Counting, Classification and Graphic Design of Sudoku

By

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Abstract

The aim of this paper is to study how to count the number of hexagonal sudokus. Firstly, using the method of Grobner basis theory of polynomials, we show the way to count the number of hexagonal sudokus and give an estimate of the number. Second, we consider the symmetry of hexagonal sudokus under the action of the cyclic group of order 6. Using the famous Burnside lemma in group theory, the number of hexagonal sudokus under the symmetry of rotation group is obtained. Lastly, we discuss the design project of the circular disc with any radius via spelling hexagonal sudokus, then the concept of spelling efficiency is introduced and its changing rule is shown.

Introduction

"Sudoku" is a Japanese word, meaning "single number" or "number which appears only once." Generally, Sudoku is a number-filling game. However, this concept did not come originally from Japan, but from Latin square, which was invented by the Swiss mathematician Euler in the eighteenth-century. Now the most widespread Sudoku is the 9×9 case, which appears in some newspapers as a puzzle for readers. There are even Sudoku competitions for "Sudoku Fans". Hexagonal Sudoku, which is studied in this paper, is one of many deformations of Sudoku. By redefining and analyzing the rules, we find that there are many interesting mathematical properties behind them. This paper is concentrated on the counting, classification and graphic design of hexagonal Sudoku.

After reviewing the references we find that there are approximately 6.671×10^{21} distinct 9×9 Sudoku (see [7]). In this paper, the counting of hexagonal Sudoku is transferred into the number of the solutions of polynomial equations; then by using the Gröbner basis theory, the number of all Sudoku is calculated. Our main approach is to calculate a certain Gröbner basis of an ideal in polynomial ring with the help of computer.

Analogous to 4×4 Sudoku has only three essentially distinct types under transformation, we find that hexagonal Sudoku also has some symmetry (rotational symmetry). Therefore, we can give the invariant under the rotational transformation. That is, with the total number of hexagonal Sudoku known, Burnside's Lemma of finite groups is used to classify Sudoku. Hence, the number of different Sudoku types and the number of Sudoku contained in each type will be found.

Hexagonal Sudoku considered in this paper is defined as follows:

① the nine numbers filled in each triangle are different from each other

② the nine numbers filled in each row are different from each other. If there are less than nine numbers in a row, then it combines the vertex angle of the corresponding large triangle, as shown in the blue area in the figure.

③ the nine numbers in each "/" hypotenuse are different from each other. If they are less than nine, similar to ②

④ the nine numbers in each "\" hypotenuse are different from each other. If they are less than nine, similar to ②

That is to say, numbers 1 to 9 are filled into each large triangle, horizontal line, "/" and "\" hypotenuses in a non-repetitive manner; the first and the last line of each level will combine with the vertex angle of the corresponding large triangle.

In terms of the geometric pattern of Sudoku, hexagonal Sudoku is symmetrical, composed of regular triangles, and can be spliced into an unlimitedly extending geometric figure, into which gaps with equal size are embedded. Thus, we consider that this geometric figure can be used to manufacture objects with uniform pattern of regular shape, such as floor tiles. By calculation, we can know the scheme of the designed objects with regular shape which consumes the least materials, and provide the method for the design of possible products.

Therefore, our discussion is divided into three parts: firstly, we use Gröbner basis theory to calculate the total number of hexagonal Sudoku. On this level, we regard the distribution of hexagonal Sudoku data in different directions to be different; secondly, by considering the rotational symmetry of hexagonal Sudoku, we give the total number of hexagonal Sudoku in the equivalent sense under the rotational symmetry using Burnside's lemma; finally, we study the scheme of splicing hexagonal Sudoku into unlimitedly extending circular geometric figure. As the number of hexagonal Sudoku contained in the circle depends on the radius and area of this circle, we propose the concept of splicing efficiency, i.e. the number of Sudoku contained in the unit disk. On this level, the variation pattern of splicing efficiency is given. This pattern serves as the scheme whereby we can design circular geometric figure with the least material and regular shape.

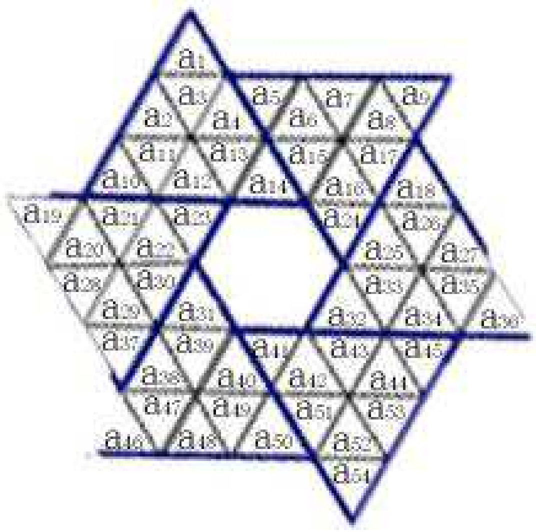
In Sudoku, obviously, the status of each number 1, 2... 9 (or for the calculation, replaced by -2,-1, 1, 2... 7) is equal. In fact, we can consider the new Sudoku obtained by number permutation as the equivalent Sudoku in contrast to the original one. If this factor is considered in counting the total number, then the equivalent Sudoku can be regarded as the same, thus the total number of Sudoku is decreased.

However, graphic design is involved here. Specifically, by matching each number of -2, -1, 1, 2... 7 to a certain color, Sudoku pattern after the permutation differs. This is precisely why we will not consider the equivalence due to data permutation.

I. Counting of Sudoku

As for hexagonal Sudoku, we first give the polynomial equation system (product-sum system) which corresponds to Sudoku's constraint conditions.

Since the number filled has no constraints itself, we may take the objects from the set S = {-2,-1,1,2,3,4,5,6,7}(this is to ensure the unique solution of polynomial equations. see below and [4]). Each position of hexagonal Sudoku is labeled as $a_1, a_2, \cdots a_{54}$, respectively, as shown in the following figure:



Due to the restriction of six triangles, we have the following constraint equation:

$$a_1 + a_2 + a_3 + a_4 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = (-2) + (-1) + 1 + 2 + \cdots + 7 = 25$$

$$a_1 a_2 a_3 a_4 a_{10} a_{11} a_{12} a_{13} a_{14} = (-2) \cdot (-1) \cdot 1 \cdot 2 \cdot \cdots \cdot 7 = 10080$$

$$a_5 + a_6 + a_7 + a_8 + a_9 + a_{15} + a_{16} + a_{17} + a_{24} = 25$$

$$a_5 a_6 a_7 a_8 a_9 a_{15} a_{16} a_{17} a_{24} = 10080$$

$$\vdots$$

12 in total

Due to the restriction of the row, we have the following constraint equation:

$$a_1 + a_2 + \cdots + a_9 = 25$$

$$a_1 \cdot a_2 \cdot \cdots \cdot a_9 = 10080$$

$$\vdots$$

12 in total

Due to the restriction of "/" hypotenuse, we have the following constraint equation:

$$a_1 + a_2 + a_3 + a_{10} + a_{11} + a_{21} + a_{20} + a_{28} + a_{19} = (-2) + (-1) + 1 + 2 + \cdots + 7 = 25$$

$$a_1 a_2 a_3 a_{10} a_{11} a_{21} a_{20} a_{28} a_{19} = (-2) \cdot (-1) \cdot 1 \cdot 2 \cdot \cdots \cdot 7 = 10080$$

$$a_5 + a_4 + a_{13} + a_{12} + a_{23} + a_{22} + a_{30} + a_{29} + a_{37} = 25$$

$$a_5 a_4 a_{13} a_{12} a_{23} a_{22} a_{30} a_{29} a_{37} = 10080$$

$$\vdots$$

12 in total

Due to the restriction of "\" hypotenuse, we have the following constraint equation:

$$a_9 + a_7 + a_8 + a_{17} + a_{18} + a_{26} + a_{27} + a_{35} + a_{36} = (-2) + (-1) + 1 + 2 + \cdots + 7 = 25$$

$$a_9 a_7 a_8 a_{17} a_{18} a_{26} a_{27} a_{35} a_{36} = (-2) \cdot (-1) \cdot 1 \cdot 2 \cdot \cdots \cdot 7 = 10080$$

$$a_5 + a_6 + a_{15} + a_{16} + a_{24} + a_{25} + a_{33} + a_{34} + a_{45} = 25$$

$$a_5 a_6 a_{15} a_{16} a_{24} a_{25} a_{33} a_{34} a_{45} = 10080$$

$$\vdots$$

12 in total

Thus, we have a total of 48 equations.

Moreover, because $a_1, a_2, \cdots a_{54}$ are taken from the set S = {-2,-1, 1, 2, 3, 4, 5, 6, 7}, we have the following constraint polynomial:

$$(a_i+2)(a_i+1)(a_i-1)(a_i-2)\cdots(a_i-7)=0 \quad (i=1,2,\cdots,54)$$

With the polynomial equations above (48 +54 = 102 equations, 54 variables), we will next illustrate how to use Gröbner basis theory to calculate the total number of Sudoku (i.e. the number of solutions of the equation).

The concept of Gröbner basis was originally proposed by B. Buchberger in 1965. Roughly speaking, a Gröbner basis for an ideal of polynomial ring is a set of generators with good properties. The good properties of Gröbner basis can be used to solve many theoretical and practical problems related to polynomial ideal, such as solving polynomial equations. Gröbner basis can be calculated from any finite generator of an ideal.

