Two Proofs of Euler-Maclaurin Formula, Its Generalizations and Applications

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Abstract

In this paper, we first give two proofs of Euler-Maclaurin formula in section 1. The estimation of the remainder term and some relevant theorems are also stated in this section. In section 2, we apply similar method to get some further results, which generalized Euler-Maclaurin formula. In section 3, we show how Euler-Maclaurin formula is applied to deal with some elementary summations. In section 4, we deal with some infinite series to prepare for the work in next section. In section 5, we apply Euler-Maclaurin formula to some series and give the orders of some finite summations.

Key Words: Euler-Maclaurin formula, infinite series, finite summations

Introduction

In eighteenth century, Euler and Maclaurin both obtained independently a formula linking discrete summations with continuous integrals almost at the same time. Maclaurin applied it to the numerical computation of definite integrals, while Euler used it to calculate series. It has an extensive application in many subjects of mathematics, such as number theory and combinatorics. And relevant research followed continuously since then.

Definitions

The difference of a function $f(x)$ is defined as $\Delta f(x) = f(x+1) - f(x)$. If there exists a function $\phi(x)$ such that $\phi(x+1) - \phi(x) = f(x)$, then we call $\phi(x)$ the inverse difference of $f(x)$, and denote it as $\sum f(x)\Delta x$. We know if $\phi(x)$ is the inverse difference of $f(x)$, so is $\phi(x) + C$, where C is a function of period 1 (including a constant), and vice versa[1]. In this paper, the 'constant' C is always omitted, since it always vanishes in a certain summation.

Bernoulli numbers B_n are defined as the coefficients of the Taylor expansion of $\frac{x}{e^x - 1}$, i.e. $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$. $\frac{\xi e^{\xi \cdot x}}{e^{\xi} - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} \xi^n$ gives Bernoulli polynomials $B_n(x)$.

Main Results

1. Proofs of Euler-Maclaurin Formula

In order to calculate a discrete summation $\sum_{x=a}^{b-1} f(x)$, the method of inverse difference is effective. If one has found a function $\phi(x)$ such that $\phi(x+1) - \phi(x) = f(x)$, then $\sum_{x=a}^{b-1} f(x)$ is simply equal to $\phi(b) - \phi(a)$. This is the main idea of the following theorem. And let us see how it is achieved.

Before we have a rigorous statement, we show the method in a rough way so that the idea will be showed more clearly.

Since our purpose is to find a function $\phi(x)$ such that $\phi(x+1) - \phi(x) = f(x)$, recall the Taylor expansion of a function at a point, we have

$$\phi(x + \Delta x) = \phi(x) + \phi'(x)\Delta x + \frac{\phi''(x)}{2!}\Delta x^2 + \ldots + \frac{\phi^{(n)}(x)}{n!}\Delta x^n + \ldots$$

Let $\Delta x = 1$, we have $\phi(x+1) = \phi(x) + \phi'(x) + \frac{\phi''(x)}{2!} + \ldots + \frac{\phi^{(n)}(x)}{n!} + \ldots$

Equivalently, $f(x) = \phi'(x) + \frac{\phi''(x)}{2!} + \ldots + \frac{\phi^{(n)}(x)}{n!} + \ldots$

We wish to express $\phi(x)$ via $f(x)$, therefore, derivative either side of the equality respect to x, and we obtain $f'(x) = \phi''(x) + \frac{\phi'''(x)}{2!} + \ldots + \frac{\phi^{(n+1)}(x)}{n!} + \ldots$

Similarly, $f''(x) = \phi'''(x) + \frac{\phi^{(4)}(x)}{2!} + \ldots + \frac{\phi^{(n+2)}(x)}{n!} + \ldots$

…

Then we multiply every of these identities by a coefficient, and add them together. The coefficients are chosen so properly that all derivatives of $f(x)$ vanish except $f'(x)$.

Therefore, we obtain $\phi'(x) = \sum_{n=0}^{\infty} a_n f^{(n)}(x)$, and $\phi(x) = \sum_{n=0}^{\infty} a_n f^{(n-1)}(x)$ is followed.

After some calculations we can figure out that $a_n = \frac{B_n}{n!}$, where B_n denote Bernoulli number.

Hence, $\sum_{x=a}^{b-1} f(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)$

Then we prove it rigorously. First, a lemma is needed.

Lemma 1: $\frac{d}{dx}\int_x^{x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt = \int_x^{x+1} f^{(m+1)}(t)\frac{(x+1-t)^n}{n!}dt$.

Proof:

$$\frac{d}{dx}\int_x^{x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{x+\Delta x}^{x+\Delta x+1} f^{(m)}(t)\frac{(x+\Delta x+1-t)^n}{n!}dt - \int_x^{x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt\right)$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{x+\Delta x}^{x+\Delta x+1} f^{(m)}(t)\frac{(x+\Delta x+1-t)^n}{n!}dt - \int_{x+\Delta x}^{x+\Delta x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt\right)$$

$$+ \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{x+\Delta x}^{x+\Delta x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt - \int_x^{x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt\right)$$

$$= \int_x^{x+1} f^{(m)}(t)\frac{\partial}{\partial x}\frac{(x+1-t)^n}{n!}\,dt + \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{x+1}^{x+\Delta x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt - \int_x^{x+\Delta x} f^{(m)}(t)\frac{(x+1-t)^n}{n!}dt\right)$$

$$= \int_x^{x+1} f^{(m)}(t)\frac{(x+1-t)^{n-1}}{(n-1)!}\,dt + \lim_{t\to x+1} f^{(m)}(t)\frac{(x+1-t)^n}{n!} - \lim_{t\to x} f^{(m)}(t)\frac{(x+1-t)^n}{n!}$$

$$= -\int_x^{x+1} f^{(m)}(t)d\,\frac{(x+1-t)^n}{n!} - \frac{f^{(m)}(x)}{n!} = -f^{(m)}(t)\frac{(x+1-t)^n}{n!}\Bigg|_x^{x+1} + \int_x^{x+1} f^{(m+1)}(t)\frac{(x+1-t)^n}{n!}\,dt - \frac{f^{(m)}(x)}{n!}$$

$$= \frac{f^{(m)}(x)}{n!} + \int_x^{x+1} f^{(m+1)}(t)\frac{(x+1-t)^n}{n!}\,dt - \frac{f^{(m)}(x)}{n!} = \int_x^{x+1} f^{(m+1)}(t)\frac{(x+1-t)^n}{n!}\,dt.$$

Then we turn to the main theorem of this paper.

Theorem 1 (Euler-Maclaurin): If $f(x)$ and its derivatives to $n-1$ are continuous in the interval $[a,b]$, then $\sum_{x=a}^{b-1} f(x) = \int_a^b f(x)dx + \sum_{k=1}^{n}\frac{B_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a)) + R_n$, where R_n is the remainder term.

Proof: Refer to Taylor formula, we have

$$F(x+\Delta x)=F(x)+F'(x)\Delta x+\frac{F''(x)}{2!}\Delta x^2+\ldots+\frac{F^{(n-1)}(x)}{(n-1)!}\Delta x^{n-1}+\int_x^{x+\Delta x}F^{(n)}(t)\frac{(x+\Delta x-t)^{n-1}}{(n-1)!}dt.$$

Let $\Delta x=1$, we obtain $F(x+1)=F(x)+F'(x)+\frac{F''(x)}{2!}+\ldots+\frac{F^{(n-1)}(x)}{(n-1)!}+\int_x^{x+1}F^{(n)}(t)\frac{(x+1-t)^{n-1}}{(n-1)!}dt$.

Namely, $f(x)=\phi(x+1)-\phi(x)=\phi'(x)+\frac{\phi''(x)}{2!}+\ldots+\frac{\phi^{(n-1)}(x)}{(n-1)!}+\int_x^{x+1}\phi^{(n)}(t)\frac{(x+1-t)^{n-1}}{(n-1)!}dt$.

Integrate and derivative either side of the equality respect to x, apply lemma 1, and make the last term before the integral remainder term in Taylor series be a multiple of $\phi^{(n-1)}(x)$, thus we obtain

$$\int_{x_0}^{x}f(t)dt+C=\phi(x)+\frac{\phi'(x)}{2!}+\ldots+\frac{\phi^{(n-1)}(x)}{n!}+\int_x^{x+1}\phi^{(n)}(t)\frac{(x+1-t)^n}{n!}dt.$$

$$f'(x)=\phi''(x)+\frac{\phi'''(x)}{2!}+\ldots+\frac{\phi^{(n-1)}(x)}{(n-2)!}+\int_x^{x+1}\phi^{(n)}(t)\frac{(x+1-t)^{n-2}}{(n-2)!}dt.$$

$$f''(x)=\phi'''(x)+\frac{\phi^{(4)}(x)}{2!}+\ldots+\frac{\phi^{(n-1)}(x)}{(n-3)!}+\int_x^{x+1}\phi^{(n)}(t)\frac{(x+1-t)^{n-3}}{(n-3)!}dt.$$

…

$$f^{(n-2)}(x)=\phi^{(n-1)}(x)+\int_x^{x+1}\phi^{(n)}(t)(x+1-t)dt.$$

$$f^{(n-1)}(x)=\int_x^{x+1}\phi^{(n)}(t)dt.$$

Multiply every of these identities by a coefficient a_k, and add them together. Assume $a_0=1$ and $\sum_{k=0}^{n-1}\frac{a_k}{(n-k)!}=0$, and we obtain $\int_{x_0}^{x}f(t)dt+C+\sum_{k=1}^{n}a_k f^{(k-1)}(x)=\phi(x)+\int_x^{x+1}\phi^{(n)}(t)\sum_{k=0}^{n}\frac{a_k(x+1-t)^{n-k}}{(n-k)!}dt$.

Then we figure out the coefficients. The recurrence equation is $\sum_{k=0}^{n-1}\frac{a_k}{(n-k)!}=0$ with the initial condition $a_0=1$. Let $a_n=\frac{b_n}{n!}$, then we have $\sum_{k=0}^{n-1}\frac{b_k}{k!\cdot(n-k)!}=0$. $\sum_{k=0}^{n-1}\binom{n}{k}b_k=0$. Therefore, b_n is Bernoulli number.

$a_n=\frac{B_n}{n!}$, $\int_{x_0}^{x}f(t)dt+C+\sum_{k=1}^{n}\frac{B_k}{k!}f^{(k-1)}(x)=\phi(x)+\int_x^{x+1}\phi^{(n)}(t)\sum_{k=0}^{n}\frac{B_k(x+1-t)^{n-k}}{k!\cdot(n-k)!}dt$.

Hence, $\sum_{x=a}^{b-1}f(x)=\phi(b)-\phi(a)=\int_a^b f(x)dx+\sum_{k=1}^{n}\frac{B_k}{k!}(f^{(k-1)}(b)-f^{(k-1)}(a))+R_n$.

Then we consider the remainder term.

Theorem 2: $R_n=-\frac{1}{n!}\int_a^b f^{(n)}(x)B_n([x]+1-x)dx=\frac{(-1)^{n-1}}{n!}\int_a^b f^{(n)}(x)B_n(x-[x])dx$.

Proof : $\int_x^{x+1}\phi^{(n)}(t)\sum_{k=0}^{n}\frac{B_k(x+1-t)^{n-k}}{k!\cdot(n-k)!}dt=\frac{1}{n!}\int_x^{x+1}\phi^{(n)}(t)B_n(x+1-t)dt=\frac{1}{n!}\int_x^{x+1}\phi^{(n)}(t)B_n([t]+1-t)dt$.

$$R_n = -\frac{1}{n!}\left(\int_b^{b+1} \phi^{(n)}(x) B_n([x]+1-x)dx - \int_a^{a+1} \phi^{(n)}(x) B_n([x]+1-x)dx\right)$$
$$= -\frac{1}{n!}\left(\sum_{k=a+1}^{b} \int_k^{k+1} \phi^{(n)}(x) B_n([x]+1-x)dx - \sum_{k=a}^{b-1} \int_k^{k+1} \phi^{(n)}(x) B_n([x]+1-x)dx\right).$$

Since $B_n([x]+1-x)$ is a periodic function of period 1, we can rewrite the summation as

$$-\frac{1}{n!}\left(\sum_{k=a}^{b-1} \int_k^{k+1} \phi^{(n)}(x+1) B_n([x]+1-x)dx - \sum_{k=a}^{b-1} \int_k^{k+1} \phi^{(n)}(x) B_n([x]+1-x)dx\right)$$
$$= -\frac{1}{n!}\sum_{k=a}^{b-1} \int_k^{k+1} f^{(n)}(x) B_n([x]+1-x)dx = -\frac{1}{n!}\int_a^b f^{(n)}(x) B_n([x]+1-x)dx .$$

If we use $B_n(1-x) = (-1)^n B_n(x)(n \ge 1)$, we can obtain $R_n = \frac{(-1)^{n-1}}{n!}\int_a^b f^{(n)}(x) B_n(x-[x])dx$

Theorem 3: $R_n = O\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)$.
Proof : Since $B_n([x]+1-x)$ is a periodic function and has no singular point, we know $B_n([x]+1-x)$ is bounded. Namely, $B_n(x-[x]) = O(1)$.

Therefore, $R_n = \frac{(-1)^{n-1}}{n!}\int_a^b f^{(n)}(x) B_n(x-[x])dx = O\left(\int_a^b f^{(n)}(x)dx\right) = O\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)$.

[5] showed that $\frac{B_n(x-[x])}{n!} = \begin{cases} (-1)^{\frac{n}{2}-1}\sum_{k=1}^{\infty}\frac{2\cos(2k\pi x)}{(2k\pi)^n}, n\ even \\ (-1)^{\frac{n-1}{2}-1}\sum_{k=1}^{\infty}\frac{2\sin(2k\pi x)}{(2k\pi)^n}, n\ odd \end{cases}$, therefore, $B_n(x-[x]) \le \frac{2\zeta(n)n!}{(2\pi)^n}$.

Lehmer[10] showed that the maximum value of $B_n(x-[x])$ obeys $M_n = \begin{cases} \frac{2\zeta(n)n!}{(2\pi)^n}, n \equiv 2(\mathrm{mod}\,4) \\ \frac{2n!}{(2\pi)^n} - \varepsilon, others \end{cases}$,

while the minimum obeys $m_n = \begin{cases} -\frac{2\zeta(n)n!}{(2\pi)^n}, n \equiv 0(\mathrm{mod}\,4) \\ -\frac{2n!}{(2\pi)^n} + \varepsilon, others \end{cases}$.

We distinguish two cases before a further consideration. If there exists an n that $f^{(n-1)}(x)$ is bounded, then $R_n = O(1)$, and the first $n-1$ terms gives the main part of this summation. If any of $f^{(n-1)}(x)$ goes to infinity, then $R_n = O\left(f^{(n-1)}(b)\right)(b \to \infty)$.

Theorem 4: If $\sum_{n=1}^{\infty}\frac{B_n}{n!}f^{(n-1)}(x)$ converges uniformly in the interval $[a,b]$, then

$$\sum_{x=a}^{b-1}f(x)=\sum_{n=0}^{\infty}\frac{B_n}{n!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right).$$

Proof : $R_n=-\frac{1}{n!}\int_a^b f^{(n)}(x)B_n([x]+1-x)dx=-\int_a^b f^{(n)}(x)\frac{B_n([x]+1-x)}{n!}dx$

$$=\int_a^b f^{(n)}(x)d\frac{B_{n+1}([x]+1-x)}{(n+1)!}=f^{(n)}(x)\frac{B_{n+1}([x]+1-x)}{(n+1)!}\Bigg|_a^b-\int_a^b df^{(n)}(x)\frac{B_{n+1}([x]+1-x)}{(n+1)!}$$

$$=\frac{B_{n+1}}{(n+1)!}(f^{(n)}(b)-f^{(n)}(a))-\int_a^b f^{(n+1)}(x)\frac{B_{n+1}([x]+1-x)}{(n+1)!}dx=\frac{B_{n+1}}{(n+1)!}(f^{(n)}(b)-f^{(n)}(a))+R_{n+1}.$$

If $\sum_{n=1}^{\infty}\frac{B_n}{n!}f^{(n-1)}(x)$ converges uniformly, so does $\sum_{i=n+1}^{\infty}\frac{B_i}{i!}f^{(i-1)}(x)$, therefore, $R_n=\sum_{i=n+1}^{\infty}\frac{B_i}{i!}f^{(i-1)}(x)$.

