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A fast and robust algorithm for image restoration with periodic boundary conditions

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Abstract

A new Tikhonov regularization method of Fuhry and Reichel [*A new Tikhonov regularization method, Numerical Algorithms*, 59:433-445, 2011] exhibits the excellent properties for ill-posed problems, but it can only deal with small or moderate size problems because of the expensive computation of singular value decomposition (SVD). In this paper, we extend the above new Tikhonov regularization method to solve large-scale problems, e.g., image restoration problem with periodic boundary conditions, and realize this extending by applying Fast Fourier Transformation (FFT) algorithm to the spectral decomposition of the block circulant with circulant blocks (BCCB) matrices. Experimental results confirm the superiority of our new method.

Key words: Periodic boundary conditions; FFT algorithm; Tikhonov regularization method; Image restoration

1 Introduction

The Fredholm integral equation of the first kind which arises from many image or signal restoration problems is formulated as follows

$$\int_a^b \kappa(s,t)f(t)dt = g(s), \tag{1}$$

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where $\kappa(s,t)$ is integral kernel and $g(s)$ is obtained by the known $\kappa(s,t)$ and $f(t)$. We can get the following linear system by discretization of integral equation (1),

$$Ax_{true} = b_{true}, \tag{2}$$

where $A \in \mathcal{R}^{m\times n}$ is blurring matrix, for simple notation, we consider $m \geq n$ and $x_{true} \in \mathcal{R}^n$ represents original signal with noise-free, blurred signal $b_{true} \in \mathcal{R}^m$ is formulated by blurring matrix A acting on original signal x_{true}.

Random noise $e \in \mathcal{R}^m$ is added to the right side of (2), so the final linear system is as follows

$$Ax = b = b_{true} + e, \tag{3}$$

where $x \in \mathcal{R}^n$ is an approximate solution of x_{true}, but it is just inaccessible to the true solution x_{true} generally. Our goal is to utilize an applicable method to make the relative error between x and x_{true} minimum. Typically, this is a large-scale ill-posed problem.

Tikhonov regularization methods are promising ways for ill-posed problems (see, e.g., [1, 2]), the general form is as follows

$$\min_{x\in R^n}\{\| Ax - b \|_2^2 + \| L_\lambda x \|_2^2\}, \tag{4}$$

where scalar $\lambda > 0$ is called regularization parameter and L_λ is regularization matrix. The regularization matrix is generally λI, where I represents identity matrix. A closely related Tikhonov regularization approach [3] by Fuhry and Reichel showed a novel construction of the regularization matrix, that is $L_\lambda = D_\lambda V^T$, where V^T is an unitary matrix and D_λ is a diagonal matrix containing the regularization parameter and some singular values. The numerical and visual experiments demonstrated that the new Tikhonov regularization method [3] is an excellent method for small or moderate size problems. However, the above Tikhonov regularization method is based on SVD which is expensive consuming for large-scale problems.

Since the fast algorithm such as FFT algorithm are good at doing spectral decomposition of structure matrices (see, e.g., [2, 4]), we can deal with the above large-scale problems using this property. We intentionally gain the BCCB matrices as the blurring operators A by setting circularly symmetry point spread functions (PSF) and assuming periodic boundary conditions (see other boundary conditions in [5, 6, 7], where the reference [6] showed a fast algorithm for deblurring models with Neumann boundary conditions, and [7] proposed a note on antireflective boundary conditions and fast deblurring models). Then we can exploit FFT algorithm (e.g., [8, 9]) to get the eigenvalues of BCCB matrices fast. FFT is an efficient algorithm which is widely used in many fields such as image filtering, image saving, image enhancement and image restoration and so on. Zhu et al. [10] introduced that FFT algorithm can be used for solving compressed sensing to accelerate the computing process. Li et al. [11] showed that FFT is an effective method in signal sparse decomposition. Since Matching Pursuit (MP) adaptively decomposes signals in the redundant of dictionary to achieve some sparse representations, and it is very time consuming, FFT-based MP implementation runs significantly faster than greedy MP implementation. Furthermore, Hu et al. [12] showed that FFT can also be used in image compression. The authors adopted Radix-4 FFT to realize the limit distortion for image coding, and to discuss the feasibility and advantage of Fourier transform for image compression. Using Radix-4 FFT can reduce data storage, computing complexity and time-consuming.

The contributions of the paper are as follows: firstly, motivated by [3], we extend the new Tikhonov regularization method to solve the large-scale ill-posed image restoration problem. Secondly, we exploit FFT algorithm to fast spectral decomposition of the BCCB matrices. Finally, we test several kinds of blurs and noises to show the robustness of our algorithms. Experimental results indicate the advantages of the proposed method.

