AN OBATA-TYPE THEOREM IN CR GEOMETRY

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Abstract

We discuss a sharp lower bound for the first positive eigenvalue of the sublaplacian on a closed, strictly pseudoconvex pseudohermitian manifold of dimension $2m + 1 \geq 5$. We prove that the equality holds iff the manifold is equivalent to the CR sphere up to a scaling. For this purpose, we establish an Obata-type theorem in CR geometry that characterizes the CR sphere in terms of a nonzero function satisfying a certain overdetermined system. Similar results are proved in dimension 3 under an additional condition.

1. Introduction

In Riemannian geometry, estimates on the first positive eigenvalue of the Laplace operator have played important roles and there have been many beautiful results. We refer the reader to the books by Chavel [**C**] and Schoen-Yau [**SY**]. The following theorem is a classic result.

Theorem 1. *(Lichnerowicz-Obata) Let (M^n, g) be a closed Riemannian manifold with $Ric \geq (n-1)\kappa$, where κ is a positive constant. Then the first positive eigenvalue of the Laplacian satisfies*

$$(1.1) \qquad \lambda_1 \geq n\kappa.$$

Moreover, equality holds iff M is isometric to a round sphere.

The estimate $\lambda_1 \geq n\kappa$ was proved by Lichnerowicz [**L**] in 1958. The characterization of the equality case was established by Obata [**O**] in 1962. In fact, he deduced it from the following more general

Theorem 2. *(Obata [**O**]) Suppose (N^n, g) is a complete Riemannian manifold and u a smooth, nonzero function on N satisfying $D^2 u = -c^2 u g$, then N is isometric to a sphere $\mathbb{S}^n(c)$ of radius $1/c$ in the Euclidean space $\mathbb{R}^{n+1}$.*

In CR geometry, we have the most basic example of a second order differential operator that is subelliptic, namely the sublaplacian Δ_b. On

a closed pseudohermitian manifold, the sublaplacian Δ_b still defines a selfadjoint operator with a discrete spectrum

$$(1.2) \qquad\qquad \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

with $\lim_{k\to\infty} \lambda_k = +\infty$. One would naturally hope that the study of these eigenvalues in CR geometry will be as fruitful as in Riemannian geometry. An analogue of the Lichnerowicz estimate for the sublaplacian on a strictly pseudoconvex pseudohermitian manifold (M^{2m+1}, θ) was proved by Greenleaf [**G**] for $m \geq 3$ and by Li and Luk [**LL**] for $m = 2$. Later it was pointed out that there was an error in the proof of the Bochner formula in [**G**]. Due to this error, the Bochner formula as well as the CR-Lichnerowicz theorem in [**G**, **LL**] are not correctly formulated. The corrected statement is

Theorem 3. *Let (M, θ) be a closed, strictly pseudoconvex pseudo-hermitian manifold of dimension $2m + 1 \geq 5$. Suppose that the Webster pseudo Ricci curvature and the pseudo torsion satisfy*

$$(1.3) \qquad Ric\,(X, X) - \frac{m+1}{2} Tor\,(X, X) \geq \kappa \,|X|^2$$

for all $X \in T^{1,0}(M)$, where κ is a positive constant. Then the first positive eigenvalue of $-\Delta_b$ satisfies

$$(1.4) \qquad\qquad \lambda_1 \geq \frac{m}{m+1} \kappa.$$

The estimate is sharp since one can verify that equality holds on the CR sphere

$$\mathbb{S}^{2m+1} = \{z \in \mathbb{C}^{m+1} : |z| = 1\}$$

with the standard pseudohermitian structure

$$\theta_0 = 2\sqrt{-1}\overline{\partial}(|z|^2 - 1).$$

The natural question whether the equality case characterizes the CR sphere was not addressed in [**G**]. The torsion appearing in (1.3) is a major new obstacle compared with the Riemanian case. This question has been recently studied by several authors and partial results have been established. Chang and Chiu [**CC1**] proved that the equality case characterizes the CR sphere if M has zero torsion. In [**CW**], Chang and Wu proved the rigidity under the condition that the torsion satisfies certain identities involving its covariant derivatives. Ivanov and Vassilev [**IV**] proved the same conclusion under the condition that the divergence of the torsion is zero. Li and Tran [**LT**] considered the special case that M is a real ellipsoid $E(A)$ in $\mathbb{C}^{m+1}$. They computed κ explicitly and proved that the equality, $\lambda_1 = m\kappa/(m+1)$ implies $E(A)$ is the sphere. But in general it is very difficult to handle the torsion.

In this paper, we provide a new method which can handle the torsion and yields an affirmative answer to this question in the general case.

Theorem 4. *If equality holds in Theorem 3, then (M, θ) is equivalent to the sphere $\mathbb{S}^{2m+1}$ with the standard pseudohermitian structure θ_0 up to a scaling, i.e., there exists a CR diffeomorphism $F : M \to \mathbb{S}^{2m+1}$ such that $F^*\theta_0 = c\theta$ for some constant $c > 0$.*

In fact our proof yields the following more general result which can be viewed as the CR analogue of Theorem 2 (for notation see Section 2).

Theorem 5. *Let M be a closed pseudohermitian manifold of dimension $2m + 1 \geq 5$. Suppose there exists a real nonzero function $u \in C^\infty(M)$ satisfying*

$$
\begin{aligned}
u_{\alpha,\beta} &= 0, \\
u_{\alpha,\overline{\beta}} &= \left(-\frac{\kappa}{2(m+1)} u + \frac{\sqrt{-1}}{2} u_0 \right) h_{\alpha\overline{\beta}},
\end{aligned}
$$

for some constant $\kappa > 0$. Then M is equivalent to the sphere $\mathbb{S}^{2m+1}$ with the standard pseudohermitian structure up to a scaling.

The 3-dimensional case is more subtle. It is not clear if these results are true in 3 dimensions. Partial results with additional conditions are discussed in the last section.

The approach in [**CC1**] is to consider a family of adapted Riemannian metrics and try to apply the Lichnerowiz-Obata theorem. This approach requires very complicated calculations to relate the various CR quantities and the corresponding Riemannian ones. In December 2010, the authors found a new approach working directly with the Riemannian Hessian of the eigenfunction. With this approach we generalized the Chang-Chiu result to show that rigidity holds provided the double divergence of the torsion vanishes (see Remark 5 in Section 4). In their preprint [**IV**], Ivanov and Vassilev found the same strategy independently and proved rigidity under the condition that the divergence of the torsion is zero. But to solve the general case requires a new ingredient. We employ a delicate integration by parts argument which requires a good understanding of the critical set of the eigenfunction.

The paper is organized as follows. In Section 2, we review some basic facts in CR geometry. In Section 3, following the argument of Greenleaf we present the proof of Theorem 2 with all the necessary corrections. Theorem 3 is proved in Sections 4 and 5. Finally, in Section 6, we discuss the 3-dimensional case.