As $n\to\infty$, $\sum_{i=n+1}^{\infty}\frac{B_i}{i!}f^{(i-1)}(x)\to 0$, $R_n\to 0$. Hence, $\sum_{x=a}^{b-1}f(x)=\sum_{n=0}^{\infty}\frac{B_n}{n!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)$.

Therefore, an effective estimation of the remainder term depends on the behavior of the derivatives of $f(x)$. Namely, we need an n such that $f^{(n-1)}(x)$ is bounded. Therefore, if a function increases so rapidly that all its derivatives tend to infinity, then Euler-Maclaurin formula doesn't work now. However, in this case, the first few terms are so inferior to the last one that they can just be dropped. For instance, one can easily find that $\sum_{n=0}^{x-1}n!\le x!$. The following theorem gives a criterion of this case.

Theorem 4: Assume $f(x)\ge 0$ and $f'(x)\ge 0$ if $x>x_0$. $f(x)=o(f'(x))(x\to\infty)$.

Then $\sum_{n=x_0}^{x-1}f(n)=o(f(x))(x\to\infty)$.

Proof : $\because f(x)=o[f'(x)](x\to\infty)$, $\therefore$ Given $\forall N>0$, $\exists x_N$, if $x>x_N$, we have $\frac{f'(x)}{f(x)}>N$.

Namely, $f'(x)>Nf(x)$.

$\therefore \int_{x_0}^{x}f'(t)dt>N\int_{x_0}^{x}f(t)dt$, $f(x)-f(x_0)>N\int_{x_0}^{x}f(t)dt$, $f(x)>N\int_{x_0}^{x}f(t)dt$.

Since $f(x)$ increases, we have $\int_{x_0}^{x}f(t)dt=\sum_{n=x_0}^{x-1}\int_n^{n+1}f(t)dt\ge\sum_{n=x_0}^{x-1}f(n)$.

Hence, $\frac{f(x)}{\sum_{n=x_0}^{x-1}f(n)}>N$, $\sum_{n=x_0}^{x-1}f(n)=o[f(x)](x\to\infty)$.

We now fix a, denote it as x_0, and consider $\sum_{n=x_0}^{x-1}f(n)$ as a function of x. We rewrite the formula as

$\sum_{n=x_0}^{x-1}f(n)=\int_{x_0}^{x}f(t)dt+C_n+\sum_{k=1}^{n}\frac{B_k}{k!}f^{(k-1)}(x)+R_n$, where C_n is a constant and R_n is the remainder term.

Sometimes $\int_{x_0}^{x} f(t)dt + C + \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x)$ diverges almost everywhere. But $\int_{x_0}^{x} f(t)dt + C + \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x)$ may be the asymptotic series of $\sum_{n=x_0}^{x-1} f(n)$. We have the following theorem:

Theorem : Assume $f^{(N-1)}(x) = o(1)(x \to \infty)$, and $f^{(n)}(x) = o(f^{(n-1)}(x))(x \to \infty)$ for $n \geq N$, then $\sum_{n=x_0}^{x-1} f(n) \sim \int_{x_0}^{x} f(t)dt + C + \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x)(x \to \infty)$.

Proof : We know the remainder term $R_n = \int_x^{\infty} f^{(n)}(t) \frac{B_n([t]+1-t)}{n!} dt$ for $n \geq N$, noticing that

$$R_n = -f^{(n)}(t) \frac{B_{n+1}([t]+1-t)}{(n+1)!}\bigg|_x^{\infty} + \int_x^{\infty} f^{(n+1)}(t) \frac{B_{n+1}([t]+1-t)}{(n+1)!} dt = \frac{B_{n+1}}{(n+1)!} f^{(n)}(x) + R_{n+1}.$$

Therefore, for $n \geq N$, $\lim_{x\to\infty} \frac{\sum_{n=x_0}^{x-1} f(n) - \int_{x_0}^{x} f(t)dt - C - \sum_{k=1}^{n} \frac{B_k}{k!} f^{(k-1)}(x)}{f^{(n-1)}(x)} = \lim_{x\to\infty} \frac{R_n}{f^{(n-1)}(x)}$

$$= \lim_{x\to\infty} \frac{\int_x^{\infty} f^{(n)}(t) \frac{B_n([t]+1-t)}{n!} dt}{f^{(n-1)}(x)} = \lim_{x\to\infty} \frac{\frac{B_{n+1}}{(n+1)!} f^{(n)}(x) + \int_x^{\infty} f^{(n+1)}(t) \frac{B_{n+1}([t]+1-t)}{(n+1)!} dt}{f^{(n-1)}(x)}$$

$$= \lim_{x\to\infty} \frac{\int_x^{\infty} f^{(n+1)}(t) \frac{B_{n+1}([t]+1-t)}{(n+1)!} dt}{f^{(n-1)}(x)} = \lim_{x\to\infty} \frac{-f^{(n+1)}(x) \frac{B_{n+1}}{(n+1)!}}{f^{(n)}(x)} = 0.$$

Hence, $\sum_{n=x_0}^{x-1} f(n) \sim \int_{x_0}^{x} f(t)dt + C + \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x)(x \to \infty)$.

Now let us consider the problem in another way. Since Weierstrass theorem states that any function continuous in a closed interval can be approximated by polynomials, now we use Bernoulli polynomials to expand a function.

Similarly, we have a rough sight on the operating first.

Assume that $f(x)$ can be expanded via Bernoulli polynomials, i.e., $f(x) = \sum_{n=0}^{\infty} a_n B_n(x)$, where a_n is undetermined coefficients. To decide a_0, we integrate either side of the expansion from $x=0$ to $x=1$, and notice that $\int_0^1 B_n(x)dx = \begin{cases} 1, n=0 \\ 0, n>0 \end{cases}$. Therefore, we obtain $a_0 = \int_0^1 f(x)dx$. To decide a_1, we derivative the expansion, and notice that $B_n'(x) = \begin{cases} 0, n=0 \\ nB_{n-1}(x), n>0 \end{cases}$. We obtain $f'(x) = \sum_{n=1}^{\infty} na_n B_{n-1}(x)$. Similarly, we integrate from $x=0$ to $x=1$, and obtain $a_1 = \int_0^1 f'(x)dx$.

Since $f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)...(n-m+1)a_n B_{n-1}(x)$, eventually we obtain $a_n = \frac{1}{n!} \int_0^1 f^{(n)}(x)dx$.

Hence, $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 f^{(n)}(x)dx\cdot B_n(x)$.

Let $x=0$, and notice that $B_n(0)=B_n$, we obtain

$$f(0)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 f^{(n)}(x)dx\cdot B_n(0)=\sum_{n=0}^{\infty}\frac{B_n}{n!}\int_0^1 f^{(n)}(x)dx=\int_0^1 f(x)dx+\sum_{n=1}^{\infty}\frac{B_n}{n!}\left(f^{(n-1)}(1)-f^{(n-1)}(0)\right).$$

We now come back to $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 f^{(n)}(x)dx\cdot B_n(x)$, and replace $f(x)$ with $f(x+i)$, we obtain

$$f(x+i)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 f^{(n)}(x+i)dx\cdot B_n(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_i^{i+1} f^{(n)}(x)dx\cdot B_n(x).$$

Therefore, $f(i)=\int_i^{i+1} f(x)dx+\sum_{n=1}^{\infty}\frac{B_n}{n!}\left(f^{(n-1)}(i+1)-f^{(n-1)}(i)\right)$.

$$\sum_{i=a}^{b-1} f(i)=\sum_{i=a}^{b-1}\left(\int_i^{i+1} f(x)dx+\sum_{n=1}^{\infty}\frac{B_n}{n!}\left(f^{(n-1)}(i+1)-f^{(n-1)}(i)\right)\right)=\int_a^b f(x)dx+\sum_{n=1}^{\infty}\frac{B_n}{n!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right).$$

And we have 'proved' Euler-Maclaurin formula in another way.

To make a rigorous statement, we should consider the remainder term in the expansion, i.e.

$$f(x)-\sum_{n=0}^{N}\frac{1}{n!}\int_0^1 f^{(n)}(x)dx\cdot B_n(x)=\int_0^1 f(x)dt-\int_0^1\sum_{n=0}^{N}\frac{B_n(x)}{n!}f^{(n)}(t)dt=\int_0^1\left(f(x)-\sum_{n=0}^{N}\frac{B_n(x)}{n!}f^{(n)}(t)\right)dt.$$

I have tried to express it as an integral, but failed. Therefore, let us consider

$\int_0^1\left(f(0)-\sum_{n=0}^{N}\frac{B_n(0)}{n!}f^{(n)}(t)\right)dt=\int_0^1\left(f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}f^{(n)}(t)\right)dt$ instead.

We use the Maclaurin expansion of $f^{(n)}(t)$: $f^{(n)}(t)=\sum_{i=0}^{N-n}\frac{f^{(n+i)}(0)}{i!}t^i+\int_0^t f^{(N+1)}(\xi)\frac{(t-\xi)^{N-n}}{(N-n)!}d\xi$,

to obtain

$$\int_0^1\left(f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}f^{(n)}(t)\right)dt=\int_0^1\left(f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}\left(\sum_{i=0}^{N-n}\frac{f^{(n+i)}(0)}{i!}t^i+\int_0^t f^{(N+1)}(\xi)\frac{(t-\xi)^{N-n}}{(N-n)!}d\xi\right)\right)dt$$

$$=\int_0^1\left(f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}\sum_{i=0}^{N-n}\frac{f^{(n+i)}(0)}{i!}t^i\right)dt-\int_0^1\sum_{n=0}^{N}\frac{B_n}{n!}\int_0^t f^{(N+1)}(\xi)\frac{(t-\xi)^{N-n}}{(N-n)!}d\xi dt$$

$$=f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}\sum_{i=0}^{N-n}\frac{f^{(n+i)}(0)}{i!}\int_0^1 t^i dt-\int_0^1\int_0^t\sum_{n=0}^{N}\frac{B_n}{n!}f^{(N+1)}(\xi)\frac{(t-\xi)^{N-n}}{(N-n)!}d\xi dt$$

$$=f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}\sum_{i=0}^{N-n}\frac{f^{(n+i)}(0)}{(i+1)!}-\frac{1}{N!}\int_0^1\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi dt.$$

Use the property of B_n, the first term eventually vanishes, therefore,

$$\int_0^1\left(f(0)-\sum_{n=0}^{N}\frac{B_n}{n!}f^{(n)}(t)\right)dt=-\frac{1}{N!}\int_0^1\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi dt,$$

$$f(0)=\sum_{n=0}^{N}\frac{B_n}{n!}\left(f^{(n-1)}(1)-f^{(n-1)}(0)\right)-\frac{1}{N!}\int_0^1\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi dt.$$

To compute $\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi$, we notice that $\frac{d}{dt}\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi$

$$=\int_0^t f^{(N+1)}(\xi)\frac{\partial}{\partial t}B_N(t-\xi)d\xi+f^{(N+1)}(\xi)B_N(t-\xi)\Big|_0^t=N\int_0^t f^{(N+1)}(\xi)B_{N-1}(t-\xi)d\xi-f^{(N+1)}(0)B_N(t).$$

$$\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi = f^{(N)}(\xi)B_N(t-\xi)\Big|_0^t - \int_0^t f^{(N)}(\xi)\frac{d}{d\xi}B_N(t-\xi)d\xi$$

$$= -f^{(N)}(0)B_N(t) + N\int_0^t f^{(N)}(\xi)B_{N-1}(t-\xi)d\xi .$$

Therefore, $\frac{d}{dt}\int_0^t f^{(N)}(\xi)B_N(t-\xi)d\xi = \int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi$

$$\int_0^1\int_0^t f^{(N+1)}(\xi)B_N(t-\xi)d\xi dt = \int_0^1 d\int_0^t f^{(N)}(\xi)B_N(t-\xi)d\xi = \int_0^1 f^{(N)}(\xi)B_N(1-\xi)d\xi$$

$$= \int_0^1 f^{(N)}(\xi)B_N([\xi]+1-\xi)d\xi .$$

Similarly, $f(i) = \sum_{n=0}^{N}\frac{B_n}{n!}\left(f^{(n-1)}(i+1) - f^{(n-1)}(i)\right) - \frac{1}{N!}\int_i^{i+1} f^{(N)}(\xi)B_N([\xi]+1-\xi)d\xi .$

Therefore, $\sum_{i=a}^{b-1} f(i) = \sum_{n=0}^{N}\frac{B_n}{n!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) - \frac{1}{N!}\int_a^b f^{(N)}(\xi)B_N([\xi]+1-\xi)d\xi .$

Since Euler-Maclaurin formula has a long history, I believe a lot of papers have been written on it, which means a probability of great numbers of proofs. I came up with these two proofs by myself before I knew there exists such a formula. Later I have read [3], [4], [5] and [6], which all give some proofs. But their proofs are not the same as 'mine'. However, I still cannot decide whether these proofs are truly 'new'. Therefore, I merely say 'two proofs' in my title rather than 'two new proofs'.

2. Generalizations of Euler-Maclaurin Formula

What we have considered in section 1 is actually the solution of recurrence equation $\phi(x+1) - \phi(x) = f(x)$ with the initial condition $\phi(a) = \phi(a)$. Applying similar idea, we can obtain solutions to some other recurrence equations.

We know the solution of an inhomogeneous linear ordinary recurrence equation $\phi(x+k) + p_{k-1}\phi(x+k-1) + \ldots + p_1\phi(x+1) + p_0\phi(x) = f(x)$ with the initial condition $\phi(a) = \phi(a)$, $\phi(a+1) = \phi(a+1)$, …, $\phi(a+k-1) = \phi(a+k-1)$, where $p_0, p_1, \ldots, p_{k-1}$ are constants is that to corresponding homogeneous equation $\phi(x+k) + p_{k-1}\phi(x+k-1) + \ldots + p_1\phi(x+1) + p_0\phi(x) = 0$ added by a special solution of the original equation. The following theorem gives a method of finding the special solution.

Similarly, we have the following lemma. Since the proof of it is almost the same as that of lemma 1, we just omit it.

Lemma 2: $\frac{d}{dx}\int_x^{x+i} f^{(m)}(t)\frac{(x+i-t)^n}{n!}dt = \int_x^{x+i} f^{(m+1)}(t)\frac{(x+i-t)^n}{n!}dt$, where i is a positive integer.

Theorem 3: We denote n_0 as the lowest non-negative integer such that $p_0 \cdot 0^n + p_1 \cdot 1^n + \ldots p_{k-1}\cdot(k-1)^n + k^n \neq 0$, where $k \geq 1$ and $p_0, p_1, \ldots, p_{k-1}$ are constants. Here we define $0^0 = 1$, and $P_n(k) = \sum_{i=0}^{k} p_{k-i}(k-i)^n$. If $f(x)$ and its derivatives to n are continuous in a subset of real numbers, then one solution of $\phi(x+k) + p_{k-1}\phi(x+k-1) + \ldots + p_1\phi(x+1) + p_0\phi(x) = f(x)$ is given by

$\phi(x)=\dfrac{n_0!}{P_{n_0}(k)}\sum_{i=0}^{n}\dfrac{b_i}{i!}f^{(i-n_0)}(x)+R_n$, where b_n is given by the generating function

$\dfrac{x^{n_0}}{\sum_{i=0}^{k}p_{k-i}e^{(k-i)\cdot x}}=\sum_{n=0}^{\infty}\dfrac{b_n}{n!}x^n$ and R_n is the remainder term.