The organization of this paper is given as follows. Section 2 is mainly a recall of the new Tikhonov regularization method proposed by Fuhry and Reichel. Section 3 exhibits our method based on the FFT algorithm. Computational results will be shown in section 4. Finally, section 5 shows a conclusion about our method.

2 New Tikhonov regularization method

For completeness, we include in this section the known new Tikhonov regularization method [3] applied to the ill-posed problem (4).

Tikhonov regularization method is a popular and classical method for ill-posed problems, the general form is to solve the following least squares problem

$$\min_{x\in R^n}\{\| Ax-b \|_2^2 + \| L_\lambda x \|_2^2\}, \tag{5}$$

where L_λ is the regularization matrix and scalar $\lambda > 0$ is called the regularization parameter. In general, the regularization matrix L_λ is chosen to be λI where I is the identity matrix, and the resulting method is called standard Tikhonov regularization method. Furthermore, the finite differential operators are also used when the desired solution x has some particular properties (see [13, 14, 15, 16]). The least squares problem (5) is equivalent to the following normal equation

$$(A^TA + L_\lambda^T L_\lambda)x = A^Tb. \tag{6}$$

A closely related approach with a novel regularization matrix has been proposed by Fuhry and Reichel [3]. We exploit singular value decomposition

$$A = U\Sigma V^T,$$

where $U \in \mathcal{R}^{m\times m}$, $V \in \mathcal{R}^{n\times n}$ are two unitary matrices to construct the new regularization matrix and $\Sigma = \mathrm{diag}[\sigma_1, \sigma_2, \cdots, \sigma_n]$ where σ_i represents the i-th singular value of A. The new regularization matrix is presented as follows

$$L_\lambda = D_\lambda V^T, \tag{7}$$

where

$$D_\lambda^2 = \begin{pmatrix} \max(\lambda^2 - \sigma_1^2, 0) & & & \\ & \max(\lambda^2 - \sigma_2^2, 0) & & \\ & & \ddots & \\ & & & \max(\lambda^2 - \sigma_n^2, 0) \end{pmatrix},$$

and matrix V^T is the unitary matrix from the SVD of matrix A, and σ_i is the i-th singular value of A.

In the light of the SVD of A and equations (6) and (7), we obtain the following equivalent equation

$$x = V^T(\Sigma^T\Sigma + D_\lambda^2)^{-1}\Sigma^T U^T b. \tag{8}$$

The solving of equation (8) needs the regularization parameter λ which is determined by discrepancy principle in [3].

It is easy to know that the regularization parameter λ satisfies $\sigma_{k+1} < \lambda < \sigma_k$ which the σ_k represents the k-th singular value and $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0$. So we have

$$\Sigma^T\Sigma + D_\lambda^2 = \mathrm{diag}[\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2, \lambda^2, \cdots, \lambda^2] \in \mathcal{R}^{n\times n}. \tag{9}$$

In order to avoid the propagation of the random noise e in (3) into the computed approximate solution x_{true}, the smallest eigenvalue of $A^TA + L_\lambda^T L_\lambda$ has to be large sufficiently. Also, since our model is minimization problem, we hope L_λ to be a small norm in order to help us decide a more accurate approximation of x. The following two properties demonstrate that the new Tikhonov method is a good one.

a. The smallest eigenvalue of the matrix $A^TA + L_\lambda^T L_\lambda$ should be λ^2 where $\lambda^2 \geq \sigma_i^2, i = k+1, k+2, \cdots, n$. Since $A^TA = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$,$L_\lambda^T L_\lambda = VD_\lambda^T D_\lambda V^T$, then $A^TA + L_\lambda^T L_\lambda = V(\Sigma^T\Sigma + D_\lambda^T D_\lambda)V^T$.

b. The regularization matrix L_λ has smaller norm than λI in Frobenius norm $||.||_F$. Since λ, σ_1 are strictly positive and $||L_\lambda||_F^2 = ||D_\lambda||_F^2 = \sum_{\sigma_j^2 \leq \lambda^2} (\lambda^2 - \sigma_j^2) < n\lambda^2 = ||\lambda I||_F^2$, then more accurate approximation of x can be reached.

The smallest eigenvalue of the matrix $A^T A + L_\lambda^T L_\lambda$ is equal to the smallest element of the diagonal matrix (9), i.e., λ^2 where $\sigma_{k+1}^2 < \lambda^2 < \sigma_k^2$. The corresponding Theorem 2.1 and Corollary 2.2 in [3] demonstrate the new Tikhonov regularization method indeed can achieve better balance for the above two aspects.

