Curve Unfolding

Han Wang　Chen Song　Jing Zi Liu

Led by the teacher　Xin De Li

Experimental Middle School of Henan Province

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Abstract

In the paper, we introduce some new geometric concepts and methds and try to prove an open problem in classical geometry.

Key Words: Y-form , convex polygon, Mathematical induction

1 Introduction

Mikhael Gromov [1] proposed the following open problem in classical geometry(see also [2] and [3]): for any Jordan curve, is there a movement such that the distance of any two points on the Jordan curve does not decrease and the length of curve does not change at any time? Moreover, the Jordan cuve becomes a convex Jordan curve eventually. In the paper, we introduce some new geometric concepts and use knowledge with our high school students and try to prove this open problem in classical geometry.

The paper is organized as follows. In section 2, we prove the open problem on polygon. We prove the open problem on Jordan cuve in section 3.

2 The Conjecture on Polygon

In this section, we will prove the conjecture on polygon.

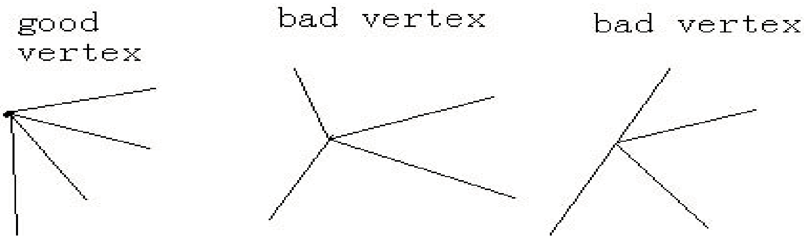
A reflex angle means the angle is larger than 180^0. Inferior angle means the angle is smaller than 180^0. Notice there is no straight angle in polygon.

For convenance, each angle degree is less than 1800, but keep calling reflex angle 'reflex' and inferior angle 'inferior'.

We call a polygon '**Y-form**' if the polygon have only 3inferior interior angle, espectially triangle if Y-form.

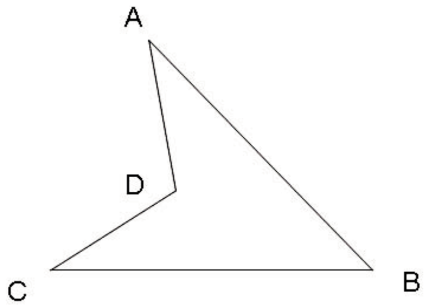
We call a movement 'articulated' if the movement only change the angle of polygon.

We call a vertex is '**good**' if the vertex contain a reflex angle.



Lemma 2.1 *When 'Y-form' is moving articulated,each inferior angle would not decrease if each reflex angle do not decrease.*

Proof. Let's begin with the the simplest quadrilateral 'Y-form'



without loss of generality we suggest $S_{ADB} > 0$,then

$$AD^2 + DC^2 - AD \times DC \times \cos D = AB^2 + BC^2 - AB \times BC \times \cos B$$

$$\Rightarrow \frac{d\cos B}{d\cos D} = \frac{AD \times DC}{AB \times BC} \tag{2.1}$$

$$\Rightarrow \frac{S_{ADC}}{S_{ABC}} = \frac{AD \times DC \times \sin D}{AB \times BC \times \sin B} = \frac{\sin D}{\sin B} \times \frac{d\cos B}{d\cos D} = \frac{\sin D}{\sin B} \times \frac{-\sin B \times d\angle B}{-\sin D \times d\angle D} = \frac{d\angle B}{d\angle D}$$

Similarly, we have

$$\frac{S_{DCB}}{S_{DAB}} = \frac{d\angle A}{d\angle C} \tag{2.2}$$

And obviously we have

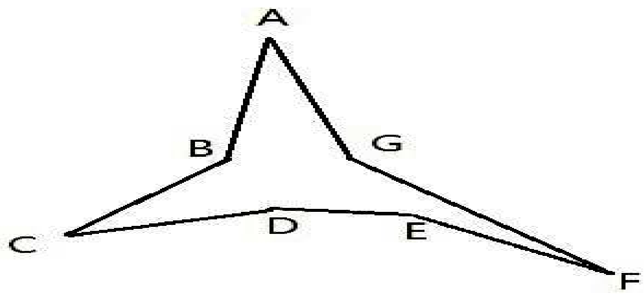
$$d\angle D = d\angle A + d\angle B + d\angle C \Rightarrow \frac{d\angle A}{d\angle D} + \frac{d\angle B}{d\angle D} + \frac{d\angle C}{d\angle D} = 1 \tag{2.3}$$

(2.1), (2.2) and (2.3) yield

$$\frac{d\angle A}{d\angle D}=\frac{S_{BDC}}{S_{ABC}}\geq 0,\quad \frac{d\angle B}{d\angle D}=\frac{S_{ADC}}{S_{ABC}}\geq 0,\quad \frac{d\angle C}{d\angle D}=\frac{S_{ADB}}{S_{ABC}}\geq 0$$

$$0\leq\frac{d\angle A}{d\angle D}<1,\quad 0\leq\frac{d\angle B}{d\angle D}<1,\quad 0\leq\frac{d\angle C}{d\angle D}<1 \tag{2.4}$$

