The relationship between the cubic function and their tangents and secants

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November 2011

Abstract

This paper contains seven sections primarily concerning the relationship between a cubic function with its tangents and secants at a point, the properties of the gradients of tangents and secants to a cubic function at a point, the properties and categorization of cubic functions, and a new definition of the cubic functions. We've found some interesting properties such as **Property**Ⅵ: $\sum_{i=1}^{n}\frac{1}{k_i}=0$ where k_i is the slope of the tangent to a polynomial function at one of its zeros. **Property**Ⅸ: $\sum_{i=1}^{n}k_i = f'(x_0)$ where k_i is the slope of the tangent to a polynomial function at $(x_0, f(x_0))$.

The structure of this paper is as follows.

Section 1: we introduce the background, some notations and some preliminary results such as definitions, lemmas and theorems.

Section 2: we investigate the questions of intersection which concern a cubic function and the tangent on this function at a point. These include the intersection point of a cubic function and a tangent and the area of the figure enclosed by the tangent and the graph of the function.

Section 3: we investigate the distance from a point on the graph of a cubic function to a fixed line, and we work out a new definition for the cubic functions.

Section 4: we talk about the symmetry cubic functions.

Section 5: we concentrate on the slopes of the tangents at the zeros for a cubic function. We also prove corollaries for some of the properties by the Vandermonde determinant.

Section 6: we investigate the slope of the line passing through both a point in the x-y plane and a zero of a cubic function. We also prove Corollaries of these properties which concern a polynomial function of degree n .

Section 7: we talk about the relationship between the types of graph for cubic functions and the roots of the cubic equation corresponding to this function.

KEY WORDS: *Cubic function Graph of cubic function Zero Inflection point Turning point Slope Tangent Secant Point of symmetry Area Polynomial function of degree* n *The Vandermonde determinant*

Contents

Section 1

Introduction

1.1 Background

For polynomial functions, we often concentrate on quadratic functions which are quite familiar to us. In this paper, we will mainly investigate cubic functions through derivatives, because the derivative of a cubic function will be a quadratic function. We also use determinants to solve some problems while we are proving the corollary about the case of degree n .

1.2 Basic Notations

1. Let R *denote the set of real numbers.*

2. Let $f'(x)$ *denote the 1st order derivative of* $f(x)$ *and* $f''(x)$ *denote the 2nd order derivative of* $f(x)$.

3. Let "$\sum_{i=1}^{n} x_i$" *denote the sum of the* x_i over $1 \le i \le n$. *That is to say*

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_{n-1} + x_n .$$

Let "$\prod_{i=1}^{n} x_i$" *denote the product of the* x_i over $1 \le i \le n$. *That is to* *say* $\prod_{i=1}^{n} x_i = x_1 \times x_2 \times \cdots \times x_{n-1} \times x_n$

1.3 Preliminary Results

Definition 1.3.1. *A cubic function is a function of the form as follows.*

$$f(x) = ax^3 + bx^2 + cx + d \, (a \neq 0)$$

The domain of it is R *, and the range is* R *.*

Definition 1.3.2. *The inflection point for a one-variable function is a point on the curve of the function at which the curve changes from being concave upwards (positive curvature) to concave downwards (negative curvature). Suppose the coordinate of the inflection point for the function* $f(x)$ *is* $\left(x_{in}, f\left(x_{in}\right)\right)$ *,if f is twice differentiable at* $\left(x_{in}, f\left(x_{in}\right)\right)$ *, then* $f''(x_{in}) = 0$

Definition 1.3.3. *A turning point is a point at which the sign of the derivative changes. For differentiable functions such as cubics, the turning point must have a zero derivative.*

Definition 1.3.4. *A zero of a function is defined as an intersection point of the curve of the function with the horizontal axis.*

Lemma 1.3.1. *A cubic function has a unique inflection point at its point* *of symmetry.*

Proof:

Suppose

$$f(x) = ax^3 + bx^2 + cx + d \ \ (a \neq 0),$$

we get $f''(x) = 6ax + 2b(a \neq 0)$ and its inflection point

$$\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right).$$

For $x \in R$, we have

$$\begin{aligned}
& f\left(-\frac{b}{3a}-x\right)+f\left(-\frac{b}{3a}+x\right) \\
&= a\left(-\frac{b}{3a}-x\right)^3+b\left(-\frac{b}{3a}-x\right)^2+c\left(-\frac{b}{3a}-x\right)+d \\
&\quad +a\left(-\frac{b}{3a}+x\right)^3+b\left(-\frac{b}{3a}+x\right)^2+c\left(-\frac{b}{3a}+x\right)+d \\
&= 2\left[a\left(-\frac{b}{3a}\right)^3+b\left(-\frac{b}{3a}\right)^2+c\left(-\frac{b}{3a}\right)+d\right] \\
&= 2f\left(-\frac{b}{3a}\right).
\end{aligned}$$

Hence, the point of symmetry of a cubic function is its inflection point.

Lemma 1.3.2. *By change of variable of the form* $y = x + k$ *, for constant* k **,** *any cubic may be written in the form*

$$f(x) = ax^3 + mx + n \ \ (a \neq 0).$$

Proof:

Suppose

$$f(x) = ax^3 + bx^2 + cx + d \ \ (a \neq 0),$$

this may be rewritten as

$$f(x)=a\left(x+\frac{b}{3a}\right)^3+m\left(x+\frac{b}{3a}\right)+n\ (a\neq 0),$$

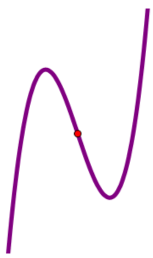
where $m=c-\frac{b^2}{3a}$, $n=d+\frac{2b^3}{27a^2}-\frac{bc}{3a}$. □

This is a very useful conclusion, which can be used to simplify the problems when exploring properties of cubics.

Lemma 1.3.3. *A cubic has either three real roots, or one real root and two complex imaginary roots. In the case of three real roots, either all of them are equal, two of them are equal, or all of them are different.*

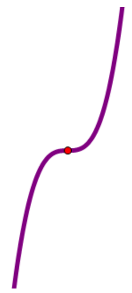
Lemma 1.3.4 *In the following discussion , the graphs of cubics are divided into three different types.*

1) *Example:* $f(x)=x^3-x$.



There are two turning points for this type of function, the cubic equation corresponding to this function may have three different real roots , three real roots with two of them equal or one real root and two complex roots.

2)*Example:* $f(x)=x^3$.



There is a stationary point but no turning point for this type of function, the cubic equation corresponding to this function may have three equal real roots or one real root and two complex roots.

3)*Example:* $f(x) = x^3 + x$.

There are no turning points or stationary points for this type of function. The cubic equation corresponding to this function would have one real root and two complex roots.

