$ such that when $\varepsilon \in (0, \varepsilon_0)$, the level curve $S_\varepsilon = \{p \in \mathbb{R}^2 : \alpha_\varepsilon(p) = 1\}$ contains flat pieces. Here, α_ε is the effective burning velocity associated with the flow velocity εV. In particular, for any $q \in \mathcal{F}_V$, there exists $\varepsilon_q > 0$ such that when $\varepsilon \in (0, \varepsilon_q)$, any $p \in S_\varepsilon$ which has q as its outward normal vector is a linear point of S_ε.*
2. (strong flow) *there exists $A_0 > 0$ such that when $A \geq A_0$, the level curve $S_A = \{p \in \mathbb{R}^2 : \alpha_A(p) = 1\}$ contains flat pieces. Here, α_A is the effective burning velocity associated with the flow velocity AV. In particular, if the flow $\dot{\xi} = V(\xi)$ has a swirl (i.e., a closed orbit that is not a single point), any $p \in S_A$ which has a rational outward normal vector is a linear point of S_A when $A \geq A_0$.*

It is interesting to point out that, in the above Case (2) with a swirl, the outer normal vector behaves like a Cantor function. We conjecture that flat segments should exist for all amplitude parameters $A \in (0, \infty)$. So far, we can only show this for some special flows. Precisely speaking,

Theorem 1.3. *Assume either*

1. (shear flow) *$V(x) = (v(x_2), 0)$ for $x = (x_1, x_2) \in \mathbb{R}^2$, where $v : \mathbb{R} \to \mathbb{R}$ is a 1-periodic smooth function with mean zero, or*
2. (cellular flow) *$V(x) = (-K_{x_2}, K_{x_1})$ with $K(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$.*

Then for any fixed $A \neq 0$, the level curve $S_A = \{p \in \mathbb{R}^2 : \alpha_A(p) = 1\}$ contains flat pieces. Here, α_A is the effective burning velocity associated with the flow velocity AV. In particular, for (1), flat pieces appear at least at the location where the outward normal vector is $(\pm 1, 0)$ and for (2), flat pieces appear at least at the location where the outward normal vector is $(\pm 1, 0)$ or $(0, \pm 1)$.

We would like to point out that for the cellular flow in part (2) of Theorem 1.3, it was derived by Xin and Yu [25] that

$$\lim_{A \to +\infty} \frac{\alpha_A(p) \log A}{A} = C(|p_1| + |p_2|).$$

for $p = (p_1, p_2) \in \mathbb{R}^2$ and a fixed constant C. See Remark 2.1 for front motion associated with the Hamiltonian $H(p) = |p_1| + |p_2|$. Moreover, it remains an interesting question to at least extend the above global result to flows which have both shear and cellular structures, e.g., the cat's eye flow. A prototypical example is $V(x) = (-K_{x_2}, K_{x_1})$ with $K(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2) + \delta\cos(2\pi x_1)\cos(2\pi x_2)$ for $\delta \in (0, 1)$.

General inverse problems. In general, the effective burning velocity cannot determine the structure of the ambient fluid. The reason is that the function $\alpha(p)$ is homogeneous of degree one and only depends on the value of $\overline{H}$ from (1.2) in a bounded domain. So $\alpha(p)$ cannot see the velocity field V in places where it rotates very fast. See Claim 1 in the proof of Theorem 1.2. See also (9.5) in [3] for a related situation. Nevertheless, we can address the following inverse type problem for the effective Hamiltonian $\overline{H}(p)$.

Question 2. Assume that $H_i(p, y) = |p|^2 + V_i(y) \cdot p$. Assume further that $\overline{H}_1 = \overline{H}_2$, where $\overline{H}_i$ is the corresponding effective Hamiltonian of H_i for $i = 1, 2$. Then what can we conclude about the relations between V_1 and V_2? Especially, can we identify some common "computable" properties shared by V_1 and V_2?

This kind of questions was posed and studied first in Luo, Tran and Yu [19] for Hamiltonians of separable forms, i.e., when $H_i(p, y) = H(p) + W_i(y)$ for $i = 1, 2$. Here $H(p)$ is the kinetic energy and W_i is the potential energy. As discussed in [19], a lot of tools from dynamical systems, e.g. KAM theory, Aubry-Mather theory, are involved in the study and the analysis of the problems. For Question 2, we conclude that if the Fourier coefficients of V_i for $i = 1, 2$ decay very fast, then

$$\overline{H}_1 = \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} |V_1|^2 \, dy = \int_{\mathbb{T}^n} |V_2|^2 \, dy.$$

This follows from the approach of "asymptotic expansion at infinity" introduced in [19]. The key idea is to expand $\overline{H}_i$ "near" ∞. Consider $\overline{H}_i(sp)$ with $s \gg 1$:

$$|sp + Dw|^2 + V_i(x) \cdot (sp + Dw) = \overline{H}_i(sp).$$

Dividing s^2 on both sides,

$$|p + Dw_s|^2 + \frac{1}{s} V_i(x) \cdot (p + Dw_s) = \frac{\overline{H}_i(sp)}{s^2},$$

where $w_s := \frac{w}{s}$. Then we can perform asymptotic expansions of w_s and $\frac{\overline{H}_i(sp)}{s^2}$ with respect to the small parameter $\varepsilon = \frac{1}{s}$ and compare coefficients of ε^k for $k = 0, 1, 2$ which involve Fourier coefficients of V_i. Since the proof is similar to that of (3) in Theorem 1.2 of [19], we omit it here.

Outline of the paper. For readers' convenience, we give a quick review of Mather sets and the weak KAM theory in Section 2. Some basic properties of $\alpha(p)$ (e.g. the C^1 regularity) will be derived as well. In Section 3, we prove Theorems 1.1 via perturbation arguments. Theorems 1.2 and 1.3 will be established in Section 4. The use of two dimensional topology is extremely essential here and we do not know yet if the results of Theorems 1.2 and 1.3 can be extended to higher dimensional spaces.

Acknowledgement. We would like to thank Sergey Bolotin and Wei Cheng for helpful discussions about Aubry-Mather theory and Hopf's rigidity theorem. We also want to thank Rafael Ruggiero for pointing out the work [6]. The authors are very grateful to Alan R. Kerstein for helping us understand the notion of turbulent flame speed in combustion literature. We also want to express our gratitude for the anonymous referees for constructive suggestions to improve the presentation of this paper.

2. Preliminaries

For the reader's convenience, we briefly review some basic results concerning the Mather sets and the weak KAM theory. See [10, 12, 14] for more details. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional flat torus and $H(p,x) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a Tonelli Hamiltonian, i.e., it satisfies that

(H1) (Periodicity) $x \mapsto H(p,x)$ is $\mathbb{T}^n$-periodic;

(H2) (Uniform convexity) There exists $c_0 > 0$ such that for all $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$, and $(p,x) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\sum_{i,j=1}^n \eta_i \frac{\partial^2 H}{\partial p_i \partial p_j} \eta_j \geq c_0 |\eta|^2.$$

Let $L(q,x) = \sup_{p\in\mathbb{R}^n}\{p\cdot q - H(p,x)\}$ be the Lagrangian associated with H. Let $\mathcal{W}$ denote the set of all Borel probability measures on $\mathbb{R}^n \times \mathbb{T}^n$ that are invariant under the corresponding Euler-Lagrangian flow.

For each fixed $p \in \mathbb{R}^n$, an element μ in $\mathcal{W}$ is called a Mather measure if

$$\int_{\mathbb{R}^n\times\mathbb{T}^n} (L(q,x) - p\cdot q)\, d\mu = \min_{\nu\in\mathcal{W}} \int_{\mathbb{R}^n\times\mathbb{T}^n} (L(q,x) - p\cdot q)\, d\nu,$$

that is, if it minimizes the action associated to $L(q,x) - p\cdot q$. Denote by $\mathcal{W}_p$ the set of all such Mather measures. The value of the minimum action turns out to be $-\overline{H}(p)$, where $\overline{H}(p)$ is the unique real number such that the following Hamilton-Jacobi equation

$$H(p + Du, x) = \overline{H}(p) \quad \text{in } \mathbb{T}^n \tag{2.5}$$

has a periodic viscosity solution $u \in C^{0,1}(\mathbb{T}^n)$. Equation (2.5) is usually called the cell problem and $\overline{H}$ is called the effective Hamiltonian.

The Mather set is defined to be the closure of the union of the support of all Mather measures, i.e.,

$$\widetilde{\mathcal{M}}_p = \overline{\bigcup_{\mu\in\mathcal{W}_p} \operatorname{supp}(\mu)}.$$

The projected Mather set $\mathcal{M}_p$ is the projection of $\widetilde{\mathcal{M}}_p$ to $\mathbb{T}^n$. The following basic and important properties of the Mather set are used frequently in this paper.