When the polygon is common 'Y-form' like:



because of degrees of 3 inferior angles can totally depend on degrees of reflex angles,so we can write a function like:

$$\angle A = f(\angle B,\angle D,\angle E,\angle G) \tag{2.5}$$

from the proof of quadrilateral 'Y-form', we have

$$\frac{\partial\angle A}{\partial\angle B}=\frac{S_{CBF}}{S_{CAF}},\quad \frac{\partial\angle A}{\partial\angle D}=\frac{S_{CDF}}{S_{CAF}},\quad \frac{\partial\angle A}{\partial\angle E}=\frac{S_{CEF}}{S_{CAF}},\quad \frac{\partial\angle A}{\partial\angle G}=\frac{S_{CGF}}{S_{CAF}} \tag{2.6}$$

This proved lemma 2.1

Lemma 2.2 *Give X, Y, A, B as 2D vector in R^2 ,and $A\neq B, X\neq 0, Y\neq 0, X\neq kY$ ($k\in$R) and suffice*

$$(X-Y)(A-B)=0 \tag{2.7}$$

$$XA=0 \tag{2.8}$$

$$YB=0 \tag{2.9}$$

then for any vector C there exist a vector Z suffice

$$(Y-Z)(B-C)=0 \tag{2.10}$$

$$(Z-X)(C-A)=0 \tag{2.11}$$

$$ZC=0 \tag{2.12}$$

Proof. Assume that A = 0 from (2.7)(2.8) we have X B = 0, because $X\neq kY$, and

(2.9), we have $B = 0$, which contradict from A≠B.the same if we assumeB = 0. so we have

$$A \neq 0,\ B \neq 0 \tag{2.13}$$

if $A = mB(m \in \mathrm{R})$ consider (2.9) we have $YA = 0$, by (2.8) and $X \neq kY$ we get $A = 0$ which contradict from (2.13). So there exist λ_1 , $\lambda_2 \in$ R such that

$$C = \lambda_1 A + \lambda_2 B \tag{2.14}$$

Let

$$Z = \lambda 1 X + \lambda 2 Y \tag{2.15}$$

From (2.14), we have

$$CZ = \lambda_1 AZ + \lambda_2 BZ$$

$$= \lambda_1 A(\lambda_1 X + \lambda_2 Y) + \lambda_2 B(\lambda_1 X + \lambda_2 Y) \tag{2.16}$$

$$= \lambda_1^{2} AX + \lambda_1 \lambda_2 (AY + BX) + \lambda_2^{2} BY$$

By (2.7),

$$XA + YB = XB + Y \tag{2.17}$$

(2.17),(2.8) and (2.9), we have

$$XB + YA = 0 \tag{2.18}$$

(2.16),(2.18), (2.8), (2.9),we have

$$CZ = 0 \tag{2.19}$$

By (2.14) and (2.15),

$$BZ + CY = B(\lambda_1 X + \lambda_2 Y) + (\lambda_1 A + \lambda_2 B)Y = \lambda_1 (BX + AY) + 2\lambda_2 BY = 0 = BY + CZ$$

$$(B - C)(Z - Y) = 0 \tag{2.20}$$

$$AZ + CX = A(\lambda_1 X + \lambda_2 \mathrm{Y}) + (\lambda_1 A + \lambda_2 B)X = \lambda_2 (AY + BX) + 2\lambda_1 AX = 0 = AX + CZ$$

$$(A - C)(Z - X) = 0 \tag{2.21}$$

(2.19),(2.20) and (2.21) means Z is what we want.

Remark 2.1 *Lemma 2.2 means if P, Q, R are vertexes of triangle and we draw line* $PQ \perp X$*,* $QO \perp Y$*.Let* $A = P - O, B = Q - O, C = R - O$*, and X, Y suffice (2.7),(2.8),(2.9), let* $X = \frac{dA}{dt}, Y = \frac{dB}{dt}, Z = \frac{dC}{dt}$*, which means the speed of vertexes. (2.7), (2.10) and (2.11) means it keep the length of edge. So, lemma 2.2 means in rigid body motion every point is rotate around a 'instantaneous center' in each moment.actully the condition can be loose for general situation, and it has been used frequently in mechanics.*

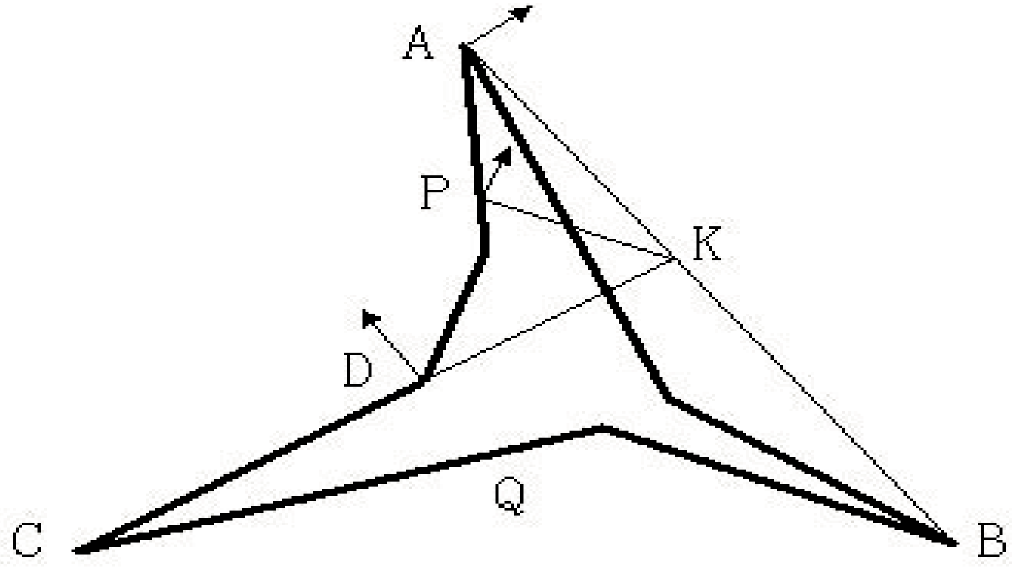
Theorem 2.1 *When 'Y-form' is moving articulated,distance between each couple of points would not be shorten if each reflex angle do not decrease.*