Lemma 1.3.5 ***Vieta's*** *theorem for a polynomial equation of degree* n

For $a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0 = 0 \ (a_n \neq 0)$, *this equation has n roots.*

Through factorization, this equation can be reformed as

$$a_n(x - x_1)(x - x_2)\cdots\cdots(x - x_n) = 0 \ (a_n \neq 0),$$

where x_i *are the roots of the equation.*

And we have

$$\begin{cases} \sum_{i=1}^{n} x_i = -\dfrac{a_{n-1}}{a_n} \\ \sum_{1 \le i < j \le n} x_i x_j = \dfrac{a_{n-2}}{a_n} \\ \sum_{1 \le i < j < k \le n} x_i x_j x_k = -\dfrac{a_{n-3}}{a_n} \\ \cdots\cdots \\ x_1 x_2 \cdots\cdots x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}.$$

Section 2

The intersection of a cubic function and a tangent to this function

2.1 Property Ⅰ

This is a property about the intersection points of a tangent to a cubic function with the function itself.

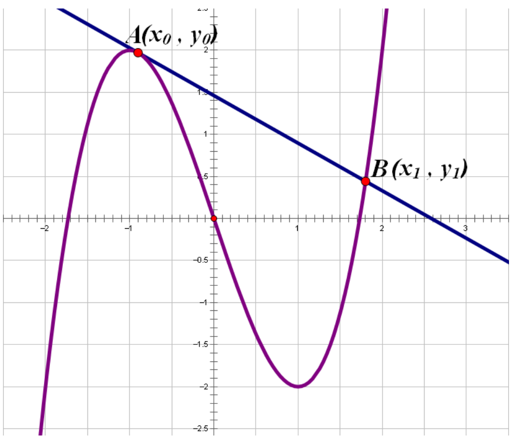
Suppose

$$f(x) = ax^3 + mx \ (a \neq 0).$$

The tangent at any point $(x_0, f(x_0))$ *on this function except the inflection point of this function crosses the graph of the cubic function at exactly two points. One is* $(x_0, f(x_0))$ *, the other is* $(x_1, f(x_1))$.

The relationship between x_0 *and* x_1 is

$$x_1 = -2x_0.$$



Proof:

As all the cubic functions can be written as

$$f(x)=ax^3+mx+n\ (a\neq 0),$$

therefore, it is only needed to prove that it is true for the case

$$f(x)=ax^3+mx+n\ (a\neq 0).$$

Suppose

$$f(x)=ax^3+mx+n\ (a\neq 0),$$

then

$$f'(x)=3ax^2+m\ (a\neq 0).$$

For a point $(x_1, f(x_1))$ on the graph of the function which is not the inflection point, the goal is to find the tangent l_{tg} to the function at $\left(x_0', f(x_0')\right)$ passing through $(x_1, f(x_1))$.We have

$$\begin{cases} l_{tg}:\ y=\left(3ax_0'^2+m\right)x-2ax_0'^3+n, \\ f(x): y=ax^3+mx+n. \end{cases}$$

Since $\begin{cases} x=x_1 \\ y=f(x_1) \end{cases}$ solves the equations above, so we have

$$ax_1{}^3+mx_1=\left(3ax_0'^2+m\right)x_1-2ax_0'^3.$$

For this equation, we know that $x_0'=x_1$ must be one of its solutions. Through factorization, we can get another solution

$$x_0'=x_0=-\frac{1}{2}x_1.$$

（we know that a cubic equation has three solutions, but in this situation, two of its solutions are same: $x_0'=x_0=-\frac{1}{2}x_1$.）

Similarly, if the point $(x_1, f(x_1))$ is the inflection point, which means $x_1 = 0$. In this case the only possible value for x_0' is 0, which means the tangent at the inflection point would pass the function through the tangent point only.

Corollary:

In general, as the point of symmetry of a cubic function $f(x) = ax^3 + bx^2 + cx + d \ (a \neq 0)$ *is* $\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right)$, *we can obtain a more general conclusion:*

The tangent at a point $(x_0, f(x_0))$ *to a cubic function, which is not the inflection point, would pass the graph of the function at* $(x_0, f(x_0))$ *and* $(x_1, f(x_1))$ *,and the relationship between these two points is*

$$x_1 + \frac{b}{3a} = -2\left(x_0 + \frac{b}{3a}\right)$$

2.2 Property Ⅱ

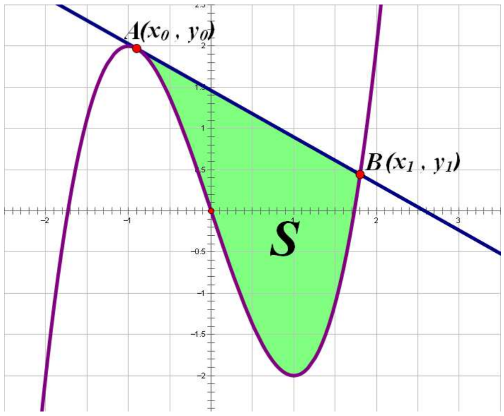
This property talks about the area enclosed by the tangent to the curve of a cubic function and the curve itself.

Suppose

$$f(x) = ax^3 + mx \ (a \neq 0).$$

The tangent at point A $(x_0, f(x_0))$ *on this function except at the point of inflection would pass the graph through another point* B. *The* *area enclosed between the tangent and the cubic function is*

$$S=\left|\frac{27}{4}a{x_0}^4\right|.$$



Proof:

Suppose

$$f(x)=ax^3+mx\ \ (a\neq 0),$$

and the tangent at point $(x_0, f(x_0))$ on this function is

$$l_{tg}: y=\left(3ax_0^2+m\right)x-2ax_0^3.$$

Let

$$F(x)=l_{tg}-f(x),$$

so we have that the area" S "of the closed graph shaped by the tangent and the function is the absolute value of the definite integral of $F(x)$ on $(x_0,-2x_0)$（or $(-2x_0,x_0)$ ）

$$\begin{aligned} S&=\left|\int_{x_0}^{-2x_0}F(x)dx\right|=\left|\int_{x_0}^{-2x_0}\left(ax^3-3ax_0^2x+2ax_0^3\right)dx\right| \\ &=\left|\frac{1}{4}ax^4\Big|_{x_0}^{-2x_0}-\frac{3}{2}ax_0^2x^2\Big|_{x_0}^{-2x_0}+2ax_0^3x\Big|_{x_0}^{-2x_0}\right| \\ &=\left|\frac{27}{4}a{x_0}^4\right|. \end{aligned}$$

Corollary:

In general ,the tangent at a point $(x_0, f(x_0))$ *on the function* $f(x)=ax^3+bx^2+cx+d\ \ (a \neq 0)$ *will enclose an area with the cubic function. This area is given by:*

$$S=\left|\frac{27}{4}a(x_0+\frac{b}{3a})^4\right|.$$

Proof:

For $f(x)=ax^3+bx^2+cx+d\ \ (a \neq 0)$,we can move its graph to the graph of $f(x)=ax^3+mx+n\ \ (a \neq 0)$ and furthermore , to $f(x)=ax^3+mx\ \ (a \neq 0)$. Since area is a translation invariant, we can prove the corollary by proving the case $f(x)=ax^3+mx(a \neq 0)$. As the case $f(x)=ax^3+mx(a \neq 0)$ has already been proved, the corollary is proved as well.

Section 3

A new definition of cubic functions

3.1 Property Ⅲ

By comparing with the polar coordinate representation of conical curves, we enquire as to the distance from a point on the curve to a fixed line which is the tangent at the inflection point for the function.