(1) For any viscosity solution u of equation (2.5), we have that

$$\widetilde{\mathcal{M}}_p \subset \{(q,x) \in \mathbb{R}^n \times \mathbb{T}^n \,:\, Du(x) \text{ exists and } p + Du(x) = D_qL(q,x)\}. \tag{2.6}$$

Moreover $u \in C^{1,1}(\mathcal{M}_p)$. More precisely, there exists a constant C depending only on H and p such that, for all $y \in \mathbb{T}^n$ and $x \in \mathcal{M}_p$,

$$|u(y) - u(x) - Du(x) \cdot (y - x)| \le C|y - x|^2,$$
$$|Du(y) - Du(x)| \le C|y - x|.$$

(2) For any orbit $\xi : \mathbb{R} \to \mathbb{T}^n$ such that $(\dot{\xi}(t), \xi(t)) \in \widetilde{\mathcal{M}}_p$ for all $t \in \mathbb{R}$, we lift ξ to $\mathbb{R}^n$ and denote the lifted orbit on $\mathbb{R}^n$ still by ξ. Then, ξ is an absolutely minimizing curve with respect to $L(q, x) - p \cdot q + \overline{H}(p)$ in $\mathbb{R}^n$, i.e., for any $-\infty < s_2 < s_1 < \infty$, $-\infty < t_2 < t_1 < \infty$ and $\gamma : [s_2, s_1] \to \mathbb{R}^n$ absolutely continuous satisfying $\gamma(s_2) = \xi(t_2)$ and $\gamma(s_1) = \xi(t_1)$ the following inequality holds,

$$\int_{s_1}^{s_2} \left(L(\dot{\gamma}(s), \gamma(s)) - p \cdot \dot{\gamma}(s) + \overline{H}(p)\right) ds \ge \int_{t_1}^{t_2} \left(L(\dot{\xi}(t), \xi(t)) - p \cdot \dot{\xi}(s) + \overline{H}(p)\right) dt$$

which is equivalent to

$$\int_{s_1}^{s_2} \left(L(\dot{\gamma}(s), \gamma(s)) + \overline{H}(p)\right) ds \ge \int_{t_1}^{t_2} \left(L(\dot{\xi}(t), \xi(t)) + \overline{H}(p)\right) dt. \tag{2.7}$$

Moreover, if ξ is a periodic orbit, then its rotation vector

$$\frac{\xi(T) - \xi(0)}{T} \in \partial\overline{H}(p). \tag{2.8}$$

Here T is the period of ξ and $\partial\overline{H}(p)$ is the subdifferential of $\overline{H}$ at p, i.e., $q \in \partial\overline{H}(p)$ if $\overline{H}(p') \ge \overline{H}(p) + q \cdot (p' - p)$ for all $p' \in \mathbb{R}^n$.

A central problem in weak KAM theory is to understand the relation between analytic properties of the effective Hamiltonian $\overline{H}$ and the underlying Hamiltonian system (e.g. structures of Mather sets). For instance, Bangert [3] gave a detailed characterization of Mather and Aubry sets on the 2-torus $\mathbb{T}^2$ for metric or mechanical Hamiltonians (i.e., $H(p, x) = \sum_{1 \le i,j \le n} a_{ij} p_i p_j + W(x)$ with a positive definite (a_{ij}).).

Let us mention some known results in this direction which are more relevant to this paper. As an immediate corollary of [7, Proposition 3], we have the following result concerning the level curves of $\overline{H}$ in two dimensional space.

Theorem 2.1. *Assume that $n = 2$. If $\overline{H}(p) = c > \min \overline{H}$, then the set $\partial\overline{H}(p)$ is a closed radial interval, i.e., there exist a unit vector $q \in \mathbb{R}^2$ and $0 < s_1 \le s_2$ such that $\partial\overline{H}(p) = [s_1 q, s_2 q] := \{sq : s \in [s_1, s_2]\}$. In particular, this implies that the level set $\{p \in \mathbb{R}^2 : \overline{H}(p) = c\}$ is a closed C^1 convex curve and q is the unit outward normal vector at p.*

The following theorem was first proven in [12, Theorem 8.1]. It says that the effective Hamiltonian is strictly convex along any direction that is not tangent to the level set.

Theorem 2.2. *Assume that p_1, $p_2 \in \mathbb{R}^n$. Suppose that $\overline{H}(p_2) > \min \overline{H}$ and $\overline{H}$ is linear along the line segment connecting p_1 and p_2. Then*

$$\overline{H}(tp_1 + (1 - t)p_2) \equiv \overline{H}(p_2) \quad \text{for all } t \in [0, 1].$$

In dynamical system literature, the effective Hamiltonian $\overline{H}$ and its Lagrangian $\overline{L}$ are often called *α and β functions* respectively. Since $Q \in \partial\overline{H}(P) \Leftrightarrow P \in \partial\overline{L}(Q)$, that $\overline{L}$ is not differentiable at Q implies that $\overline{H}$ is linear along any two vectors in $\partial\overline{L}(Q)$. Accordingly, as an immediate outcome of [21, Corollary 1], we have that

Theorem 2.3. *Let $n=2$, $c>\min\overline{H}$ and $p\in\Gamma_c=\{\overline{H}=c\}$. If the unit normal vector of Γ_c at p is a rational vector and p is not a linear point of Γ_c, then $\mathcal{M}_p$ consists of periodic orbits which foliate $\mathbb{T}^2$.*

For metric or mechanical Hamiltonians (i.e., $H(p,x)=\sum_{1\le i,j\le n}a_{ij}p_ip_j+W(x)$), the above result was first established in [4]. See also [22] for a closely related result about twist maps.

In this paper we have the Mañé Hamiltonian $H(p,x)=|p|^2+V(x)\cdot p$, with smooth periodic velocity field V satisfying (1.3), and the main objective is to study the properties of the effective burning velocity $\alpha(p)$ given by (1.1) and (1.2). In particular, H satisfies (H1)–(H2) and, hence, the previous theorems in this section apply. We conclude this section with some useful properties of α.

Lemma 2.1. *Fix $p\in\mathbb{R}^n\backslash\{0\}$. The followings hold.*

(1) *$\alpha:\mathbb{R}^n\to\mathbb{R}$ is convex, and there exists a unique $\lambda_p>0$ such that*
$$\alpha(p)=\frac{1+\overline{H}(\lambda_p p)}{\lambda_p}.$$
Moreover, there exists $q\in\partial\overline{H}(\lambda_p p)$ such that
$$q\cdot\lambda_p p=\overline{H}(\lambda_p p)+1.$$

(2) *Assume that $n=2$. Then $\alpha(p)\in C^1(\mathbb{R}^n\backslash\{0\})$.*

(3) *Assume that $n=2$. Then p is a linear point of the level curve $\{\alpha=1\}$ if and only if $\lambda_p p$ is a linear point of the level curve $\{\overline{H}=\lambda_p-1\}$.*

Proof. (1) Taking integration on both sides of (1.2), since V is incompressible and has zero mean, we have that
$$\overline{H}(p)\ge|p|^2.$$
The existence of λ_p is clear. For the convexity of α, fix p_0, $p_1\in\mathbb{R}^n\backslash\{0\}$ and choose λ_0, $\lambda_1>0$ such that
$$\alpha(p_0)=\frac{1+\overline{H}(\lambda_0p_0)}{\lambda_0}\quad\text{and}\quad\alpha(p_1)=\frac{1+\overline{H}(\lambda_1p_1)}{\lambda_1}.$$
For $\theta\in[0,1]$, write $p_\theta=\theta p_1+(1-\theta)p_0$. If $p_\theta=0$, the convexity is obvious since $\alpha(0)=0$ and $\alpha(p)\ge 2|p|$. So we assume $p_\theta\ne0$. Choose $\lambda_\theta>0$ such that $\frac{1}{\lambda_\theta}=\frac{\theta}{\lambda_1}+\frac{1-\theta}{\lambda_0}$. It follows immediately from the definition of α and the convexity of $\overline{H}$ that
$$\alpha(p_\theta)\le\frac{1+\overline{H}(\lambda_\theta p_\theta)}{\lambda_\theta}\le\theta\alpha(p_1)+(1-\theta)\alpha(p_0).$$
The convexity of α is proved.

