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Stability of Traveling Waves of Nonlinear Schrödinger Equation with Nonzero Condition at Infinity

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Abstract

We study the stability of traveling waves of the nonlinear Schrödinger equation with nonzero condition at infinity obtained via a constrained variational approach. Two important physical models for this are the Gross–Pitaevskii (GP) equation and the cubic-quintic equation. First, under a non-degeneracy condition we prove a sharp instability criterion for 3D traveling waves of (GP), which had been conjectured in the physical literature. This result is also extended for general nonlinearity and higher dimensions, including 4D (GP) and 3D cubic-quintic equations. Second, for cubic-quintic type nonlinearity, we construct slow traveling waves and prove their nonlinear instability in any dimension. For dimension two, the non-degeneracy condition is also proved for these slow traveling waves. For general traveling waves without vortices (that is nonvanishing) and with general nonlinearity in any dimension, we find a sharp condition for linear instability. Third, we prove that any 2D traveling wave of (GP) is transversally unstable, and we find the sharp interval of unstable transversal wave numbers. Near unstable traveling waves of all of the above cases, we construct unstable and stable invariant manifolds.

1. Introduction

Consider the Gross–Pitaevskii (GP) equation

$$i\frac{\partial u}{\partial t} + \Delta u + (1 - |u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1}$$

where u satisfies the boundary condition $|u| \to 1$ when $|x| \to \infty$. Equation (1), with the considered non-zero conditions at infinity, arises in lots of physical problems such as superconductivity, superfluidity in Helium II, and Bose–Einstein condensate (for example [1,10]). On a formal level, the Gross–Pitaevskii equation is a Hamiltonian PDE. The conserved Hamiltonian is the energy defined by

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbf{R}^3} \frac{1}{4}(1-|u|^2)^2 \,\mathrm{d}x$$

and the energy space is defined by

$$X_0 = \{u \in H^1_{\mathrm{loc}}(\mathbf{R}^3) : E(u) < +\infty\}.$$

The momentum

$$\vec{P}(u) = \frac{1}{2}\int_{\mathbf{R}^3} \langle i\nabla u, u-1\rangle$$

is also formally conserved, due to the translation invariance of (GP). We denote

$$P(u) = \frac{1}{2}\int_{\mathbf{R}^3} \langle i\partial_{x_1} u, u-1\rangle \,\mathrm{d}x = -\int_{\mathbf{R}^3} (u_1 - 1)\,\partial_{x_1} u_2 \,\mathrm{d}x \tag{2}$$

to be the first component of $\vec{P}$. The global existence of the Cauchy problem for (GP) in the energy space X_0 was proved in [29,30]. Some studies on the asymptotic behavior of a solution with regard to (1) can be found in, for example, [35,36].

Traveling waves are solutions to (GP) of the form $u(t,x) = U_c(x - ce_1 t)$, where $e_1 = (1,0,0)$ and U_c satisfies the equation

$$-ic\partial_{x_1} U_c + \Delta U_c + (1 - |U_c|^2)U_c = 0. \tag{3}$$

Such traveling waves of finite energy play an important role in the dynamics of the Gross–Pitaevskii equation. In a series of papers including [39,40], JONES, PUTTERMAN and ROBERTS used formal expansions and numerics to construct traveling waves and studied their properties for both 2D and 3D (GP). For 3D, they found a branch of traveling waves with the travel speed in the subsonic interval $(0, \sqrt{2})$. These traveling waves tend to a pair of vortex rings when $c \to 0$ and to solitary waves of the Kadomtsev–Petviashvili (KP) equation when $c \to \sqrt{2}$. Starting in late 1990s [13], BÉTHUEL and SAUT initiated a rigorous mathematical study of the program of Jones, Putterman and Roberts. Since then, there have been lots of mathematical studies on this subject. We refer to the survey [14] and two recent papers [49,51] on the existence and properties of traveling waves of (GP). In particular, the existence of 3D traveling waves in the full subsonic range $(0, \sqrt{2})$ was proved in [49]; non-existence of supersonic and sonic traveling waves was shown in [50]; symmetry, decay and regularity of both 2D and 3D traveling waves were studied in [14,31]. However, the stability and dynamics of these traveling waves have not been well studied. Recently, CHIRON and MARIS [51] constructed both 2D and 3D traveling waves of (GP) by minimizing the energy under the constraint of fixed momentum. They showed the compactness of the minimizing sequence and as a corollary the orbital stability of these traveling waves was obtained. However the range of traveling speeds that these stable traveling waves cover is not clear. Moreover, for 3D (GP) it is known that only one part of the traveling waves branch could be constructed as energy minimizers subject to fixed momentum.

In the physical literature [12,39], the following linear stability criterion for 3D traveling waves was conjectured based on numerics and heuristic arguments: there is linear stability of the branch of traveling waves U_c satisfying $\frac{\mathrm{d}P(U_c)}{\mathrm{d}c} > 0$, commonly referred as the lower branch, and linear instability on the branch with $\frac{dP(U_c)}{dc} < 0$, commonly referred as the upper branch. More specifically, numerical evidence [12,39] suggests that there exists $c^* \in (0, \sqrt{2})$ such that $\frac{dP(U_c)}{dc} > 0$ for $c \in (0, c^*)$ and $\frac{dP(U_c)}{dc} < 0$ for $c \in (c^*, \sqrt{2})$. Here, our definition of $P(u)$ follows the notation in [49] and differs from that of [12,39] by a negative sign. In this paper, we rigorously justify this stability criterion under a non-degeneracy condition (24) or its cylindrical symmetric version (56). Roughly, we show the following main theorem for (4), a more general (than (1)) nonlinear Schrödinger equation with non-vanishing condition at infinity:

Main Theorem 1. *Let* $0 < c_0 < \sqrt{2}$ *and* U_{c_0} *be a traveling wave solution of* (4) *radial in* (x_2, x_3) *directions constructed in* [49].

- *Suppose the nonlinearity* F *in* (4) *satisfies* $(F1\text{-}2)$ *and a non-degeneracy condition* (24) *holds. Then for c in a neighborhood of* c_0, *there exists a locally unique* C^1 *family of traveling waves* U_c. *If, in addition,* U_c *satisfies* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$, *then the traveling wave* U_{c_0} *is orbitally stable in the energy space* X_0.
- *Suppose* $F \in C^5$ *and that a cylindrical version of the non-degeneracy condition* (56) *holds. Then for c in a neighborhood of* c_0, *there exists a* C^1 *family of traveling waves* U_c *locally unique in cylindrically symmetric function spaces. If, in addition,* U_c *satisfies* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$, *the linearized equation at* U_{c_0} *has an unstable eigenvalue and locally* U_{c_0} *has a* 1*-dim* C^2 *unstable manifold and a* 1*-dim* C^2 *stable manifold, which yields the nonlinear instability.*

Here assumptions (F1-2) are given in Section 2.5. The existence of the local C^1 family of traveling waves are due to the Implicit Function Theorem based on the non-degeneracy assumption, see Theorem 5.3. We refer to Theorems 2.1, 2.2, 3.1 and Corollary 2.2 for more precise statements on the stability/instability, where the exact meaning of the orbital stability is also given. In fact we do not have to limit ourselves to those traveling waves constructed in [49]. The main properties on U_c we really need are that, as critical points of the energy-momentum functional $E_c \triangleq E + cP$, the Hessian E_c'' of E_c at U_c has exactly one negative direction, in addition to the non-degeneracy (24) of E_c''.

Condition (24) states that the kernel of the Hessian E_c'' of the energy-momentum functional E_c is spanned by the translation modes $\{\partial_{x_i} U_c\}$ only. Equivalently, the linearization of the (elliptic) traveling wave equation has only solutions of translation modes. Such a condition is commonly assumed in the stability analysis of dynamical systems (for example [32,33]). It is a nontrivial task to confirm the non-degeneracy condition for a given traveling wave associated to a specific nonlinearity, which mainly involves the analysis of the linearized elliptic equation of traveling waves. In Appendix 2, we verify such kinds of conditions in certain cases.

Remark 1.1. Assume U_c is a family of traveling waves C^1 in c. If the stability sign condition $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} \geqq 0$ is satisfied, actually we can still obtain the spectral stability of U_{c_0} even if the non-degeneracy condition (24) is not satisfied. Here the spectral stability means that the spectrum of the linearized equation at U_{c_0} is contained in the imaginary axis on the complex plane. This is a consequence of the results in a more general setting in a forthcoming paper [46]. However, the linear stability is not guaranteed as linear solutions may grow like $O(t)$.

We give a brief description of key ideas in the proof. The troubles from the non-zero condition at infinity can be seen from the linearized operator, which is of the form JL_c, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and L_c (defined by (20)) is the second variation operator of the Hamiltonian $E + cP$. When $|x| \to \infty$, the operator L_c has the asymptotic form

$$\begin{pmatrix} -\Delta + 2 & -c\partial_{x_1} \\ c\partial_{x_1} & -\Delta \end{pmatrix},$$

which implies that the essential spectrum of L_c is $[0, +\infty)$ for any $c \in (0, \sqrt{2})$. Therefore, there is no spectral gap for L_c between the discrete spectrum (negative and zero eigenvalues) and the rest of the spectrum, so we cannot use the standard stability theory for Hamiltonian PDEs as in [32,33], which requires such a spectral gap condition. To overcome this issue, we observe that the quadratic form $\langle L_c\cdot, \cdot\rangle$ has the right spectral structure in the space $X_1 = H^1(\mathbf{R}^3) \times \dot{H}^1(\mathbf{R}^3)$. More precisely, the quadratic form of L_c is uniformly positive definite modulo a finite dimensional negative and zero modes. However, another issue arises since the operator $J^{-1} = -J$ does not map X_1 to its dual $(X_1)^* = H^{-1}(\mathbf{R}^3) \times \dot{H}^{-1}(\mathbf{R}^3)$. The boundedness of $J^{-1} : X_1 \to (X_1)^*$ is required in [32,33] and is true for Schrödinger equation with vanishing condition where $X_1 = H^1 \times H^1$. We use a new argument to avoid using the boundedness of J^{-1} and prove the linear instability criterion $\frac{\mathrm{d}P(U_c)}{\mathrm{d}c} < 0$ (Proposition 3.3) under the non-degeneracy condition (56).

To study the nonlinear dynamics, we use a coordinate system of the (non-flat) energy space X_0 over the Hilbert space X_1. More precisely, there exists a bi-continuous mapping $\psi : X_1 \to X_0$ as defined in (11), which was first introduced in [30] to understand the structure of the energy space X_0. The nonlinear stability on the lower branch with $\frac{\mathrm{d}P(U_c)}{\mathrm{d}c} > 0$ is proved by the Taylor expansions of Hamiltonian functional $(E + c\tilde{P})(\psi(w))$ for $w \in X_1$ near w_c, where $U_c = \psi(w_c)$ and $\tilde{P}(u)$ is the extended momentum (defined in (12)) in the energy space X_0. The proof of stability (Theorem 2.1) implies that the stable traveling waves are local energy minimizers with a fixed momentum.

To study the nonlinear dynamics near the linearly unstable traveling waves on the upper branch, we rewrite the (GP) equation in terms of the coordinate function $w \in X_1$, where $u = \psi(w)$ satisfies the (GP) Equation (1). We construct stable (unstable) manifolds near unstable traveling waves by this new equation for $w \in X_3 = H^3 \times \dot{H}^3$, on which the nonlinear term of the w-equation is shown to be semilinear in Appendix 1. The linearized operator for w is similar to the operator JL_c, that is, of the form $K^{-1}JL_cK$, where K is an isomorphism of X_1 defined in (22). Thus the study of the linearized w equation is reduced to the study of the semigroup e^{tJL_c}. To show the existence of unstable (stable) manifolds, first we establish an exponential dichotomy estimate for e^{tJL_c} in X_3. That is, to decompose X_3 into the direct sum of two invariant subspaces, on one the linearized solutions have an exponential growth and on the other one have strictly slower growth. It is highly nontrivial to get such an exponential dichotomy for e^{tJL_c} from the spectra of JL_c due to the issue of spectral mapping (see Remark 3.3). In this paper, we develop a new approach to proving the exponential dichotomy of e^{tJL_c}, which might be useful for very general Hamiltonian PDEs. The idea is very simple and natural. We observe that the quadratic form of $\langle L_c u(t), v(t)\rangle$ is invariant for any two linearized solutions $u(t)$ and $v(t)$. This implies that the orthogonal complement (in the inner product $\langle L_c\cdot, \cdot\rangle$) to the unstable and stable modes defines a subspace invariant under the linearized flow e^{tJL_c}. The quadratic form $\langle L_c\cdot, \cdot\rangle$ restricted to the above defined space is shown to be positive definite modulo the translation modes. By using this positivity estimate and the invariance of $\langle L_c\cdot, \cdot\rangle$ under e^{tJL_c}, the solutions on this subspace are shown to have at most polynomial growth. Therefore it serves as the invariant center subspace of the linearized flow and the exponential trichotomy of the linearized flow between the stable, unstable, and the center subspaces is established. Consequently, the existence of unstable (stable) manifolds follows from the standard invariant manifold theory for semilinear equations (for example [7,22]). In a future work we will construct center manifolds near the orbital neighborhood of the unstable traveling waves in the energy space X_0. The positivity of L_c on the center space (modulo the translation modes) then implies the orbital stability and local uniqueness of the center manifold.

The above study of stability of traveling waves can be generalized to the nonlinear Schrödinger equation with general nonlinear terms or in higher dimensions. Consider

$$i\frac{\partial u}{\partial t} + \Delta u + F(|u|^2)u = 0, \tag{4}$$

where $(t,x) \in \mathbf{R}\times\mathbf{R}^n$ $\left(n \geqq 3\right)$ and u satisfies the boundary condition $|u| \to 1$ as $|x| \to \infty$. Assume that the nonlinear term $F(u)$ satisfies the assumptions (F1)–(F2) or (F1)–(F2′) in Section 2.5 for $n = 3$ and in Section 6 for $n \geqq 4$. These include the 4D (GP) and 3D cubic-quintic equations, which have the critical nonlinearity. Then the sharp linear instability criterion $\frac{d}{dc}P(U_c) < 0$ can be proved in the same way (see Theorems 2.2, 3.1 and Corollary 2.2), by studying the quadratic form $\langle L_c\cdot, \cdot\rangle$ in the same space $X_1 = H^1(\mathbf{R}^n) \times \dot{H}^1(\mathbf{R}^n)$ for $n \geqq 4$. The unstable (stable) manifolds can then be constructed near unstable traveling waves by using the Equation (4). To prove orbital stability when $\frac{d}{dc}P(U_c) > 0$ for dimensions $n \geqq 4$, a coordinate mapping ψ relating X_1 and the energy space X_0 is required. For $n = 4$, such a mapping is simply given by $\psi(w) = 1 + w,\ w \in X_1$ and the global existence of 4D (GP) was recently shown in [41]. We refer to Section 6 for more details on the extensions.

The above approach does not work for dimensions $n = 1, 2$. First, the quadratic form $\langle L_c\cdot, \cdot\rangle$ is not well-defined in the space $X_1 = H^1(\mathbf{R}^n)\times\dot{H}^1(\mathbf{R}^n)$ for $n = 1, 2$. Second, the energy space X_0 cannot be written as a metric space homeomorphism to X_1, due to the oscillations of functions in X_0 at infinity (see [30]). However, when the traveling wave U_c has no vortices, that is, $U_c \neq 0$, we can study the linear instability of U_c by the following hydrodynamic formulation. By the Madelung transformation $u = \sqrt{\rho}e^{i\theta}$, the Equation (4) becomes

$$\begin{cases} \theta_t + |\nabla\theta|^2 - \frac{1}{2}\frac{1}{\rho}\Delta\rho + \frac{1}{4}\frac{1}{\rho^2}|\nabla\rho|^2 - F(\rho) = 0 \\ \rho_t + 2\nabla.(\rho\nabla\theta) = 0. \end{cases} \tag{5}$$

Define the velocity $\vec{v} = \nabla\theta$. Then the first equation of (5) is the Bernoulli equation for the vector potential θ and the second equation is the continuity equation for the density ρ. Define the energy functional

$$E(\rho,\theta) = \frac{1}{2}\int_{\mathbf{R}^n}\left(\frac{|\nabla\rho|^2}{4\rho} + \rho|\nabla\theta|^2 + V(\rho)\right)\mathrm{d}x. \tag{6}$$

The Equation (5) is also formally Hamiltonian as

$$\partial_t\begin{pmatrix}\rho\\ \theta\end{pmatrix} = JE'(\rho,\theta).$$

Linearizing above equation at the traveling wave (ρ_c, θ_c), we get

$$\partial_t\begin{pmatrix}\rho\\ \theta\end{pmatrix} = JM_c\begin{pmatrix}\rho\\ \theta\end{pmatrix}, \tag{7}$$

where J is as before and M_c is defined in (90). We show that for any dimension $n \geqq 1$, the quadratic form $\langle M_c\cdot,\cdot\rangle$ has the right spectral structure for $(\rho,\theta) \in H^1(\mathbf{R}^n) \times \dot{H}^1(\mathbf{R}^n)$, that is, it is positive definite modulo with a one-dimensional negative mode and translation modes. By the same proof as in the 3D (GP) case, the linear instability criterion $\frac{\mathrm{d}}{\mathrm{d}c}P(U_c) < 0$ is obtained under the non-degeneracy assumption (see Proposition 5.3). As an example, we consider the cubic-quintic equation with $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, where $\alpha_1, \alpha_3, \alpha_5$ are positive constants satisfying

$$\frac{3}{16} < \alpha_1\alpha_5/\alpha_3^2 < \frac{1}{4}. \tag{8}$$

This equation has many interpretations in physics. For example, in the context of a Boson gas, it describes two-body attractive and three-body repulsive interactions [4,5]. Different from the (GP) equation, the cubic-quintic type equations have unstable stationary solutions for any dimension $n \geqq 1$ [4,5,24]. First, by using the hydrodynamic formulation we show the existence of traveling waves for cubic-quintic type equations with small traveling speeds (Theorem 5.2) in any dimension $n \geqq 2$. This gives a simplified proof of the previous results on the existence of slow traveling waves in the work of Maris [48] for $n \geqq 4$ and in an unpublished manuscript of Lin [45] for $n = 2, 3$. Moreover, our proof implies the local uniqueness and differentiability of the traveling wave branch. For $n = 2$, we are also able to show that the non-degeneracy condition (56) is satisfied for these slow traveling waves (see Appendix 2 and Proposition 5.2). Then, we show that the slow traveling waves are linearly unstable (Theorem 5.4). This follows from the computation of the sign of $\frac{\mathrm{d}P(U_c)}{\mathrm{d}c}|_{c=0}$ for stationary solutions. To construct unstable (stable) manifolds, it is not convenient to use the hydrodynamic formulation (5) which has the loss of derivative in the nonlinear terms. Our strategy is to construct unstable

(stable) manifolds by the original Equation (4), based on the linear exponential dichotomy in $(H^k(\mathbf{R}^n))^2$ which is first obtained in the (ρ, θ) coordinates. This is possible due to the observation that the unstable (stable) eigenfunctions do have the L^2 estimate for θ.

Lastly, we show that any 2D traveling wave of (GP) is transversely unstable. In Theorems 4.1 and 4.2, we find the sharp range of transverse wave numbers for linear instability, and construct unstable and stable manifolds under 3D perturbations. For the proof, we observe that the linearized problem with transversal wave number k is reduced to the study of the spectrum of the operator $J\left(L_c + k^2\right)$, where L_c is defined by (44) for 2D traveling waves. For $k > 0$, the spectrum of $L_c + k^2$ has the gap structure in the usual space $(H^m(\mathbf{R}^2))^2$ and thus the proof of linear instability follows by that of Proposition 3.3 in a much simpler version. In the physical literature [11,43], the transversal instability of 2D traveling waves of (GP) was studied by the asymptotic expansions and numerics, in the long wavelength (or small wave number) limit.

This paper is organized as follows. In Section 2, we study the spectral structures of the second variation of energy-momentum functional and then prove the orbital stability on the lower branch of 3D traveling waves of (GP). In Section 3, we prove linear instability of 3D traveling waves on the upper branch and then construct unstable (stable) manifolds. Section 4 is to show the transversal instability of 2D traveling waves of (GP). In Section 5, we construct slow traveling waves of cubic-quintic type equations and then prove their instability. Section 6 extends the main results to other dimensions and more general nonlinear terms. In the appendix, we give the proof of several technical lemmas.

We list some notations and function spaces used in the paper. For any integer $k \geqq 1, n \geqq 1$, denote the space

$$\dot{H}^k\left(\mathbf{R}^n\right) = \left\{u \mid \nabla u, \ldots, \nabla^k u \in L^2\left(\mathbf{R}^n\right)\right\},$$

with the norm $\|u\|_{\dot{H}^k} = \sum_{j=1}^k \|\nabla^j u\|_{L^2}$. For $n \geqq 3$, by Sobolev embedding we can impose the condition $u \in L^{\frac{2n}{n-2}}(\mathbf{R}^n)$ when $u \in \dot{H}^1(\mathbf{R}^n)$. Let $H^k_{\mathbf{R}}(\mathbf{R}^n)$ ($\dot{H}^k_{\mathbf{R}}(\mathbf{R}^n)$) be all the real valued functions in $H^k(\mathbf{R}^n)$ ($\dot{H}^k(\mathbf{R}^n)$). Let $X_k(\mathbf{R}^n) = H^k_{\mathbf{R}}(\mathbf{R}^n) \times \dot{H}^k_{\mathbf{R}}(\mathbf{R}^n)$ be equipped with the norm

$$\|w\|_{X_k} = \|w_1\|_{H^k} + \|w_2\|_{\dot{H}^k}, \quad w = (w_1, w_2) \in X_k.$$

So as to avoid confusion, we write $X_k\left(\mathbf{R}^n\right)$, $H^k_{\mathbf{R}}(\mathbf{R}^n)$ ($\dot{H}^k_{\mathbf{R}}(\mathbf{R}^n)$) simply as X_k, H^k $\left(\dot{H}^k\right)$. For $n \geqq 2$, denote $L^2_{r_\perp}$, $\dot{H}^{-1}_{r_\perp}$ and $H^k_{r_\perp}$ $\left(\dot{H}^k_{r_\perp}\right)$ to be the cylindrically symmetric subspaces of L^2, $\dot{H}^{-1}$ $\left(\text{the dual of } \dot{H}^1\right)$ and H^k $\left(\dot{H}^k\right)$. A function u is cylindrically symmetric if $u = u\left(x_1, r_\perp\right)$ with $x_\perp = \left(x_2, \ldots, x_n\right)$, $r_\perp = |x_\perp|$. Denote X^s_k to be the cylindrically symmetric subspaces of X_k, that is, $X^s_k = H^k_{r_\perp} \times \dot{H}^k_{r_\perp}$.

2. Orbital Stability of Lower Branch Traveling Waves

In this section, we prove nonlinear orbital stability for 3D traveling waves obtained via a constrained variational approach on the lower branch with $\frac{\mathrm{d}}{\mathrm{d}c} P\left(U_c\right) >$

0. The proof is to expand the energy-momentum functional near the traveling wave and show that the second variation is positive definite and dominant. A corollary of this proof is that the stable traveling waves are local energy minimizers with fixed momentum. We give the detailed proof for (GP) and then discuss briefly the extensions for general nonlinearity.

The energy functional of (GP)

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3}\left[|\nabla u|^2 + \frac{1}{2}\left(|u|^2-1\right)^2\right]\mathrm{d}x$$

is defined on the energy space

$$X_0 = \left\{u \in H^1_{\mathrm{loc}}(\mathbf{R}^3;\mathbf{C}) \mid \nabla u \in L^2(\mathbf{R}^3),\, |u|^2-1 \in L^2(\mathbf{R}^3)\right\}. \tag{9}$$

By [30], for any $u \in X_0$, we can write $u = c(1+v)$ where $c \in \mathbb{S}^1$ and $v \in \dot{H}^1\left(\mathbf{R}^3\right)$. Given $u = c(1+v)$ and $\tilde{u} = \tilde{c}(1+\tilde{v})$, we define the natural distance in X_0 by

$$d_1(u,\tilde{u}) = |c-\tilde{c}| + \|\nabla v - \nabla\tilde{v}\|_{L^2} + \left\||v|^2 + 2\,\mathrm{Re}\, v - |\tilde{v}|^2 - 2\,\mathrm{Re}\,\tilde{v}\right\|_{L^2}. \tag{10}$$

The global well-posedness of (GP) equation on X_0 was proved in [29]. Moreover, if $u(t)$ is the solution of (GP) with $u(0) = c(1+v_0)$ where $c \in \mathbb{S}^1$ and $v_0 \in \dot{H}^1\left(\mathbf{R}^3\right)$, then $u(t) = c(1+v(t))$ with $v(t) \in \dot{H}^1\left(\mathbf{R}^3\right)$. Thus, for stability considerations, we only need to consider $c = 1$, which will be assumed for the rest of this paper.

2.1. Momentum

Besides the energy, another invariant of (GP) is the momentum which is due to the translation invariance in x_1 of the equation. For

$$u = u_1 + iu_2 \in 1 + H^1(\mathbf{R}^3) \subset X_0$$

the momentum is defined by (2). However, that form of momentum is not defined for an arbitrary function u in the energy space X_0. So, first we need to extend the definition of P to all functions in X_0. For this and the proof of main Theorem, we use the following manifold structure of X_0 given in [30]. Let $\chi \in C_0^\infty(\mathbf{R}^3, [0,1])$ be a real valued and radial function such that $\chi(\xi) = 1$ near $\xi = 0$, and consider the Fourier multiplier $\chi(D)$ defined on $\mathcal{S}'(\mathbf{R}^3)$ by

$$(\widehat{\chi(D)u})(\xi) = \chi(\xi)\hat{u}(\xi).$$

Define

$$\psi(w) = 1 + w_1 - \chi(D)\left(\frac{w_2^2}{2}\right) + iw_2, \quad \text{for } w = (w_1, w_2) \in X_1. \tag{11}$$

By Proposition 1.3 in [30], the mapping $w \to \psi(w)$ is locally bi-Lipschitz between X_1 and (X_0, d_1). So the space X_0 can be considered as a manifold over the coordinate space X_1. For any $u = \psi(w) \in X_0$ with $w \in X_1$, we define the momentum by

$$\tilde{P}(u) = -\int_{\mathbf{R}^3} \left[w_1 + (1 - \chi(D)) \left(\frac{w_2^2}{2} \right) \right] \partial_{x_1} w_2 \, \mathrm{d}x. \tag{12}$$

By Proposition 1.3 of [30], we have

$$\|\chi(D) f\|_{L^p \cap L^\infty} \leqq C \|f\|_{L^p}, \quad \forall f \in L^p, 1 \leqq p \leqq \infty, \tag{13}$$

$$\|(1 - \chi(D))(fg)\|_{L^2} \leqq C \|\nabla f\|_{L^2} \|\nabla g\|_{L^2}, \quad \forall f, g \in \dot{H}^1(\mathbf{R}^3). \tag{14}$$

So the right hand side of (12) is well-defined. First, we show that when $u \in 1 + H^1\left(\mathbf{R}^3\right)$, $\tilde{P}(u) = P(u)$, that is, $\tilde{P}$ is an extension of P. Indeed, when $u \in 1 + H^1\left(\mathbf{R}^3\right)$, or equivalently, $u = \psi(w)$ with $w \in H^1\left(\mathbf{R}^3\right)$, we have

$$\begin{aligned} P(u) &= \frac{1}{2} \int_{\mathbf{R}^3} \langle i \partial_{x_1} \psi(w), \psi(w) - 1 \rangle \, \mathrm{d}x \\ &= -\int_{\mathbf{R}^3} \left(w_1 - \chi(D) \left(\frac{w_2^2}{2} \right) \right) \partial_{x_1} w_2 \, \mathrm{d}x \\ &= -\int_{\mathbf{R}^3} \left[w_1 + (1 - \chi(D)) \left(\frac{w_2^2}{2} \right) \right] \partial_{x_1} w_2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbf{R}^3} w_2^2 \partial_{x_1} w_2 \, \mathrm{d}x. \\ &= \tilde{P}(u) + \frac{1}{6} \int_{\mathbf{R}^3} \partial_{x_1} w_2^3 \, \mathrm{d}x. \end{aligned}$$

Since $w_2^3 \in L^2$ and $\partial_{x_1} w_2^3 = 3 w_2^2 \partial_{x_1} w_2 \in L^1$, thus $\tilde{P}(u) = P(u)$ by the following lemma.

Lemma 2.1. *Let* $\mathcal{X} = \{\partial_{x_1} \phi \mid \phi \in L^2(\mathbf{R}^3)\}$. *If* $v \in L^1(\mathbf{R}^3) \cap \mathcal{X}$, *then* $\int_{\mathbf{R}^3} v(x) \, \mathrm{d}x = 0$.

Proof. The proof is similar to that of Lemma 2.3 in [49], so we skip it. □

We collect the main properties of $\tilde{P}(u)$.

Lemma 2.2. (i) *The functional* $\bar{P}(w) := \tilde{P} \circ \psi(w)$ *is* C^∞ *for* $w \in X_1$.
(ii) $\tilde{P}(u)$ *is the unique continuous extension of* $P(u)$ *from* $1 + H^1\left(\mathbf{R}^3\right)$ *to* (X_0, d_1).
(iii) *When* $u(t)$ *is the solution of* (GP) *with* $u(0) \in X_0$, $\tilde{P}(u(t)) = \tilde{P}(u(0))$.