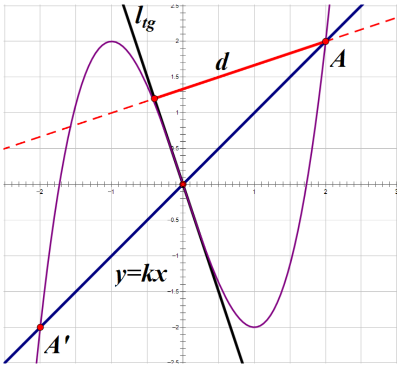
Suppose

$$f(x) = ax^3 + mx \ (am < 0)$$

Let a tangent l_{tg} *pass across the inflection point of the function. For any line* $y = kx(if\ \ a > 0\ \ then\ \ k \geq m.\ if\ \ a < 0\ \ then\ \ k \leq m)$, *the distance "*$d$*" from the intersection point of* $y = kx$ *and* $f(x)$ *to the tangent* l_{tg} is

$$d = \left(\frac{m-k}{m+\frac{1}{m}} \right)^{\frac{3}{2}} d_0 ,$$

where d_0 *is defined as the distance from the intersection point of* $f(x)$ *and the normal at the inflection point to the tangent* l_{tg}.



Proof:

Suppose

$$f(x) = ax^3 + mx \ (am < 0)$$

and

$$y = kx(if \ \ a > 0 \ \ then \ \ k \geq m. \ if \ \ a < 0 \ \ then \ \ k \leq m),$$

Two intersection points of $f(x)$ and $y = kx$ are

$$(\sqrt{\frac{k-m}{a}}, f\left(\sqrt{\frac{k-m}{a}}\right)) \text{ and } (-\sqrt{\frac{k-m}{a}}, f\left(-\sqrt{\frac{k-m}{a}}\right)).$$

According to the formula of the distance from a point to a line

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} = \frac{\left|a\left(\sqrt{\frac{k-m}{a}}\right)^3\right|}{\sqrt{m^2+1}},$$

taking $k = -\frac{1}{m}$, we obtain

$$d_0 = \frac{\left|a\left(\sqrt{\frac{-\frac{1}{m}-m}{a}}\right)^3\right|}{\sqrt{m^2+1}} = \left|\left(m + \frac{1}{m}\right)\sqrt{-\frac{1}{am}}\right|.$$

Hence

$$d = \left(\frac{m-k}{m+\frac{1}{m}}\right)^{\frac{3}{2}} d_0 .$$

3.2 Property Ⅳ

This property may be used as a definition of cubic functions.

Given a fixed line $y = mx$ *, for any variable line* $y = kx\left(m\times(m-k)>0\right)$ *, there are two special points on this line satisfying the equality*

$$d = p\left(|k-m|\right)^{\frac{3}{2}},$$

where d *is defined as the distance from the special point to the line* $y = mx$ *,and* p *is a positive constant. The set of these points would form the graph of a cubic function.*

Proof:

Suppose (x_0, kx_0) is one of the points on the graph, by the formula of the distance from a point to a line

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}},$$

where $y_0 = kx_0$, $A = m$, $B = -1$, $C = 0$, $d = p\left(|k-m|\right)^{\frac{3}{2}}$.

Thereby

$$|x_0| = p\sqrt{m^2+1}\times\sqrt{|k-m|}.$$

Since

$$k = \frac{y_0}{x_0} \text{ and } |x_0| = p\sqrt{m^2+1}\times\sqrt{|k-m|},$$

we have

$$y_0 = \frac{-\operatorname{sgn}(m)}{p^2(m^2+1)}x_0^3 + mx_0,$$

where $$\mathrm{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}.$$

This means that those points form a graph of a cubic function.

Summarize

According to this property, we have a new representation of a cubic function:

Suppose a fixed line $y=mx$. For any line $y=kx\left(m\times(m-k)>0\right)$, there are two special points on this variable line satisfying the equality

$$d=p\left(|k-m|\right)^{\frac{3}{2}},$$

where d is defined as the distance from the special point to the line $y=mx$, and p is a constant number. All these points would form the graph of a cubic function, whose point of symmetry is the origin. This can be seen as a new definition of a cubic function.

Comparing with the polar coordinate representation of conical curves, this representation has more confining conditions. This is possibly a result of the fact cubic functions do not form closed graphs, while conical curves are.

Section 4

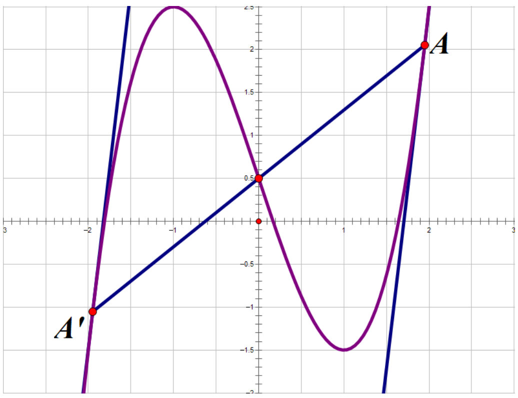
The symmetry of a cubic function

4.1 Property Ⅴ

Suppose

$$f(x) = ax^3 + bx^2 + cx + d (a \neq 0).$$

If there are two different points for which the derivatives are equal to each other on the function, then the line passing the two points will pass through the inflection point for this function.



Proof :

Suppose

$$f(x) = ax^3 + bx^2 + cx + d \ \ (a \neq 0),$$

then

$$f'(x) = 3ax^2 + 2bx + c \ \ (a \neq 0).$$

If the derivative of the function at point A is equal to the derivative at point B , which means

$$f'(x_A) = f'(x_B).$$

Since $f'(x)$ is symmetrical about the axis $x = -\frac{b}{3a}$, we obtain that

$$\frac{x_A + x_B}{2} = -\frac{b}{3a}.$$

Because $\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right)$ is the point of symmetry of this cubic function, the x coordinate of A and B are symmetrical about the point of symmetry.

Thus the line passing the two points must pass through the point of symmetry of this cubic function. □

This property is the result of the fact that an odd function has a even function as its derivative function.

Section 5

The tangents to a cubic function at its zeros

5.1 Property Ⅵ

This property concentrates on the slopes of the tangents at each zero for a cubic function. We also prove a corollary for the polynomial function of degree n by the Vandermonde determinant.

Given a cubic function

$$f(x) = ax^3 + bx^2 + cx + d \ (a \neq 0),$$

if there are three different zeros for this function

$$A\ (x_1, 0)\ , B\ (x_2, 0), C\ (x_3, 0),$$

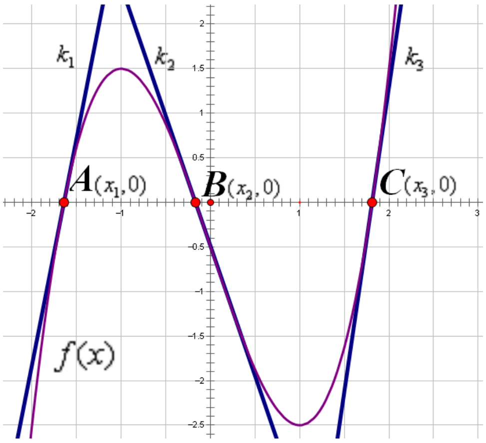
then we have

$$\sum_{i=1}^{3} \frac{1}{k_i} = 0$$

and

$$\sum_{i=1}^{3} \frac{x_i}{k_i} = 0,$$

where $k_i (1 \le i \le 3)$ *is the slope of the tangent at* $(x_i, 0)(1 \le i \le 3)$.