Proof. Choose two point *P* and *Q* randomly,just like lemma 2.1,the distance between *P* and *Q* are completely depend on degrees of reflex angles , so if every $\frac{\partial |PQ|}{\partial \angle D} \geq 0$ which *D* is an reflex angles.

The lemma will be proved.

We divide the 'Y-form' into 4 broken line which endpoint is 3 inferior angle and D.

If *P*, *Q* are in the same broken line or in the 2 broken line which are Neighboring, the proposition is obviously.



Else, the only situation rest on the picture above. P, Q are in the different,apart broken line.and D is the only increasing reflex angle (the rest degree of reflex angle retain fix for calculate the $\frac{\partial |PQ|}{\partial \angle D}$). K is intersection of line CD and line AB. $\angle D$ is increase means $\frac{d\angle D}{dt} > 0$. We denote the position vector of X as X, and movement vector $\frac{dX}{dt}$ as X' which X means an angle such as A, B, C, D, P, Q, K. Then

$$(D - C)^2 = a^2$$

which a is an constant.differentiate it we have

$$(D - C)(D' - C') = 0 \qquad (2.22)$$

The same we have

$$(A - B)(A' - B') = 0 \qquad (2.23)$$

$$(D - A)(D' - A') = 0 \qquad (2.24)$$

$$(P - D)(P' - D') = 0 \qquad (2.25)$$

$$(A - P)(A' - P') = 0 \qquad (2.26)$$

as we fix the broken line CB (i.e C'= 0, Q' = 0, B' = 0), (2.22) and (2.23) reduce to

$$D'(D - C) = 0 \qquad (2.27)$$

$$A'(A - B) = 0 \qquad (2.28)$$

C, D, K are in the same line, B, K, A are in the same line. By(2.27) and (2.28), we have

$$D'(D-K)=0 \quad (2.29)$$

$$A'(A-K)=0 \quad (2.30)$$

$S_{CDA}>0 \Rightarrow A' \neq 0$. If $D'=0$ from lemma 2.1, $S_{ADB}=0$, so DPA collinear ,

$$P'=kA' \quad (k>0, \in \mathrm{R}) \quad (2.31)$$

$$P'(P-Q)=P'(P-A+A-Q)=kA'(P-A)+kA'(A-Q) \quad (2.32)$$

by (2.26)and (2.31), we have

$$(k-1)A'(P-A)=0 \quad (2.33)$$

$$k-1 \neq 0 \Rightarrow A'(P-A)=0$$

$$k-1=0 \Rightarrow P=A \Rightarrow A'(P-A)=0$$

and $\frac{d\angle B}{dt}>0 \Rightarrow A'(A-Q)>0$ with (2.32) we have $P'(P-Q)$ which means theorem proved in this situation.

If $D' \neq 0$ and K exist infer $D' \neq kA'$ $(k \in \mathrm{R})$. Let

$$P'=\lambda_1 D' + \lambda_2 A' \quad (2.34)$$

which λ_1, λ_2 suffice

$$(P-K)=\lambda 1(D-K)+\lambda 2(A-K) \quad (2.35)$$

by lemma 2.2, this P fit (2.25), (2.26). Combine (2.24),it means DPA is rigid.

Because of the position of P and (2.35), we have $\lambda_1 \geq 0$, $\lambda_2 \geq 0$,combine (2.35)

$$P'=\lambda_1 D'+\lambda_2 A'$$

$$\Rightarrow P'(P-Q)=P'(P-K+K-Q)=P'(K-Q)$$

$$=(\lambda_1 D'(K-Q)+\lambda_2 A'(K-Q))$$

$$=(\lambda_1 D'(K-D+D-Q)+\lambda_2 A'(K-A+A-Q)) \quad (2.36)$$

$$= \lambda_1 D'(D - Q) + \lambda_2 A'(A - Q)$$

as D, Q are in neighboring broken line,we have $D'(D - Q) \geq 0$. as A, Q are in neighboring broken line,we have $A'(A - Q) \geq 0$.and by (2.35), (2.36)

$$P'(P - Q) = \lambda_1 D'(D - Q) + \lambda_2 A'(A - Q) \geq 0$$

And $\frac{d\angle D}{dt} > 0$. we have

$$\frac{\partial (P-Q)^2}{\partial \angle D} > 0$$

Theorem proved.

Lemma 2.3 *Each polygon can divide into n − 2 piece of 'Y-form',and every vertex is 'good'. Here n is number of polygon's reflex angle.*

Proof. When n = 3, lemma obviously.When n > 3,we numbered the inferior angle sequentially.denote inferior angle numbered 1 and inferior angle numbered 3 as A, C , we can draw a broken line from A to C, which completely in the polygon (can touch or partly coincides with the boundary). Let $A, P_1, P_2, \ldots, P_k, C$ be the shortest one. connect 2 point from different segment of this broken line,it must cross the boundary of polygon,or the broken line can be shorter,contradict with the assumption.So P_i are coincide with the reflex vertex of polygon, and P_i is good vertex. Because inferior angle number 2 and number 4 does not on the broken line ,so the broken line split the polygon at least 2 part.choose that segment of broken line that can split the polygon.