Next we prove the uniqueness of λ_p. Assume that for λ, $\bar{\lambda}>0$, we have that
$$\alpha(p)=\frac{1+\overline{H}(\lambda p)}{\lambda}=\frac{1+\overline{H}(\bar{\lambda}p)}{\bar{\lambda}}.$$

Then $\partial\alpha(p) \subseteq \partial\overline{H}(\lambda p)$ and $\partial\alpha(p) \subseteq \partial\overline{H}(\bar{\lambda}p)$. Therefore, $\partial\overline{H}(\lambda p) \cap \partial\overline{H}(\bar{\lambda}p) \neq \emptyset$. So $\overline{H}$ is linear along the line segment connecting λp and $\bar{\lambda}p$. Then by Theorem 2.2, $\overline{H}(\lambda p) = \overline{H}(\bar{\lambda}p)$, which immediately leads to $\lambda = \bar{\lambda}$.

Next we prove the second equality in Claim (1). For $\lambda > 0$, denote by $w(\lambda) = \overline{H}(\lambda p) \geq \lambda^2|p|^2$ and

$$h(\lambda) = \frac{1 + w(\lambda)}{\lambda}.$$

Since $w(\lambda)$ is convex, there exists a decreasing sequence $\{\lambda_m\}$ such that $\lambda_m \downarrow \lambda_p$ and w is differentiable at λ_m and $h'(\lambda_m) \geq 0$. Clearly,

$$w'(\lambda_m) = q_m \cdot p \quad \text{for any } q_m \in \partial\overline{H}(\lambda_m p).$$

Up to a subsequence, we may assume that $q_m \to q^+ \in \partial\overline{H}(\lambda_p p)$. Then in light of the fact that $h'(\lambda_m) \geq 0$, we deduce

$$q^+ \cdot \lambda_p p \geq \overline{H}(\lambda_p p) + 1.$$

Similarly, by considering an increasing sequence that converges to λ_p, we can pick $q^- \in \partial\overline{H}(\lambda_p p)$ such that

$$q^- \cdot \lambda_p p \leq \overline{H}(\lambda_p p) + 1.$$

Since $\partial H(\lambda_p p)$ is a convex set, we can find $q \in \partial\overline{H}(\lambda_p p)$ which satisfies

$$q \cdot \lambda_p p = \overline{H}(\lambda_p p) + 1.$$

(2) Apparently,

$$\hat{q} \in \partial\alpha(p) \quad \Rightarrow \quad \hat{q} \in \partial\overline{H}(\lambda_p p). \tag{2.9}$$

Owing to Theorem 2.1, $\partial\alpha(p)$ is also a closed radial interval. Since $\alpha(p)$ is homogeneous of degree 1, any $q \in \partial\alpha(p)$ satisfies $p \cdot q = \alpha(p)$. Since $p \neq 0$ and $\alpha(p) > 0$, this interval can only contain a single point; it follows that α is differentiable at p.

(3) "$\Rightarrow$": This part is true in any dimension. Clearly, that p is a linear point of $S = \{\alpha = 1\}$ implies that there exists $p' \in S$ such that $p \neq p'$ and

$$\partial\alpha(p) \cap \partial\alpha(p') \neq \emptyset.$$

By (2.9), $\partial\alpha(p) \subseteq \partial\overline{H}(\lambda_p p)$ and $\partial\alpha(p') \subseteq \partial\overline{H}(\lambda_{p'} p')$. Hence $\overline{H}$ is linear along the line segment connecting $\lambda_p p$ and $\lambda_{p'} p'$. Then Theorem 2.2 implies that $\overline{H}(\lambda_p p) = \overline{H}(\lambda_{p'} p')$ and $\lambda_p = \lambda_{p'}$. The necessity then follows.

Now we prove the sufficiency which relies on the 2-dimensional topology. For $p \in \mathbb{R}^2$, assume that $\lambda_p p$ is a linear point of the level curve $C_p = \{\overline{H} = \lambda_p - 1\}$, i.e., there exists a distinct vector $\lambda_p p' \in C_p$ such that the line segment $l_{p,p'} = \{sp + (1-s)p' \,:\, s \in [0,1]\}$, which connects p and p', satisfies $l_{p,p'} \subset \{G = 1\}$. Here for $q \in \mathbb{R}^2$,

$$G(q) = \frac{1 + \overline{H}(\lambda_p q)}{\lambda_p} \geq \alpha(q).$$

By Theorem 2.1 and $D\alpha(p) \in \partial G(p) = \partial\overline{H}(\lambda_p p)$, we have that

$$\partial G(p) = \{sD\alpha(p) \,:\, s \in [\theta_1, \theta_2]\}$$

for some $0 < \theta_1 \leq \theta_2$. Therefore $D\alpha(p) \cdot (p' - p) = 0$, which implies that

$$1 = G(q) \geq \alpha(q) \geq \alpha(p) + D\alpha(p) \cdot (q - p) = \alpha(p) = 1$$

for any $q \in l_{p,p'}$. Hence $l_{p,p'} \subset \{\alpha = 1\}$ and p is a linear point.

□

The following result characterizes the shape of the moving front when the initial front is the unit circle in $\mathbb{R}^2$.

Theorem 2.4. *Suppose that $n = 2$ and $\alpha : \mathbb{R}^2 \to \mathbb{R}$ is convex, coercive and positive homogeneous of degree 1. Let $u \in C(\mathbb{R}^2 \times [0, +\infty))$ be the unique viscosity solution to*

$$\begin{cases} u_t + \alpha(Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x, 0) = |x| - 1. \end{cases}$$

Then $u(x, t) = \max\{-t\alpha(p) + x \cdot p \,:\, |p| \le 1\} - 1$ and its zero level set is

$$\Gamma_t := \{x \in \mathbb{R}^2 \,:\, u(x, t) = 0\} = \{p + tq \,:\, p \in S^1,\ q \in \partial\alpha(p)\}. \tag{2.10}$$

Also Γ_t is C^1. Moreover,

$$\alpha \in C^1(\mathbb{R}^2 \backslash \{0\}) \Longleftrightarrow \Gamma_t \text{ is strictly convex.} \tag{2.11}$$

Proof. We first prove the representation (2.10). Due to the 1-homogeneity of $\alpha(p)$, $p \cdot q = \alpha(p)$ for any $q \in \partial\alpha(p)$. The formula of $u(x, t)$ then follows directly from Theorem 3.1 in [5]. Clearly, if $u(x, t) > -1$, then

$$u(x, t) = \max\{-t\alpha(p) + x \cdot p \,:\, |p| = 1\} - 1.$$

Now fix $x \in \mathbb{R}^2$ such that $u(x, t) = 0$. Choose $|\bar{p}| = 1$ such that

$$u(x, t) = \overline{p} \cdot x - t\alpha(\overline{p}) - 1 = 0. \tag{2.12}$$

By the Lagrange multiplier method, we get $x - tq = s\overline{p}$ for some $q \in \partial\alpha(\overline{p})$ and some $s \in \mathbb{R}$. We use (2.12) to deduce further that $s = 1$, and hence $x = \overline{p} + tq$.

Conversely, if $x = \overline{p} + tq$ for some $\overline{p} \in S^1$ and $q \in \partial\alpha(\overline{p})$, we want to show that $u(x, t) = 0$. In fact, in the representation formula of u, choosing $p = \bar{p}$ immediately leads to $u(x, t) \ge 0$. On the other hand, for any $|p| = 1$, $q \in \partial\alpha(\overline{p})$ implies

$$\alpha(p) \ge \alpha(\bar{p}) + q \cdot (p - \bar{p}).$$

Therefore

$$-t\alpha(p) + x \cdot p \le -t\alpha(\bar{p}) - tq \cdot (p - \bar{p}) + x \cdot p = p \cdot \bar{p} \le 1.$$

So $u(x, t) \le 0$. Hence we proved that $u(x, t) = 0$.

Next we show that Γ_t is C^1. Fix $t > 0$. Owing to the above arguments, given $x \in \Gamma_t$, there exists a unique unit vector p_x such that $x = p_x + q_x$ for some $q_x \in \partial\alpha(p_x)$ and

$$u(x, t) = -t\alpha(p_x) + p_x \cdot x - 1.$$

The uniqueness is due to the convexity of α which implies that $(p - p') \cdot (q - q') \ge 0$ for $q \in \partial\alpha(p)$ and $q' \in \partial\alpha(p')$. Hence $x \to p_x$ is a continuous map from Γ_t to the unit circle. Combining with $p_x \in \partial_x u(x, t)$, p_x is the outward unit normal vector of Γ_t at x and Γ_t is C^1.