Proof:

Since by assumption, there are three different zeros

$$A\ (x_1,0),\ B\ (x_2,0),\ C\ (x_3,0),$$

then

$$f(x) = ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3).$$

Thus we obtain

$$f'(x) = a(x - x_1)(x - x_2) + a(x - x_1)(x - x_3) + a(x - x_2)(x - x_3).$$

Because $x_1 \neq x_2 \neq x_3$,

it follows

$$\frac{1}{k_1} = \frac{1}{a(x_1 - x_2)(x_1 - x_3)}\ ,$$

$$\frac{1}{k_2} = \frac{1}{a(x_2 - x_1)(x_2 - x_3)}\ ,$$

$$\frac{1}{k_3} = \frac{1}{a(x_3 - x_1)(x_3 - x_2)}\ ,$$

and

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{a(x_1 - x_2)(x_1 - x_3)} + \frac{1}{a(x_2 - x_1)(x_2 - x_3)} + \frac{1}{a(x_3 - x_1)(x_3 - x_2)}$$

$$= \frac{(x_2 - x_3) - (x_1 - x_3) + (x_1 - x_2)}{a(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$= \frac{0}{a(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$= 0 \cdot$$

Likewise

$$\sum_{i=1}^{3} \frac{x_i}{k_i} = \frac{x_1}{a(x_1 - x_2)(x_1 - x_3)} + \frac{x_2}{a(x_2 - x_1)(x_2 - x_3)} + \frac{x_3}{a(x_3 - x_1)(x_3 - x_2)}$$

$$= \frac{x_1(x_2 - x_3) - x_2(x_1 - x_3) + x_3(x_1 - x_2)}{a(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$= 0 \cdot$$

Corollary:

For any polynomial function of degree n with n different zeros ,the equation

$$\sum_{i=1}^{n} \frac{1}{k_i} = 0$$

holds,

where $k_i (1 \le i \le n)$ *is defined as the slope of the tangent to this function at one of the zeros* $(x_i, 0)(1 \le i \le n)$.

Definition 5.1.3.1 *A matrix is a rectangular array of numbers. The individual items in a matrix are called its elements or entries.*

Definition 5.1.3.2 *The determinant is a value associated with an n-by-n matrix.The determinant of a matrix A is denoted as det(A), det A, or |A|.*

In the case where the matrix entries are written out in full, the determinant is denoted by surrounding the matrix entries by vertical bars instead of the brackets or parentheses of the matrix. For instance, the determinant of the matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ is written } \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Lemma 5.1.3.1 Laplace's *formula*

Laplace's formula expresses the determinant of a matrix in terms of its minors. The minor M_{ij} *is defined to be the determinant of the (n−1)×(n−1)-matrix that results from A by removing the* i^{th} *row and the* j^{th} *column. The expression* $(-1)^{i+j} M_{ij}$ *is known as the cofactor of* a_{ij}*.* a_{ij} *is defined as the entry from the* i^{th} *row and* j^{th} *column of A. The determinant of A is given by*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} .$$

Lemma 5.1.3.2 *The **Vandermonde** determinant*

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} \left(x_i - x_j \right).$$

Lemma 5.1.3.3 *If two or more columns or rows of a determinant are equal，then the value of the determinant is equal to zero.*

Proof:

Suppose

$$f(x)=a(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)(a\neq 0),$$

then

$$f'(x)=a(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)\times\sum_{i=1}^{n}\frac{1}{x-x_i}.$$

Thus we can obtain

$$\begin{cases} k_1=f'(x_1)=a(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n), \\ k_2=f'(x_2)=a(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n), \\ \quad\cdots \\ k_i=f'(x_i)=a(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n), \\ \quad\cdots \\ k_n=f'(x_n)=a(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1}), \end{cases}$$

therefore,

$$\sum_{i=1}^{n}\frac{1}{k_i}=\frac{1}{a}\times\frac{\sum_{i=1}^{n}\left\{(-1)^{i-1}\times\prod_{1\leq j<k\leq n,j,k\neq i}\left(x_j-x_k\right)\right\}}{\prod_{1\leq i<j\leq n}\left(x_i-x_j\right)}.$$

We only need to prove

$$\sum_{i=1}^{n}\left\{(-1)^{i-1}\times\prod_{1\leq j<k\leq n,j,k\neq i}\left(x_j-x_k\right)\right\}=0.$$

Considering the determinant as follows

$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

According to ***Lemma 5.1.3.3***

$$D = 0.$$

Expanding the last row of D , we obtain

$$D = (-1)^{n+1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_2^{n-3} & x_3^{n-3} & \cdots & x_n^{n-3} \\ 1 & 1 & \cdots & 1 \end{vmatrix} + (-1)^{n+2} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_3 & \cdots & x_n \\ x_1^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-3} & x_3^{n-3} & \cdots & x_n^{n-3} \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$+ \cdots + (-1)^{2n} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_{n-1}^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-3} & x_2^{n-3} & \cdots & x_{n-1}^{n-2} \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

According to the ***Vandermonde*** *determinant*

$$D=(-1)^{n+1}\prod_{2\le j<k\le n}\left(x_k-x_j\right)+(-1)^{n+2}\prod_{1\le j<k\le n,j,k\ne 1}\left(x_k-x_j\right)+(-1)^{n+3}\prod_{1\le j<k\le n,j,k\ne 2}\left(x_k-x_j\right)$$

$$+\cdots+(-1)^{2n}\prod_{1\le j<k\le n-1}\left(x_k-x_j\right)$$

$$=(-1)^{n+1}(-1)^{\binom{n-1}{2}}\sum_{i=1}^{n}\left\{(-1)^{i-1}\times\prod_{1\le j<k\le n,j,k\ne i}\left(x_j-x_k\right)\right\},$$

where $\begin{pmatrix} n-1 \\ 2 \end{pmatrix}$ is the number of ways of selecting 2 things from (n-1) .

$$\sum_{i=1}^{n}\left\{(-1)^{i-1}\times\prod_{1\le j<k\le n,j,k\ne i}\left(x_j-x_k\right)\right\}=0.$$

5.2 Property Ⅶ

A geometric property about the normals to a cubic function at zeros.