Proof. (i) Since

$$\begin{aligned} \tilde{P} \circ \psi(w) &= -\int_{\mathbf{R}^3} w_1 \partial_{x_1} w_2 \, \mathrm{d}x - \frac{1}{2} \int_{\mathbf{R}^3} (1 - \chi(D)) w_2^2 \partial_{x_1} w_2 \, \mathrm{d}x \\ &= B_1(w_1, w_2) + B_2(w_2, w_2, w_2), \end{aligned}$$

where

$$B_1(w_1, w_2) = -\int_{\mathbf{R}^3} w_1 \partial_{x_1} w_2 \, \mathrm{d}x : H^1 \times \dot{H}^1 \to R$$

and

$$B_2(w_1, w_2, w_3) = -\frac{1}{2}\int_{\mathbf{R}^3}(1-\chi(D))(w_1w_2)\,\partial_{x_1}w_3\,dx : (\dot{H}^1)^3 \to R$$

are multi-linear and bounded, so it follows that $\tilde{P}\circ\psi(w)$ is C^∞ on X_1.

(ii) follows from (i), the bicontinuity of $\psi : \left(X_1, \|\|_{X_1}\right) \to (X_0, d_1)$, and the density of $1 + H^1\left(\mathbf{R}^3\right)$ in (X_0, d_1).

(iii): When $u(0) \in 1 + H^1\left(\mathbf{R}^3\right)$, we have $u(t) \in 1 + H^1\left(\mathbf{R}^3\right)$. The global existence in this case was first proved in [13]. It is straightforward to show that $\tilde{P}(u(t)) = P(u(t))$ is invariant in time by using the translation invariance of (GP). For general $u(0) \in X_0$, we can choose a sequence $\{u_n(0)\} \subset 1 + H^1\left(R^3\right)$ such that $\|u_n(0) - u(0)\|_{d_1} \to 0$ when $n \to \infty$. Then for any $t \in \mathbf{R}$, $P(u_n(t)) = P(u_n(0))$, letting $n \to \infty$, we get $\tilde{P}(u(t)) = \tilde{P}(u(0))$ due to the continuous dependence of solutions to (GP) on the initial with respect to the distance d_1 (see [29]). □

Remark 2.1. In [49], the momentum was extended from $1+H^1\left(\mathbf{R}^3\right)$ to the energy space X_0 in a different way and it was shown that such extended momentum is continuous on (X_0, d_1). So by Lemma 2.2(ii), the extended momentum in [49] gives the same functional as $\tilde{P}(u)$, but the form of $\tilde{P}(u)$ given in (12) is more explicit.

2.2. The Energy-Momentum Functional

First, we show that the functional $E\circ\psi : X_1 \to \mathbf{R}$ is smooth.

Lemma 2.3. *The functional*

$$\bar{E}(w) := E\circ\psi(w) = \frac{1}{2}\int_{\mathbf{R}^3}\left[|\nabla\psi(w)|^2 + \frac{1}{2}(|\psi(w)|^2-1)^2\right]dx \tag{15}$$

is C^∞ on X_1.

Proof. For $w = (w_1, w_2) \in X_1$, that is, $w_1 \in H^1$, $w_2 \in \dot{H}^1$,

$$\nabla\psi(w) = \nabla w_1 - \chi(D)(w_2\nabla w_2) + i\nabla w_2,$$
$$|\psi(w)|^2 - 1 = \left(w_1 - \chi(D)\left(\frac{w_2^2}{2}\right)\right)^2 + 2w_1 + (1-\chi(D))(w_2^2),$$

and by (13) and (14),

$$\nabla w_1,\ \chi(D)(w_2\nabla w_2),\ \nabla w_2,\ (1-\chi(D))(w_2^2) \in L^2$$
$$w_1, \chi(D)\left(\frac{w_2^2}{2}\right) \in L^4.$$

We can write the right hand side of (15) as a sum of multilinear forms, as in the proof of Lemma 2.2. The C^∞ property of $E\circ\psi(w)$ thus follows. □

Define the energy-momentum functional $E_c(u) = E(u) + c\tilde{P}(u)$ on X_0 and

$$\bar{E}_c(w) = E_c \circ \psi(w) = \bar{E}(w) + c\bar{P}(w), \quad w \in X_1 \tag{16}$$

which is a smooth functional on space X_1. Let $U_c = u_c + iv_c$ be a finite energy traveling wave solution of (GP) equation, that is, (u_c, v_c) satisfies

$$\begin{cases} \Delta u_c + c\partial_{x_1} v_c = -\left(1 - |U_c|^2\right) u_c, \\ \Delta v_c - c\partial_{x_1} u_c = -\left(1 - |U_c|^2\right) v_c. \end{cases} \tag{17}$$

Lemma 2.4. *Let $w_c = (w_{1c}, w_{2c}) \in X_1$ be such that $\psi(w_c) = U_c = u_c + iv_c$. Then U_c solves the traveling wave equation if and only if $\bar{E}_c'(w_c) = 0$.*

Proof. Since $\bar{E}_c'(w_c) \in X_1^*$ and the Schwartz class is dense in X_1, we have $\bar{E}_c'(w_c) = 0$ if and only if $\langle \bar{E}_c{}'(w_c), \phi\rangle = 0$ for all ϕ in Schwartz class. One may compute by integration by parts that, $\bar{E}_c'(w_c)$ satisfies, for any $\phi = (\phi_1, \phi_2)$ in Schwartz class,

$$\begin{aligned} \langle \bar{E}_c{}'(w_c), \phi\rangle = &\int_{\mathbf{R}^3} [-\Delta u_c - (1 - |U_c|^2)u_c - c\partial_{x_1} v_c](\phi_1 - \chi(D)(v_c\phi_2))\mathrm{d}x \\ &+ \int_{\mathbf{R}^3} [-\Delta v_c - (1 - |U_c|^2)v_c + c\partial_{x_1} u_c]\phi_2 \,\mathrm{d}x. \end{aligned} \tag{18}$$

Therefore, it is clear that $\bar{E}_c'(w_c) = 0$ if and only if (17) holds. □

We now compute the second variation of $\bar{E}_c(\psi)$. By straightforward computations using the criticality of w_c, we have, for any ϕ in Schwartz class,

$$\begin{aligned} &\langle \bar{E}_c{}''(w_c)\phi, \phi\rangle := q_c(\phi) \\ &= \int_{\mathbf{R}^3} [|\nabla(\phi_1 - \chi(D)(v_c\phi_2))|^2 + |\nabla\phi_2|^2 \\ &\quad + (3u_c^2 + v_c^2 - 1)|\phi_1 - \chi(D)(v_c\phi_2)|^2 + (u_c^2 + 3v_c^2 - 1)|\phi_2|^2 \\ &\quad + 4u_cv_c(\phi_1 - \chi(D)(v_c\phi_2))\phi_2 - 2c(\phi_1 - \chi(D)(v_c\phi_2))\partial_{x_1}\phi_2]\,\mathrm{d}x. \end{aligned} \tag{19}$$

Since the functional $\bar{E}_c(w)$ is smooth on X_1, its second variation at w_c which is given by the quadratic form q_c of (19) is well-defined and bounded on X_1.

Define the operator

$$L_c := \begin{pmatrix} -\Delta - 1 + 3u_c^2 + v_c^2 & -c\partial_{x_1} + 2u_cv_c \\ c\partial_{x_1} + 2u_cv_c & -\Delta - 1 + u_c^2 + 3v_c^2 \end{pmatrix}, \tag{20}$$

then formally we can write

$$q_c(\phi) = \left\langle L_c \begin{pmatrix} \phi_1 - \chi(D)(v_c\phi_2) \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 - \chi(D)(v_c\phi_2) \\ \phi_2 \end{pmatrix} \right\rangle. \tag{21}$$

Here we use $\langle\cdot,\cdot\rangle$ for the dual product of $X_1 = H^1 \times \dot{H}^1$ and its dual $X_1^* = H^{-1} \times \dot{H}^{-1}$, and $(\cdot,\cdot)$ is used for the inner product in X_1. By [14], the traveling wave solutions (u_c, v_c) of (GP) equation satisfy: $u_c - 1, v_c \in H^k$ for any $k \geqq 0$, and $u_c - 1 = O\left(\frac{1}{|x|^3}\right)$, $v_c = O\left(\frac{1}{|x|^2}\right)$ for $|x| \to \infty$. Since $\phi_2 \in \dot{H}^1$ implies that $\chi(D)(v_c\phi_2) \in L^2$, the mapping $K : X_1 \to X_1$ defined by

$$K\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1 - \chi(D)(v_c\phi_2)\\ \phi_2\end{pmatrix} \tag{22}$$

is an isomorphism on X_1. To study the quadratic form $q_c(\phi)$ on X_1, it is equivalent to study the quadratic form

$$\tilde{q}_c(\phi) = \langle L_c(\phi_1, \phi_2)^{\mathrm{T}}, (\phi_1, \phi_2)^{\mathrm{T}}\rangle$$

on X_1. To simplify notations, we write $\langle L_c(\phi_1, \phi_2)^{\mathrm{T}}, (\phi_1, \phi_2)^{\mathrm{T}}\rangle$ as $\langle L_c\phi, \phi\rangle$. The quadratic form $\langle L_c\phi, \phi\rangle$ is well defined and bounded on X_1, by the boundedness of $q_c(\phi)$ and the isomorphism of K on X_1. This can also be seen directly by using the Hardy inequality

$$\left\|\frac{u}{|x|}\right\|_{L^2(\mathbf{R}^N)} \leqq \frac{2}{N-2}\|\nabla u\|_{L^2(\mathbf{R}^N)}, \quad \text{for any } N \geqq 3. \tag{23}$$

Since $|u_c^2 + 3v_c^2 - 1|,\ |v_c| \leqq \frac{C}{|x|^2}$, we have

$$\left|\int_{\mathbf{R}^3}(u_c^2 + 3v_c^2 - 1)\phi_2^2\,\mathrm{d}x\right| \leqq C\int_{\mathbf{R}^3}\frac{\phi_2^2}{|x|^2}\,\mathrm{d}x \leqq C\int_{\mathbf{R}^3}|\nabla\phi_2|^2\,\mathrm{d}x$$

and

$$\begin{aligned}\left|\int_{\mathbf{R}^3}u_c v_c\,\phi_1\phi_2\,\mathrm{d}x\right| &\leqq C\,\|\phi_1\|_{L^2}\left\|\frac{\phi_2}{|x|}\right\|_{L^2}\\ &\leqq C\left(\|\phi_1\|_{L^2}^2 + \|\nabla\phi_2\|_{L^2}^2\right).\end{aligned}$$

Remark 2.2. The quadratic form $q_c(\phi) = \langle \bar{E}_c{}''(w_c)\phi, \phi\rangle$ given in (19) and (21) can be seen in the following way. Suppose $w \in H^1$, then $u = \psi(w) \in 1 + H^1$ and

$$\bar{E}_c(w) := E_c \circ \psi(w) = E(u) + cP(u).$$

If the first order variation of w at w_c is $\delta w = \phi$, then $\delta u = K\phi$ and $\delta^2 u = -\chi(D)(\phi_2^2)$. So

$$\begin{aligned}\left\langle \bar{E}_c''(w_c)\phi, \phi\right\rangle &= \left\langle E_c''(U_c)\delta u, \delta u\right\rangle + \left\langle E_c'(U_c), \delta^2 u\right\rangle\\ &= \langle L_c(K\phi), K\phi\rangle,\end{aligned}$$

since $E_c''(U_c) = L_c$ and $E_c'(U_c) = 0$ by the Equation (17).

2.3. Spectral Properties of Second Order Variation

Differentiating the traveling wave equation (17) in x_i, we get $L_c \partial_{x_i} U_c = 0$. We assume that these are all the kernels of L_c, that is,

$$\ker L_c = Z = \text{span}\{\partial_{x_i} U_c, i = 1, 2, 3\}. \tag{24}$$

Remark 2.3. The non-degeneracy condition (24) for $c = c_0$ implies that the traveling wave U_{c_0} is locally unique. More precisely, there exists a unique C^1 curve of traveling waves passing through (c_0, U_{c_0}). See Theorem 5.3 for the proof.

In [49], traveling wave solutions to (GP) were found by minimizing $\bar{E}_c$ subject to a Pohozaev type constraint. Our main result of this section is to give a spectral decomposition of the quadratic form $\tilde{q}_c(\phi)$ which is the quadratic part of E_c at U_c.

Proposition 2.1. *For $0 < c < \sqrt{2}$, let U_c be a traveling wave solution of (GP) constructed in [49] and L_c be the operator defined by (20). Assume the non-degeneracy condition (24). The space X_1 is decomposed as a direct sum*

$$X_1 = N \oplus Z \oplus P,$$

where Z is defined in (24), N is one-dimensional and such that $\tilde{q}_c(u) = \langle L_c u, u\rangle < 0$ for $0 \neq u \in N$, and P is a closed subspace such that

$$\tilde{q}_c(u) \geqq \delta \|u\|_{X_1}^2, \quad \forall u \in P,$$

for some constant $\delta > 0$.

Proof. Define the isomorphism $G : L^2 \to X_1$ by

$$G\varphi = (-\Delta + 1)^{-\frac{1}{2}}\varphi_1 + i(-\Delta)^{-\frac{1}{2}}\varphi_2, \tag{25}$$

for $\varphi = \varphi_1 + i\varphi_2 \in L_2$. Let $\tilde{L}_c := \tilde{G} \circ L_c \circ \tilde{G}$ with

$$\tilde{G} = \begin{pmatrix} (-\Delta + 1)^{-\frac{1}{2}} & 0 \\ 0 & (-\Delta)^{-\frac{1}{2}} \end{pmatrix}, \tag{26}$$

and define the quadratic form on L_2 by

$$p_c(\varphi) = \tilde{q}_c(G\varphi) = \langle \tilde{L}_c (\varphi_1, \varphi_2)^{\mathrm{T}}, (\varphi_1, \varphi_2)^{\mathrm{T}}\rangle := \langle \tilde{L}_c\varphi, \varphi\rangle. \tag{27}$$

Then

$$\frac{\tilde{q}_c(\varphi)}{\|\varphi\|_{X_1}^2} = \frac{p_c(G^{-1}\varphi)}{\|G^{-1}\varphi\|_{L^2}^2}, \quad \text{for any } \varphi \in X_1,$$

and it is equivalent to prove the conclusions of proposition for the quadratic form $p_c(\varphi)$ on L^2. Since $u_c \to 1$, $v_c \to 0$ as $|x| \to \infty$, let

$$L_{c,\infty} := \begin{pmatrix} -\Delta + 2 & -c\partial_{x_1} \\ c\partial_{x_1} & -\Delta \end{pmatrix} \tag{28}$$

and $q_{c,\infty}(\phi) = \langle L_{c,\infty}\phi, \phi\rangle$. Correspondingly, let

$$\tilde{L}_{c,\infty} := \tilde{G} \circ L_{c,\infty} \circ \tilde{G}$$

and $p_{c,\infty}(\varphi) = \langle \tilde{L}_{c,\infty}\varphi, \varphi\rangle$. The properties of the quadratic form $p_c(\varphi)$ on L^2 follow from the spectral properties of the operator $\tilde{L}_c$. We claim that:

(i) $\tilde{L}_c : L^2 \to L^2$ is self-adjoint and bounded.
(ii) $\tilde{L}_c$ has one-dimensional negative eigenspace,

$$\ker \tilde{L}_c = \{G^{-1}\partial_{x_i} U_c, i = 1, 2, 3\},$$

and the rest of the spectrum are uniformly positive.

The proof of these claims will be split into a few lemmas to be proved later and we outline the rest of the proof of the Proposition based on these lemmas.

To prove claim (i), first we note that the constant coefficient operator $\tilde{L}_{c,\infty} : L^2 \to L^2$ is self-adjoint and bounded. We shall show that the operator $\tilde{L}_c - \tilde{L}_{c,\infty}$ is compact on L^2. Indeed,

$$\begin{aligned} &\tilde{L}_c - \tilde{L}_{c,\infty} \\ &= \begin{pmatrix} (-\Delta+1)^{-\frac{1}{2}} \left(3u_c^2 - 3 + v_c^2\right) (-\Delta+1)^{-\frac{1}{2}} & 2(-\Delta+1)^{-\frac{1}{2}} u_c v_c (-\Delta)^{-\frac{1}{2}} \\ 2(-\Delta)^{-\frac{1}{2}} u_c v_c (-\Delta+1)^{-\frac{1}{2}} & (-\Delta)^{-\frac{1}{2}} \left(u_c^2 - 1 + 3v_c^2\right) (-\Delta)^{-\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}. \end{aligned} \tag{29}$$

By Lemma 2.5, the operators L_{ij} $(i, j = 1, 2)$ are all compact on $L^2\left(\mathbf{R}^3\right)$. Moreover, L_{11}, L_{22} are symmetric and $L_{21} = L_{12}^*$, thus $\tilde{L}_c - \tilde{L}_{c,\infty}$ is bounded and self-adjoint and (i) is proved.

To prove claim (ii), first we note that by Lemma 2.9, there exists $\delta_0 > 0$, such that

$$\left\langle \tilde{L}_{c,\infty}\varphi, \varphi \right\rangle \geqq \delta_0 \left\|\varphi\right\|_{L^2}^2 .$$

Thus $\sigma_{\text{ess}}(\tilde{L}_{c,\infty}) \subset [\delta_0, +\infty)$. By Weyl's theorem and the compactness of $\tilde{L}_c - \tilde{L}_{c,\infty}$, we have $\sigma_{\text{ess}}(\tilde{L}_c) = \sigma_{\text{ess}}(\tilde{L}_{c,\infty}) \subset [\delta_0, +\infty)$. This shows that the negative eigenspace of $\tilde{L}_c$ is finite-dimensional. By assumption (24), $\ker \tilde{L}_c = \{G\partial_{x_i} U_c, i = 1, 2, 3\}$. By Lemmas 2.7 and 2.8, the negative eigenspace of $\tilde{L}_c$ is one-dimensional. This proves claim (ii) and finishes the proof of the proposition. □

From the above proposition and the relation

$$\langle \bar{E}_c{}''(w_c)\phi, \phi\rangle = \langle L_c\left(K\phi\right), K\phi\rangle , \tag{30}$$

where K is defined by (22), we immediately get the following.

Corollary 2.1. *Under the conditions of Proposition 2.1, the space X_1 is decomposed as a direct sum*

$$X_1 = N' \oplus Z' \oplus P',$$

where $Z' = \{\partial_{x_i} w_c, i = 1, 2, 3\}$, N' *is a one-dimensional subspace such that* $q_c(u) = \langle \bar{E}_c{}''(w_c)u, u\rangle < 0$ *for* $0 \neq u \in N'$, *and* P' *is a closed subspace such that*

$$q_c(u) \geqq \delta \|u\|_{X_1}^2, \quad \forall u \in P'$$

for some constant $\delta > 0$.

Proof. We define $N' = K^{-1}N$, $Z' = K^{-1}Z$ and $P' = K^{-1}P$, where N, Z, P are defined in Proposition 2.1. Then the conclusion follows by (30). In particular, $Z' = K^{-1}Z = \left\{\partial_{x_i} w_c, i = 1, 2, 3\right\}$ since

$$\partial_{x_i} U_c = \partial_{x_i} \psi(w_c) = K \partial_{x_i} w_c.$$

□

Now we prove several lemmas used in the proof of Proposition 2.1. We use C for a generic constant in the estimates.

Lemma 2.5. *The operators* L_{ij} $(i, j = 1, 2)$ *defined in* (29) *are compact on* $L^2(\mathbf{R}^3)$.

Proof. Since $V_1(x) = 3u_c^2 - 3 + v_c^2 \to 0$ when $|x| \to \infty$, and the operator $(-\Delta+1)^{-\frac{1}{2}} : L^2 \to H^1$ is bounded, thus $\left(3u_c^2 - 3 + v_c^2\right)(-\Delta+1)^{-\frac{1}{2}}$ is compact on L^2 by the local compactness of $H^1 \to L^2$. So L_{11} is compact on L^2.

Take a sequence $\{v_k\} \subset L^2(\mathbf{R}^3)$ and $v_k \to v_\infty$ weakly in L^2. To show an operator T is compact on L^2, it suffices to prove that $Tv_k \to Tv_\infty$ strongly in L^2. By Hardy's inequality in the Fourier space,

$$\begin{aligned}
\|L_{21}(v_k - v_\infty)\|_{L^2} &= \left\| \frac{1}{|\xi|} \left(2u_c v_c(-\Delta+1)^{-\frac{1}{2}}(v_k - v_\infty)\right)^{\wedge}(\xi) \right\|_{L^2} \\
&\leqq C \left\| \nabla_\xi \left(2u_c v_c(-\Delta+1)^{-\frac{1}{2}}(v_k - v_\infty)\right)^{\wedge}(\xi) \right\|_{L^2} \\
&\leqq C \left\| |x| v_c(-\Delta+1)^{-\frac{1}{2}}(v_k - v_\infty) \right\|_{L^2} \to 0,
\end{aligned}$$

since the operator $|x| v_c(-\Delta+1)^{-\frac{1}{2}}$ is compact by using $|x| v_c = O\left(\frac{1}{|x|}\right)$. Then $L_{12} = L_{21}^*$ is also compact.

To show the compactness of L_{22}, first note that $V_2(x) = \left(u_c^2 - 1 + 3v_c^2\right) = O\left(\frac{1}{|x|^3}\right)$. Let $\chi \in C_0^\infty(\mathbf{R}^3, [0, 1])$ be a radial cut-off function such that $\chi(\xi) = 1$ when $|\xi| \leqq \frac{1}{2}$ and $\chi(\xi) = 0$ when $|\xi| \geqq 1$. For any $R > 0$, let $\chi_R = \chi\left(\frac{x}{R}\right)$. Denote $u_k = (-\Delta)^{-\frac{1}{2}}(v_k - v_\infty)$, then $\|u_k\|_{\dot{H}^1} \leqq \|v_k - v_\infty\|_{L^2} \leqq C$. Thus

$$\begin{aligned}
&\|L_{22}(v_k - v_\infty)\|_{L^2} \\
&\leqq C \, \||x| V_2(x) u_k\|_{L^2} \\
&\leqq C \left(\||x| V_2(x) \chi_R u_k\|_{L^2} + \||x| V_2(x)(1 - \chi_R) u_k\|_{L^2}\right) \\
&\leqq C \left(\||x| V_2(x) \chi_R u_k\|_{L^2} + \frac{1}{R} \left\| \frac{1}{|x|}(1 - \chi_R) u_k \right\|_{L^2}\right).
\end{aligned}$$

We have

$$\left\| \frac{1}{|x|} (1-\chi_R)\, u_k \right\|_{L^2} \leqq C\, \|\nabla [(1-\chi_R)\, u_k]\|_{L^2}$$
$$\leqq C \left(\|\nabla u_k\|_{L^2} + \left\| \frac{1}{R} \nabla \chi \left(\frac{x}{R} \right) \right\|_{L^3} \|u_k\|_{L^6} \right)$$
$$\leqq C\, \|\nabla u_k\|_{L^2},$$

so $\frac{1}{R} \left\| \frac{1}{|x|} (1-\chi_R)\, u_k \right\|_{L^2}$ can be made arbitrarily small by taking R sufficiently large. For fixed R, by the compactness of $H_0^1(\{|x| < R\}) \to L^2$, we get that

$$\| |x|\, V_2(x)\, \chi_R u_k \|_{L^2} \to 0, \quad \text{when } k \to \infty.$$

Thus $\|L_{22}(v_k - v_\infty)\|_{L^2} \to 0$ when $k \to \infty$ and this finishes the proof of the lemma. □

Define the functional $\bar{P}_c : X_1 \to \mathbf{R}$ by

$$\bar{P}_c(w) = \int_{\mathbf{R}^3} |\partial_{x_1} \psi(w)|^2 \,\mathrm{d}x + 2c\tilde{P} \circ \psi(w) + \int_{\mathbf{R}^3} \frac{1}{2}(1-|\psi(w)|^2)^2 \,\mathrm{d}x. \quad (31)$$

Lemma 2.6. *Assume that* $\varphi \in X_1$ *satisfies* $\langle \bar{P}_c{}'(w_c), \varphi \rangle = 0$, *where* $w_c = \psi^{-1}(U_c)$, *then there holds* $q_c(\varphi) = \langle \bar{E}_c{}''(w_c)\varphi, \varphi \rangle \geqq 0$.

Proof. First, we note that $\bar{P}_c{}'(w_c) \neq 0$. Indeed, suppose $\bar{P}_c{}'(w_c) = 0$. Define

$$A(u) = \int_{\mathbf{R}^3} |\partial_{x_2} u|^2 + |\partial_{x_3} u|^2 \,\mathrm{d}x.$$

Then since $\bar{E}_c{}'(w_c) = 0$ and $\bar{E}_c(w) - \frac{1}{2}\bar{P}_c(w) = A \circ \psi(w)$, we have $(A \circ \psi)'(w_c) = 0$, that is, $K^* \left(\Delta_{x_2 x_3} U_c \right) = 0$. Here,

$$K^* \phi = \begin{pmatrix} \phi_1 \\ \phi_2 - v_c\, (\chi(D)\phi_1) \end{pmatrix} \quad (32)$$

is the adjoint of K defined in (22). Thus $\Delta_{x_2 x_3} U_c = 0$ which implies that $U_c \equiv 1$, a contradiction.

Thus we can choose $\phi \in X_1$ such that $\langle \bar{P}_c{}'(w_c), \phi \rangle \neq 0$. Set $G(\sigma, s) = \bar{P}_c(w_c + \sigma\phi + s\varphi)$. Then

$$\begin{aligned} G(0,0) &= \bar{P}_c(w_c) \\ &= \int_{\mathbf{R}^3} |\partial_{x_1} U_c|^2 \,\mathrm{d}x + 2c \int_{\mathbf{R}^3} \langle i\partial_{x_1}(1-U_c), 1-U_c \rangle \,\mathrm{d}x \\ &\quad + \int_{\mathbf{R}^3} \frac{1}{2}(1-|U_c|^2)^2 \,\mathrm{d}x \\ &= 0, \end{aligned}$$

by the Pohozaev-type identity (see Proposition 4.1 of [50]). Since

$$\frac{\partial G}{\partial \sigma}(0,0) = \langle \bar{P}_c{}'(w_c), \phi \rangle \neq 0,$$

by the implicit function theorem, there exists a C^1 function $\sigma(s)$ near $s = 0$ such that $\sigma(0) = 0$ and

$$G(\sigma(s), s) = \bar{P}_c(w_c + \sigma(s)\phi + s\varphi) = 0.$$

Then from $\frac{\mathrm{d}}{\mathrm{d}s} G(\sigma(s), s)|_{s=0} = 0$, we get $\langle \bar{P}_c{}'(w_c), \sigma'(0)\phi + \varphi\rangle = 0$. Since $\langle \bar{P}_c{}'(w_c), \varphi\rangle = 0$ and $\langle \bar{P}_c{}'(w_c), \phi\rangle \neq 0$, we get $\sigma'(0) = 0$. Let $w(s) = w_c + \sigma(s)\phi + s\varphi$ and $g(s) = \bar{E}_c(w(s))$. Then we have $w(0) = w_c$, $w'(0) = \varphi$ and $\bar{P}_c(w(s)) = 0$. By the variational characterization of traveling wave solution [49], we know that $s = 0$ is a local minimum point of $g(s)$. So, we get $g''(0) \geqq 0$. This implies that $\langle \bar{E}_c{}''(w_c)\varphi, \varphi\rangle \geqq 0$. □

The above lemma implies the following:

Lemma 2.7. *For any $0 < c_0 < \sqrt{2}$, $\tilde{L}_{c_0}$ has at most one-dimensional negative eigenspace.*

Proof. We assume by contradiction that $\varphi, \tilde{\varphi} \in L^2$ are two linearly independent eigenfunctions of $\tilde{L}_{c_0}$corresponding to negative eigenvalues. Since $\tilde{L}_{c_0}$ is self-adjoint, we can assume that $\langle \tilde{L}_{c_0}\varphi, \tilde{\varphi}\rangle = 0$. Let $w_{c_0} \in X_1$ be such that $\psi(w_{c_0}) = U_{c_0}$. From the definition of $\tilde{L}_{c_0}$, for any $\phi, \tilde{\phi} \in L^2$, we have

$$\left\langle \tilde{L}_{c_0}\phi, \tilde{\phi}\right\rangle = \langle \bar{E}_{c_0}{}''(w_{c_0})K^{-1}G\phi, K^{-1}G\tilde{\phi}\rangle, \tag{33}$$

where the mappings G, K are defined in (25) and (22). Let $w = K^{-1}G\varphi$, $\tilde{w} = K^{-1}G\tilde{\varphi}$, then $\langle \bar{E}_{c_0}{}''(w_{c_0})w, \tilde{w}\rangle = 0$ and

$$\langle \bar{E}_{c_0}{}''(w_{c_0})w, w\rangle, \langle \bar{E}_{c_0}{}''(w_{c_0})\tilde{w}, \tilde{w}\rangle < 0.$$

By Lemma 2.6 we have

$$\langle (P_{c_0} \circ \psi)'(w_{c_0}), w\rangle \neq 0, \langle (P_{c_0} \circ \psi)'(w_{c_0}), \tilde{w}\rangle \neq 0.$$

Thus there exists $\alpha \neq 0$ such that

$$\langle (P_{c_0} \circ \psi)'(w_{c_0}), \xi_0\rangle = 0, \text{ for } \xi_0 = w + \alpha\tilde{w}.$$

Again by lemma 2.6, we get

$$\left\langle \bar{E}_{c_0}{}''(w_{c_0})\xi_0, \xi_0\right\rangle \geqq 0.$$

This is in contradiction to

$$\left\langle \bar{E}_{c_0}{}''(w_{c_0})\xi_0, \xi_0\right\rangle = \langle \bar{E}_{c_0}{}''(w_{c_0})w, w\rangle + \alpha^2\langle \bar{E}_{c_0}{}''(w_{c_0})\tilde{w}, \tilde{w}\rangle < 0,$$

so $\tilde{L}_{c_0}$ has at most a one-dimensional negative eigenspace. □

Lemma 2.8. *For any $0 < c_0 < \sqrt{2}$, $\tilde{L}_{c_0}$ has at least one negative eigenvalue.*

Proof. By (33), it suffices to find a test function $w_0 \in X_1$ such that $q_c(w_0) = \langle \bar{E}_{c0}''(w_{c0})w_0, w_0 \rangle < 0$. By (30), it is equivalent to find $\phi \in X_1$ such that $\left\langle L_{c0}\phi, \phi \right\rangle < 0$. We note that the traveling wave solutions of (17) constructed in [49] are cylindrical symmetric, that is, $U_{c_0} = U_{c_0}(x_1, r_\perp)$ with $r_\perp = \sqrt{x_2^2 + x_3^2}$. Differentiating (17) to $r_\perp$, we get

$$L_{c0}\partial_{r_\perp} U_{c0} = -\frac{1}{r_\perp^2}\partial_{r_\perp} U_{c0}.$$

In Appendix 3, we show that $\partial_{r_\perp} U_c \in H^1(\mathbf{R}^3)$ and $\frac{1}{r_\perp}\partial_{r_\perp} U_{c0} \in L^2\left(\mathbf{R}^3\right)$. Thus

$$\left\langle L_{c0}\partial_{r_\perp} U_{c0}, \partial_{r_\perp} U_{c0}\right\rangle = -\left\| \frac{1}{r_\perp}\partial_{r_\perp} U_{c0} \right\|_{L^2}^2 < 0.$$

This proves the lemma. □

Lemma 2.9. *For any* $0 < c_0 < \sqrt{2}$, *there exists* $\delta_0 > 0$ *such that*

$$\frac{p_{c0,\infty}(\varphi)}{\|\varphi\|_{L^2}^2} \geqq \delta_0, \quad \forall\, \varphi \in L^2. \tag{34}$$

Proof. By (27), it suffices to prove that there exists $\delta_0 > 0$ such that

$$\frac{q_{c0,\infty}(w)}{\|w_1\|_{H^1}^2 + \|w_2\|_{\dot{H}^1}^2} \geqq \delta_0, \quad \forall\, w = w_1 + i w_2 \in X_1. \tag{35}$$

Since $0 < c_0 < \sqrt{2}$, there exists $0 < a_0 < 1$ such that $2 - \frac{c_0^2}{a_0^2} > 0$. Then for $w = w_1 + i w_2 \in X_1$, we have

$$\begin{aligned} q_{c0,\infty}(w) &= \int_{\mathbf{R}^3} \left[|\nabla w_1|^2 + 2w_1^2 + |\nabla w_2|^2 - 2c_0(\partial_{x_1} w_2) w_1 \right] \mathrm{d}x \\ &= \int_{\mathbf{R}^3} \Bigg(|\nabla w_1|^2 + \left(2 - \frac{c_0^2}{a_0^2}\right) w_1^2 + (1 - a_0^2)(\partial_{x_1} w_2)^2 \\ &\qquad + (\partial_{x_2} w_2)^2 + (\partial_{x_3} w_2)^2 + \left(\frac{c_0}{a_0} w_1 - a_0 \partial_{x_1} w_2\right)^2 \Bigg) \mathrm{d}x. \\ &\geqq \min\left\{2 - \frac{c_0^2}{a_0^2}, 1 - a_0^2\right\} \left(\|w_1\|_{H^1}^2 + \|w_2\|_{\dot{H}^1}^2\right). \end{aligned}$$

Thus (35) holds for $\delta = \min\left\{2 - \frac{c_0^2}{a_0^2}, 1 - a_0^2\right\}$. □

2.4. Proof of Nonlinear Stability

We can now prove the orbital stability of traveling waves on the lower branch (that is when $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$).

