Generalization of the Topological Sum

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Abstract:
In topology, a number of theorems have the following form: A, B are two subspaces of X. A, B satisfy certain conditions, and then X has a certain property. This paper aims to study the general method of dealing with such problems. We will generalize the concept of topological sum to achieve this goal.

Generalization of the Topological Sum

Note: The regularity axiom and normality axiom in this paper are both stronger than the Hausdorff axiom.

First we show the definition of topological sum in reference 【2】:

Suppose $\{X_\alpha\}_{\alpha\in I}$ **is a family of disjoint spaces.** We define a topology on the set $X=\bigcup_{\alpha\in I} X_\alpha$ **as follows: X's subset O is an open set iff** $\forall\alpha\in I$, $O\cap X_\alpha$ **is an open set of** X_α. Space **X given such a topology is called the topological sum of** $\{X_\alpha\}_{\alpha\in I}$ **.**

Easy to see, this definition has the following deficiencies: the family of spaces is required to be disjoint, which limits most of its applications; more than one nonempty spaces' sum is not connected.

We start the generalization with a lemma.

Lemma 1: $\{X_\alpha\}_{\alpha\in I}$ **is a family of topological spaces meeting the following conditions:**

a. $\forall\alpha\in I$**, only finite** $\beta\in I$ **exist, such that** $X_\alpha\cap X_\beta\neq\varnothing$**.**

b. $\forall\alpha,\ \beta\in I$, $X_\alpha\cap X_\beta$ **is closed in** X_α **, and the induced topologies from** X_α **and** X_β **are the same.**

Then there exists a unique topology on the set $X=\bigcup_{\alpha\in I} X_\alpha$**, such that**

a. $\forall\alpha\in I, X_\alpha$ **is a closed subspace of X.**

b. $\{X_\alpha\}_{\alpha\in I}$ **is locally finite.**

Proof: We define a topology on X *:* A *is a closed set of* X *iff* $\forall\alpha\in I, X_\alpha\cap A$ *is a closed set of* X_α*. Trivial calculation shows it's really a topology. According to condition (a) and (b), a subset of X, called A, is closed iff it can be represented as* $\bigcup_{\alpha\in I} A_\alpha$*, in which* A_α *is closed in* X_α*.*

Conclusion (a) is obvious. We'll verify (b): Suppose $x\in X_\alpha$*, let* N *be the union of spaces disjoint with* X_α*,* $U=X-N$ *, according to condition (a),* U *has nonempty* *intersection with only finite spaces in this family.*

The verification of the uniqueness is given as follows: Let Y and Z be the set X with two topologies meeting conclusions (a) and (b), clearly $id: X_\alpha \to X_\alpha$ *is continuous. According to conclusion (b), the identical map on X,* $id: Y \to Z$ *is continuous. Similarly* $id^{-1}: Z \to Y$ *is also continuous, so id is a homeomorphism between Y and Z. Thus the two topologies are in fact equal.* ■

Note 1: Condition (b) can also be found on wikipedia.

Note 2: The topology we just construct can also be treated as the quotient space of $\{X_\alpha\}_{\alpha\in I}$ **'s disjoint union (with the topology given by the topological sum in the narrow sense).**

As a result, we can easily generalize the topological sum.

Definition 1(Compatibility): A family of spaces $\{X_\alpha\}_{\alpha\in I}$ **meeting the conditions (a) and (b) in Lemma 1 is called to be compatible.**

Definition 2(Topological Sum): $\{X_\alpha\}_{\alpha\in I}$ **is a compatible family of spaces,we call the set** $X=\bigcup_{\alpha\in I} X_\alpha$ **given the topology meeting the conclusions (a) and (b) in Lemma 1 the topological sum(or sum for short) of** $\{X_\alpha\}_{\alpha\in I}$**.We write it as** $X=\bigoplus_{\alpha\in I} X_\alpha$**.In particular, when I is finite({1,2,⋯,n} for example), we also write** $X = X_1\oplus\cdots\oplus X_n$**.The family** $\{X_\alpha\}_{\alpha\in I}$ **is called a topological division of X.**

Obviously, this is the generalization of the concept given earlier.

We first give two basic properties.

Lemma 2: The sum of a compatible family of finite spaces can be put on brackets arbitrarily. Or rather, if $\{X_i\}_{i=1,2,3}$ **is compatible,** $(X_1\oplus X_2)\oplus X_3 = X_1\oplus(X_2\oplus X_3)$.

Lemma 3: $\{X_\alpha\}_{\alpha\in I}$ **is compatible.** $I = J\cup K, J\cap K=\varnothing$**, then**

$\bigoplus_{\alpha\in I} X_\alpha = (\bigoplus_{\alpha\in J} X_\alpha)\oplus(\bigoplus_{\alpha\in K} X_\alpha)$.