If P_i, P_{i+1} split the polygon,because P_i, P_{i+1} is good vertex,a reflex angle in P_i split into one inferior angle,one reflex angle.another reflex angle in P_{i+1} split into one inferior angle,one reflex angle too.

If A, P_1 split the polygon, a inferior angle of A split into 2 inferior angle, a reflex angle in P_1 split into one inferior angle,one reflex angle.situation P_k, C is the same.

If A, C split the polygon.it means no P_i between them,a inferior angle of A split into 2 inferior angle,a inferior angle of C split into 2 inferior angle.

All the situation above shows that the count of inferior angle increase 2,and reflex angle remain the same.

Now we get 2 polygon which total inferior angle in $n+2$ and each polygon at least have 3 inferior angle,it means each polygon at most have $n-1$ inferior angle.then we can assume polygon have $n-1$ inferior angle can suffice the lemma by mathematical induction.then we know that polygon have n inferior angle suffice the lemma.

Lemma 2.4 *For each non-convex polygon,grouped by $N(N \geq 2)$ pieces of 'Y-form',and each vertex is 'good',we choose any one of these 'Y-form',there exist one 'Y-form' which different from what we chosen,and without this 'Y-form',rest 'Y-form' are still grouping as one polygon.*

Proof. N=2, lemma obviously , when $N > 2$, we denote the chosen 'Y-form' as Y_1 .first we can choose a 'Y-form' Y_2 different from $Y-1$ and at lest one edge is the polygon's boundary.If the rest 'Y-form' can grouping as one olygon, $Y2$ is what we want.else, infer that $Y2$ cut the polygon into 2 piece (denote as P_1, P_2). We choose the piece which doesn't contain Y_1 (denote as P_1). $P_1 \cup Y_2$ is one polygon, which grouped by M ($M \geq 2$) 'Y-form', because $Y1$ doesn't in it, so $M < N$. By mathematical induction, we know there exist a $Y-form$ (denote as Y_3) different from $Y-2$ and let $(P_1 \cup Y_2) \setminus Y_3$ be one polygon, becauseY_2 cut original polygon into unattached P_1,P_2,so $Y_3 \in P_1$ dose no effect to P_2.So $(P_1 \cup Y_2 \cup P_2) \setminus Y_3$ is one polygon, which means Y_3 is what we want.

Lemma 2.5 F*or each non-convex polygon,grouped by N pieces of 'Y-form',If an articulated movement keep each reflex angle of polygon doesn*'t *decrease then this articulated movement keep angle of each 'Y-form' not decrease.*

Proof. In lemma 2.1, we have already proved situation $N=1$. When $N>1$, denote the polygon as O_1.we fix the reflex angle of polygon except one (denote as P_1).P_1 can contain many angle of 'Y-form'. Because P_1 is good, so it contain one reflex angle of 'Y-form'.denote this 'Y-form' as Y_1 .By lemma 2.4,we can find a 'Y-form' Y_2 different from Y_1 and $O_1 \backslash Y_2$ is polygon.denote $O_1 \backslash Y_2$ as O_2. denote inferior angle of Y_2 as P_2, P_3, P_4.Denote reflex angle of Y_2 which is also the inferior angle of O_2,as $Q_1, Q_2, \ldots, Q_s$.denote $\frac{d\angle P_j}{dt}$ as p_j ($j = 1, 2, 3, 4$).denote $\frac{d\angle Q_j}{dt}$ as q_j ($j = 1, 2, \ldots, s$).

Situation 1: 3 inferior angle of Y_2 are both the reflex angle of O_2 .and P_1 is not one of P_2, P_3, P_4.

Because we fix all the reflex angle of O_1 except P_1 which means we fix all the reflex angle of O_2 except P_j ($j = 1, 2, 3, 4$). By mathematical induction on O_2,we have

$$q_i = \sum_{j=1}^{4} k_{ij} p_j \quad (i = 1,2,\cdots,s) \tag{2.37}$$

here $k_{ij} \geq 0$ suffice

$$\sum_{i=1}^{s} k_{ij} \leq 1 \quad (j = 1,2,3,4) \tag{2.38}$$

Because we fix all the reflex angle of Y except Q_i,by lemma 2.1 on Y_2 we have

$$p_j = \sum_{i=1}^{s} m_{ij} q_i \quad (j = 2,3,4) \tag{2.39}$$

here $m_{ij} \geq 0$ suffice

$$\sum m_{ij} = 1 \quad (i = 1,2,\cdots,s) \tag{2.40}$$

(2.37) substitute (2.39) we get

$$p_j = \sum_{t=1}^{4} u_{tj} p_t, \quad u_{tj} = \sum_{i=1}^{s} k_{it} m_{ij} \quad (t = 1,2,3,4; j = 2,3,4) \tag{2.41}$$