Proof : Refer to Taylor formula, we have

$$f(x)=\phi(x+k)+p_{k-1}\phi(x+k-1)+\ldots+p_1\phi(x+1)+p_0\phi(x)$$

$$=\sum_{i=0}^{n}\frac{P_i(k)}{i!}\phi^{(i)}(x)+\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\frac{(x+i-t)^n}{n!}dt$$

$$=\sum_{i=n_0}^{n}\frac{P_i(k)}{i!}\phi^{(i)}(x)+\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\frac{(x+i-t)^n}{n!}dt.$$

We apply the same method in theorem 1, and obtain

$$f'(x)=\sum_{i=n_0}^{n-1}\frac{P_i(k)}{i!}\phi^{(i+1)}(x)+\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\frac{(x+i-t)^{n-1}}{(n-1)!}dt.$$

$$f''(x)=\sum_{i=n_0}^{n-2}\frac{P_i(k)}{i!}\phi^{(i+2)}(x)+\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\frac{(x+i-t)^{n-2}}{(n-2)!}dt.$$

…

$$f^{(n-n_0)}(x)=\frac{P_n(k)}{n!}\phi^{(n)}(x)+\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\frac{(x+i-t)^{n_0}}{n_0!}dt.$$

$$f^{(n-n_0+1)}(x)=\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\frac{(x+i-t)^{n_0-1}}{(n_0-1)!}dt.$$

…

$$f^{(n-1)}(x)=\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)(x+i-t)dt$$

$$f^{(n)}(x)=\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)dt.$$

And similarly, $\dfrac{P_{n_0}(k)\phi^{(n_0)}(x)}{n_0!}=\sum_{i=0}^{n}a_if^{(i)}(x)+\sum_{j=0}^{n}a_j\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1)}(t)\dfrac{(x+i-t)^{n-j}}{(n-j)!}dt$

$$=\sum_{i=0}^{n}a_if^{(i)}(x)+\sum_{i=1}^{k}p_i\sum_{j=0}^{n}\int_x^{x+i}\phi^{(n+1)}(t)a_j\frac{(x+i-t)^{n-j}}{(n-j)!}dt.$$

$$\frac{P_{n_0}(k)\phi(x)}{n_0!}=\sum_{i=0}^{n}a_if^{(i-n_0)}(x)+\sum_{i=1}^{k}p_i\sum_{j=0}^{n}\int_x^{x+i}\phi^{(n+1-n_0)}(t)a_j\frac{(x+i-t)^{n-j}}{(n-j)!}dt.$$

Then we deduce the generating function of a_i.

The recurrence equation is $\sum_{i=0}^{n}P_{n-i}(k)\dfrac{a_i}{(n-i)!}=0$ with the initial condition $a_0=1$. Since $P_n(k)=0(n<n_0)$, it can also be written as $\sum_{i=0}^{n-n_0}P_{n-i}(k)\dfrac{a_i}{(n-i)!}=0$.

Let $a_i = \dfrac{b_i}{i!}$, then $\sum_{i=0}^{n-n_0} P_{n-i}(k)\dfrac{b_i}{i!\cdot(n-i)!} = 0$. $\sum_{i=0}^{n-n_0}\binom{n}{i} b_i P_{n-i}(k) = 0$.

Consider the sequence b_i' given by $\dfrac{\dfrac{P_{n_0}(k)}{n_0!}x^{n_0}}{\sum_{i=0}^{k} p_{k-i}e^{(k-i)\cdot x}} = \sum_{n=0}^{\infty}\dfrac{b_n'}{n!}x^n$, we have $\dfrac{\dfrac{P_{n_0}(k)}{n_0!}x^{n_0}}{\sum_{n=n_0}^{\infty}\dfrac{P_n(k)}{n!}x^n} = \sum_{n=0}^{\infty}\dfrac{b_n'}{n!}x^n$,

$$\frac{P_{n_0}(k)}{n_0!}x^{n_0} = \sum_{n=n_0}^{\infty}\frac{P_n(k)}{n!}x^n \cdot \sum_{n=0}^{\infty}\frac{b_n'}{n!}x^n = x^{n_0}\sum_{n=0}^{\infty}\frac{P_{n+n_0}(k)}{(n+n_0)!}x^n \cdot \sum_{n=0}^{\infty}\frac{b_n'}{n!}x^n .$$

$$\frac{P_{n_0}(k)}{n_0!} = \sum_{n=0}^{\infty}\frac{\sum_{i=0}^{n}\binom{n}{i}b_i'\dfrac{(n-i)!P_{n+n_0-i}(k)}{(n+n_0-i)!}}{n!}x^n = \sum_{n=0}^{\infty}\frac{\dfrac{n!}{(n+n_0)!}\sum_{i=0}^{n}\binom{n+n_0}{i}b_i'P_{n+n_0-i}(k)}{n!}x^n .$$

$\therefore b_0' = 1$. $\sum_{i=0}^{n}\binom{n+n_0}{i}b_i'P_{n+n_0-i}(k) = 0 (n \ge 1)$.

Hence, $b_i = b_i'$, the generating function of b_n is $\dfrac{\dfrac{P_{n_0}(k)}{n_0!}x^{n_0}}{\sum_{i=0}^{k} p_{k-i}e^{(k-i)\cdot x}} = \sum_{n=0}^{\infty}\dfrac{b_n}{n!}x^n$.

We define $b_n(\xi)$ a polynomial of ξ by $\dfrac{\dfrac{P_{n_0}(k)}{n_0!}x^{n_0}}{\sum_{i=0}^{k} p_{k-i}e^{(k-i)\cdot x}}e^{x\cdot\xi} = \sum_{n=0}^{\infty}\dfrac{b_n(\xi)}{n!}x^n$. Therefore, we have

$$\sum_{n=0}^{\infty}\frac{b_n(\xi)}{n!}x^n = \frac{\dfrac{P_{n_0}(k)}{n_0!}x^{n_0}}{\sum_{i=0}^{k} p_{k-i}e^{(k-i)\cdot x}}e^{x\cdot\xi} = \sum_{n=0}^{\infty}\frac{b_n}{n!}x^n \cdot \sum_{n=0}^{\infty}\frac{\xi^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{\sum_{i=0}^{n}\binom{n}{i}b_i\xi^{n-i}}{n!}x^n .$$

Hence, $b_n(\xi) = \sum_{i=0}^{n}\binom{n}{i}b_i\xi^{n-i}$.

$$\frac{P_{n_0}(k)\phi(x)}{n_0!} = \sum_{i=0}^{n}a_i f^{(i-n_0)}(x) + \sum_{i=1}^{k}p_i\sum_{j=0}^{n}\int_x^{x+i}\phi^{(n+1-n_0)}(t)a_j\frac{(x+i-t)^{n-j}}{(n-j)!}dt$$
$$= \sum_{i=0}^{n}\frac{b_i}{i!}f^{(i-n_0)}(x) + \sum_{i=1}^{k}p_i\sum_{j=0}^{n}\int_x^{x+i}\phi^{(n+1-n_0)}(t)b_j\frac{(x+i-t)^{n-j}}{j!\cdot(n-j)!}dt$$
$$= \sum_{i=0}^{n}\frac{b_i}{i!}f^{(i-n_0)}(x) + \frac{1}{n!}\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1-n_0)}(t)b_n(x+i-t)dt .$$

Hence, $\phi(x) = \dfrac{n_0!}{P_{n_0}(k)}\sum_{i=0}^{n}\dfrac{b_i}{i!}f^{(i-n_0)}(x) + R_n$, where R_n is the remainder term and

$$R_n = \frac{n_0!}{n!}\frac{1}{P_{n_0}(k)}\sum_{i=1}^{k}p_i\int_x^{x+i}\phi^{(n+1-n_0)}(t)b_n(x+i-t)dt .$$

We sometimes encounter the summation $\sum_{i=0}^{n}\frac{f(i)}{i!}$. It is somewhat difficult to deal with by means of Euler-Maclaurin formula, as $\frac{f(i)}{i!}$ is not a continuous function. If we replace $\frac{f(i)}{i!}$ with $\frac{f(i)}{\Gamma(i+1)}$, then its derivatives are complicated. Therefore, we come back to the former method.

Assume $\frac{\phi(x+1)}{x!}-\frac{\phi(x)}{(x-1)!}=\frac{f(x)}{x!}$, i.e. $\phi(x+1)-x\cdot\phi(x)=f(x)$.

Refer to Taylor formula,

$$\phi(x+\Delta x)=\phi(x)+\phi'(x)\Delta x+\frac{\phi''(x)}{2!}\Delta x^2+\ldots+\frac{\phi^{(n)}(x)}{n!}\Delta x^n+\int_x^{x+\Delta x}\phi^{(n+1)}(t)\frac{(x+\Delta x-t)^n}{n!}dt.$$

Let $\Delta x=1$, we have $\phi(x+1)=\phi(x)+\phi'(x)+\frac{\phi''(x)}{2!}+\ldots+\frac{\phi^{(n)}(x)}{n!}+\int_x^{x+1}\phi^{(n+1)}(t)\frac{(x+1-t)^n}{n!}dt$.

Therefore,

$$f(x)=\phi(x+1)-x\cdot\phi(x)=(1-x)\phi(x)+\phi'(x)+\frac{\phi''(x)}{2!}+\ldots+\frac{\phi^{(n)}(x)}{n!}+\int_x^{x+1}\phi^{(n+1)}(t)\frac{(x+1-t)^n}{n!}dt.$$

Derivative it, we obtain

$$f'(x)=-\phi(x)+(1-x)\phi'(x)+\phi''(x)+\frac{\phi'''(x)}{2!}+\ldots+\frac{\phi^{(n)}(x)}{(n-1)!}+\int_x^{x+1}\phi^{(n+1)}(t)\frac{(x+1-t)^{n-1}}{(n-1)!}dt.$$

$$f''(x)=-2\phi'(x)+(1-x)\phi''(x)+\phi'''(x)+\frac{\phi^{(4)}(x)}{2!}+\ldots+\frac{\phi^{(n)}(x)}{(n-2)!}+\int_x^{x+1}\phi^{(n+1)}(t)\frac{(x+1-t)^{n-2}}{(n-2)!}dt.$$

…

$$f^{(n-1)}(x)=-(n-1)\phi^{(n-2)}(x)+(1-x)\phi^{(n-1)}(x)+\phi^{(n)}(x)+\int_x^{x+1}\phi^{(n+1)}(t)(x+1-t)dt.$$

$$f^{(n)}(x)=-n\phi^{(n-1)}(x)+(1-x)\phi^{(n)}(x)+\int_x^{x+1}\phi^{(n+1)}(t)dt.$$

Suppose

$$\sum_{k=0}^{n}\alpha_k(x)f^{(k)}(x)=\sum_{k=0}^{n-1}\alpha_k(x)\frac{\phi^{(n)}(x)}{(n-k)!}+(1-x)\alpha_n(x)\phi^{(n)}(x)+\int_x^{x+1}\phi^{(n+1)}(t)\sum_{k=0}^{n}\alpha_k(x)\frac{(x+1-t)^{n-k}}{(n-k)!}dt.$$

The recurrence equation is $\sum_{i=0}^{n-1}\frac{\alpha_i(x)}{(n-i)!}+(1-x)\alpha_n(x)-(n+1)\alpha_{n+1}(x)=0$ with the initial condition $\alpha_0(x)=1$. Let $\alpha_n(x)=\frac{\beta_n(x)}{n!}$, then we can rewrite the recurrence equation as

$$\sum_{i=0}^{n-1}\frac{\beta_i(x)}{i!\cdot(n-i)!}+(1-x)\frac{\beta_n(x)}{n!}-(n+1)\frac{\beta_{n+1}(x)}{(n+1)!}=0,\ \sum_{i=0}^{n-1}\frac{\beta_i(x)}{i!\cdot(n-i)!}+(1-x)\frac{\beta_n(x)}{n!}-\frac{\beta_{n+1}(x)}{n!}=0,$$

$$\sum_{i=0}^{n-1}\frac{\beta_i(x)}{i!\cdot(n-i)!}+\frac{\beta_n(x)}{n!}=x\frac{\beta_n(x)}{n!}+\frac{\beta_{n+1}(x)}{n!},\ \sum_{i=0}^{n}\frac{\beta_i(x)}{i!\cdot(n-i)!}=x\frac{\beta_n(x)}{n!}+\frac{\beta_{n+1}(x)}{n!},$$

$$\sum_{i=0}^{n}\binom{n}{i}\beta_i(x)=x\beta_n(x)+\beta_{n+1}(x).$$

Assume $\mathrm{B}(x,\xi)=\sum_{n=0}^{\infty}\frac{\beta_n(x)}{n!}\xi^n$, then we have $\mathrm{B}_\xi(x,\xi)=\sum_{n=1}^{\infty}\frac{\beta_n(x)}{(n-1)!}\xi^{n-1}=\sum_{n=0}^{\infty}\frac{\beta_{n+1}(x)}{n!}\xi^n$.

$$e^{\xi}\cdot \mathrm{B}(x,\xi)=\sum_{n=0}^{\infty}\frac{1}{n!}\xi^n\cdot\sum_{n=0}^{\infty}\frac{\beta_n(x)}{n!}\xi^n=\sum_{n=0}^{\infty}\frac{\sum_{i=0}^{n}\binom{n}{i}\beta_i(x)}{n!}\xi^n=\sum_{n=0}^{\infty}\frac{x\beta_n(x)+\beta_{n+1}(x)}{n!}\xi^n=x\mathrm{B}(x,\xi)+\mathrm{B}_{\xi}(x,\xi).$$

$$\frac{d\mathrm{B}(x,\xi)}{d\xi}=(e^{\xi}-x)\mathrm{B}(x,\xi),\ \frac{d\mathrm{B}(x,\xi)}{\mathrm{B}(x,\xi)}=(e^{\xi}-x)d\xi,\ \log\mathrm{B}(x,\xi)=e^{\xi}-x\cdot\xi+C,\ \mathrm{B}(x,\xi)=e^{e^{\xi}-x\cdot\xi+C}.$$

Since $\mathrm{B}(x,0)=e^{1+C}=\beta_0(x)=1$, we obtain $C=-1$, $\mathrm{B}(x,\xi)=e^{e^{\xi}-x\cdot\xi-1}$.

Notice that $\mathrm{B}(0,\xi)=e^{e^{\xi}-1}=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}\xi^n$ gives Bell number, and $\beta_n=\sum_{k=0}^{\infty}\frac{k^n}{k!}$. There seems no wonder why $\mathrm{B}(x,\xi)$ appears here.