3 The new method combined FFT algorithm with new Tikhonov regularization method

Motivated by the idea of new Tikhonov regularization method proposed in [3], and due to the fast FFT algorithm, we use FFT algorithm to accelerate the spectral decomposition of BCCB matrices in the process of new Tikhonov regularization method. Particularly, where the BCCB matrix which is gained by imposing circularly symmetric PSF and periodic boundary conditions (see [2, 8, 9]). The detailed FFT algorithm is showed in this section.

3.1 FFT algorithm applied to BCCB matrices

It is well known that BCCB matrices which are normal matrices have the particular spectral decomposition

$$A = \mathcal{F}^* \Lambda \mathcal{F}, \tag{10}$$

where $\mathcal{F} \in \mathcal{C}^{n\times n}$ is 2D unitary discrete Fourier transform (DFT) matrix, $*$ represents conjugate transpose and the diagonal matrix $\Lambda = \text{diag}[\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n]$ contains all eigenvalues of $A \in \mathcal{R}^{n\times n}$. This matrix $\mathcal{F}$ has a very convenient property which can perform fast matrix-vector multiplications without constructing $\mathcal{F}$ explicitly. In MATLAB, the function `fft2` and `ifft2` are used for matrix-vector multiplications of $\mathcal{F}$ and $\mathcal{F}^*$, respectively.

Since the implicit matrix $\mathcal{F}$ is a unitary matrix, we have the following equation according to the properties of Fourier transforms,

$$A = \mathcal{F}^* \Lambda \mathcal{F} \Rightarrow \mathcal{F} A = \Lambda \mathcal{F} \Rightarrow \mathcal{F} a_1 = \Lambda f_1 = \boldsymbol{\lambda}/\sqrt{N}, \tag{11}$$

where $\boldsymbol{\lambda} \in \mathcal{R}^{n\times 1}$ is a vector which contains all eigenvalues of A. It is well known that the first column of $\mathcal{F}$, f_1, is a vector of all ones, and the first column of A, a_1, can be gained by PSF and MATLAB function `circshift` (see [2]). We assume the matrix A^{-1} exists, so the final computing form is as follows

$$b = Ax = \mathcal{F}^* \Lambda \mathcal{F} x \Rightarrow x = A^{-1} b = \mathcal{F}^* \Lambda^{-1} \mathcal{F} b, \tag{12}$$

where b is the observed image and A is the BCCB matrix which can exploit the FFT algorithm.

3.2 The new Tikhonov regularization method using FFT algorithm (NTRF)

Tikhonov regularization method is a classical and promising method for image deblurring, but it shows disadvantages if we impose the random noise on the images. Fuhry and Reichel recently proposed a novel construction of the regularization matrix $L_\lambda = D_\lambda V^T$ introduced in section 2 (see [3]), called new Tikhonov regularization method.

Combining section 2 with section 3.1, we extend the new Tikhonov regularization method for small or moderate size problems to new Tikhonov regularization method for large-scale problems. Similar to section 2, for solving the least squares problem (5), we get the normal equation (6) easily. Differently, the regularization matrix is as follows

$$\tilde{L}_\mu = \tilde{D}_\mu \mathcal{F}, \tag{13}$$

where $\mathcal{F} \in \mathcal{C}^{n\times n}$ is the 2D unitary discrete Fourier transform (DFT) matrix and $*$ represents conjugate transpose. And regularization matrix $\tilde{D}_\mu \in \mathcal{R}^{n\times n}$ is as follows

$$\tilde{D}_\mu^2 = \begin{pmatrix} \max(\mu^2-\lambda_1^2,0) & & & \\ & \max(\mu^2-\lambda_2^2,0) & & \\ & & \ddots & \\ & & & \max(\mu^2-\lambda_n^2,0) \end{pmatrix},$$

where $\mu \in \mathcal{R}$ is also the regularization parameter just like the λ in section 2 and λ_i is the i-th eigenvalue of matrix A.

From least square problem (5), the following equation can be gained and A is a real matrix,

$$(A^*A + \tilde{L}_\mu^* \tilde{L}_\mu)x = A^*b.$$

We exploit spectral decomposition $A = \mathcal{F}^*\Lambda\mathcal{F}$ and $\tilde{L}_\mu = \tilde{D}_\mu\mathcal{F}$ to gain the following eqution

$$\mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)\mathcal{F}x = \mathcal{F}^*\Lambda^*\mathcal{F}b.$$

Similar to equation (12), it is easy to get the following equation

$$x = \mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)^{-1}\Lambda^*\mathcal{F}b. \tag{14}$$

The above equation (14) is our final computing scheme. The following experiments in section 4 demonstrate the equation (14) is indeed a promising way for image restoration. The following Theorem shows that if the smallest eigenvalue of $A^*A + \tilde{L}_\mu^*\tilde{L}_\mu^*\tilde{L}_\mu$ is sufficiently large, this can avoid propagation of the noise. Moreover, since our goal is to get smaller norm, the choosing of the regularization matrix $\tilde{L}_\mu^*\tilde{L}_\mu$ is proper which can help to determine the approximation solution.