Next we prove the duality (2.11). Again fix $t > 0$. This direction "$\Leftarrow$" follows immediately from the representation formula (2.10). So let us prove "$\Rightarrow$". We argue by contradiction. Assume that α is C^1 away from the origin. If Γ_t is not strictly convex, then there exist x, $y \in \Gamma_t$ such that $x \ne y$ and $p_x = p_y$. Hence $q_x \ne q_y$.

However, $q_x = D\alpha(p_x) = D\alpha(p_y) = q_y$, which is a contradiction. This proves that (2.11) holds. □

Remark 2.1. *As mentioned in Remark 1.1, when $n = 2$, a flat piece on the level set $\{\alpha(p) = 1\}$ leads to a translated arc of the unit circle on Γ_t. Moreover, singular points of α (i.e., points where $\partial\alpha(p)$ contains a line segment) generate flat pieces on Γ_t. For example, if $\alpha(p) = |p_1| + |p_2|$ for $p = (p_1, p_2)$, then the front Γ_1 at $t = 1$ is the closed curve shown in Fig. 1:*

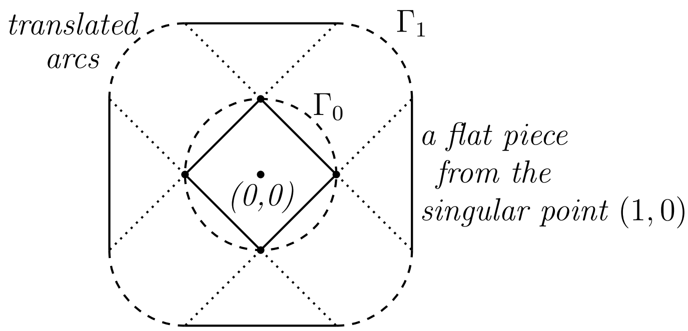

FIGURE 1. Front propagation and the shape of Γ_1.

3. The Proof of Theorem 1.1

Fix $p \in \mathbb{R}^n$ to be an irrational vector satisfying a Diophantine condition, i.e., there exist $c = c(p) > 0$ and $\gamma > 0$ such that

$$|p \cdot k| \geq \frac{c}{|k|^\gamma} \quad \text{for all } k \in \mathbb{Z}^n \setminus \{0\}.$$

For small ε, let $\overline{H}_\varepsilon(p)$ be the effective Hamiltonian associated with $|p|^2 + \varepsilon V \cdot p$, i.e.,

$$|p + Du^\varepsilon|^2 + \varepsilon V \cdot (p + Du^\varepsilon) = \overline{H}_\varepsilon(p). \tag{3.13}$$

We now perform a formal asymptotic expansion in term of ε, which will be proved rigorously by using the viscosity solution techniques. Suppose that

$$u^\varepsilon = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$$
$$\overline{H}_\varepsilon(p) = a_0(p) + \varepsilon a_1(p) + \varepsilon^2 a_2(p) + \cdots$$

We then get that

$$\begin{aligned} a_0(p) &= |p|^2 \\ a_1(p) &= 2p \cdot D\phi_1 + V \cdot p \Rightarrow a_1(p) = 0 \\ a_2(p) &= 2p \cdot D\phi_2 + |D\phi_1|^2 + V \cdot D\phi_1 \Rightarrow a_2(p) = \int_{\mathbb{T}^n} |D\phi_1|^2\,dx \\ &\cdots \end{aligned} \tag{3.14}$$

Set

$$V = \sum_{k \neq 0} v_k e^{i2\pi k \cdot x}.$$

We need $v_k \cdot k = 0$ for all k to have that $\mathrm{div} V = 0$. Then we get

$$D\phi_1 = -\frac{1}{2}\sum_{k\neq 0} \frac{(p\cdot v_k)e^{i2\pi k\cdot x}k}{p\cdot k},$$

and

$$a_2(p) = \frac{1}{4}\sum_{k\neq 0} \frac{|p\cdot v_k|^2|k|^2}{|p\cdot k|^2}.$$

Thus, formally, we can conclude that

$$\overline{H}_\varepsilon(p) \approx |p|^2 + \varepsilon^2 \frac{1}{4}\sum_{k\neq 0} \frac{|p\cdot v_k|^2|k|^2}{|p\cdot k|^2} + O(\varepsilon^3).$$

We now prove this expansion formula rigorously. See related computations in [16].

Lemma 3.1. *There exists $\tau > 0$, such that for all p satisfying a Diophantine condition and $|p| \in \left[\tau, \frac{1}{\tau}\right]$, we have*

$$\overline{H}_\varepsilon(p) = |p|^2 + \varepsilon^2 \frac{1}{4}\sum_{k\neq 0} \frac{|p\cdot v_k|^2|k|^2}{|p\cdot k|^2} + O(\varepsilon^3) \tag{3.15}$$

as $\varepsilon \to 0$. Here, the error term satisfies $|O(\varepsilon^3)| \le K\varepsilon^3$ for some K depending only on τ, V and $\frac{p}{|p|}$.

Proof. As p satisfies a Diophantine condition, we are able to solve the following two equations explicitly in $\mathbb{T}^n$ by computing the Fourier coefficients

$$\begin{cases} p\cdot D\phi_1 = -\frac{1}{2}V\cdot p \\ p\cdot D\phi_2 = \frac{1}{2}\left(a_2(p) - |D\phi_1|^2 - V\cdot D\phi_1\right). \end{cases}$$

Here $\phi_1, \phi_2 : \mathbb{T}^n \to \mathbb{R}$ are unknown functions.

Set $w^\varepsilon = \varepsilon\phi_1 + \varepsilon^2\phi_2$. Then, in light of the properties of ϕ_1, ϕ_2, w^ε satisfies

$$|p + Dw^\varepsilon|^2 + \varepsilon V\cdot(p + Dw^\varepsilon) = |p|^2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

By looking at places where $u^\varepsilon - w^\varepsilon$ attains its maximum and minimum and using the definition of viscosity solutions, we derive that

$$\overline{H}_\varepsilon(p) = |p|^2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

The error estimate can be read from the proof easily. □

It is obvious that $a_2(p)$ is not a constant function of p. Hence Theorem 1.1 follows immediately from the following lemma.

Lemma 3.2. *Let $p \in \mathbb{R}^n$ be a vector satisfying a Diophantine condition. For each $\varepsilon \in (0, 1]$, let α_ε be the effective burning velocity function defined by* (1.1) *with $\overline{H}$ replaced by associated to $\overline{H}_\varepsilon$. Then*

$$\lim_{\varepsilon\to 0} \frac{\alpha_\varepsilon(p) - 2|p|}{\varepsilon^2|p|} = a_2(p). \tag{3.16}$$

Proof. Since $\alpha_\varepsilon(p)$ is homogeneous of degree 1 and $a_2(p)$ is homogeneous of degree 0 ($a_2(p) = a_2(\lambda p)$ for all $\lambda > 0$), we may assume that $|p| = 1$. Thanks to Lemma 3.1, we can write

$$\overline{H}_\varepsilon(p) = |p|^2 + \varepsilon^2 \frac{1}{4} \sum_{k \neq 0} \frac{|p \cdot v_k|^2 |k|^2}{|p \cdot k|^2} + O(\varepsilon^3) = 1 + \varepsilon^2 a_2(p) + O(\varepsilon^3). \tag{3.17}$$

Owing to Lemma 2.1, there exists a unique constant $\lambda_\varepsilon = \lambda_\varepsilon(p) > 0$ such that

$$\alpha_\varepsilon(p) = \frac{1 + \overline{H}_\varepsilon(\lambda_\varepsilon p)}{\lambda_\varepsilon} \geq \lambda_\varepsilon + \frac{1}{\lambda_\varepsilon}.$$

The second inequality is due to $\overline{H}_\varepsilon(q) \geq |q|^2$. By the definition, it is obvious that

$$\alpha_\varepsilon \leq 1 + \overline{H}_\varepsilon(p) = 2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

Hence it is easy to see that $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0$. Then by Lemma 3.1 and $\lambda_\varepsilon + \frac{1}{\lambda_\varepsilon} \geq 2$,

$$\alpha_\varepsilon(p) \geq 2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

Therefore, the conclusion of the lemma holds. □

4. Proofs of Theorems 1.2 and 1.3

Before proceeding to the proofs, we would like to point out some connection but more importantly a crucial difference between the studies of the flat pieces of the level curves of H and those of α. Clearly, if the level curve $\{\alpha = 1\}$ of α is strictly convex at p, the level curve $\{\overline{H} = \lambda_p - 1\}$ of $\overline{H}$ must be strictly convex at $\lambda_p p$, where λ_p is determined by Lemma 2.1. Nevertheless, our results do not follow from any rigidity result for $\overline{H}$, namely that of [6]. Indeed, different p's in $\{\alpha = 1\}$ might correspond to different λ_p, which corresponds to different energy levels of $\overline{H}$, but the rigidity result from [6] can be applied only on the same energy level. A key point of our proofs is to identify the exact location of at least one flat piece.