Theorem 2.1. *For* $0 < c_0 < \sqrt{2}$, *let* U_{c_0} *be a traveling wave solution of* (GP) *constructed in* [49], *satisfying* (24) *and* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$. *Then the traveling wave* U_{c_0} *is orbitally stable in the following sense: There exists constants* $\varepsilon_0, M > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$, *if*

$$u(0) \in X_0, \quad d_1\left(u(0), U_{c_0}\right) < M\varepsilon, \tag{36}$$

then

$$\sup_{0<t<\infty} \inf_{y \in \mathbf{R}^3} d_1\left(u(\cdot, t), U_{c_0}(\cdot + y)\right) < \varepsilon.$$

The proof of this theorem basically follows the line in [32]. However, the more precise stability estimate (36) was not given there. The proof given below is to modify the proof of Theorem 3.4 in [32] and get (36). First, we need the following:

Lemma 2.10. *Let* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$. *If* $\phi \in X_1$ *is such that*

$$\left\langle \bar{P}'(w_c), \phi \right\rangle = \left(\partial_{x_i} w_{c_0}, \phi\right) = 0, \quad i = 1, 2, 3,$$

then

$$\langle \bar{E}_c''(w_{c_0})\phi, \phi\rangle \geqq \delta \|\phi\|_{X_1}^2,$$

for some $\delta > 0$.

The proof of this lemma is essentially the same as in [32], by using Corollary 2.1 on the spectral properties of the quadratic form $\langle \bar{E}_c''(w_{c_0})\cdot, \cdot\rangle$.

Proof of Theorem 2.1. Let $u(t) = \psi(w(t))$. Since the mapping

$$\psi : \left(X_1, \|\cdot\|_{X_1}\right) \to (X_0, d_1)$$

is locally bi-Lipschitz with the local Lipschitz constant invariant under translation, it suffices to show the following statement: if $w(0) \in X_1$, $\left\|w(0) - w_{c_0}\right\|_{X_1} < M\varepsilon$, then

$$\sup_{0<t<\infty} \inf_{y \in \mathbf{R}^3} \left\|w(t, \cdot) - w_{c_0}(\cdot + y)\right\|_{X_1} < \varepsilon.$$

Let $y(w(t)) \in \mathbf{R}^3$ be such that the infimum

$$\inf_{y \in \mathbf{R}^3} \left\|w(t, \cdot) - w_{c_0}(\cdot + y)\right\|_{X_1} = \inf_{y \in \mathbf{R}^3} \left\|w(t, \cdot - y) - w_{c_0}(\cdot)\right\|_{X_1}$$

is obtained. Below, we use $\|\cdot\|$ for $\|\cdot\|_{X_1}$ for simplicity and denote $T(y)w(t) = w(t, \cdot + y)$. Then by definition

$$\left(T(y(w(t)))w(t) - w_{c_0}, \partial_{x_i} w_{c_0}\right) = 0, \quad i = 1, 2, 3.$$

Denote $u(t) = T(y(w(t)))w(t) - w_{c_0}$, and $d(t) = \|u(t)\|^2$. Since

$$\begin{aligned}\left|\bar{P}(T(y(w))w(t)) - \bar{P}\left(w_{c_0}\right)\right| &= \left|\bar{P}(w(0)) - \bar{P}\left(w_{c_0}\right)\right| \\ &\leqq C\left\|w(0) - w_{c_0}\right\| = C\|d(0)\|^{\frac{1}{2}}\end{aligned}$$

and

$$\bar{P}\left(T\left(y\left(w\right)\right)w\left(t\right)\right)-\bar{P}\left(w_{c0}\right)=\left\langle \bar{P}'\left(w_{c0}\right),u\left(t\right)\right\rangle +O\left(\left\|u\left(t\right)\right\|^{2}\right),$$

so

$$\left|\left\langle \bar{P}'\left(w_{c0}\right),u\left(t\right)\right\rangle \right|\leqq C\left(d\left(t\right)+\left\|d\left(0\right)\right\|^{\frac{1}{2}}\right). \tag{37}$$

Let $I : X_1 \to (X_1)^*$ be the isomorphism defined by $\langle Iu, v\rangle = (u, v)$ for any $u, v \in X_1$. Define $q = I^{-1}\bar{P}'\left(w_{c0}\right)$ and decompose $u\left(t\right) = v + aq$, where $a = (u, q) / (q, q)$ and $(v, q) = \left\langle \bar{P}'\left(w_{c0}\right), v\right\rangle = 0$. Then (37) implies that

$$|a|=\frac{\left|\left\langle \bar{P}'\left(w_{c0}\right),u\left(t\right)\right\rangle \right|}{\left(q,q\right)}\leqq C\left(d\left(t\right)+\left\|d\left(0\right)\right\|^{\frac{1}{2}}\right).$$

Moreover,

$$\left(v,\partial_{x_i}w_{c0}\right)=\left(u\left(t\right),\partial_{x_i}w_{c0}\right)-a\left\langle \bar{P}'\left(w_{c0}\right),\partial_{x_i}w_{c0}\right\rangle =0.$$

So by Lemma 2.10, we get $\langle \bar{E}_c''(w_{c0})v, v\rangle \geqq \delta \left\|v\right\|^2$. We start with

$$\bar{E}_c\left(T\left(y\left(w\right)\right)w\left(t\right)\right)-\bar{E}_c\left(w_{c0}\right)=\bar{E}_c\left(w\left(0\right)\right)-\bar{E}_c\left(w_{c0}\right). \tag{38}$$

The Taylor expansion of the left hand side of (38) yields

$$\begin{aligned}
&\frac{1}{2}\left\langle \bar{E}_c''(w_{c0})u\left(t\right),u\left(t\right)\right\rangle +O\left(\left\|u\left(t\right)\right\|^{3}\right)\\
&\quad=\frac{1}{2}\left\langle \bar{E}_c''(w_{c0})v\left(t\right),v\left(t\right)\right\rangle +O\left(|a|^{2}+a\left\|v\right\|+\left\|u\right\|^{3}\right)\\
&\quad\geqq\frac{1}{2}\delta\left\|v\right\|^{2}-C\left(|a|^{2}+a\left\|v\right\|+\left\|u\right\|^{3}\right)\\
&\quad\geqq\frac{1}{2}\delta\left\|u\right\|^{2}-C'\left(|a|^{2}+a\left\|u\right\|+\left\|u\right\|^{3}\right)\\
&\quad=\frac{1}{2}\delta d-C'\left(\left(d+\sqrt{d\left(0\right)}\right)^{2}+\left(d+\sqrt{d\left(0\right)}\right)\sqrt{d}+d^{\frac{3}{2}}\right)\\
&\quad\geqq\frac{1}{4}\delta d-C''\left(d^{2}+d^{\frac{3}{2}}+d\left(0\right)\right),
\end{aligned}$$

here, in the second inequality above we use

$$\left\|u\right\|-|a|\left\|q\right\|\leqq\left\|v\right\|\leqq\left\|u\right\|+|a|\left\|q\right\|$$

and in the last inequality we use

$$\sqrt{d\left(0\right)}\sqrt{d}\leqq\frac{1}{2}\left(\eta d+\frac{1}{\eta}d\left(0\right)\right),\quad\eta=\frac{1}{2}\delta.$$

The right hand side of (38) is controlled by $Cd\left(0\right)$. Combining above, we get

$$d\left(t\right)-C_1F\left(d\left(t\right)\right)\leqq C_2d\left(0\right), \tag{39}$$

for some $C_1, C_2 > 0$ and $F\left(d\right) = d^2 + d^{\frac{3}{2}}$. The stability and the estimate (36) follows easily from (39) by taking $M = \frac{2}{C_2}$. □

Remark 2.4. In [51], Chiron and Maris constructed 3D traveling waves of (4) with a nonnegative potential function $V(s)$, by minimizing the energy functional under the constraint of constant momentum. They proved the compactness of the minimizing sequence and as a corollary the orbital stability of these traveling waves is obtained. There are two differences between their result and Theorem 2.1. First, in [51], the orbital stability is for the set of all minimizers which are not known to be unique. Moreover, the more precise stability estimate (36) cannot be obtained by such compactness approach. Second, the stability criterion $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$ obtained in Theorem 2.1 (under the non-degeneracy assumption) confirmed the conjecture in the physical literature [12,39]. No such stability criterion was obtained in [51]. In our proof, the variational characterization (such as in [49]) is only used in Lemma 2.6 to show that the second variation of energy-momentum functional has at most one negative direction. We do not need the compactness of the minimizing sequence and the traveling waves constructed by other variational arguments (for example [15]) also fit into our approach.

2.5. The Case of General Nonlinearity

In this section, we extend Theorem 2.1 on nonlinear stability to general nonlinearity F satisfying the following conditions:

(F1) $F \in C^1(\mathbf{R}^+) \cap C^0([0,\infty))$, C^2 in a neighborhood of 1, $F(1) = 0$ and $F'(1) = -1$.

(F2) There exists $C > 0$ and $0 < p_1 \leqq 1 \leqq p_0 < 2$ such that $|F'(s)| \leqq C(1 + s^{p_1-1} + s^{p_0-1})$ for all $s \geqq 0$.

Remark 2.5. The exponent p_0 in condition (F2) restricts the growth of F' at infinity and p_1 is the order of singularity allowed for F' at $s = 0$, which means F is only assumed to be Hölder near $s = 0$. Condition (F2) implies that $|F(s)| \leqq C(1+s^{p_0})$ for all $s \geqq 0$. The nonlinearity of Gross–Pitaevskii equation is $F(s) = 1-s$ which certainly satisfies (F1)(F2).

The energy function is now given by

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} [|\nabla u|^2 + V(|u|^2)]\,\mathrm{d}x,$$

where $V(s) = \int_s^1 F(\tau)d\tau$. By the proof of Lemma 4.1 in [49], $E(u) < \infty$ if and only if $u \in X_0$ (defined in (9)). So we can use the same coordinate mapping $u = \psi(w)$ $(w \in X_1)$ for the energy space. For $w \in X_1$, define

$$\bar{E}(w) := E \circ \psi(w) = \frac{1}{2}\int_{\mathbf{R}^3} [|\nabla \psi(w)|^2 + V(|\psi(w)|^2)]\mathrm{d}x. \tag{40}$$

In order to prove the smoothness of $\bar{E}$, we need the following standard properties of Nemytskii operators:

Lemma 2.11. *Suppose* $g \in C(\mathbf{R}^m, \mathbf{R})$ *and* $|g(s)| \leqq |s|^{q_0}$ *for some* $q_0 > 0$ *and all* $s \in \mathbf{R}^m$, *then the mapping* $G(\phi) \triangleq g \circ \phi$ *is continuous from* $L^{q_1}(\mathbf{R}^n, \mathbf{R}^m)$ *to* $L^{\frac{q_1}{q_0}}(\mathbf{R}^n, \mathbf{R})$ *where* $q_1 \in [\min\{1, \frac{1}{q_0}\}, \infty]$.

The proof is simply a modification of the one of Theorem 2.2 of [3] based on Theorem 4.9 in [17], the latter of which is valid on $\mathbf{R}^n$ in particular.

Lemma 2.12. *Assume* $(F1)(F2)$. *Then the functional* $\bar{E}(w) : X_1 \to \mathbf{R}$ *is* C^2.

Proof. For $w = w_1 + iw_2 \in X_1$, we set

$$J_1(w) = \int_{\mathbf{R}^3} |\nabla\psi(w)|^2 \, dx = \int_{\mathbf{R}^3} |\nabla w_1 - \nabla\chi(D)\left(\frac{w_2^2}{2}\right)|^2 + |\nabla w_2|^2 \, dx,$$

$$J_2(w) = \int_{\mathbf{R}^3} V(|\psi(w)|^2) \, dx.$$

Then $\bar{E}(w) = \frac{1}{2}(J_1(w) + J_2(w))$. Since $J_1(w) \in C^\infty(X_1, \mathbf{R})$ as shown in the proof of Lemma 2.3, it suffices to show that $J_2 \in C^2(X_1, \mathbf{R})$. In the sequel, let $C(\|w\|_{X_1})$ be a positive constant depending on $\|w\|_{X_1}$ increasingly.

Following the notation in Appendix 1, we denote

$$\begin{aligned}\Psi_2(w) &= |\psi(w)|^2 - 1 \\ &= \left(w_1 - \chi(D)\left(\frac{w_2^2}{2}\right)\right)^2 + (1 - \chi(D))\, w_2^2 + 2w_1.\end{aligned}$$

Then by (13) and (14), it is easy to show that $\Psi_2 \in C^\infty\left(X_1, L^2 \cap L^3\right)$. By (F1),

$$F(s) = F(1) + F'(1)(s-1) + (s-1)\varepsilon(s-1),$$

where $\varepsilon(t) \to 0$ as $t \to 0$. Thus there exists $\beta \in (0,1)$ such that

$$|F(s)| = |F(s) - F(1)| \leqq 2|s-1|, \quad \text{for all } s \in (1-\beta, 1+\beta). \tag{41}$$

We choose three cut-off functions ξ_1, ξ_2, ξ_3 with supports in

$$[0, 1-\beta/2), \quad (1-\beta, 1+\beta) \quad \text{and} \quad (1+\beta/2, \infty)$$

respectively, and $0 \leqq \xi_i \leqq 1$, $\sum_{i=1}^3 \xi_i = 1$. Denote $F_i(s) = F(s)\xi_i(s)$, and $V_i(s) = \int_s^1 F_i(\tau)d\tau$. Then by (41), $|F_2(s)| \leqq 2|s-1|$ and by (F2)

$$|F_1(s)| \leqq C,\ |F_3(s)| \leqq C\left(1 + s^{p_0}\right) \implies |F_1(s)|, |F_3(s)| \leqq C'|s-1|^{p_0},$$

since $|s-1| \geqq \beta/2$ on the supports of F_1, F_3. By Lemma 2.11 we have

$$F_1(|\psi(w)|^2),\ F_3(|\psi(w)|^2) \in C(X_1, L^{\frac{3}{2}})$$

and $F_2\left(|\psi(w)|^2\right) \in C\left(X_1, L^2\right)$. Thus the Gateau derivative of $J_2(w)$ at $\phi \in X_1$

$$\langle J_2'(w), \phi\rangle = -\sum_{i=1}^3 \int_{\mathbf{R}^3} F_i(|\psi(w)|^2)\left(\Psi_2'(w)\phi\right) dx$$

is continuous in $w \in X_1$ and thus $J_2 \in C^1(X_1, \mathbf{R})$.

Now we consider the Gateau derivative of $J_2'(w)$. For any $\phi = \phi_1 + i\phi_2, h = h_1 + ih_2 \in X_1$, we have

$$\begin{aligned} J_2''(w)(\phi, h) &= -\sum_{i=1}^{3} \int_{\mathbf{R}^3} F_i(|\psi(w)|^2) \left(\Psi_2''(w)(\phi, h)\right) \mathrm{d}x \\ &\quad - \sum_{i=1}^{3} \int_{\mathbf{R}^3} F_i'(|\psi(w)|^2) \left(\Psi_2'(w)\phi\right) \left(\Psi_2'(w)h\right) \mathrm{d}x \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

It is not difficult to verify that the above is indeed the Gateau derivative of $J_2'(w)$ and we skip the details. Now we show the continuity of $J_2''(w)(\phi, h)$ in w, which implies that it is the Fréchet derivative of $J_2'(w)$. The continuity of I to $w \in X_1$ follows by the same reasoning for $J_2'(w)$. We write

$$\mathrm{II} = -\sum_{i=1}^{3} \int_{\mathbf{R}^3} F_i'(|\psi(w)|^2) \left(\Psi_2'(w)\phi\right) \left(\Psi_2'(w)h\right) \mathrm{d}x = -\sum_{i=1}^{3} \mathrm{II}_i(w)(\phi, h).$$

Since $F_{2,3}'$ are continuous on $\mathbf{R}$ and satisfy $|F_2'(s)|,\ |F_3'(s)| \leqq C|s-1|$, Lemma 2.11 and the smoothness of $\Psi_2 : X_1 \to L^3$ imply $F_{2,3}'(|\psi(w)|^2)$ is continuous from X_1 to L^3, and consequently the uniform continuity of the quadratic forms $\mathrm{II}_{2,3}(w)$ on X_1 with respect to $w \in X_1$. To see the uniform continuity of the quadratic forms $\mathrm{II}_1(w)$ in w, we write it more explicitly:

$$\mathrm{II}_1(w)(\phi, h) = \int_{\mathbf{R}^3} \left(\psi'(w)h\right)^{\mathrm{T}} \left(F_1'(|\psi(w)|^2)\psi(w)\psi(w)^{\mathrm{T}}\right) \left(\psi'(w)\phi\right) \mathrm{d}x,$$

where in the above the complex valued $\psi(w)$, $\psi'(w)h$, $\psi'(w)\phi$ are viewed as 2-dim column vectors. Since F_1' is supported on $[0, 1 - \frac{\beta}{2})$ with $\beta \in (0, 1)$ and satisfies $|F_1'| \leqq C(1 + s^{p_1 - 1})$, $p_1 \in (0, 1]$, we have

$$\left|F_1'(|\psi(w)|^2)\psi(w)\psi(w)^{\mathrm{T}}\right| \leqq C_p \left||\psi(w)|^2 - 1\right|^p, \quad \forall\ p \geqq 0.$$

As $\Psi_2(w) = |\psi(w)|^2 - 1$ is a smooth mapping from X_1 to $L^2 \cap L^3$, Lemma 2.11 implies that $w \to F_1'(|\psi(w)|^2)\psi(w)\psi(w)^{\mathrm{T}}$ is a continuous mapping from X_1 to $L^{\frac{3}{2}}$. Therefore, the uniform continuity with respect to w of the quadratic form $II_1(w)$ on X_1 follows from the smoothness of $\psi : X_1 \to \dot{H}^1$ and this completes the proof of the lemma. □

A traveling wave $U_c = u_c + iv_c = \psi(w_c)$ of (4) satisfies the equation

$$-ic\partial_{x_1}U_c + \Delta U_c + F(|U_c|^2)U_c = 0. \tag{42}$$

Under (F1)–(F2), for any $0 < c < \sqrt{2}$, traveling waves were constructed in [49] as an energy minimizer under the constraint of Pohozaev type identity. As in the (GP) case, w_c is a critical point of the momentum functional $\bar{E}_c(w) = \bar{E}(w) + c\bar{P}(w)$. The second variation functional can be written in the form

$$\langle \bar{E}_c{}''(w_c)\phi, \phi\rangle = \langle L_c(K\phi), K\phi\rangle, \tag{43}$$

where K is defined in (22) and

$$L_c := \begin{pmatrix} -\Delta - F\left(|U_c|^2\right) - F'\left(|U_c|^2\right) 2u_c^2 & -c\partial_{x_1} - 2F'\left(|U_c|^2\right) u_c v_c \\ c\partial_{x_1} - 2F'\left(|U_c|^2\right) u_c v_c & -\Delta - F\left(|U_c|^2\right) - F'\left(|U_c|^2\right) 2v_c^2 \end{pmatrix}. \tag{44}$$

Assuming that the traveling wave solution $U_c = u_c + iv_c$ satisfies the decay estimate

$$u_c - 1 = o\left(\frac{1}{|x|^2}\right), \quad v_c = o\left(\frac{1}{|x|}\right), \tag{45}$$

and the non-degeneracy condition (24) as in the (GP) case, we can show the same decomposition result for the quadratic form $\langle \bar{E}_c{}''(w_c)\phi, \phi\rangle$, as in Proposition 2.1 and Corollary 2.1. Then by the proof of Theorem 2.1, we get the same nonlinear stability criterion for traveling waves of (4). That is, we have:

Theorem 2.2. *Assume* (F1-2). *For* $0 < c < \sqrt{2}$, *let* U_c *be a traveling wave solution of* (4) *constructed in* [49]. *Assume the* (24) *type non-degeneracy condition*:

$$\ker(L_c) = span\{\partial_{x_j} U_c \mid j = 1, 2, 3\}.$$

Then the traveling wave U_c *satisfying* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$ *is orbitally stable in the same sense* (*in terms of the distance* d_1) *as in Theorem* 2.1.

In fact, the above theorem also holds for some cases when $p_0 = 2$ in the assumption (F2). More precisely, assume that:

(F2′) There exists $C, \alpha_0, s_0 > 0$, and $0 < p_1 \leqq 1 \leqq p_0 \leqq 2$, such that $|F'(s)| \leqq C(1 + s^{p_1 - 1} + s^{p_0 - 1})$ for all $s \geqq 0$ and $F(s) \leqq -Cs^{\alpha_0}$ for all $s > s_0$.

Corollary 2.2. *Assume* (F1) *and* (F2′). *For* $0 < c < \sqrt{2}$, *let* U_c *be a traveling wave solution of* (4) *constructed in* [49], *satisfying* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$. *Assume the* (24) *type non-degeneracy condition*: $\ker(L_c) = span\{\partial_{x_j} U_c \mid j = 1, 2, 3\}$. *Then the traveling wave* U_c *is orbitally stable.*

Remark 2.6. This corollary applies to the cubic-quintic nonlinear Schrödinger equation where the nonlinearity corresponds to

$$F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2, \quad \alpha_{1,2,3} > 0.$$

For 3D, the cubic-quintic equation is critical and its global existence in the energy space was shown recently in [41]. For dimension $n \leqq 4$ and rather general subcritical nonlinear terms, the global existence in the energy space was shown in [27].

Remark 2.7. The decay property (45) for traveling waves was proved for (GP) equation in [31]. It seems possible to use the arguments of [31] to get the same decay (45) for general nonlinear terms.

In fact, if $p_0 = 2$, the energy and momentum functional E and P are still C^2 on X_1. Supposed U_c is a traveling wave, that is, a critical point of the energy-momentum functional E_c, such that $E_c''(U_c)$ is uniformly positive as in the sense of Lemma 2.10, then the same proof as the one of Theorem 2.1 applies and we obtain the orbital stability of U_c.

In assumption (F2), $p_0 < 2$ is assumed so that the existence of traveling waves is obtained through a constrained minimization approach as in Theorem 1.1 in [49], where the compactness of the embedding is needed.

Fortunately, with assumptions (F1) and (F2′), Corollary 1.2 in [49] applies and thus traveling waves exist through constrained minimization. The idea is that (F2′) allows us to carefully modify the nonlinearity F to F_M such that

$$F_M(s) = F(s), \quad \forall\, s \in [0, s_1], \qquad F_M(s) = -C_1 s^{\beta} \quad \forall\, s \geqq s_2,$$

where C_1, β, s_1, s_2 are some constants satisfying $s_1 \geqq s_0$, $s_2 >> s_1$, and $\beta \in (0, 2)$. The construction of F_M ensures that (F1–2) are satisfied, which implies the existence of a constrained minimizer U_c of the energy-momentum functional $E_{c,M}$ associated to F_M and $L_{c,M} \triangleq E_{c,M}''(U_c)$ can be analyzed as in the above. Moreover, one can prove that the range of U_c is contained in $[0, s_1]$. Therefore, U_c is also a traveling wave of the original equation. More details on the existence through the calculus of variation can be found in [49]. Finally, due to the fact $E_c''(U_c) = E_{c,M}''(U_c)$ as $F_M = F$ on the range of U_c, we obtain the uniform positivity of $E_c''(U_c)$ in the sense of Lemma 2.10 and the nonlinear stability follows subsequently.

3. Instability of Traveling Waves on the Upper Branch

In this section, we prove the instability of $3D$ traveling waves obtained via a constrained variational approach when $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$. First, we prove linear instability by studying the linearized problem. Then, instead of passing linear instability to nonlinear instability, we will prove a much stronger statement by constructing stable and unstable manifolds near the unstable traveling waves.

3.1. Linear Instability

In the traveling frame $(t, x - ce_1 t)$, the nonlinear equation (4) becomes

$$i\partial_t U - ic\partial_{x_1} U + \Delta U + F(|U|^2)U = 0, \tag{46}$$

where $u\,(t, x) = U\,(t, x - ce_1 t)$.

Near the traveling wave solution $U_c = u_c + iv_c$ satisfying (42), the linearized equation can be written as

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = JL_c \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{47}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and L_c is defined by (44).

We construct invariant manifolds by using the nonlinear equation for $w \in X_1$, where $u = \psi(w)$ satisfies the (GP) equation. The reason is two-fold. First, we need to use the spectral properties of the quadratic form $\langle L_c \cdot, \cdot \rangle$ in the space X_1 (Proposition 2.1) to prove the exponential dichotomy of the semigroup e^{tJL_c} in Lemma 3.1 below. Second, to ensure that the constructed invariant manifolds lie in the energy space (see Remark 3.4). Denote $U_c = \psi(w_c)$ and $w_c = w_{1c} + i w_{2c}$. Let

$$\begin{aligned} U &= \psi(w_{1c} + w_1, w_{2c} + w_2) \\ &= U_c + w_1 - \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) + i w_2. \end{aligned} \tag{48}$$

Plugging (48) into (46), we get

$$\begin{aligned} \partial_t w_2 = \Delta w_1 - \Delta \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) + c\partial_{x_1} w_2 + [F(|U|^2) - F(|U_c|^2)] u_c \\ + F(|U|^2)\left[w_1 - \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) \right], \end{aligned} \tag{49}$$

$$\begin{aligned} \partial_t w_1 = -\Delta w_2 + \chi(D)((w_{2c} + w_2)\partial_t w_2) + c\partial_{x_1}\left[w_1 - \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) \right] \\ + [F(|U_c|^2) - F(|U|^2)] v_c - F(|U|^2) w_2. \end{aligned} \tag{50}$$

The above two equations can be written as

$$i\partial_t w - ic\partial_{x_1} w + \Delta w = \Psi(w), \tag{51}$$

where

$$\begin{aligned} \operatorname{Re}\Psi(w) = \Delta \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) + [F(|U_c|^2) - F(|U|^2)] u_c \\ - F(|U|^2)\left[w_1 - \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) \right], \end{aligned} \tag{52}$$

and

$$\begin{aligned} \operatorname{Im}\Psi(w) = \chi(D)((w_{2c} + w_2)\partial_t w_2) - c\partial_{x_1} \chi(D)\left(w_{2c} w_2 + \frac{w_2^2}{2} \right) \\ + [F(|U_c|^2) - F(|U|^2)] v_c - F(|U|^2) w_2. \end{aligned} \tag{53}$$

Instead of linearizing the nonlinear term $\Psi(w)$ at $w = 0$ directly, we derive the linearized equation of (51) by relating it with the linearized equation (47) for u. The linearization of the coordinate mapping $u = \psi(w)$ at w_c yields $u = Kw$, where K is defined by (22). Thus, the linearized equation of (51) at $w = 0$ takes the form

$$\partial_t \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = K^{-1} J L_c K \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{54}$$

which implies that

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(t) = K^{-1} e^{tJL_c} K \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(0)\,. \tag{55}$$

So it suffices to study the spectrum of JL_c and the semigroup e^{tJL_c}. Note that the traveling wave U_c is cylindrical symmetric, that is, $U_c = U_c\left(x_1, \left|x^{\perp}\right|\right)$ with $x^{\perp} = (x_2, x_3)$. We will prove linear instability in X_1^s, the cylindrical symmetric subspace of X_1. Assume the non-degeneracy condition in the cylindrical symmetric space, that is,

$$\ker L_c \cap X_1^s = \left\{\partial_{x_1} U_c\right\}. \tag{56}$$

We have the following analogue of Proposition 2.1:

Proposition 3.1. *For $0 < c < \sqrt{2}$, let $U_c = \psi(w_c)$ be a traveling wave solution of (4) constructed in [49] and L_c be the operator defined by (44). Assume (56). The space X_1^s is decomposed as a direct sum*

$$X_1^s = N \oplus Z \oplus P,$$

where $Z = \left\{\partial_{x_1} U_c\right\}$, N is a one-dimensional subspace such that $\langle L_c u, u\rangle < 0$ for $0 \neq u \in N$, and P is a closed subspace such that

$$\langle L_c u, u\rangle \geqq \delta \|u\|_{X_1}^2 \quad \textit{for any } u \in P,$$

for some constant $\delta > 0$.