Now we introduce the basic concept of polynomial ring first, and then give the precise definition of Gröbner basis of an ideal as follows:

Definition 1: Let Q be the set of all rational numbers. For any number field R, the set containing all polynomials of n variables with respect to $x_1, x_2, \cdots x_n$ with coefficients on R is denoted as $R[x_1, x_2, \cdots x_n]$. It is usually referred to as the polynomial ring with respect to variables $x_1, x_2, \cdots x_n$ on R. Moreover, addition and multiplication is defined as the summation and multiplication of polynomials. In this paper, we always take the set R = Q (rational number field), i.e. Gröbner basis we consider is for the ideal in $Q[x_1, x_2, \cdots x_n]$.

Definition 2: An ideal I of polynomial ring $R[x_1, x_2, \cdots x_n]$ is a subset which satisfies the following conditions:

For any $r \in R[x_1, x_2, \cdots x_n]$, $a \in I$, we always have $ra \in I$, i.e. $R[x_1, x_2, \cdots x_n]I \subseteq I$.

Example: Let $I = \{f(x) \in Q[x],\ f(x)$ has no constant term$\}$, then I is an ideal of $Q[x]$.

Definition 3: Let $I \subset R[x_1, x_2, \cdots x_n]$ be an ideal. The radical of I is the set $\sqrt{I} = \{g \in R[x_1, x_2, \cdots x_n] : g^m \in I$ for some $m \geq 1\}$. An ideal I is said to be a radical ideal if $\sqrt{I} = I$.

Definition 4: An ideal $I \subset R[x_1, x_2, \cdots x_n]$ is said to be zero-dimensional if $A = R[x_1, x_2, \cdots x_n]/I$ is finite-dimensional over the field R.

Definition 5: If an ideal I can be generated by a finite number of elements, i.e. $\exists a_1, a_2, \cdots a_n \in I$ such that $I = \{\sum_{i=1}^{s} r_i a_i, r_i \in R[x_1, x_2, \cdots x_n], i = 1, 2, \cdots s\}$, then I is said to be finitely generated by $a_1, a_2, \cdots a_n$, and the set $\{a_1, a_2, \cdots a_n\}$ is the generator of I.

Next we always consider the polynomial ring over the rational number field in 54 variables, i.e. $Q[a_1, a_2, \cdots, a_{54}]$. Before defining Gröbner basis, we first introduce the concepts about variable sequence and the first item.

Definition 6: The total ordering "<" on the set S is called term order if the following conditions are satisfied:

(1) $\forall a, b, c \in S$, if $a < b$, then $ca < cb$

(2) "<" is well-ordered, i.e. any non-empty subset of S has a minimum element with respect to "<".

In this paper, we always take the dictionary order for the monomials generated by the set $\{a_1, a_2, \cdots, a_{54}\}$, i.e. the term order is defined as $a_1 < a_2 < \cdots < a_{54}$.

Example: $a_{54} a_{53} > a_{53}{}^2, a_7 a_1 > a_3{}^2$

Definition 7: According to dictionary order, we can arrange the monomials of a polynomial in a descending order. The leading term of a polynomial f is the largest monomial with respect to this ordering after combining like terms, denoted as $lt(f)$. Moreover, we define the leading term ideal of S to be the ideal $Lt(S)$ generated by the leading term of all polynomials in S, i.e. $Lt(S) = < \{lt(f) | f \in S >$.

Example: the leading term of $f = a_7{}^2 - a_6 a_5 + a_5{}^3 + a_8{}^2$ is $a_8{}^2$. According to dictionary order, it is written as $a_8{}^2 + a_7{}^2 - a_6 a_5 + a_5{}^3$, i.e. $lt(f) = a_8{}^2$.

Gröbner basis is defined as follows:

Definition 8: Let I be an ideal of $Q[a_1, a_2, \cdots, a_{54}]$, and S is a generator set of I. If the leading term ideal of the set S is equal to the leading term ideal of the ideal I, i.e. $Lt(S) = Lt(I)$, then S is called a Gröbner basis of I.

Example: (1) When $I = < a_1^2 - 1, a_1^5 - a_1^4 >$, then $S = \{a_1 - 1\}$ is a Gröbner basis of I;

(2) When $I = < a_1 - 1, a_1 >$, then $S = \{a_1 - 1, a_1\}$ is not a Gröbner basis of I.

This is because $Lt(S) = \langle a_1 \rangle$, $1 \in I \Rightarrow Lt(I) \supseteq \langle 1 \rangle = Q[a_1, a_2, \cdots, a_{54}]$ and hence $Lt(I) = Q[a_1, a_2, \cdots, a_{54}]$.

Remarks on Gröbner Basis:

(1) The existence and algorithm of Gröbner basis:

In the polynomial ring $Q[a_1, a_2, \cdots, a_{54}]$, the existence of Gröbner basis is guaranteed by Buchberger algorithm and Hilbert Basis Theorem.

Hilbert Basis Theorem:

Every ideal I in $Q[a_1, a_2, \cdots, a_{54}]$ has a finite generating set, i.e. there exists a finite collection of polynomials $\{g_1, g_2, \cdots, g_s\} \subset Q[a_1, a_2, \cdots, a_{54}]$ such that $I = \langle g_1, g_2, \cdots, g_s \rangle$.

Since we have already known that for any ideal I in $Q[a_1, a_2, \cdots, a_{54}]$, there exists $g_1, g_2, \cdots, g_n \in I$ such that $I = \langle g_1, g_2, \cdots, g_n \rangle$, then the Buchberger algorithm listed below tells us that how to get Gröbner basis of the ideal I from $g_1, g_2, \cdots, g_n$.

Buchberger Algorithm:

Input: $I = \{g_1, g_2, \cdots g_n\} \subseteq Q[a_1, a_2, \cdots, a_{54}]$

Output: Gröbner basis S of I to make $I \subseteq S$.

$S := I$;

$X := \{\{f, g\} : f, g \in S$ such that $f \neq g\}$;

While $X \neq \varnothing$ do

$\{f, g\} := pop(X)$;

$R := S(f, g)$—a paradigm of S;

If $R \neq 0$ then

$S := S \cup \{\{h, R\} : h \in S\}$;

$S := S \cup \{R\}$;

End

End

Return S;

For more details, see [2].

(2) The good properties of Gröbner basis:

① It can keep all of the information of the root of each polynomial of the generating set of the original ideal;

② To some extent it can be "triangulated", i.e. making the polynomial solving convenient, which is required in this paper.

(3) Several applications of Gröbner basis:

① solving linear equations by Gauss elimination;

② obtaining the greatest common factor of polynomials by Euclidean algorithm.

Now let us return to hexagonal Sudoku. Gröbner basis of the ideal I generated by the corresponding polynomials of 102 constraint equations is calculated.

Lemma 1.1: With the other numbers in hexagonal Sudoku are remain unchanged, the number in the two grids of each vertex angle of large triangle (e.g. a_1 and a_3) can be interchanged and still satisfy the rules of Sudoku.

Proof: According to the rules of hexagonal Sudoku, a_1 and a_3 should satisfy the following conditions:

① $a_1, a_2, \cdots, a_9$ are numbers from $\{1,2,3,4,5,6,7,8,9\}$ without repetition;

② $a_1, a_2, a_3, a_{10}, a_{11}, a_{21}, a_{20}, a_{28}, a_{19}$ are numbers from $\{1,2,3,4,5,6,7,8,9\}$ without repetition;

③ $a_1, a_3, a_4, a_{13}, a_{14}, a_{32}, a_{43}, a_{44}, a_{53}$ are numbers from $\{1,2,3,4,5,6,7,8,9\}$ without repetition;

④ $a_1, a_2, a_3, a_4, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$ are numbers from $\{1,2,3,4,5,6,7,8,9\}$ without repetition.

Under these conditions, we can see that a_1 and a_3 are having the same status and can be interchanged. Similarly, a_{19} and a_{20} can also be interchanged. Therefore, under the condition that the other number in hexagonal Sudoku remain unchanged, the numbers in the two grids of each vertex angle of large triangle (e.g. a_1 and a_3) can be interchanged and still satisfy the rules of Sudoku. ※

Lemma 1.2: The numbers in the small triangle next to each side of the regular hexagon in the middle are equal to the numbers in the hourglass position (e.g. a_5 and a_{14}).

Here we give two proofs of this lemma.

Proof 1: Let $a_1, a_2, \cdots, a_9$ are equal to the numbers 1,2,3,4,5,6,7,8 and 9, respectively. In particular, $a_5 = 5$.

Then, the position of $a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$ are numbers from $\{5,6,7,8,9\}$

without repetition.

Because $a_5=5$, so a_{12},a_{13} can not be 5. If one of a_{10},a_{11} is 5, then 5 can not be filled in the positions of $a_{21},a_{20},a_{28},a_{19}$. Because $a_5=5$, 5 also can not be filled in $a_{22},a_{23},a_{29},a_{30},a_{37}$. Therefore, large triangle of $a_{19},a_{20},a_{21},a_{22},a_{23},a_{28},a_{29},a_{30},a_{37}$ lacks 5, so both a_{10},a_{11} are not 5, therefore, $a_{14}=5$.