We first prove Claim (1) of Theorem 1.2.

Proof of Theorem 1.2 (1). We carry out the proof in a few steps.

Step 1: Due to Claim (2) of Lemma 2.1, the level curve S_ε is C^1. It is worth keeping in mind that $\alpha_\varepsilon(p)$ is homogeneous of degree 1. For each $p \in S_\varepsilon$, denote n_p the outward unit normal vector at p to S_ε.

Step 2: Fix $q_0 \in \mathcal{F}_V$ from (1.4). Then there exists $k_0 \in \mathbb{Z}^2 \setminus \{0\}$ such that k_0 is a frequency of V and $q_0 = k_0^\perp/|k_0|$. We claim that if V is not constantly zero, then there exists $x_0 \in \mathbb{R}^2$ satisfying that

$$\int_0^{|k_0|} q_0 \cdot DV(x_0 + q_0 t)\, dt = \int_0^1 k_0^\perp \cdot DV(x_0 + k_0^\perp t)\, dt \neq 0. \tag{4.18}$$

Here $q_0 \cdot DV = D(q_0 \cdot V)$. Caution: $q_0 \cdot DV(x + q_0 t) \neq \frac{dV(x+q_0 t)}{dt}$.

In fact, assume that $V(y) = \sum_{k \in \mathbb{Z}^2} v_k e^{i2\pi k \cdot y}$, where $\{v_k\} \subset \mathbb{R}^2$ are the Fourier coefficients of V. Since $\mathrm{div}(V) = 0$ and $\int_{\mathbb{T}^2} V\, dx = 0$, we have that $v_0 = 0$ and

$$k \cdot v_k = 0 \quad \text{for all } k \in \mathbb{Z}^2. \tag{4.19}$$

Also, $v_{k_0} \neq 0$ since k_0 is a frequency. Then for any $q \in \mathbb{R}^2$,

$$q \cdot DV(y) = D(q \cdot V) = 2\pi i \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} (q \cdot v_k) e^{i2\pi k \cdot y} k.$$

Now for the vector $q_0 = k_0^\perp/|k_0|$ that is fixed earlier, we deduce from the results above that

$$\int_0^1 q_0 \cdot DV(x + k_0^\perp t)\, dt = 0 \quad \text{for all } x \in \mathbb{R}^2 \quad \Rightarrow \quad q_0 \cdot v_{k_0} = 0.$$

Combining with (4.19), we deduce that $v_{k_0} = 0$. This is a contradiction. So our claim holds.

Step 3: For each $\varepsilon > 0$, choose $p_\varepsilon \in S_\varepsilon$ such that $n_{p_\varepsilon} = q_0 = k_0^\perp/|k_0|$. To simplify notations, we write $n_\varepsilon = n_{p_\varepsilon}$. We claim that when ε is small enough, p_ε is a linear point of the set $\{\alpha_\varepsilon = 1\}$. Suppose this is false, then there exists a decreasing sequence $\varepsilon_m \downarrow 0$ and a sequence $\{p_{\varepsilon_m}\}$ such that p_{ε_m} is not a linear point of the set $\{\alpha_{\varepsilon_m} = 1\}$. By (3) of Lemma 2.1, $\tilde{p}_{\varepsilon_m} = \lambda_{\varepsilon_m} p_{\varepsilon_m}$ is not a linear point of the level curve $\{\overline{H}_{\varepsilon_m} = \lambda_{\varepsilon_m} - 1\}$ either. Here $\lambda_{\varepsilon_m} > 0$ is from Lemma 2.1. Clearly, the outward unit normal vector of the level curve $\{\overline{H}_{\varepsilon_m} = \lambda_{\varepsilon_m} - 1\}$ at $\tilde{p}_{\varepsilon_m}$ is also q_0. According to Theorem 2.3, the projected Mather set $\mathcal{M}_{\tilde{p}_{\varepsilon_m}}$ is the whole torus $\mathbb{T}^2$. Moreover, by (2.8), there is a periodic minimizing orbit $\xi_m : \mathbb{R} \to \mathbb{R}^2$ passing through x_0 from Step 2 such that $\xi_m(0) = x_0$, $\xi_m(t_m) = x_0 + |k_0| q_0$ for some $t_m > 0$ and ξ satisfies the Euler-Lagrange equation associated with the Lagrangian $L(q,x) = \frac{1}{4}|q - \varepsilon_m V|^2$:

$$\frac{d\left(\dot{\xi}_m(t) - \varepsilon_m V(\xi_m(t))\right)}{dt} = -\left(\dot{\xi}_m(t) - \varepsilon_m V(\xi_m(t))\right) \cdot \varepsilon_m DV(\xi_m).$$

Taking the integration on both sides over $[0, t_m]$, and by periodicity, we get

$$\int_0^{t_m} \left(\dot{\xi}_m(t) - \varepsilon_m V(\xi_m(t))\right) \cdot DV(\xi_m)\, dt = 0.$$

Sending $m \to +\infty$, we find

$$\int_0^{|k_0|} q_0 \cdot DV(x_0 + q_0 t)\, dt = 0.$$

This contradicts to (4.18). As a result, we identified a flat piece of S_ε. □

Next we prove Claim (2) of Theorem 1.2.

Proof of Theorem 1.2(2). Recall that, for $A \geq 0$ and $p \in \mathbb{R}^2$, $\alpha_A(p)$ is defined as

$$\alpha_A(p) = \inf_{\lambda > 0} \frac{\overline{H}_A(\lambda p) + 1}{\lambda}.$$

Here $\overline{H}_A$ is the effective Hamiltonian associated with $H_A(p,x) = |p|^2 + AV \cdot p$. The corresponding Lagrangian is

$$L_A(q,x) = \frac{1}{4}|q - AV(x)|^2.$$

For $p \in \mathbb{R}^2 \backslash \{0\}$ and $A \geq 0$, by Lemma 2.1, denote $\lambda_{p,A} > 0$ as the unique positive number such that

$$\alpha_A(p) = \frac{\overline{H}_A(\lambda_{p,A} p) + 1}{\lambda_{p,A}}.$$

Since V is divergence-free and has zero mean, there exists a smooth periodic function K (stream function) such that $V = (-K_{x_2}, K_{x_1})$. Clearly, we have that $DK \cdot V \equiv 0$. We consider the dynamical system $\dot{\xi} = V(\xi)$. The flow $\xi_t : \mathbb{T}^2 \to \mathbb{T}^2$, at each $t \geq 0$, is then Lebesgue-measure preserving and has zero mean translation. It follows that, for each $t \geq 0$, ξ_t has fixed points; see for instance [8]. Since $V \not\equiv 0$, the system must have non-critical periodic orbits on $\mathbb{T}^2$. Note also that the system can also be viewed as defined on the whole space $\mathbb{R}^2$; we adopt both views in the following proof. We have the following two cases.

Case 1: $\dot{\xi} = V(\xi)$ has a non-critical contractable periodic orbit on $\mathbb{T}^2$. This means the orbit on $\mathbb{T}^2$ can be continuously shrunk to a point or, equivalently, the lift of this orbit to $\mathbb{R}^2$ is a closed curve in some compact set. By the stability in 2d, there exists a strip of closed periodic orbits in its neighborhood. Without loss of generality, we may label them as $\gamma_s(t)$ for $s \in [0, \delta]$ for some $\delta > 0$ sufficiently small such that $K(\gamma_s(t)) \equiv s$ and $\gamma_s(0) = \gamma_s(T_s)$ for some $T_s > 0$ (minimum period). See the following figure. Denote $\Gamma = \bigcup_{s \in [0,\delta]} \{\gamma_s(t) : t \in [0, T_s]\}$ as the union of these closed curves and

$$\tau = \max_{x \in \Gamma} |DK(x)| > 0.$$

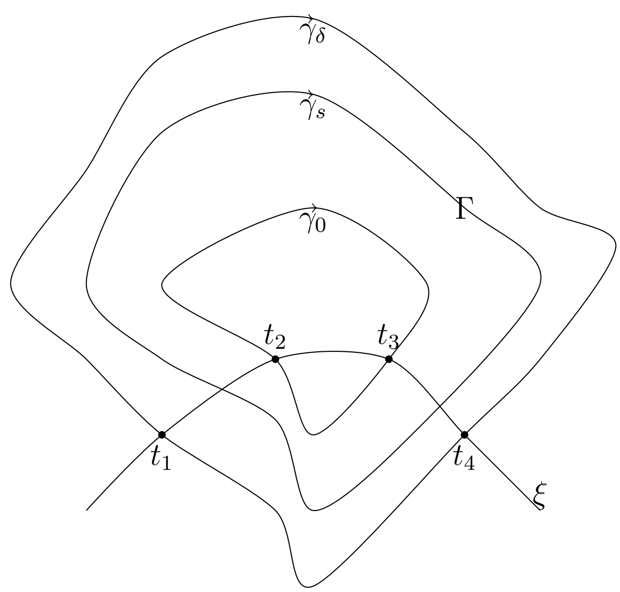

FIGURE 2. Closed periodic orbits in Γ

Claim 1: For $p \in \mathbb{R}^2$, if $\overline{H}_A(p) < \bar{c}A^2$ for $\bar{c} = \frac{4\delta^2}{T_\delta^2 \tau^2}$, then any unbounded absolutely minimizing trajectory associated with $L_A + \overline{H}_A(p)$ cannot intersect γ_0.