2. Preliminaries

Let (M, θ, J) be a strictly pseudoconvex pseudohermitian manifold of dimension $2m + 1$. Thus $G_\theta = d\theta(\cdot, J\cdot)$ defines a Riemannian metric on the contact distribution $H(M) = \ker \theta$. As usual, we set $T^{1,0}(M) = \{w - \sqrt{-1}Jw : w \in H(M)\} \subset T(M) \otimes \mathbb{C}$ and $T^{0,1}(M) = \overline{T^{1,0}(M)}$. Let T be the Reeb vector field and extend J to an endomorphism ϕ on TM by defining $\phi(T) = 0$. We have a natural Riemannian metric g_θ on M such that $TM = \mathbb{R}T \oplus H(M)$ is an orthogonal decomposition and $g_\theta(T, T) = 1$. In the following, we will simply denote g_θ by $\langle \cdot, \cdot \rangle$. Let $\widetilde{\nabla}$ be the Levi-Civita connection of g_θ while ∇ is the Tanaka-Webster connection. For basic facts on CR geometry, one can consult the recent book [**DT**] or the original papers by Tanaka [**T**] and Webster [**W**].

Recall that the Tanaka-Webster connection is compatible with the metric g_θ, but it has a non-trivial torsion. The torsion τ satisfies

$$
\begin{aligned}
\tau(Z, W) &= 0, \\
\tau(Z, \overline{W}) &= \omega(Z, \overline{W})T, \\
\tau(T, J\cdot) &= -J\tau(T, \cdot)
\end{aligned}
$$

for any $Z, W \in T^{1,0}(M)$, where $\omega = d\theta$. We define $A : T(M) \to T(M)$ by $AX = \tau(T, X)$. It is customary to simply call A the torsion of the CR manifold. It is easy to see that A is symmetric. Moreover $AT = 0, AH(M) \subset H(M)$ and $A\phi X = -\phi AX$.

The following formula gives the difference between the two connections $\widetilde{\nabla}$ and ∇. The proof is based on straightforward calculation and can be found in [**DT**].

Proposition 1. *We have*

$$
\begin{aligned}
\widetilde{\nabla}_X Y &= \nabla_X Y + \theta(Y) AX + \frac{1}{2}\left(\theta(Y)\phi X + \theta(X)\phi Y\right) \\
&\quad - \left[\langle AX, Y \rangle + \frac{1}{2}\omega(X, Y)\right]T.
\end{aligned}
$$

Remark 1. We have

$$
\widetilde{\nabla}_X T = AX + \frac{1}{2}\phi(X).
$$

In particular, $\widetilde{\nabla}_T T = 0$. If X and Y are both horizontal i.e., $X, Y \in H(M)$, then

$$
\widetilde{\nabla}_X Y = \nabla_X Y - \left[\langle AX, Y \rangle + \frac{1}{2}\omega(X, Y)\right]T.
$$

In the following, we will always work with a local frame $\{T_\alpha : \alpha = 1, \cdots, m\}$ for $T^{1,0}(M)$. Then $\{T_\alpha, T_{\overline{\alpha}} = \overline{T_\alpha}, T_0 = T\}$ is a local frame for $T(M) \otimes \mathbb{C}$. Let $h_{\alpha\overline{\beta}} = -i\omega\left(T_\alpha, T_{\overline{\beta}}\right) = g_\theta\left(T_\alpha, T_{\overline{\beta}}\right)$ be the components of

the Levi form. For a smooth function u on M, we will use notations such as $u_{\alpha,\overline{\beta}}$ to denote its covariant derivatives with respect to the Tanaka-Webster connection ∇. Let D^2u be the Hessian of u with respect to the Riemannian metric g_θ.

By straightforward calculation using Proposition 1, one can derive

Proposition 2. *We have the following formulas*

$$D^2u\,(T,T) = u_{0,0},$$

$$D^2u\,(T,T_\alpha) = u_{\alpha,0} - \frac{\sqrt{-1}}{2}u_\alpha,$$

$$D^2u\,(T_\alpha,T_\beta) = u_{\alpha,\beta} + A_{\alpha\beta}u_0,$$

$$D^2u\left(T_\alpha,T_{\overline{\beta}}\right) = u_{\alpha,\overline{\beta}} - \frac{\sqrt{-1}}{2}h_{\alpha\overline{\beta}}u_0.$$

In doing calculations we will need to use repeatedly the following formulas which can be found in [**DT**] or [**Lee**].

Proposition 3. *We have the following formulas*

$$
\begin{aligned}
u_{0,\alpha} &= u_{\alpha,0} + A_\alpha^{\overline{\beta}}u_{\overline{\beta}}, \\
u_{\alpha,\beta} &= u_{\beta,\alpha}, \\
u_{\alpha,\overline{\beta}} &= u_{\overline{\beta},\alpha} + \sqrt{-1}h_{\alpha\overline{\beta}}u_0, \\
u_{\alpha,0\beta} &= u_{\alpha,\beta0} + A_\beta^{\overline{\gamma}}u_{\alpha,\overline{\gamma}} + R_{\beta0\alpha}^\sigma u_\sigma \\
&= u_{\alpha,\beta0} + A_\beta^{\overline{\gamma}}u_{\alpha,\overline{\gamma}} - A_{\alpha\beta,\overline{\gamma}}h^{\sigma\overline{\gamma}}u_\sigma, \\
u_{\alpha,0\overline{\beta}} &= u_{\alpha,\overline{\beta}0} + u_{a,\gamma}h^{\gamma\overline{\nu}}A_{\overline{\nu}\overline{\beta}} + h^{\gamma\overline{\nu}}A_{\overline{\nu}\overline{\beta},\alpha}u_\gamma, \\
u_{\alpha,\overline{\beta}\overline{\gamma}} &= u_{\alpha,\overline{\gamma}\overline{\beta}} + \sqrt{-1}\left(h_{\alpha\overline{\beta}}A_{\overline{\gamma}}^\sigma - h_{\alpha\overline{\gamma}}A_{\overline{\beta}}^\sigma\right)u_\sigma, \\
u_{\alpha,\beta\overline{\gamma}} &= u_{\alpha,\overline{\gamma}\beta} + \sqrt{-1}h_{\beta\overline{\gamma}}u_{\alpha,0} - R_{\beta\overline{\gamma}\alpha}^\sigma u_\sigma, \\
u_{\alpha,\beta\gamma} &= u_{\alpha,\gamma\beta} - R_{\beta\gamma\alpha}^\sigma u_\sigma \\
&= u_{\alpha,\gamma\beta} + \sqrt{-1}\left(A_{\alpha\gamma}u_\beta - A_{\alpha\beta}u_\gamma\right).
\end{aligned}
$$

Remark 2. Our convention for the curvature tensor is

$$R\,(X,Y,Z,W) = \left\langle -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z, W\right\rangle.$$

3. The estimate on λ_1

In this section, we prove the estimate on λ_1 following Greenleaf [**G**]. This serves two purposes. First, there is a mistake in [**G**] as pointed out in [**GL**] and [**CC1**]. This has caused some confusion (e.g., see the presentation in [**DT**]) and we hope to clarify the whole situation. Secondly, we need to analyze the proof when we address the equality case.

From now on, we always work with a local unitary frame $\{T_\alpha : \alpha = 1, \cdots, m\}$ for $T^{1,0}(M)$. Given a smooth function u, its sublaplacian is given by

$$\Delta_b u = \sum_\alpha u_{\alpha,\overline{\alpha}} + u_{\overline{\alpha},\alpha}.$$

We have the following Bochner formula.