Let

$$K_j = 1 - \sum_{i=1}^{s} k_{ij} \quad (j = 1,2,3,4) \tag{2.42}$$

Since (2.38), we have

$$K_j \geq 0 \tag{2.43}$$

(2.42) and (2.40) infer

$$\sum_{t=2}^{4} u_{jt} + K_j = \sum_{i=1}^{s} k_{ij} + K_j = 1 \tag{2.47}$$

Form (2.41)

$$p_2 = u_{12} P_1 + u_{22} P_2 + u_{32} P_3 + u_{42} P_4$$

$$(u_{22} + u_{23} + u_{24} + K_2) P_2 = u_{12} P_1 + u_{22} P_2 + u_{32} P_3 + u_{42} P_4 \tag{2.44}$$

$$(u_{23} + u_{24} + K_2) P_2 = u_{12} P_1 + u_{32} P_3 + u_{42} P_4$$

The same we have

$$(u_{32} + u_{34} + K_3) p_3 = u_{13} p_1 + u_{23} p_2 + u_{43} p_4 \tag{2.45}$$

$$(u_{42} + u_{43} + K_4) p_4 = u_{14} p_1 + u_{24} p_2 + u_{34} p_3 \tag{2.46}$$

(2.44),(2.45) and (2.46) get

$$K_2 p_2 + K_3 p_3 + K_4 p_4 = (u_{12} + u_{13} + u_{14}) p_1 \tag{2.47}$$

(2.44) and (2.45) get

$$(u_{24}+K_2)p_2+(u_{34}+K_3)p_3=(u_{12}+u_{13})p_1+(u_{42}+u_{43})p_4 \tag{2.48}$$

(2.44) and (2.46) get

$$(u_{23}+K_2)p_2+(u_{43}+K_4)p_4=(u_{12}+u_{14})p_1+(u_{32}+u_{34})p_3 \tag{2.49}$$

(2.44) and (2.45) get

$$(u_{32}+K_3)p_3+(u_{42}+K_4)p_4=(u_{13}+u_{14})p_1+(u_{23}+u_{24})p_2 \tag{2.50}$$

(2.37) (2.40) have $s+3$ linear equations,and $s+4$ variable, so it has not all zero solutions. Let P_j, q_i be this not all zero solution.

If $P_1=P_2=P_3=P_4=0$ from (2.37) we know $q_i=0$ ($i=1, 2, ..., s$) it means all variable equal to zero.Contradict by assumption.

To complete the prove of this situation,we first prove that $K_j>0$ ($j=2, 3, 4$). If for one j suffice $\sum k_{ij}=1$,it is to say in O_2 it exist a movement as p_j increasing, q_i ($i=1, 2, ..., s$) non-decreasing (but sum is equal to p_j) and the boundary which also is O_1's boundary unchanged and p_t ($t\neq j$) unchanged. It infer that in Y_2 exist a movement as p_j increasing,q_i ($i=1, 2, ..., s$) non-decreasing(but sum is equal to p_j) and all the rest angle unchanged.but from (2.4), we knows that if one reflex angle is increasing,at lest 2 inferior angle is increasing.but the movement just before only one inferior angle is increasing.Contradict occur.so $\sum k_{ij}<1$ .from (2.45) we have

$$K_j>0 \quad (j=2,3,4) \tag{2.51}$$

from (2.47), (2.51), if $p_1=0$ then $p_2=p_3=p_4=0$.

so in the no all zero solution $p_1\neq 0$ if $p_1>0$ from (2.47) we know it at lest one of p_2, p_3, p_4 must ≥ 0,then at lest one right hand side of (2.48)(2.49)(2.50)$\geq$ 0,so at lest one of the left hand side of (2.48),(2.49)(2.50)$\geq$ 0.then at lest 2 of p_2, p_3, p_4 are ≥ 0 then at lest one of right hand side of (2.48)(2.49)(2.50) is ≥ 0. then all p_j

$\geq 0(j = 2, 3, 4)$ then all $q_i \geq 0(i = 1, 2, ..., s)$.

Above all ,there exist $a_j \geq 0(j = 2, 3, 4)$, $b_i \geq 0(i = 1, 2, ..., s)$ suffice $p_j = a_j p_1$, $q_i = b_i p_1$,it means $O1$ suffice the lemma.

Situation 2: not all inferior angle of Y_2 is reflex angle of O_2 and P_1 is not inferior angle of Y_2.

If P_t is inferior angle of Y_2 but not reflex angle of O_2,then the equation is strictly the same as (2.37) - (2.42) which $k_{it} = 0$ $(i = 1, 2, ..., s)$ and prove $K_j > 0(j = 2, 3, 4, j \neq t)$ the same as situation 1.and $K_t = 0$.and the rest prove is the same as situation 1.

Situation 3: P_1 is one of P_j $(j = 2, 3, 4)$.

Without loss of generality, let P_2 in Y_2 is P_1 in O_2.here p_2 needn't equal to p_1 .so we treat P_2 in Y_2 as it doesn't attach O_2 .then following proof is the same as situation 2 $k_{i2} = 0(i = 1, 2, ..., s)$.but different from situation 2 ,we shall prove $p_2 < p_1$. From (2.37),(2.42),(2.51)and $k_{i2} = 0(i = 1, 2, ..., s)$

$$\sum_{i=1}^{s} q_i = \sum_{i=1}^{s} \sum_{j=1}^{4} k_{ij} p_j = \sum_{j=1}^{4} (1 - K_j) p_j < p_1 + p_3 + p_4$$

$$p_2 < p_1 + p_3 + p_4 - p_3 - p_4 = p_1$$

Lemma is proved.