We argue by contradiction. If not, let $\xi : \mathbb{R} \to \mathbb{R}^2$ be an unbounded absolutely minimizing trajectory with $\xi \cap \gamma_0 \neq \emptyset$. Then there must exist $t_1 < t_2 \leq t_3 < t_4$ such that

$$\xi(t_1), \xi(t_4) \in \gamma_\delta, \quad \xi(t_2), \xi(t_3) \in \gamma_0 \quad \text{and} \quad \xi([t_1, t_2]) \cup \xi([t_3, t_4]) \subset \Gamma.$$

See Figure 2 for demonstration. Set

$$E_1 = \int_{t_1}^{t_2} \frac{1}{4}|\dot{\xi} - AV(\xi)|^2 + \overline{H}_A(p)\, ds \quad \text{and} \quad E_2 = \int_{t_3}^{t_4} \frac{1}{4}|\dot{\xi} - AV(\xi)|^2 + \overline{H}_A(p)\, ds.$$

Since

$$|\dot{\xi} - AV(\xi)| \geq \frac{1}{\tau}|\dot{\xi} - AV(\xi)| \cdot |DK(\xi)| \geq \frac{1}{\tau}|(\dot{\xi} - AV(\xi)) \cdot DK(\xi)| = \frac{1}{\tau}|\dot{w}(t)|$$

for $w(t) = K(\xi(t))$ and $t \in [t_1, t_2] \cup [t_3, t_4]$, we have that

$$\begin{aligned} E_1 + E_2 &\geq \overline{H}_A(p)(t_2 - t_1 + t_4 - t_3) + \frac{1}{4\tau^2}\left(\int_{t_1}^{t_2} |\dot{w}(t)|^2\, dt + \int_{t_3}^{t_4} |\dot{w}(t)|^2\, dt\right) \\ &\geq \overline{H}_A(p)(t_2 - t_1 + t_4 - t_3) + \frac{1}{4\tau^2}\left(\frac{\delta^2}{t_4 - t_3} + \frac{\delta^2}{t_2 - t_1}\right) \\ &\geq \frac{2\delta}{\tau}\sqrt{\overline{H}_A(p)}. \end{aligned}$$

However, if we travel from $\xi(t_1)$ to $\xi(t_4)$ along the route $\gamma(s) = \gamma_\delta(sA)$, the cost is at most $\frac{T_\delta}{A}\overline{H}_A(p) < \frac{2\delta}{\tau}\sqrt{\overline{H}_A(p)}$. This contradicts to the assumption that ξ is a minimizing trajectory. Hence our above claim holds.

Now choose $\varepsilon_0 > 0$ such that $\varepsilon_0^2 + \overline{M}\varepsilon_0 < \bar{c}$ for $\overline{M} = \max_{\mathbb{T}^2} |V|$. Owing to Lemma 4.2, there exists A_0 such that if $A \geq A_0$, then

$$\lambda_{p,A} \leq \frac{\varepsilon_0}{2} A$$

for any unit vector p.

Claim 2: Assume that $p_A \in S_A = \{\alpha_A = 1\}$ has a rational outward normal vector. Then p_A is a linear point of S_A if $A \geq A_0$.

In view of item (2) in Lemma 2.1, S_A is a convex C^1 curve, and the set of outward normal vectors attached to S_A is the whole S^1. The claim above, hence, locates countably many flat pieces of S_A, provided that Case 1 occurs.

We prove Claim 2 by contradiction. Clearly, $p_A \neq 0$, so the above is equivalent to say that $\bar{p} = \frac{p_A}{|p_A|}$ is a linear point of the level curve $\left\{\alpha_A(p) = \frac{1}{|p_A|}\right\}$. Suppose $\bar{p}$ is not a linear point, by Theorem 2.3 and (3) of Lemma 2.1, $\overline{H}_A$ is strictly convex at $\lambda_{\bar{p},A}\, \bar{p}$ and the associated projected Mather set $\mathcal{M}_{\lambda_{\bar{p},A}\, \bar{p}}$ is the whole Torus. Due to Claim 1, we must have that

$$\overline{H}_A(\lambda_{\bar{p},A}\bar{p}) \geq \bar{c}A^2.$$

Since $\overline{H}_A(p) \leq |p|^2 + A\overline{M}|p|$, we have that $\lambda_{\bar{p},A} > \varepsilon_0 A$. This contradicts to the choice of A. Therefore, the above claim holds and the result of this theorem follows.

Case 2: Next we consider the case when $\dot{\xi} = V(\xi)$ has a non-contractible periodic orbit on $\mathbb{T}^2$ or, equivalently, the lifted system on $\mathbb{R}^2$ has a solution $\eta : \mathbb{R} \to \mathbb{R}^2$, such that $\eta(T_0) - \eta(0) \in \mathbb{Z}^2\backslash\{(0,0)\}$ for some $T_0 > 0$. Denote q_0 as a rotation vector of η. Clearly, q_0 is a rational vector. Since $\int_{\mathbb{T}^2} V\,dx = 0$, there also exists a non-contractible periodic orbit $\dot{\tilde{\eta}}(t) = V(\tilde{\eta}(t))$ with a rotation vector $-cq_0$ for some $c > 0$. See the following Figure 3.

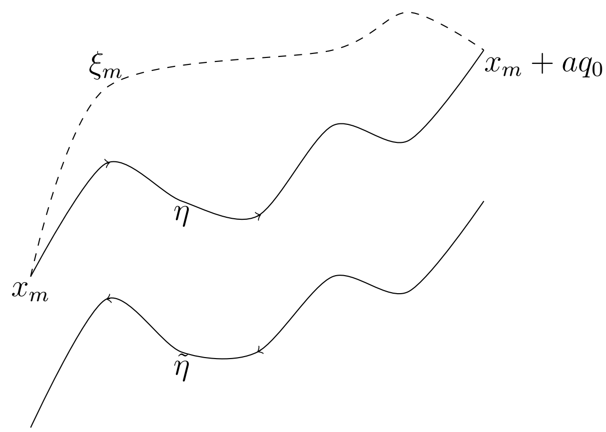

FIGURE 3. Unbounded periodic orbits η and $\tilde{\eta}$

Claim 3: Choose $p_A \in S_A$ such that the unit outward normal vector at p_A is $\frac{q_0}{|q_0|}$. Then when A is large enough, p_A is a linear point of S_A.

This is consistent with the last statement in Remark 1.1: when A is very large, we expect the shear structure to dominate the flame propagation. Again, the set of outward unit normal vectors of S_A is the whole circle S^1, so the p_A above exists. Claim 3 hence locates at least one flat piece of S_A, provided that Case 2 occurs.