Similarly, a_{10},a_{23} and other cases are equal. Therefore, the numbers in the small triangle next to each side of the regular hexagon in the middle are equal to the numbers in the hourglass position (e.g. a_5 and a_{14}). ※

Proof 2: For the two "/" types hypotenuse on the left of hexagonal Sudoku, we have

$$a_1+a_2+a_3+a_{10}+a_{11}+a_{21}+a_{20}+a_{28}+a_{19}=25 \quad \cdots\cdots ①$$

$$a_4+a_5+a_{12}+a_{13}+a_{22}+a_{23}+a_{29}+a_{30}+a_{37}=25 \quad \cdots\cdots ②$$

For the two large triangles on the left of hexagonal Sudoku, we have:

$$a_1+a_2+a_3+a_4+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}=25 \quad \cdots\cdots ③$$

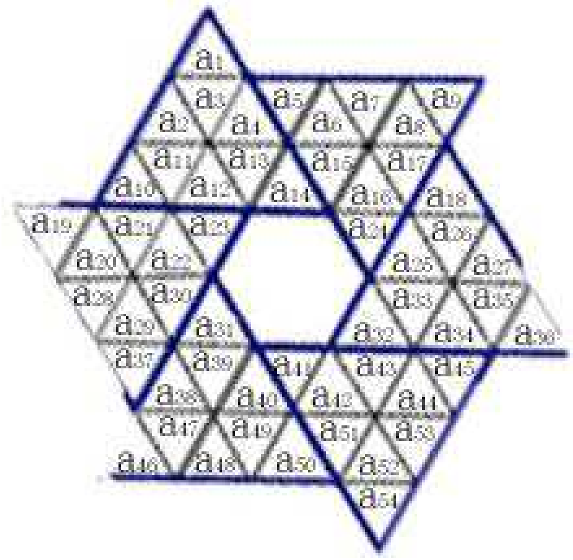
$$a_{19}+a_{20}+a_{21}+a_{22}+a_{23}+a_{28}+a_{29}+a_{30}+a_{37}=25 \cdots\cdots ④$$

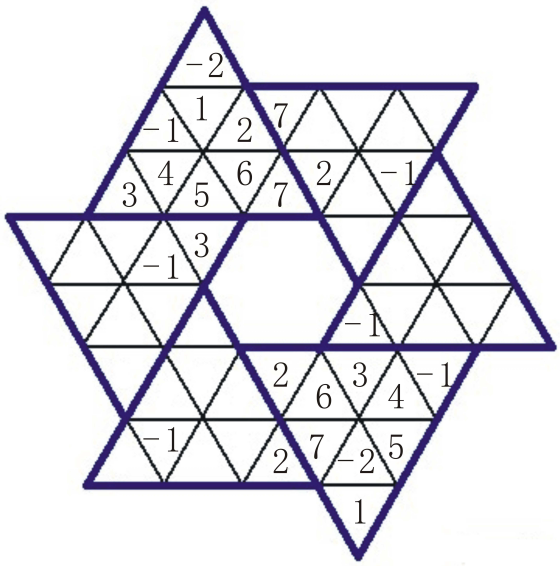
By equation calculation（①+②）−（③+④）, we obtain that $a_{14}=a_5$. Therefore, the numbers in the small triangle next to each side of the regular hexagon in the middle are equal to the numbers in the hourglass position (e.g. a_5 and a_{14}). ※

Remarks: Proof 1 is based on the observation of the hexagonal Sudoku, and the number-filling rules are appropriately used to give this descriptive proof. In Proof 2, we express hexagonal Sudoku rules by mathematical language—equation, which makes proof more clear and concise.

According to Lemma 1.1, 1.2, we use Maple software to calculate the Gröbner basis required.

First, let $a_1=-2,a_2=-1,a_3=1,a_4=2,a_{10}=3,a_{11}=4,a_{12}=5,a_{13}=6,a_{14}=7$. Moreover, when $a_{32}=-1,a_{43}=3,a_{44}=4,a_{53}=5,a_{41}=2,a_{42}=6,a_{51}=7,a_{52}=-2,a_{54}=1$, we can get the Sudoku as shown in the following figure according to the rules of hexagonal Sudoku:





According to Lemma 1.2, $a_5 = 7, a_{32} = -1, a_{50} = 2, a_{23} = 3$

We can know that a_7, a_8 can not be -1 because $a_3 = -1$; since $a_{32} = -1$, we know that $a_{25}, a_{26}, a_{27}, a_{32}, a_{33}, a_{34}, a_{35}, a_{36}$ can not be -1, from which $a_{17} = -1$ can be derived.

Since $a_{32} = -1, a_{45} = -1, a_2 = -1$, we know that $a_{19}, a_{20}, a_{21}, a_{28}, a_{29}, a_{30}, a_{37}$ can not be -1. Moreover, $a_{23} = 3$, from which $a_{22} = -1$ can be derived.

Similarly, either a_{46} or a_{47} is equal to -1. According to Lemma 1.1, we can know that a_{46}, a_{47} can be interchanged. Therefore, we may let $a_{47} = -1$.

By increasing the conditions of hexagonal Sudoku, the number of polynomials of hexagonal Sudoku has been reduced, thus improving the operation efficiency of the computer. Then by using the Gröbner Basis package of Maple software in programming, we obtain the leading terms of Gröbner basis (totaling 39) for ideal I, determined by the polynomials corresponding to the above 102 equations. Now we list them as follows:

a_6^2, a_7^2, a_8^2, a_9, a_{16}^2, a_{18}, $a_{19}^3 a_6$, $a_7 a_{19}^3$, a_{19}^4, $a_{20} a_{19} a_6$, $a_{20}^2 a_6$, $a_7 a_{20}^2$, $a_{20}^2 a_{19}^2$, a_{20}^3, a_{21}, a_{25}, a_{26}, a_{27}, a_{28}, $a_{29} a_7 a_6$, $a_{29} a_{19}^3$, $a_{29} a_{20}^2$, a_{29}^2, $a_{30} a_7$, $a_{30} a_{19}^3$, $a_{30} a_{20} a_{19}$, $a_{30} a_{20}^2$, $a_{29} a_{30}$, a_{30}^2, a_{31}, a_{33}, a_{34}, a_{35}^2, a_{36}, a_{38}, a_{39}, a_{46}, a_{48}, a_{49}

We have already known that how to obtain the Gröbner basis of an ideal I by the foregoing discussion, from now on we will establish the relationship between the number of hexagonal Sudoku N and the leading terms of Gröbner basis $\{s_1, s_2, \cdots, s_n\} \subseteq Q[a_1, a_2, \cdots, a_{54}]$, according to [2][4][5][6]:

Lemma 1.3 [6] (Finiteness Theorem) Let $I \subseteq C[x_1, x_2, \cdots, x_n]$ be an ideal. Then the following conditions are equivalent:

(1) $A = C[x_1, x_2, \cdots, x_n]/I$ is finite-dimensional over C, where C is the complex number field.

(2) the variety $V(I)$ (the set of common zero points of all polynomials in I) is a finite set.

Lemma 1.4 [6, Proposition 2.7] Let $I \subseteq C[x_1, x_2, \cdots, x_n]$ be a zero-dimensional ideal. For each $i = 1, 2, \cdots, n$, let p_i be the unique monic generator of $I \cap C[x_i]$ and let $p_{i,red}$ be the square-free part of p_i. Then $\sqrt{I} = I + \langle p_{1,red}, \cdots, p_{n,red} \rangle$.

Lemma 1.5 [6, Proposition 2.10] Let I be a zero-dimensional ideal in $C[x_1, x_2, \cdots, x_n]$, and let $A = C[x_1, x_2, \cdots, x_n]/I$. Then $\dim_C A$ is greater than or equal to the number of points in $V(I)$. Moreover, equality occurs if and only if I is a radical ideal.

Lemma 1.6 [5, Proposition 2.1.6] Let $I = \langle s_1, s_2, \cdots, s_m \rangle$ be an ideal of $C[x_1, x_2, \cdots, x_n]$, then a basis for the C-vector space $C[x_1, x_2, \cdots, x_n]/I$ consists of the cosets $\overline{f} = f + I$ of monomials $f = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ such that $lt(s_i)$ does not divide f for all $i = 1, 2, \cdots, m$.

With the Lemma 1.3 to 1.6, now we list our main theorem in this section below.

Theorem 1.7 The number of hexagonal Sudoku N and the leading terms of Gröbner basis $\{s_1, s_2, \cdots, s_n\} \subseteq Q[a_1, a_2, \cdots, a_{54}]$ have the following relationship:

(*) $N = \#\{f = a_1^{i_1} a_2^{i_2} \cdots a_{54}^{i_{54}} \mid f \text{ can not be divided by } lt(s_i) \text{ for all } i = 1, 2, \cdots n\}$

where $lt(s_i)$ is the leading terms of polynomial s_i under the dictionary order, # is the number of elements in a set.

Proof: We have concluded the constraint conditions of hexagonal Sudoku to 102 equations, so that we have the following relations:

the number of hexagonal Sudoku N = the number of common solutions of 102 equations = the number of common zero points of 102 polynomials

Let I be the ideal generated by the 102 polynomials, we should prove that I is a radical ideal and zero-dimensional ideal first.

Whatever the number of hexagonal Sudoku N is, it is a finite number, so that the variety $V(I)$ corresponding to the ideal I is a finite set. By lemma 1.3 $A=Q[a_1,a_2,\cdots,a_{54}]/I$ is finite-dimensional over the rational field Q, i.e. I is a zero-dimensional ideal.