Theorem 2.2 *For each non-convex polygon,there exist an articulated movement suffice the con-junction.*

Proof. Denote the polygon as O_1 . make a convex hull of O_1 (denote as HO),then we get some more polygon $S_1, S_2, ..., S_z$. split all of these polygon to 'Y-form' as lemma 2.3 ,the movement go on by these step below:

Step 1 : we choose one edge on HO 's boundary but not O_1 's boundary.This edge belong to a 'Y-form' (denote as Y_1).because all the angle on HO 's boundary is inferior, so Y_1 has 2 inferior angle on HO's boundary, and has one edge on HO's boundary (if has two, HO is not the convex hull).we erase this edge.then expose the rest one Y_1's inferior angle (denote the vertex as P_1 angle as $\angle AP1\,B$) and all reflex angle to the boundary of $H \backslash Y_1$.Denote $HO \backslash Y_1$ as O_2.

Step 2 : increase $\angle AP1\,B$ until any vertex reach 180^o. O_2 has only one reflex angle P_1,by lemma 2.5,when P_1 is increasing all the angle will not decrease.polygon fit lemma 2.5 until one vertex reach 180^o.

Step 3: divide into few situation:

Situation 1: angle in vertex P_1 reach 180^0 (denote the angle as $\angle CP_1D$).divide into 3 situations:

Situation 1.1: both of CP_1, DP_1 are O_2's boundary.

Because only AP_1, $BP1$ is $O2$'s boundary at vertex P_1.It is to say C, D is the same as A, B,in another word, $\angle AP_1B = 180^0$.So O_2 is convex hull of O_1 and amount of 'Y-form' compare to HO decrease 1.If $O_2 = O_1$ movement finish. else denote O_2 as HO and go to step 1.

Situation 1.2: both of $CP1$, $DP1$ are NOT O_2's boundary.

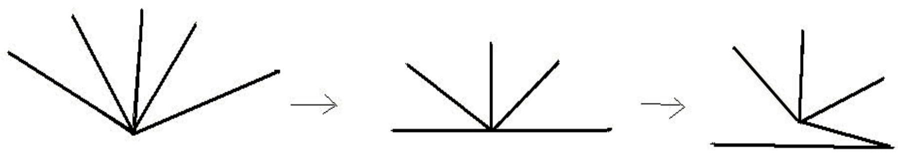
First we draw a new edge connect CD directly,then if CP_1 is edge of O_1,we erase line segment $DP1$,else we erase line segment $CP1$.This operation keep boundary of O_1, O_2,and new splitting make each part still a 'Y-form' and amount of 'Y-form' doesn't change,and the angle in vertex P_1 is not 180^o anymore,and each vertex is 'good'.So we can go to step 2.

Situation 1.3 : has one and only one of $CP1$, $DP1$ is O_2's boundary. Without loss of generality,let $CP1$ is O_2's boundary.

Situation 1.3.1: $CP1$ is NOT O_1's boundary and DP_1 is O_1s boundary.First we erase edge $CP1$, then we draw a new edge connect CD directly.This operation keep boundary of O_1 ,but let $CD, DP1$ replace $CP1$ become the boundary of O_2 ,new splitting make each part still a 'Y-form' and amount of 'Y-form' doesn't change,and each vertex is 'good',but AP_1B no longer exist. Replace $\angle AP1B$ by $\angle CDP_1$. If operation cut O_2 into 2 part connected by only one vertex D, because O_2 as only one reflex angle D,so 2 part are both convex.then erase the part doesn't contain O_1 and go to step 1. else go to step 2.

Situation 1.3.2: otherwise, First we erase edge DP_1 then we draw a new edge connect CD directly.This operation keep boundary of O_1 , O_2 ,and new splitting make each part still a 'Y-form' and amount of 'Y-form' doesn't change,and the angle in vertex P_1 is not 180^o anymore,and each vertex is 'good'.So we can go to step 2.

Situation 2: angle in vertex V ($V \neq P_1$) reach 180^0 (denote the angle as $\angle EVF$). First we draw a new line segment connect EF directly,then if EV is edge of O_1 we erase line segment FV ,else we erase line segment EV .This operation keep bundary of O_1 O_2 ,New splitting make each part still a 'Y-form' and amount of 'Y-form' doesn't change,and the angle V is not 180^o anymore,and each vertex is 'good'.So we can go to step 2.



The fig above is about operation of reconnect the vertex.

Only step 3 situation 1.1 and situation 1.3.1 can go to step 1,from step 3 situation 1.1 to step 1,the amount of 'Y-form' decrease 1,from step 3 situation 1.3.1 to step 1,the amount of ‘Y-form’ decrease at least 1.So step 1 execute limit times.

Without step 1,only step 3 situation 1.3.1 can change O_2’s boundary. if step 3 situation 1.3.1 happen limit times,the total length of O_2’s boundary is limit,each operation in step 3 will increase one edge’s length at least the shortest edge’s length.and the shortest edge’s length does not decrease.And the amount of edge is fixed,and the maximum length of edge less then length of O_2’s boundary,So operation in step 3 execute limit times,So movement will finish in limit step.