The proof is the same as that of Proposition 2.1, by observing that the negative mode constructed in Lemma 2.8 is cylindrical symmetric. Now we show the linear instability of traveling waves on the upper branch.

Proposition 3.2. *Let U_c, $c \in [c_1, c_2] \subset (0, \sqrt{2})$, be a C^1 (with respect to the wave speed c) family of traveling waves of (4) in the energy space X_0. For $c_0 \in (c_1, c_2)$, assume*

1. *$F \in C^1$ on $U_{c_0}(\mathbf{R}^n)$;*
2. *L_{c_0} satisfies (56);*
3. $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$;

then there exists $0 \neq w_u \in X_1^s$ and $\lambda_u > 0$, such that $e^{\lambda_u t} w_u(x)$ is a solution of (54).

In particular, this proposition applies to those traveling waves obtained in [49] via a constrained variational approach.

Proposition 3.3. *Assume* (*F*1–2) *or* (*F*1)–(*F*2′). *For* $0 < c_0 < \sqrt{2}$, *let* U_{c_0} *be a traveling wave solution of* (4) *constructed in* [49], *satisfying* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$. *Assume* (56). *Then there exists a linearly unstable mode of* (54). *That is, there exists* $0 \neq w_u \in X_1^s$ *and* $\lambda_u > 0$, *such that* $e^{\lambda_u t} w_u(x)$ *is a solution of* (54).

Proof of Proposition 3.2. By (55), it suffices to show that the operator JL_{c_0} has an unstable eigenvalue in the space X_1^s. The proof is to modify that of Theorem 5.1 in [33], as explained in Remark 3.2 below. Define the following subspace of X_1^s by

$$Y_1^s = \left\{ u \in X_1^s \mid \left\langle u, J^{-1}\partial_{x_1} U_{c_0} \right\rangle = \left\langle u, J^{-1}\partial_c U_c|_{c=c_0} \right\rangle = 0 \right\}. \tag{57}$$

We show that the quadratic form $\langle L_{c_0}\cdot, \cdot\rangle$ restricted to Y_1^s is non-degenerate. Indeed, any $u \in X_1^s$ can be uniquely written as

$$u = a\partial_{x_1} U_{c_0} + b\partial_c U_c|_{c=c_0} + v, \tag{58}$$

where $v \in Y_1^s$,

$$a = -\left\langle u, J^{-1}\partial_c U_c|_{c=c_0} \right\rangle / \frac{\partial P(U_c)}{\partial c}\Big|_{c=c_0}, \tag{59}$$

and

$$b = \left\langle u, J^{-1}\partial_{x_1} U_{c_0} \right\rangle / \frac{\partial P(U_c)}{\partial c}\Big|_{c=c_0}. \tag{60}$$

Here, we use the identity

$$L_c\partial_c U_c = -P'(U_c) = -J^{-1}\partial_{x_1} U_c. \tag{61}$$

Suppose $\langle L_{c_0}\cdot, \cdot\rangle$ is degenerate on Y_1^s, then there exists $\phi \in Y_1^s$ such that $\langle L_{c_0}\phi, v\rangle = 0$ for any $v \in Y_1^s$. This implies that $\langle L_{c_0}\phi, u\rangle = 0$ for any $u \in X_1^s$, by the decomposition (58). So $\phi \in \ker L_c \cap X_1^s$ and by assumption (56), $\phi = c\partial_{x_1} U_{c_0}$ which implies $\phi = 0$ since $\partial_{x_1} U_{c=c_0} \notin Y_1^s$.

Moreover, since $L_{c_0}\partial_{x_1} U_{c_0} = 0$, (61) and the definition of Y_1^s imply a.) the splitting of X_1^s into Y_1^s and $\operatorname{span}\{\partial_{x_1} U_{c_0},\ \partial_c U_c|_{c=c_0}\}$ is orthogonal with respect to the quadratic form L_{c_0} and b.) $\operatorname{span}\{\partial_{x_1} U_{c_0},\ \partial_c U_c|_{c=c_0}\}$ is invariant under JL_{c_0} and thus so is Y_1^s, which also imply their invariance under the linearized flow $e^{tJL_{c_0}}$.

Denote $n(L_{c_0}|_X)$ to be the number of negative modes of the quadratic form $\langle L_{c_0}\cdot, \cdot\rangle$ restricted to a subspace $X \subset X_1^s$. We show that $n(L_{c_0}|_{Y_1^s}) = 1$. Indeed, for any $u \in X_1^s$, by (58) and (61), we have

$$\left\langle L_{c_0}u, u\right\rangle = b^2 \left\langle L_{c_0}\partial_c U_c|_{c=c_0}, \partial_c U_c|_{c=c_0}\right\rangle + \langle Lv, v\rangle .$$

Since $n(L_{c_0}|_{X_1^s}) = 1$ and

$$\left\langle L_{c_0}\partial_c U_c|_{c=c_0}, \partial_c U_c|_{c=c_0}\right\rangle = -\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0,$$

so $n(L_{c0}|_{Y_1^s}) = 1$. Let $Y_1^s = N \oplus P$, where on P and N, the quadratic form $\langle L_{c0}\cdot, \cdot\rangle$ is positive and negative definite respectively, dim $N = 1$, and N, P are orthogonal in the inner product $[\cdot, \cdot] := \langle L_{c0}\cdot, \cdot\rangle$.

It can be verified that

$$D\left(JL_{c0}\right) = D\left(L_{c0}\right) = X_3.$$

Indeed, $JL_{c0}, L_{c0} : X_3 \to H^1$. Since Y_1^s is separable, there is an increasing sequence of subspaces $P^{(n)} \subset P$ of odd dimension n such that $\cup X^{(n)}$ is dense in Y_1^s, where $X^{(n)} = N + P^{(n)}$. We can choose N and $P^{(n)}$to lie in X_3. Denote by π^-, π^+ and $\pi^{(n)}$ the orthogonal projections of Y_1^s to N, P and $X^{(n)}$ respectively in the inner product $[\cdot, \cdot]$. Consider the set

$$\mathcal{C} = \left\{u \in Y_1^s \mid \left[\pi^- u, u\right] = -1, \ \langle L_{c0} u, u\rangle = 0\right\}$$

and $\mathcal{C}^n = \mathcal{C} \cap X^{(n)}$. For $v \in X^{(n)}$, consider the mapping

$$f_n(v) = \pi^{(n)}\left(JL_{c0}v\right) + \left[\pi^-\left(JL_{c0}v\right), v\right] v. \tag{62}$$

In the above definition, we use the observation that $JL_{c0}v \in Y_1^s$ for any $v \in Y_1^s$. It is easy to check that $\left[\pi^- f_n(v), v\right] = 0$ and

$$\begin{aligned}\langle f_n(v), L_{c0}v\rangle &= [f_n(v), v] = \left[JL_{c0}v, v\right] + \left[\pi^-\left(JL_{c0}v\right), v\right][v, v] \\ &= \langle JL_{c0}v, L_{c0}v\rangle + \left[\pi^-\left(JL_{c0}v\right), v\right]\langle v, L_{c0}v\rangle = 0.\end{aligned}$$

Therefore, f_n is a tangent vector field on the manifold $\mathcal{C}^n$, which is the union of two spheres S^{n-1} and thus has non-vanishing Euler characteristic. Thus f_n must vanish at some $y_n \in \mathcal{C}^n$. That is, there is a real scalar $a_n = -\left[\pi^-\left(JL_{c0}v_n\right), v_n\right]$, such that

$$\left[JL_{c0}y_n, w\right] = a_n[y_n, w], \quad \text{for any } w \in X^{(n)}. \tag{63}$$

Let $y_n = y_n^- + y_n^+$, where $y_n^- \in N$ and $y_n^- \in P^{(n)}$. Let $y_n^- = b_n\chi_-$ with $\langle L_{c0}\chi_-, \chi_-\rangle = -1$, then $\left[y_n^-, y_n^-\right] = -1$ implies that $|b_n| = 1$. We can normalize $b_n = 1$. Since

$$0 = \langle L_{c0}y_n, y_n\rangle = -1 + \langle L_{c0}y_n^+, y_n^+\rangle$$

and $\langle L_{c0}\cdot, \cdot\rangle|_P$ is positive, $\left\|y_n^+\right\|_{X_1}$ is uniformly bounded. So $y_n \rightharpoonup y_\infty \in Y_1^s$ weakly in X_1. We note that $y_\infty \neq 0$ since $\pi^- y_\infty = \chi_- \neq 0$. We claim that $\{a_n\}$ is bounded. Suppose otherwise, $a_n \to \infty$ when $n \to \infty$. For any integer $k \in \mathbf{N}$ and a fixed $w \in X^{(k)}$, when $n \geqq k$, by (63) we have

$$\begin{aligned}[y_n, w] &= \frac{1}{a_n}\left[JL_{c0}y_n, w\right] = \frac{1}{a_n}\langle JL_{c0}y_n, L_{c0}w\rangle \\ &= -\frac{1}{a_n}\langle L_{c0}y_n, JL_{c0}w\rangle = -\frac{1}{a_n}\left[y_n, JL_{c0}w\right].\end{aligned} \tag{64}$$

Let $n \to \infty$ in (64), we have $[y_\infty, w] = 0$. By the density argument, this is also true for any $w \in Y_1^s$ and thus $y_\infty = 0$ since $[\cdot, \cdot]$ is non-degenerate on Y_1^s. This contradiction shows that $\{a_n\}$ is bounded. So we can pick a subsequence $\{n_k\}$ such that $a_{n_k} \to a$, for some $a \in \mathbf{R}$. For convenience, we still denote the subsequence by a_n. By (63),

$$-\left[y_n, JL_{c0}w\right] = \left[JL_{c0}y_n, w\right] = a_n\left[y_n, w\right].$$

Passing to the limit of above, we have

$$-\left[y_\infty, JL_{c0}w\right] = a\left[y_\infty, w\right],$$

for any fixed $w \in X^{(k)}$. For any $v \in X_3 \cap Y_1^s$, $JL_{c_0}v \in X_1$ and by density argument, we have

$$-\left[y_\infty, JL_{c_0}v\right] = a\left[y_\infty, v\right]. \tag{65}$$

It is easy to see that (65) is also satisfied when $v \in \left\{\partial_{x_1}U_{c0}, \partial_c U_c|_{c=c0}\right\}$. Thus by the decomposition (58), the Equation (65) is satisfied for any $v \in X_3 \cap X_1^s$. Thus,

$$-\left\langle y_\infty, L_{c_0}Jw\right\rangle = a\left\langle y_\infty, w\right\rangle$$

for any $w \in R\left(L_{c0}\right)$, which is the orthogonal complement of $\partial_{x_1}U_c$. Therefore, there exists a constant d, such that $y_\infty \in Y_1^s$ is the weak solution of the equation

$$JL_{c0}y_\infty = ay_\infty + d\partial_{x_1}U_{c0}. \tag{66}$$

We must have $a \neq 0$, since $0 \neq y_\infty \in Y_1^s$ and $Y_1^s \cap \left\{\partial_{x_1}U_{c0}, \partial_c U_c|_{c=c0}\right\} = \varnothing$. By elliptic regularity, $y_\infty \in D\left(JL_{c0}\right) = X_3$ and then $y_\infty = \frac{1}{a}\left(JL_{c0}y_\infty - \frac{d}{a}\partial_{x_1}U_{c0}\right) \in H^1$, so $y_\infty \in H^3$. If $U_c \in 1 + H^k$ for some integer k, then it can be shown that $y_\infty \in H^k$. Since (66) implies that

$$JL_{c_0}\left(y_\infty + \frac{d}{a}\partial_{x_1}U_{c0}\right) = a\left(y_\infty + \frac{d}{a}\partial_{x_1}U_{c0}\right),$$

$a \neq 0$ is an eigenvalue of JL_{c0}.

For any nonzero eigenvalue λ of JL_{c_0} with an eigenfunction y, we must have $u = L_{c0}y \neq 0$, so we obtain from $L_{c_0}Ju = \lambda u$ that λ is also an eigenvalue of $L_{c0}J = -(JL_{c0})^*$. Therefore $-\lambda$ is an eigenvalue of JL_{c_0} as well. This and the above argument imply that $\pm a$ are eigenvalues of JL_{c0}. This finishes the proof that JL_{c_0} must have a positive eigenvalue. □

Remark 3.1. By the above proof, there also exists a stable eigenvalue $\lambda_s < 0$ of JL_{c_0} which gives an exponentially decaying solution $e^{\lambda_s t}w_s(x)$ ($w_s(x) \in X_1^s$) of the linearized equation (47). This is due to the Hamiltonian nature of the equation.

Remark 3.2. The invariant subspace Y_1^s is used to remove the generalized kernel $\left\{\partial_{x_1}U_{c0}, \partial_c U_{c0}\right\}$ of L_{c0} in X_1^s. This space also plays an important role in proving the exponential dichotomy of the semigroup $e^{tJL_{c0}}$ below. In [32,33], a general theory was developed for studying stability of standing waves (traveling waves etc.) of an abstract Hamiltonian PDE $\frac{\mathrm{d}u}{\mathrm{d}t} = JE'(u)$. In this framework, the symplectic operator J should be invertible in the sense that $J^{-1} : X \to X^*$,is bounded, where X is the energy space. In our case, the space X is $X_1 = H^1 \times \dot{H}^1$, X^* is $X_1^* = H^{-1} \times \dot{H}^{-1}$ and the operator

$$J^{-1} = -J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so $J^{-1} : X_1 \to X_1^*$ is not bounded since $\dot{H}^1 \subsetneq H^{-1}$ and we cannot apply the theory of [32,33] directly. In [47], an abstract theorem was given for the case when J is not onto. However, as also commented in [19], it would take substantial effort to verify some of the assumptions in [47], particularly the semigroup estimates, for our current case and the instability of slow traveling waves in Section 5.

To handle this issue, we modify the proof of linear instability in [33] (Theorem 5.1) to avoid using the invertibility of J. Our modified argument could be used for general Hamiltonian PDEs with a non-invertible symplectic operator. We do not need to assume the semigroup estimates as in [47].

3.2. Linear Exponential Dichotomy of Semigroup

To construct invariant manifolds, the first step is to establish the exponential dichotomy of the linearized semigroup. First, we prove this for the semigroup generated by JL_c.

Lemma 3.1. *For $0 < c < \sqrt{2}$, let U_c be a traveling wave solution of (4) constructed in [49] and L_c be the operator defined by (44). Assume (56) and $\frac{\partial P(U_c)}{\partial c} < 0$. The space X_1^s is decomposed as a direct sum*

$$X_1^s = E^u \oplus E^{cs}, \tag{67}$$

satisfying: (i) *Both $E^u = span\{w_u\}$ and E^{cs} are invariant under the linear semigroup e^{tJL_c}*. (ii) *there exist constants $M > 0$ and $\lambda_u > 0$, such that*

$$\left| e^{tJL_c}|_{E^{cs}} \right|_{X_1} \leqq M(1+t), \quad \forall\, t \geqq 0 \qquad and \qquad |e^{tJL_c}|_{E^u}|_{X_1} \leqq Me^{\lambda_u t}, \quad \forall\, t \leqq 0.$$

Proof. Let $w_u, w_s \in X_1^s$ be the unstable and stable eigenfunctions of JL_c as constructed in Proposition 3.3 and its subsequent remark. Denote

$$E^s = \text{span}\{w_s\}, \quad E^u = \text{span}\{w_u\}, \quad E^{us} = \text{span}\{w_u, w_s\}.$$

First, we claim that $\langle L_c w_u, w_s\rangle \neq 0$ and the quadratic form $\langle L_c\cdot, \cdot\rangle|_{E^{us}}$ has one positive and one negative mode. Suppose, otherwise, that $\langle L_c w_u, w_s\rangle = 0$, then $\langle L_c\cdot, \cdot\rangle|_{E^{us}}$ is identically zero since $\langle Lw_u, w_u\rangle = \langle Lw_s, w_s\rangle = 0$ due to the skew-symmetry of J. By Proposition 3.3, $\langle L_c\cdot, \cdot\rangle|_{Y_1^s}$ is non-degenerate and has exactly one negative mode. Let $Y_1^s = N \oplus P$ be such that $N = \{u_-\}$ with $\langle L_c u_-, u_-\rangle < 0$ and $\langle L_c\cdot, \cdot\rangle|_P > 0$. Then we can decompose $w_u = a_1 u_- + b_1 p_1$ and $w_s = a_2 u_- + b_2 p_2$ where $p_1, p_2 \in P$. Since $a_1, a_2 \neq 0$, we define $\tilde{w} = w_u - \frac{a_2}{a_1} w_s \in E^{us} \cap P$. This is a contradiction since $\tilde{w} \neq 0$. Thus $\langle L_c\cdot, \cdot\rangle|_{E^{us}}$ is represented by the 2×2 matrix

$$\begin{pmatrix} 0 & \langle L_c w_u, w_s\rangle \\ \langle L_c w_u, w_s\rangle & 0 \end{pmatrix},$$

which has one positive and one negative mode.

We define the subspace E^e by

$$E^e = \left\{u \in Y_1^s \mid \langle u, L_c w_u \rangle = \langle u, L_c w_s \rangle = 0\right\}.$$

Let $E_g^{\ker} = \text{span}\left\{\partial_{x_1} U_{c_0}, \partial_c U_c\right\}$ be the generalized kernel of JL_c in X_1^s. For any two solutions $u(t), v(t)$ of the linearized equation $du/dt = JL_c u$, the quadratic form $\langle u(t), L_c v(t)\rangle$ is independent of t, since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), L_c v(t)\rangle &= \langle JL_c u, L_c v\rangle + \langle u, L_c JL_c v\rangle \\ &= \langle JL_c u, L_c v\rangle + \langle L_c u, JL_c v\rangle = 0.\end{aligned}$$

By using this observation and the invariance of Y_1^s and E^{us}, it is easy to show that the subspace E^e is invariant under the semigroup e^{tJL_c}. Furthermore, we show that there exists $C > 0$, such that for any $u \in E^e$ and $t \in \mathbf{R}$,

$$\left\| e^{tJL_c} u \right\|_{X_1} \leqq C \|u\|_{X_1}. \tag{68}$$

In fact, we note that any $v \in Y_1^s$ can be decomposed as

$$v = c_u w_u + c_s w_s + v_1, \tag{69}$$

where

$$c_u = \langle L_c v, w_s\rangle / \langle L_c w_u, w_s\rangle, \; c_s = \langle L_c v, w_u\rangle / \langle L_c w_u, w_s\rangle$$

and $v_1 \in E^e$. Thus we have

$$\langle L_c \cdot, \cdot\rangle |_{Y_1^s} = \langle L_c \cdot, \cdot\rangle |_{E^{us}} + \langle L_c \cdot, \cdot\rangle |_{E^e},$$

and a counting of negative modes on both sides shows that $\langle L_c \cdot, \cdot\rangle |_{E^e} > 0$. Then the estimate (68) follows by the invariance of the quadratic form $\langle L_c \cdot, \cdot\rangle$. Combining the decompositions (69) and (58), we have

$$X_1^s = E^u \oplus E^s \oplus E^e \oplus E_g^{\ker} = E^u \oplus E^{cs},$$

where

$$E^{cs} = E^s \oplus E^e \oplus E_g^{\ker}.$$

For any $t \geqq 0$, we have

$$\left| e^{tJL_c}|_{E^s} \right|_{X_1} \leqq M e^{-\lambda_u t}, \quad \left| e^{tJL_c}|_{E_g^{\ker}} \right|_{X_1} \leqq M(1+t),$$

and this finishes the proof. □

Next, we prove the exponential dichotomy in X_3^s, the cylindrical symmetric subspace of X_3 which is the domain of L_c and JL_c.

Lemma 3.2. *Under the conditions of Lemma 3.1, the space X_3^s can be written as*

$$X_3^s = E^u \oplus E_3^{cs}, \quad \textit{where } E_3^{cs} = X_3^s \cap E^{cs} \tag{70}$$

satisfying: (i) *Both* E^u *and* E_3^{cs} *are invariant under* e^{tJL_c}. (ii) *There exist constants* $M > 0$ *and* $\lambda_u > 0$, *such that*

$$\left| e^{tJL_c}|_{E_3^{cs}} \right|_{X_3} \leqq M(1+t), \quad \forall\, t \geqq 0 \quad \text{and} \quad |e^{tJL_c}|_{E^u}|_{X_3} \leqq Me^{\lambda_u t}, \quad \forall\, t \leqq 0. \tag{71}$$

Proof. Since the eigenvectors $w_u, w_s \in X_3^s$, we have $E^u \subset X_3^s$. The invariance of E_3^{cs} clearly follows from the invariance of X_3^s and E^{cs} under e^{tJL_c}. The direct sum decomposition of X_3^s is a direct consequence of that of X_1^s.

To complete the proof, we only need to show estimate (71) on E_3^{cs}. It is easy to check that the norm $\|u\|_{X_3}$ is equivalent to the norm $\|u\|_{X_1} + \|JL_c u\|_{X_1}$, so we only need to estimate the growth of $\left\|e^{tJL_c}u\right\|_{X_1} + \left\|JL_c e^{tJL_c}u\right\|_{X_1}$. For any $u \in E_3^{cs}$, by Lemma 3.1, we have

$$\left\|e^{tJL_c}u\right\|_{X_1} \leqq M(1+t)\,\|u\|_{X_1}$$

and

$$\left\|JL_c e^{tJL_c}u\right\|_{X_1} = \left\|e^{tJL_c}JL_c u\right\|_{X_1} \leqq M(1+t)\,\|JL_c u\|_{X_1}\,.$$

This finishes the proof of the lemma. □

By using the relation (55), we get the exponential dichotomy for solutions of the linearized equation (54).

Corollary 3.1. *Under the conditions of Lemma* 3.1, *the space* X_3^s *can be decomposed as a direct sum*

$$X_3^s = \tilde{E}^u \oplus \tilde{E}^{cs},$$

satisfying: (i) *Both* $\tilde{E}^u$ *and* $\tilde{E}^{cs}$ *are invariant under the linear semigroup* $S(t)$ *defined by* (54). (ii) *there exist constant* $M > 0$ *and* $\lambda_u > 0$, *such that*

$$\left|S(t)\,|_{\tilde{E}^{cs}}\right|_{X_3} \leqq M(1+t), \quad \forall\, t \geqq 0 \quad \text{and} \quad |S(t)\,|_{\tilde{E}^u}|_{X_1} \leqq Me^{\lambda_u t}, \quad \forall\, t \leqq 0.$$

Remark 3.3. The linear exponential dichotomy is the first step in constructing invariant manifolds. In general, it is rather tricky to get the exponential dichotomy of the semigroup even if its generator has a spectral gap. This is due to the issue of spectral mapping. More precisely, let $\sigma(L)$ and $\sigma\left(e^L\right)$ be the spectra of the generator L and its exponential e^L. In general, it is not true that $\sigma\left(e^L\right) = e^{\sigma(L)}$. In the literature, the exponential dichotomy was proved by using resolvent estimates [28] or compact perturbation theory of semigroups [55,56] or dispersive estimates [34,52]. In the current case, it seems difficult to apply these approaches. In Lemma 3.1, we prove the exponential dichotomy of e^{tJL_c} by using the energy estimates and the invariant quadratic form $\langle L_c\cdot, \cdot\rangle$ due to the Hamiltonian structure. This could provide a general approach to get the exponential dichotomy for lots of Hamiltonian PDEs.

In the construction of unstable (stable) manifolds, we only need to establish the exponential dichotomy in the cylindrical symmetric space X_1^s since the unstable modes are cylindrical symmetric. This also yields cylindrical symmetric invariant manifolds. By using the non-degeneracy condition (24) and the proof of Lemma 3.1, we can also get the exponential trichotomy in the whole space X_1. This will be important in a future work for the construction of center manifolds in the energy space.

Lemma 3.3. *For $0 < c < \sqrt{2}$, let U_c be a traveling wave solution of* (4) *constructed in* [49] *and L_c be the operator defined by* (44)*. Assume* (24) *and $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$. Then the space X_1 is decomposed as a direct sum*

$$X_1 = E^u \oplus E^c \oplus E^s,$$

satisfying: (i) *E^u, E^s and E^c are invariant under the linear semigroup e^{tJL_c}.* (ii) *There exist constant $M > 0,\ \lambda_u > 0$, such that*

$$\left|e^{tJL_c}|_{E^s}\right|_{X_1} \leqq Me^{-\lambda_u t}, \quad \forall\, t \geqq 0, \qquad |e^{tJL_c}|_{E^u}|_{X_1} \leqq Me^{\lambda_u t}, \quad \forall\, t \leqq 0.$$

and

$$|e^{tJL_c}|_{E^c}|_{X_1} \leqq M(1+t), \quad \forall\, t \in \mathbf{R},. \tag{72}$$

Proof. Denote $E^s = \{w_s\}$ and $E^u = \{w_u\}$, where w_u, w_s are the unstable and stable eigenfunctions of JL_c. Let $w^1 = -\partial_c U_c$, then $JL_c w^1 = \partial_{x_1} U_c$. Let w^2, w^3 be such that

$$JL_c w^2 = \partial_{x_2} U_c, \quad JL_c w^3 = \partial_{x_3} U_c.$$

The above two equations are solvable since the kernel of $-L_c J = (JL_c)^*$ is spanned by $J^{-1}\partial_{x_i} U_c$, $i = 1, 2, 3$, and

$$\left\langle J^{-1}\partial_{x_j} U_c, \partial_{x_i} U_c \right\rangle = 0, \quad \text{for } i, j = 1, 2, 3,$$

by the translation invariance of the momentum $\vec{P}(U_c) = \left(\left\langle J^{-1}\partial_{x_i} U_c, U_c - 1\right\rangle\right)$. Denote the generalized kernel of JL_c by

$$E_g^{\ker} = \operatorname{span}\left\{\cup_{i=1}^3 \{\partial_{x_i} U_{c_0}, w^i\}\right\}.$$

Define

$$Y_1 = \{u \in X_1 |\ \langle u, L_c w_u \rangle = \langle u, L_c w_s \rangle = 0\}.$$

Clearly $E_g^{\ker} \subset Y_1$, due to the symmetry of L_c and the skew-symmetry of J. Moreover Y_1 is invariant under e^{tJL_c} due to the invariance of $\operatorname{span}\{w^u, w^s\}$ and the invariance of the quadratic form given by L_c. Let

$$E^e = \left\{u \in Y_1 |\ \left\langle u, J^{-1} w\right\rangle = 0,\ \forall w \in E_g^{\ker}\right\}.$$

It is straightforward to check that E^e is invariant under e^{tJL_c} due to the invariance of Y_1 and $E_g^{\ker}$.