Hence,

$$\sum_{k=0}^{n}\frac{\beta_k(x)}{k!}f^{(k)}(x)=\sum_{k=0}^{n-1}\beta_k(x)\frac{\phi^{(n)}(x)}{k!\cdot(n-k)!}+(1-x)\frac{\beta_n(x)}{n!}\phi^{(n)}(x)+\int_x^{x+1}\phi^{(n+1)}(t)\sum_{k=0}^{n}\beta_k(x)\frac{(x+1-t)^{n-k}}{k!\cdot(n-k)!}dt,$$

$$\sum_{k=0}^{n}\frac{\beta_k(x)}{k!}f^{(k)}(x)=\frac{\beta_{n+1}(x)}{n!}\phi^{(n)}(x)+\frac{1}{n!}\int_x^{x+1}\phi^{(n+1)}(t)\sum_{k=0}^{n}\frac{\beta_k(x)}{k!}\frac{(x+1-t)^{n-k}}{(n-k)!}dt.$$

$$e^{e^{\xi}-x\cdot\xi-1}\cdot e^{(x+1-t)\cdot\xi}=\sum_{n=0}^{\infty}\frac{\beta_n(x)}{n!}\xi^n\cdot\sum_{n=0}^{\infty}\frac{(x+1-t)^n}{n!}\xi^n=\sum_{n=0}^{\infty}\frac{\sum_{i=0}^{n}\binom{n}{i}\beta_i(x)(x+1-t)^{n-i}}{n!}\xi^n=e^{e^{\xi}-(t-1)\cdot\xi-1}.$$

Hence, $\sum_{k=0}^{n}\frac{\beta_k(x)}{k!}f^{(k)}(x)=\frac{\beta_{n+1}(x)}{n!}\phi^{(n)}(x)+\frac{1}{n!}\int_x^{x+1}\phi^{(n+1)}(t)\beta_n(t-1)dt$.

$$\frac{1}{\beta_{n+1}(x)}\sum_{k=0}^{n}\frac{\beta_k(x)}{k!}f^{(k)}(x)=\frac{\phi^{(n)}(x)}{n!}+\frac{1}{\beta_{n+1}(x)}\frac{1}{n!}\int_x^{x+1}\phi^{(n+1)}(t)\beta_n(t-1)dt.$$

$$\frac{\phi^{(n)}(x)}{n!}=\frac{1}{\beta_{n+1}(x)}\sum_{k=0}^{n}\frac{\beta_k(x)}{k!}f^{(k)}(x)-\frac{1}{\beta_{n+1}(x)}\frac{1}{n!}\int_x^{x+1}\phi^{(n+1)}(t)\beta_n(t-1)dt.$$

Integrate the equation, and we obtain the result.

3. Applications of Euler-Maclaurin Formula to Elementary Summations

We list some elementary summations in this section. All these results can be established in elementary methods. In spite of it, we want to show how Euler-Maclaurin formula can be used to deal with these problems in a common way.

Refer to [5], we know the convergence radius of $\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$ is 2π. Therefore, in the disk center at $z=0$ with radius 2π, we can derivative the series term by term.

Theorem 2: (1) $\sum_{n=0}^{\infty}\frac{B_n}{n!}=\frac{1}{e-1}$;

(2) $\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}=\frac{e}{e-1}$;

(3) $\sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}=\frac{1}{2}\cot\left(\frac{1}{2}\right)$;

(4) $\sum_{n=0}^{\infty}\frac{B_{n+1}}{n!}=-\frac{1}{(e-1)^2}$; $\sum_{n=0}^{\infty}\frac{B_{n+2}}{n!}=-\frac{e(e-3)}{(e-1)^3}$; $\sum_{n=0}^{\infty}\frac{B_{n+3}}{n!}=\frac{2e(e^2-2e-2)}{(e-1)^4}$; ...

Proof : From $\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n$, we obtain $\sum_{n=0}^{\infty}\frac{B_n}{n!}=\frac{1}{e-1}$.

Replace x with $-x$, we obtain $\frac{-x}{e^{-x}-1}=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}x^n$. Therefore, $\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}=\frac{-1}{e^{-1}-1}=\frac{e}{e-1}$.

$\frac{x}{e^x-1}+\frac{x}{2}=\frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right)=\frac{x}{2}\left(\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)=\frac{x}{2}\coth\left(\frac{x}{2}\right)=\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n}$. Replace x with ix , we obtain

$\frac{ix}{2}\coth\left(\frac{ix}{2}\right)=\frac{x}{2}\cot\left(\frac{x}{2}\right)=\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}(ix)^{2n}=\sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}x^{2n}$. Therefore, $\sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}=\frac{1}{2}\cot\left(\frac{1}{2}\right)$.

$\frac{d}{dx}\frac{x}{e^x-1}=-\frac{xe^x-e^x+1}{(e^x-1)^2}=\frac{d}{dx}\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n=\sum_{n=0}^{\infty}B_n\frac{d}{dx}\frac{x^n}{n!}=\sum_{n=1}^{\infty}\frac{B_n}{(n-1)!}x^{n-1}=\sum_{n=0}^{\infty}\frac{B_{n+1}}{n!}x^n$.

Therefore, $\sum_{n=0}^{\infty}\frac{B_{n+1}}{n!}=-\frac{e-e+1}{(e-1)^2}=-\frac{1}{(e-1)^2}$.

Similarly, $\sum_{n=0}^{\infty}\frac{B_{n+2}}{n!}=-\frac{e(e-3)}{(e-1)^3}$, $\sum_{n=0}^{\infty}\frac{B_{n+3}}{n!}=\frac{2e(e^2-2e-2)}{(e-1)^4}$, ...

Then we consider some most basic summations. We focus on the application of Euler-Maclaurin formula.

(1) $\sum_{n=1}^{x-1}n^{k-1}$

The inverse difference of x^{k-1} is given by

$$\sum x^{k-1}\Delta x=\sum_{n=o}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}x^{k-1}=\sum_{n=o}^{k}\frac{B_n}{n!}\frac{(k-1)!x^{k-n}}{(k-n)!}=\sum_{n=o}^{k}\frac{\binom{k}{n}B_n x^{k-n}}{k}.$$

Hence, we have $\sum_{n=1}^{x-1}n^{k-1}=\sum_1^x n^{k-1}\Delta n=\sum_{i=o}^{k}\frac{\binom{k}{i}B_i x^{k-i}}{k}\Bigg|_1^x=\sum_{i=o}^{k}\frac{\binom{k}{i}B_i x^{k-n}}{k}-\sum_{i=o}^{k}\frac{\binom{k}{i}B_i}{k}$.

Since $\sum_{i=o}^{k-1}\frac{\binom{k}{i}B_i}{k}=0$, we can simplify it as follow,

$$\sum_{i=o}^{k}\frac{\binom{k}{i}B_i x^{k-n}}{k}-\frac{B_k}{k}=\frac{\sum_{i=o}^{k}\binom{k}{i}B_i x^{k-n}-B_k}{k}.$$

A similar method can be used as follow.

$$\sum_{n=1}^{x-1} n^{k-1} = \sum_{n=0}^{x-1} n^{k-1} = \sum_0^x n^{k-1}\Delta n = \sum_{i=o}^{k}\frac{\binom{k}{i}B_i x^{k-i}}{k}\Bigg|_0^x = \sum_{i=o}^{k}\frac{\binom{k}{i}B_i x^{k-n}}{k} - \frac{B_k}{k} = \frac{\sum_{i=o}^{k}\binom{k}{i}B_i x^{k-n} - B_k}{k}.$$

Suppose $P(x)$ is a polynomial of degree $k-1$, we have $P^{(n)}(x)(n \ge k) = 0$. Therefore, its inverse difference is a polynomial of degree k. The method of undetermined coefficients is applicable, Newton series is also effective. However, I think Euler-Maclaurin formula is more direct and convenient.

(2) $\sum_{n=0}^{x-1} e^n$.

The inverse difference of e^x is given by $\sum e^x \Delta x = \sum_{n=o}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}e^x = \sum_{n=o}^{\infty}\frac{B_n}{n!}e^x = e^x \cdot \sum_{n=o}^{\infty}\frac{B_n}{n!} = \frac{e^x}{e-1}$.

Hence, we have $\sum_{n=0}^{x-1} e^n = \sum_0^x e^n \Delta n = \frac{e^n}{e-1}\Big|_0^x = \frac{e^x-1}{e-1}$.

(3) $\sum_{n=1}^{x-1} n\cdot e^n$.

We prove a lemma first.

Lemma 2: $\frac{d^n}{dx^n}\left(x\cdot e^x\right) = (x+n)\cdot e^x (n\in N)$.

Proof : If $n=0$, then $\frac{d^0}{dx^0}\left(x\cdot e^x\right) = x\cdot e^x = (x+0)\cdot e^x$.

Notice that $\frac{d^{k+1}}{dx^{k+1}}\left(x\cdot e^x\right) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left(x\cdot e^x\right)\right) = \frac{d}{dx}\left((x+n)\cdot e^x\right) = e^x + (x+n)\cdot e^x = (x+n+1)\cdot e^x$.

Hence, from mathematical induction we know it is true for $n\in N$.

The inverse difference of $\sin(x)$ is given by

$$\sum x\cdot e^x\Delta x = \sum_{n=o}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}x\cdot e^x = \sum_{n=o}^{\infty}\frac{B_n}{n!}(x+n-1)\cdot e^x = (x-1)\cdot e^x\cdot\sum_{n=o}^{\infty}\frac{B_n}{n!} + \sum_{n=o}^{\infty}\frac{B_n}{n!}n\cdot e^x$$

$$= \frac{(x-1)\cdot e^x}{e-1} + e^x\cdot\sum_{n=o}^{\infty}\frac{B_{n+1}}{n!} = \frac{(x-1)\cdot e^x}{e-1} - \frac{e^x}{(e-1)^2}.$$

Hence, we have

$$\sum_{n=1}^{x-1} n\cdot e^n = \sum_1^x n\cdot e^n\Delta n = \left(\frac{(n-1)\cdot e^n}{e-1} - \frac{e^n}{(e-1)^2}\right)\Bigg|_1^x = \frac{(x-1)\cdot e^x}{e-1} - \frac{e^x}{(e-1)^2} - \left(-\frac{e}{(e-1)^2}\right)$$

$$= \frac{(x-1)\cdot e^x}{e-1} - \frac{e^x}{(e-1)^2} + \frac{e}{(e-1)^2} = \frac{(x-1)\cdot e^x}{e-1} - \frac{e(e^{x-1}-1)}{(e-1)^2}.$$

Similarly, we have $\sum_{n=1}^{x-1} n^2 \cdot e^n = \frac{(x-1)^2 \cdot e^x}{e-1} - \frac{2(x-1)\cdot e^x}{(e-1)^2} + \frac{e(e+1)(e^{x-1}-1)}{(e-1)^3}$,

$\sum_{n=1}^{x-1} n^3 \cdot e^n = \frac{(x-1)^3 \cdot e^x}{e-1} - \frac{3(x-1)^2 \cdot e^x}{(e-1)^2} + \frac{3(x-1)\cdot e^x (e+1)}{(e-1)^3} - \frac{e(e^2+4e+1)(e^{x-1}-1)}{(e-1)^3}$, ...

(4) $\sum_{n=1}^{x-1} \sin(n)$.

The inverse difference of $\sin(x)$ is given by

$$\sum \sin(x)\Delta x = \sum_{n=o}^{\infty} \frac{B_n}{n!} \frac{d^{n-1}}{dx^{n-1}} \sin(x) = \sum_{n=o}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} \cos(x) - \frac{1}{2}\sin(x) = -\frac{1}{2}\cot\left(\frac{1}{2}\right)\cdot\cos(x) - \frac{1}{2}\sin(x)$$

$$= -\frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)}\cdot\cos(x) + \frac{\sin\left(\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)}\cdot\sin(x) \right) = -\frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\right)\cdot\cos(x) + \sin\left(\frac{1}{2}\right)\cdot\sin(x)}{\sin\left(\frac{1}{2}\right)} \right) = -\frac{\cos\left(x-\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}.$$

Hence, we have

$$\sum_{n=1}^{x-1} \sin(n) = \sum\nolimits_1^x \sin(n)\Delta n = -\frac{\cos\left(n-\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}\Bigg|_1^x = -\frac{\cos\left(x-\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)} + \frac{\cos\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)} = -\frac{\cos\left(x-\frac{1}{2}\right) - \cos\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}$$

$$= -\frac{-2\sin\left(\frac{x-1}{2}\right)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{1}{2}\right)} = \frac{\sin\left(\frac{x-1}{2}\right)\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{1}{2}\right)}.$$

(5) $\sum_{n=0}^{x-1} \cos(n)$.

The inverse difference of $\cos(x)$ is given by

$$\sum \cos(x)\Delta x = \sum_{n=o}^{\infty} \frac{B_n}{n!} \frac{d^{n-1}}{dx^{n-1}} \cos(x) = \sum_{n=o}^{\infty} (-1)^{n} \frac{B_{2n}}{(2n)!} \sin(x) - \frac{1}{2}\cos(x) = \frac{1}{2}\cot\left(\frac{1}{2}\right)\cdot\sin(x) - \frac{1}{2}\cos(x)$$

$$= \frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)}\cdot\sin(x) - \frac{\sin\left(\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)}\cdot\cos(x) \right) = \frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\right)\cdot\sin(x) - \sin\left(\frac{1}{2}\right)\cdot\cos(x)}{\sin\left(\frac{1}{2}\right)} \right) = \frac{\sin\left(x-\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}.$$

Hence, we have

$$\sum_{n=0}^{x-1}\cos(n)=\sum\nolimits_0^x\cos(n)\Delta n=\left.\frac{\sin\left(n-\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}\right|_0^x=\frac{\sin\left(x-\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}+\frac{\sin\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}=\frac{\sin\left(x-\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}$$

$$=\frac{2\cos\left(\frac{x-1}{2}\right)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{1}{2}\right)}=\frac{\cos\left(\frac{x-1}{2}\right)\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{1}{2}\right)}.$$

4. Some Infinite Series

We consider some infinite series in this section. They may seem to have little to do with our subject. However, it is these series that led me to the discovery of Euler-Maclaurin formula. Moreover, some of the results are relevant to the issues in section 6.

The relevant research is so plentiful that I can hardly decide whether the results have already been in literature. I will point it out if I know it has already been known.

(1) $\sum_{x=1}^{n}\frac{1}{x^s}(s>1)$.

We know it is the well known zeta function on the real axis $s>1$. We show how it can be transformed into a definite integral. The result has already been known.

$$\sum_{x=1}^{n}\frac{1}{x^s}=\sum_{x=1}^{n}\frac{1}{x^s}\cdot\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-\xi}\xi^{s-1}d\xi=\frac{1}{(s-1)!}\cdot\sum_{x=1}^{n}\int_0^\infty e^{-\xi}\left(\frac{\xi}{x}\right)^{s-1}d\frac{\xi}{x}.$$

Let $t=\frac{\xi}{x}$, we obtain $\frac{1}{(s-1)!}\cdot\sum_{x=1}^{n}\int_0^\infty e^{-\xi}\left(\frac{\xi}{x}\right)^{s-1}d\frac{\xi}{x}=\frac{1}{(s-1)!}\cdot\sum_{x=1}^{n}\int_0^\infty e^{-x\cdot t}t^{s-1}dt$

$$=\frac{1}{(s-1)!}\cdot\int_0^\infty\sum_{x=1}^{n}e^{-x\cdot t}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-t}\frac{1-e^{-n\cdot t}}{1-e^{-t}}t^{s-1}dt.$$

Let $n\to\infty$, we obtain $\sum_{x=1}^{\infty}\frac{1}{x^s}=\frac{1}{(s-1)!}\cdot\int_0^\infty\frac{e^{-t}}{1-e^{-t}}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty\frac{t^{s-1}}{e^t-1}dt=\frac{1}{\Gamma(s)}\cdot\int_0^\infty\frac{t^{s-1}}{e^t-1}dt$.

The last formula can serve as an equivalent definition of $\zeta(s)$ on the real axis $s>1$.