Theorem 3.1 *Let $M = A^*A + \tilde{L}_\mu^* \tilde{L}_\mu, M \in R^{n\times n}$, where $A \in R^{n\times n}$ satisfies equation (10) and $\tilde{L}_\mu \in C^{n\times n}$ satisfies equation (13). Let $\mu > 0$ be the regularization parameter, then*

i) *The smallest eigenvalue of the matrix M is μ^2 where $\mu^2 \geq \lambda_i^2, i \in \bar{S}$ where index set $S = \{j|\lambda_j^2 > \mu^2, j = 1, 2, \cdots, n\}$.*

ii) *The regularization matrix $\tilde{L}_\mu$ has smaller Frobenius norm than μI, where μ here is the regularization parameter of Tikhonov model.*

Proof. i) According to the definitions of M, A, $\tilde{L}_\mu$, we have

$$M = A^*A + \tilde{L}_\mu^* \tilde{L}_\mu = \mathcal{F}^*\Lambda^*\Lambda\mathcal{F} + \mathcal{F}^*\tilde{D}_\mu^*\tilde{D}_\mu\mathcal{F} = \mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)\mathcal{F} = \mathcal{F}^*D\mathcal{F},$$

where D is a diagonal matrix (i.e., $\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu$) that includes diagonal elements λ_i^2 and μ^2, $i \in S, S = \{j|\lambda_j^2 > \mu^2, j = 1, 2, \cdots, n\}$.

Due to the symmetric matrix M and unitary matrix $\mathcal{F}$, matrix M has the smallest eigenvalue μ^2.

ii) We have

$$||\tilde{L}_\mu||_F^2 = ||\tilde{D}_\mu\mathcal{F}||_F^2 = ||\tilde{D}_\mu||_F^2 = \Sigma_{\lambda_j^2<\mu^2}(\mu^2 - \lambda_j^2), j \in \bar{S},$$

and $0 < \mu^2 - \lambda_j^2 < \mu^2$,

$$||\tilde{L}_\mu||_F^2 = \Sigma_{\lambda_j^2<\mu^2}(\mu^2 - \lambda_j^2) < n\mu^2 = ||\mu I||_F^2.$$

□

The new algorithm is shown as follows:

Algorithm 1 (New Tikhonov regularization method using FFT algorithm (NTRF))

1. Compute Λ by spectral decomposition $A = \mathcal{F}^*\Lambda\mathcal{F}$ where A is a BCCB matrix.
2. Compute parameter μ where $\mu = 5\mu_{gcv}$, μ_{gcv} is obtained by GCV method.
3. Construct
$$\tilde{D}_\mu^2 = \mathrm{diag}[\max(u^2 - \lambda_1^2, 0), \max(u^2 - \lambda_2^2, 0), \cdots, \max(u^2 - \lambda_n^2, 0)].$$
4. Directly compute
$$x = \mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)^{-1}\Lambda^*\mathcal{F}b,$$
where $\mathcal{F}$ is not explicit, but matrix-vector multiplication $\mathcal{F}b$ and $\mathcal{F}^*b$ can be obtained by fft2(b) and ifft2(b) fastly in practical MATLAB implementation.

4 Experimental results

In this section, we present four different images *synthetic*, *cameraman*, *lena*, *einstein* in Figure 1 which are all of size 256×256 pixels to show the effectiveness and feasibility of our proposed method. The Tikhonov regularization method based on FFT (TRF), called `tik_ fft` in the MATLAB package HON[1] from [2], is compared with our method NTRF by imposing periodic boundary conditions. Particularly, the traditional Tikhonov regularization method based on FFT method (TRF) is different from NTRF method. Mainly due to the different construction $\tilde{D}_\mu^2$ where the diagonal matrix of TRF method is diag$[\mu_{gcv}^2, \mu_{gcv}^2, \cdots, \mu_{gcv}^2]$, and the diagonal matrix of NTRF method is diag$[\max(u^2 - \lambda_1^2, 0), \max(u^2 - \lambda_2^2, 0), \cdots, \max(u^2 - \lambda_n^2, 0)]$.

In the following examples, we mainly compare visual quality of restored image and the peak signal-to-noise ratio ($PSNR$) value which is defined as follows:

$$PSNR(u, v) = 10 \cdot log_{10} \frac{255^2}{\frac{1}{mn} \sum_{i,j} (u_{i,j} - v_{i,j})^2}$$

where $v_{i,j}$ and $u_{i,j}$ denote the pixel values of the restored and the original images, respectively. Mainly we have that larger $PSNR$ means better restored image.