Moreover, for any $i=1,2,\cdots,54$, $I\cap Q[a_i]=\langle(a_i+2)(a_i+1)(a_i-1)\cdots(a_i-7)\rangle$, so $p_i=(a_i+2)(a_i+1)(a_i-1)\cdots(a_i-7)$, $p_{i,red}=(a_i+2)(a_i+1)(a_i-1)\cdots(a_i-7)=p_i$, then by lemma 1.4,

$$\sqrt{I}=I+\langle p_{1,red},\cdots,p_{n,red}\rangle=I+\langle(a_1+2)\cdots(a_1-7),\cdots,(a_{54}+2)\cdots(a_{54}-7)\rangle=I$$

Thus I is a radical ideal.

Because I is a radical ideal and zero-dimensional ideal, by lemma 1.5 we obtain the following equation:

$$\text{the number of common zero points of 102 polynomials} = \dim_Q A$$

where $A=Q[a_1,a_2,\cdots,a_{54}]/I$.

The Gröbner basis $\{s_1,s_2,\cdots,s_n\}$ is obviously a generating set of ideal I, so at last by lemma 1.6 we have

$$\dim_Q A=\#\{f=a_1^{i_1}a_2^{i_2}\cdots a_{54}^{i_{54}}\mid f \text{ can not be divided by } lt(s_i) \text{ for all } i=1,2,\cdots n\}$$

Combining the equations above, we can derive our result. i.e. the number of hexagonal Sudoku $N=\#\{f=a_1^{i_1}a_2^{i_2}\cdots a_{54}^{i_{54}}\mid f$ can not be divided by $lt(s_i)$ for all $i=1,2,\cdots n\}$, which completes the proof.

For $A=Q[a_1,a_2,\cdots,a_{54}]/I$ and $\dim_Q A$ which occurs in the above lemmas and theorem, we interpret as follows:

(i) A is called the quotient ring of $Q[a_1,a_2,\cdots,a_{54}]$ modulo ideal I, consists of the elements in the set $\{\underline{h}=h+I: h\in\{Q[a_1,a_2,\cdots,a_{54}]\}$.

Its addition and multiplication are defined by $\underline{h_1}+\underline{h_2}=\underline{h_1+h_2}, \underline{h_1}\cdot\underline{h_2}=\underline{h_1\cdot h_2}$.

(ii) If there are p elements $\underline{h_1},\underline{h_2},\cdots,\underline{h_p}$ in A such that for any $\underline{h}\in A$, there exists a unique linear combination $\underline{h}=c_1\underline{h_1}+c_2\underline{h_2}+\cdots+c_p\underline{h_p}$, where $c_1,c_2,\cdots,c_p\in Q$, then p is called the dimension of A, denoted as $p=\dim_Q A$.

In this paper, we prepare to use the Inclusion-Exclusion Principle to calculate N .

Lemma 1.8 (Inclusion-Exclusion Principle) Let $A_1, A_2, \cdots, A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_1 \cap A_n| - \cdots |A_{n-1} \cap A_n|$$
$$+ |A_1 \cap A_2 \cap A_3| + \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n|$$

where $|A|$ means the number of elements in the set A.

In particular, when $n = 3$, i.e. there are A, B, C three classes of things to be counted, and then Inclusion-Exclusion Principle can be described as follows:

The number of elements in either class A, B or C = the number of elements in class A + the number of elements in class B + the number of elements in class C—the number of elements in both classes A and B—the number of elements in both classes B and C—the number of elements in both classes A and C+ the number of elements in all three classes A, B and C, that is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

To apply the Inclusion-Exclusion principle, we define the following notations:

$$A_i = \{f = a_1^{i_1} a_2^{i_2} \cdots a_{54}^{i_{54}} \mid f \text{ can be divided by } lt(s_i)\}$$

now we consider the set $U = \{f = a_1^{i_1} a_2^{i_2} \cdots a_{54}^{i_{54}} \mid f$ can be divided by some $lt(s_i), i = 1, 2, \cdots n\}$. Obviously, $U = A_1 \cup A_2 \cup \cdots \cup A_n$.

According to Inclusion-Exclusion principle, we have the following formula:

$$|U| = |A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_1 \cap A_n| - \cdots |A_{n-1} \cap A_n|$$
$$+ |A_1 \cap A_2 \cap A_3| + \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n|$$

then by taking the appropriate universal set S (we may choose different universal set S according to different leading terms of Gröbner basis, for the specific selecting method, see remark below), we get $N = |\overline{U}| = \overline{|A_1 \cup A_2 \cup \cdots \cup A_n|}$ ($\overline{U}$ is the complement set of U in S).

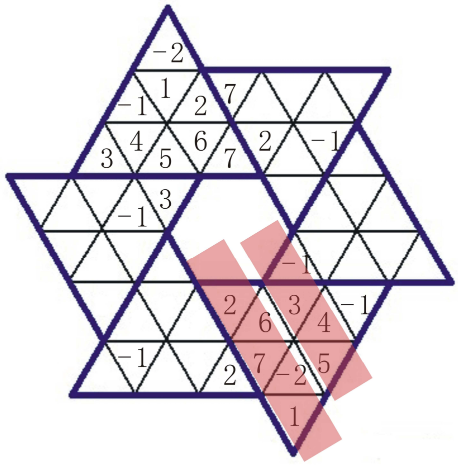
Remark: For general leading terms of Gröbner basis, the universal set S will be taken by following rules:

$$S = \{a_1^{i_1} a_2^{i_2} a_3^{i_3} \cdots a_{52}^{i_{52}} a_{53}^{i_{53}} a_{54}^{i_{54}} \mid 0 \le i_j \le l_j\text{-}1, j = 1, 2, 3, \cdots, 54\}$$

If a_j appears in the leading terms of Gröbner basis, then define l_j be the highest degree of a_j in the basis;

If a_j does not appear in the leading terms of Gröbner basis, then define $l_j = 1$;

In particular, for the above-mentioned $\{-1,3,4,5\}\{2,6,7,-2,1\}$ (where the labeling rules of $\{-1,3,4,5\}\{2,6,7,-2,1\}$ is shown in the right figure), the universal set of hexagonal Sudoku is



$$S = \{a_6^{i_1} a_7^{i_2} a_8^{i_3} a_{16}^{i_4} a_{19}^{i_5} a_{20}^{i_6} a_{29}^{i_7} a_{30}^{i_8} a_{35}^{i_9}$$
$$| 0 \le i_1, i_2, i_3, i_4, i_7, i_8, i_9 \le 1, 0 \le i_5 \le 3, 0 \le i_6 \le 2\}$$

Finally, by using the Inclusion-Exclusion Principle, we calculate our results in the table below. Under the premise of $\{-1,3,4,5\}$, the number of hexagonal Sudoku of each permutation corresponding to the five-tuple $\{2,6,7,-2,1\}$ as well as the total number are calculated.

Sudoku Types	Counting	Multiple	Number	Sudoku Types	Counting	Multiple	Number
{2, 6, 7, −2, 1}	384	×2×2	1536	{7, 2, 1, −2, 6}	192	×2×2×2	1536
{2, 7, 6, −2, 1}	384	×2×2	1536	{1, 2, 6, −2, 7}	208	×2×2×2	1664
{6, 2, 7, −2, 1}	320	×2×2	1280	{1, 6, 2, −2, 7}	48	×2×2×2	384
{6, 7, 2, −2, 1}	528	×2×2	2112	{2, 1, 6, −2, 7}	16	×2×2×2	128
{7, 2, 6, −2, 1}	384	×2×2	1536	{2, 6, 1, −2, 7}	80	×2×2×2	640
{7, 6, 2, −2, 1}	592	×2×2	2368	{6, 1, 2, −2, 7}	64	×2×2×2	512
{1, 6, 7, −2, 2}	48	×2×2×2	384	{6, 2, 1, −2, 7}	64	×2×2×2	512
{1, 7, 6, −2, 2}	64	×2×2×2	512	{7, −2, 1, 2, 6}	0	×2×2×2	0
{6, 1, 7, −2, 2}	64	×2×2×2	512	{−2, 7, 1, 2, 6}	0	×2×2×2	0
{6, 7, 1, −2, 2}	136	×2×2×2	1088	{−2, 1, 7, 2, 6}	0	×2×2×2	0
{7, 1, 6, −2, 2}	96	×2×2×2	768	{6, −2, 1, 2, 7}	0	×2×2×2	0
{7, 6, 1, −2, 2}	152	×2×2×2	1216	{−2, 6, 1, 2, 7}	0	×2×2×2	0
{1, 2, 7, −1, 6}	0	×2×2×2	0	{−2, 1, 6, 2, 7}	0	×2×2×2	0
{1, 7, 2, −2, 6}	112	×2×2×2	896	{2, −2, 1, 6, 7}	0	×2×2×2	0
{2, 7, 1, −2, 6}	56	×2×2×2	448	{−2, 2, 1, 6, 7}	0	×2×2×2	0
{2, 1, 7, −2, 6}	16	×2×2×2	128	{−2, 1, 2, 6, 7}	0	×2×2×2	0
{7, 1, 2, −2, 6}	208	×2×2×2	1664				
Total	23360						

Remark 1: The 33 situations listed above have already covered all the permutations corresponding to five-tuple $\{2,6,7,-2,1\}$.

This is because for the first 6 situations, there are two types of permutation, while for the last 27 situations, there are four types of permutation.