Similarly, we can obtain

$$\sum_{x=0}^{n}\frac{1}{(x+a)^s}=\sum_{x=0}^{n}\frac{1}{(x+a)^s}\cdot\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-\xi}\xi^{s-1}d\xi=\frac{1}{(s-1)!}\cdot\sum_{x=0}^{n}\int_0^\infty e^{-\xi}\left(\frac{\xi}{x+a}\right)^{s-1}d\frac{\xi}{x+a}$$

$$=\frac{1}{(s-1)!}\cdot\sum_{x=0}^{n}\int_0^\infty e^{-(x+a)\cdot t}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-a\cdot t}\sum_{x=o}^{n}e^{-t\cdot x}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-a\cdot t}\frac{1-e^{-n\cdot t}}{1-e^{-t}}t^{s-1}dt.$$

Let $n\to\infty$, we obtain

$$\sum_{x=0}^{\infty}\frac{1}{(x+a)^s}=\frac{1}{(s-1)!}\cdot\int_0^\infty\frac{e^{-a\cdot t}}{1-e^{-t}}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty\frac{e^{(1-a)\cdot t}}{e^t-1}t^{s-1}dt=\frac{1}{\Gamma(s)}\cdot\int_0^\infty\frac{e^{(1-a)\cdot t}}{e^t-1}t^{s-1}dt.$$

We know it is Hurwitz zeta function $\zeta(s,a)$.

Lastly,

$$\sum_{x=0}^{n}\frac{(-1)^x}{(x+a)^s}=\sum_{x=0}^{n}\frac{(-1)^x}{(x+a)^s}\cdot\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-\xi}\xi^{s-1}d\xi=\frac{1}{(s-1)!}\cdot\sum_{x=0}^{n}\int_0^\infty(-1)^x e^{-\xi}\left(\frac{\xi}{x+a}\right)^{s-1}d\frac{\xi}{x+a}$$

$$=\frac{1}{(s-1)!}\cdot\sum_{x=0}^{n}\int_0^\infty(-1)^x e^{-(x+a)\cdot t}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-a\cdot t}\sum_{x=o}^{n}(-1)^x e^{-t\cdot x}t^{s-1}dt$$

$$=\frac{1}{(s-1)!}\cdot\int_0^\infty e^{-a\cdot t}\frac{1+(-e^{-t})^{n+1}}{1+e^{-t}}t^{s-1}dt.$$

Let $n\to\infty$, we obtain

$$\sum_{x=0}^{\infty}\frac{(-1)^x}{(x+a)^s}=\frac{1}{(s-1)!}\cdot\int_0^\infty\frac{e^{-a\cdot t}}{1+e^{-t}}t^{s-1}dt=\frac{1}{(s-1)!}\cdot\int_0^\infty\frac{e^{(1-a)\cdot t}}{e^t+1}t^{s-1}dt=\frac{1}{\Gamma(s)}\cdot\int_0^\infty\frac{e^{(1-a)\cdot t}}{e^t+1}dt.$$

(2) $\sum_{x=1}^{\infty}\frac{1}{x^x}$

As usual, we begin from

$$\sum_{x=1}^{\infty}\frac{1}{x^x}=\sum_{x=1}^{\infty}\frac{1}{x^x}\cdot\frac{1}{(x-1)!}\cdot\int_0^\infty e^{-\xi}\xi^{x-1}d\xi=\sum_{x=1}^{\infty}\int_0^\infty\frac{1}{(x-1)!}e^{-\xi}\left(\frac{\xi}{x}\right)^{x-1}d\frac{\xi}{x}=\sum_{x=1}^{\infty}\int_0^\infty\frac{1}{(x-1)!}e^{-x\cdot t}t^{x-1}dt.$$

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$\sum_{x=1}^{\infty}\int_0^\infty\frac{1}{(x-1)!}e^{-x\cdot t}t^{x-1}dt=\int_0^\infty\sum_{x=1}^{\infty}\frac{1}{(x-1)!}\cdot e^{-x\cdot t}t^{x-1}dt=\int_0^\infty e^{-t}\cdot\sum_{x=1}^{\infty}\frac{1}{(x-1)!}\cdot e^{-(x-1)\cdot t}t^{x-1}dt=\int_0^\infty e^{-t}\cdot e^{e^{-t}\cdot t}dt.$$

Let $x=e^{-t}$, i.e. $t=-\ln(x)$, we obtain $\int_0^\infty e^{-t}\cdot e^{e^{-t}\cdot t}dt=-\int_0^1 x\cdot x^{-x}\cdot\left(-\frac{1}{x}\right)dx=\int_0^1 x^{-x}dx$.

Thus we obtain an amazing identity: $\sum_{x=1}^{\infty}\frac{1}{x^x}=\int_0^1\frac{dx}{x^x}$.

(3) $\sum_{x=1}^{\infty}\frac{1}{x\cdot x!}$

We prove a relevant theorem before we consider this series. A lemma is needed first.

Lemma 3: $\frac{1}{\prod_{i=0}^{k}(n+i)}=\sum_{i=0}^{k}(-1)^i\frac{1}{i!\cdot(n-i)!}\cdot\frac{1}{(n+i)}(k\geq 1,k\in N)$.

Proof : Suppose $\frac{1}{\prod_{i=0}^{k}(n+i)}=\sum_{i=0}^{k}a_i\frac{1}{(n+i)}$, then we have $\sum_{i=0}^{k}a_i\frac{\prod_{i=0}^{k}(n+i)}{n+i}=\sum_{i=0}^{k}a_i\prod_{j\neq i}(n+i)=1$ $(j\in\{1,2,\ldots k\})$.

Let $n=-i(i=0,1,\dots k)$, then we obtain $a_i\prod_{j\neq i}(j-i)=1$.

Hence, $a_i=\frac{1}{\prod_{j\neq i}(j-i)}=\frac{1}{\prod_{j<i}(j-i)}\cdot\frac{1}{\prod_{j>i}(j-i)}=(-1)^i\frac{1}{i!}\cdot\frac{1}{(n-i)!}$.

Theorem 4: $\sum_{n=1}^{\infty}\frac{1}{\prod_{i=0}^{k}(n+i)}=\frac{1}{k\cdot k!}\ (k\geq 1,k\in N)$.

Proof :

$$\sum_{n=1}^{\infty}\frac{1}{\prod_{i=0}^{k}(n+i)}=\sum_{n=1}^{\infty}\left(\sum_{i=0}^{k}(-1)^i\frac{1}{i!\cdot(n-i)!}\cdot\frac{1}{(n+i)}\right)=\sum_{n=1}^{\infty}\left(\sum_{i=0}^{k}(-1)^i\frac{1}{k!}\binom{k}{i}\frac{1}{(n+i)}\right)=\frac{1}{k!}\sum_{n=1}^{\infty}\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{1}{(n+i)}\right)$$

$$=\frac{1}{k!}\cdot\left(\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}\left(\sum_{n=i}^{k-1}\frac{1}{n+1}\right)+\sum_{n=k+1}^{\infty}\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{1}{n}\right)\right).$$

Notice that $\sum_{i=0}^{k}(-1)^i\binom{k}{i}=0$, thus the latter summation vanishes, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{\prod_{i=0}^{k}(n+i)}=\frac{1}{k!}\cdot\left(\sum_{i=0}^{k\text{-}1}(-1)^i\binom{k}{i}\left(\sum_{n=i}^{k-1}\frac{1}{n+1}\right)\right).$$

The formula can be simplified as follow.

$$\frac{1}{k!}\cdot\left(\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}\left(\sum_{n=i}^{k-1}\frac{1}{n+1}\right)\right)=\frac{1}{k!}\cdot\left(\sum_{n=1}^{k}\left(\sum_{i=0}^{n-1}(-1)^i\binom{k}{i}\frac{1}{n}\right)\right)=\frac{1}{k!}\cdot\left(\sum_{n=1}^{k}(-1)^n\binom{k-1}{n-1}\frac{1}{n}\right)$$

$$=\frac{1}{k!}\cdot\left(\sum_{n=1}^{k}(-1)^n\binom{k}{n}\cdot\frac{1}{k}\right)=\frac{1}{k\cdot k!}\left(\sum_{n=1}^{k}(-1)^n\binom{k}{n}\right)=\frac{1}{k\cdot k!}.$$

$$\sum_{x=1}^{\infty}\frac{1}{x\cdot x!}=\sum_{x=1}^{\infty}\frac{1}{x^2\cdot(x-1)!}=\sum_{x=1}^{\infty}\frac{1}{x^2\cdot(x-1)!}\int_0^{\infty}e^{-\xi}\xi d\xi=\sum_{x=1}^{\infty}\int_0^{\infty}\frac{e^{-\xi}}{(x-1)!}\left(\frac{\xi}{x}\right)d\frac{\xi}{x}=\sum_{x=1}^{\infty}\int_0^{\infty}\frac{e^{-x\cdot t}}{(x-1)!}tdt.$$

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$\sum_{x=1}^{\infty}\int_0^{\infty}\frac{e^{-x\cdot t}}{(x-1)!}tdt=\int_0^{\infty}\sum_{x=1}^{\infty}\frac{1}{(x-1)!}\cdot e^{-x\cdot t}tdt=\int_0^{\infty}e^{-t}\cdot e^{e^{-t}}\cdot tdt.$$

Let $x=e^{-t}$, we obtain

$$\sum_{x=1}^{\infty}\frac{1}{x\cdot x!}=-\int_0^1 x\cdot e^x\cdot(-\ln(x))d(-\ln(x))=\int_0^1 x\cdot e^x\cdot\ln(x)\cdot\left(-\frac{1}{x}\right)dx=-\int_0^1 e^x\cdot\ln(x)dx.$$

Similarly, $\sum_{x=1}^{\infty}\frac{1}{x^{n-1}\cdot x!}=\sum_{x=1}^{\infty}\frac{1}{x^n\cdot(x-1)!}=\sum_{x=1}^{\infty}\frac{1}{x^n\cdot(x-1)!}\frac{1}{(n-1)!}\int_0^{\infty}e^{-\xi}\xi^{n-1}d\xi$

$$=\frac{1}{(n-1)!}\sum_{x=1}^{\infty}\int_0^{\infty}\frac{e^{-\xi}}{(x-1)!}\left(\frac{\xi}{x}\right)^{n-1}d\frac{\xi}{x}=\frac{1}{(n-1)!}\cdot\sum_{x=1}^{\infty}\int_0^{\infty}\frac{e^{-x\cdot t}}{(x-1)!}t^{n-1}dt.$$

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$\frac{1}{(n-1)!}\cdot\sum_{x=1}^{\infty}\int_0^{\infty}\frac{e^{-x\cdot t}}{(x-1)!}t^{n-1}dt=\frac{1}{(n-1)!}\cdot\int_0^{\infty}\sum_{x=1}^{\infty}\frac{e^{-x\cdot t}}{(x-1)!}\cdot t^{n-1}dt=\frac{1}{(n-1)!}\cdot\int_0^{\infty}e^{-t}\cdot\sum_{x=1}^{\infty}\frac{e^{-(x-1)\cdot t}}{(x-1)!}\cdot t^{n-1}dt$$

$$=\frac{1}{(n-1)!}\cdot\int_0^{\infty}e^{-t}\cdot e^{e^{-t}}\cdot t^{n-1}dt\,.$$

Let $x=e^{-t}$, we obtain $\sum_{x=1}^{\infty}\frac{1}{x^{n-1}\cdot x!}=-\frac{1}{(n-1)!}\cdot\int_0^1 x\cdot e^x\cdot(-\ln(x))^{n-1}d(-\ln(x))$

$$=\frac{(-1)^n}{(n-1)!}\cdot\int_0^1 x\cdot e^x\cdot(\ln(x))^{n-1}\cdot\left(-\frac{1}{x}\right)dx=\frac{(-1)^{n-1}}{(n-1)!}\cdot\int_0^1 e^x\cdot(\ln(x))^{n-1}dx=\frac{(-1)^{n-1}}{\Gamma(n)}\cdot\int_0^1 e^x\cdot(\ln(x))^{n-1}dx\,.$$

Let us have some further consideration on it. If we define $\varepsilon_n=\sum_{x=1}^{\infty}\frac{1}{x^{n-1}\cdot x!}$, then we know $\varepsilon_0=e-1$, $\varepsilon_{-1}=e$, … In general, $\varepsilon_{-n}=B_n\cdot e$ for $n\ge 1$, where B_n denotes nth Bell number, whose exponential generating function is $e^{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n$. Hence the generating function of ε_{-n} for $n\ge 0$ is $e^{e^x}=1+\sum_{n=0}^{\infty}\frac{\varepsilon_{-n}}{n!}x^n$. As for the generating function of ε_n for $n\ge 0$, we have the following result:

Theorem : The generating function of ε_n for $n\ge 0$ is given by $\int_0^1 e^t\cdot t^x dt=\sum_{n=0}^{\infty}(-1)^n\varepsilon_n x^n$.

Proof : $\sum_{n=0}^{\infty}(-1)^n\varepsilon_n x^n=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\int_0^1 e^t\cdot(\ln(t))^n dt\cdot x^n=\sum_{n=0}^{\infty}\int_0^1 e^t\cdot\frac{(\ln(t)\cdot x)^n}{n!}dt$.

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$\sum_{n=0}^{\infty}\int_0^1 e^t\cdot\frac{(\ln(t)\cdot x)^n}{n!}dt=\int_0^1\sum_{n=0}^{\infty}e^t\cdot\frac{(\ln(t)\cdot x)^n}{n!}dt=\int_0^1 e^t\cdot e^{\ln(t)\cdot x}dt=\int_0^1 e^t\cdot t^x dt\,.$$

(4) $\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}$

We use the identity $\Gamma(\alpha)\Gamma(\beta)=\Gamma(\alpha+\beta)B(\alpha,\beta)$ to transform the summand into

$$\frac{1}{\binom{2n}{n}}=\frac{\Gamma^2(n+1)}{\Gamma(2n+1)}=\frac{\Gamma(2n+2)}{\Gamma(2n+1)}\cdot B(n+1,n+1)=(2n+1)B(n+1,n+1)=(2n+1)\int_0^1 x^n(1-x)^n dx$$

$$=\int_0^1(2n+1)(x-x^2)^n dx\,.$$

Therefore, $\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}=\sum_{n=1}^{\infty}\int_0^1(2n+1)(x-x^2)^n dx$.

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$\sum_{n=1}^{\infty}\int_0^1 (2n+1)(x-x^2)^n\,dx = \int_0^1 \sum_{n=1}^{\infty}(2n+1)(x-x^2)^n\,dx = -\int_0^1 \frac{x^4-2x^3+4x^2-3x}{(x-x^2-1)^2}dx = \frac{1}{3}+\frac{2\sqrt{3}\pi}{27}.$$

Similarly, we can calculate $\sum_{n=1}^{\infty}\frac{1}{n^s\binom{2n}{n}}$ at where $s \le 1$. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s\binom{2n}{n}}$ at where $s \le 1$ can also be computed in this way. Some other identities includes $\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = \frac{1}{3}\zeta(2) = \frac{\pi^2}{18}$,

$\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}} = \frac{17}{36}\zeta(4) = \frac{17\pi^4}{3240}$, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} = \frac{2}{5}\zeta(3)$. etc. Since these identities appeared in Apery's paper proving $\zeta(2)$ and $\zeta(3)$ are irrational, we have some further consideration.

Let us begin from $\sum_{n=1}^{\infty}\frac{x^n}{n^2\binom{2n}{n}}$, we transform it into $\sum_{n=1}^{\infty}\frac{x^n}{n^2\binom{2n}{n}} = \sum_{n=1}^{\infty}\frac{(n!)^2}{n^2(2n)!}x^n = \sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}x^n$

$$= \sum_{n=1}^{\infty}\frac{\Gamma^2(n)}{\Gamma(2n+1)}x^n = \sum_{n=1}^{\infty}\frac{\Gamma(2n)\mathrm{B}(n,n)}{\Gamma(2n+1)}x^n = \sum_{n=1}^{\infty}\frac{\mathrm{B}(n,n)}{2n}x^n = \sum_{n=1}^{\infty}\int_0^1\frac{(t-t^2)^{n-1}}{2n}x^n\,dt.$$

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$\sum_{n=1}^{\infty}\int_0^1\frac{(t-t^2)^{n-1}}{2n}x^n\,dt = \int_0^1\sum_{n=1}^{\infty}\frac{(t-t^2)^{n-1}}{2n}x^n\,dt = -\int_0^1\frac{\log\bigl(1-(t-t^2)x\bigr)}{2(t-t^2)}dt = 2\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)^2.$$

Hence, we obtain $\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = 2\sin^{-1}\left(\frac{1}{2}\right)^2 = \frac{\pi^2}{18} = \frac{1}{3}\zeta(2)$.