They are immediate corollaries from the uniqueness in Lemma 1.

From now on, we always assume that $\{X_\alpha\}_{\alpha\in I}$ is a given compatible family of topological spaces and $X=\bigoplus_{\alpha\in I} X_\alpha$ without special declaration. We always omit the index range (default is I) when it won't lead to confusion. For example, the former formula can be rewritten as $X=\oplus X_\alpha$. We point out that our notation is in fact flawed. After all the sum depends on the topology on X_α.However, in this paper, they won't cause ambiguity, so we'll continue using this notation. In addition, we call one problem regional, if it involves spaces' sum.

We'll use the topological sum in this way: When we're dealing with a problem about a space X, we represent it by the sum of some subspaces, and the properties on X can always become stronger on the subspaces. After dealing with them regionally, we can translate the result into the properties of the space X.

To this aim, we'll study some theorems on the relationship between X and $\{X_\alpha\}$.

1、 Basic concepts

Lemma 4(Closed set):

(a) A is closed in X iff $A=\bigcup A_\alpha$**, in which** $\forall\alpha$**,** A_α **is closed in** X_α**.**

(b) A is closed in X iff $\forall\alpha$**,** $X_\alpha\cap A$ **is closed in** X_α**.**

Lemma 5(Open set): U is open in X iff $\forall\alpha$**,** $U\cap X_\alpha$ **is open in** X_α**.**

They are trivial from the topological construction of X.

Corollary 1: I={1,2}, then X_1-X_2 **is open in X.**

Though it can be proved trivially, this corollary shows the importance of the first half of condition (b) in Lemma 1, which will become clearer when we are dealing with the Hausdorff axiom later.

Lemma 6(Subspace): $Y\subset X$**, then** $\{X_\alpha\cap Y\}$ **is compatible,** $Y=\oplus(X_\alpha\cap Y)$**.**

In the next lemma α and β don't have to be taken from the same set of indices.

Lemma 7(Product space): $\{X_\alpha\}$ **and** $\{Y_\beta\}$ **are both compatible, then** $\{X_\alpha\times Y_\beta\}$ **is compatible,** $\oplus(X_\alpha\times Y_\beta)=\oplus X_\alpha\times\oplus Y_\beta$**.**

These two lemmas are also trivial corollaries of the uniqueness of topological sum.

Lemma 8(Continuous function): $f:X\to Y$ **is continuous iff** $\forall\alpha, f|_{X_\alpha}$ **is continuous.**

Note: The case where I={1,2} is known as the pasting lemma.

2、 Separation axioms

Lemma 8: I={1,2}, $x \in X_1 - X_2$**, for any neighborhood U of x in** X_1**.Let**

$V = U \cap (X_1 - X_2)$**, then V is open in X.**

It's clear from Corollary 1 and Lemma 5.

Lemma 9: x belongs to only one space in the family $\{X_\alpha\}$**, called** X_α**, then there exists a neighborhood of x in X, called U, such that** $\forall \beta \neq \alpha, U \cap X_\beta = \varnothing$**.**

Proof: $U = X - \bigcup_{\beta \neq \alpha} X_\beta$. ■

Theorem 1: X meets T_i **axiom iff** $\forall \alpha$**,** X_α **meets** T_i**.（i=1,2,3）**

Proof：Only the proof of sufficiency with i=2 or i=3 is needed.

Hausdorff axiom: We first consider the case where I is finite. We can assume I={1,2} according to Lemma 2. Suppose $x, y \in X$.

Case 1, neither x and y belongs to $X_1 \cap X_2$ *, x and y belong different sets. Lemma 8 gives the result.*

Case 2, neither x and y belongs to $X_1 \cap X_2$*, x and y belong to the same set,* X_2 *for example. As a subspace of* X_2*,* $X_2 - X_1$ *is Hausdorff. According to Corollary 1, the disjoint neighborhoods in* $X_2 - X_1$ *are also open in X.*

Case 3, only one of x,y belongs to $X_1 \cap X_2$*, they are from the same space. Suppose* $x \in X_1 - X_2$*,* $y \in X_1 \cap X_2$*.Take disjoint neighborhoods of x and y in* X_1*, called U and V. We may also assume that U is also open in X according to Lemma 8. Let* $Z = X - (X_1 - V)$*.Z is clearly to be a neighborhood of y, which is disjoint with U.*

Case 4, both x and y belong to $X_1 \cap X_2$*.Take the disjoint neighborhoods of x,y in* X_i*, called* U_i, V_i *（i=1,2）.Define*

$$U = X - (X_1 - U_1) \cup (X_2 - U_2) \quad V = X - (X_1 - V_1) \cup (X_2 - V_2)$$