By the arguments in the proof of Lemma 3.1, we have $\langle L_c \cdot, \cdot \rangle|_{Y_1} \geqq 0$. This implies that

$$\left\langle L_c w^i, w^i \right\rangle > 0, \quad \text{for } i = 1, 2, 3,$$

by noting that $w^i \in Y_1$. Indeed, supposing otherwise, we would then have

$$\left\langle L_c w^i, w^i \right\rangle = 0 = \min_{w \in Y_1} \langle L_c w, w \rangle .$$

Thus $\left\langle L_c w^i, w \right\rangle = 0$ for any $w \in Y_1$, and it follows that $\left\langle L_c w^i, w \right\rangle = 0$ for any $w \in X_1$. Thus, $L_c w^i = 0$, a contradiction, so for any $u \in X_1$, we can write

$$u = c_u w_u + c_s w_s + \sum_{i=1}^{3} \left(a_i \partial_{x_i} U_c + b_i w^i \right) + v_1,$$

where $v_1 \in E^e$,

$$c_u = \langle L_c u, w_s \rangle / \langle L_c w_u, w_s \rangle, \; c_s = \langle L_c u, w_u \rangle / \langle L_c w_u, w_s \rangle,$$

and

$$a_i = -\left\langle u, J^{-1} w^i \right\rangle / \left\langle L_c w^i, w^i \right\rangle, \; b_i = \left\langle u, J^{-1} \partial_{x_i} U_c \right\rangle / \left\langle L_c w^i, w^i \right\rangle.$$

Here we used the facts that

$$\left\langle w^j, J^{-1} \partial_{x_i} U_c \right\rangle = \left\langle w^i, J^{-1} w^j \right\rangle = 0 \quad \text{and} \quad i \neq j,$$

due to the even or odd symmetry of w^i and $\partial_{x_i} U_c$ in x_j. Thus we get the direct sum decomposition

$$X_1 = E^u \oplus E^s \oplus E^e \oplus E_g^{\ker}.$$

By the proof of Lemma 3.1, it follows that the quadratic form $\langle L_c \cdot, \cdot \rangle|_{E^e}$ is positive definite. This implies that

$$|e^{tJL_c}|_{E^e}|_{X_1} \leqq C, \; \forall\, t \in \mathbf{R}$$

for some constant C. Define $E^c = E^e \oplus E_g^{\ker}$. Since $|e^{tJL_c}|_{E_g^{\ker}}|_{X_1}$ has only linear growth, the estimate (72) follows. This finishes the proof. □

3.3. Invariant Manifolds and Orbital Instability

In Appendix 1, we prove that the nonlinear term $\Psi(w)$ in the Equation (51) is $C^2(X_3, X_3)$. Thus, by the standard invariant manifold theory for semilinear PDEs (for example [7,22]), we get the following:

Theorem 3.1. *For* $0 < c_0 < \sqrt{2}$, *let* $U_{c_0} = \psi\left(w_{c_0}\right)$ *be a traveling wave solution of* (4) *constructed in* [49], *satisfying* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$. *Assume in addition* $F \in C^5$ *in a neighborhood of the set* $|U_{c_0}(\mathbf{R}^3)|^2$ *and the non-degeneracy condition* (56). *Then there exists a unique* C^2 *local unstable manifold* W^u *of* w_{c_0} *in* X_3^s *which satisfies*:

1. *It is one-dimensional and tangential to* $\tilde{E}^u$ *at* w_{c_0}.
2. *It can be written as the graph of a* C^2 *mapping from a neighborhood of* w_{c_0} *in* $\tilde{E}^u$ *to* $\tilde{E}^{cs}$.
3. *It is locally invariant under the flow of the Equation* (51), *that is solutions starting on* W^u *can only leave* W^u *through its boundary.*
4. *Solutions starting on* W^u *converge to* w_{c_0} *at the rate* $e^{\lambda_u t}$ *as* $t \to -\infty$.

The same results hold for the local stable manifold of w_{c_0} *as the Equation* (51) *is time-reversible.*

Corollary 3.2. *By using the transformation* $u = \psi(w)$, *we get the stable and unstable manifolds* $\tilde{W}^{u,s} = \psi(W^{u,s})$ *near* U_{c_0} *in the metric*

$$d_3(u,\tilde{u}) = d_1(u,\tilde{u}) + \left\|\nabla^2(u-\tilde{u})\right\|_{L^2} + \left\|\nabla^3(u-\tilde{u})\right\|_{L^2},$$

which is equivalent to the metric $\|w-\tilde{w}\|_{X_3}$ *for* w. *Since* W^u *is one-dimensional, the* d_3 *topology and* d_1 *topology are equivalent on* W^u. *Then an immediate consequence of the above theorem is the nonlinear instability in* d_1 *metric with initial data slightly perturbed from* U_{c_0} *in the* d_3 *metric.*

To compare with the orbital stability result, it is more desirable to get an orbital instability result as follows:

Corollary 3.3. *Under the assumptions of Theorem* 3.1, *the traveling wave solution* U_{c_0} *is nonlinearly unstable in the following sense*:

$\exists\,\theta, C > 0$, *such that for any* $\delta > 0$, *there exists a solution* $u_\delta(t)$ *of Equation* (4) *satisfying*

$$d_3\left(u_\delta(0), U_{c_0}\right) \leqq \delta, \tag{73}$$

and

$$\sup_{0<t\leqq C|\ln\delta|}\ \inf_{y\in\mathbf{R}^3}\left\|\nabla\left(u_{\delta,i}(t) - U_{c_0,i}(\cdot+y)\right)\right\|_{L^2} \geqq \theta, \quad i=1,2. \tag{74}$$

Here, $u_\delta(t) = u_{\delta,1}(t) + iu_{\delta,2}(t)$ *and* $U_{c_0}(x) = U_{c_0,1}(x) + iU_{c_0,2}(x)$.

Proof. First, we observe that if $u_g = u_{g,1} + iu_{g,2} \in H^3$ is an unstable eigenfunction of JL_c, then $u_{g,i} \neq 0$ for $i = 1, 2$. Suppose otherwise, that is, that $u_{g,1} = 0$, then from the equation $JL_c u_g = \lambda_u u_g$ $(\lambda_u > 0)$, we get

$$c_0\partial_{x_1}u_{g,2} - \left(2F'\left(|U_c|^2\right)u_c v_c + \lambda_u\right)u_{g,2} = 0.$$

This implies that $u_{g,2} = 0$, thus $u_g = 0$, which is a contradiction. Similarly, we can show that $u_{g,2} \neq 0$. The nonlinear instability in $\left\| \nabla \left(u_i - U_{c_0,i} \right) \right\|_{L^2}$ follows directly from the existence of unstable manifold and the above observation. To show orbital instability, we follow the proof of Theorem 6.2 in [33]. We only show the orbital instability in the norm $\left\| \nabla \left(u_1 - U_{c_0,1} \right) \right\|_{L^2}$, since the proof for $\left\| \nabla \left(u_2 - U_{c_0,2} \right) \right\|_{L^2}$ is the same. Let $u_{g,1}^{\perp}$ be the projection of $u_{g,1}$ onto the space $Z_1^{\perp}$-the orthogonal complement space of $Z_1 = \text{span} \left\{ \partial_{x_i} U_{c_0,1}, i = 1, 2, 3 \right\}$ in the inner product $\langle\langle u_1, v_1 \rangle\rangle = (\nabla u_1, \nabla v_1)$. Fix ε_0 sufficiently small and for any $\delta > 0$, we can choose the solution $u_\delta(t)$ on the unstable manifold $\tilde{W}^u$, such that $d_3 \left(u_\delta(0), U_{c_0} \right) \leqq \delta$,

$$\begin{aligned} &\left\| \nabla \left(u_{\delta,1}(t) - U_{c_0,1} \right) \right\|_{L^2} \leqq C\varepsilon_0, \quad \text{for } 0 < t < T_1 \\ &\left\langle\!\left\langle u_{\delta,1}(T_1) - U_{c_0,1}, u_{g,1}^{\perp} \right\rangle\!\right\rangle \geqq \varepsilon_0, \end{aligned}$$

where $T_1 = C |\ln \delta|$. Here C may depend on ε_0, but is independent of $\delta > 0$. Let $h = h(t) \in \mathbf{R}^3$ be such that

$$\begin{aligned} &\left\| \nabla \left(u_{\delta,1}(t) - U_{c_0,1}(\cdot + h) \right) \right\|_{L^2} \leqq 2\theta, \\ &\theta = \inf_{y \in \mathbf{R}^3} \left\| \nabla \left(u_{\delta,1}(t) - U_{c_0,1}(\cdot + y) \right) \right\|_{L^2}. \end{aligned}$$

Then

$$\left\| \nabla \left(U_{c_0,1}(\cdot) - U_{c_0,1}(\cdot + h) \right) \right\|_{L^2} \leqq 3 \left\| \nabla \left(u_{\delta,1}(t) - U_{c_0,1} \right) \right\|_{L^2} \leqq 2C\varepsilon_0,$$

thus $|h| = O(\varepsilon_0)$. Therefore we can write

$$U_{c_0,1}(x + h) = U_{c_0,1}(x) + h \cdot \nabla U_{c_0,1}(x) + O\left(\varepsilon_0^2 \right).$$

This implies that

$$\begin{aligned} C\theta &\geqq \left\langle\!\left\langle u_{\delta,1} - U_{c_0,1}(\cdot + h), u_{g,1}^{\perp} \right\rangle\!\right\rangle \\ &\geqq \left\langle\!\left\langle u_{\delta,1} - U_{c_0,1}, u_{g,1}^{\perp} \right\rangle\!\right\rangle - O\left(\varepsilon_0^2 \right) \geqq \varepsilon_0 / 2, \end{aligned}$$

by using the orthogonal property of $u_{g,1}^{\perp}$ and Z_1. This finishes the proof. □

Remark 3.4. By using the exponential dichotomy for the semigroup e^{tJL_c} (Lemma 3.2), we can construct unstable (stable) manifolds near U_c directly from Equation (46) in the space $H^3\left(\mathbf{R}^3\right) \times \dot{H}^3\left(\mathbf{R}^3\right)$. However, the functions in $U_c + H^3\left(\mathbf{R}^3\right) \times \dot{H}^3\left(\mathbf{R}^3\right)$ are not guaranteed to be in the energy space X_0. To get the invariant manifolds lying on X_0, we use the coordinate mapping $U = \psi(w)$ to rewrite the Equation (46) as (51) for $w \in H^3\left(\mathbf{R}^3\right) \times \dot{H}^3\left(\mathbf{R}^3\right)$.

Remark 3.5. Since the eigenfunctions of JL_c actually belong to H^k, instead of constructing the unstable/stable manifolds of traveling waves through the coordinate change $U = \psi(w)$ and working on (51), one can also work on (46) directly in the space $U_c + H^k$. The details are similar to the proof of Proposition 5.4 and Corollary 5.1. However, that approach, based on the improved properties of unstable eigenfunctions, would not be useful when we construct the center manifolds in the energy space in the forthcoming work.

Remark 3.6. For the (GP) equation, numerical computations [12,39] suggested that $\frac{\partial P(U_c)}{\partial c} < 0$ iff $c \in (c^*, \sqrt{2})$ for some $c^* \in (0, \sqrt{2})$, so for the 3D traveling waves of (GP), the instability sets in at a critical velocity c^*. By contrast, for the cubic-quintic equation, we have $\frac{\partial P(U_c)}{\partial c} < 0$, and thus the instability when c is near 0 and $\sqrt{2}$. Thus there may not exist a critical speed for instability. The case for small c is proved in Theorem 5.4 and $\frac{\partial P(U_c)}{\partial c} < 0$ for c near $\sqrt{2}$ can be seen from the transonic limit [6,20,39] of traveling waves of (4) to solitary waves of the Kadomtsev–Petviashvili (KP) equation.

4. Transversal Instability of $2D$ Traveling Waves

In this section, we prove the transversal instability of 2D traveling waves of (4). Unlike the 3D instability result (Proposition 3.3 and Theorem 3.1), we do not need to assume the non-degeneracy condition (24) for the 2D traveling waves.

To state the result, first we introduce some notations. Assume $F \in C^1(\mathbf{R}^+)$. For $0 < c < \sqrt{2}$, consider the operator L_c defined by (44), where $U_c(x_1 - ct, |x_2|)$ is a 2D traveling wave solution of (4). Then it is easy to show that $L_c : \left(H^2\left(\mathbf{R}^2\right)\right)^2 \to \left(L^2\left(\mathbf{R}^2\right)\right)^2$ is self-adjoint and

$$\sigma_{\text{ess}}(L_c) = \sigma_{\text{ess}}\left(L_{c,\infty}\right) = [0, +\infty), \quad \text{for any } c \in \left(0, \sqrt{2}\right),$$

where $L_{c,\infty}$ is defined in (28). Let $\lambda_0(L_c)$ be the first eigenvalue of L_c.

Theorem 4.1. *For $0 < c < \sqrt{2}$, let $U_c(x_1 - ct, |x_2|)$ be a 2D traveling wave of (4). Suppose $\lambda_0(L_c) < 0$. Let $\lambda_1 \leqq 0$ be the second eigenvalue. Then U_c is transversely unstable in the following sense: for any*

$$k \in \left(\sqrt{-\lambda_1}, \sqrt{-\lambda_0}\right),$$

there exists an unstable solution

$$e^{\lambda_u t + ikx_3} u_g(x_1, x_2), \quad \textit{with } \lambda_u > 0, u_g \in (H^3(\mathbf{R}^2))^2 \tag{75}$$

for the linearized equation (76). If $k > \sqrt{-\lambda_0}$, then no such solution with $\lambda_u > 0$ exists, that is, there is spectral stability.

Remark 4.1. Denote the momentum by

$$P(u) = \frac{1}{2}\int_{\mathbf{R}^2} \langle i\partial_{x_1} u, u\rangle \,\mathrm{d}x = -\int_{\mathbf{R}^2} u_1 \partial_{x_1} u_2 \,\mathrm{d}x.$$

When $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$, the instability condition $\lambda_0(L_c) < 0$ is satisfied by the traveling wave U_{c_0}. This is due to the identity

$$\langle L_c \partial_c U_c, \partial_c U_c\rangle = -\left\langle P'(U_c), \partial_c U_c\right\rangle = -\frac{\partial P(U_c)}{\partial c},$$

by (61). Numerical evidence [11,39] shows that the condition $\frac{\partial P(U_c)}{\partial c} > 0$ is satisfied for 2D traveling waves of (GP). Moreover, 2D traveling waves of (GP) were constructed in [15,51] as energy minimizers subject to a fixed momentum. This implies (by a similar proof to Lemma 2.7) that L_c can have at most one negative eigenvalue. Thus, for any $0 < c < \sqrt{2}$, we have $\lambda_0(L_c) < 0$ and $\lambda_1(L_c) = 0$ for any 2D traveling wave of (GP). By Theorem 4.1, any 2D traveling wave of (GP) is transversely unstable if and only if the transversal wave number $k \in \left(0, \sqrt{-\lambda_0}\right)$. When $k \to 0+$, see the asymptotic analysis in [11,43]. In the limit $c \to 0$, the 2D traveling waves of (GP) consist of an antiparallel vortex pair [13,40]. In this case, the mechanism of transversal instability is analogous to the Crow instability of an antiparallel vortex pair of incompressible fluid [23].

Proof. The linearized equation of (46) near $U_c(x_1, x_2)$ can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = J\tilde{L}_c u, \tag{76}$$

where

$$\tilde{L}_c = L_c + \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x_3^2} & 0 \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x_3^2} \end{pmatrix}.$$

Finding an unstable solution of the form (75) for the linearized equation (76) is equivalent to solving the eigenvalue problem $J\left(L_c + k^2\right)u_g = \lambda_u u_g$. Denote $L_{c,k} = L_c + k^2$, then for $k \in \left(\sqrt{-\lambda_1}, \sqrt{-\lambda_0}\right)$, the operator $L_{c,k}$ has one negative eigenvalue, no kernel and the rest of the spectrum is contained in (δ_0, ∞) with $\delta_0 = k^2 + \lambda_1 > 0$. The existence of an unstable eigenvalue of $JL_{c,k}$ follows by the line of proof of Proposition 3.3, in a much simplified way since $\sigma\left(L_{c,k}\right)$ does not contain 0. When $k > \sqrt{-\lambda_0}$, the operator $L_{c,k}$ is positive. This implies the non-existence of unstable modes since any such mode satisfies $\left\langle L_{c,k}u_g, u_g\right\rangle = 0$. $\square$

We now prove nonlinear transversal instability under the instability condition in Theorem 4.1. For any $k_0 \in (\sqrt{-\lambda_1}, \sqrt{-\lambda_0})$, denote $H^m(\mathbf{R}^2 \times S_{\frac{2\pi}{k_0}})$ to be all functions in $H^m(\mathbf{R}^2 \times [0, \frac{2\pi}{k_0}])$ which are $\frac{2\pi}{k_0}$-periodic in x_3. Let

$$X_{1,k_0} = \left\{ u(x_1, x_2, x_3) \in \left(H^1\left(\mathbf{R}^2 \times S_{\frac{2\pi}{k_0}}\right)\right)^2 \mid u \text{ is odd in } x_3 \right\},$$

and

$$X_{3,k_0} = X_{1,k_0} \cap \left(H^3\left(\mathbf{R}^2 \times S_{\frac{2\pi}{k_0}}\right)\right)^2.$$

From Theorem 4.1, we have a linearly unstable mode of the form

$$e^{\lambda_u t}\sin(k_0 x_3)\, u_g(x_1, x_2)$$

in the space X_{3,k_0}. We will construct unstable manifold near the traveling wave $U_c(x_1, x_2)$ in the space $1 + X_{3,k_0}$. First, we show the exponential dichotomy of $e^{tJ\tilde{L}_c}$ in the space X_{3,k_0}.

Lemma 4.1. *For any*

$$k_0 \in \left(\max\left\{\sqrt{-\lambda_1}, \frac{\sqrt{-\lambda_0}}{4}\right\}, \sqrt{-\lambda_0}\right), \tag{77}$$

the space X_{3,k_0} *is decomposed as a direct sum*

$$X_{3,k_0} = E^u \oplus E^{cs},$$

satisfying that: (i) *both* E^u *and* E^{cs} *are invariant under the linear semigroup* $e^{tJ\tilde{L}_c}$. (ii) *There exist constants* $M > 0$ *and* $\lambda_u > 0$, *such that*

$$\left|e^{tJ\tilde{L}_c}|_{E^{cs}}\right|_{H^3\left(\mathbf{R}^2\times S_{\frac{2\pi}{k_0}}\right)} \leqq M, \quad \forall\, t \geqq 0,$$

and

$$|e^{tJ\tilde{L}_c}|_{E^u}|_{H^3\left(\mathbf{R}^2\times S_{\frac{2\pi}{k_0}}\right)} \leqq Me^{\lambda_u t}, \quad \forall\, t \leqq 0.$$

Proof. First, we show the exponential dichotomy in the space X_{1,k_0}. Any function $u \in X_{1,k_0}$ can be written as

$$u\left(x_1, x_2, x_3\right) = \sum_{j=1}^{\infty} \sin\left(jk_0 x_3\right) u_j\left(x_1, x_2\right),$$

and

$$\|u\|^2_{H^1\left(\mathbf{R}^2\times S_{\frac{2\pi}{k_0}}\right)} = \sum_{j=1}^{\infty}\left(\left\|u_j\right\|^2_{H^1(\mathbf{R}^2)} + j^2\left\|u_j\right\|^2_{L^2(\mathbf{R}^2)}\right). \tag{78}$$

We have

$$u\left(t\right) = e^{tJ\tilde{L}_c}u = \sum_{j=1}^{\infty}\sin\left(jk_0x_3\right)\left(e^{tJL_{c,jk_0}}u_j\right)\left(x_1, x_2\right).$$

By assumption (77), the operator L_{c,k_0} has one negative eigenvalue and the rest of the spectrum lies in the positive axis, and $\left\{L_{c,jk_0}\right\}$ $\left(j \geqq 2\right)$ are positive, so by the proof of Theorem 4.1, there exist a pair of stable and unstable modes of the form $e^{\pm\lambda_u t}u^{\pm}$ $(\lambda_u > 0)$, where $u^{\pm} = \sin\left(k_0x_3\right)u_{\pm}\left(x_1, x_2\right)$. Define $E^u = \operatorname{span}\left\{u^+\right\}$, $E^s = \operatorname{span}\left\{u^-\right\}$ and

$$E^e = \left\{u = \sum_{j=1}^{\infty}\sin\left(jk_0x_3\right)u_j\left(x_1, x_2\right) \in X_{1,k_0} \mid \left\langle L_{c,k_0}u_1, u^{\pm}\right\rangle = 0\right\}.$$

Then by the arguments in the proof of Lemma 3.1, for any $u \in E^e$, we have

$$\left\|e^{tJL_{c,k_0}}u_1\right\|_{H^1(\mathbf{R}^2)} \leqq C\left\|u_1\right\|_{H^1(\mathbf{R}^2)}, \quad \text{for some constant } C,$$

and by the positivity of L_{c,jk_0} $\left(j \geqq 2\right)$,

$$\left\| e^{tJL_{c,jk_0}} u_j \right\|^2_{H^1(\mathbf{R}^2)} + j^2 \left\| e^{tJL_{c,jk_0}} u_j \right\|^2_{L^2(\mathbf{R}^2)}$$
$$\leqq C \left(\left\| u_j \right\|^2_{H^1(\mathbf{R}^2)} + j^2 \left\| u_j \right\|^2_{L^2(\mathbf{R}^2)} \right)$$

with some constant C independent of j. So by (78), we have

$$\left\| e^{tJ\tilde{L}_c} u \right\|_{H^1\left(\mathbf{R}^2 \times S_{\frac{2\pi}{k_0}}\right)} \leqq C \left\| u \right\|_{H^1\left(\mathbf{R}^2 \times S_{\frac{2\pi}{k_0}}\right)}, \quad \text{for } u \in E^e.$$

Define $E^{cs} = E^s \cup E^e$. Then $X_{1,k_0} = E^u \oplus E^{cs}$ is a direct sum decomposition for the exponential dichotomy of $e^{tJ\tilde{L}_c}$. Define

$$E_3^{cs} = \left\{ u \in X_{3,k_0} |\ J\tilde{L}_c u \in E^{cs} \right\}.$$

Then $X_{3,k_0} = E^u \oplus E_3^{cs}$ and the exponential dichotomy follow by the same argument in the proof of Lemma 3.2. □

In the Equation (46), we let $U = U_c + u$, with $u \in X_{3,k_0}$. Then the equation can be written as

$$u_t = J\tilde{L}_c u + \Psi(u).$$

If $F \in C^5(\mathbf{R})$, it is easy to show that the nonlinear term $\Psi(u)$ is $C^2\left(X_{3,k_0}, X_{3,k_0}\right)$. By using Lemma 4.1, we have the following:

Theorem 4.2. *For* $0 < c_0 < \sqrt{2}$, *let* U_{c_0} *be a 2D traveling wave solution of* (4), *satisfying* $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$ *or more generally* $\lambda_0(L_c) < 0$. *For any* k_0 *satisfying* (77), *there exists a unique* C^1 *local unstable manifold* W^u *of* U_{c_0} *in* X_{3,k_0} *which satisfies that*:

1. *It is one-dimensional and tangential to* E^u *at* U_{c_0}.
2. *It can be written as the graph of a* C^1 *mapping from a neighborhood of* U_{c_0} *in* E^u *to* E_3^{cs}.
3. *It is locally invariant under the flow of the equation* (46).
4. *Solutions starting on* W^u *converge to* U_{c_0} *at the rate* $e^{\lambda_u t}$ *as* $t \to -\infty$.

As a corollary of the above theorem, we get nonlinear transversal instability of any 2D traveling wave of the (GP) equation.

Remark 4.2. Assumption (77) ensures that the unstable subspace of the linearized equation in X_{3,k_0} is 1-dimensional. In fact this assumption can be generalized to

$$\exists\ j_0 \geqq 1 \quad \text{such that } j_0 k_0 \text{ satisfies (77).} \tag{79}$$

Since the subspace corresponding to the j_0-th mode is decoupled in the linearized equation, this assumption ensures that there exists a 1-dimensional unstable subspace in the j_0-th mode which implies the linear instability with possibly multiple dimensional unstable subspaces. $\tilde{L}_c$ is uniformly positive in all but finitely many directions, one may prove the linear exponential dichotomy and the existence of unstable manifolds through a similar procedure.

5. Slow Traveling Waves of Cubic-Quintic Type Equations

In this section, we assume that the nonlinear term of (4) satisfies the following:

(H1) $F \in C^1([0,\infty))$, $F(r_0) = 0$, and $F'(r_0) < 0$, where r_0 is a positive constant.

(H2) $\exists\, C > 0$ such that $\left|F'(s)\right| \leqq C\,|s|^{p_0-1}$, for $s \geqq 1$, where $p_0 = \frac{2}{n-2}$.

(H3) $\exists\, r_1$ such that $0 \leqq r_1 < r_0$ and $V(r_1) < 0$, where $V(r) = \int_0^r F(s)\,\mathrm{d}s$.

A typical example is the so-called cubic-quintic (or $\psi^3 - \psi^5$) nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi - \alpha_1\psi + \alpha_3\psi|\psi|^2 - \alpha_5\psi|\psi|^4 = 0, \quad x \in \mathbf{R}^3, \tag{80}$$

where $\alpha_1, \alpha_3, \alpha_5$ are positive constants satisfying (8). The main result of this section is to show the existence and instability of traveling waves with small speeds.

5.1. Existence of Slow Traveling Waves

First we recall the result of stationary solutions.

Theorem 5.1. [24] *Under assumptions* $(H1)$–$(H3)$, *there exists a real-valued function* $\phi_0 \in C^2(\mathbf{R}^n)$ *satisfying*:

1. $\phi_0(x) = \phi_0(|x|)$ (*that is* ϕ *is radially symmetric*)
2. $$\Delta\phi_0 + F(\phi_0^2)\phi_0 = 0, \quad \textit{in } \mathbf{R}^n \ (n \geqq 2). \tag{81}$$
3. $0 < \phi_0(r) < \sqrt{r_0}$, $\forall r \in [0,\infty)$, *and* $\lim_{r\to\infty}\phi_0(r) = \sqrt{r_0}$
4. $\phi_0'(0) = 0$, $\phi_0'(r) > 0\ \forall r \in (0,+\infty)$
5. *There exist* $C > 0$, $\delta > 0$ *such that*: $\forall \alpha \in \mathbf{N}^n$ *with* $|\alpha| < 2$,

$$|\partial_x^\alpha(\phi_0(x) - \sqrt{r_0})| \leqq Ce^{-\delta|x|}, \quad \forall x \in \mathbf{R}^n.$$

The steady solution ϕ_0 constructed above is called a stationary bubble of (4). To simplify notations, below we assume $r_0 = 1$, $F'(1) = -1$. Denote the operator $A : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ by

$$A := -\Delta - F(\phi_0^2) - 2F'\left(\phi_0^2\right)\phi_0^2. \tag{82}$$

Note that A is the linearized operator with the steady equation (81). Differentiating (81) to x_i, we get $\partial_{x_i}\phi_0 \in \ker A$. We state the following non-degeneracy condition

$$\ker A = \left\{\partial_{x_i}\phi_0, i = 1,\ldots,n\right\}. \tag{83}$$

First, we study the two and three dimensional cases.

Theorem 5.2. *Let* $n = 2, 3$. *Under assumptions* $(H1)$–$(H3)$, *and the condition* (83), *there exists* $b_0 > 0$, *such that for any* $c \in (-b_0, b_0)$, *there exist* $(\rho_c, \theta_c) \in X_2^s$ *such that*

$$\phi^c\left(x_1 - ct, x^\perp\right) = \left((\rho_0 + \rho_c)^{\frac{1}{2}} e^{i\theta_c}\right)\left(x_1 - ct, x^\perp\right)$$

is a cylindrically symmetric traveling wave solution of Equation (4). *Here,* $\sqrt{\rho_0} = \phi_0(r)$ *is the stationary solution to* (4). *Moreover,* (ρ_c, θ_c) *is* C^1 *for* $c \in (-b_0, b_0)$,

$$\|\rho_c\|_{H^2} + \|\theta_c\|_{\dot{H}^2} \leqq K\,|c|\,, \quad \textit{for some } K > 0.$$

For $n = 2$, the non-degeneracy condition (83) is proved for cubic-quintic nonlinearity in Appendix 2.

To prove the existence of traveling waves, we use the hydrodynamic formulation (5). The traveling wave solution

$$\psi(x_1 - ct, x_\perp) = \sqrt{\rho}e^{i\theta}(x_1 - ct, x_\perp)$$

satisfies

$$\begin{cases} -c\theta_{x_1} + |\nabla\theta|^2 - \frac{1}{2}\frac{1}{\rho}\Delta\rho + \frac{1}{4}\frac{1}{\rho^2}|\nabla\rho|^2 - F(\rho) = 0 \\ c\rho_{x_1} - 2\nabla.(\rho\nabla\theta) = 0. \end{cases} \tag{84}$$

We define $S(\rho, \theta; c)$ to be the left-hand side of (84), then (84) becomes $S(\rho, \theta; c) = 0$. First, we define several function spaces. Define the spaces

$$Z := L^2_{r_\perp} \cap \dot{H}^{-1}_{r_\perp}, \quad \text{with norm } \|\cdot\|_Z = \|\cdot\|_2 + \|\cdot\|_{\dot{H}^{-1}}\,,$$

and $Y := L^2_{r_\perp} \times Z$. The energy functional is defined by (6) and the momentum is

$$P(\rho, \theta) = -\frac{1}{2}\int_{\mathbf{R}^n} (\rho - 1)\,\theta_{x_1}\,\mathrm{d}x,$$

where $(\rho, \theta) \in (\rho_0, 0) + B_{\varepsilon_0}$ with

$$B_{\varepsilon_0} = \left\{(\rho, \theta) \in X_2^s \mid \|\rho\|_{H^2} + \|\theta\|_{\dot{H}^2} \leqq \varepsilon_0\right\}.$$

Since $H^2(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ for $n = 2, 3$ and $\rho_0(r) \geqq \rho_0(0) > 0$, thus when ε_0 is small enough, the functional E and P are well-defined.