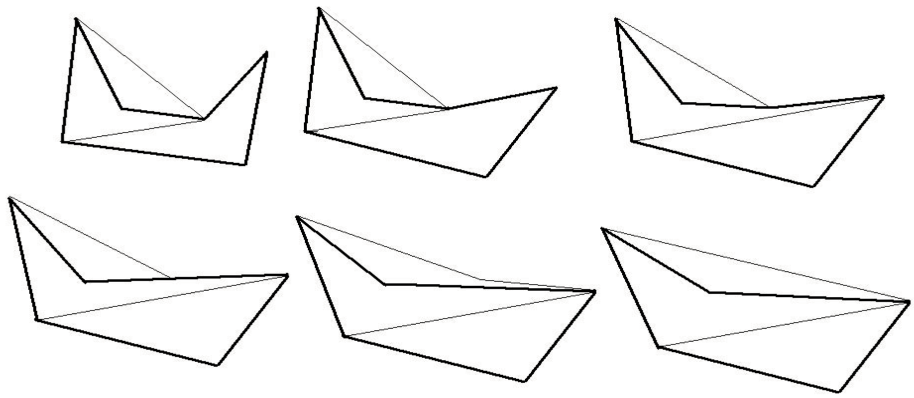
Situation 1.3.1 replace $\angle AP_1B$ by another angle $CDP1$ which $DP1$ is O_1’s boundary.

If vertex D has a reflex angle in O_1 (denote as $\angle JDL$),then before CD become a edge of 180^0 angle, $\angle JDL$ will become 180^0 first.then by Situation 1.3.2,operation will let vertex D detach O_2 and every operation in step 3 will not attach a vertex to O_2.So Situation 1.3.1 will not happen again.

Else vertex D has NO reflex angle in O_1. denote the other edge on D as K D .Because D is ‘good’,so $\angle CDK > \angle CDP1$,So $\angle CDP1$ will not reach 180^0 before $\angle CDK$ reach 180^0 ,So as situation 1.3.1 happen again,the reflex angle of O_2 transfer from D to K.

Denote the transfer route of O_2’s reflex angle as P_1, P_2, P_3.....Transfer will terminate when reach a P_i which has a reflex angle in O_1 or the transfer cut O_2 into 2 part connected only one public vertex P_i.from previous description,step 3 situation 1.3.1 will not happen before execute step 1.so situation 1.3.1 execute limit times between two execution of step 1.

Above all,the movement will stop in limit step and make O_1 become convex.If we give a positive number to $\frac{d\angle AP_1B}{dt}$,the movement will stop in limit time.



Now we shall prove distance between each 2 point in O_1 will not decrease by movement above. Denote the whole broken line which is on the boundary of O_2 and not on the boundary of O_2's convex hull as l.Let the vertex on l arrange as $P_2, Q_1, Q_2, ..., Q_t, P_1, Q_{t+1}, Q_{t+2}, ..., Q_S, P_3$,here $P_j\ (j = 1, 2, 3), Q_i(i = 1, 2, ..., s)$ is vertex on O_2 and P_1 is the only reflex angle of O_2, P_2, P_3 is vertex on convex hull of O_2, $Q_i\ (i = 1, 2, ..., s)$ is vertex between P_2, P_3. Suppose A on l between P_1P_2, B on l between P_1P_3,and the vertex arrange as

$A, Q_m, Q_{m+1,} \cdots, Q_t, P_1, Q_{t+1,}, Q_{n-1}, \cdots, Q_n, B$

then

$$\frac{\partial AB}{\partial \angle P_1} = \frac{AP_1 \times BP_1 \sin \angle AP_1\ B}{AB} = \frac{S_{AP_1B}}{AB} \tag{2.52}$$

$$\frac{\partial AB}{\partial \angle Q_i} = \frac{-AQ_i \times BQ_i \sin \angle AQ_i\ B}{AB} = \frac{S_{AQ_1B}}{AB} \tag{2.53}$$

$$AB\frac{dAB}{dt} = S_{AP_1B}\frac{d\angle P_1}{dt} - \sum_{i=m}^{n} S_{AQ_iB}\frac{d\angle Q_i}{dt} \tag{2.54}$$

$$S_{AP_1B} > S_{AQ_iB} \quad (i = m, \cdots, n) \tag{2.55}$$

By lemma 2.5 we have

$$\frac{d\angle P_1}{dt} > \sum_{i=m}^{n} \frac{d\angle Q_i}{dt} \tag{2.56}$$

If $\frac{d\angle P_1}{dt} > 0$,then from lemma 2.5 we have $\frac{d\angle Q_i}{dt} > 0$.Then (2.54), (2.55) and (2.55) lead to

$$\frac{dAB}{dt} > 0 \tag{2.57}$$

So as movement go by AB will not decrease.

Let movement act from t_0 to t_1 ,at moment t_1 give any 2 point K_1, K_2 from O_2 ,at t_1 moment line segment K_1K_2 intersect 'Y-form' at T_1, T_2, ..., T_S ,and T_i in time t0 is T', K_1, K_2 in moment t_0 is K', K' then from theorem 2.1 we have

$$K_1T_1 > K_1^{'}T_1^{'},\quad T_iT_{i+1} > T_i^{'}T_{i+1}^{'}\quad (i = 1,2,\cdots,s-1), T_sK_2 > T_s^{2}K_2^{'} \tag{2.58}$$

so

$$K_1K_2 = K_1T_1 + \sum_{i=1}^{s-1} T_iT_{i+1} + T_sK_2 \geq K_1^{'}T_1^{'} + \sum_{i=1}^{s-1} T_i^{'}T_{i+1}^{'} + T_s^{'}K_2^{'} = K_1^{'}K_2^{'} \tag{2.59}$$

It means distance between $K_1 K_2$ will not decrease from t0 to t1 theorem proved.