Then $U = (U_1 \cap (X_1 - X_2)) \cup (U_1 \cap U_2) \cup (U_2 \cap (X_2 - X_1))$,

$$V = (V_1 \cap (X_1 - X_2)) \cup (V_1 \cap V_2) \cup (V_2 \cap (X_2 - X_1))$$

We assert that U and V are disjoint. Suppose $z \in V$ *, then*

a. $z \in V_1 \cap (X_1 - X_2)$. *Since* $U_1 \cap V_1 = \varnothing$, $(X_1 - X_2) \cap (X_2 - X_1) = \varnothing$, *so* $z \notin U$.

b. $z \in V_2 \cap (X_2 - X_1)$, *the proof is similar.*

c. $z \in V_1 \cap V_2$, *It holds since* $U_1 \cap V_1 = \varnothing$, $U_2 \cap V_2 = \varnothing$.

Hence $U \cap V = \varnothing$ *.X is then Hausdorff.*

The proof for the general case goes as follows. Suppose $x, y \in X$ *, we take their neighborhoods which only intersect with finite* X_α*.The discuss of finite index case, Lemma 3 and Lemma 9 give the result.*

Regularity axiom: Suppose x$\in X$ *, V is a neighborhood of x. Take a neighborhood W of x, which intersects only finite* X_α, $X_1, \cdots, X_n$ *for example. If* $x \in X_i$*,take a neighborhood of x, called* U_i *in* X_i*,which fits* $\overline{U_i} \subset V \cap X_i$*,otherwise take* $U_i = \varnothing$*.Let*

$U = W - \bigcup_{i=1}^{n} (X_i - U_i)$*,and then U is a neighborhood of x and* $\overline{U} \subset V$ *.Considering with the* T_1 *part in this theorem, the result is got.* ■

We can see that topological sum has a very good performance on T_1 axiom, Hausdorff axiom and regularity axiom. In the case of normal axiom, however, it seems to be more complicated, and I cannot give the necessary and sufficient conditions of it, but some sufficient ones with some kind of compact conditions will be obtained easily later.

3、 Covering properties

We'll deduce some nontrivial theorems in this section. As a result, we can get a method to study the properties of paracompact and locally compact spaces.

Theorem 2: X is a paracompact and locally compact space iff X has a topological division $X = \oplus X_\alpha$**, such that** $\forall \alpha$**,** X_α **is compact.**

Proof:

Sufficiency: Suppose that every X_α *is compact and* $\{A_\beta\}_{\beta \in K}$ *is an open covering of X. Then for any given* α, $\{A_\beta \cap X_\alpha\}_{\beta \in K}$ *is an open covering of* X_α*.We may take finite of them to cover* X_α*,written as* $A_{\alpha 1}, \cdots, A_{\alpha m_\alpha}$*.Suppose spaces intersecting with* X_α *in* $\{X_\alpha\}$ *are* $X_1, \cdots, X_n$*.We may require that* $A_{\alpha 1}, \cdots, A_{\alpha m_\alpha}$ *also only intersect with* $X_1, \cdots, X_n$*(Otherwise, let* $A_{\alpha i}$ *minis extra* X_α*).Repeat this for any* $\alpha \in I$ *,and* *we will get a family of* $A_{\alpha i}$*.Suppose* $x \in X$ *,U is a neighborhood of X that only intersects* $X_1, \cdots, X_n$*.Clearly U only intersects finite* $A_{\alpha i}$*.So the family of* $A_{\alpha i}$ *is locally finite, thus X is paracompact. Clearly* $\overline{U}$ *is compact, hence X is also locally compact.*

Necessity: For any $x \in X$ *,take a neighborhood* U_x *with* $\overline{U_x}$ *compact, then* $\{U_x\}_{x \in X}$ *is an open covering of* X*.We may take its locally finite open refinement* $\{A_\beta\}_{\beta \in J}$*.We assert* $X = \bigoplus_{\beta \in J} \overline{A_\beta}$*.Clearly,* $\{\overline{A_\beta}\}_{\beta \in J}$ *is compatible since the locally finite property of* $\{A_\beta\}_{\beta \in J}$ *implies that of* $\{\overline{A_\beta}\}_{\beta \in J}$*.* $\overline{A_\beta}$ *is apparently closed in X. The uniqueness of topological sum implies the result.* ■

Definition 3: Space X is called to be locally Lindelöf, if $\forall x \in X$**, there exists a neighborhood U, such that** $\overline{U}$ **is Lindelöf.**

Theorem 3: X is a paracompact and locally Lindelöf space iff X has a topological division $X = \oplus X_\alpha$**, such that** $\forall \alpha$**,** X_α **is Lindelöf.**

The proof is just trivial modification of Theorem 2.

Theorem 4: X is a paracompact Hausdorff space iff every X_α **is paracompact and Hausdorff.**

Proof: Necessity is clear.