For example: ① in the case of $\{2,6,7,-2,1\}$, the two permutations included are $\{2,6,7,-2,1\}$ and $\{2,6,7,1,-2\}$;

② in the case of $\{1,6,7,-2,2\}$, the four permutations included are $\{1,6,7,-2,2\}$, $\{1,6,7,2,-2\}$, $\{-2,6,7,1,2\}$ and $\{-2,6,7,2,1\}$.

Since 6 × 2 +27 × 4 = 120, we have considered all of the 120 permutations.

We have already calculated the total number of standard hexagonal Sudoku is 23360 by using Gröbner basis theory when $a_{32}=-1, a_{43}=3, a_{44}=4, a_{53}=5$. For the remaining 23 types, the same method applies similarly (the total number of permutations in the four-tuple $\{-1,3,4,5\}$ is 24). Therefore, after we finding that the total numbers of hexagonal Sudoku when $a_{32}=-1, a_{43}=3, a_{44}=5, a_{53}=4$ and $a_{32}=-1, a_{43}=4, a_{44}=5, a_{53}=3$ are 47424 and 384, respectively, we give the estimation of the total number of standard hexagonal Sudoku D_1, that is

$$D_1 \approx \frac{23360+47424+384}{3}\times 24 = 569344$$

The reason of our estimation is as follows: Firstly we divide the standard hexagonal Sudoku into four groups and different groups with different numbers filled in the a_{32} position. Secondly, each group is divided into 2 types, respectively, every type with $C_3^1=3$ different ways filling in the a_{53} position. For instance, in the first group with $\{-1,3,4,5\},\{-1,3,5,4\},\{-1,4,5,3\},\{-1,4,3,5\},\{-1,5,3,4\},\{-1,5,4,3\}$, we calculate the total number of standard hexagonal Sudoku for the first type $\{-1,3,4,5\},\{-1,3,5,4\},\{-1,4,5,3\}$, and then give our estimation.

Unfortunately, this estimation is rough. It is because due to constraints of hexagonal Sudoku itself, it does not have good symmetry properties, so that swapping any two adjacent numbers will lead to different results. Limited by time constraints, we have not yet calculated the total number of standard hexagonal Sudoku of all 24 types. The remaining 21 types will be calculated in our subsequent study with the use of Gröbner basis theory and finally we will obtain the precise value of the total numbers of standard hexagonal Sudoku.

Now we will give the estimation of the total number of all the hexagonal Sudoku D as follows:

$$D = D_1 \times 9! = D_1 \times 362880 \approx 206603550720 \approx 2.066\times 10^{11}$$

II. Classification of Sudoku

We have already estimated the total number of hexagonal Sudoku in the last section. Next, by applying group theory (the group action on a set, Burnside's Lemma), we will do some classification of hexagonal Sudoku.

First, we will introduce the concept and examples of group.

Defining a binary operation "•" on a non-empty set G ($a \bullet b$ is abbreviated as ab), if the following four conditions $(i),(ii),(iii),(iv)$ are satisfied, then the set G is called a group.

(i) For any $a,b \in G$, there is $ab \in G$;

(ii) For any $a,b,c \in G$, there is $(ab)c = a(bc) \in G$, i.e. the associative law holds.

(iii) There is an element $e \in G$ such that $ae = ea = a$ for any $a \in G$, moreover, this element is called the identity element of G;

(iv) For any $a \in G$, there is inverse element a^{-1}, i.e. $aa^{-1} = a^{-1}a = e$.

If $|G| < +\infty$, G is called a finite group, and then $|G| = n$ is called the order of group G. If there is an element g in group G such that any element in G is a certain power of g, then G is called a cyclic group with g the generator, denoted as $G = \langle g \rangle$.

The concept of group is the best tool to understand the symmetric structure in nature, such as the rotation, reflection, central symmetry and axisymmetric of some pattern, all of which can be interpreted by group. This is our main method discussed in this section.

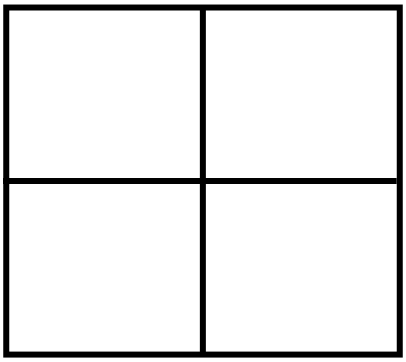
Now we introduce the Burnside's Lemma:

Let G be a finite group, X be a set. Denote X^g as the fixed-points of X acted by $g \in G$, that is, $X^g = \{x \in X \mid g \cdot x = x\}$.

Burnside's Lemma states as follows: The number of orbits for group G acting on set X can be calculated by the formula $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$, i.e. the average number of fixed-points of X acted by the elements of G.

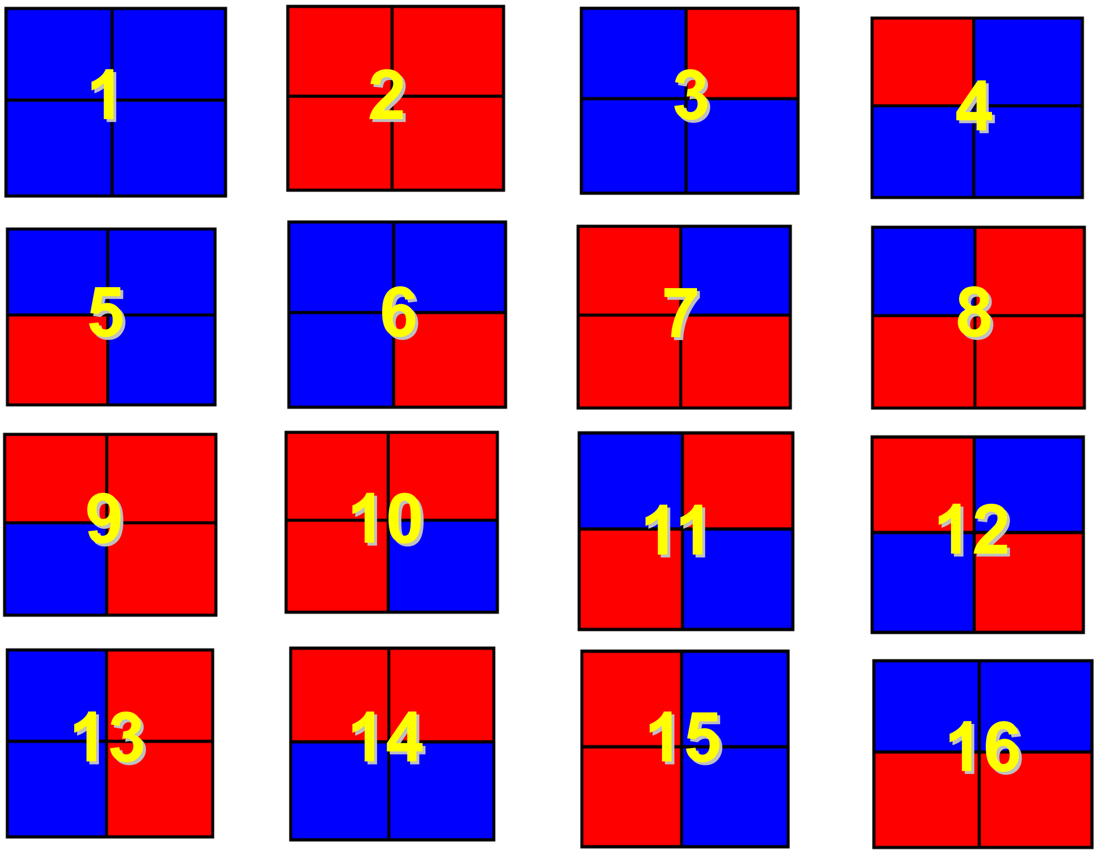
Here we give a simple example:

Example: Coloring a 2×2 chessboard with two colors, i.e. one color for each grid. Under the equivalence of counterclockwise rotation, how many essential different coloring plans there?



Solution: If rotation is not allowed, then there are $2^4 = 16$ different coloring plans.

Now we will number these 16 plans from 1 to 16. Denote the set of 16 coloring plans as $X = \{1, 2, \cdots, 16\}$, and list them as follows;



We will call two coloring plans equivalent if they coincide under the rotation permutation.

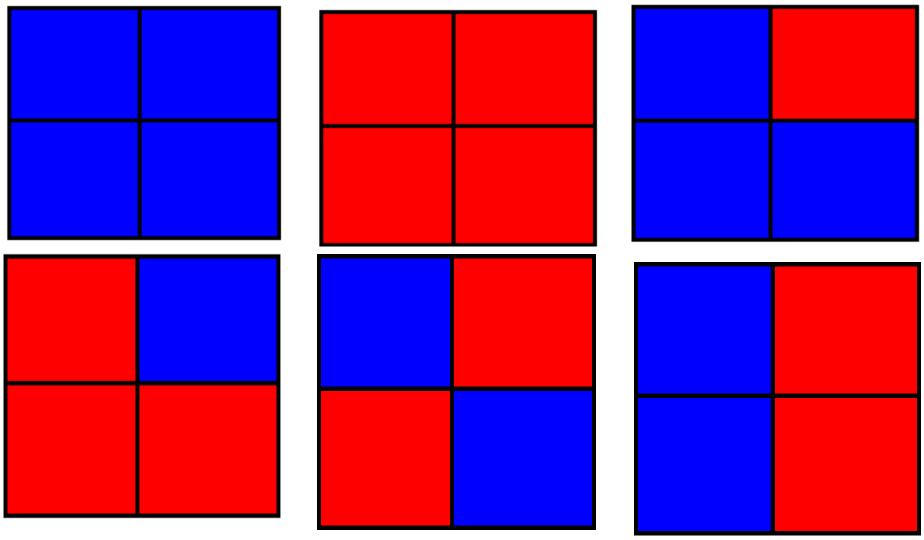
Next we discuss the permutation happens in the above 16 coloring plans under the counterclockwise rotation of angles 0^o, 90^o, 180^o and 270^o, respectively.