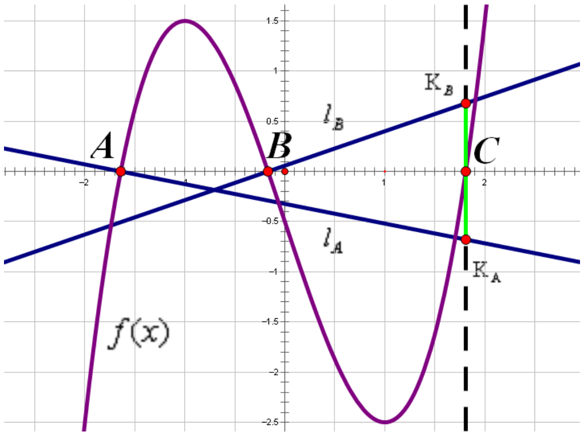
Suppose we have

$$f(x)=ax^3+bx^2+cx+d(a\ne 0).$$

If there are three different zeros for this function

$$A\ (x_1,0)\quad B\ (x_2,0)\quad C\ (x_3,0),$$

choose any two of these three points, such as A *and* B *, and we denote the normals at these two points as* l_A *and* l_B *respectively, then* C *is the middle point of the segment* $\mathrm{K_A K_B}$, *where* $\mathrm{K_A}$ *is defined as the intersection point of* l_A *and the vertical line passing through* C, *with* K_B *defined in the same way.*



Proof:

Given a cubic function $f(x)$ with three different zeros for this function. The zeros are

$$A\ (x_1,0),\ B\ (x_2,0),\ C\ (x_3,0).$$

Let k_i to be the slope of the tangent at $(x_i,0)$.

By the **Property** Ⅵ, we have

$$\sum_{i=1}^{3}\frac{1}{k_i}=0$$

and

$$\sum_{i=1}^{3}\frac{x_i}{k_i}=0.$$

Thus

$$x_3\times\sum_{i=1}^{3}\frac{1}{k_i}-\sum_{i=1}^{3}\frac{x_i}{k_i}=0$$

$$\Rightarrow\quad \frac{x_3-x_1}{k_1}=\frac{x_2-x_3}{k_2}.$$

Since k is the slope of the tangent at a point, $-\frac{1}{k}$ is the slope of the normal at the same point. Note that $|x_3-x_1|$ is the length of AC and

$\left|x_2 - x_3\right|$ is the length of BC, so we can obtain $\left|\frac{x_3 - x_1}{k_1}\right|$ is the length of CK_A and $\left|\frac{x_2 - x_3}{k_2}\right|$ is the length of CK_B

$$\frac{x_3 - x_1}{k_1} = \frac{x_2 - x_3}{k_2}$$
$$\Rightarrow \left|\frac{x_3 - x_1}{k_1}\right| = \left|\frac{x_2 - x_3}{k_2}\right|.$$

Thus the length of CK_A is equal to the length of CK_B.

For any sequence of positions for A , B and C, points K_A, K_B are always on the opposite sides of C , along with the fact that the length of CK_A is equal to the length of CK_B ,then we obtain that C is the midpoint of the segment K_AK_B.

5.3 Property Ⅷ

Though the zeros change when we move by translation the graph of the cubic function, the sum of the three slopes of the tangents at each zero will be a constant number unless we change the curve.

Suppose we have a cubic function with three zeros.

The equality

$$k_0 + \frac{\sum_{i=1}^{3} k_i}{3} = 0$$

holds,

where k_0 *is defined as the slope of the tangent at the inflection point, and* $k_i (1 \le i \le 3)$ *is the slope of the tangent at the point* $(x_i, 0)$ *which is one of the three zeros.*

Proof:

Suppose

$$f(x) = ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3) \ (a \ne 0).$$

Let $k_i (1 \le i \le 3)$ be the slope of the tangent at $(x_i, 0)$,

$$k = f'(x) = a(x - x_1)(x - x_2) + a(x - x_1)(x - x_3) + a(x - x_2)(x - x_3)$$

$$\sum_{i=1}^{3} k_i = a(x_1 - x_2)(x_1 - x_3) + a(x_2 - x_1)(x_2 - x_3) + a(x_3 - x_1)(x_3 - x_2)$$

$$= a(x_1^2 + x_2^2 + x_3^2) - a(x_1 x_2 + x_2 x_3 + x_1 x_3)$$

$$= a(x_1 + x_2 + x_3)^2 - 3a(x_1 x_2 + x_2 x_3 + x_1 x_3).$$

By ***Vieta's*** *theorem (**Lemma 1.3.5**)*, we can obtain that

$$\sum_{i=1}^{3} k_i = a\left(-\frac{b}{a}\right)^2 - 3a\left(\frac{c}{a}\right)$$

$$= \frac{b^2}{a} - 3c.$$

Since

$$f'(x_0) = 3ax_0^2 + 2bx_0 + c$$

$$= 3a\left(-\frac{b}{3a}\right)^2 + 2b\left(-\frac{b}{3a}\right) + c$$

$$= -\frac{b^2}{3a} + c,$$

where $(x_0, f(x_0))$ is the inflection point, then we have

$$\sum_{i}^{3} k_i = -3f'(x_0).$$

Therefore

$$k_0 + \frac{\sum_{i=1}^{3} k_i}{3} = 0.$$

5.3.3 Corollary:

The zeros for a function are the intersection points of both the curve of the function and $y=0$. *If* $y=0$ *is replaced by* $y = px + q$ *,we can obtain*

$$\frac{\sum_{i=1}^{3} k_i}{3} + f'(x_0) = p,$$

where $(x_0, f(x_0))$ *is the inflection point.*

Proof:

Similarly, by ***Vieta's*** *theorem,* we can prove

$$\sum_{i=1}^{3} k_i = \frac{b^2}{a} - 3c + 3p.$$

$$\begin{aligned} f'(x_0) &= 3ax_0^2 + 2bx_0 + c \\ &= 3a\left(-\frac{b}{3a}\right)^2 + 2b\left(-\frac{b}{3a}\right) + c \\ &= -\frac{b^2}{3a} + c. \end{aligned}$$

Hence

$$\frac{\sum_{i=1}^{3} k_i}{3} + f'(x_0) = p \,.$$

Section 6

The slope of the line passing through an arbitrary point and one of the zeros for a cubic function

6.1 Property Ⅸ

This property concentrates on the slope of the line passing through both a point on the function and a zero for this function.

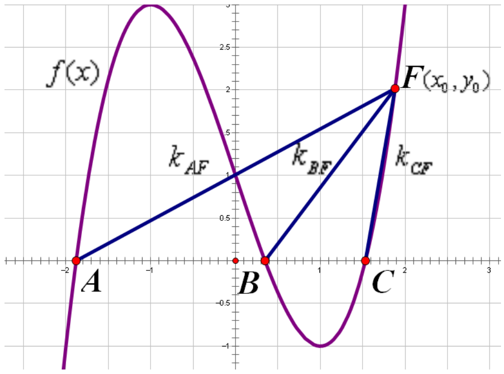
Take a cubic function with three different zeros. For any point $F(x_0, f(x_0))$ *on the graph of the function,*

the equality

$$k_{AF} + k_{BF} + k_{CF} = f'(x_0)$$

holds,

where k_{AF} *is defined as the slope of the line passing through both* F *and* A *, with* k_{BF} *and* k_{CF} *defined similarly. Here* A *,* B *and* C *are the three zeros for this function.*



Proof:

Suppose we have

$$f(x)=ax^3+bx^2+cx+d \ \ (a\neq 0).$$

For any point $F(x_0, f(x_0))$ on the graph of the function, we can denote

$$k_i=\frac{f(x_0)-f(x_i)}{x_0-x_i} \quad (1\leq i\leq 3),$$

where $(x_i,0)(1\leq i\leq 3)$ is a zero of this function, k_i is the slope of the line passing through both $(x_i,0)(1\leq i\leq 3)$ and point F .