Proof of Theorem 5.2. When c is small enough, we look for solutions of (84) in the form $(\rho+\rho_0, \theta)$ where $(\rho, \theta) \in B_{\varepsilon_0}$ with ε_0 small enough such that $\rho+\rho_0 > 0$. First, we note the following variational structure of (84):

$$S(\rho + \rho_0, \theta; c) = 2D_{(\rho,\theta)}\left(E(\rho + \rho_0, \theta) + cP(\rho + \rho_0, \theta)\right).$$

Since the functionals E, P are translation invariant to x_1, the above implies that

$$\left\langle S(\rho + \rho_0, \theta; c), \begin{pmatrix} \partial_{x_1}(\rho + \rho_0) \\ \partial_{x_1}\theta \end{pmatrix} \right\rangle = 0 \tag{85}$$

for any $(\rho, \theta) \in B_{\varepsilon_0}$. Define $K_0 = \text{span}\left\{(\partial_{x_1}\rho_0(x), 0)\right\}$. Let $K_0^\perp$ be the orthogonal complement of K_0 in Y, and $\Pi^\perp : Y \mapsto K_0^\perp$ be the L^2 orthogonal projection. We solve the equation

$$\Pi^\perp S(\rho_0 + \rho, \theta; c) = 0, \quad (\rho, \theta) \in K_0^\perp, \tag{86}$$

near (0, 0; 0) by the implicit function theorem. The linearized operator of S with respect to (ρ, θ) at (0, 0; 0) is

$$D_{(\rho,\theta)}S(\rho_0, 0; 0) := M_0 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} : \quad X_2^s \mapsto Y \tag{87}$$

where

$$\begin{aligned} M_1 &= -\nabla \cdot \left(\frac{\nabla}{2\rho_0} \right) - \frac{1}{2}\frac{1}{\rho_0^3}|\nabla\rho_0|^2 + \frac{\Delta\rho_0}{2\rho_0^2} - F'(\rho_0), \\ M_2 &= -2\nabla \cdot (\rho_0 \nabla). \end{aligned} \tag{88}$$

The linearized mapping of $\Pi^\perp S(\rho_0+\rho, \theta; c)|_{K_0^\perp \cap X_2^s}$ at (0, 0; 0) is $\Pi^\perp M_0|_{X_2^s \cap K_0^\perp} = M_0|_{X_2^s \cap K_0^\perp}$. It can be checked that for any $\rho \in H^2$,

$$M_1 \rho = A\left(\frac{\rho}{2\sqrt{\rho_0}} \right) \frac{1}{\sqrt{\rho_0}}, \tag{89}$$

which also follows from (99) below. So by the assumption (83),

$$\ker M_1 = \operatorname{span}\left\{ \sqrt{\rho_0}\partial_{x_1}\phi_0 \right\} = \operatorname{span}\left\{ \partial_{x_1}\rho_0 \right\}, \quad \text{on } H^2_{r_\perp}.$$

Moreover, $(M_2\theta, \theta) > 0$ for any $\theta \in \dot{H}^1$. Thus $\ker M = \left\{ (\partial_{x_1}\rho_0(x), 0) \right\} = K_0$. By Lemma 5.1 below, the operator $M_0 : X_2^s \cap K_0^\perp \to K_0^\perp$ is bounded with a bounded inversion. Moreover, it is easy to show that $S(\rho+\rho_0, \theta; c) \in C^1\left(B_{\varepsilon_0} \times \mathbf{R}; Y\right)$. Thus by the Implicit Function Theorem [21], there exists a neighborhood $B_{\delta_0} \times (-b_0, b_0)$ of (0, 0; 0) in $\left(X_2^s \cap K_0^\perp\right) \times \mathbf{R}$ such that

$$(\rho(c), \theta(c)) : (-b_0, b_0) \to B_{\delta_0}$$

is the unique solution to (86) near (0, 0; 0) which is C^1 in c. Moreover, as implied by the proof of IFT, we have

$$\|\rho(c)\|_{H^2} + \|\theta(c)\|_{\dot{H}^2} \leqq K \|S(\rho_0, 0; c)\|_Y \leqq K|c|$$

for some constant K. We claim that $(\rho(c), \theta(c))$ solves the original problem, that is, $S(\rho(c) + \rho_0, \theta(c); c) = 0$. Indeed, by equation (86), we have

$$S(\rho(c) + \rho_0, \theta(c); c) = k(\partial_{x_1}\rho_0(x), 0)$$

for some constant k. We claim that $k = 0$. Suppose otherwise, that is, that $k \neq 0$, then by (85),

$$\left\langle \begin{pmatrix} \partial_{x_1}\rho_0 \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_{x_1}(\rho(c) + \rho_0) \\ \partial_{x_1}\theta(c) \end{pmatrix} \right\rangle = 0,$$

or $\left\| \partial_{x_1}\rho_0 \right\|_{L^2}^2 + O(c) = 0$, which is a contradiction. This finishes the proof of the Theorem. □

It remains to show that the operator $M|_{X_2^s \cap K_0^\perp}$ has a bounded inversion. We study this in a more general setting. For $0 < c_0 < \sqrt{2}$, suppose $(\rho_{c_0}, \theta_{c_0})$ is a traveling wave solution satisfying (84) and $\min \rho_{c_0} > 0$. The linearized operator of $S(\rho + \rho_{c_0}, \theta + \theta_{c_0}; c)$ at $(0, 0; c_0)$ is

$$D_{(\rho,\theta)} S(\rho_{c_0}, \theta_{c_0}; c_0) := M_{c_0} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} : \quad X_2^s \mapsto Y. \tag{90}$$

Here,

$$\begin{aligned} M_{11} &= -\nabla \cdot \left(\frac{\nabla}{2\rho_{c_0}} \right) - \frac{1}{2} \frac{1}{\rho_{c_0}^3} |\nabla \rho_{c_0}|^2 + \frac{\Delta \rho_{c_0}}{2\rho_{c_0}^2} - F'(\rho_{c_0}), \\ M_{22} &= -2\nabla \cdot (\rho_{c_0} \nabla), \end{aligned} \tag{91}$$

and

$$\begin{aligned} M_{21} &= c_0 \partial_{x_1} - 2\nabla \cdot \left(\nabla \theta_{c_0} \cdot \right), \\ M_{12} &= M_{21}^* = -c_0 \partial_{x_1} + 2\nabla \theta_{c_0} \cdot \nabla. \end{aligned} \tag{92}$$

Define $K_{c_0} = \left\{ \left(\partial_{x_1} \rho_{c_0}, \partial_{x_1} \theta_{c_0} \right) \right\}$. Let $K_{c_0}^\perp$ be the orthogonal complement of K_{c_0} in Y, and $\Pi_{c_0}^\perp : Y \mapsto K_{c_0}^\perp$ be the orthogonal projection. Note that $\Pi_{c_0}^\perp M_{c_0}|_{K_{c_0}^\perp \cap X_2^s} = M_{c_0}|_{K_{c_0}^\perp \cap X_2^s}$.

Lemma 5.1. *Assume*

$$\ker M_{c_0} = span \left\{ \left(\partial_{x_1} \rho_{c_0}, \partial_{x_1} \theta_{c_0} \right) \right\} \quad on\ X_2^s. \tag{93}$$

Then there exists $\gamma > 0$, *such that for any* $(\rho, \theta) \in K_{c_0}^\perp \cap X_2^s$,

$$\left\| M_{c_0}(\rho, \theta) \right\|_Y \geqq \gamma \left\| (\rho, \theta) \right\|_{X_2^s}. \tag{94}$$

In particular, $M_{c_0}|_{K_{c_0}^\perp \cap X_2^s} : K_{c_0}^\perp \cap X_2^s \to K_{c_0}^\perp$ *is invertible and*

$$\left\| M_{c_0}|_{K_{c_0}^\perp \cap X_2^s}^{-1} \right\| \leqq \gamma^{-1}.$$

Proof. We follow the arguments of the proof of Proposition 2.3 in [25]. Suppose (94) is not true. Then there exists a sequence

$$\psi_n = (\rho_n, \theta_n) \in K_{c_0}^\perp \cap X_2^s, \quad n = 1, 2, \ldots,$$

such that $\|\psi_n\|_{X_2} = 1$ and $\left\| M_{c_0} \psi_n \right\|_Y \to 0$ when $n \to \infty$. Since $\|\psi_n\|_{X_2} = 1$, we may assume (by passing to a subsequence) that $\psi_n \to \psi_\infty$ weakly in X_2^s for some $\psi_\infty \in X_2^s$. The fact that $\{\psi_n\}$ is orthogonal to K_{c_0} implies that $\psi_\infty \in K_{c_0}^\perp \cap X_2^s$. The weak convergence of ψ_n to ψ_∞ in X_2^s implies the weak convergence of $M_{c_0} \psi_n$ to $M_{c_0} \psi_\infty$ in L^2. It follows from $\left\| M_{c_0} \psi_n \right\|_Y \to 0$ that $M_{c_0} \psi_\infty = 0$. Thus $\psi_\infty = 0$ since ψ_∞ is orthogonal to $\ker M_{c_0} = K_{c_0}$.

Denote the operator

$$M_{c_0}^{\infty} = \begin{pmatrix} M_{11}^{\infty} & -c_0\partial_{x_1} \\ c_0\partial_{x_1} & M_{22} \end{pmatrix}, \quad \text{with } M_{11}^{\infty} = -\nabla.\left(\frac{\nabla}{2\rho_{c_0}}\right) + \frac{1}{\rho_{c_0}}. \tag{95}$$

We claim that $\left\|M_{c_0}^{\infty}\psi_n\right\|_Y \to 0$ when $n \to \infty$. First, $\psi_n \to 0$ weakly in X_2 implies that $\rho_n \to 0$ weakly in H^2 and $\theta_n \to 0$ weakly in $\dot{H}^2$. Thus, for any bounded function $a(x)$ decaying at infinity, we have $a(x)\rho_n,\ a(x)\nabla\rho_n,\ a(x)\nabla\theta_n \to 0$ strongly in L^2, since the restriction of $\rho_n, \nabla\rho_n, \nabla\theta_n$ to a bounded domain implies strong convergence. Thus, we have

$$\begin{aligned}
&\left\|\left(M_{11} - M_{11}^{\infty}\right)\rho_n\right\|_{L^2} \to 0,\\
&\left\|\left(M_{21} - c_0\partial_{x_1}\right)\rho_n\right\|_{L^2} = 2\left\|\nabla\cdot\left(\nabla\theta_{c_0}\rho_n\right)\right\|_{L^2} \to 0,\\
&\left\|\left(M_{12} + c_0\partial_{x_1}\right)\theta_n\right\|_{L^2} = 2\left\|\nabla\theta_{c_0}\cdot\nabla\theta_n\right\|_{L^2} \to 0,
\end{aligned}$$

and

$$\left\|\left(M_{21} - c_0\partial_{x_1}\right)\rho_n\right\|_{\dot{H}^{-1}} \leqq C\left\|\nabla\theta_{c_0}\rho_n\right\|_{L^2} \to 0.$$

This shows that $\left\|\left(M_{c_0} - M_{c_0}^{\infty}\right)\psi_n\right\|_Y \to 0$ and thus $\left\|M_{c_0}^{\infty}\psi_n\right\|_Y \to 0$. By Lemma 5.2 below, there exists $\eta > 0$ such that

$$\left\|M_{c_0}^{\infty}\psi_n\right\|_Y \geqq \eta\left\|\psi_n\right\|_{X_2} = \eta.$$

This contradiction proves the lemma. □

Lemma 5.2. *Assume* $0 < c_0 < \sqrt{2}$ *and* $\inf\rho_{c_0}(x) = \delta_0 > 0$. *Then there exists* $\eta > 0$ *such that*

$$\left\|M_{c_0}^{\infty}\psi\right\|_Y \geqq \eta\left\|\psi\right\|_{X_2}, \tag{96}$$

for any $\psi \in X_2^s$.

Proof. Take any $\psi = (\rho,\theta) \in X_2^s$. First, we estimate $\|\psi\|_{H^1\times\dot{H}^1}$ as in the proof of Lemma 2.9. Since $0 < c_0 < \sqrt{2}$, we can choose $0 < a_0 < 1$ such that $2 - \frac{c_0^2}{a_0^2} > 0$. Then,

$$\begin{aligned}
\left\langle M_{c_0}^{\infty}\psi,\psi\right\rangle &= \int_{\mathbf{R}^n}\left[\frac{1}{2\rho_{c_0}}|\nabla\rho|^2 + \frac{1}{\rho_{c_0}}\rho^2 - 2c_0\rho\partial_{x_1}\theta + 2\rho_{c_0}|\nabla\theta|^2\right]\mathrm{d}x \qquad (97)\\
&= \int_{\mathbf{R}^n}\left[\frac{1}{2\rho_{c_0}}|\nabla\rho|^2 + \frac{1}{\rho_{c_0}}\left(1 - a_0^2\right)\rho^2 + 2\rho_{c_0}\left|\nabla^{\perp}\theta\right|^2\right.\\
&\quad\left. + \left(2 - \frac{c_0^2}{a_0^2}\right)\rho_{c_0}\left|\partial_{x_1}\theta\right|^2 + \left(\frac{a_0\rho}{\sqrt{\rho_{c_0}}} - \frac{c_0}{a_0}\partial_{x_1}\theta\sqrt{\rho_{c_0}}\right)^2\right]\mathrm{d}x\\
&\geqq \eta_0\left(\|\rho\|_{H^1}^2 + \|\theta\|_{\dot{H}^1}^2\right), \quad \text{for some } \eta_0 > 0.
\end{aligned}$$

Therefore,

$$\eta_0 \left(\|\rho\|_{H^1}^2 + \|\theta\|_{\dot{H}^1}^2 \right) \leqq \left\| M_{c_0}^{\infty} \psi \right\|_{L^2 \times \dot{H}^{-1}} \left(\|\rho\|_{L^2} + \|\theta\|_{\dot{H}^1} \right),$$

and thus

$$\frac{1}{2} \eta_0 \left(\|\rho\|_{H^1} + \|\theta\|_{\dot{H}^1} \right) \leqq \left\| M_{c_0}^{\infty} \psi \right\|_{L^2 \times \dot{H}^{-1}}.$$

From the standard elliptic estimates, there exists $C > 0$ such that

$$\left\| \nabla^2 \rho \right\|_{L^2} + \left\| \nabla^2 \theta \right\|_{L^2} \leqq C \left(\|\rho\|_{H^1} + \|\theta\|_{\dot{H}^1} + \left\| M_{c_0}^{\infty} \psi \right\|_{L^2 \times L^2} \right).$$

Combining above two inequalities, we get (96). □

For dimension $n \geqq 4$, we need to study the equation in the function space of higher regularity. Choose $k > \frac{n}{2}$ such that $H^k(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$. Let $Y_k = H_{r_\perp}^{k-2} \times \left(H_{r_\perp}^{k-2} \cap \dot{H}_{r_\perp}^{-1} \right)$. We construct traveling waves near stationary bubbles by solving the equation $S(\rho + \rho_0, \theta; c) = 0$ in the space X_k^s. Assuming that $F \in C^{k-1}$, from Lemma 5.1, by bootstrapping we get the estimate

$$\left\| M_{c_0}(\rho, \theta) \right\|_{Y_k} \geqq \gamma \left\| (\rho, \theta) \right\|_{X_k},$$

and thus $M_{c_0}|_{K_{c_0}^{\perp} \cap X_k^s} : K_{c_0}^{\perp} \cap X_k^s \to K_{c_0}^{\perp} \cap Y_k$ is invertible. Then by the same proof of Theorem 5.2, we get the existence of slow traveling waves near ρ_0 for $n \geqq 4$ in the space X_k^s.

Remark 5.1. The two non-degeneracy conditions (93) and (56) are equivalent. This can be seen from the relation of operators M_{c_0} and L_{c_0}. For a traveling wave $U_{c_0} = \sqrt{\rho_{c_0}} e^{i\theta_{c_0}}$ with no vortices, denote the matrix operator

$$T_{c_0} = \begin{pmatrix} \frac{1}{2} \frac{1}{\sqrt{\rho_{c_0}}} \cos\theta_{c_0} & -\sqrt{\rho_{c_0}} \sin\theta_{c_0} \\ \frac{1}{2} \frac{1}{\sqrt{\rho_{c_0}}} \sin\theta_{c_0} & \sqrt{\rho_{c_0}} \cos\theta_{c_0} \end{pmatrix}. \tag{98}$$

Then

$$M_{c_0} = 2 T_{c_0}^t L_{c_0} T_{c_0}. \tag{99}$$

Since T_{c_0} is obviously an isomorphism of X, we have

$$T_{c_0}\left(\ker M_{c_0} \right) = \ker L_{c_0}$$

which implies the equivalence of (93) and (56). To show (99), we note that: 1) M_{c_0} and L_{c_0} are from the second variation of the energy-momentum functional $2(E + cP)$ in (ρ, θ) and $E + cP$ in (u, v) respectively and 2) the first order variations of (u, v) and (ρ, θ) are related by the matrix T_{c_0}.

Remark 5.2. The existence of slow traveling waves for cubic-quintic type equations was proved for $n \geqq 4$ in [48] by using the critical point theory, and for $n = 2, 3$ in an unpublished manuscript of LIN [45] by using the hydrodynamic formulation and Lyapunov–Schmidt reduction. The proof we give here adapts the formulation of [45], but it is much simpler and works for any dimension $n \geqq 2$. The new observation is to use the variational structure of the traveling wave equation (84) in hydrodynamic variables to reduce it to equation (86) which is solved by the implicit function theorem. Moreover, as a corollary of the proof we get the local uniqueness and differentiability of the traveling wave branch.

5.2. Continuation of Traveling Waves

By using Lemma 5.1 and the proof of Theorem 5.2, we get the following result on the continuation of traveling waves without vortices (that is $|U_c| \neq 0$):

Proposition 5.1. *For $n \geqq 2$, fix $k > \frac{n}{2}$ and assume $F \in C^{k-1}$, $0 < c_0 < \sqrt{2}$, $\left(\rho_{c_0}, \theta_{c_0}\right)$ is a cylindrically symmetric traveling wave of* (84) *satisfying* $\inf \rho_{c_0}(x) > 0$ *and the non-degeneracy condition* (93). *Then* $\exists\, \varepsilon_0 > 0$, *such that for*

$$c \in (-\varepsilon_0 + c_0, \varepsilon_0 + c_0) \subset \left(0, \sqrt{2}\right),$$

there exists a locally unique C^1 solution curve (ρ_c, θ_c) of (84), *with $\left(\rho_c - \rho_{c_0}, \theta_c\right) \in X_k^s$. That is,*

$$\phi^c\left(x_1 - ct, , x^\perp\right) = \left(\sqrt{\rho_c} e^{i\theta_c}\right)\left(x_1 - ct, , x^\perp\right)$$

are the only traveling wave solutions of (4) *near $\left(\rho_{c_0}, \theta_{c_0}\right)$.*

For $n = 3$, we can prove the continuation of general traveling waves even with vortices, under the non-degeneracy condition (56). Instead of using the hydrodynamic formulation, this is achieved by using the original equation (42). First, we need an analogue of Lemma 5.1. We still use $X,\ Y$ for the cylindrical symmetric spaces defined before.

Lemma 5.3. *For $n = 3$ and $0 \leqq c_0 < \sqrt{2}$, let $U_{c_0} = u_{c_0} + i v_{c_0}$ be a traveling wave solution of* (42) *satisfying the decay condition* (45). *Let $L_{c_0} : X_2^s \to Y$ be the operator defined in* (44). *Assume* (56), *that is,*

$$\ker L_{c_0} = \bar{K}_{c_0} = \left\{\partial_{x_1} U_{c_0}\right\}, \quad \text{on } X_2^s,$$

and denote $\bar{K}_{c_0}^\perp$ to be the L^2 orthogonal complement of $\bar{K}_{c_0}$ in Y. Then, there exists $\gamma > 0$, such that

$$\left\|L_{c_0}\phi\right\|_Y \geqq \gamma \left\|\phi\right\|_{X_2^s}, \quad \text{for any } \phi \in \bar{K}_{c_0}^\perp \cap X_2^s.$$

In particular, $L_{c_0} : \bar{K}_{c_0}^\perp \cap X_2^s \to \bar{K}_{c_0}^\perp$ is invertible and

$$\left\|\left(L_{c_0}|_{\bar{K}_{c_0}^\perp \cap X_2^s}\right)^{-1}\right\| \leqq \gamma^{-1}.$$

Proof. The proof is almost the same as that of Lemma 5.1. So we only point out some key points in the proof. For any sequence $\{\psi_n\} \in X_2^s$ with $\|\psi_n\|_{X_2} = 1$ and $\psi_n \to 0$ weakly in X_2^s, we show that

$$\left\| \left(L_{c_0} - L_{c0,\infty}\right) \psi_n \right\|_Y \to 0,$$

where

$$L_{c0,\infty} := \begin{pmatrix} -\Delta + 2 & -c_0 \partial_{x_1} \\ c_0 \partial_{x_1} & -\Delta \end{pmatrix}.$$

Let $L_{c_0} = \left(L^{ij}\right)$ and $L_{c0,\infty} = \left(L_\infty^{ij}\right)$, $i, j = 1, 2$, and $\psi_n = u_n + i v_n$. By (45),

$$L^{ij} - L_\infty^{ij} = a^{ij}(x) = o\left(\frac{1}{|x|}\right), \quad a^{22}(x) = o\left(\frac{1}{|x|^2}\right).$$

Then, since $u_n \to 0$ weakly in H^2,

$$\begin{aligned} \left\| \left(L^{11} - L_\infty^{11}\right) u_n \right\|_{L^2} &= \left\| a^{11}(x) u_n \right\|_{L^2} \to 0, \\ \left\| \left(L^{21} - L_\infty^{21}\right) u_n \right\|_{L^2} &= \left\| a^{21}(x) u_n \right\|_{L^2} \to 0, \\ \left\| \left(L^{21} - L_\infty^{21}\right) u_n \right\|_{\dot{H}^{-1}} &\leqq \left\| |x| \, a^{21}(x) u_n \right\|_{L^2} \to 0, \end{aligned}$$

by the local compactness of $H^2 \hookrightarrow L^2$. Since $v_n \to 0$ weakly in $\dot{H}^2$, we have

$$\begin{aligned} \left\| \left(L^{12} - L_\infty^{12}\right) v_n \right\|_{L^2} &= \left\| a^{12}(x) v_n \right\|_{L^2} \to 0, \\ \left\| \left(L^{22} - L_\infty^{22}\right) v_n \right\|_{L^2} &= \left\| a^{22}(x) v_n \right\|_{L^2} \to 0, \\ \left\| \left(L^{22} - L_\infty^{22}\right) v_n \right\|_{\dot{H}^{-1}} &\leqq \left\| |x| \, a^{22}(x) v_n \right\|_{L^2} \to 0, \end{aligned}$$

by the arguments in the proof of Lemma 2.5.

By Lemma 2.9, there exists $\eta_0 > 0$ such that

$$\left\langle L_{c0,\infty} \phi, \phi \right\rangle \geqq \eta_0 \|\phi\|^2_{H^1 \times \dot{H}^1}, \quad \text{for any } \phi \in H^1 \times \dot{H}^1.$$

Then by the same proof of Lemma 5.2, for some $\eta > 0$,

$$\left\| L_{c0,\infty} \phi \right\|_Y \geqq \eta \|\phi\|_{X_2}, \quad \text{for any } \phi \in X_2^s.$$

The rest of the proof is the same as Lemma 5.1. □

Theorem 5.3. *For* $0 < c_0 < \sqrt{2}$, *assume that* $U_{c_0} = \psi\left(w_{c_0}\right)$ *is a cylindrical symmetric 3D traveling wave solution of* (4) *satisfying the non-degeneracy condition* (56). *Then* $\exists\, \varepsilon_0 > 0$, *such that for*

$$c \in (-\varepsilon_0 + c_0, \varepsilon_0 + c_0) \subset \left(0, \sqrt{2}\right),$$

there exists a locally unique C^1 *solution curve* $U_c = \psi(w_c)$ *of* (42) *near* U_{c_0}, *where* $w_c \in H^2_{r_\perp}(\mathbf{R}^3, \mathbf{R}) \times \dot{H}^2_{r_\perp}(\mathbf{R}^3, \mathbf{R})$ *and* $\left\| w_c - w_{c_0} \right\| = O(|c - c_0|)$.

Proof. By Lemma 2.4, for a traveling wave solution $U_c = \psi(w_c)$, it is equivalent to solve $(\bar{E}_c)'(w_c) = 0$ for w_c. Define $K_{c_0} = \operatorname{span}\{\partial_{x_1} w_{c_0}\}$. Let $K_{c_0}^{\perp}$ be the L^2 orthogonal complement of K_{c_0} in Y, and $\Pi_{c_0}^{\perp} : Y \mapsto K_{c_0}^{\perp}$ be the orthogonal projection. Let $\tilde{K}_{c_0}^{\perp}$ be orthogonal complement of K_{c_0} in X_2^s, in the inner product $[\cdot,\cdot] = (K\cdot, K\cdot)$, where the operator

$$K\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1 - \chi(D)(v_c\phi_2)\\ \phi_2\end{pmatrix}$$

is defined in (22). We use the implicit function theorem to find solutions $(w_{c_0} + w, c)$ near (w_{c_0}, c_0) of the equation

$$\Pi_{c_0}^{\perp}\bar{E}_c{}'(w_{c_0} + w) = 0, \quad w \in \tilde{K}_{c_0}^{\perp}.$$

The linearized operator with respect to w of the left hand side above at (w_{c_0}, c_0) is

$$\Pi_{c_0}^{\perp}\bar{E}_{c_0}{}''(w_{c_0})|_{\tilde{K}_{c_0}^{\perp}} = \bar{E}_{c_0}{}''(w_{c_0})|_{\tilde{K}_{c_0}^{\perp}} : \tilde{K}_{c_0}^{\perp} \to K_{c_0}^{\perp},$$

which will be shown to be invertible below. In fact, by (30), we have

$$\bar{E}_{c_0}{}''(w_{c_0}) = K^* L_{c_0} K,$$

where K^* is given by (32). So by (56),

$$\ker \bar{E}_{c_0}{}''(w_{c_0}) = K_{c_0} = \{\partial_{x_1} w_{c_0}\}$$

and

$$\Pi_{c_0}^{\perp}\bar{E}_{c_0}{}''(w_{c_0})|_{\tilde{K}_{c_0}^{\perp}} = \bar{E}_{c_0}{}''(w_{c_0})|_{\tilde{K}_{c_0}^{\perp}}.$$

By the definition of $\tilde{K}_{c_0}^{\perp}$, $\phi \in \tilde{K}_{c_0}^{\perp}$ iff $K\phi \in \bar{K}_{c_0}^{\perp} \cap X_2^s$ where $\bar{K}_{c_0}$ is defined in Lemma 5.3. By Lemma 5.3, there exists $\gamma > 0$, such that

$$\left\|L_{c_0} K\phi\right\|_Y \geqq \gamma \left\|K\phi\right\|_{X_2}, \quad \text{for any } \phi \in \tilde{K}_{c_0}^{\perp}. \tag{100}$$

It is easy to show that the mappings $K : X_2^s \to X_2^s$ and $K^* : Y \to Y$ are isometric, that is, there exist $C_1, C_2 > 0$, such that

$$C_1 \left\|\phi\right\|_{X_2} \leqq \left\|K\phi\right\|_{X_2} \leqq C_2 \left\|\phi\right\|_{X_2}$$

and

$$C_1 \left\|\phi\right\|_Y \leqq \left\|K^*\phi\right\|_Y \leqq C_2 \left\|\phi\right\|_Y.$$

Thus by (100), there exists some $\gamma_1 > 0$, such that

$$\left\|K^* L_{c_0} K\phi\right\|_Y \geqq \gamma_1 \left\|\phi\right\|_{X_2}, \quad \text{for any } \phi \in \tilde{K}_{c_0}^{\perp}.$$

That is, the operator

$$\bar{E}_{c_0}{}''(w_{c_0})|_{\tilde{K}_{c_0}^{\perp}} = K^* L_{c_0} K\phi|_{\tilde{K}_{c_0}^{\perp}} : \tilde{K}_{c_0}^{\perp} \to K_{c_0}^{\perp}$$

has a bounded inverse. The rest of the proof is the same as that of Theorem 5.2, so we skip it. □

5.3. Instability of Slow Traveling Waves

In this subsection, we prove the instability of slow traveling waves constructed in Theorem 5.2. The approach is the same as developed in Section 3. The linearized equation is (7). To find unstable eigenvalues of JM_c, we first study the quadratic form $\langle M_c u, u\rangle$, where $u = (\rho, \theta) \in X_1^s$, the cylindrical symmetric subspace of $X_1 = H^1(\mathbf{R}^n) \times \dot{H}^1(\mathbf{R}^n)$.

Proposition 5.2. *Assume* $(H1)$–$(H3)$ *and the non-degeneracy condition* (83) *on the stationary bubble. Then* $\exists\, a_0 \in (0, \sqrt{2})$, *such that for any* $0 \leqq c < a_0$, *there exists a traveling wave solution* $U_c = \sqrt{\rho_c}e^{i\theta_c}$ *of* (4) *without vortices and* (56) *is satisfied. Moreover, the space* X_1^s *can be decomposed as a direct sum*

$$X_1^s = N \oplus Z \oplus P,$$

where $Z = \left\{\left(\partial_{x_1}\rho_c, \partial_{x_1}\theta_c\right)\right\}$, N *is a one-dimensional subspace such that* $\langle M_c u, u\rangle < 0$ *for* $0 \neq u \in N$, *and* P *is a closed subspace such that*

$$\langle M_c u, u\rangle \geqq \delta \left\|u\right\|_{X_1}^2, \quad \forall\, u \in P,$$

for some constant $\delta > 0$.

Proof. The proof is similar to that of Proposition 2.1, so we only sketch it. The existence of traveling waves is shown in Theorem 5.2, for $c \in [0, b_0),\ b_0 > 0$. Define the operator

$$\tilde{M}_c := \tilde{G} \circ M_c \circ \tilde{G} : L^2_{r_\perp} \to L^2_{r_\perp},$$

where $\tilde{G}$ is defined in (26). We will show that there exists $a_0 > 0$, such that when $c \in [0, a_0)$:

(i) $\tilde{M}_c : L^2 \to L^2$ is self-adjoint and bounded.

(ii) $\tilde{L}_c$ has one-dimensional cylindrical symmetric negative eigenspace,

$$\ker \tilde{M}_c \cap L^2_{r_\perp} = \left\{\tilde{G}^{-1}\left(\partial_{x_1}\rho_c, \partial_{x_1}\theta_c\right)\right\},$$

and the rest of the spectrum is positive.