Theorem 6. *Let* $|\partial_b u|^2 = u_\alpha u_{\overline{\alpha}}$. *Then*

$$
\begin{aligned}
\frac{1}{2}\Delta_b |\partial_b u|^2 &= |u_{\alpha,\beta}|^2 + \left|u_{\alpha,\overline{\beta}}\right|^2 + \frac{1}{2}\left[(\Delta_b u)_\alpha u_{\overline{\alpha}} + (\Delta_b u)_{\overline{\alpha}} u_\alpha\right] \\
&\quad + R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}} + \frac{m}{2}\sqrt{-1}\left[A_{\alpha\sigma} u_{\overline{\alpha}} u_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} u_\sigma u_\alpha\right] \\
&\quad + \sqrt{-1}\left(u_{\overline{\beta}} u_{\beta,0} - u_\beta u_{\overline{\beta},0}\right).
\end{aligned}
$$

Remark 3. This was first derived by Greenleaf [**G**]. But due to a mistake in calculation pointed out in [**GL**] and [**CC1**], the coefficient $\frac{m}{2}$ on the RHS was mistaken to be $\frac{m-2}{2}$.

The following formulas are also derived in [**G**].

Lemma 1. *Let* u *be a smooth function on a closed pseudohermitian manifold* M *of dimension* $2m + 1$. *We have the following integral equalities*

$$\int_M \sqrt{-1}\left(u_{\overline{\beta}} u_{\beta,0} - u_\beta u_{\overline{\beta},0}\right) = \frac{2}{m}\int_M \left|u_{\alpha,\overline{\beta}}\right|^2 - |u_{\alpha,\beta}|^2 - R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}},$$

$$
\begin{aligned}
&\int_M \sqrt{-1}\left(u_{\overline{\beta}} u_{\beta,0} - u_\beta u_{\overline{\beta},0}\right) \\
&= \int_M \frac{1}{m}(\Delta_b u)^2 - \frac{4}{m}\left|\sum u_{\alpha,\overline{\alpha}}\right|^2 - \sqrt{-1}\left(A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} - A_{\overline{\alpha\beta}} u_\alpha u_\beta\right),
\end{aligned}
$$

$$
\begin{aligned}
&(m-2)\int_M \sqrt{-1}\left(u_{\overline{\beta}} u_{\beta,0} - u_\beta u_{\overline{\beta},0}\right) \\
&= \int_M 4\left|u_{\alpha,\overline{\beta}}\right|^2 - (\Delta_b u)^2 + \sqrt{-1}m\left(A_{\alpha\sigma} u_{\overline{\alpha}} u_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} u_\sigma u_\alpha\right).
\end{aligned}
$$

We can now state the main estimate on λ_1. For completion and future application in the next section, we also provide the detail of the proof here.

Theorem 7. *Let* M *be a closed pseudohermitian manifold of dimension* $2m + 1 \geq 5$. *Suppose for any* $X \in T^{1,0}(M)$

$$(3.1) \qquad Ric(X, X) - \frac{m+1}{2}Tor(X, X) \geq \kappa |X|^2,$$

where κ *is a positive constant. Then the first eigenvalue of* $-\Delta_b$ *satisfies*

$$\lambda_1 \geq \frac{m}{m+1}\kappa.$$

Remark 4. In terms of our local unitary frame, the assumption (3.1) means that for any $X = f_\alpha T_\alpha$

$$R_{\alpha\overline{\sigma}} f_\sigma f_{\overline{\alpha}} + \frac{m+1}{2} \sqrt{-1} \left[A_{\alpha\sigma} f_{\overline{\alpha}} f_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} f_\sigma f_\alpha \right] \geq \kappa \sum_\alpha |f_\alpha|^2 .$$

Proof. Suppose $-\Delta_b u = \lambda_1 u$. Applying the Bochner formula, we have

$$\begin{aligned}
0 &= \int |u_{\alpha,\beta}|^2 + \left| u_{\alpha,\overline{\beta}} \right|^2 - \lambda_1 |\partial_b u|^2 \\
&\quad + R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}} + \frac{m}{2} \sqrt{-1} \left[A_{\alpha\sigma} u_{\overline{\alpha}} u_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} u_\sigma u_\alpha \right] \\
&\quad + \sqrt{-1} \left(u_{\overline{\beta}} u_{\beta,0} - u_\beta u_{\overline{\beta},0} \right).
\end{aligned}$$

We write the last term as c times the first identity plus $(1-c)$ times the second identity of Lemma 1,

$$\begin{aligned}
0 &= \int |u_{\alpha,\beta}|^2 + \left| u_{\alpha,\overline{\beta}} \right|^2 - \lambda_1 |\partial_b u|^2 \\
&\quad + R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}} + \frac{m}{2} \sqrt{-1} \left[A_{\alpha\sigma} u_{\overline{\alpha}} u_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} u_\sigma u_\alpha \right] \\
&\quad + \frac{2c}{m} \int \left(\left| u_{\alpha,\overline{\beta}} \right|^2 - |u_{\alpha,\beta}|^2 - R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}} \right) \\
&\quad + \int \frac{1-c}{m} \lambda_1^2 u^2 - \frac{4(1-c)}{m} \left| \sum u_{\alpha,\overline{\alpha}} \right|^2 \\
&\quad - (1-c) \sqrt{-1} \left(A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} - A_{\overline{\alpha\beta}} u_\alpha u_\beta \right) \\
&= \int \left(1 - \frac{2c}{m} \right) |u_{\alpha,\beta}|^2 + \left(1 + \frac{2c}{m} \right) \left| u_{\alpha,\overline{\beta}} \right|^2 + \left(-1 + \frac{2(1-c)}{m} \right) \lambda_1 |\partial_b u|^2 \\
&\quad + \left(1 - \frac{2c}{m} \right) R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}} + \left(\frac{m}{2} - 1 + c \right) \sqrt{-1} \left[A_{\alpha\sigma} u_{\overline{\alpha}} u_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} u_\sigma u_\alpha \right] \\
&\quad - \frac{4(1-c)}{m} \left| \sum u_{\alpha,\overline{\alpha}} \right|^2 .
\end{aligned}$$

Since $\sum_{\alpha,\beta=1}^m \left| u_{\alpha,\overline{\beta}} \right|^2 \geq \left| \sum u_{\alpha,\overline{\alpha}} \right|^2 / m$, we have

$$\begin{aligned}
0 &\geq \int \left(1 - \frac{2c}{m} \right) |u_{\alpha,\beta}|^2 + \left(-1 + \frac{2(1-c)}{m} \right) \lambda_1 |\partial_b u|^2 \\
&\quad + \left(1 - \frac{2c}{m} \right) R_{\alpha\overline{\sigma}} u_\sigma u_{\overline{\alpha}} + \left(\frac{m}{2} - 1 + c \right) \sqrt{-1} \left[A_{\alpha\sigma} u_{\overline{\alpha}} u_{\overline{\sigma}} - A_{\overline{\alpha\sigma}} u_\sigma u_\alpha \right] + \\
&\quad + \left[\left(1 + \frac{2c}{m} \right) \frac{1}{m} - \frac{4(1-c)}{m} \right] \left| \sum u_{\alpha,\overline{\alpha}} \right|^2 .
\end{aligned}$$

We choose c such that

$$\left(1 + \frac{2c}{m} \right) \frac{1}{m} - \frac{4(1-c)}{m} = 0,$$

i.e., $c = 3m/(4m+2)$. Then

$$0 \geq \int \frac{2(m-1)}{2m+1}|u_{\alpha,\beta}|^2 - \frac{2(m^2-1)}{m(2m+1)}\lambda_1|\partial_b u|^2$$
$$+\frac{2(m-1)}{2m+1}\left\{R_{\alpha\bar\sigma}u_\sigma u_{\bar\alpha} + \frac{m+1}{2}\sqrt{-1}[A_{\alpha\sigma}u_{\bar\alpha}u_{\bar\sigma} - A_{\overline{\alpha\sigma}}u_\sigma u_\alpha]\right\}.$$

Therefore

$$\frac{(m-1)}{2m+1}\int 2\left(\kappa - \frac{m+1}{m}\lambda_1\right)|\partial_b u|^2 + |u_{\alpha,\beta}|^2 \leq 0.$$

It follows that $\lambda_1 \geq m\kappa/(m+1)$ when $m \geq 2$.   q.e.d.