1) When $F(x_0, f(x_0))$ is not one of the three zeros.

Since $$x_0-x_i\neq 0,$$

then we get

$$\begin{aligned}k_i&=\frac{f(x_0)-f(x_i)}{x_0-x_i}\\&=\frac{(ax_0{}^3+bx_0{}^2+cx_0+d)-\left(ax_i{}^3+bx_i{}^2+cx_i+d\right)}{x_0-x_i}\\&=\frac{a(x_0-x_i)(x_0^2+x_0x_i+x_i{}^2)+b(x_0-x_i)(x_0+x_i\ )+c(x_0-x_i)}{x_0-x_i}\\&=a(x_0^2+x_0x_i+x_i{}^2)+b(x_0+x_i\ )+c.\end{aligned}$$

Therefore

$$\sum_{i=1}^{3}k_i=3ax_0^2+ax_0\sum_{i=1}^{3}x_i+a\sum_{i=1}^{3}x_i{}^2+3bx_0+b\sum_{i=1}^{3}x_i+3c.$$

By ***Vieta's*** *theorem*

$$\begin{cases} \sum_{i}^{3} x_i = -\frac{b}{a}, \\ \sum_{1 \le i < j \le 3} x_i x_j = \frac{c}{a}, \end{cases}$$

and

$$\sum_{i=1}^{3} x_i^{\ 2} = \left(\sum_{i=1}^{3} x_i \right)^2 - 2 \sum_{1 \le i < j \le 3} x_i x_j ,$$

we have

$$\sum_{i=1}^{3} k_i = 3ax_0^2 + 2bx_0 + c = f'\left(x_0\right).$$

2) When $F(x_0, f(x_0))$ is one of the three zeros, it is obvious that the property holds.

Corollary:

This property holds not only for cubic functions, but also functions of degree n *(with* n *different zeros).*

Suppose we have a polynomial function

$$f(x) = a(x - x_1)(x - x_2)\cdots\cdots(x - x_n) \ (a \ne 0),$$

where $x_1 \ne x_2 \ne \cdots \ne x_n$.

Given a point $F\left(x_0, f\left(x_0\right)\right)$ *on the curve and* $(x_i, 0)$ $(1 \le i \le n, i \in \mathrm{N})$ *is one of the zeros for this function.*

The equality

$$\sum_{i=1}^{n} k_i = f'(x_0)$$

holds,

where k_i $(1 \le i \le n, i \in \mathrm{N})$ *is defined as the slope of the line passing across both* $(x_i, 0)$ *and* $F\left(x_0, f\left(x_0\right)\right)$.

6.2 Property Ⅹ

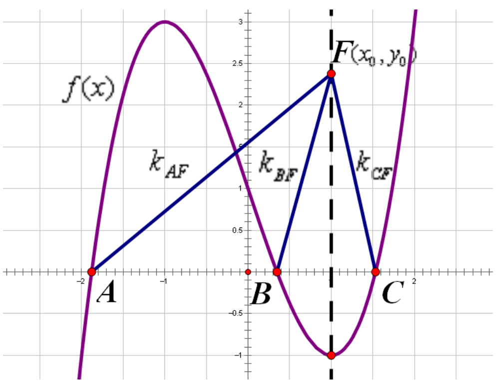
According to **Property** Ⅸ, we have a property about a point on the curve. What about the case of a point not on the curve, like the case of a point on the vertical line passing through a turning point.

Take a cubic function with three different zeros, for any point $F(x_0, y_0)$ *on the vertical line passing through a turning point for the function, the equality*

$$k_{AF} + k_{BF} + k_{CF} = 0$$

holds,

where k_{AF} *is defined as the slope of the line passing through both* F *and* A *, with* k_{BF} *and* k_{CF} *defined similarly. Here* A , B and C *are the three zeros for this function.*



Proof:

Suppose

$$f(x)=ax^3+bx^2+cx+d \ (a\neq 0).$$

Given a point $F(x_0, y_0)$ on the vertical line passing through a turning point and $(x_i,0)(1\le i\le 3)$ is the zero for this function .

Denote

$$k_i=\frac{y_0-f(x_i)}{x_0-x_i}(1\le i\le 3),$$

where k_i is the slope of the line passing through both $(x_i,0)(1\le i\le 3)$ and point F .

Because $f(x_i)=0$, then we get

$$\begin{aligned}\sum_{i=1}^{3}k_i &= y_0\sum_{i=1}^{3}\frac{1}{x_0-x_i}\\ &= y_0\left[\frac{(x_0-x_1)(x_0-x_2)+(x_0-x_2)(x_0-x_3)+(x_0-x_1)(x_0-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right]\\ &= y_0\left[\frac{3x_0^2-2x_0(x_1+x_2+x_3)+(x_1x_2+x_2x_3+x_1x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right].\end{aligned}$$

By

$$\begin{cases}\sum_{i}^{3}x_i=-\dfrac{b}{a},\\ \sum_{1\le i<j\le 3}x_ix_j=\dfrac{c}{a},\end{cases}$$

we can get

$$\sum_{i=1}^{3}k_i=\frac{y_0}{a}\left[\frac{3ax_0^2+2bx_0+c}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right].$$

According to $f'(x)=3ax^2+2bx+c$, we have

$$\sum_{i=1}^{3}k_i=\frac{y_0}{a}\left[\frac{f'(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right].$$

Since $(x_0, f(x_0))$ is a turning point for this function, which means that

$f'(x_0)=0$, therefore

$$\sum_{i=1}^{3} k_i = 0.$$

Corollary:

This property holds not only for cubic functions, but also functions of degree n *(with* n **different** *zeros).*

Suppose we have a polynomial function

$$f(x)=a(x-x_1)(x-x_2)\cdots\cdots(x-x_n)\ (a\neq 0),$$

where $x_1 \neq x_2 \neq \cdots \neq x_n$.

Given a point $F(x_0, y_0)$ *on the vertical line passing through a turning point for this polynomial function and* $(x_i, 0)$ $(1\leq i\leq n, i\in \mathrm{N})$ *is one of the zeros for the function.*

The equality

$$\sum_{i=1}^{n} k_i = 0$$

holds,

where k_i $(1\leq i\leq n, i\in \mathrm{N})$ *is defined as the slope of the line passing through both* $(x_i, 0)$ *and* $F(x_0, y_0)$.