3 The conjecture on Jordan curve

Lemma 3.1 *Give a Jordan curve ,There exist a polygon sequence $\{A_n\}, n \in$*N *that vertex are on C and A_n 's vertex are also A_{n+1} 's vertex and suffice for any neighborhood Gp of any point $p \in C$,there exist a N* $\in$ N *that for each m> N,A_m has a vertex in Gp.*

Now we denote vertex of A_n as $p_{n,i}$ $(i = 1, 2, ...M_n)$,here M_s is amount of vertex of A_n .by lemma 3.1,$p_{n,i} \in C$.so we denote $C(\rho_{n,i}) = p_{n,i}$.

Lemma 3.2 *Let $\alpha = \oint |k|\, ds$. Here k is curvature of C. let*

$$\alpha_n = \sum_{i=1}^{M_n} (\pi - \angle P_{n,i})\ (s = 1,2,\cdots)$$

Then we have $\alpha > \alpha_n$.

Proof. If C is of class C^1,then use Cauchy mean value theorem can prove it easily.

Let $G_n(i, t)$ be the position of vertex $p_{n,i}$ at time t when move follow Theorem 2.2,we can fix $G_n(1, t) = (0, 0)$ and set

$$\sum_{i=1}^{Ms}\frac{d\angle P_{s,i}}{dt} = \alpha_s - 2\pi \tag{3.1}$$

Then when $t = 1, \sum \angle P_{n,i} = 2\pi$. It is to say the polygon is convex.

Let $f_n(\rho, t) : (R/2\pi, [0, 1]) \to R^2$ be the movement that:

1) $f_n(\rho, 0) = C(\rho)$;

2) $f_n(\rho_{n,i}, t) = G_n(i, t)$;

3) for each point ρ between 2 vertex $p_{n,i}, p_{n,j}$,we have $|f_n(\rho, t) - f_n(p_{n,i}, t)| = |f_n(\rho, 0) - f_n(p_{n,i}, 0)|$ and $|f_n(\rho, t) - f_n(p_{n,j}, t)| = |f_n(\rho, 0) - f_n(p_{n,j}, 0)|$ for any t.

From condition 2), $|f_n(p_{n,j}, t) - f_n(p_{n,i}, t)| = |f_n(p_{n,j}, 0) - f_n(p_{n,i}, 0).|$ So condition 3) is to say each piece of curve $p_{i,n}p_{i+1,n}, pM_n, n\, p_{1,n}$ is rigid during the movement.The whole movement seems like binding the curve on the polygon.

$\{f_n\}$ keep the curve's length,so it is uniformly bounded.

Form (3.1) and Lemma 3.2 we have $\sum \frac{d\angle p_{s,i}}{dt} < \alpha - 2\pi$. By Theorem 2.2 we have

$\frac{d\angle p_{s,i}}{dt} > 0$. So $\{f_n\}$ is equicontinuous.

By Arzela-Ascoli theorem $\{f_n\}$, there exists a subsequence $\{f_{nk}\}$ that conver -ges uniformly.

Let f be the function that subsequence $\{f_{nk}\}$ converge to,because the conver

-gence is uniformly , f is continuous.

For any $0 \le t_1 < t_2 \le 1$, if there exist 2 point ρ_1, ρ_2 that

$$|f(\rho_1, t_1) - f(\rho_2, t_1)| - |f(\rho_1, t_2) - f(\rho_2, t_2)| = \epsilon > 0 \tag{3.2}$$

Then exist a N_1 that when $k > N_1$

$$|f_{nk}(\rho_1, t_1) - f_{nk}(\rho_2, t_1)| - |f_{nk}(\rho_1, t_2) - f_{nk}(\rho_2, t_2)| > \epsilon/2 \tag{3.3}$$

From lemma 3.1 we can find ρ_3 in ρ_1's neighborhood and ρ_4 in ρ_2's neighborhood suffice $|\int_{\rho_1}^{\rho_3} C(\rho)ds| < \frac{\varepsilon}{16}, \int_{\rho_2}^{\rho_4} C(\rho)ds| < \frac{\varepsilon}{16},$ and for every $k > N_2, \rho_3$ and ρ_4 is vertex of A_{nk} ·then for every $k > N_2$

$$| f_{nk}(\rho_3, t_1) - f_{nk}(\rho_4, t_1) | < | f_{nk}(\rho_3, t_2) - f_{nk}(\rho_4, t_2) | \tag{3.4}$$

let $N_{\max}$ be the maximum one of $N_1.N_2$, for every $k > N_{\max}$,

$$| f_{nk}(\rho_1, t_1) - f_{nk}(\rho_2, t_1) | - | f_{nk}(\rho_1, t_2) - fn_k(\rho_2, t_2) | < \frac{\varepsilon}{4} \tag{3.5}$$

Contradict from (3.3).

So for every $0 \le t_1 < t_2 \le 1$ and ρ_1, ρ_2 we have

$$| f(\rho_1, t_1) - f(\rho_2, t_1) | < | f(\rho_1, t_2) - f(\rho_2, t_2) | \tag{3.6}$$

Let $d(\rho_1, \rho_2, t)$ be the curve length between ρ_1, ρ_2 at time t in $f(\rho_1, t_1)$. The same we can prove.

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