4. Equality case

We now discuss the equality case. By scaling, we can assume $\kappa = (m+1)/2$ and thus $\lambda = m/2$.

Proposition 4. *If equality holds in Theorem 7, we must have*

$$(4.1) \qquad\qquad u_{\alpha,\beta} = 0,$$

$$(4.2) \qquad\qquad u_{\alpha,\bar\beta} = \left(-\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0\right)\delta_{\alpha\beta},$$

$$(4.3) \qquad\qquad u_{0,\alpha} = 2A_{\alpha\sigma}u_{\bar\sigma} + \frac{\sqrt{-1}}{2}u_\alpha,$$

$$(4.4) \qquad\qquad u_{0,0} = -\frac{1}{4}u + \frac{4}{m}\mathrm{Im}A_{\alpha\sigma,\bar\alpha}u_{\bar\sigma}.$$

Moreover, at any point where $\partial_b u \neq 0$

$$(4.5) \qquad\qquad \sqrt{-1}A_{\alpha\beta} = \frac{Q}{|\partial_b u|^4}u_\alpha u_\beta,$$

where $Q = \sqrt{-1}A_{\alpha\sigma}u_{\bar\alpha}u_{\bar\sigma}$.

Proof. If equality holds, we must have $u_{\alpha,\beta} = 0$ and

$$(4.6) \qquad\qquad u_{\alpha,\bar\beta} = f\delta_{\alpha\beta},$$

where f is a complex-valued function. Taking the conjugate of (4.6) yields

$$\begin{aligned}
\bar f\delta_{\alpha\beta} &= u_{\bar\alpha,\beta} \\
&= u_{\beta,\bar\alpha} - \sqrt{-1}\delta_{\alpha\beta}u_0 \\
&= \left(f - \sqrt{-1}u_0\right)\delta_{\alpha\beta}.
\end{aligned}$$

Hence $\mathrm{Im}f = \frac{1}{2}u_0$. We also have

$$\begin{aligned}
\frac{m}{2}u &= -\Delta_b u \\
&= -u_{\alpha,\bar\alpha} - u_{\bar\alpha,\alpha} \\
&= -m\left(f + \bar f\right).
\end{aligned}$$

Thus $\mathrm{Re} f = -\frac{1}{4}u$. Therefore

(4.7)
$$f = -\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0.$$

This proves (4.2).

Differentiating (4.1), we have

$$\begin{aligned}
0 &= u_{\alpha,\beta\gamma} - u_{\alpha,\gamma\beta} \\
&= \sqrt{-1}\left(A_{\alpha\gamma}u_\beta - A_{\alpha\beta}u_\gamma\right).
\end{aligned}$$

Therefore

$$A_{\alpha\gamma}u_\beta - A_{\alpha\beta}u_\gamma = 0.$$

From this we easily obtain (4.5).

Differentiating (4.6), we have

$$\begin{aligned}
f_{\bar{\gamma}}\delta_{\alpha\beta} &= u_{\alpha,\bar{\beta}\bar{\gamma}} \\
&= u_{\alpha,\bar{\gamma}\bar{\beta}} + \sqrt{-1}\left(h_{\alpha\bar{\beta}}A_{\bar{\gamma}}^\sigma - h_{\alpha\bar{\gamma}}A_{\bar{\beta}}^\sigma\right)u_\sigma \\
&= f_{\bar{\beta}}\delta_{\alpha\gamma} + \sqrt{-1}\left(\delta_{\alpha\beta}A_{\bar{\gamma}}^\sigma - \delta_{\alpha\gamma}A_{\bar{\beta}}^\sigma\right)u_\sigma,
\end{aligned}$$

i.e., $\left(f_{\bar{\gamma}} - \sqrt{-1}A_{\bar{\gamma}}^\sigma u_\sigma\right)\delta_{\alpha\beta} = \left(f_{\bar{\beta}} - \sqrt{-1}A_{\bar{\beta}}^\sigma u_\sigma\right)\delta_{\alpha\gamma}$. It follows $f_{\bar{\gamma}} - \sqrt{-1}A_{\bar{\gamma}}^\sigma u_\sigma = 0$. Using (4.7), this yields

(4.8)
$$\sqrt{-1}A_{\alpha\sigma}u_{\bar{\sigma}} = \frac{1}{2}u_\alpha + \sqrt{-1}u_{\alpha,0}.$$

This then implies (4.3) by using the first identity of Proposition 3.

To prove the last identity, taking the trace of (4.2) we obtain

$$m\left(-\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0\right) = u_{\alpha,\bar{\alpha}}.$$

Differentiating and using Proposition 3 yields

$$\begin{aligned}
m\left(-\frac{1}{4}u_0 + \frac{\sqrt{-1}}{2}u_{0,0}\right) &= u_{\alpha,\bar{\alpha}0} \\
&= u_{\alpha,0\bar{\alpha}} - A_{\bar{\alpha}\bar{\sigma},\alpha}u_\sigma \\
&= \left(A_{\alpha\sigma}u_{\bar{\sigma}} + \frac{\sqrt{-1}}{2}u_\alpha\right)_{,\bar{\alpha}} - A_{\bar{\alpha}\bar{\sigma},\alpha}u_\sigma \\
&= A_{\alpha\sigma,\bar{\alpha}}u_{\bar{\sigma}} + \frac{\sqrt{-1}}{2}u_{\alpha,\bar{\alpha}} - A_{\bar{\alpha}\bar{\sigma},\alpha}u_\sigma \\
&= 2\sqrt{-1}\mathrm{Im}A_{\alpha\sigma,\bar{\alpha}}u_{\bar{\sigma}} + m\frac{\sqrt{-1}}{2}\left(-\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0\right) \\
&= -\frac{m}{4}u_0 + \sqrt{-1}\left(2\mathrm{Im}A_{\alpha\sigma,\bar{\alpha}}u_{\bar{\sigma}} - \frac{m}{8}u\right).
\end{aligned}$$

Taking the imaginary part yields (4.4).   q.e.d.