The noise-free blurred image b_{true} is computed as $b_{true} = Ax_{true}$ (see equation (2)). The elements of the noise vector e are normally distributed with zero mean, and if we set $b = b_{true} + \alpha \cdot \|b_{true}\|_2 \cdot e$ where b_{true} is blurred signal. In this case, we say that the level of noise is α. For example, if $b = b_{true} + 0.01 \cdot ||b_{true}|| \cdot e$, the level of noise is 1%

The corresponding regularization parameters α of all examples are generated by Generalized Cross Validation (GCV) for TRF, for simplicity, five times α for NTRF due to empirical estimation. The following numerical examples are all implemented with MATLAB (R2010a) and the computer of test has 1G RAM and Intel(R) Pentium(R) D CPU @2.80GHz @2.79GHz.

Here, we consider two kinds of blur in our experiments, i.e., Gaussian blur and Moffat blur. The blurred images also are corrupted by additive noise-Gaussian noise. We not only compare the visual quality, but also compare the $PSNR$ values of TRF method and NTRF method. From the following tables and restored images, we can easily get the fair comparisons of our NTRF method and TRF method. We also get that our method is more effective and stable.

4.1 Example 1

We consider images which are corrupted by blur and noise, where the blur is 10×10 pixels Gaussian-shaped PSFs with standard deviation(σ^2) with 1, 1.5 and 2, meanwhile, two kinds of Gaussian noise level are 0.5% and 1% . Table 1 shows the results obtained by TRF method and NTRF method. From the table, we can see that NTRF method gets larger $PSNR$ values than TRF method. That

[1] www.siam.org/books/fa03.

Fig. 1: Original images.

demonstrates the better numerical results of our NTRF method. Furthermore, from the *PSNR* comparisons of the two methods, we obtain that the numerical difference of *PSNR* in our NTRF method becomes larger with the standard deviation σ^2 decreasing and Gaussian noise increasing. For example, the *PSNR* value difference of $\sigma^2 = 1$ and 1% noise between TRF and NTRF is larger than $\sigma^2 = 2$ and 0.5% noise. It demonstrates that our NTRF method behaves better under the condition of lower blur and higher noise. Figure 2 shows the images degenerated by $\sigma^2 = 1.5$ blur and 0.5% noise and the restored results by TRF method and NTRF method. Figure 3 displays the images degenerated by $\sigma^2 = 2$ blur and 1% noise and the restored images by TRF method and NTRF method. Evidently, the visual results with TRF method leave more noise (see the black region of second column in Figure 2) than our method. And our NTRF method shows the favorable denoising ability.

Table 1: Corresponding PSNR values using TRF method and NTRF method under the different Gaussian blurs and Gaussian white noises.

Examples	Variance	$\sigma^2 = 1$		$\sigma^2 = 1.5$		$\sigma^2 = 2$	
	Noise	0.5%	1%	0.5%	1%	0.5%	1%
synthetic	TRF	74.9845	72.2842	74.4708	72.7537	73.0333	71.6071
	NTRF	78.7617	77.2222	75.7437	75.0221	74.0365	73.5287
lena	TRF	79.1305	76.7826	77.3220	76.0470	75.1652	74.2478
	NTRF	81.2388	79.8669	77.4890	76.8047	75.4911	74.9820
einstein	TRF	79.8850	77.3747	78.0915	76.7371	76.0327	74.9479
	NTRF	82.3276	80.8633	78.2457	77.4736	76.1387	75.6452