6.3 Property Ⅺ

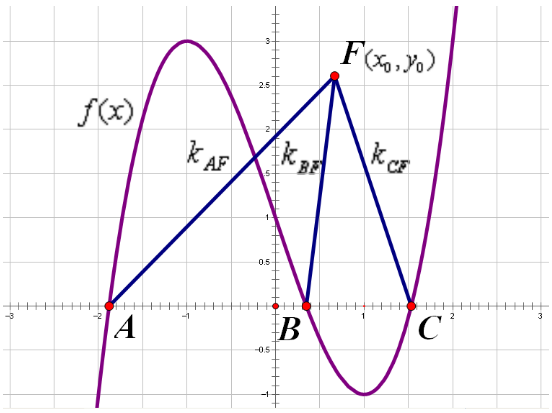
From **Property** Ⅹ, we have a property regarding points on a special line, so we consider the case of a general point in the plane.

Take a cubic function with three different zeros. For any point $F(x_0, y_0)$ *in the plane and* $(x_0, 0)$ *is not any one of the zeros for the function, the equality*

$$k_{AF} + k_{BF} + k_{CF} = y_0 \frac{f'(x_0)}{f(x_0)}$$

holds,

where k_{AF} *is defined as the slope of the line passing through both* F *and* A *, with* k_{BF} *and* k_{CF} *defined similarly. Here* A *,* B *and* C *are the three zeros for the function.*



Proof:

Suppose

$$f(x) = ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3) \ (a \neq 0).$$

Given a point $F(x_0, y_0)$ while $(x_0, 0)$ is not any one of the zeros for the function and $(x_i, 0)(1 \le i \le 3)$ is a zero for the function.

Then we obtain

$$k_i = \frac{y_0 - f(x_i)}{x_0 - x_i} (1 \le i \le 3) \quad ,$$

where k_i is the slope of the line passing through both $(x_i,0)$ and $F(x_0,y_0)$.

Because $f(x_i)=0$, then we get

$$\sum_{i=1}^{3}k_i = y_0\sum_{i=1}^{3}\frac{1}{x_0-x_i}$$

$$= y_0\left[\frac{(x_0-x_1)(x_0-x_2)+(x_0-x_2)(x_0-x_3)+(x_0-x_1)(x_0-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right]$$

$$= y_0\left[\frac{3x_0^2-2x_0(x_1+x_2+x_3)+(x_1x_2+x_2x_3+x_1x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right].$$

We have

$$\begin{cases}\sum_{i}^{3}x_i = -\dfrac{b}{a},\\ \sum_{1\le i<j\le 3}x_ix_j = \dfrac{c}{a},\end{cases}$$

therefore

$$\sum_{i=1}^{3}k_i = \frac{y_0}{a}\left[\frac{3ax_0^2+2bx_0+c}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right].$$

But $f'(x)=3ax^2+2bx+c$,

hence we have

$$\sum_{i=1}^{3}k_i = \frac{y_0}{a}\left[\frac{f'(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right].$$

Since $f(x)=a(x-x_1)(x-x_2)(x-x_3)(a\neq 0)$, we have

$$\sum_{i=1}^{3}k_i = y_0\frac{f'(x_0)}{f(x_0)}.$$

Corollary:

This property holds not only for cubic functions, but also functions of degree n (with n different zeros).

Suppose we have a polynomial function

$$f(x) = a(x - x_1)(x - x_2)\cdots\cdots(x - x_n)\ (a \neq 0),$$

where $x_1 \neq x_2 \neq \cdots \neq x_n$.

Given a point $F(x_0, y_0)$ *in the coordinate plane while* $(x_0, 0)$ *is not any one of the zeros for the function and* $(x_i, 0)$ $(1 \leq i \leq n, i \in \mathrm{N})$ *is one of the zeros for the function.*

The equality

$$\sum_{i=1}^{n} k_i = y_0 \frac{f'(x_0)}{f(x_0)}$$

holds,

where k_i $(1 \leq i \leq n, i \in \mathrm{N})$ *is defined as the slope of the line passing through both* $(x_i, 0)$ *and* $F(x_0, y_0)$.

Summarize

Obviously, **Property** Ⅸ and **Property** Ⅹ are special cases of **Property** Ⅺ. When (x_0, y_0) is on the graph of the function, i.e. $y_0 = f(x_0)$, we can obtain **Property**Ⅸ from **Property**Ⅺ. On the other hand, when (x_0, y_0) is on the vertical line passing through a turning point for the function, i.e. $f'(x_0) = 0$, we can obtain **Property** Ⅹ from **Property**Ⅺ.

Section 7

The division of the graph of a cubic function

7.1 Property Ⅻ

This property concentrates on how to categorize the graphs of a cubic function based on its roots.

If the three roots to a cubic equation are known, one of which is a real root x_0 *, while the others are complex roots* $p \pm qi$*. Then we can deduce the type of the graph of the function corresponding to this equation.*

Proof:

Take

$$f(x) = ax^3 + bx^2 + cx + d(a \neq 0),$$

with roots x_0 , $p \pm qi$.

The inflection point can be obtained as follows

$$\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right).$$

By ***Vieta's*** *theorem*

$$-\frac{b}{3a} = \frac{x_0 + (p+qi) + (p-qi)}{3} = \frac{x_0 + 2p}{3},$$

which is the x coordinate of the inflection point on the coordinate plane.

Then we can obtain

$$f'(x) = 3ax^2 + 2bx + c(a \neq 0),$$

and the x coordinate of the zero for the derivative function is equal to the x coordinate of the turning point for the cubic function.

Thereby

$$x_{turn} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} = \frac{-b}{3a} \pm \frac{\sqrt{\frac{b^2}{a^2} - 3\frac{c}{a}}}{3}.$$

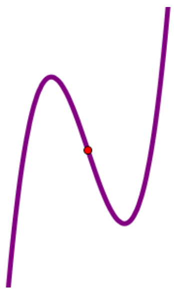
By Vieta' theorem , we can obtain

$$x_{turn} = \frac{-b}{3a} \pm \frac{\sqrt{\left(\sum_{i=1}^{3} x_i\right)^2 - 3\sum_{1 \le i < j \le 3} x_i x_j}}{3}.$$

1） When

$$\left(\sum_{i=1}^{3} x_i\right)^2 - 3\sum_{1 \le i < j \le 3} x_i x_j > 0 \quad \text{or} \quad |x_0 - p| > \left|\sqrt{3}q\right|$$

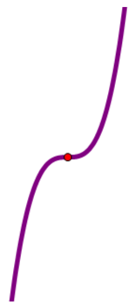
The function has two turning points (such as $f(x) = x^3 - x$).



2） when

$$\left(\sum_{i=1}^{3} x_i\right)^2 - 3\sum_{1 \le i < j \le 3} x_i x_j = 0 \quad \text{or} \quad |x_0 - p| = \left|\sqrt{3}q\right|$$

The function has no turning point but it has a stationary point (such as $f(x) = x^3$).



3）When

$$\left(\sum_{i=1}^{3} x_i\right)^2 - 3\sum_{1\le i<j\le 3} x_i x_j < 0 \quad \text{or} \quad |x_0 - p| < \left|\sqrt{3}q\right|$$

The function has no turning point (such as $f(x) = x^3 + x$).