The conclusions in the Proposition follow from the above properties of the operator $\tilde{M}_c$. Denote $\tilde{M}_c^\infty := \tilde{G} \circ M_c^\infty \circ \tilde{G}$, where M_c^∞ is defined in (95). Then it is easy to see that $\tilde{M}_c^\infty$ is bounded and self-adjoint, and by the estimate (97), the essential spectrum of $\tilde{M}_c^\infty \subset [\delta_0, \infty)$ for some $\delta_0 > 0$. We show that $\tilde{M}_c$ is a compact perturbation of $\tilde{M}_c^\infty$. Indeed, $\tilde{M}_c - \tilde{M}_c^\infty = (\tilde{M}_{ij})$, where $M_{22} = 0$,

$$\begin{aligned}
\tilde{M}_{11} &= (-\Delta + 1)^{-\frac{1}{2}} a_1(x) (-\Delta + 1)^{-\frac{1}{2}},\\
\tilde{M}_{21} &= -2(-\Delta)^{-\frac{1}{2}} \nabla \cdot \left(\vec{a}_2(x) (-\Delta + 1)^{-\frac{1}{2}}\right), \quad \tilde{M}_{12} = \tilde{M}_{21}^*,
\end{aligned}$$

with

$$\begin{aligned}
a_1(x) &= \frac{1}{2}\frac{1}{\rho_c^3}\left|\nabla\rho_{c_0}\right|^2 - \frac{\Delta\rho_{c_0}}{2\rho_{c_0}^2} - F'(\rho_{c_0}) - \frac{1}{\rho_{c_0}} \to 0,\\
\vec{a}_2(x) &= \nabla\theta_{c_0} \to 0,
\end{aligned}$$

when $|x| \to \infty$. Thus by the local compactness of $H^1 \hookrightarrow L^2$, the operators $\tilde{M}_{11}$, $\tilde{M}_{21}$ and $\tilde{M}_{12}$ are compact, so by the perturbation theory of self-adjoint operators, $\tilde{M}_c$ is self-adjoint and bounded, with its essential spectrum in $[\delta_0, \infty)$. By Lemma 5.4 below, $\ker \tilde{M}_0 \cap L^2_{r_\perp} = \{\tilde{G}^{-1}(\partial_{x_1}\rho_0, 0)\}$ and $\tilde{M}_0$ has exactly one negative eigenvalue which is simple with radially symmetric eigenfunction. Since the traveling wave solution (ρ_c, θ_c) is C^1 for $c \in [-b_0, b_0)$, the discrete spectrum of $\tilde{M}_c$ is continuous in c. Thus, there exists $a_0 > 0$ such that for $c \in [0, a_0)$, $\tilde{M}_c$ has a one-dimensional kernel in $L^2_{r_\perp}$ spanned by $\tilde{G}^{-1}\left(\partial_{x_1}\rho_c, \partial_{x_1}\theta_c\right)$ and exactly one negative eigenvalue which is simple with a cylindrically symmetric eigenfunction. □

Lemma 5.4. *Assume* (*H*1)–(*H*3). *Let* $\phi_0 = \sqrt{\rho_0}$ *be a stationary bubble of* (4) *obtained in* [24] (*via constrained minimization*) *satisfying* (83). *Then*

$$\ker \tilde{M}_0 \cap L^2_{r_\perp} = span\left\{\tilde{G}^{-1}\left(\partial_{x_1}\rho_0, 0\right)\right\},$$

and $\tilde{M}_0$ *has exactly one negative eigenvalue which is simple and with a radially symmetric eigenfunction.*

Proof. Since $\tilde{M}_0 = \tilde{G} \circ M_0 \circ \tilde{G}$, it is equivalent to show that

$$\ker M_0 \cap L^2_{r_\perp} = \left\{\left(\partial_{x_1}\rho_0, 0\right)\right\},$$

and M_0 has only one negative eigenvalue which is simple with a radially symmetric eigenfunction. Recall that

$$M_0 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where

$$M_1 = A\left(\frac{1}{2\sqrt{\rho_0}}\cdot\right)\frac{1}{\sqrt{\rho_0}}, \quad M_2 = -2\nabla\cdot(\rho_0\nabla).$$

Since $M_2 > 0$, by (83) the cylindrically symmetric kernel of M_1 is spanned by $\partial_{x_1}\rho_0$. It remains to show that the operator A has only one negative eigenvalue which is simple with a radially symmetric eigenfunction. This property was shown in Lemmas 2.1 and 2.2 of [48] for $n \geqq 3$. The proof for $n = 2$ is almost the same and we sketch it below. By Lemma 3.3 of [24] or Lemma 2.1 of [48], A has at least one negative eigenvalue with radial symmetric eigenfunction. It was shown in [2,9] that ϕ_0 minimizes the functional

$$T(u) = \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 \, dx$$

subject to the constraint

$$I(u) = \int_{\mathbf{R}^2} V\left(|u|^2\right) dx = 0.$$

By the arguments in the proof of Lemma 2.7 or the proof of Lemma 2.2 in [48], it follows that A has at most one-dimensional negative eigenspace. Thus A has exactly one-dimensional negative eigenspace. □

Remark 5.3. In Appendix 2, we prove the non-degeneracy condition (83) for cubic-quintic nonlinearity and $n = 2$. For $n \geqq 3$, the condition (83) was proved in [48] for nonlinearity satisfying some additional condition ((H5) in P. 1209 of [48]). However, our computation indicates that this additional condition appears to be not satisfied by the cubic-quintic nonlinearity.

Much as Proposition 3.2 and Lemma 3.2, we have the same linear instability criterion $\frac{\partial P(U_c)}{\partial c} < 0$ and the subsequent linear exponential dichotomy.

Proposition 5.3. *Let* $U_c = \sqrt{\rho_c} e^{i\theta_c}$, $c \in [c_1, c_2]$, *be a* C^1 (*with respect to the wave speed* c) *family of traveling waves of* (4). *For* $c_0 \in (c_1, c_2)$, *assume that*:

1. $U_{c_0}(x) \neq 0$, *for all* $x \in \mathbf{R}^n$;
2. *non-degeneracy condition* (93) *is satisfied*;
3. $F \in C^1$ *on* $U_{c_0}(\mathbf{R}^n)$;
4. M_{c_0} *satisfies the decomposition result stated in Proposition* 5.2, *and*
5. $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$;

then there exists $w_u \in X_1^s$ *and* $\lambda_u > 0$, *such that* $e^{\lambda_u t} w_u(x)$ *is a solution of* (7). *Moreover*, *the linearized semigroup* $e^{tJM_{c_0}}$ *also has an exponential dichotomy in the space* X_3^s.

After these preparations, we show the linear instability of slow traveling waves of (4) with cubic-quintic type nonlinear terms.

Theorem 5.4. *Assume* (*H*1)–(*H*3) *and* (83). *For any* $n \geqq 2$, $\exists\, \varepsilon_0 > 0$, *such that for all* $0 \leqq c < \varepsilon_0$, *the traveling wave solutions* $U_c = \sqrt{\rho_c} e^{i\theta_c}$ *constructed in Theorem* 5.2 *are linearly unstable in the following sense: there exists an unstable solution* $e^{\lambda_u t} w_u(x)$ *with*

$$w_u = (\rho_u, \theta_u) \in X_3^s, \quad \lambda_u > 0,$$

of the linearized equation (7). *Moreover*, *the linearized semigroup* e^{tJM_c} *also has an exponential dichotomy in the space* X_3^s.

Proof. Since the traveling wave branch U_c constructed in Theorem 5.2 is C^1 for $c \in (-b_0, b_0)$, according to Proposition 5.3, it is reduced to show that $\frac{\partial P(U_c)}{\partial c}|_{c=0} < 0$. We note that

$$\frac{1}{2} M_c \partial_c(\rho_c, \theta_c) = -P'(\rho_c, \theta_c) = -\frac{1}{2} J^{-1} \partial_{x_1}(\rho_c, \theta_c),$$

since (ρ_c, θ_c) satisfies $\left(E' + cP'\right)(\rho_c, \theta_c) = 0$ and $\left(E'' + cP''\right)(\rho_c, \theta_c) = \frac{1}{2} M_c$. Here, we use $'$ to denote the functional derivative in (ρ, θ). Thus

$$\begin{aligned}\frac{\partial P(U_c)}{\partial c}|_{c=0} &= \langle P'(\rho_c, \theta_c), \partial_c(\rho_c, \theta_c)\rangle|_{c=0} \\ &= \frac{1}{2}\left(\partial_c(\rho_c, \theta_c), \left(0, \partial_{x_1}\rho_0\right)\right) = -\frac{1}{2}\left(M_2^{-1}\partial_{x_1}\rho_0, \partial_{x_1}\rho_0\right) < 0\end{aligned}$$

since $M_2 = -2\nabla.(\rho_0 \nabla) > 0$.

By Proposition 5.2, we can show the exponential dichotomy for the semigroup e^{tJM_c} in the space X_3^s, as in Lemma 3.2. The proof is the same as in Lemmas 3.1 and 3.2, thus we skip it. □

Due to the presence of derivative terms in the nonlinearity of the hydrodynamic equation (5), it is much easier to obtain the unstable manifolds and thus the nonlinear instability of traveling waves by working with the original form of the nonlinear Schrödinger equation (4), which is semilinear, based on the linear instability obtained in the above theorems. We first state the following proposition:

Proposition 5.4. *For any dimension $n \geqq 1$, let $k > \frac{n}{2}$ be an integer and $\Omega_\perp \subset \mathbf{R}^{n-1}$ be a smooth domain. Consider (4) for $x \in \Omega = \mathbf{R} \times \Omega_\perp$ subject to homogeneous Dirichlet, Neumann, or periodic boundary condition on $\partial\Omega$. Suppose $F \in C^{k+l}$ and $U_c = u_c + i v_c$ be a traveling wave of (4) on Ω such that*

$$u_c - 1 \in H^k(\Omega) \quad \text{and} \quad v_c \in \dot{H}^k(\Omega).$$

Assume the linearized flow e^{tJL_c}, where L_c is defined in (44), has an exponential dichotomy in $H^k(\Omega) \times H^k(\Omega)$, that is there exist closed subspaces $E^{cs,u} \subset H^k(\Omega) \times H^k(\Omega)$ and $\lambda_{u,cs}$, $M \geqq 0$ such that $\lambda_u > \lambda_{cs}$, $e^{tJL_c} E^{u,cs} = E^{u,cs}$, and

$$\left| e^{tJL_c}|_{E^u} \right| \leqq M e^{\lambda_u t}, \quad \forall t \leqq 0 \quad \text{and} \quad \left| e^{tJL_c}|_{E^{cs}} \right| \leqq M e^{\lambda_{cs} t}, \quad \forall t \geqq 0.$$

Then there exists a unique C^l locally invariant unstable manifold $W^u \subset U_c + H^k(\Omega) \times H^k(\Omega)$ of U_c in the sense as described in Theorem 3.1.

Remark 5.4. If $\Omega_\perp$ and U_c are invariant under certain symmetries like rotations or reflections, then one may work in the subspace of $H^k(\Omega)$ with the same symmetries and thus the unstable manifold would consist of functions in $U_c + H^k(\Omega) \times H^k(\Omega)$ with the same symmetries.

The proposition can be obtained from the general theorems in [7,22] simply based on the observation that, under the above assumptions, the dynamic equation of $z(t,x) = U(t, x - ct\vec{\mathbf{e}}_1) - U_c(x - ct\vec{\mathbf{e}}_1)$, where $U(t,x)$ is a solution of (4), has the linear part $JL_c z$ and its nonlinear part defines a C^l transformation from $H^k(\Omega) \times H^k(\Omega)$ to itself. As a corollary, we have

Corollary 5.1. *Let $n \geqq 2$ and $U_c = \sqrt{\rho_c} e^{i\theta_c}$ be a traveling wave of (4), radially symmetric in $x_\perp$ and linearly unstable with the linear exponential dichotomy for $(\rho, \theta) \in H^k_{r_\perp}(\mathbf{R}^n) \times \dot{H}^k_{r_\perp}(\mathbf{R}^n)$ $\left(k \geqq 2\right)$, including those proved in Theorem 5.4. Assume $F \in C^{k+l}$, then there exists a unique C^l local unstable (stable) manifolds W^u (W^s) of U_c in $U_c + H^k_{r_\perp}(\mathbf{R}^n) \times H^k_{r_\perp}(\mathbf{R}^n)$.*

Proof. In order to prove the corollary, we only need to establish the linear exponential dichotomy of e^{tJL_c} in $H^k_{r_\perp} \times H^k_{r_\perp}$. Since JL_c and JM_c, where M_c is defined in (90), are conjugate through T_c defined in (98) and T_c is an isomorphism on $H^k_{r_\perp} \times H^k_{r_\perp}$, we only need to obtain the exponential dichotomy of e^{tJM_c} on $H^k_{r_\perp} \times H^k_{r_\perp}$. Based on Theorem 5.4, it is straightforward to repeat the arguments to derive the exponential dichotomy of e^{tJM_c} in $H^k_{r_\perp} \times \dot{H}^k_{r_\perp}$. The form of JM_c implies that its unstable and stable eigenfunctions in $H^k_{r_\perp} \times \dot{H}^k_{r_\perp}$ actually belong to $H^k_{r_\perp} \times H^k_{r_\perp}$, that is, $E^u, E^s \subset H^k_{r_\perp} \times H^k_{r_\perp}$. Indeed, for an unstable eigenvalue $\lambda_u > 0$, the eigenfunction $(\rho_u, \theta_u) \in H^k_{r_\perp} \times \dot{H}^k_{r_\perp}$ satisfies

$$\theta_u = -\frac{1}{\lambda_u}\left(M_{11}\rho_u + M_{12}\theta_u\right) \in L^2,$$

and the same is true for the stable eigenfunction. Here, M_{11} and M_{12} are defined in (91)–(92) and we use the observation that $M_{12}\theta$ contains only $\partial\theta(t)$, instead of $\theta(t)$ itself. It is easy to see that $\tilde{E}^{cs} = E^{cs} \cap H^k_{r_\perp} \times H^k_{r_\perp}$ is a closed subspace of $H^k_{r_\perp} \times H^k_{r_\perp}$, invariant under e^{tJM_c}, and $H^k_{r_\perp} \times H^k_{r_\perp} = E^u \oplus \tilde{E}^{cs}$. To complete the proof, we only need to obtain the following growth estimate of $|\theta(t)|_{L^2}$ where $z(t) = (\rho(t), \theta(t)) = e^{tJM_c}z^0$ for $z^0 \in \tilde{E}^{cs}$. When $z^0 = (\rho(0), \theta(0)) \in \tilde{E}^{cs}$, we have that for any $t \geqq 0$,

$$\begin{aligned}
|\theta(t)|_{L^2} &\leqq |\theta(0)|_{L^2} + \int_0^t |\theta_t(s)|\ \mathrm{d}s \\
&\leqq |\theta(0)|_{L^2} + \int_0^t \left(|M_{11}\rho(s)|_{L^2} + |M_{12}\theta(s)|_{L^2}\right) \mathrm{d}s \\
&\leqq |\theta(0)|_{L^2} + C\int_0^t \left(|\rho(s)|_{H^2} + |\theta(s)|_{\dot{H}^1}\right) \mathrm{d}s \\
&\leqq |\theta(0)|_{L^2} + C\int_0^t Me^{\lambda_{cs}s}|z^0|_{H^k \times \dot{H}^k}\ \mathrm{d}s \\
&\leqq M'(1+t)e^{\lambda_{cs}t}|z^0|_{H^k \times H^k}.
\end{aligned}$$

In the above, we use the special form of M_{12} again and the exponential dichotomy of the linear equation $z_t(t) = JM_cz(t)$ in $H^k_{r_\perp} \times \dot{H}^k_{r_\perp}$. Therefore the exponential dichotomy of e^{tJM_c} holds in $H^k_{r_\perp} \times H^k_{r_\perp}$ with λ_u and any $\tilde{\lambda}_{cs} \in (\lambda_{cs}, \lambda_u)$. □

As a corollary, we get nonlinear orbital instability for slow traveling waves in dimension $n \geqq 2$.

Remark 5.5. The instability of stationary bubbles was proved in [24]. It is possible to show the instability of slow traveling waves by certain perturbation argument. However, the instability proof of Theorem 5.4 contains more information than what can be obtained from a perturbation theory. First, it yields the exponential dichotomy of the semigroup which is essential for constructing invariant manifolds. The unstable manifold theorem automatically implies the optimal orbital instability result, that is, the instability is measured in a weak norm with initial deviation in a strong norm and the growth is exponentially fast. By contrast, the nonlinear instability proof in [24] used an abstract theorem of [37] (see [54] for a similar theorem). It does not require the exponential dichotomy or the precise growth estimate of the semigroup, but the instability was proved in a strong norm H^k $\left(k > \frac{n}{2}\right)$ and no estimate of the growth time scale was given. Second, the proof of Theorem 5.4 actually gives a instability criterion $\frac{\partial P(U_c)}{\partial c} < 0$ under the non-degeneracy condition (56). In particular, the instability of stationary solution persists until the first travel speed c at which either $\frac{\partial P(U_c)}{\partial c} = 0$ or the condition (56) fails.

6. Extensions and Future Problems

In this section, we discuss the extensions of the results in previous sections. We also mention some remaining issues on stability of traveling waves of (4).

In the one-dimensional case, when $0 < c < \sqrt{2}$, the traveling waves U_c of (4) is nonvanishing. In this case, a sharp stability criterion was obtained in [44] by using the hydrodynamic formulation and the theory of [32]. The traveling waves are stable if and only if $\frac{\mathrm{d}}{\mathrm{d}c} P\left(U_c\right) > 0$, where

$$P\left(u\right) = -\int_{\mathbf{R}} \operatorname{Im}\left(\bar{u}u'\right)\left(1 - \frac{1}{|u|^2}\right) \mathrm{d}x.$$

However, in [44] nonlinear instability was only proved in the energy space and without any estimate of the growth time scale, as in [16,32], where the linear instability problem was bypassed and the nonlinear instability was proved by a contradiction argument. Recently, the nonlinear orbital instability with exponential growth was proved in [19] by studying the linearized problem. In Remark 6.1 below, we comment on some possible gap in the proof of [19]. By using the methods in Sections 3 and 5.3, we can prove the existence of unstable (stable) manifolds near U_c and thus obtain the optimal nonlinear instability result when $\frac{\mathrm{d}}{\mathrm{d}c} P\left(U_c\right) < 0$. In fact, for 1D traveling waves, the spectral property of the quadratic form $\langle M_c \cdot, \cdot\rangle$ as in Proposition 5.2 was essentially proved in [44]. By using this spectral property, the linear instability when $\frac{\mathrm{d}}{\mathrm{d}c} P\left(U_c\right) < 0$ and the exponential dichotomy of the linear semigroup e^{tJM_c} can be proved by the same approach as in Section 3. Then the unstable (stable) manifolds can be constructed via Theorem 5.4 as in corollary 5.1.

Another extension is to prove the transversal instability of any 1D traveling waves of (4) by the approach of Section 4. Let

$$L_c := \begin{pmatrix} -\partial_x^2 - F\left(|U_c|^2\right) - F'\left(|U_c|^2\right) 2u_c^2 & -c\partial_x - 2F'\left(|U_c|^2\right) u_c v_c \\ c\partial_x - 2F'\left(|U_c|^2\right) u_c v_c & -\partial_x^2 - F\left(|U_c|^2\right) - F'\left(|U_c|^2\right) 2v_c^2 \end{pmatrix}. \tag{101}$$

When $0 < c < \sqrt{2}$, the operator L_c has exactly one negative eigenvalue $\lambda_0 < 0$ and the second eigenvalue is zero. Since by the analogue of the formula (99) in 1D, the number of negative modes of the quadratic form with L_c equals that of the quadratic form $\langle M_c \cdot, \cdot\rangle$. The latter number was shown to be the one in [44]. Thus, by the same proof of Theorem 4.1, when $k \in \left(0, \sqrt{-\lambda_0}\right)$ the 1D traveling wave is transversal unstable with the transversal period $\frac{2\pi}{k}$ and is transversely stable when $k \geqq \sqrt{-\lambda_0}$. Similar to the proof of Lemma 4.1 and Theorem 4.2, we can construct unstable (stable) manifolds from the above transversal instability. For (GP), when $c = 0$, the stationary solution $U_0 = \tanh\left(\frac{x}{\sqrt{2}}\right)$ vanishes at $x = 0$, the operator

$$L_0 = \begin{pmatrix} -\partial_x^2 - 1 + 3U_0^2 & 0 \\ 0 & -\partial_x^2 - 1 + U_0^2 \end{pmatrix}$$

can be verified to have exactly one negative eigenvalue, so the above discussions on transversal instability are also valid for U_0. We should note that in [53], the linear transversal instability of 1D traveling waves of (GP) was shown for k near $\sqrt{-\lambda_0}$ by a different method. Our results are novel in the following two respects: 1) locating the sharp interval for unstable transversal wave numbers; 2) constructing the unstable (stable) manifolds.

Remark 6.1. In [19], the proof of nonlinear orbital instability of 1D traveling waves (Theorem 4 and Corollary 3) consists of several steps. For linear instability when $\frac{\mathrm{d}}{\mathrm{d}c} P(U_c) < 0$, the author cited the result in [8]. Then, an abstract result (Theorem B.3 in [19]) is used to get the growth estimate (and actually exponential dichotomy) of the semigroup e^{tJL_c} in $H^1(\mathbf{R}) \times H^1(\mathbf{R})$, where the operator L_c is defined in (101). By using this semigroup estimate, the nonlinear orbital instability follows since the equation (4) is semilinear in $H^1 \times H^1$. However, the operator L_c does not satisfy Assumption (A) for Theorem B.3, which requires that $\sigma_{\text{ess}}(L_c) = [\delta_0, +\infty)$ for some $\delta_0 > 0$. Since by (99) we have $\sigma_{\text{ess}}(L_c) = \sigma_{\text{ess}}(M_c) = [0, +\infty)$, where M_c is the 1D version of the operator defined in (90). Such a lack of spectral gap is exactly one of the main difficulty for studying stability of traveling waves of (4) with nonvanishing condition at infinity. To overcome this difficulty in 1D, first we get the exponential dichotomy of e^{tJM_c} on $H^k \times \dot{H}^k$ $\left(k \geqq 2\right)$ where M_c has a spectral gap. Then we lift this exponential dichotomy of e^{tJM_c} to $H^k \times H^k$ by noting that the unstable eigenfunction lies in this space. The exponential dichotomy of e^{tJL_c} on $H^k \times H^k$ then follows.

Theorems 2.1, 3.1 and Corollary 2.2 in Sections 2 and 3 give a completed theory for the orbital stability (instability) of 3D traveling waves. We briefly comment on the extensions to higher dimensions $n \geqq 4$. Assume that the nonlinear term $F(u)$ in (4) satisfies:

(F1) $F \in C^1(\mathbf{R})$, C^2 in a neighborhood of 1, $F(1) = 0$ and $F'(1) = -1$;

(F2) there exists $0 \leqq p_1 \leqq 1 \leqq p_0 < \frac{2}{n-2}$ such that $|F'(s)| \leqq C(1 + s^{p_1-1} + s^{p_0-1})$ for all $s \geqq 0$.

Under the assumption (F1)-(F2), as in 3D case, the traveling waves have been constructed in [49] by minimizing the energy subject to Pohozaev type constraint, for $n \geqq 4$. We can prove the spectral property for the quadratic form $\langle L_c \cdot, \cdot \rangle$ as in Proposition 2.1, in the space $X_1 = H^1(\mathbf{R}^n) \times \dot{H}^1(\mathbf{R}^n)$. This allows us to prove a linear instability criterion that $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} < 0$ where

$$P(u) = \frac{1}{2} \int_{\mathbf{R}^n} \langle i \partial_{x_1} u, u - 1 \rangle \, \mathrm{d}x = -\int_{\mathbf{R}^n} (u_1 - 1) \, \partial_{x_1} u_2 \, \mathrm{d}x.$$

To pass to nonlinear results, first we note that when $n \geqq 4$, it was shown in [48] that the energy-momentum functional $E_c = E + cP$ is C^2 on the space $1 + X_1$. Thus, by the proof of Theorem 2.1, we can prove that the traveling waves with $\frac{\partial P(U_c)}{\partial c}|_{c=c_0} > 0$ is orbitally stable in the distance $\|u - U_c\|_{X_1}$. Moreover, when $n = 4$, it can be shown that the energy space

$$\begin{aligned} X_0 &= \left\{ u \, | \nabla u \in L^2\left(\mathbf{R}^n\right), \ V(u) \in L^1\left(\mathbf{R}^n\right) \right\} \\ &= \left\{ u \, | \nabla u \in L^2\left(\mathbf{R}^n\right), \ 1 - |u|^2 \in L^1\left(\mathbf{R}^n\right) \right\} \end{aligned}$$

exactly consists of functions of the form $\{c\,(1+w)\mid c\in\mathbb{S}^1,\ w\in X_1\}$. Thus, when $n=4$, $\frac{\partial P(U_c)}{\partial c}|_{c=c_0}>0$ is sharp for orbital stability in the energy space. When $n>4$, the energy space X_0 might be strictly larger than the set $\{c\,(1+X_1)\}$. To show the orbital stability in X_0 for $n>4$, we need to find a coordinate mapping $u=g\,(w)\,,w\in X_1$ for $u\in X_0$, as for the 3D case. To construct unstable (stable) manifolds under the instability criterion $\frac{\partial P(U_c)}{\partial c}|_{c=c_0}<0$, we first note that the exponential dichotomy is still true in X_1 and then in $X_k=H^k\times\dot{H}^k$ for any integer $k>1$, by the proof of Lemmas 3.1 and 3.2. Then we write $U=U_c+w$ $(w\in X_k)$ in (46). It can be shown that, assuming $F\in C^{k+2}$, the nonlinear term $F\left(|U|^2\right)U\in C^2\left(X_k,X_k\right)$ for k large. Therefore, the unstable (stable) manifolds can be constructed in the space U_c+X_k, which is contained in the energy space by [48] as mentioned earlier.

We notice that equation (GP) for $n=4$ is just the borderline case and does not satisfy (F2) for nonlinear stability (keep in mind (F1–2) are not needed for unstable manifolds). Exactly as in Corollary 2.2, we may instead assume (F); there exists $C,\alpha_0,s_0>0$, and $0<p_1\leqq 1\leqq p_0\leqq\frac{2}{n-2}$ such that $|F'(s)|\leqq C(1+s^{p_1-1}+s^{p_0-1})$ for all $s\geqq 0$ and $F(s)\leqq -Cs^{\alpha_0}$ for all $s>s_0$.

Following the same argument for Corollary 2.2, we obtain the nonlinear instability of traveling waves obtained in [49].

Now we discuss the 2D case. When the traveling wave U_c has no vortices ($U_c\neq 0$), we can use the Madelung transform to derive the instability criterion $\frac{\partial P(U_c)}{\partial c}|_{c=c_0}<0$ and construct stable (unstable) manifolds. See Theorems 5.4 and 5.1. However, things get more tricky for traveling waves with vortices. The Madelung transform is not applicable. Also, it is improper to use the base space X_1 to study the linearized problem (47) for two reasons. First, the Hardy's inequality (23) is not valid for $n=2$, so we cannot even define the quadratic form $\langle L_c\cdot,\cdot\rangle$ on X_1. Secondly, due to the oscillations at infinity of functions in X_0, we don't have a manifold structure of X_0 with the base X_1. In [30], a manifold structure of X_0 is given with the base space

$$X'=H^1_{\mathbf{R}}\left(\mathbf{R}^2\right)\times\left(X^1_{\mathbf{R}}\left(\mathbf{R}^2\right)+H^1_{\mathbf{R}}\left(\mathbf{R}^2\right)\right),$$

where $X^1_{\mathbf{R}}=\left\{u\in L^\infty\mid\nabla u\in L^2\right\}$. However, it is unclear how to use X' in place of X_1 in our approach.

In an ongoing work, we construct center manifolds near the unstable 3D traveling waves of (GP) as proved in Proposition 3.3. The linear exponential trichotomy of the semigroup e^{tJL_c} has been established for the space X_1 in Lemma 3.3. This trichotomy can also be lifted to the space X_3, similar to Lemma 3.2. Then we can construct center manifold in the orbital neighborhood of w_c in X_3. However, it is more desirable to construct center manifold in the energy space X_1. We note that in Lemma 3.3 it is shown that the second variation of the energy-momentum is positive definite when restricted on the center space and modulo the generalized kernel, so the construction of the center manifold in X_1 would imply that the orbital stability is restricted there and also the local uniqueness of the center manifold. Together with the unstable (stable) manifolds in Theorem 3.1, these will give a foliation of the local dynamics near the unstable 3D traveling waves.

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Appendix 1

In this appendix, we prove the C^2 smoothness of the nonlinear term $\Psi(w)$ (defined in (52) and (53)) on X_3. Indeed, we will show that $\Psi \in C^2\left(X_3, H^3\left(\mathbf{R}^3; \mathbf{C}\right)\right)$. Let $U_c = u_c + iv_c$ be a finite energy traveling wave solution of Equation (4), that is, $|U_c|^2 - 1,\ \nabla U_c \in L^2(\mathbf{R}^3)$. Let $U_c = \psi(w_c)$, then $w_c \in X_1 = H^1 \times \dot{H}^1$. Moreover, by the proof of Lemma 5.5 of [49], $u_c - 1, v_c \in \dot{H}^3(\mathbf{R}^3)$, so it follows from the definition of the coordinate mapping ψ that

$$w_{1c} \in H^3(\mathbf{R}^3), \quad w_{2c} \in \dot{H}^3(\mathbf{R}^3).$$

Lemma 7.1. *Assume that $F \in C^5(\mathbf{R})$ and $F(1) = 0$. Then $\Psi \in C^2(X_3, H^3)$.*

Remark 7.1. For the construction of unstable (stable) manifolds, we work on a small X_3 neighborhood of a linearly unstable traveling wave U_c. By Sobolev embedding, $X_3 \hookrightarrow L^\infty\left(\mathbf{R}^3\right)$, so we only need to assume the smoothness of F in a finite interval $\left[\min|U_c|^2 - \varepsilon_0, \max|U_c|^2 + \varepsilon_0\right]$ for some $\varepsilon_0 > 0$. In particular, for traveling waves with no vortices ($U_c \neq 0$), we do not need to assume the smoothness of $F(s)$ near $s = 0$.