Lemma 2. *We also have*

(4.9) $$R_{\alpha\overline{\sigma}}u_\sigma + (m+1)\sqrt{-1}A_{\alpha\sigma}u_{\overline{\sigma}} = \frac{m+1}{2}u_\alpha.$$

Proof. This follows easily from the fact that equality is achieved by $X = u_{\overline{\sigma}}T_\sigma$ in (3.1). We can also derive it in the following way. Differentiating (4.1) and using (4.6) yields

$$
\begin{aligned}
0 &= u_{\alpha,\beta\overline{\gamma}} \\
&= u_{\alpha,\overline{\gamma}\beta} + \sqrt{-1}\delta_{\beta\gamma}u_{\alpha,0} - R_{\beta\overline{\gamma}\alpha\overline{\sigma}}u_\sigma \\
&= \left(-\frac{1}{4}u_\beta + \frac{\sqrt{-1}}{2}u_{0,\beta}\right)\delta_{\alpha\gamma} + \sqrt{-1}u_{\alpha,0}\delta_{\beta\gamma} - R_{\beta\overline{\gamma}\alpha\overline{\sigma}}u_\sigma.
\end{aligned}
$$

Taking the trace over β and γ, we obtain

$$
\begin{aligned}
0 &= -\frac{1}{4}u_\alpha + \frac{\sqrt{-1}}{2}u_{0,\alpha} + m\sqrt{-1}u_{\alpha,0} + R_{\alpha\overline{\sigma}}u_\sigma \\
&= -\frac{1}{4}u_\alpha + \frac{\sqrt{-1}}{2}A_{\alpha\sigma}u_{\overline{\sigma}} + \left(m + \frac{1}{2}\right)\sqrt{-1}u_{\alpha,0} + R_{\alpha\overline{\sigma}}u_\sigma \\
&= -\frac{(m+1)}{2}u_\alpha + (m+1)\sqrt{-1}A_{\alpha\sigma}u_{\overline{\sigma}} + R_{\alpha\overline{\sigma}}u_\sigma,
\end{aligned}
$$

where in the last step we have used (4.8) to replace $u_{\alpha,0}$.   q.e.d.

Lemma 3. Q *is real and nonpositive.*

Remark 5. This lemma will not be needed in the proof of the rigidity. However, it yields a quick proof if we assume the following extra condition

$$A_{\alpha\beta,\overline{\alpha}\overline{\beta}} = 0,$$

i.e., the double divergence of the torsion is zero. Indeed, integrating by parts and using (4.1) we obtain

$$
\begin{aligned}
\int_M Q &= -\sqrt{-1}\int_M A_{\alpha\beta,\overline{\alpha}}u_{\overline{\beta}}u \\
&= \frac{\sqrt{-1}}{2}\int_M A_{\alpha\beta,\overline{\alpha}\overline{\beta}}u^2 \\
&= 0.
\end{aligned}
$$

As Q is nonpositive, this implies that $Q = 0$. Therefore $A = 0$. See the discussion on the torsion-free case below.

Proof. From (4.9) we have

$$R_{\alpha\overline{\sigma}}u_{\overline{\alpha}}u_\sigma + (m+1)Q = \frac{m+1}{2}|\partial_b u|^2.$$

This shows that Q is real. Taking the conjugate, we also have $Q = -\sqrt{-1}A_{\overline{\alpha}\overline{\sigma}}u_\alpha u_\sigma$. In the inequality (3.1), taking $X = e^{it}u_{\overline{\alpha}}T_\alpha$ yields

$$R_{\alpha\overline{\sigma}}u_{\overline{\alpha}}u_\sigma + (m+1)Q\cos 2t \geq \frac{m+1}{2}|\partial_b u|^2.$$

Therefore $Q \leq 0$.   q.e.d.

Theorem 4 follows from

Lemma 4. *The torsion $A = 0$.*

The proof of this statement will be presented in the next section.

Assuming this lemma, Theorem 4 then follows from Chang and Chiu [**CC1**]. In the following, we present a simpler and more direct argument. Since A vanishes, we have

$$(4.10) \qquad u_{0,\alpha} \;=\; u_{\alpha,0} = \frac{\sqrt{-1}}{2} u_\alpha,$$

$$(4.11) \qquad u_{\alpha,\beta} \;=\; 0, u_{0,0} = -\frac{1}{4} u$$

$$(4.12) \qquad u_{\alpha,\overline{\beta}} \;=\; \left(-\frac{1}{4} u + \frac{\sqrt{-1}}{2} u_0 \right) \delta_{\alpha\beta}.$$

By Proposition 2, we obtain

Proposition 5. *Let $D^2 u$ be the Hessian of u with respect to the Riemannian metric g_θ. Then*

$$D^2 u = -\frac{1}{4} u g_\theta.$$

By Obata's theorem (Theorem 2), (M, g_θ) is isometric to the sphere $\mathbb{S}^{2m+1}$ with the metric $g_0 = 4g_c$, where g_c is the canonical metric. Without loss of generality, we can take (M, g_θ) to be $(\mathbb{S}^{2m+1}, g_0)$. Then θ is a pseudohermitian structure on $\mathbb{S}^{2m+1}$ whose adapted metric is g_0 and the associated Tanaka-Webster connection is torsion-free. It is a well known fact that the Reeb vector field T is then a Killing vector field for g_0 (this can be easily proved by the first formula in Remark 1). Therefore there exists a skew-symmetric matrix A such that for all $X \in \mathbb{S}^{2m+1}, T(X) = AX$; here we use the obvious identification between $z = (z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1}$ and $X = (x_1, y_1, \ldots, x_{m+1}, y_{m+1}) \in \mathbb{R}^{2m+2}$. Changing coordinates by an orthogonal transformation we can assume that A is of the following form

$$A = \begin{bmatrix} 0 & a_1 & & & \\ a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{m+1} \\ & & & a_{m+1} & 0 \end{bmatrix}$$

where $a_i \geq 0$. Therefore

$$T = \sum_i a_i \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right)$$

Since T is of unit length we must have

$$4 \sum_i a_i^2 (x_i^2 + y_i^2) = 1$$

on $\mathbb{S}^{2m+1}$. Therefore all the a_i's are equal to $1/2$. It follows that

$$\theta = g_0(T, \cdot) = 2\sqrt{-1} \partial (|z|^2 - 1).$$

This finishes the proof of Theorem 4.

5. Proof of Lemma 4

Let $\psi = |A|$. In local unitary frame we have

$$\psi = \sqrt{\sum |A_{\alpha\beta}|^2}.$$

We note that ψ is continuous and ψ^2 is smooth. Let $K = \{\partial_b u = 0\}$. By (4.5), on $M \backslash K$ ψ is smooth and

$$\psi = -\frac{Q}{|\partial_b u|^2},$$

or

$$A_{\alpha\beta} = \sqrt{-1} \psi \frac{u_\alpha u_\beta}{|\partial_b u|^2}.$$

Lemma 5. *The compact set K is of Hausdorff dimension at most $n-2$ ($n = 2m+1 = \dim M$). More precisely we have a countable union $K = \cup_{i=1}^{\infty} E_i$, where each E_i has finite $n-2$ dimensional Hausdorff measure: $\mathcal{H}^{n-2}(E_i) < \infty$.*

Remark 6. Here the Hausdorff dimension is defined using the distance function of the Riemannian metric g_θ.