- Rotation 0^o : p_1=(1)(2)…(16);
- Rotation 90^o: p_2=(1)(2)(3 4 5 6)(7 8 9 10)(11 12)(13 14 15 16);
- Rotation 180^o: p_3=(1)(2)(3 5)(4 6)(7 9)(8 10) (11)(12)(13 15)(14 16);

- Rotation 270°: p_4=(1)(2)(6 5 4 3)(10 9 8 7)(11 12)(16 15 14 13);
- Obviously, $G=\{p_1,p_2,p_3,p_4\}$ is a group of order four;
- Moreover, the coloring plans in the same orbit are exactly the same plan.
- Therefore, the number of orbits of G acting on X is equal to the number of coloring plans. By using Burnside's lemma, we have:

$$T=\frac{c_1(p_1)+c_1(p_2)+c_1(p_3)+c_1(p_4)}{|G|}=\frac{16+2+4+2}{4}=6$$

These 6 essential different coloring plans are given as follows:



Now we calculate the number of hexagonal Sudoku under symmetrical equivalence using Burnside's Lemma.

According to the characteristic of hexagonal Sudoku, its symmetric raises only through counterclockwise rotation by an angle of 0°, 60°, 120°, 180°, 240° and 300°, and hence making the equivalence of Sudoku. Hexagonal Sudoku has no other symmetry.

① **Calculation of Group G of hexagonal Sudoku**

Denote the counterclockwise rotation by an angle of 0°, 60°, 120°, 180°, 240° and 300° as P_0,P_1,P_2,P_3,P_4,P_5, respectively. Moreover, let $G=\{P_0,P_1,P_2,P_3,P_4,P_5\}$. By the definition of rotation, there are the following equations:

$$P_0P_1=P_1P_0=P_1, \forall\ i=0,1,2,3,4,5;$$

$$P_i=(P_1)^i, \forall\ i=0,1,2,3,4,5.$$

Then, G becomes a cyclic group of degree 6 whose composition is the composition of rotations and P_1 is its generator, i.e. $G=<P_1>$, and $|G|=6$, in other words, $G\cong Z_6$.

② Calculation of fixed-points χ^g of hexagonal Sudoku.

First, we may denote the total number of hexagonal Sudoku as D. Due to the formula of Burnside's Lemma, for $g \in Z_6$, we discuss into four cases respectively:

① g is identical transformation, i.e. $g = [0]$

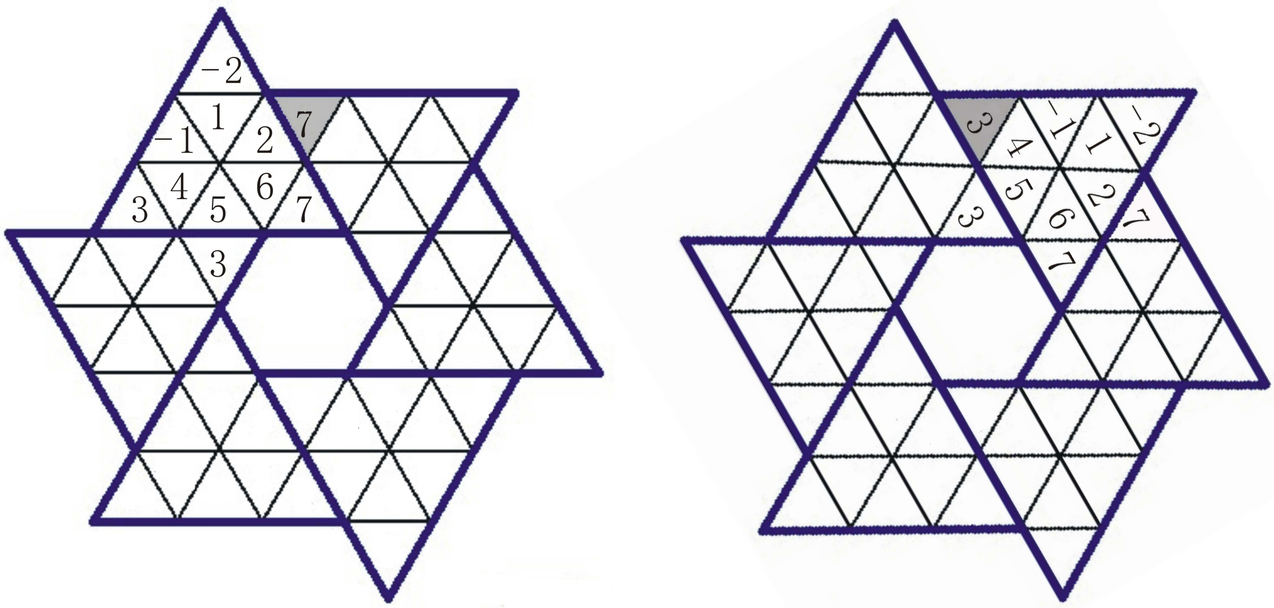
Clearly, all hexagonal Sudoku remain unchanged under the transformation, i.e. all hexagonal Sudoku are fixed-points, thus $\chi_g = D$.

② g is the rotation 60°or 300°, i.e. $g = [1]$ or $[5]$.

Next we mainly discuss the case when g is the rotation 60°; when rotation 300 °, the case is similar.

We may assume that the original hexagonal Sudoku is standard, so we can fill in the upper left triangle with numbers $-2,-1,1,2,3,4,5,6,7$, respectively. After rotation by 60°, we will find that in the a_5 position, the original hexagonal Sudoku have number $a_5 = 7$ according to Lemma 1.2, however in the new one $a_5 = 3$, which is a contradiction. Therefore, when $g = [1]$, $\chi_g = 0$, i.e. there is no fixed point (see the figure below):

The original one The new one

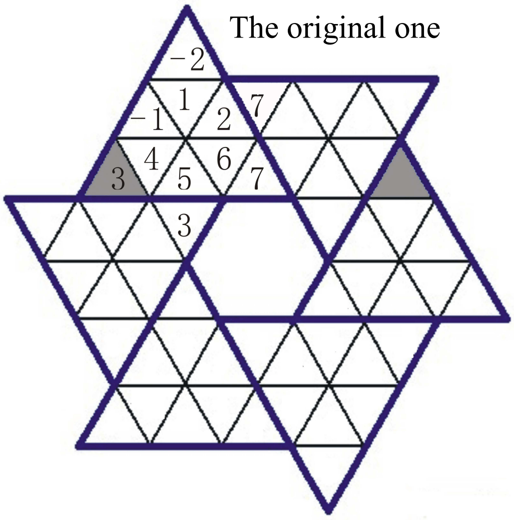


When $g = [5]$, we only need to check the position a_{23}. Before rotation, $a_{23} = 3$; after rotation, $a_{23} = 7$. In this case, χ_g is also equal to 0.

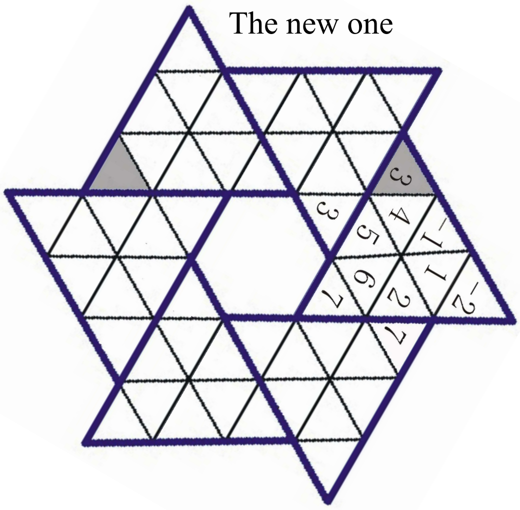
③ g is the rotation 120°or 240°, i.e. $g=[2]$ or $[4]$.

Similarly, we will discuss the case when g is the rotation 120 °; when rotation 240 °, the case is similar.

We may again assume that the original hexagonal Sudoku is standard, so we can fill in the upper left triangle with numbers $-2,-1,1,2,3,4,5,6,7$, respectively. After rotation by 120 °, we focus on the a_{10} and a_{18} position. In the original hexagonal Sudoku, because $a_{10}=3$, so that $a_{18}\neq 3$ according to the number-filling rules. But in the new one, we find $a_{18}=3$, which indicate that after the rotation by 120°, hexagonal Sudoku does not remain constant, i.e. when $g=[2], \chi_g=0$.