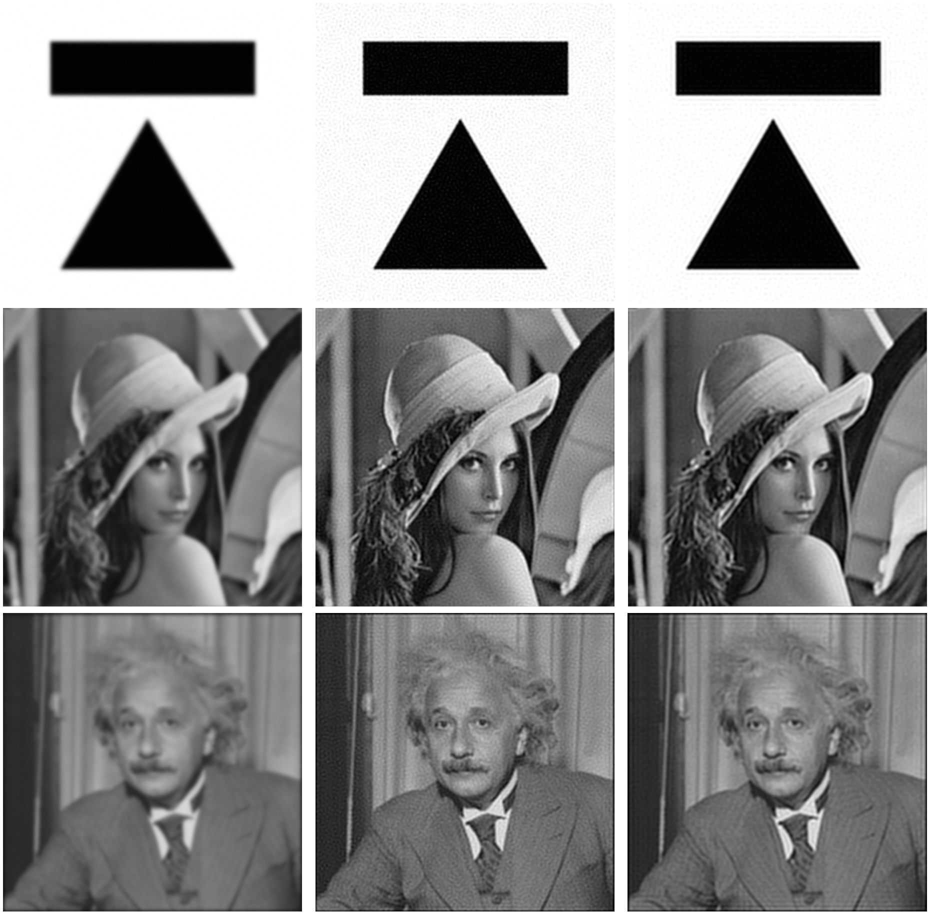

Fig. 2: First column: blurred and noisy images with Gaussian blur($\sigma^2 = 1.5$) and Gaussian noise(0.5%); Second column: restored images using TRF method ; Third column: restored images using NTRF method.

4.2 Example 2

We add Moffat blur into the images in this subsection. The detail of Moffat blur can be got from [17, 18]. And we use (x, y, z) to denote blur, where x represents the size of Moffat blur, y denotes standard deviation (σ^2) of the blur, z is a parameter. Here the noisy-blurry images have $\sigma^2 = 1$ and $\sigma^2 = 1.5$ blur and 0.5% and 1% Gaussian noise. Table 2 shows the excellence of NTRF method duo to the larger *PSNR* values using NTRF method for all test images. Similar as example 1, the numerical difference of *PSNR* in our NTRF method becomes larger with the standard deviation decreasing and Gaussian noise increasing. So we can also conclude that our NTRF method behaves better under the condition of lower blur and higher noise. Figure 4 shows the blurred and noisy images with