Appendix

The initial proof of the corollary of Property Ⅵ

Suppose

$$f(x)=a(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)(a\neq 0),$$

then

$$f'(x)=a(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)\times\sum_{i=1}^{n}\frac{1}{x-x_i},$$

we can obtain

$$\begin{cases} k_1=f'(x_1)=a(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n), \\ k_2=f'(x_2)=a(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n), \\ \quad\cdots \\ k_i=f'(x_i)=a(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n), \\ \quad\cdots \\ k_n=f'(x_n)=a(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1}), \end{cases}$$

$$\sum_{i=1}^{n}\frac{1}{k_i}=\frac{1}{a}\times\frac{\sum_{i=1}^{n}\left\{(-1)^{i-1}\times\prod_{1\le j<k\le n,\,j,k\neq i}\left(x_j-x_k\right)\right\}}{\prod_{1\le i<j\le n}\left(x_i-x_j\right)}.$$

According to ***Lemma 5.1.3.2***, the determinant $D_{n/i}$ is defined as follows

$$D_{n/i} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_{i-1}^2 & x_{i+1}^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_{i-1}^{n-2} & x_{i+1}^{n-2} & \cdots & x_n^{n-2} \end{vmatrix} = \prod_{1\le j<k\le n,\, j,k\ne i} \left(x_k - x_j\right),$$

thus

$$\sum_{i=1}^{n} \frac{1}{k_i} = \frac{1}{a} \times \frac{(-1)^{\binom{n-1}{2}} \times \sum_{i=1}^{n} \left[\left(-1\right)^{i-1} \times D_{n/i} \right]}{\prod_{1\le i<j\le n} \left(x_i - x_j\right)},$$

where $\binom{n-1}{2}$ is the number of ways of selecting 2 things from (n-1).

Our goal is to prove

$$\sum_{i=1}^{n} \left[\left(-1\right)^{i-1} \times D_{n/i} \right] = 0.$$

According to induction, suppose for the case $n = k$ the equation holds, then for the case $n = k + 1$:

According to ***Lemma 5.1.3.1***,

$$D_{n/i} = \sum_{1\le j\le k+1,\, j\ne i} x_j^{k-1} \times A_j,$$

where A_j is defined as the cofactor of x_j^{k-1}.

$$\sum_{i=1}^{k+1}(-1)^{i-1}\times D_{n/i}=\sum_{i=1}^{k+1}\left[(-1)^{i-1}\times\sum_{1\le j\le k+1,\,j\ne i}x_j^{k-1}\times A_j\right]$$

$$=\sum_{1\le j\le k+1}\left\{x_j^{k-1}\times\sum_{1\le i\le k+1,i\ne j}\left[(-1)^{i-1}\times A_{j(D_{n/i})}\right]\right\},$$

where $A_{j(D_{n/i})}$ is defined as the cofactor of x_j^{k-1} which is an element of the determinant $D_{n/i}$.

The sum $\sum_{1\le i\le k+1,i\ne j}(-1)^{i-1}A_{j(D_{n/i})}$ is exactly the sum of some $(k\text{-}1)$ order ***Vandermonde*** *determinant* and the number of those determinants is k.

1) while $i>j$

$$A_{j(D_{n/i})}=(-1)^{k+j}\begin{vmatrix}1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_{j-1} & x_{j+1} & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{k+1}\\ x_1^2 & x_2^2 & \cdots & x_{j-1}^2 & x_{j+1}^2 & \cdots & x_{i-1}^2 & x_{i+1}^2 & \cdots & x_{k+1}^2\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ x_1^{k-2} & x_2^{k-2} & \cdots & x_{j-1}^{k-2} & x_{j+1}^{k-2} & \cdots & x_{i-1}^{k-2} & x_{i+1}^{k-2} & \cdots & x_{k+1}^{k-2}\end{vmatrix}$$

2) while $i<j$

$$A_{j(D_{n/i})}=(-1)^{k-1+j}\begin{vmatrix}1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{j-1} & x_{j+1} & \cdots & x_{k+1}\\ x_1^2 & x_2^2 & \cdots & x_{i-1}^2 & x_{i+1}^2 & \cdots & x_{j-1}^2 & x_{j+1}^2 & \cdots & x_{k+1}^2\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ x_1^{k-2} & x_2^{k-2} & \cdots & x_{i-1}^{k-2} & x_{i+1}^{k-2} & \cdots & x_{j-1}^{k-2} & x_{j+1}^{k-2} & \cdots & x_{k+1}^{k-2}\end{vmatrix}$$

Hence we have, when $i = j-1$, the sign of the determinant $A_{j(D_{n/i})}$ is

$$(-1)^{j-2+k-1+j} = (-1)^{2j+k-3},$$

when $i = j+1$, the sign of the determinant $A_{j(D_{n/i})}$ is

$$(-1)^{j+k+j} = (-1)^{2j+k}.$$

So we can prove that the signs of any two of the determinants which are contiguous are opposite.

Let $$x_1, x_2 \cdots x_{j-1}, x_{j+1} \cdots x_{k+1}$$

correspond separately to

$$x_1^j, x_2^j \cdots x_{j-1}^j, x_j^j \cdots x_k^j.$$

Hence

$$D_{n/i}^j = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1^j & x_2^j & \cdots & x_{i-1}^j & x_{i+1}^j & \cdots & x_k^j \\ \left(x_1^j\right)^2 & \left(x_2^j\right)^2 & \cdots & \left(x_{i-1}^j\right)^2 & \left(x_{i+1}^j\right)^2 & \cdots & \left(x_k^j\right)^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \left(x_1^j\right)^{k-2} & \left(x_2^j\right)^{k-2} & \cdots & \left(x_{i-1}^j\right)^{k-2} & \left(x_{i+1}^j\right)^{k-2} & \cdots & \left(x_k^j\right)^{k-2} \end{vmatrix},$$

so

$$\sum_{1 \le i \le k+1, i \ne j} (-1)^{i-1} A_{j(D_{n/i})} = \sum_{i=1}^{k} \left[(-1)^{i-1} \times D_{n/i}^j \right].$$

According to induction, we have

$$\sum_{i=1}^{k} \left[(-1)^{i-1} \times D_{n/i}^j \right] = 0.$$

Therefore

$$\sum_{i=1}^{k+1}\left[(-1)^{i-1}\times D_{n/i}\right]=0 .$$

Clearly, when $n=2$

$$\sum_{i=1}^{2}\left[(-1)^{i-1}\times D_{n/i}\right]=1-1=0 .$$

By induction, we prove that for any $n\geq 2$,

$$\sum_{i=1}^{n}\left[(-1)^{i-1}\times D_{n/i}\right]=0 ,$$

then

$$\sum_{i=1}^{n}\frac{1}{k_i}=\frac{1}{a}\times\frac{(-1)^{\binom{n-1}{2}}\times\sum_{i=1}^{n}\left[(-1)^{i-1}\times D_{n/i}\right]}{\prod_{1\leq i<j\leq n}\left(x_i-x_j\right)}=0 ,$$

and the conclusion is proved.

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Acknowledgements

Thanks for my teachers' help and encouragement, they gave me some suggestions while I was writing this paper. Thanks to my parents and classmates for support and encouragement. Thanks to them, I have the courage and confidence to finish this paper.

There are undoubtedly some mistakes in this paper which remain uncorrected. I hope you can point them out. Thanks a lot.