Again, the claim is equivalent to say that $\bar{p} = \frac{p_A}{|p_A|}$ is a linear point of the level curve $\left\{\alpha_A(p) = \frac{1}{|p_A|}\right\}$ for sufficiently large A. We argue by contradiction. If not, then by (3) of Lemma 2.1, there exist a sequence $A_m \to +\infty$ as $m \to +\infty$ and $|p_m| = 1$ such that $\overline{H}_{A_m}(\lambda_m p_m)$ is strictly convex near $\lambda_m p_m$. Here $\lambda_m > 0$ is the unique number satisfying (Lemma 2.1)

$$\alpha_{A_m}(p_m) = \frac{1 + \overline{H}_{A_m}(\lambda_m p_m)}{\lambda_m}.$$

Therefore, by Theorem 2.3, the associated projected Mather set $\mathcal{M}_{\lambda_m p_m}$ is the whole torus. So there exists a unique periodic C^1 solution v_m (up to additive constants) to

$$|\lambda_m p_m + Dv_m|^2 + A_m V \cdot (\lambda_m p_m + Dv_m) = \overline{H}_{A_m}(\lambda_m p_m) \quad \text{in } \mathbb{R}^2.$$

Let T_0 and $\tilde{T}_0$ be the minimal period of η and $\tilde{\eta}$ respectively. Then $q_0 = \frac{\eta(T_0)-\eta(0)}{T_0}$ and $-cq_0 = \frac{\tilde{\eta}(T_0)-\tilde{\eta}(0)}{T_0}$. Taking integration along η and $\tilde{\eta}$, we obtain that

$$\frac{1}{T_0}\int_0^{T_0} |\lambda_m p_m + Dv_m(\eta(s))|^2\,ds + A_m q_0 \cdot \lambda_m p_m = \overline{H}_{A_m}(\lambda_m p_m)$$

and

$$\frac{1}{\tilde{T}_0}\int_0^{\tilde{T}_0} |\lambda_m p_m + Dv_m(\tilde{\eta}(s))|^2\, ds - cA_m q_0 \cdot \lambda_m p_m = \overline{H}_{A_m}(\lambda_m p_m).$$

Accordingly, without loss of generality, we may assume that for all $m \geq 1$,

$$\max_{s\in\mathbb{R}} |\lambda_m p_m + Dv_m(\eta(s))| \geq \sqrt{\overline{H}_{A_m}(\lambda_m p_m)}.$$

So there exists $x_m \in \eta(\mathbb{R}) \cap [0,1]^n$ such that

$$|\lambda_m p_m + Dv_m(x_m)| \geq \sqrt{\overline{H}_{A_m}(\lambda_m p_m)}. \tag{4.20}$$

Since the projected Mather set $\mathcal{M}_{\lambda_m p_m}$ is the whole torus and the unit outward normal vector of $\{\overline{H}_{A_m} = \overline{H}_{A_m}(\lambda_m p_m)\}$ at $\lambda_m p_m$ is also $\frac{q_0}{|q_0|}$, by (2.8), we may find an non-contractable periodic minimizing trajectory $\xi_m : \mathbb{R} \to \mathbb{R}^2$ such that $\xi_m(0) = x_m$ and $\xi_m(t_m) = x_m + a_0 q_0$ (see Figure 3). Here $t_m > 0$ is the minimal period of ξ_m and $a_0 > 0$ is the smallest positive number such that $a_0 q_0 \in \mathbb{Z}^2$. Note that $\eta(T_0) - \eta(0) = a_0 q_0$ as well. Moreover, by (2.6),

$$\dot{\xi}_m = 2(\lambda_m p_m + Dv_m(\xi_m)) + A_m V(\xi_m) \tag{4.21}$$

and, by evaluating the cell problem along the curve ξ_m and differentiating in time, we get, after some cancellation,

$$\frac{d(\dot{\xi}_m(s) - A_m V(\xi_m(s)))}{ds} = -(\dot{\xi}_m - A_m V(\xi_m))A_m DV(\xi_m).$$

Since ξ_m is an absolutely minimizing trajectory, we must have that

$$\frac{T_0}{A_m}\overline{H}_{A_m}(\lambda_m p_m) \geq \int_0^{t_m} \frac{1}{4}|\dot{\xi}_m(s) - A_m V(\xi_m)|^2\, ds + t_m \overline{H}_{A_m}(\lambda_m p_m). \tag{4.22}$$

The left hand side of the above is the cost of traveling along the route $\gamma(s) = \eta(sA_m)$ from x_m to $x_m + aq_0$. So $t_m \leq \frac{T_0}{A_m}$. Consider

$$w_m(s) = \xi_m\left(\frac{s}{A_m}\right).$$

Then w_m is an non-contractable periodic curve with a minimal period $A_m t_m \leq T_0$,

$$\frac{1}{4}|\dot{w}_m - V(w_m)|^2 + \frac{1}{2}V(w_m)\cdot(\dot{w}_m - V(w_m)) = \frac{\overline{H}_{A_m}(\lambda_m p_m)}{A_m^2} \tag{4.23}$$

and

$$\frac{d(\dot{w}_m(s) - V(w_m(s)))}{ds} = -(\dot{w}_m(s) - V(w_m(s)) \cdot DV(w_m(s)).$$

Denote $G(s) = |\dot{w}_m(s) - V(w_m(s))|^2$. Then

$$G'(s) \leq CG(s)$$

for $C = 2\max_{\mathbb{R}^2}|DV|$. So $\frac{d\log(G(s))}{ds} \leq C$. This implies that for $0 \leq s_1 \leq s_2 \leq t_m A_m \leq T_0$,

$$\frac{G(s_2)}{G(s_1)} \leq e^{C(s_2 - s_1)} \leq e^{CT_0}$$

and

$$\frac{G(s_1)}{G(s_2)} = \frac{G(s_1 + A_m t_m)}{G(s_2)} \le e^{C(s_1 + A_m t_m - s_2)} \le e^{CT_0}.$$

Combining with the periodicity of G, we obtain that

$$\frac{\min_{s\in\mathbb{R}} |\dot{w}_m(s) - V(w_m(s))|}{\max_{s\in\mathbb{R}} |\dot{w}_m(s) - V(w_m(s))|} \ge \theta_0 := e^{-\frac{CT_0}{2}}. \tag{4.24}$$

Owing to (4.22), we obtain that

$$\int_0^{A_m t_m} |\dot{w}_m(s) - V(w_m(s))|^2 \, ds \le \frac{4T_0 \overline{H}_{A_m}(\lambda_m p_m)}{A_m^2}. \tag{4.25}$$

Owing to (4.23) and Lemma 4.2, $\max_{s\in\mathbb{R}} |\dot{w}_m(s)|$ is uniformly bounded. Since $w_m(A_m t_m) - w_m(0) = a_0 q_0$, it is clear that

$$\liminf_{m\to+\infty} A_m t_m > 0.$$

Now combining (4.25), (4.24) and Lemma 4.2, it is not hard to show that

$$\lim_{m\to+\infty} A_m t_m = T_0$$

and

$$\lim_{m\to+\infty} w_m(s) = \eta(s) \quad \text{uniformly in } C^1(\mathbb{R}^1). \tag{4.26}$$

Write $c_m = \max_{s\in\mathbb{R}} |\dot{w}_m(s) - V(w_m(s))|$. Note that

$$\dot{w}_m(s) = \frac{2(\lambda_m p_m + Dv_m(w_m(s)))}{A_m} + V(w_m(s)).$$

and by (4.23),

$$V(w_m) \cdot \frac{\dot{w}_m - V(w_m)}{c_m} = \frac{2\overline{H}_{A_m}(\lambda_m p_m)}{A_m^2 c_m} - \frac{1}{2c_m} |\dot{w}_m - V(w_m)|^2.$$

Due to (4.20) and (4.21), $c_m A_m \ge 2\sqrt{\overline{H}_{A_m}(\lambda_m p_m)}$. Combining with (4.26) and Lemma 4.2, we have that

$$\lim_{m\to+\infty} V(w_m(s)) \cdot \frac{\dot{w}_m(s) - V(w_m(s))}{c_m} = 0 \quad \text{uniformly in } \mathbb{R}^1. \tag{4.27}$$

Note that

$$\frac{dK(w_m(s))}{ds} = DK(w_m(s)) \cdot (\dot{w}_m(s) - V(w_m(s))).$$

Taking integration from 0 to $t_m A_m$, due to periodicity, we have that

$$\int_0^{t_m A_m} DK(w_m(s)) \cdot \left(\frac{\dot{w}_m(s) - V(w_m(s))}{c_m} \right) ds = 0.$$

By (4.24), $\frac{|\dot{w}_m(s) - V(w_m(s))|}{c_m} \in [\theta_0, 1]$. Combining with (4.27), by sending $m \to +\infty$, we obtain that

$$\int_0^{T_0} a(s) |DK(\eta(s))| \, ds = 0$$

for some $a(t) > 0$. This is a contradiction. So our claim holds.

Combining Case 1 and Case 2, we obtain the desired result.

□

Lemma 4.1. *Let $\overline{H}$ be the effective Hamiltonian of*

$$|p+Dv|^2 + V(x)\cdot(p+Dv) = \overline{H}(p).$$

Then for $|p|\geq\theta>0$, there exists $\mu_\theta>0$ depending only on θ and V such that

$$\min_{q\in\partial\overline{H}(p)} q\cdot p \geq \overline{H}(p)+\mu_\theta.$$

Proof. This follows easily from a compactness argument, $\overline{H}(0)=0$, $\overline{H}(p)\geq|p|^2$ and the strict convexity of $\overline{H}$ along the radial direction (Theorem 2.2). □

Due to the simple equality $\overline{H}(p)=\frac{\overline{H}_A(Ap)}{A^2}$, we immediately derive the following corollary. Recall that $\overline{H}_A$ is the effective Hamiltonian from (1.2) with V replaced by AV.