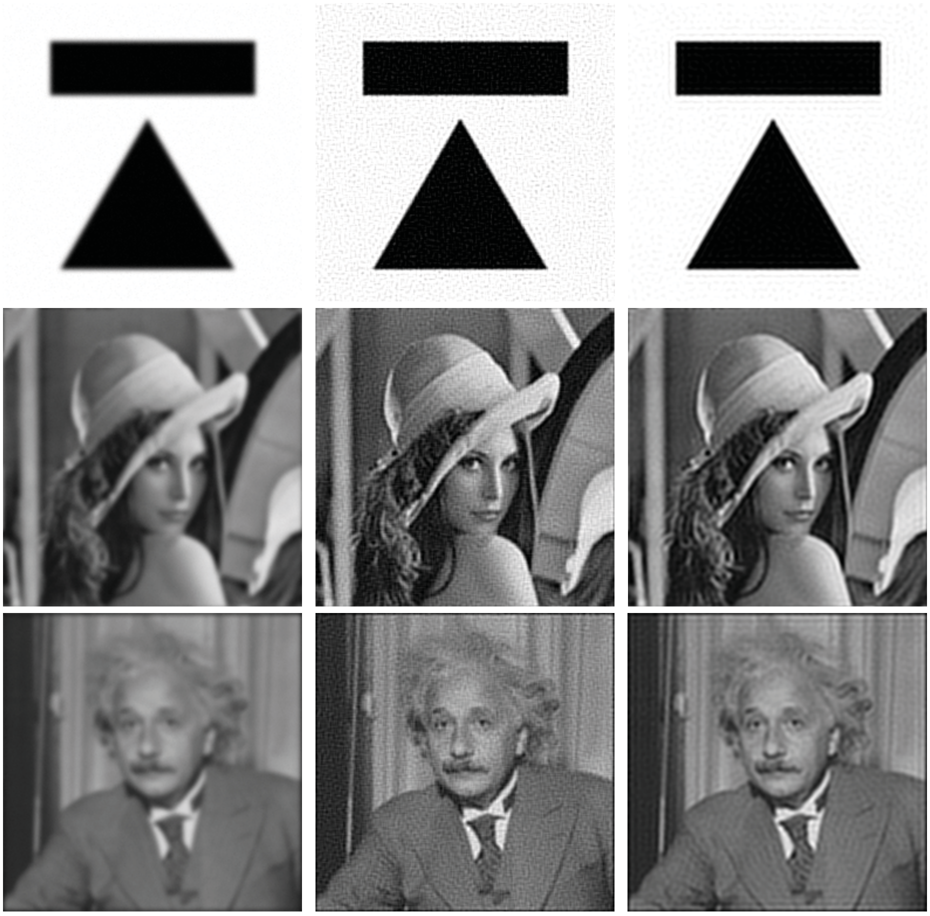

Fig. 3: First column: Blurred and noisy images with Gaussian blur ($\sigma^2 = 2$) and Gaussian noise (1%); Second column: Restored images using TRF method ; Third column: Restored images using NTRF method.

Moffat blur ($\sigma^2 = 1$) and Gaussian noise (0.5%). Figure 5 shows the blurred and noisy images with Moffat blur ($\sigma^2 = 1.5$) and Gaussian noise (0.5%). From the restored images in Figure 4 and 5, the more residual noise using TRF and less residual noise using our NTRF method demonstrate the better visual results of our method.

5 Conclusions

In this paper, we apply the new Tikhonov regularization method with FFT algorithm to generate a novel method , i.e. NTRF method, for dealing with large-scale ill-posed image restoration problems, since FFT algorithm is good at computing

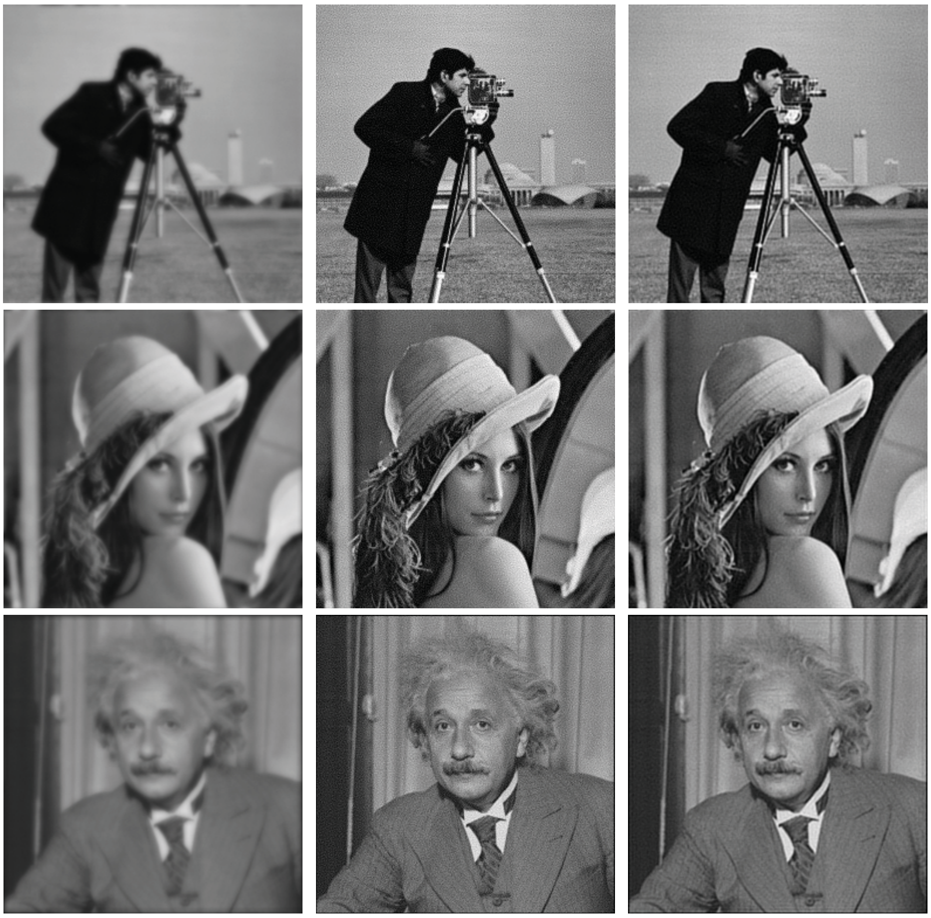

Fig. 4: First column: blurred and noisy images with Moffat blur ($\sigma^2 = 1$) and Gaussian noise (0.5%); Second column: restored images using TRF method ; Third column: restored images using NTRF method.