Sufficiency: Suppose $\{U_\beta\}_{\beta \in J}$ *is an open covering of X, then for any given* α*,* $\{X_\alpha \cap U_\beta\}_{\beta \in J}$ *is an open covering of* X_α*. Let* $\{A_{\alpha\gamma}\}_{\gamma \in K}$ *be a locally finite closed refinement which covers* X_α *(Michael's Theorem).Repeat this for every* α*.We assert* $\{A_{\alpha\gamma}\}_{\alpha \in I, \gamma \in K}$ *is a locally finite closed refinement of* $\{U_\beta\}_{\beta \in J}$*.*

Suppose $x \in X$ *,take a neighborhood U which only intersects finite* X_α *,noted as* $X_1, \cdots, X_n$*.Take a neighborhood of x in* X_i*, called* U_i*, which only intersects finite* $A_{i\gamma}$*.Let* $W = U - \bigcup_{i=1}^{n}(X_i - U_i)$ *,and then W only intersect finite* $A_{i\gamma}$*.* ■

Corollary 2: X is a locally Lindelöf paracompact Hausdorff space iff there exists

a topological division $X=\oplus X_{\alpha}$, such that every X_{α} is a regular Lindelöf space.

Theorem 5: X is locally compact iff every X_{α} is locally compact.

Proof: Necessity is clear.

Sufficiency: Suppose $x\in X$,take its neighborhood U which only intersects finite X_{α},noted as $X_1,\cdots,X_n$.Take a neighborhood of x in X_i,called U_i,such that $\overline{U_i}$ is compact. Let $W=U-\bigcup_{i=1}^{n}(X_i-U_i)$, and then W is a neighborhood of x with a compact closure. ■

It's clear that though the normality axiom may not keep very well under the topological sum, it can be kept if it's strengthen into the paracompact Hausdorff condition.

4、 Metric properties

We'll use a lemma from reference【1】,which originally said:

Let X be a compact Hausdorff space that is the union of the closed subspaces X_1,X_2.If X_1,X_2 are metrizable, X is metrizable.

We adapt it into the following form.

Lemma 10: $\{X_1,X_2\}$ is a compatible family of metrizable compact spaces, and then $X=X_1\oplus X_2$ is metrizable.

Theorem 6: X is a metrizable and locally compact space iff every X_{α} is metrizable and locally compact.

Proof: Necessity is clear.

Sufficiency: According to Smirnov's theorem, every X_{α} is a paracompact Hausdorff space, thus X is also one. We only need to prove that X is locally metrizable.

First we consider the finite case, we may suppose I={1,2}.Let X_1 and X_2 both be locally metrizable and locally compact spaces. Suppose $x\in X_1\cup X_2$, take a neighborhood of x in X_i,called U_i,with $\overline{U_i}$ compact and metrizable (If it doesn't exist, we take it to be the empty set),then $\overline{U_1\cup U_2}=\overline{U_1}\oplus\overline{U_2}$ is metrizable(Lemma 10).Let $W=X-(X_1-U_1)\cup(X_2-U_2)$,clearly W is a neighborhood of x and

$W \subset \overline{U_1 \cup U_2}$.Thus W is metrizable, and X is locally metrizable.

The general case can be proved just as that in Theorem 1. ■

We will end this paper with the application of our method in studying a kind of spaces.

Definition 4: X is called to be half-compact, if X is a paracompact and locally compact space.

These are some definitions that may not be well-known: A map is called to be perfect if it's a continuous closed surjection and the preimage of a single point set is compact. A map is called to be keeping the locally finite property of closed sets family(KLC for short in this paper),if the image of any locally finite closed sets family is locally finite. A space X is called to be compactly generated if A is open iff for any compact subspace C in X, $A \cap C$ is open in C.

Theorem 7:

a,A space X is half-compact iff it has a topological division $X = \oplus X_\alpha$,such that every X_α is compact.

b,$f: X \to Y$ is perfect and X is a half-compact Hausdorff space, then Y is also half-compact and Hausdorff.

c,X is a compactly generated space iff X is a quotient space of a half-compact space.

Note:(c) can also be found in reference【2】,in which it's claimed to be Gale's work.

Proof:(a)It's just Theorem 2.

(b)Clearly perfect map keeps the compact property of a space.(In fact, a surjective and continuous map is OK.)

When X is half-compact, let $X = \oplus X_\alpha$ with every X_α compact.A perfect map fits the KLC condition[2],so the image family is also locally finite. It's easy to prove the compatibility from their compact property. Also a perfect map keeps the Hausdorff property.

(c)The sufficiency is trivial.

Necessity: X is clearly homeomorphous with the quotient space of the disjoint union of X's compact spaces. ■

This theorem just shows how we can use the topological sum.

Reference：

【1】J.R.Munkres,Topology
【2】Shou Lin,the topologies on metric spaces and function spaces.