Then let us consider $\sum_{n=1}^{\infty}\frac{x^n}{n^3\binom{2n}{n}}$. We can obtain $\sum_{n=1}^{\infty}\frac{x^n}{n^3\binom{2n}{n}} = 2\int_0^x \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)^2\frac{dx}{x}$.

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = 2\int_0^1 \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)^2\frac{dx}{x}$.

Apply the variable substitution $u=\sqrt{x}$, we obtain $2\int_0^1 \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)^2\frac{dx}{x} = 4\int_0^1 \sin^{-1}\left(\frac{u}{2}\right)^2\frac{du}{u}$.

Similarly, $\sum_{n=1}^{\infty}\frac{x^n}{n^{2+s}\binom{2n}{n}}(s\ge 1)=2\int_0^x\ldots\int_0^{x_3}\int_0^{x_2}\sin^{-1}\left(\frac{\sqrt{x_1}}{2}\right)^2\frac{1}{x_1}dx_1dx_2\ldots dx_s$

$=2\int_0^x\ldots\int_0^{x_3}\int_0^1\sin^{-1}\left(\frac{\sqrt{x_1x_2}}{2}\right)^2\frac{1}{x_1x_2}dx_1dx_2\ldots dx_s$ $=2\int_0^x\ldots\int_0^1\int_0^1\sin^{-1}\left(\frac{\sqrt{x_1x_2x_3}}{2}\right)^2\frac{1}{x_1x_2x_3}dx_1dx_2\ldots dx_s$

$=\ldots=2\int_0^x\ldots\int_0^1\int_0^1\sin^{-1}\left(\frac{\sqrt{x_1x_2\ldots x_s}}{2}\right)^2\frac{1}{x_1x_2\ldots x_s}dx_1dx_2\ldots dx_s$

$=2\int_\Omega\sin^{-1}\left(\frac{\sqrt{x_1x_2\ldots x_s}}{2}\right)^2\frac{1}{x_1x_2\ldots x_s}dx_1dx_2\ldots dx_s$.

Hence, $\sum_{n=1}^{\infty}\frac{1}{n^{2+s}\binom{2n}{n}}(s\ge 1)=2\int_\Omega\sin^{-1}\left(\frac{\sqrt{x_1x_2\ldots x_s}}{2}\right)^2\frac{1}{x_1x_2\ldots x_s}dx_1dx_2\ldots dx_s$.

The integral area is a s-dimensional unit cube $[0,1]^s$.

We apply the variable substitution $\begin{pmatrix}u_1\\u_2\\u_3\\\ldots\\u_s\end{pmatrix}=\begin{pmatrix}\sqrt{x_1x_2\ldots x_s}\\x_1\\x_2\\\ldots\\x_{s-1}\end{pmatrix}$, i.e. $\begin{pmatrix}x_1\\x_2\\x_3\\\ldots\\x_s\end{pmatrix}=\begin{pmatrix}u_2\\u_3\\u_4\\\ldots\\\frac{u_1^2}{u_2u_3\ldots u_s}\end{pmatrix}$.

To calculate the Jacobi determinant of it, we may notice that

$$\frac{\partial(x_1,x_2,\ldots,x_s)}{\partial(u_1,u_2,\ldots,u_s)}=\begin{vmatrix}0&1&0&\ldots&0\\0&0&1&\ldots&0\\\ldots&\ldots&\ldots&\ldots&\ldots\\0&0&0&\ldots&1\\\frac{2u_1}{u_2u_3\ldots u_s}&-\frac{u_1^2}{u_2^2u_3\ldots u_s}&-\frac{u_1^2}{u_2u_3^2\ldots u_s}&\ldots&-\frac{u_1^2}{u_2u_3\ldots u_s^2}\end{vmatrix}$$

$$=(-1)^{s-1}\left(\frac{2u_1}{u_2u_3\ldots u_s}\begin{vmatrix}1&0&\ldots&0\\0&1&\ldots&0\\\ldots&\ldots&\ldots&\ldots\\0&0&\ldots&1\end{vmatrix}-\left(-\frac{u_1^2}{u_2^2u_3\ldots u_s}\right)\begin{vmatrix}0&0&\ldots&0\\0&1&\ldots&0\\\ldots&\ldots&\ldots&\ldots\\0&0&\ldots&1\end{vmatrix}+\ldots+\left(-\frac{u_1^2}{u_2u_3\ldots u_s^2}\right)\begin{vmatrix}0&1&\ldots&0\\0&0&\ldots&0\\\ldots&\ldots&\ldots&\ldots\\0&0&\ldots&0\end{vmatrix}\right).$$

Notice that if there is a line filled by 0, then the determinant vanishes. Therefore, only the first term remain, and we obtain $\frac{\partial(x_1,x_2,\ldots,x_s)}{\partial(u_1,u_2,\ldots,u_s)}=(-1)^{s-1}\frac{2u_1}{u_2u_3\ldots u_s}$.

The new integral area is $[0,1]\times[u_1^2,1]\times[\frac{u_1^2}{u_2},1]\times[\frac{u_1^2}{u_2u_3},1]\times\ldots\times[\frac{u_1^2}{u_2u_3\ldots u_{s-1}},1]$.

$$2\int_{\Omega}\sin^{-1}\left(\frac{\sqrt{x_1x_2\ldots x_s}}{2}\right)^2\frac{1}{x_1x_2\ldots x_s}dx_1dx_2\ldots dx_s=2\int_{\Omega}\sin^{-1}\left(\frac{u_1}{2}\right)^2\frac{1}{u_1^2}\left|\frac{\partial(x_1,x_2,\ldots,x_s)}{\partial(u_1,u_2,\ldots,u_s)}\right|du_1du_2\ldots du_s$$

$$=2\int_{\Omega}\sin^{-1}\left(\frac{u_1}{2}\right)^2\frac{1}{u_1^2}\frac{2u_1}{u_2u_3\ldots u_s}du_1du_2\ldots du_s=4\int_{\Omega}\sin^{-1}\left(\frac{u_1}{2}\right)^2\frac{1}{u_1u_2u_3\ldots u_s}du_1du_2\ldots du_s$$

$$=4\int_0^1\sin^{-1}\left(\frac{u_1}{2}\right)^2\frac{du_1}{u_1}\int_{u_1^2}^1\frac{du_2}{u_2}\int_{\frac{u_1^2}{u_2u_3}}^1\frac{du_3}{u_3}\ldots\int_{\frac{u_1^2}{u_2u_3\ldots u_{s-1}}}^1\frac{du_s}{u_s}.$$

To compute $\int_{u_1^2}^1\frac{du_2}{u_2}\int_{\frac{u_1^2}{u_2u_3}}^1\frac{du_3}{u_3}\ldots\int_{\frac{u_1^2}{u_2u_3\ldots u_{s-1}}}^1\frac{du_s}{u_s}$, we denote it as $I_s(u_1)$. Then

$$I_{s+1}(u_1)=\int_{u_1^2}^1\frac{du_2}{u_2}\int_{\frac{u_1^2}{u_2u_3}}^1\frac{du_3}{u_3}\ldots\int_{\frac{u_1^2}{u_2u_3\ldots u_s}}^1\frac{du_{s+1}}{u_{s+1}}=\int_{u_1^2}^1\frac{du_2}{u_2}\int_{\left(\frac{u_1}{\sqrt{u_2}}\right)^2\frac{1}{u_3}}^1\frac{du_3}{u_3}\ldots\int_{\left(\frac{u_1}{\sqrt{u_2}}\right)^2\frac{1}{u_3u_4\ldots u_s}}^1\frac{du_{s+1}}{u_{s+1}}=\int_{u_1^2}^1 I_s\left(\frac{u_1}{\sqrt{u_2}}\right)\frac{du_2}{u_2}.$$

$I_1(u_1)=1$, $I_2(u_1)=-\log(u_1)$, assume $I_s(u_1)=p_s\log(u_1)^{s-1}$, where p_s is a constant, then

$$I_{s+1}(u_1)=\int_{u_1^2}^1 I_s\left(\frac{u_1}{\sqrt{u_2}}\right)\frac{du_2}{u_2}=\int_{u_1^2}^1 p_s\log\left(\frac{u_1}{\sqrt{u_2}}\right)^{s-1}\frac{du_2}{u_2}=-2p_s\int_{u_1^2}^1\log\left(\frac{u_1}{\sqrt{u_2}}\right)^{s-1}d\log\left(\frac{u_1}{\sqrt{u_2}}\right)$$

$$=\frac{-2p_s}{s}\log\left(\frac{u_1}{\sqrt{u_2}}\right)^s\Bigg|_{u_1^2}^1=\frac{-2p_s}{s}\log(u_1)^s=p_{s+1}\log(u_1)^s.$$

Therefore, $p_{s+1}=\frac{-2p_s}{s}$, $s!p_{s+1}=-2(s-1)!p_s$, $(s-1)!p_s=(-2)^{s-1}$, $p_s=\frac{(-2)^{s-1}}{(s-1)!}$.

Hence, $\sum_{n=1}^{\infty}\frac{1}{n^{2+s}\binom{2n}{n}}(s\geq 1)=\frac{(-2)^{s+1}}{(s-1)!}\int_0^1\sin^{-1}\left(\frac{u}{2}\right)^2\frac{(\log(u))^{s-1}}{u}du$.

Apply the variable substitution $\xi=\sin^{-1}\left(\frac{u}{2}\right)$, we obtain

$$\frac{(-2)^{s+1}}{(s-1)!}\int_0^1\sin^{-1}\left(\frac{u}{2}\right)^2\frac{(\log(u))^{s-1}}{u}du=\frac{(-2)^{s+1}}{(s-1)!}\int_0^{\frac{\pi}{6}}\frac{\xi^2}{2\sin(\xi)}(\log(2\sin(\xi)))^{s-1}2\cos(\xi)d\xi$$

$$=\frac{(-2)^{s+1}}{(s-1)!}\int_0^{\frac{\pi}{6}}\xi^2\cot(\xi)(\log(2\sin(\xi)))^{s-1}d\xi.$$

Notice that $\cot(\xi)=\frac{d}{d\xi}\log(2\sin(\xi))$. Therefore, we have

$$\frac{(-2)^{s+1}}{(s-1)!}\int_0^{\frac{\pi}{6}}\xi^2\cot(\xi)(\log(2\sin(\xi)))^{s-1}d\xi=\frac{(-2)^{s+1}}{(s-1)!}\int_0^{\frac{\pi}{6}}\xi^2\frac{d}{d\xi}(\log(2\sin(\xi)))^s$$

$$=\frac{(-2)^{s+1}}{(s-1)!}\left(\xi^2(\log(2\sin(\xi)))^s\Big|_0^{\frac{\pi}{6}}-\int_0^{\frac{\pi}{6}}2\xi(\log(2\sin(\xi)))^s d\xi\right)=-\frac{(-2)^{s+2}}{(s-1)!}\int_0^{\frac{\pi}{6}}\xi(\log(2\sin(\xi)))^s d\xi.$$

Let us consider the case when $s=1$.

We have $\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = 4\int_0^{\frac{\pi}{6}}\xi^2\cot(\xi)d\xi$. Apply the Fourier expansion of $\cot(\xi)$,

$\cot(\xi) = 2\sum_{n=1}^{\infty}\sin(2nx)$, we have $4\int_0^{\frac{\pi}{6}}\xi^2\cot(\xi)d\xi = 8\int_0^{\frac{\pi}{6}}\sum_{n=1}^{\infty}\xi^2\sin(2nx)d\xi$.

Notice that the series converges uniformly. Change the order of series and integral, we obtain

$$8\int_0^{\frac{\pi}{6}}\sum_{n=1}^{\infty}\xi^2\sin(2nx)d\xi = 8\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{6}}\xi^2\sin(2nx)d\xi$$

$$= \sum_{n=1}^{\infty}\left(-\frac{\pi^2}{9}\frac{\cos\left(\frac{n\pi}{3}\right)}{n} + \frac{2\pi}{3}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} + 2\frac{\cos\left(\frac{n\pi}{3}\right)}{n^3} - \frac{2}{n^3}\right)$$

Since $\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{3}\right)}{n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n} = 0$,

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{3}\right)}{n^3} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(3n)^3} = \frac{1}{2}\left(1-2\cdot\frac{1}{8}\right)\zeta(3) - \frac{3}{2}\cdot\frac{1}{27}\left(1-2\cdot\frac{1}{8}\right)\zeta(3) = \frac{1}{3}\zeta(3).$$

We eventually obtain $\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = \frac{2\pi}{3}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} - \frac{4}{3}\zeta(3)$.

The summation $\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}$ can be written via polylogarithm as $\frac{1}{2i}\left(Li_2\left(e^{\frac{i\pi}{3}}\right) - Li_2\left(e^{-\frac{i\pi}{3}}\right)\right)$.

5. Applications of Euler-Maclaurin Formula to Series

The applications of Euler-Maclaurin formula should not stop here. We continue to use it to deal with some series in this section.

(1) $\sum_{x=1}^{\infty}\frac{1}{x^2}$.

It is the well known series as zeta function at $s=2$, and we know it is precisely $\frac{\pi^2}{6}$. We have transformed it into a definite integral in section 4. This time we apply Euler-Maclaurin formula directly.

Lemma 4: $\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x^2} = (-1)^{n-1}\frac{n!}{x^{n+1}}$, where n is a non-negative integer.

Proof : If $n=0$, we have $\frac{d^{-1}}{dx^{-1}}\frac{1}{x^2} = -\frac{1}{x}$.

Notice that $\frac{d^k}{dx^k}\frac{1}{x^2}=\frac{d}{dx}\left(\frac{d^{k-1}}{dx^{k-1}}\frac{1}{x^2}\right)=\frac{d}{dx}\left((-1)^{k-1}\frac{k!}{x^{k+1}}\right)=(-1)^{k-1}\cdot k!\cdot\frac{-(k+1)}{x^{k+2}}=(-1)^k\cdot\frac{(k+1)!}{x^{k+2}}$.
Hence, from mathematical induction we know it is true for all $n\in N$.

We may use Euler-Maclaurin formula to obtain its inverse difference as follow.
$\sum\frac{1}{x^2}\Delta x=\sum_{n=0}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x^2}=\sum_{n=0}^{\infty}\frac{B_n}{n!}(-1)^{n-1}\frac{n!}{x^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}B_n}{x^{n+1}}$.
But this series diverges everywhere except when $x\to\infty$ it tends to 0.

We may deal with this series as follow. Anyone who first see it may think it too lax. However, since theory of divergent series has already been established, a strict foundation may also be built up.

We begin from the identity $\frac{-t}{e^{-t}-1}=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}t^n$, and apply Laplace transform to either side of the equality, we have $\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt=\int_0^{\infty}e^{-x\cdot t}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}t^n dt$.
Regardless of the divergence of the series in the integrand, we change the order of series and integral, and obtain $\int_0^{\infty}e^{-x\cdot t}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}t^n dt=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^{\infty}e^{-x\cdot t}\cdot t^n dt$.
Then the integral in summand is easy to deal with. Let $\xi=x\cdot t$, we have

$$\sum_{n=o}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^{\infty}e^{-x\cdot t}\cdot t^n dt=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^{\infty}e^{-\zeta}\cdot\left(\frac{\xi}{x}\right)^n d\frac{\xi}{x}=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^{n+1}}\int_0^{\infty}e^{-\zeta}\cdot\xi^n d\xi$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^{n+1}}n!=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{x^{n+1}}.$$

Thus we obtain $\sum\frac{1}{x^2}\Delta x=-\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt$.
Therefore,

$$\sum_{x=1}^{\infty}\frac{1}{x^2}=\sum\nolimits_1^{\infty}\frac{1}{x^2}\Delta x=-\lim_{x\to\infty}\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt+\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt\Bigg|_{x=1}=\int_0^{\infty}e^{-t}\cdot\frac{-t}{e^{-t}-1}dt=\int_0^{\infty}\frac{t}{e^t-1}dt.$$

Amazingly, the result is true.