the spectral decomposition. Our new method retains the stability and effectiveness of the method in [3], and reduces time-consuming by using FFT algotithm. The structure of blurring matrix is a key step and should be BCCB structure that generated by circularly symmetric PSF and periodic boundary conditions. In the numerical tests, we employed different variances of different types blur and Gaussian noise to compare the effectiveness of TRF method and NTRF method, respectively. Meanwhile, the comparison results show that our NTRF method works better than TRF method under different blurs and noises. Furthermore, it is easy to discover that the difference of *PSNR* values using our NTRF method becomes bigger if we set more Gaussian noise and the smaller standard deviation of blur. It demonstrates that our NTRF method behaves better under the condition of lower blur and higher noise, and shows the favourable denoising abil-

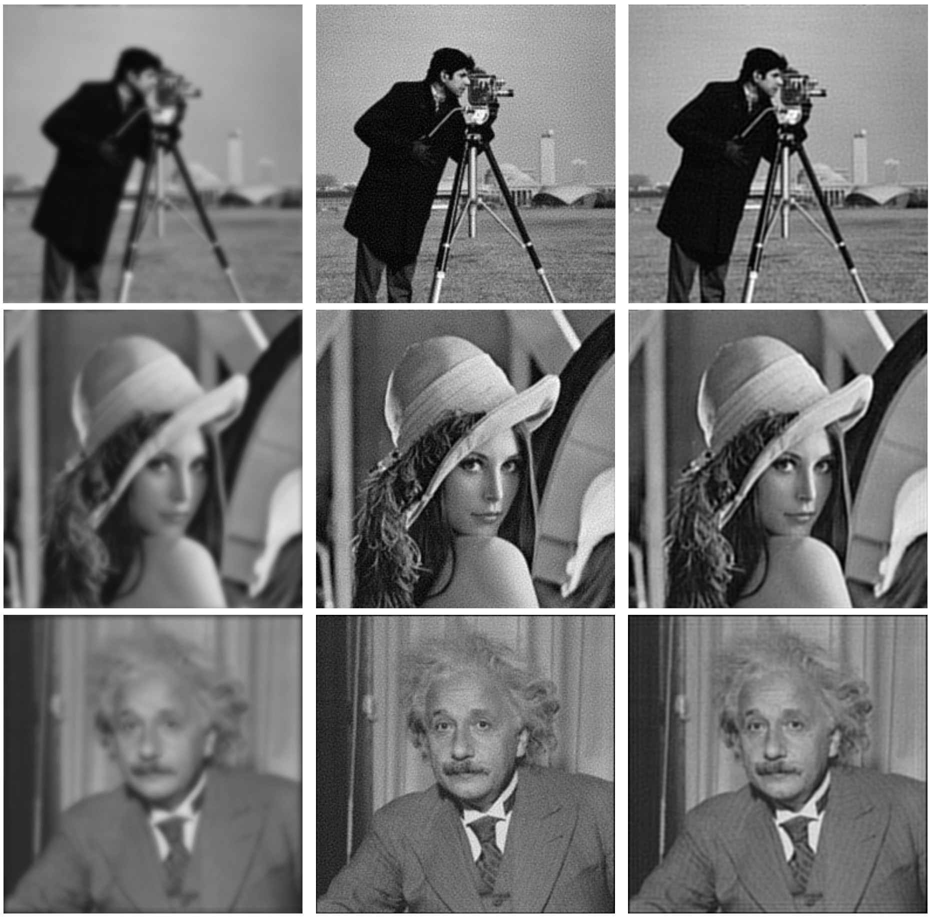

Fig. 5: First column: blurred and noisy images with Moffat blur ($\sigma^2 = 1, 5$) and Gaussian noise (0.5%); Second column: restored images using TRF method ; Third column: restored images using NTRF method.

ity. Moreover, the restored images processed by TRF method contain more noise from the visual results while our NTRF method is not. Due to this, the proposed NTRF method performs better than the TRF method in the denoising process.

The reason why our new method for large-scale problems can be implemented is that we can exploit the fast algorithm of structure matrix to gain the spectral decomposition, e.g., FFT algorithm. Similar as the idea of our method, the another structure matrix which can also gain the spectral decomposition by fast algorithm discrete cosine transformation (DCT) will be gained if we impose reflexive boundary conditions and circularly symmetric PSF on the images. This work will be considered in the following paper.

Table 2: Corresponding PSNR values using TRF method and NTRF method under the different Moffat blurs and Gaussian white noises.

Examples	Variance	$\sigma^2 = 1$		$\sigma^2 = 1.5$	
	Noise	0.5%	1%	0.5%	1%
cameraman	TRF	74.6245	72.1259	74.8573	73.0050
	NTRF	78.5629	76.2789	75.5309	73.8735
lena	TRF	75.9755	73.3959	76.5289	74.5310
	NTRF	80.5133	78.3322	77.6106	75.9278
einstein	TRF	76.5182	73.9135	77.2480	75.1816
	NTRF	81.3081	79.2094	78.7393	76.7393

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