The original one



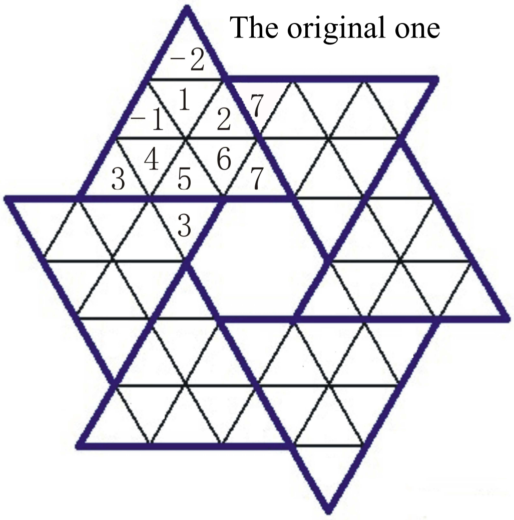
The new one

When $g=[4]$, we only need to check two positions a_{10}, a_{50}. In the original hexagonal Sudoku, $a_{10}=3, a_{50}\neq 3$, however, after rotation $a_{50}=3$. Therefore, in this case, χ_g is also equal to 0.

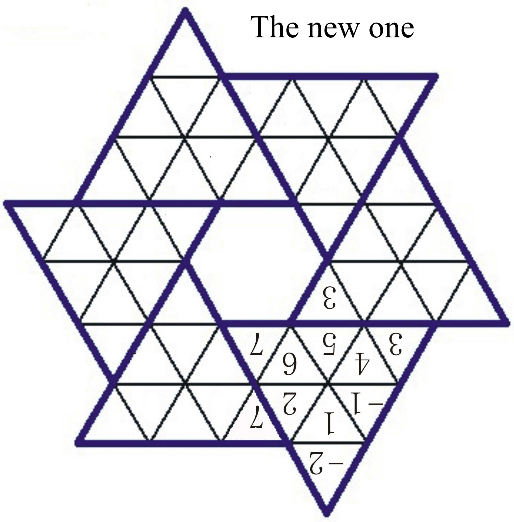
④ g is the rotation 180°, i.e. $g=[3]$.

After ①, ②, ③, we will examine the case when the rotation angle is 180°. We again assume that the original hexagonal Sudoku is a standard one.

After rotation by 180°, we get the new Sudoku as shown below:



The original one



The new one

After rotating by180°, Sudoku is required to remain the same. By comparing the numbers in the two corresponding positions between the original and new Sudoku, we get the following conditions for the original Sudoku to satisfy:

(1) $a_{32} = a_{45} = 3, a_{41} = a_{50} = 7, a_{42} = 6, a_{43} = 5, a_{44} = 4, a_{51} = 2, a_{52} = 1, a_{53} = -1, a_{54} = -2$

(2) $a_6 = a_{49}, a_7 = a_{48}, a_8 = a_{47}, a_9 = a_{46}, a_{15} = a_{40}, a_{16} = a_{39}, a_{17} = a_{38}, a_{24} = a_{31}$

(3) $a_{18} = a_{37}, a_{25} = a_{30}, a_{26} = a_{29}, a_{27} = a_{28}, a_{33} = a_{22}, a_{34} = a_{21}, a_{35} = a_{20}, a_{36} = a_{19}$

(4) $a_{18} = a_{24} = a_{31} = a_{37}$

Under these constraints, using the theory discussed in the first section and Maple14 software, we calculate the leading terms of the corresponding Gröbner basis are $a_6^2, a_7^3, a_8^2, a_9, a_{15}^4, a_{15}\,a_{16}, a_{16}^3$.

Then by using of theorem 1.7, we obtain the relationship between the number of hexagonal Sudoku N and the leading terms of Gröbner basis $\{s_1, s_2, \cdots, s_n\} \in Q[a_1, a_2, \cdots, a_{54}]$, it is shown as below:

$$N = \#\{f = a_1^{i_1} a_2^{i_2} \cdots a_{54}^{i_{54}} \mid f \text{ can not be divided by } lt(s_i) \text{ for all } i = 1, 2, \cdots n\}$$

Now the number of Sudoku $N = 72$ is obtained using Mathematica5.0 software, i.e. when $g = [3]$, the number of fixed-points $\chi_g = 72 \times 9! = 72 \times 362880 = 26127360$ (here 9! is due to number permutation, under assuming that the original Sudoku is standard).

In summary, from ①, ②, ③, ④, we obtain the following table:

Degree of rotations	Number of fixed-points	Degree of rotations	Number of fixed-points
0°	D	180°	26127360
60°	0	240°	0
120°	0	360°	0

Finally, by the formula of Burnside's Lemma, the number of equivalent classes of hexagonal Sudoku under the rotation group Z_6 is

$$M = \frac{1}{|Z_6|} \sum_{g \in Z_6} \chi_g = \frac{D + 26127360}{6} \approx 34438279680$$

where D is the total number of hexagonal Sudoku. Here D is substituted by the estimated value 206603550720 .

III. Graphic Design of Sudoku

The above two sections are focused on the algebraic and combinatoric properties of Sudoku. Through the observation of Sudoku pattern, we make some analysis on graphic design.

Proposition 3.1: In splicing of Sudoku, the number of Sudoku on the $n-th$ level $(n \geq 2)$ is $a_n = 6(n-1)$; for all n levels, the total number is $T_n = 3n^2 - 3n + 1$.

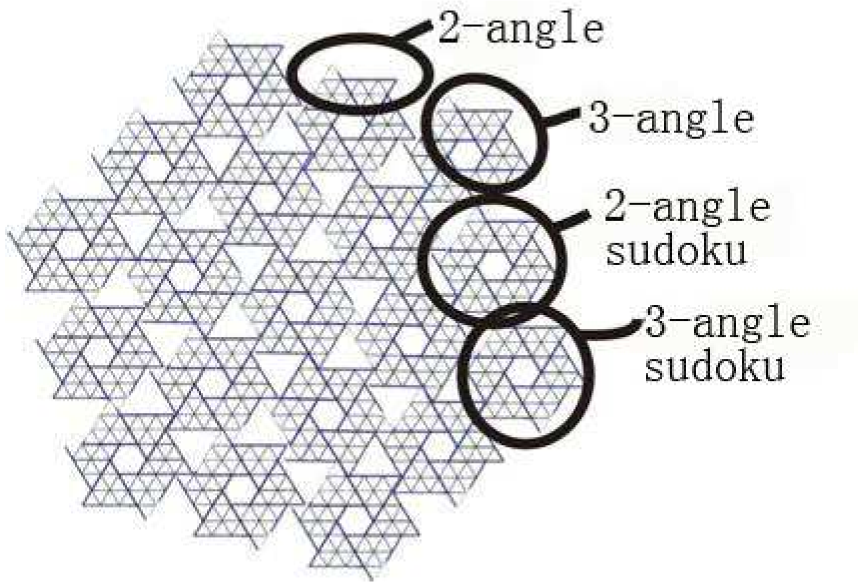
Proof: We denote v_n as the number of the external angles on the $n-th$ level. We will use mathematical induction to prove that $v_n = 18 + 12(n-2)$.

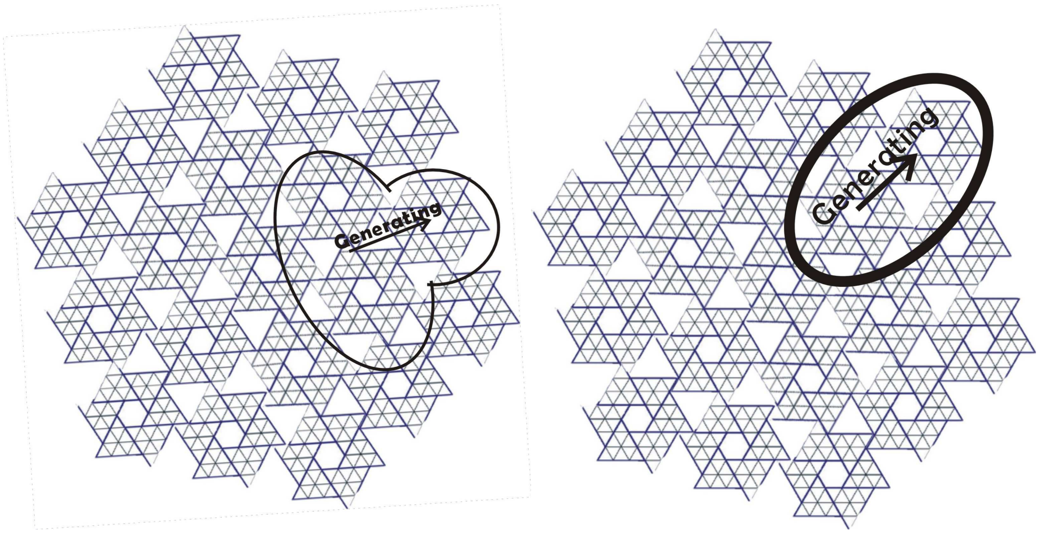
(1) When $n = 1, 2$, we can see the number of external angles $v_1 = 6, v_2 = 18$ directly.

(2) When $n \geq 3$, assume that the equation holds when $n = k$, i.e. $v_k = 18 + 12(k-2)$.

By observation we can find that among the three angles of the Sudoku on the second level not connected to the Sudoku on the outer level, the two side angles and the adjacent side angle of the adjacent Sudoku can form a new Sudoku; the middle angle can form a new Sudoku itself. Moreover, the three new Sudoku formed must be connected, such that there are two unconnected angles of the Sudoku on the two sides. These two angles are called 2-angle; there are three unconnected angles of Sudoku in the middle, we call them 3-angle.