Then let us consider the 'identity' $\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{x^{n+1}}$. The '$=$' may seem unpleasant here, we change it to '$\sim$'.

Notice that $\frac{-t}{e^{-t}-1}=\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}t^n=1+\frac{1}{2}t+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}t^{2n}$ and the sign of $\frac{B_{2n}}{(2n)!}$ is $(-1)^{n-1}$ for $n\ge 1$.
Therefore, $\left|\frac{-t}{e^{-t}-1}-\left(1+\frac{1}{2}t+\sum_{n=1}^{N}\frac{B_{2n}}{(2n)!}t^{2n}\right)\right|\le(-1)^N\frac{B_{2N+2}}{(2N+2)!}t^{2N+2}$ for $t\ge 0$.

$$\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt-\sum_{n=0}^{2N}(-1)^n\frac{B_n}{x^{n+1}}=\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt-\sum_{n=0}^{2N}(-1)^n\frac{B_n}{n!}\int_0^{\infty}e^{-x\cdot t}\cdot t^n dt$$

$$=\int_0^{\infty}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt-\int_0^{\infty}e^{-x\cdot t}\cdot\sum_{n=0}^{2N}(-1)^n\frac{B_n}{n!}t^n dt=\int_0^{\infty}e^{-x\cdot t}\cdot\left(\frac{-t}{e^{-t}-1}-\sum_{n=0}^{2N}(-1)^n\frac{B_n}{n!}t^n\right)dt$$

$$\le \int_0^\infty e^{-x\cdot t}\cdot\left((-1)^N \frac{B_{2N+2}}{(2N+2)!}t^{2N+2}\right)dt = (-1)^N \frac{B_{2N+2}}{(2N+2)!}\int_0^\infty e^{-x\cdot t}\cdot t^{2N+2}dt = (-1)^N \frac{B_{2N+2}}{(2N+2)!}\frac{(2N+2)!}{x^{2N+3}}.$$

Therefore, $\lim_{x\to\infty} x^{2N}\left(\int_0^\infty e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt - \sum_{n=0}^{2N}(-1)^n \frac{B_n}{x^{n+1}}\right) = 0$.

Hence, $\sum_{n=0}^{\infty}(-1)^n \frac{B_n}{x^{n+1}}$ is the asymptotic series of $\int_0^\infty e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt$.

(2) $\sum_{x=1}^{\infty}\frac{1}{x^s}$, where s is an integer ≥ 2.

It is zeta function on the real axis $s>1$.

Lemma 5: $\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x^s} = (-1)^{n-1}\frac{(s)^{(n-2)}}{x^{s+n-1}} = (-1)^{n-1}\frac{(s+n-2)!}{(s-1)!\cdot x^{\upsilon+n-1}} (s\ge 2, s\in N, n\in N)$, where $(s)^{(n-2)} = s\cdot(s+1)\ldots(s+n-2)$ denote rising factorial, and we define $(s)^{(-n)} = 1 (n\in N)$.

Since the proof is almost the same as that of lemma 4, we will just omit it.

$$\sum\frac{1}{x^s}\Delta x = \sum_{n=o}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x^s} = \sum_{n=o}^{\infty}\frac{B_n}{n!}(-1)^{n-1}\frac{(s+n-2)!}{(s-1)!\cdot x^{s+n-1}} = \frac{1}{(s-1)!}\sum_{n=o}^{\infty}\frac{(-1)^{n-1}(s+n-2)!B_n}{n!\cdot x^{s+n-1}}$$
$$= \frac{1}{(s-1)!}\sum_{n=o}^{\infty}\frac{(-1)^{n-1}\cdot(n+1)^{(s-3)}\cdot B_n}{x^{s+n-1}}.$$

As is shown in theorem , we have $\sum_{n=o}^{\infty}(-1)^n\frac{B_n}{x^{n+1}} = -\int_0^\infty e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt$.

Derivative it respect to x, we obtain

$$\frac{d^{s-2}}{dx^{s-2}}\sum_{n=o}^{\infty}(-1)^n\frac{B_n}{x^{n+1}} = \sum_{n=o}^{\infty}(-1)^n B_n\frac{d^{s-2}}{dx^{s-2}}\frac{1}{x^{n+1}} = \sum_{n=o}^{\infty}(-1)^n B_n\frac{(n+1)^{(s-3)}}{x^{n+s-1}} = -\frac{d^{s-2}}{dx^{s-2}}\int_0^\infty e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt$$
$$= -\int_0^\infty\frac{\partial^{s-2}}{\partial x^{s-2}}e^{-x\cdot t}\cdot\frac{-t}{e^{-t}-1}dt = -\int_0^\infty e^{-x\cdot t}\cdot(-t)^{s-2}\cdot\frac{-t}{e^{-t}-1}dt = -\int_0^\infty e^{-x\cdot t}\cdot\frac{(-t)^{s-1}}{e^{-t}-1}dt.$$

$$\sum_{x=1}^{\infty}\frac{1}{x^s} = \sum\nolimits_1^\infty\frac{1}{x^s}\Delta x = (-1)^{s-1}\left(\lim_{x\to\infty}\frac{1}{(s-1)!}\int_0^\infty e^{-x\cdot t}\cdot\frac{(-t)^{s-1}}{e^{-t}-1}dt - \frac{1}{(s-1)!}\int_0^\infty e^{-x\cdot t}\cdot\frac{(-t)^{s-1}}{e^{-t}-1}dt\Bigg|_{x=1}\right)$$
$$= (-1)^{s-1}\frac{1}{(s-1)!}\int_0^\infty e^{-t}\cdot\frac{(-t)^{s-1}}{e^{-t}-1}dt = \frac{1}{(s-1)!}\int_0^\infty\frac{t^{s-1}}{e^t-1}dt = \frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}}{e^t-1}dt.$$

(3) $\sum_{n=1}^{x-1}\frac{1}{n}$.

It is known as harmony series and has been solved at least since Euler. We just have a look at how Euler-Maclaurin formula is applied in this case.

Lemma 6: $\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x}=(-1)^{n-1}\frac{(n-1)!}{x^n}(n\geq 1)$.
The proof is omitted.

We only take the first term of Euler-Maclaurin formula, but that is enough to obtain
$\sum_{n=1}^{x-1}\frac{1}{n}=\int_1^x\frac{d\xi}{\xi}+O\left(\frac{1}{x}\right)+O(1)=\log(x)+O\left(\frac{1}{x}\right)+O(1)$.

To know more about the constant, we take all of the last terms,
$\sum_{n=1}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x}=\sum_{n=1}^{\infty}\frac{B_n}{n!}(-1)^{n-1}\frac{(n-1)!}{x^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{n\cdot x^n}$.

Similarly, we begin from the identity $\frac{1}{\xi}\left(\frac{-\xi}{e^{-\xi}-1}-1\right)=\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n!}\xi^{n-1}$.

Therefore, $\int_0^\infty\frac{e^{-x\cdot\xi}}{\xi}\cdot\left(\frac{-\xi}{e^{-\xi}-1}-1\right)d\xi=\int_0^\infty e^{-x\cdot\xi}\cdot\left(\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n!}\xi^{n-1}\right)d\xi=\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^\infty e^{-x\cdot\xi}\cdot\xi^{n-1}d\xi$

$=\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^\infty e^{-y}\cdot\left(\frac{y}{x}\right)^{n-1}d\frac{y}{x}=\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^n}\int_0^\infty e^{-y}\cdot y^{n-1}dy$

$=\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^n}(n-1)!=\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n\cdot x^n}$.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{n\cdot x^n}\bigg|_{x=1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{n}=\int_0^\infty\frac{e^{-\xi}}{\xi}\cdot\left(\frac{-\xi}{e^{-\xi}-1}-1\right)d\xi=\int_0^\infty\left(\frac{-e^{-\xi}}{e^{-\xi}-1}-\frac{e^{-\xi}}{\xi}\right)d\xi=\int_0^\infty\left(\frac{1}{e^{\xi}-1}-\frac{1}{\xi e^{\xi}}\right)d\xi.$$

We denote $\gamma=\int_0^\infty\left(\frac{1}{e^{\xi}-1}-\frac{1}{\xi e^{\xi}}\right)d\xi$, and it is known as Euler-Mascheroni constant.
$\gamma=0.5772156649\ldots$
Hence, $\sum_{n=1}^{x-1}\frac{1}{n}\sim\log(x)+\gamma(x\to\infty)$, which has already been known to Euler.

(4) $\sum_{n=1}^{x-1}\log(n)$.

Lemma 7: $\frac{d^{n-1}}{dx^{n-1}}\log(x)=(-1)^n\frac{(n-2)!}{x^{n-1}}(n\geq 2)$.
The proof is omitted.

We take the first two terms of Euler-Maclaurin formula to obtain
$$\sum_{n=1}^{x-1}\log(n)=\int_1^x\log(\xi)d\xi-\frac{1}{2}\log(\xi)\bigg|_1^x+O\left(\frac{1}{x}\right)+O(1)=\left(x-\frac{1}{2}\right)\log(x)-x+1+O\left(\frac{1}{x}\right)+O(1).$$

To know more about the constant, we take all of the last terms,

$$\sum_{n=2}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dx^{n-1}}\log(x)=\sum_{n=2}^{\infty}\frac{B_n}{n!}(-1)^n\frac{(n-2)!}{x^{n-1}}=\sum_{n=2}^{\infty}\frac{(-1)^n B_n}{n\cdot(n-1)\cdot x^{n-1}}.$$

Similarly,

$$\int_0^{\infty}\frac{e^{-x\cdot\xi}}{\xi^2}\cdot\left(\frac{-\xi}{e^{-\xi}-1}-1-\frac{\xi}{2}\right)d\xi=\int_0^{\infty}e^{-x\cdot\xi}\cdot\left(\sum_{n=2}^{\infty}(-1)^n\frac{B_n}{n!}\xi^{n-2}\right)d\xi=\sum_{n=2}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^{\infty}e^{-x\cdot\xi}\cdot\xi^{n-2}d\xi$$

$$=\sum_{n=2}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^{\infty}e^{-y}\cdot\left(\frac{y}{x}\right)^{n-2}d\frac{y}{x}=\sum_{n=2}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^{n-1}}\int_0^{\infty}e^{-y}\cdot y^{n-2}dy$$

$$=\sum_{n=2}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^{n-1}}(n-2)!=\sum_{n=2}^{\infty}(-1)^n\frac{B_n}{n\cdot(n-1)\cdot x^{n-1}}.$$

Therefore,

$$\sum_{n=2}^{\infty}\frac{(-1)^n B_n}{n\cdot(n-1)\cdot x^{n-1}}\bigg|_{x=1}=\sum_{n=2}^{\infty}\frac{(-1)^n B_n}{n\cdot(n-1)}=\int_0^{\infty}\frac{e^{-\xi}}{\xi^2}\cdot\left(\frac{-\xi}{e^{-\xi}-1}-1-\frac{\xi}{2}\right)d\xi=\int_0^{\infty}\left(\frac{1}{\xi(e^{\xi}-1)}-\frac{1}{\xi^2e^{\xi}}-\frac{1}{2\xi e^{\xi}}\right)d\xi.$$

After some further calculation, we may know $\int_0^{\infty}\left(\frac{1}{\xi(e^{\xi}-1)}-\frac{1}{\xi^2e^{\xi}}-\frac{1}{2\xi e^{\xi}}\right)d\xi=\frac{1}{2}\log(2\pi)-1$.

Therefore, $\log\Gamma(x)=\sum_{n=1}^{x-1}\log(n)\sim\left(x-\frac{1}{2}\right)\log(x)-x+\frac{1}{2}\log(2\pi)(x\to\infty)$, which has already been known to de Moivre and Stirling.

(5) $\sum_{n=1}^{x-1}n\log(n)$.

Lemma 8: $\frac{d^{n-1}}{dx^{n-1}}x\log(x)=(-1)^{n-1}\frac{(n-3)!}{x^{n-2}}(n\geq 3)$.
The proof is omitted.

We take the first three terms of Euler-Maclaurin formula to obtain

$$\sum_{n=1}^{x-1}n\log(n)=\int_1^x\xi\log(\xi)d\xi-\frac{1}{2}\xi\log(\xi)\bigg|_1^x+\frac{1}{12}\frac{d}{d\xi}\xi\log(\xi)\bigg|_1^x+O\left(\frac{1}{x}\right)+O(1)$$

$$=\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{12}\right)\log(x)-\frac{x^2}{4}+\frac{1}{4}+O\left(\frac{1}{x}\right)+O(1).$$

To know more about the constant, we take all of the last terms,

$$\sum t\ln(t)\Delta t=\sum_{n=0}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{dt^{n-1}}t\log(t)=\frac{t^2}{2}\log(t)-\frac{t^2}{4}-\frac{1}{2}t\log(t)+\frac{1}{12}(\log(t)+1).$$

Similarly,

$$\frac{-\xi}{e^{-\xi}-1}=\sum_{n=o}^{\infty}(-1)^n\frac{B_n}{n!}\xi^n,\ \frac{1}{\xi^3}\left(\frac{-\xi}{e^{-\xi}-1}-1-\frac{\xi}{2}-\frac{\xi^2}{12}\right)=\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n!}\xi^{n-3}.$$

$$\int_0^\infty \frac{e^{-x\cdot\xi}}{\xi^3}\left(\frac{-\xi}{e^{-\xi}-1}-1-\frac{\xi}{2}-\frac{\xi^2}{12}\right)d\xi=\int_0^\infty e^{-x\cdot\xi}\cdot\left(\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n!}\xi^{n-3}\right)d\xi=\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^\infty e^{-x\cdot\xi}\cdot\xi^{n-3}d\xi$$
$$=\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n!}\int_0^\infty e^{-y}\cdot\left(\frac{y}{x}\right)^{n-3}d\frac{y}{x}=\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^{n-2}}\int_0^\infty e^{-y}\cdot y^{n-3}dy$$
$$=\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n!}\frac{1}{x^{n-2}}(n-3)!=\sum_{n=3}^{\infty}(-1)^n\frac{B_n}{n\cdot(n-1)\cdot(n-2)\cdot x^{n-1}}.$$

Therefore,

$$\sum_{n=3}^{\infty}\frac{(-1)^{n-1}B_n}{n\cdot(n-1)\cdot(n-2)x^{n-1}}\Bigg|_{x=1}=\sum_{n=3}^{\infty}\frac{(-1)^{n-1}B_n}{n\cdot(n-1)\cdot(n-2)}=-\int_0^\infty\frac{e^{-t\cdot\xi}}{\xi^3}\cdot\left(\frac{-\xi}{e^{-\xi}-1}-1-\frac{\xi}{2}-\frac{\xi^2}{12}\right)d\xi$$
$$=-\int_0^\infty\left(\frac{1}{\xi^2(e^{\xi}-1)}-\frac{1}{\xi^3e^{\xi}}-\frac{1}{2\xi^2e^{\xi}}-\frac{1}{12\xi e^{\xi}}\right)d\xi.$$

Hence, $\sum_{n=1}^{x-1}n\log(n)\sim\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{12}\right)\log(x)-\frac{x^2}{4}+C(x\to\infty)$.