Corollary 4.1. *If $|p|\geq\theta A$, then*

$$\min_{q\in\partial\overline{H}_A(p)} q\cdot p \geq \overline{H}_A(p)+\mu_\theta A^2.$$

Lemma 4.2. *For $|p|=1$ and $A\geq1$, denote $\lambda_{p,A}$ such that*

$$\alpha_A(p)=\frac{\overline{H}_A(\lambda_{p,A}p)+1}{\lambda_{p,A}}.$$

Then

$$\lim_{A\to+\infty}\frac{\max_{|p|=1}\lambda_{p,A}}{A}=\lim_{A\to+\infty}\frac{\max_{|p|=1}\overline{H}_A(\lambda_{p,A}p)}{A^2}=0.$$

Proof. Since $\overline{H}_A(p)\leq|p|^2+A\overline{M}|p|$ for $\overline{M}=\max_{\mathbb{T}^2}|V|$, the second limit holds true immediately once we prove the validity of the first limit.

We prove the first limit by contradiction. If not, then there exists a sequence $A_m\to+\infty$ as $m\to+\infty$ and $|p_m|=1$ such that for $\lambda_m=\lambda_{p_m,A_m}$,

$$\lim_{m\to+\infty}\frac{\lambda_m}{A_m}=b_0>0.$$

So by Lemma 2.1, there is $q_m\in\partial\overline{H}_{A_m}(\lambda_m p_m)$ such that

$$q_m\cdot\lambda_m p_m=\overline{H}_A(\lambda_m p_m)+1.$$

This contradicts to Corollary 4.1 when m is large enough. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. It suffices to show that there exists a unit vector p_0 such that sp_0 is a linear point of $\{\overline{H}_A=\overline{H}_A(sp_0)\}$ for any $s>0$.

(1) Assume that V is the shear flow, i.e. $V=(v(x_2),0)$. Without loss of generality, we omit the dependence on A. Then the cell problem is reduced to 1d:

$$|p_1|^2+|p_2+u'(y)|^2+p_1v(y)=\overline{H}(p).$$

So $\overline{H}(p) - |p_1|^2$ is the effective Hamiltonian associated with the 1d mechanical Hamiltonian $H_1(p_2, y) = |p|^2 + p_1 v(y)$, which is given by a simple explicit formula ([18]). Accordingly, $\overline{H}_A(p)$ has the following explicit formulas: for $p = (p_1, p_2) \in \mathbb{R}^2$,

$$\overline{H}(p) = |p_1|^2 + h(p_1, p_2)$$

and $h : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\begin{cases} h(p) = M(p_1) = \max_{y\in\mathbb{T}} p_1 v(y) & \text{if } |p_2| \le \int_0^1 \sqrt{M(p_1) - p_1 v(y)}\, dy \\ |p_2| = \int_0^1 \sqrt{h(p) - p_1 v(y)}\, dy & \text{otherwise.} \end{cases}$$

Hence $\overline{H}$ is linear near the point $p = (s, 0)$ as long as $s \neq 0$ and the corresponding outward normal vector is either $(1, 0)$ or $(-1, 0)$.

(2) Now let $V = (-K_{x_2}, K_{x_1})$ for $K(x) = \sin(2\pi x_1)\sin(2\pi x_2)$. Recall that the cell problem is

$$|p + Dv|^2 + AV \cdot (p + Dv) = \overline{H}_A(p) \ge |p|^2. \tag{4.28}$$

Due to the symmetry, the proof for the cellular flow case of Theorem 1.3 follows directly from the result of the following proposition.

□

Proposition 4.1. *Fix $s > 0$ and let $Q = (s, 0) \in \mathbb{R}^2$. If $A \neq 0$, then Q is a linear point of the level set $\{\overline{H}_A = \overline{H}_A(Q)\}$.*

Proof. Let $\mathcal{M}_Q$ be the projected Mather set at the point Q. By symmetry, it is easy to see that $\partial\overline{H}(Q)$ is parallel to (1,0). Then due to Theorem 2.3, it suffices to show that

$$\mathcal{M}_Q \cap \{y \in \mathbb{T}^2 \,:\, y_2 = 0\} = \emptyset. \tag{4.29}$$

Step 1: We claim that there is a viscosity solution v to (4.28) which satisfies that $v(y_1, y_2) = v(y_1, -y_2)$. In fact, let us now look at the discounted approximation of (4.28) with $p = Q$. For each $\varepsilon > 0$, consider

$$\varepsilon v^\varepsilon + |Q + Dv^\varepsilon|^2 + AV \cdot (Q + Dv^\varepsilon) = 0 \quad \text{in } \mathbb{T}^2, \tag{4.30}$$

which has a unique viscosity solution $v^\varepsilon \in C^{0,1}(\mathbb{T}^2)$. By the fact that $Q = (s, 0)$ and the special structure of V, it is clear that $(y_1, y_2) \mapsto v^\varepsilon(y_1, -y_2)$ is also a solution to the above. Therefore, $v^\varepsilon(y_1, y_2) = v^\varepsilon(y_1, -y_2)$ for all $(y_1, y_2) \in \mathbb{T}^2$. Clearly, any convergent subsequence of $v^\varepsilon - v^\varepsilon(0)$ tends to a v which is a solution of (4.28) and is even in the y_2 variable. We would like to point out that a recent result of Davini, Fathi, Iturriaga and Zavidovique [9] (see also Mitake and Tran [23]) gives the convergence of the full sequence $v^\varepsilon - v^\varepsilon(0)$ as $\varepsilon \to 0$.

Step 2: Assume by contradiction that (4.29) is not correct. Suppose that

$$(\mu_0, 0) \in \mathcal{M}_Q \cap \{y \in \mathbb{T}^2 \,:\, y_2 = 0\}.$$

Then v is differentiable at $(\mu_0, 0)$ and $v_{x_2}(\mu_0, 0) = 0$. Due to (2.6), the flow-invariance of the Mather set and the Euler-Lagrangian equation, it is easy to see that

$$\{y \in \mathbb{T}^2 \,:\, y_2 = 0\} \subset \mathcal{M}_Q.$$

Hence v is C^1 along the y_1 axis and

$$v_{y_2}(y_1,0)=0 \quad \text{for all } y_1\in\mathbb{T}. \tag{4.31}$$

Set $w(y_1)=v(y_1,0)$. Plug this into the equation (4.28) of v with $y_2=0$ and use (4.31) to get that

$$|s+w'|^2-A(s+w')\sin(2\pi y_1)=\overline{H}_A(Q)\ge s^2 \quad \text{in } \mathbb{T}. \tag{4.32}$$

Clearly, $s+w'(y_1)\neq 0$ for all $y_1\in\mathbb{T}$ in light of (4.32). Note further that $w'\in C(\mathbb{T})$ and

$$\int_0^1 (s+w'(y_1))\,dy_1=s>0.$$

Thus, $s+w'>0$ in $\mathbb{T}$ and for all $y_1\in\mathbb{T}$,

$$s+w'(y_1)=\frac{1}{2}\left(A\sin(2\pi y_1)+\sqrt{A^2\sin^2(2\pi y_1)+4\overline{H}_A(Q)}\right).$$

Integrate this over $\mathbb{T}$ to deduce that

$$\begin{aligned} s=\int_0^1 (s+w'(y_1))\,dy_1 &= \int_0^1 \frac{1}{2}\left(A\sin(2\pi y_1)+\sqrt{A^2\sin^2(2\pi y_1)+4\overline{H}_A(Q)}\right)dy_1\\ &=\int_0^1\frac{1}{2}\sqrt{A^2\sin^2(2\pi y_1)+4\overline{H}_A(Q)}\,dy_1\ge\sqrt{\overline{H}_A(Q)}\ge s.\end{aligned}$$

Therefore, all inequalities in the above must be equalities. In particular, the second last inequality must be an equality, which yields that $A=0$. □

Remark 4.1. *Theorem 2.3 is not really necessary to get the above proposition. In fact, using the same argument, we can derive that the Aubry set has no intersection with the y_1 axis. Then by* [15]*, there is a strict subsolution to* (4.28) *near the y_1 axis. The linearity of $\overline{H}$ near Q will follow from some elementary calculations.*

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