The concepts of 2-angle, 3-angle, 2-angle Sudoku, 3-angle Sudoku and how to generate new Sudoku on the $(k+1)-th$ level from the $k-th$ level can be shown in the following figures:





Therefore, when $n=2$, there are 6 3-angle Sudoku and 0 2-angle Sudoku;

when $n=3$, there are 6 3-angle Sudoku (generated by middle angles of 6 Sudoku on the second level), $\frac{6\times(3-1)}{2}\times 2=12$ 2-angles Sudoku.

Moreover, since 3-angle Sudoku can only be generated by the middle angle of Sudoku on the previous level, the number of 3-angles Sudoku is always 6. Therefore, when $n=k$, i.e. when $v_k=18+12(k-2)$, there are 6 3-angles Sudoku, and $\frac{18+12(\mathrm{k}-2)-6\times 3}{2}=6(k-2)$ 2-angle Sudoku.

When $n=k+1$,

$$v_{k+1}=6\times 3+\frac{6\times(3-1)+2\times 6(k-2)}{2}\times 2=18+12+12k-24$$

$$=18+12k-12=18+12(k+1-2)$$

which completes the induction and $v_n=18+12(n-2)$.

Moreover, because the number of 3-angles Sudoku is always 6, so that

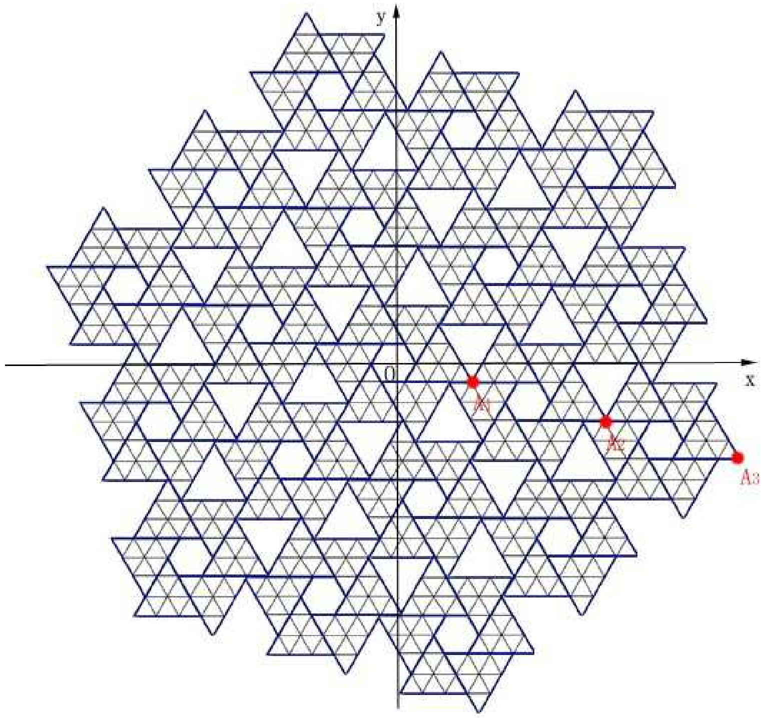
$$a_n=6+\frac{18+12(n-2)-6\times 3}{2}=6n-6$$

Finally, by the summation formula of arithmetic sequence (starting from the second term), we calculate the total number of Sudoku for all n levels is

$$S_n=a_1+a_2+\cdots+a_n=1+\frac{(a_n+a_2)(n-1)}{2}=\frac{(6n-6+6)(n-1)}{2}+1=3n^2-3n+1$$

Proposition 3.2: Suppose the length of the side of any small triangle in hexagonal Sudoku is 1. The radius of the disk with n levels whose center just is the center of the median hexagonal Sudoku is equal to $r_n=\sqrt{39n^2-33n+7}$, the area of this disk is $S_n=\left(39n^2-33n+7\right)\pi$.

Proof: We establish Cartesian coordinate system $O-xy$ as shown in the figure, where $O(0,0)$ is the center of the original hexagonal Sudoku. By observation we can find that point A_n is the point farthest away from the original point O of Sudoku on the $n-th$ level, as shown in the figure below:



Hence $r_n = |A_n O|$

Suppose $A_n(x_n, y_n)$, then from the figure we can see:

when $n=1$, $x_1 = 3\times1+0.5 = \frac{7}{2}$, $y_1 = -\frac{\sqrt{3}}{2}$;

when $n\geq 2$, $x_n - x_{n-1} = 6$, $y_n - y_{n-1} = -\frac{\sqrt{3}}{2}\times 2 = -\sqrt{3}$;

so $x_n = \frac{7}{2} + 6(n-1) = \frac{12n-5}{2}$, $y_n = -\frac{\sqrt{3}}{2} - \sqrt{3}(n-1) = -\frac{(2n-1)\sqrt{3}}{2}$;

hence $r_n = \sqrt{x_n^2 + y_n^2} = \sqrt{39n^2 - 33n + 7}$, $S_n = \pi r_n^2 = (39n^2 - 33n + 7)\pi$

Now we define the concept of splicing efficiency, i.e. $\frac{\mathrm{T_n}}{\mathrm{S_n}}$, the number of Sudoku contained in the unit area of disk, and consider the variation rules of splicing efficiency.

Theorem 3.3: In the splicing figure composed by hexagonal Sudoku, when the number of levels n is 1 (i.e. a single Sudoku), splicing efficiency $\frac{T_1}{S_1}$ reaches its maximum value $\frac{1}{13\pi}$; when the number of levels n is 2, splicing efficiency $\frac{T_2}{S_2}$ reaches its minimum value $\frac{1}{13\pi}\times\frac{91}{97}$; when n tends to ∞, the splicing efficiency $\frac{T_n}{S_n}$ is strictly increasing with limit $\frac{1}{13\pi}$.

Proof: By Proposition 3.1 and 3.2, we have obtained that

$$T_n = 3n^2 - 3n + 1, S_n = \left(39n^2 - 33n + 7\right)\pi;$$

So $\frac{T_n}{S_n} = \frac{3n^2 - 3n + 1}{(39n^2 - 33n + 7)\pi} = \frac{1}{13\pi}(1 - \frac{6n - 6}{39n^2 - 33n + 7})$;

When $n = 1$, $\frac{T_1}{S_1} = \frac{1}{13\pi}$.

Moreover, for any $n = 2,3,\cdots$, we have $0 < \frac{6n-6}{39n^2 - 33n + 7} < 1$, thus $0<\frac{T_n}{S_n} < \frac{1}{13\pi}$, $n = 2,3,\cdots$.

It is easy to calculate that

$$\frac{T_n}{S_n} - \frac{T_{n-1}}{S_{n-1}} = \frac{6}{13\pi}\frac{39n^2 - 117n + 65}{(39n^2 - 33n + 7)(39(n-1)^2 - 33(n-1) + 7)}$$

When $n = 2$, the numerator is $39\times 2^2 - 117\times 2 + 65 < 0$; when $n \ge 3$, it is easy to see that the numerator $39n^2 - 117n + 65 > 0$. Thus, there is always $1 > \frac{T_n}{S_n} > \frac{T_{n-1}}{S_{n-1}}$.

Hence, for any $n \in N$, the minimum value of splicing efficiency $\frac{T_n}{S_n}$ is

$$\left.\frac{T_n}{S_n}\right|_{min} = \frac{T_2}{S_2} = \frac{1}{13\pi}(1 - \frac{6\times 2 - 6}{39\times 2^2 - 33\times 2 + 7}) = \frac{1}{13\pi}\times\frac{91}{97}.$$

Therefore, the maximum splicing efficiency occurs on the first level, i.e. the single hexagonal Sudoku. The minimum splicing efficiency occurs on the second level; Moreover, when n tends to ∞, the splicing efficiency is always increasing, and $\lim_{n\to\infty}\frac{T_n}{S_n} = \lim_{n\to\infty}\frac{1}{13\pi}(1 - \frac{6n-6}{39n^2 - 33n + 7}) = \frac{1}{13\pi}$, that is to say, it will approach $\frac{1}{13\pi}$ but never reaching.

IV. Further Research

As the further research, we will focus on the following three aspects:

(1) We will continue to use Gröbner basis theory to find the total number of standard hexagonal Sudoku of the remaining 21 types, to obtain the precise total number of standard hexagonal Sudoku of all 24 types and the precise value of D.

(2) In the application of Burnside's Lemma, we focus on the action of rotation group Z_6. However, due to the equivalence of number status, we also know that there is another permutation group S_9 of hexagonal Sudoku apart from Z_6. In this case, Sudoku can also be considered as equivalent. By taking Z_6 and S_9 into account at the same time, we can first classify all hexagonal Sudoku due to the number equivalence, so that only standard Sudoku is considered. Finally, when we use Burnside's Lemma, the fixed points considered are not the fixed points in the original sense, but in the sense of equivalence (or fixed orbits, if the equivalent class is seen as an orbit), i.e. $\chi_g = \left|\{x \in X \mid g \cdot \overline{x} = \overline{x}\}\right|$, where X is the set of all standard hexagonal Sudoku; $\overline{x}$ is the equivalent class of hexagonal Sudoku in number permutation, i.e. $\overline{x} = \{\sigma \cdot x \mid \sigma \in S_9\}$.

(3) When studying splicing efficiency in the disk of hexagonal Sudoku, we can further fix the radius of the circle, and then give the maximum number of Sudoku contained in this circle. Moreover, by combining (1) and (2), we can give the total number of all hexagonal Sudoku in the sense of equivalence with the given circle of fixed radius. This is meaningful for graphic design.

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