Proof. In the sequel, let $C(\|w\|_{X_3})$ be a constant depending (increasingly) on $\|w\|_{X_3}$. In the proof, we will use the following basic facts:
(i) Let $n \in \mathbb{N}$, $F \in C^{3+n}(\mathbf{R})$ with $F(0) = 0$, and $g \in H^3(\mathbf{R}^3)$. Then

$$F(g) \in H^3(\mathbf{R}^3) \quad \text{and} \quad F \in C^n\left(H^3, H^3\right); \tag{102}$$

(ii)

$$\|fg\|_{H^3} \leqq C\|f\|_{H^3}\|g\|_{\dot{H}^3}, \quad \forall\, f \in H^3(\mathbf{R}^3),\ g \in \dot{H}^3(\mathbf{R}^3); \tag{103}$$

(iii)

$$\|fg\|_{\dot{H}^3} \leqq C\|f\|_{\dot{H}^3}\|g\|_{\dot{H}^3}, \quad \forall\, f, g \in \dot{H}^3(\mathbf{R}^3); \tag{104}$$

(iv) Let $\chi \in C_0^\infty(\mathbf{R}^3, [0, 1])$, then $\forall\, f, g \in \dot{H}^3(\mathbf{R}^3)$,

$$\|\Delta\chi(D)(fg)\|_{H^3} \leqq C\|(|\xi| + |\xi|^3)\widehat{fg}\|_{L^2} \leqq C\|f\|_{\dot{H}^3}\|g\|_{\dot{H}^3} \tag{105}$$

and

$$\|D^\alpha\chi(D)(fg)\|_{\dot{H}^3} \leqq C\|f\|_{\dot{H}^3}\|g\|_{\dot{H}^3}, \quad \forall\, 0 \leqq |\alpha| < +\infty. \tag{106}$$

Here, (i) is by Moser's composition inequality, (ii)-(iv) can be shown by Sobolev embedding and Fourier transforms.

Step 1. Show $\operatorname{Re}\Psi(w) \in C^2(X_3, H^3)$.
Let $\tilde{F}(s) = F(s+1)$, then $\tilde{F} \in C^5(\mathbf{R})$ and $\tilde{F}(0) = 0$. Denote $U = \psi(w + w_c)$, we write

$$\begin{aligned} Re\Psi(w) &= \Delta\chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right) + [F(|U_c|^2) - F(|U|^2)]u_c \\ &\quad - F(|U|^2)\left[w_1 - \chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right)\right] \\ &= F(|U_c|^2)u_c + \Delta\chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right) \\ &\quad - F(|U|^2)\left[u_c + w_1 - \chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right)\right]. \\ &= \tilde{F}(|U_c|^2 - 1)u_c + \Psi_1(w) + \tilde{F}(\Psi_2(w + w_c))\Psi_3(w), \end{aligned}$$

where

$$\begin{aligned} \Psi_1(w) &= \Delta\chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right), \\ \Psi_2(w) &= |\psi(w)|^2 - 1 = \left(1 + w_1 - \chi(D)\left(\frac{w_2^2}{2}\right)\right)^2 + w_2^2 - 1 \\ &= \left(w_1 - \chi(D)\left(\frac{w_2^2}{2}\right)\right)^2 + (1 - \chi(D))\,w_2^2 + 2w_1, \end{aligned}$$

and

$$\Psi_3(w) = u_c + w_1 - \chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right).$$

Since $\tilde{F} \in C^2\left(H^3, H^3\right)$, by (104) and (103) it suffices to show that $\Psi_1, \Psi_2 \in C^2\left(X^3, H^3\right)$ and $\Psi_3 \in C^2\left(X^3, \dot{H}^3\right)$. It follows from (105) and (106) that $\Psi_1 \in C^\infty\left(X^3, H^3\right)$ and $\Psi_3 - 1 \in C^\infty\left(X^3, \dot{H}^3\right)$. Let

$$\Psi_2(w) = (\Psi_4(w))^2 + \Psi_5(w),$$

where

$$\Psi_4(w) = w_1 - \chi(D)\left(\frac{w_2^2}{2}\right), \quad \Psi_5(w) = (1 - \chi(D))\,w_2^2 + 2w_1.$$

By (14), (103) and (106), for any $f, g \in \dot{H}^3(\mathbf{R}^3)$,

$$\|(1-\chi(D))\,(fg)\|_{H^3} \leqq C\|f\|_{\dot{H}^3}\|g\|_{\dot{H}^3}.$$

This implies that $\Psi_5 \in C^\infty\left(X^3, H^3\right)$. By (14),

$$\|\chi(D)\,(fg)\|_{L^4} \leqq C\|f\|_{\dot{H}^1}\|g\|_{\dot{H}^1}.$$

Combining with (106), this implies that

$$\Psi_4\,(w) \in C^\infty\left(X^3, L^4 \cap \dot{H}^3\right)$$

and thus $(\Psi_4)^2 \in C^\infty\left(X^3, H^3\right)$ by (104). This finishes the proof for $\operatorname{Re}\Psi(w)$.
Step 2. Show $\operatorname{Im}\Psi(w) \in C^2(X_3, H^3)$.
We write

$$\begin{aligned}
\operatorname{Im}\Psi(w) &= \chi(D)((w_{2c}+w_2)\partial_t w_2) - c\partial_{x_1}\chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right) \\
&\quad + [F(|U_c|^2) - F(|U|^2)]v_c - F(|U|^2)w_2 \\
&= F(|U_c|^2)v_c + \Psi_6\,(w) + \Psi_7\,(w) - \tilde{F}\,(\Psi_2\,(w+w_c))\,(v_c+w_2)\,,
\end{aligned} \tag{107}$$

where

$$\begin{aligned}
\Psi_6\,(w) &= \chi(D)((w_{2c}+w_2)\Psi_8\,(w)), \\
\Psi_8\,(w) &= \partial_t w_2 = \Delta w_1 + c\partial_{x_1}w_2 - \operatorname{Re}\Psi(w),
\end{aligned}$$

and

$$\Psi_7\,(w) = -c\partial_{x_1}\chi(D)\left(w_{2c}w_2 + \frac{w_2^2}{2}\right).$$

By the proof in Step 1, the last term in (107) is in $C^2(X_3, H^3)$ and $\Psi_8 \in C^2\left(X_3, H^1\right)$. Since for any $f \in \dot{H}^1$, $g \in H^1$,

$$\|\chi(D)(fg)\|_{H^3} \leqq C\,\|fg\|_{L^2} \leqq C\,\|f\|_{\dot{H}^1}\,\|g\|_{H^1}\,,$$

so $\Psi_6 \in C^2\left(X_3, H^3\right)$. For any $f, g \in \dot{H}^1$, by (13) we have

$$\begin{aligned}
&\left\|\partial_{x_1}\chi(D)\,(fg)\right\|_{L^2} \\
&\quad = \left\|\chi(D)\left(f\partial_{x_1}g + g\partial_{x_1}f\right)\right\|_{L^2} \leqq C\left\|f\partial_{x_1}g + g\partial_{x_1}f\right\|_{L^{\frac{3}{2}}} \\
&\quad \leqq C\left(\|f\|_{L^6}\left\|\partial_{x_1}g\right\|_{L^2} + \|g\|_{L^6}\left\|\partial_{x_1}f\right\|_{L^2}\right) \leqq C\,\|f\|_{\dot{H}^1}\,\|g\|_{\dot{H}^1}\,.
\end{aligned}$$

Combined with (106), above implies that

$$\left\|\partial_{x_1}\chi(D)\,(fg)\right\|_{H^3} \leqq C\|f\|_{\dot{H}^3}\|g\|_{\dot{H}^3},$$

for any $f, g \in \dot{H}^3(\mathbf{R}^3)$. Thus $\Psi_7 \in C^\infty\left(X_3, H^3\right)$. This finishes the proof for $\operatorname{Im}\Psi(w)$. □

Appendix 2

In this Appendix, we show the non-degeneracy condition (83) of stationary bubbles ϕ_0 of the cubic-quintic equation for $N = 2$. Consider the cubic-quintic nonlinear Schrödinger equation (80). Denote

$$F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2, \quad \alpha_1, \alpha_3, \alpha_5 > 0,$$

with

$$\frac{\alpha_1\alpha_5}{\alpha_3^2} \in \left(\frac{3}{16}, \frac{1}{4}\right). \tag{108}$$

Set $\rho_0 = \frac{\alpha_3+\sqrt{\alpha_3^2-4\alpha_1\alpha_5}}{2\alpha_5}$, then $F(\rho_0) = 0$ and $F'(\rho_0) < 0$. Define

$$g(s) = \begin{cases} -F((\sqrt{\rho_0}-s)^2)(\sqrt{\rho_0}-s), & 0 \leqq s \leqq \sqrt{\rho_0}, \\ 0, & s \geqq \sqrt{\rho_0}, \\ -g(-s), & s \leqq 0. \end{cases} \tag{109}$$

According to Theorem 2.1 of [24], if (108) holds, the semilinear elliptic equation

$$-\Delta u = g(u), \quad u \in H^1(\mathbb{R}^N),\ u \neq 0 \tag{110}$$

has a positive radially symmetric, decreasing solution $Q(|x|) \in (0, \sqrt{\rho_0})$, which is usually called a *ground state*. Then $\phi_0 = \sqrt{\rho_0} - Q(|x|)$ is a stationary bubble of (80) with the nonzero boundary condition $|\phi_0| \to \sqrt{\rho_0}$ as $|x| \to \infty$. See Theorem 5.1 for more properties of ϕ_0.

Theorem 8.1. *For $N = 2$, let $Q(|x|)$ be the ground state of (110) with the cubic-quintic nonlinear term $g(u)$ defined in (109). Consider the operator*

$$L_0 = -\Delta - g'(Q) : H^2\left(\mathbf{R}^2\right) \to L^2\left(\mathbf{R}^2\right).$$

Then

$$\ker L_0 = span\left\{\partial_{x_1} Q, \partial_{x_2} Q\right\}. \tag{111}$$

First, we check some properties of the cubic-quintic nonlinear term.

Lemma 8.1. *Let $\frac{\alpha_1\alpha_5}{\alpha_3^2} \in (\frac{3}{16}, \frac{1}{4})$, then g satisfies the following conditions*
$(G1)$ $g(0) = 0$ and $g'(0) < 0$,
$(G2)$ there exists $u_0 \in (0, \sqrt{\rho_0})$ such that $g(u_0) = 0$, $g'(u_0) > 0$, $g(u) < 0$ for all $0 < u < u_0$, and $g(u) > 0$ for all $u_0 < u < \sqrt{\rho_0}$.
Furthermore, there exists $c_0 \in (\frac{3}{16}, \frac{21}{100})$ and $u_1 \in (u_0, \sqrt{\rho_0})$ such that if $\frac{\alpha_1\alpha_5}{\alpha_3^2} = c_0$, then g satisfies
$(G3)$ $g'(u_1) = 0$, $g'(u) < 0$ for all $u_1 < u < \sqrt{\rho_0}$, $g'(u) > 0$ for all $u_0 < u < u_1$ and $G(u_1) = \int_0^{u_1} g(s)\,ds = 0$, $G(u) < 0$ for all $0 < u < u_1$, and $G(u) > 0$ for all $u_1 < u \leqq \sqrt{\rho_0}$.
$(G4)$ for any $\beta > 0$, $\Phi_\beta(u) = \beta(ug'(u) - g(u)) - 2g(u)$ has exactly one zero in $(u_0, \sqrt{\rho_0})$.

Proof. By the definition of g, we have for $u \in [0, \sqrt{\rho_0})$,

$$g'(u) = (-1)[\alpha_1 - 3\alpha_3(\sqrt{\rho_0} - u)^2 + 5\alpha_5(\sqrt{\rho_0} - u)^4],$$
$$g''(u) = -6\alpha_3(\sqrt{\rho_0} - u) + 20\alpha_5(\sqrt{\rho_0} - u)^3.$$

Let $\rho_1 = \frac{\alpha_3 - \sqrt{\alpha_3^2 - 4\alpha_1\alpha_5}}{2\alpha_5}$, $\tilde{\rho}_0 = \frac{3\alpha_3 + \sqrt{9\alpha_3^2 - 20\alpha_1\alpha_5}}{10\alpha_5}$, $\tilde{\rho}_1 = \frac{3\alpha_3 - \sqrt{9\alpha_3^2 - 20\alpha_1\alpha_5}}{10\alpha_5}$. For all $\frac{\alpha_1\alpha_5}{\alpha_3^2} \in (\frac{3}{16}, \frac{1}{4})$, we have

$$\tilde{\rho}_1 < \rho_1 < \tilde{\rho}_0 < \rho_0.$$

Choose $u_0 = \sqrt{\rho_0} - \sqrt{\rho_1}$. Then (G1)(G2) hold.
Choose $u_1 = \sqrt{\rho_0} - \sqrt{\tilde{\rho}_1}$. Then $u_1 \in (u_0, \sqrt{\rho_0})$,

$$g'(u_1) = (-1)[\alpha_1 - 3\alpha_3\tilde{\rho}_1 + 5\alpha_5\tilde{\rho}_1^2] = 0,$$

$g'(u) < 0$ for all $u_1 < u < \sqrt{\rho_0}$ and $g'(u) > 0$ for all $u_0 \leqq u < u_1$.
Let $c = \frac{\alpha_1\alpha_5}{\alpha_3^2}$. If $c \in (\frac{3}{16}, \frac{21}{100})$, we have $g''(u) < 0$ for all $u_0 \leqq u < \sqrt{\rho_0}$. Then for any $\beta > 0$,

$$\Phi'_\beta(u) = \beta u g''(u) - 2g'(u) < 0, \quad \forall u \in [u_0, u_1].$$

Moreover, $\Phi_\beta(u_0) > 0$, $\Phi_\beta(u_1) < 0$ and $\Phi_\beta(u) < 0$ for all $u_1 < u < \sqrt{\rho_0}$.
By direct calculations, we have

$$G(u_1) = \frac{-\alpha_3^3}{2\alpha_5^2}\left\{\frac{3 - \sqrt{9 - 20c}}{10}\left(\frac{14c}{15} - \frac{9 - 9\sqrt{9 - 20c}}{100}\right) - \frac{1 + \sqrt{1 - 4c}}{24}[8c - (1 + \sqrt{1 - 4c})]\right\} \triangleq \frac{-\alpha_3^3}{2\alpha_5^2} h(c).$$

Since $h(\frac{3}{16}) > 0$, $h(\frac{21}{100}) < 0$, there exists $c_0 \in (\frac{3}{16}, \frac{21}{100})$ such that $G(u_1) = 0$ if $\frac{\alpha_1\alpha_5}{\alpha_3^2} = c_0$. Then (G3)(G4) hold. $\square$

Let $u(\alpha, r)$ be the solution of the initial value problem

$$\begin{cases} u''(r) + \frac{N-1}{r}u'(r) + g(u) = 0 \\ u(0) = \alpha, \ u'(0) = 0. \end{cases} \tag{112}$$

Then $\phi(\alpha, r) = \frac{\partial u(\alpha, r)}{\partial \alpha}$ solves

$$\begin{cases} \phi''(r) + \frac{N-1}{r}\phi'(r) + g'(u(\alpha, r))\phi = 0 \\ \phi(0) = 1, \ \phi'(0) = 0. \end{cases} \tag{113}$$

Let $Q(|x|) = u(\alpha_0, |x|)$ be a ground state of (110). To show the non-degeneracy condition

$$\ker L_0 = \left\{\partial_{x_1} Q, \ldots, \partial_{x_N} Q\right\},$$

it suffices to show that the function $\phi(\alpha_0, r)$ does not vanish at infinity. (See [57] or [48] for the proof). When $N = 2$, such a result is provided in the following lemma, which was motivated by [18] and [38].

Lemma 8.2. *Suppose that* $(G1)$–$(G4)$ *hold and let* $u(\alpha_0, r)$ *be a ground state of* (110), *then* $\lim_{r\to+\infty} \phi(\alpha_0, r) \neq 0$ *when* $N = 2$.

Proof. To simplify notations, we denote $u(\alpha_0, r)$, $\phi(\alpha_0, r)$ by $u(r)$, $\phi(r)$ respectively. Since $u(r)$ is a ground state of (110), then $u(r) > 0$, $u'(r) < 0$ for all $r > 0$ and $u_1 < u(0) < \sqrt{\rho_0}$. Moreover, by (G1), it follows from Lemma 6 of [42] that ϕ becomes monotone for large r. Therefore $\lim_{r\to+\infty} \phi(r)$ exists. In order to prove this lemma, we suppose to the contrary that $\lim_{r\to+\infty} \phi(r) = 0$.

Claim 1. ϕ has exactly one zero in $(0, +\infty)$.
Let $A_0 = -\partial_r^2 - \frac{N-1}{r}\partial_r - g'(u(r))$. From $A_0 u' = -(N-1)\,r^{-2}u'$, we deduce that the first eigenvalue of A_0 is negative. By Proposition B.1 of [26], the second eigenvalue of A_0 is nonnegative. Since $A_0\phi = 0$ and $\lim_{r\to+\infty} \phi(r) = 0$, 0 must be the second eigenvalue of A_0. Thus, ϕ has exactly one zero $z_1 \in (0, +\infty)$.

Claim 2. Let $r_0 \in (0, +\infty)$ be such that $u(r_0) = u_0$, then $0 < z_1 < r_0$. Here, u_0 is defined in (G2).
For $\beta \geqq 0$, let $v_\beta(r) = ru'(r) + \beta u(r)$. Then v_β solves

$$v_\beta''(r) + \frac{N-1}{r} v_\beta'(r) + g'(u)v_\beta = \Phi_\beta(u), \tag{114}$$

where $\Phi_\beta(u) = \beta(ug'(u) - g(u)) - 2g(u)$. By (114) and Green's Theorem, for any $0 \leqq r_1 < r_2$, we have

$$\int_{r_1}^{r_2} r^{N-1}\Phi_\beta(u)\phi \, \mathrm{d}r = r_2^{N-1}[\phi(r_2)v_\beta'(r_2) - v_\beta(r_2)\phi'(r_2)] - r_1^{N-1}[\phi(r_1)v_\beta'(r_1) - v_\beta(r_1)\phi'(r_1)]. \tag{115}$$

Set $H(\beta) = \phi(r_0)v_\beta'(r_0) - v_\beta(r_0)\phi'(r_0)$. By the proof of lemma 2.8 in [38], we deduce that $H(0) > 0$. Then from (115) we get

$$\int_0^{r_0} r^{N-1}\Phi_0(u)\phi \, \mathrm{d}r = r_0^{N-1}H(0) > 0. \tag{116}$$

By (G2) we know that $\Phi_0(u) = -2g(u) < 0$ for all $u_0 < u < \sqrt{\rho_0}$. If $\phi > 0$ on $[0, r_0)$, it is impossible by (116). Thus we must have $\phi(r_0) < 0$ and $0 < z_1 < r_0$.

Claim 3. $\theta(r) = \frac{-ru'(r)}{u(r)}$ is increasing in $(0, r_0)$.
For the proof of this claim, we need $N = 2$. In fact, by (112) we have $(-ru'(r))' = rg(u(r))$ for $N = 2$. Thus, from (G2) we know that $(-ru'(r))' > 0$ in $(0, r_0)$. Since $u(r)$ is decreasing in $(0, +\infty)$, we get that $\theta(r) = \frac{-ru'(r)}{u(r)}$ is increasing in $(0, r_0)$.
Set $\beta_0 = \frac{-z_1u'(z_1)}{u(z_1)}$, then $\beta_0 > 0$ and $v_{\beta_0}(z_1) = 0$. From (115), we get

$$\int_0^{z_1} r^{N-1}\Phi_{\beta_0}(u)\phi \, \mathrm{d}r = 0. \tag{117}$$

By (G2)(G3), we have $\Phi_{\beta_0}(u) < 0$ for all $u_1 \leqq u < \sqrt{\rho_0}$. Note that $\phi > 0$ on $[0, z_1)$, $u'(r) < 0$ for all $r > 0$ and $u_1 < u(0) < \sqrt{\rho_0}$, then from (117) we deduce that $u(z_1) < u_1$ and $\Phi_{\beta_0}(u(z_1)) > 0$. Furthermore, by (G4) we have $\Phi_{\beta_0}(u(r)) > 0$ for all $r \in (z_1, r_0)$. Since $\phi < 0$ on $(z_1, r_0]$, we have

$$\int_{z_1}^{r_0} r^{N-1} \Phi_{\beta_0}(u)\phi \, \mathrm{d}r < 0. \tag{118}$$

On the other hand, from (115) we get

$$\int_{z_1}^{r_0} r^{N-1} \Phi_{\beta_0}(u)\phi \, \mathrm{d}r = r_0^{N-1}[\phi(r_0)v'_{\beta_0}(r_0) - v_{\beta_0}(r_0)\phi'(r_0)]. \tag{119}$$

Claim 4. $\phi(r_0)v'_{\beta_0}(r_0) - v_{\beta_0}(r_0)\phi'(r_0) > 0$.
By Claim 4 and (118) (119), we get a contradiction.
Proof of Claim 4. Let $H(\beta) = \phi(r_0)v'_{\beta}(r_0) - v_{\beta}(r_0)\phi'(r_0)$, then

$$H(\beta) = H(0) + \beta[\phi(r_0)u'(r_0) - \phi'(r_0)u(r_0)]. \tag{120}$$

We show $H(\beta_0) > 0$ in two cases.

Case 1. $\phi(r_0)u'(r_0) - \phi'(r_0)u(r_0) \geqq 0$. In this case, by $H(0) > 0$ and (120) we obviously have $H(\beta_0) > 0$.

Case 2. $\phi(r_0)u'(r_0) - \phi'(r_0)u(r_0) < 0$. Since $\phi(r_0) < 0, u'(r_0) < 0, u(r_0) > 0$, we must have $\phi'(r_0) > 0$.
Let $b_1 = \frac{-r_0u'(r_0)}{u(r_0)}$. Since $z_1 < r_0$ and $\theta(r) = \frac{-ru'(r)}{u(r)}$ is increasing in $(0, r_0)$, we have $\beta_0 < b_1$. Then by $\phi(r_0)u'(r_0) - \phi'(r_0)u(r_0) < 0$ and (120) we get $H(\beta_0) > H(b_1)$. Note that $v'_{N-2}(r_0) = -r_0 g(u_0) = 0$ by (G2) and $v_{b_1}(r_0) = 0$. If $b_1 \geqq N-2$, we have $v'_{b_1}(r_0) \leqq v'_{N-2}(r_0) = 0$ and

$$H(b_1) = \phi(r_0)v'_{b_1}(r_0) - v_{b_1}(r_0)\phi'(r_0) = \phi(r_0)v'_{b_1}(r_0) \geqq 0.$$

This finishes the proof of the lemma for $N = 2$. □

Appendix 3

Consider a function in the form of $U(x_1, x_\perp) = U(x_1, r_\perp)$, where $x_\perp = (x_2, x_3)$ and $r_\perp = |x_\perp|$, and assume $\nabla U \in H^s(\mathbf{R}^3)$, $s > 1$, not necessarily an integer. In this appendix, we prove

$$\frac{1}{r_\perp}\partial_{r_\perp} U \in L^2(\mathbf{R}^3) \quad \text{and} \quad \partial_{r_\perp} U \in H^1(\mathbf{R}^3),$$

which are needed in Lemma 2.8 to show that the Hessian L_c of the energy functional has a negative mode.
Due to the density of Schwartz class functions, we will work on Schwartz class functions, but keep tracking of the norms carefully. Denote $\vec{\mathbf{e}}_\perp = \frac{1}{r_\perp}(0, x_\perp)$, then

$$\partial_{r_\perp} U(x_1, r_\perp) = \nabla_{x_\perp} U(x_1, x_\perp) \cdot \frac{x_\perp}{r_\perp} = DU(x_1, x_\perp) \cdot \vec{\mathbf{e}}_\perp$$

and

$$\partial_{x_1}\partial_{r_\perp}U(x_1,r_\perp) = D^2U(x_1,x_\perp)(\vec{\mathbf{e}}_1,\vec{\mathbf{e}}_\perp).$$

Moreover, since $\partial_{r_\perp}U(x_1,x_\perp)$ is radial in $x_\perp$, its gradient in $x_\perp$ must be in the radial direction and thus

$$\nabla_{x_\perp}\partial_{r_\perp}U(x_1,x_\perp) = D^2U(x_1,x_\perp)(\vec{\mathbf{e}}_\perp,\vec{\mathbf{e}}_\perp)\frac{x_\perp}{r_\perp}.$$

Therefore $\partial_{r_\perp}U \in H^1(\mathbf{R}^3)$ is obvious. Computing higher order derivatives in a similar fashion and applying an interpolation argument if s is not an integer, one can prove $\partial_{r_\perp}U \in H^s(\mathbf{R}^3)$.
To show $\frac{1}{r_\perp}\partial_{r_\perp}U \in L^2(\mathbf{R}^3)$, we first observe that the radial symmetry of U implies its linearization at $x_\perp = 0$ is also a radially symmetric linear function, which can only be 0, and thus

$$\nabla_{x_\perp}U(x_1,0) = 0 \implies \partial_{r_\perp}U(x_1,0) = 0.$$

Fix x_1, on the one hand, one may estimate by using the Cauchy–Schwarz inequality

$$\begin{aligned}
\big(\partial_{r_\perp}U(x_1,r_\perp)\big)^2 &= 2\int_0^{r_\perp}\partial_{r_\perp}U(x_1,r'_\perp)\partial_{r_\perp r_\perp}U(x_1,r'_\perp)\,\mathrm{d}r'_\perp \\
&\leqq Cr_\perp^{\frac{p-2}{p}}|\nabla U(x_1,\cdot)|_{L^\infty(\mathbf{R}^2)}\left(\int_0^{r_\perp} r'_\perp\left|\partial_{r_\perp r_\perp}U\left(x_1,r'_\perp\right)\right|^p\,\mathrm{d}r'_\perp\right)^{\frac{1}{p}} \\
&\leqq Cr_\perp^{\frac{p-2}{p}}|\nabla U(x_1,\cdot)|_{H^s(\mathbf{R}^2)}|D^2U(x_1,\cdot)|_{L^p(\{|x_\perp|<r_\perp\})} \\
&\leqq Cr_\perp^{\frac{p-2}{p}}|\nabla U(x_1,\cdot)|^2_{H^s(\mathbf{R}^2)}
\end{aligned}$$

for some $p > 2$. Integrating in x_1 we obtain

$$\int_{\mathbf{R}}\big(\partial_{r_\perp}U(x_1,r_\perp)\big)^2\,\mathrm{d}x_1 \leqq Cr_\perp^{1-\frac{2}{p}}|\nabla U|^2_{H^s(\mathbf{R}^3)}.$$

On the other hand, via integration by parts, we have

$$\begin{aligned}
\int_{\tilde r_\perp}^1\frac{1}{r'_\perp}\big(\partial_{r_\perp}U(x_1,r'_\perp)\big)^2\,\mathrm{d}r'_\perp &= -2\int_{\tilde r_\perp}^1(\log r'_\perp)\partial_{r_\perp}U(x_1,r'_\perp)\partial_{r_\perp r_\perp}U(x_1,r'_\perp)\,\mathrm{d}r'_\perp \\
&\quad -(\log\tilde r_\perp)\big(\partial_{r_\perp}U(x_1,\tilde r_\perp)\big)^2.
\end{aligned}$$

Integrating it with respect to x_1, letting $\tilde r_\perp \to 0+$, and using the above inequality, we obtain

$$\left|\frac{1}{r_\perp}\partial_{r_\perp}U\right|^2_{L^2(\{|x_\perp|<1\})} = -2\int_{|x_\perp|<1}(\log r_\perp)\partial_{r_\perp}U(x_1,r_\perp)\partial_{r_\perp r_\perp}U(x_1,r_\perp)\,\mathrm{d}x.$$

Splitting the integrand on the right side into the product of $r_\perp^{-\frac{1}{2}}\partial_{r_\perp}U$, $r_\perp^{\frac{1}{p}}\partial_{r_\perp r_\perp}U$, and $r_\perp^{\frac{1}{2}-\frac{1}{p}}\log r_\perp$ and applying the Hölder inequality first with indices $\frac{1}{2}$, $\frac{1}{p}$, and

$\frac{1}{2} - \frac{1}{p}$ to the integral in $x_\perp$ and then the Cauchy–Schwarz inequality to the x_1 integral, we have

$$\begin{aligned}
&\left|\frac{1}{r_\perp}\partial_{r_\perp} U\right|^2_{L^2(\{|x_\perp|<1\})} \\
&\leqq C \int_{\mathbf{R}} \left|\frac{1}{r_\perp}\partial_{r_\perp} U(x_1,\cdot)\right|_{L^2(\{|x_\perp|<1\})} |D^2 U(x_1,\cdot)|_{L^p(\{|x_\perp|<1\})}\,\mathrm{d}x_1 \\
&\leqq C \left|\frac{1}{r_\perp}\partial_{r_\perp} U\right|_{L^2(\{|x_\perp|<1\})} |\nabla U|_{H^s(\mathbf{R}^3)}.
\end{aligned}$$

Therefore we obtain

$$\left|\frac{1}{r_\perp}\partial_{r_\perp} U\right|_{L^2(\{|x_\perp|<1\})} \leqq C|\nabla U|_{H^s(\mathbf{R}^3)}.$$

As the estimate is trivially true on $\{|x_\perp| > 1\}$, the proof is complete.

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