Proof. We have $K = K_1 \sqcup K_2$, where

$$K_1 = \{p \in K : u(p) \neq 0 \text{ or } u_0(p) \neq 0\},$$
$$K_2 = \{p \in M : u(p) = 0 \text{ and } du(p) = 0\}.$$

We first prove that K_1 is of Hausdorff dimension $n-2$. Suppose $p \in K_1$. In a local unitary frame $\{T_\alpha\}$ we have by Proposition 4

$$u_{\alpha,\overline{\beta}} = \left(-\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0\right)\delta_{\alpha\beta}.$$

We write $T_\alpha = X_\alpha - \sqrt{-1}Y_\alpha$ in terms of the real and imaginary parts. Then we have $2m$ real local vector fields $\{Z_i\}$, where $Z_i = X_i$ for $i \leq m$ and $Z_i = Y_{i-m}$ for $i > m$. Along K_1 the above equation takes the following form

$$X_\beta X_\alpha u + Y_\beta Y_\alpha u = -\frac{1}{4}u\delta_{\alpha\beta},$$
$$Y_\beta X_\alpha u - X_\beta Y_\alpha u = \frac{1}{2}u_0\delta_{\alpha\beta}.$$

Since either $u(p) \neq$ or $u_0(p) \neq 0$, from the above equation it is straight-forward to check that there exists $i < j$ such that the local map $F :$ $q \to (Z_i u(q), Z_j u(q))$ from M to $\mathbb{R}^2$ is of rank 2 at p. By the implicit function theorem, $F^{-1}(0)$ is a codimension 2 submanifold at p. As $K_1 \subset F^{-1}(0)$, we conclude that K_1 is of Hausdorff dimension at most $n-2$.

To handle K_2, we note that u satisfies the following 2nd order elliptic equation by Proposition 4

$$\Delta u = -\left(\frac{m}{2} + \frac{1}{4}\right) u + \frac{4}{m} \mathrm{Im} A_{\alpha\sigma,\overline{\alpha}} u_{\overline{\sigma}}.$$

As K_2 is the singular nodal set of u, we have (see, e.g., [**HHL**])

$$\mathcal{H}^{n-2}(K_2) < \infty.$$

<div align="right">q.e.d.</div>

Lemma 6. *We have on $M \backslash K$*

$$\mathrm{Re} \sum \psi_\alpha u_{\overline{\alpha}} = 0.$$

Proof. Let $v = Tu = u_0$. By the second formula of Lemma 1

$$(5.1) \quad \int_M \sqrt{-1}\left(v_{\overline{\beta}} v_{\beta,0} - v_\beta v_{\overline{\beta},0}\right) = \int_M \frac{1}{m}(\Delta_b v)^2 - \frac{4}{m}\left|\sum v_{\alpha,\overline{\alpha}}\right|^2$$
$$- \sqrt{-1}\left(A_{\alpha\beta} v_{\overline{\alpha}} v_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}} v_\alpha v_\beta\right).$$

We will use Proposition 4 to simplify both sides. On $M \backslash K$

$$v_\alpha = u_{0,\alpha} = \sqrt{-1}\left(2\psi + \frac{1}{2}\right) u_\alpha,$$

$$v_{\alpha,\overline{\beta}} = \sqrt{-1} 2\psi_{\overline{\beta}} u_\alpha + \left(2\psi + \frac{1}{2}\right)\left(-\frac{1}{2} u_0 - \frac{\sqrt{-1}}{4} u\right) \delta_{\alpha\beta}.$$

Differentiating the first equation yields $v_{\alpha,\beta} = 2\sqrt{-1}\psi_\beta u_\alpha$. As $v_{\alpha,\beta} = v_{\beta,\alpha}$, we have $\psi_\beta u_\alpha = \psi_\alpha u_\beta$. As a result, on $M \backslash K$ there are smooth real functions a, b such that

$$\psi_\alpha = (a + ib) u_\alpha.$$

A simple calculation shows

$$\sqrt{-1}\left(v_{\overline{\beta}} v_{\beta,0} - v_\beta v_{\overline{\beta},0}\right) = \sqrt{-1}\left(2\psi + \frac{1}{2}\right)^2 \left(u_{\overline{\beta}} u_{\beta,0} - u_\beta u_{\overline{\beta},0}\right)$$
$$= -\left(2\psi + \frac{1}{2}\right)^2 (2\psi + 1) |\partial_b u|^2.$$

The integrand of the right hand side can be simplified as follows:

$$\frac{1}{m}\left(\Delta_b v\right)^2 - \frac{4}{m}\left|\sum v_{\alpha,\overline{\alpha}}\right|^2 - \sqrt{-1}\left(A_{\alpha\beta}v_{\overline{\alpha}}v_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}v_\alpha v_\beta\right)$$

$$= -\frac{4}{m}\left|\mathrm{Im}\sum v_{\alpha,\overline{\alpha}}\right|^2 - \sqrt{-1}\left(A_{\alpha\beta}v_{\overline{\alpha}}v_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}v_\alpha v_\beta\right)$$

$$= -\frac{4}{m}\left(2a\left|\partial_b u\right|^2 - \frac{m}{4}\left(2\psi + \frac{1}{2}\right)u\right)^2 - \left(2\psi + \frac{1}{2}\right)^2 2\psi\left|\partial_b u\right|^2$$

$$= -\frac{m}{4}\left(2\psi + \frac{1}{2}\right)^2 u^2 + 4a\left|\partial_b u\right|^2\left(2\psi + \frac{1}{2}\right)u - \frac{16}{m}a^2\left|\partial_b u\right|^4$$

$$\quad - \left(2\psi + \frac{1}{2}\right)^2 2\psi\left|\partial_b u\right|^2.$$

Integrating by parts (see the remark below) yields

$$\int -\frac{m}{4}\left(2\psi + \frac{1}{2}\right)^2 u^2 = \mathrm{Re}\int\left(2\psi + \frac{1}{2}\right)^2 u_{\alpha,\overline{\alpha}}u$$

$$= -\int\left(2\psi + \frac{1}{2}\right)^2\left|\partial_b u\right|^2 - 4\mathrm{Re}\int\left(2\psi + \frac{1}{2}\right)u\psi_{\overline{\alpha}}u_\alpha$$

$$= -\int\left(2\psi + \frac{1}{2}\right)^2\left|\partial_b u\right|^2 - 4\int\left(2\psi + \frac{1}{2}\right)au\left|\partial_b u\right|^2.$$

Plugging these calculations in (5.1), we obtain

$$\frac{16}{m}\int_M a^2\left|\partial_b u\right|^4 = 0.$$

Therefore $\mathrm{Re}\sum\psi_\alpha u_{\overline{\alpha}} = a\left|\partial_b u\right|^2 = 0.$ q.e.d.