We denote $A=e^C=\exp\left(\frac{1}{4}-\int_0^\infty e^{-\xi}\cdot\left(\frac{1}{\xi^2(e^{-\xi}-1)}+\frac{1}{\xi^3}+\frac{1}{2\xi^2}+\frac{1}{12\xi}\right)d\xi\right)$, and it is known as Glaisher-Kinkelin constant. $A=1.2824271291\ldots$

(5) $\sum_{n=1}^{x-1}n^s\log(n)(s\geq 0)$.

Lemma 4: $\frac{d^{n-1}}{dx^{n-1}}x^s\log(x)=(s)_{(n-1)}x^{s-n+1}\log(x)+C_{n-1}x^{s-n+1}(1\leq n\leq s+1)$, where C_{n-1} is a constant and $(s)_{(n-1)}$ denote falling factorial, $(s)_{(n-1)}=s(s-1)...(s-k+2)$.

Proof : If $n=1$, we have $\frac{d^0}{dx^0}x^s\log(x)=x^s\log(x)$.

Notice that $\frac{d^k}{dx^k}x^s\log(x)=\frac{d}{dx}\left(\frac{d^{k-1}}{dx^{k-1}}x^s\log(x)\right)=\frac{d}{dx}\left((s)_{(k-1)}x^{s-k+1}\log(x)+C_{k-1}x^{s-k+1}\right)$

$=(s)_{(k)}x^{s-k}\log(x)+(s)_{(k-1)}x^{s-k+1}\frac{1}{x}+(s-k+1)C_{k-1}x^{s-k}=(s)_{(k)}x^{s-k}\log(x)+C_kx^{s-k}$.

Hence, from mathematical induction we know it is true for all $(1\leq n\leq s+1)$.

Then we consider the constant C_n. From lemma 4 we know that $C_n=(s-n+1)C_{n-1}+(s)_{(n-1)}$, therefore, $\frac{C_n}{(s)_{(n)}}=\frac{C_{n-1}}{(s)_{(n-1)}}+\frac{1}{s+1-n}$. $C_0=0$, $C_n=(s)_{(n)}\sum_{i=1}^{n}\frac{1}{s+1-i}=(s)_{(n)}\sum_{i=0}^{n-1}\frac{1}{s-i}$.

Lemma 4: $\int_1^x\xi^s\log(\xi)d\xi=\frac{x^{s+1}}{s+1}\log(x)+\frac{1-x^{s+1}}{(s+1)^2}$.

Proof : $\frac{d}{dx}\left(\frac{x^{s+1}}{s+1}\log(x)+\frac{1-x^{s+1}}{(s+1)^2}\right)=x^s\log(x)+\frac{x^{s+1}}{s+1}\frac{1}{x}-\frac{x^s}{s+1}=x^s\log(x)$,

$\left.\frac{x^{s+1}}{s+1}\log(x)+\frac{1-x^{s+1}}{(s+1)^2}\right|_{x=1}=0$. Therefore, $\int_1^x \xi^s\log(\xi)d\xi=\frac{x^{s+1}}{s+1}\log(x)+\frac{1-x^{s+1}}{(s+1)^2}$.

We take the first $s+2$ terms of Euler-Maclaurin formula to obtain

$$\sum_{n=1}^{x-1}n^s\log(n)=\int_1^x\xi^s\log(\xi)d\xi+\sum_{n=1}^{s+1}\frac{B_n}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\xi^s\log(\xi)\Big|_1^x+O\left(\frac{1}{x}\right)+O(1)$$

$$=\frac{x^{s+1}}{s+1}\log(x)+\frac{1-x^{s+1}}{(s+1)^2}+\sum_{n=1}^{s+1}\frac{B_n}{n!}\left((s)_{(n-1)}x^{s-n+1}\log(x)+C_{n-1}x^{s-n+1}-C_{n-1}\right)+O\left(\frac{1}{x}\right)+O(1).$$

$$=\left(\frac{x^{s+1}}{s+1}+\sum_{n=1}^{s+1}\frac{B_n}{n!}(s)_{(n-1)}x^{s-n+1}\right)\log(x)+P_{s+1}(x)+O\left(\frac{1}{x}\right)+O(1)$$

$$=\left(\frac{x^{s+1}}{s+1}+\sum_{n=1}^{s+1}\frac{B_n}{n!}\frac{s!}{(s-n+1)!}x^{s-n+1}\right)\log(x)+P_{s+1}(x)+O\left(\frac{1}{x}\right)+O(1)$$

$$=\frac{1}{s+1}\left(x^{s+1}+\sum_{n=1}^{s+1}\binom{s+1}{n}B_nx^{s+1-n}\right)\log(x)+P_{s+1}(x)+O\left(\frac{1}{x}\right)+O(1)$$

$=\frac{1}{s+1}B_{s+1}(x)\log(x)+P_{s+1}(x)+O\left(\frac{1}{x}\right)+O(1)$, where $P_{s+1}(x)$ is a polynomial of x of degree $s+1$.

$P_{s+1}(x)=xP_s(x)+\frac{1}{(s+1)^2}-\sum_{n=1}^{s}\frac{B_n}{n!}C_{n-1}$, $\frac{1}{(s+1)^2}-\sum_{n=1}^{s}\frac{B_n}{n!}C_{n-1}=\frac{1}{(s+1)^2}-\sum_{n=1}^{s}\frac{B_n}{n!}(s)_{(n)}\sum_{i=0}^{n-1}\frac{1}{s-i}$

$=\frac{1}{(s+1)^2}-\sum_{n=1}^{s}\frac{B_n}{n!}\frac{s!}{(s-n)!}\sum_{i=0}^{n-1}\frac{1}{s-i}=\frac{1}{(s+1)^2}-\sum_{n=1}^{s}\binom{s}{n}B_n\sum_{i=0}^{n-1}\frac{1}{s-i}$.

Hence, $\sum_{n=1}^{x-1}n^s\log(n)\sim\frac{1}{s+1}B_{s+1}(x)\log(x)+P_{s+1}(x)+C_s\ (x\to\infty)$.

In a similar way, we obtain $C_s=\int_0^\infty\frac{e^{-\xi}}{\xi^{s+2}}\left(\frac{-\xi}{e^{-\xi}-1}-\sum_{n=0}^{s+2}(-1)^n\frac{B_n}{n!}\xi^n\right)d\xi$.

(6) $\sum_{n=1}^{x-1}\tan^{-1}(n)$.

Lemma 12: $\left.\frac{d^{n-1}}{dx^{n-1}}\tan^{-1}(x)\right|_{x=0}=\begin{cases}0,\ n\ odd\\(-1)^{\frac{n}{2}-1}\cdot(n-2)!,\ n\ even\end{cases}$, where $n\ge 2$.

Proof : Recall the Taylor expansion of $\tan^{-1}(x)$, we have $\tan^{-1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$.

Meanwhile, we have $\tan^{-1}(x)=\sum_{n=0}^{\infty}\frac{\left.\frac{d^n}{dx^n}\tan^{-1}(x)\right|_{x=0}}{n!}x^n$.

Equate the coefficients of the two equalities, we obtain

$$\left.\frac{d^{n-1}}{dx^{n-1}}\tan^{-1}(x)\right|_{x=0}=\begin{cases}0,\ n\ odd\\ (-1)^{\frac{n}{2}-1}\cdot(n-2)!,\ n\ even\end{cases}.$$

We take the first two terms of Euler-Maclaurin formula to obtain

$$\sum_{n=1}^{x-1}\tan^{-1}(n)=\sum_{n=0}^{x-1}\tan^{-1}(n)=\int_0^x\tan^{-1}(\xi)d\xi-\frac{1}{2}\tan^{-1}(\xi)\Big|_0^x+O\left(\frac{1}{x^2}\right)+O(1)$$

$$=\left(x-\frac{1}{2}\right)\tan^{-1}(x)-\frac{1}{2}\log(x^2+1)+O\left(\frac{1}{x^2}\right)+O(1)$$

To know more about the constant, we take all of the last terms,

$$\sum_{n=2}^{\infty}\frac{B_n}{n!}\left.\frac{d^{n-1}}{dx^{n-1}}\tan^{-1}(x)\right|_{x=0}=\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(-1)^{n-1}(2n-2)!=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{2n(2n-1)}.$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}x^{2n}=\frac{x}{2}\cot(\frac{x}{2}),\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{(2n)!}x^{2n-2}=\frac{1}{x^2}\left(1-\frac{x}{2}\cot\left(\frac{x}{2}\right)\right).$$

$$\int_0^\infty\frac{e^{-x\cdot\xi}}{\xi^2}\left(1-\frac{\xi}{2}\cot\left(\frac{\xi}{2}\right)\right)d\xi=\int_0^\infty e^{-x\cdot\xi}\cdot\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{(2n)!}\xi^{2n-2}=\right)d\xi=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{(2n)!}\int_0^\infty e^{-x\cdot\xi}\cdot\xi^{2n-2}d\xi$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{(2n)!}\int_0^\infty e^{-y}\cdot\left(\frac{y}{x}\right)^{2n-2}d\frac{y}{x}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{(2n)!}\frac{1}{x^{2n-1}}\int_0^\infty e^{-y}\cdot y^{2n-2}dy$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{(2n)!}\frac{1}{x^{2n-1}}(2n-2)!=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{2n(2n-1)}\frac{1}{x^{2n-1}}.$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{2n(2n-1)}=\int_0^\infty\frac{e^{-\xi}}{\xi^2}\left(1-\frac{\xi}{2}\cot\left(\frac{\xi}{2}\right)\right)d\xi.$$

Hence, $\sum_{n=1}^{x-1}\tan^{-1}(n)\sim\left(x-\frac{1}{2}\right)\tan^{-1}(x)-\frac{1}{2}\log(x^2+1)+C$, where $C=\int_0^\infty\frac{e^{-\xi}}{\xi^2}\left(1-\frac{\xi}{2}\cot\left(\frac{\xi}{2}\right)\right)d\xi$

We know in (4) that $\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}=\int_0^\infty\frac{e^{-\xi}}{\xi^2}\left(\frac{\xi}{2}\coth\left(\frac{\xi}{2}\right)-1\right)d\xi=\frac{1}{2}\log(2\pi)-1$, but I failed to figure out a closed form of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{2n(2n-1)}=\int_0^\infty\frac{e^{-\xi}}{\xi^2}\left(1-\frac{\xi}{2}\cot\left(\frac{\xi}{2}\right)\right)d\xi$.

(7) $\sum_{n=1}^{x-1}n^n$.

Lemma 13: $\frac{d^{n-1}}{dt^{n-1}}x^x=x^{x-n+1}\cdot P_{n-1}(x(\log(x)+1))\sim x^x(\log(x)+1)^{n-1}(x\to\infty)$, where $n\geq 1$ and $P_{n-1}(x(\log(x)+1))$ denotes a polynomial of $x(\log(x)+1)$ of degree $n-1$.

Proof : If $n=1$, we have $\frac{d^0}{dt^0}x^x=x^x=x^x(\log(x)+1)^0$.

Notice that $\frac{d^k}{dt^k}x^x = \frac{d}{dx}\left(\frac{d^{k-1}}{dx^{k-1}}x^x\right) = \frac{d}{dx}\left(x^{x-k+1}\cdot P_{k-1}(x(\log(x)+1))\right)$
$= x^{x-k}(x(\log(x)+1)-k+1)\cdot P_{k-1}(x(\log(x)+1)) + x^{x-k+1}\cdot P_{k-2}(x(\log(x)+1))\cdot(\log(x)+2)$
$= x^{x-k}\cdot P_k(x(\log(x)+1)) \sim x^x(\log(x)+1)^k$.
Hence, from mathematical induction we know it is true for all $n \geq 1$.

From lemma 13 we know all derivatives of x^x goes to infinity as x increases. Therefore, the remainder term can never be dropped if x is sufficient large, which means we cannot use Euler-Maclaurin formula directly. Let us have a observation of the following operating.

$$\sum \xi^\xi \Delta\xi = \sum_{n=o}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\xi^\xi = \int \xi^\xi d\xi + \sum_{n=1}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\xi^\xi \sim \int \xi^\xi d\xi + \sum_{n=1}^{\infty}\frac{B_n}{n!}\xi^\xi(\log(\xi)+1)^{n-1}$$
$$= \int \xi^\xi d\xi + \frac{\xi^\xi}{\log(\xi)+1}\sum_{n=1}^{\infty}\frac{B_n}{n!}(\log(\xi)+1)^n = \int \xi^\xi d\xi + \frac{\xi^\xi}{\log(\xi)+1}\left(\frac{\log(\xi)+1}{e^{\log(\xi)+1}-1}-1\right)$$
$$= \int \xi^\xi d\xi + \xi^\xi\left(\frac{1}{e\cdot\xi-1}-\frac{1}{\log(\xi)+1}\right)(\xi\to\infty).$$

$$\sum_{n=1}^{x-1} n^n = \sum\nolimits_1^x n^n \Delta n = \int_1^x \xi^\xi d\xi + \sum_{n=1}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\xi^\xi\Bigg|_1^x \sim \int_1^x \xi^\xi d\xi + \xi^\xi\left(\frac{1}{e\cdot\xi-1}-\frac{1}{\log(\xi)+1}\right) - \sum_{n=1}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\xi^\xi\Bigg|_{\xi=1}$$
$$= \int_1^x \xi^\xi d\xi + \xi^\xi\left(\frac{1}{e\cdot\xi-1}-\frac{1}{\log(\xi)+1}\right) + C, \text{ where } C = -\sum_{n=1}^{\infty}\frac{B_n}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\xi^\xi\Bigg|_{\xi=1} \text{ is a constant.}$$

I have computed the summation to some magnitude, and found that the last formula did approximate to it. It's also one of the problem I will consider further afterwards.

(8) $\zeta(s) - \sum_{n=1}^{x-1}\frac{1}{n^s} = \sum_{n=x}^{\infty}\frac{1}{n^s}(s>1)$.

We take the first term of Euler-Maclaurin formula to obtain
$$\sum_{n=x}^{\infty}\frac{1}{n^s} = \int_x^{\infty}\frac{1}{\xi^s}d\xi + O\left(\frac{1}{x^s}\right) = \frac{1}{(s-1)x^{s-1}} + O\left(\frac{1}{x^s}\right).$$
Hence, $\sum_{n=x}^{\infty}\frac{1}{n^s} \sim \frac{1}{(s-1)x^{s-1}}(x\to\infty)$ or $\sum_{n=x}^{\infty}\frac{1}{n^s} = O\left(\frac{1}{x^{s-1}}\right)$.

This theorem is also commonly used in number theory.

(9) $\gamma - \left(\sum_{n=1}^{x-1}\frac{1}{n} - \log(x)\right) = \gamma - \sum_{n=1}^{x-1}\left(\frac{1}{n} - \log\left(\frac{n+1}{n}\right)\right) = \sum_{n=x}^{\infty}\left(\frac{1}{n} - \log\left(\frac{n+1}{n}\right)\right)$.

We take the first term of Euler-Maclaurin formula to obtain

$$\sum_{n=x}^{\infty}\left(\frac{1}{n}-\log\left(\frac{n+1}{n}\right)\right)=\int_{x}^{\infty}\left(\frac{1}{\xi}-\log\left(\frac{\xi+1}{\xi}\right)\right)d\xi+O\left(\frac{1}{x}-\log\left(\frac{x+1}{x}\right)\right)=(x+1)\log\left(\frac{x+1}{x}\right)-1+O\left(\frac{1}{x^2}\right).$$

Hence, $\sum_{n=x}^{\infty}\left(\frac{1}{n}-\log\left(\frac{n+1}{n}\right)\right)\sim(x+1)\log\left(\frac{x+1}{x}\right)-1(x\to\infty)$ or $\sum_{n=x}^{\infty}\left(\frac{1}{n}-\log\left(\frac{n+1}{n}\right)\right)=O\left(\frac{1}{x}\right)$.

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