Remark 7. We can justify the integration by parts in the following way. We note that the compact set K has zero 2-capacity by Lemma 5 (cf. [**EG**, **HKM**]). Therefore there exists a sequence $\chi_k \in C_c^\infty(M\backslash K)$ s.t. $\chi_k \to 1$ in $W^{1,2}(M)$. Then

$$-\int\left(2\phi + \frac{1}{2}\right)^2 u_{\alpha,\overline{\alpha}}u\left(\chi_k\right)^2 = \int\left(2\phi + \frac{1}{2}\right)^2\left|\partial_b u\right|^2\left(\chi_k\right)^2$$

$$+ 2\int\left(2\phi + \frac{1}{2}\right)\phi_{\overline{\alpha}}u_\alpha u\left(\chi_k\right)^2 + E_k,$$

where

$$E_k = 2\int\left(2\phi + \frac{1}{2}\right)^2 u_\alpha u\chi_k\left(\chi_k\right)_{\overline{\alpha}}.$$

It is easy to see that $\lim_{k\to\infty}E_k = 0$. Therefore, letting $k\to\infty$ yields

$$-\int\left(2\phi + \frac{1}{2}\right)^2 u_{\alpha,\overline{\alpha}}u = \int\left(2\phi + \frac{1}{2}\right)^2\left|\partial_b u\right|^2 + 4\int\left(2\phi + \frac{1}{2}\right)\phi_{\overline{\alpha}}u_\alpha u.$$

We now prove Lemma 4. Suppose ψ^2 is not identically zero. Let ε^2 be a regular value of ψ^2 such that $\{\psi \geq \varepsilon\}$ is a nonempty domain with smooth boundary. Define

$$F = \begin{cases} \psi(\psi - \varepsilon)^2 & \text{if } \psi \geq \varepsilon, \\ 0 & \text{if } \psi < \varepsilon. \end{cases}$$

Then $F \in W^{1,2}(M)$. Integrating by parts, we obtain

$$
\begin{aligned}
(5.2) \qquad \int_M F u^{2(k+1)} &= -\frac{4}{m} \mathrm{Re} \int_M F \, |u|^{2k+1} \, u_{\alpha, \overline{\alpha}} \\
&= \frac{4(2k+1)}{m} \mathrm{Re} \int_M F u^{2k} \, |\partial_b u|^2 \\
&\quad + \frac{4}{m} \int_{\{\psi \geq \varepsilon\}} \left(3\psi^2 - 4\varepsilon\psi + \varepsilon^2\right) |u|^{2k+1} \, \mathrm{Re} u_\alpha \psi_{\overline{\alpha}} \\
&= \frac{4(2k+1)}{m} \int_M F u^{2k} \, |\partial_b u|^2,
\end{aligned}
$$

by Lemma 6. Integrating by parts again, we have

$$
\begin{aligned}
\int_M & F u^{2k} \, |\partial_b u|^2 \\
&= \int_M u^{2k} (\psi - \varepsilon)_+^2 \, \psi \, |\partial_b u|^2 \\
&= -\mathrm{Re}\sqrt{-1} \int_M u^{2k} (\psi - \varepsilon)_+^2 \, A_{\alpha\beta} u_{\overline{\alpha}} u_{\overline{\beta}} \\
&= -\mathrm{Re}\frac{\sqrt{-1}}{2k+1} \int_M (\psi - \varepsilon)_+^2 \, A_{\alpha\beta} \left(u^{2k+1}\right)_{\overline{\alpha}} u_{\overline{\beta}} \\
&= \mathrm{Re}\frac{\sqrt{-1}}{2k+1} \Bigg(\int_M (\psi - \varepsilon)_+^2 \, A_{\alpha\beta,\overline{\alpha}} u^{2k+1} u_{\overline{\beta}} \\
&\qquad\qquad + 2 \int_{\{\psi \geq \varepsilon\}} (\psi - \varepsilon) \, u^{2k+1} \psi_{\overline{\alpha}} A_{\alpha\beta} u_{\overline{\beta}} \Bigg) \\
&= \mathrm{Re}\frac{\sqrt{-1}}{2k+1} \int_M (\psi - \varepsilon)_+^2 \, A_{\alpha\beta,\overline{\alpha}} u^{2k+1} u_{\overline{\beta}} \\
&\qquad\qquad - \frac{2}{2k+1} \int_{\{\psi \geq \varepsilon\}} (\psi - \varepsilon) \, \psi u^{2k+1} \mathrm{Re} \psi_{\overline{\alpha}} u_\alpha \\
&= \mathrm{Re}\frac{\sqrt{-1}}{2k+1} \int_M (\psi - \varepsilon)_+^2 \, A_{\alpha\beta,\overline{\alpha}} u^{2k+1} u_{\overline{\beta}},
\end{aligned}
$$

by Lemma 6 again. Let C be the super norm of $\operatorname{div} A = A_{\alpha\beta,\overline{\alpha}}$. Then by the Hölder inequality

$$\int_M F u^{2k} |\partial_b u|^2 \leq \frac{C}{\varepsilon(2k+1)} \int_M F |u|^{2k+1} |\partial_b u|$$

$$\leq \frac{C}{\varepsilon(2k+1)} \left(\int_M F |u|^{2(k+1)}\right)^{1/2} \left(\int_M F |u|^{2k} |\partial_b u|^2\right)^{1/2}.$$

Hence

$$\int_M F u^{2k} |\partial_b u|^2 \leq \left[\frac{C}{\varepsilon(2k+1)}\right]^2 \int_M F |u|^{2(k+1)}$$

$$\leq \frac{4C^2}{\varepsilon^2 m(2k+1)} \int_M F u^{2k} |\partial_b u|^2,$$

where in the last step we used (5.2). Choosing k such that $\frac{4C^2}{\varepsilon^2 m(2k+1)} \leq \frac{1}{2}$ yields

$$\int_M F u^{2k} |\partial_b u|^2 = 0.$$

This is a contradiction. Therefore Lemma 4 is proved.

Inspecting the proof of the rigidity, it is clear that we only need to have a nonconstant function u satisfying (4.1) and (4.2) as all the other identities used in the proof are derived from these two. In summary, we have proved the following theorem.

Theorem 8. *Let M be a closed pseudohermitian manifold of dimension $2m + 1 \geq 5$. Suppose there exists a nonconstant function $u \in C^\infty(M)$ satisfying*

$$u_{\alpha,\beta} = 0,$$

$$u_{\alpha,\overline{\beta}} = \left(-\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0\right)\delta_{\alpha\beta}.$$

Then M is equivalent to the sphere $(\mathbb{S}^{2m+1}, 2\sqrt{-1}\overline{\partial}(|z|^2 - 1))$.

This is equivalent to Theorem 5 by scaling.

6. Remarks for the case $2m + 1 = 3$

Generally speaking, 3-dimensional CR manifolds are more subtle to understand than higher dimensional ones. A famous example is the CR embedding problem. In our situation, it is not clear if Theorem 8 is true in 3 dimensions. The reason is that (4.3) does not follow from (4.1) and (4.2) in 3 dimensions (In deriving (4.8) we need at least 2 indices). The arguments in Section 5 do yield the following weaker rigidity theorem in dimension 3 with (4.3) as an extra condition.

Theorem 9. *Let M be a 3-dimensional closed pseudohermitian manifold. Suppose there exists a nonconstant function $u \in C^\infty(M)$ satisfying*

$$
\begin{aligned}
u_{1,1} &= 0, \\
u_{1,\overline{1}} &= \left(-\frac{1}{4}u + \frac{\sqrt{-1}}{2}u_0\right), \\
u_{0,1} &= 2A_{11}u_{\overline{1}} + \frac{\sqrt{-1}}{2}u_1.
\end{aligned}
$$

Then M is equivalent to the sphere $(\mathbb{S}^3, 2\sqrt{-1}\partial(|z|^2 - 1))$.

In fact, the eigenvalue estimate (Theorem 3) is not known in the 3-dimensional case without any extra condition. Chang and Chiu in [**CC2**] proved the estimate under the extra condition that the Panietz operator is nonnegative. They also proved that M is CR equivalent to the sphere if equality holds and the torsion is zero.

Recall that the Panietz operator P_0 acting on functions on a pseudohermitian manifold M of dimension $2m + 1$ is defined by

$$
P_0 u = 4\,\mathrm{Re}\left(u_{\overline{\beta},\beta\alpha} + m\sqrt{-1}A_{\alpha\beta}u_{\overline{\beta}}\right)_{,\overline{\alpha}}.
$$

It is proved by Graham and Lee [**GL**] that P_0 is always nonnegative if M is closed and of dimension ≥ 5 in the sense

$$
\int_M uP_0 u \geq 0
$$

for any smooth function u. In 3 dimensions, P_0 is known to be nonnegative if the torsion is zero.

With our method, we can remove the torsion-free condition in the characterization of the equality case in Chang and Chiu's work.

Theorem 10. *Let (M^3, θ) be a closed pseudohermitian manifold such that for any $X = cT_1$*

(6.1) $$R_{1\overline{1}}|c|^2 - \sqrt{-1}(A_{11}\overline{c}^2 - A_{\overline{11}}c^2) \geq |c|^2.$$

If $P_0 \geq 0$, then $\lambda_1 \geq \frac{1}{2}$ and the equality holds if and only if (M^3, θ) is equivalent to $(\mathbb{S}^3, 2\sqrt{-1}\partial(|z|^2 - 1))$.

Remark 8. The first part of the theorem was proved by Chiu [**Ch**] and the second part of the theorem was proved by Chang and Chiu [**CC2**] under the extra condition that M is torsion free.

We sketch the proof here. By Lemma 2.2 in [**CC1**], one has

$$
\int_M u_0^2 = \int_M (\Delta_b u)^2 - 2\sqrt{-1}(A_{11}u_{\overline{1}}^2 - A_{\overline{11}}u_1^2)] - \frac{1}{2}\int_M uP_0 u.
$$

Using Proposition 3 and integrating by parts yields

$$\text{Re}\left[\sqrt{-1}\int_M (u_{\bar{1}}u_{1,0} - u_1 u_{\bar{1},0})\right] = -\int_M (u_0)^2 - \int_M \sqrt{-1}(A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2).$$

Let u be a non-zero first eigenfunction, $\Delta_b u = -\lambda_1 u$. Then

$$\int_M (\Delta_b u)^2 = 2\lambda_1 \int_M |\partial_b u|^2.$$

By the Bochner formula (Theorem 6), Lemma 1, and the above two identities, we have

$$0 = \int_M |u_{1,1}|^2 + |u_{1,\bar{1}}|^2 - \lambda_1|\partial u|^2 + R_{1\bar{1}}u_1 u_{\bar{1}} + \frac{1}{2}\sqrt{-1}[A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2]$$

$$+ \text{Re}\sqrt{-1}\int_M (u_{\bar{1}}u_{1,0} - u_1 u_{\bar{1},0})$$

$$= \int_M |u_{1,1}|^2 + \frac{\lambda_1^2 u^2}{4} + \frac{u_0^2}{4} - \lambda_1|\partial u|^2 + R_{1\bar{1}}|u_1|^2 + \frac{1}{2}\sqrt{-1}[A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2]$$

$$- \int_M (u_0)^2 - \int_M \sqrt{-1}(A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2)$$

$$= \int_M |u_{1,1}|^2 - \frac{\lambda_1}{2}|\partial u|^2 + R_{1\bar{1}}|u_1|^2 - \frac{1}{2}\sqrt{-1}[A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2] - \frac{3}{4}\int_M (u_0)^2$$

$$= \int_M |u_{1,1}|^2 - \frac{\lambda_1}{2}|\partial u|^2 + R_{1\bar{1}}|u_1|^2 - \frac{1}{2}\sqrt{-1}[A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2]$$

$$- \frac{3}{4}\int_M [2\lambda_1|\partial u|^2 - 2\sqrt{-1}[A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2] - \frac{1}{2}u P_0 u]$$

$$= \int_M |u_{1,1}|^2 + \frac{3}{2}u P_0 u + \int_M [-2\lambda_1|\partial u|^2 + R_{1\bar{1}}|u_1|^2 + \sqrt{-1}[A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2]$$

$$\geq \int_M [-2\lambda_1|\partial u|^2 + |\partial u|^2]$$

This implies that $\lambda_1 \geq \frac{1}{2}$. If equality holds, we must have

$$(6.2) \qquad\qquad u_{1,1} = 0, \quad u_{1,\bar{1}} = -\frac{u}{4} + \frac{\sqrt{-1}}{2}u_0$$

$$(6.3) \qquad\qquad R_{1\bar{1}}|u_1|^2 + \sqrt{-1}(A_{11}u_{\bar{1}}^2 - A_{\overline{11}}u_1^2) = |u_1|^2.$$

Writing $A_{11} = e^{\sqrt{-1}\theta_1}|A_{11}|$, $u_{\bar{1}}^2 = e^{\sqrt{-1}\theta_2}|u_1|^2$ and $X = e^{\sqrt{-1}\theta}|X|$, by (6.1), we have
(6.4)
$$R_{1\bar{1}}|X|^2 + \sqrt{-1}|A_{\overline{11}}||X|^2(e^{\sqrt{-1}(\theta_1+\theta_2+\theta)} - e^{-\sqrt{-1}(\theta_1+\theta_2+\theta)})| \geq |X|^2,$$

for all $\theta \in [0, 2\pi)$. Choosing $\theta = -\theta_1 + \frac{\pi}{2}$, we have

$$(6.5) \qquad\qquad R_{1\bar{1}}|X|^2 - 2|A_{11}||X|^2 \geq |X|^2.$$

Therefore, by comparing (6.3) and (6.5) with $X = u_1 T_{\overline{1}}$,

$$\sqrt{-1}(A_{11}u_{\overline{1}}^2 - A_{\overline{1}\overline{1}}u_1^2) = -2|A_{11}||u_1|^2, \quad A_{11} = \sqrt{-1}|A_{11}|\frac{u_1^2}{|u_1|^2}.$$

Notice that

$$R_{1\overline{1}}|u_1 + tX|^2 - 2|A_{11}||u_1 + tX|^2 - |u_1 + tX|^2 \geq 0, \quad \text{on} \quad M,$$

and equality holds at $t = 0$. Therefore we obtain, by differentiating at $t = 0$,

$$R_{1\overline{1}}u_1 + 2\sqrt{-1}A_{11}u_{\overline{1}} = u_1.$$

Using the 7th formula of Proposition 3, we have

$$
\begin{aligned}
\frac{-u_1}{4} + \frac{\sqrt{-1}}{2}u_{0,1} &= u_{1,\overline{1}1} - u_{1,1\overline{1}} \\
&= -\sqrt{-1}u_{1,0} - R_{1\overline{1}}u_1 \\
&= -\sqrt{-1}u_{0,1} + \sqrt{-1}A_{11}u_{\overline{1}} - R_{1\overline{1}}u_1 \\
&= -\sqrt{-1}u_{0,1} + 3\sqrt{-1}A_{11}u_{\overline{1}} - u_1
\end{aligned}
$$

Therefore,

$$\frac{1}{2}u_1 + \sqrt{-1}u_{0,1} = 2\sqrt{-1}A_{11}u_{\overline{1}}, \quad u_{0,1} = \frac{\sqrt{-1}}{2}u_1 + 2A_{11}u_{\overline{1}}.$$

Applying Theorem 9, the proof of Theorem 10 is complete.

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