Journal of Combinatorial Theory, Series B 144 (2020) 225–264

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

The Kelmans-Seymour conjecture II: 2-Vertices in K_4^-

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ARTICLE INFO

Article history:
Received 24 February 2016
Available online 11 December 2019

Keywords:
Subdivision of graph
Independent paths
Nonseparating path
Planar graph

ABSTRACT

We use K_4^- to denote the graph obtained from K_4 by removing an edge, and use TK_5 to denote a subdivision of K_5. Let G be a 5-connected nonplanar graph and $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct. We show that G contains a TK_5 in which y_2 is not a branch vertex, or $G - y_2$ contains K_4^-, or G has a special 5-separation, or $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5. This result will be used to prove the Kelmans-Seymour conjecture that every 5-connected nonplanar graph contains TK_5.

1. Introduction

We use notation and terminology from [3]. In particular, for a graph K, we use TK to denote a *subdivision* of K. The vertices in a TK corresponding to the vertices of K are its *branch vertices*. Kelmans [5] and, independently, Seymour [10] conjectured that

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[1] Partially supported by NSF grant through X. Yu.
[2] Partially supported by NSF grants DMS-1265564 and DMS-1600738.

https://doi.org/10.1016/j.jctb.2019.11.007
0095-8956/

every 5-connected nonplanar graph contains TK_5. In [6,7], this conjecture is shown to be true for graphs containing K_4^-. (When we say that a graph G contains another graph H, denoted as $H \subseteq G$, we mean that G has a subgraph isomorphic to H.)

This paper is the second in a series of four papers in which we give a proof of the Kelmans-Seymour conjecture. In the first paper [3] of the series, we outline the strategy of our proof, which we briefly sketch below. Let G be a 5-connected nonplanar graph, and let M be a maximal connected subgraph of G such that G/M (the graph obtained from G by contracting M) is 5-connected and nonplanar. (Note that we allow M to be a single vertex graph.) Let z denote the vertex of G/M representing the contraction of M, and let $H = G/M$. Then one of the following holds:

(a) H contains a subgraph K such that $K \cong K_4^-$ and z has degree 2 in K.
(b) H contains a subgraph K such that $K \cong K_4^-$ and z has degree 3 in K.
(c) H does not contain K_4^- and there exists $T \subseteq H$, with $z \in V(T)$ and either $T \cong K_2$ or $T \cong K_3$, such that H/T is 5-connected and planar.
(d) H does not contain K_4^- and, for any $T \subseteq H$ with $z \in V(T)$ and either $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

We remark that if (c) occurs then by applying Proposition 4.1 in [3] we can conclude that $H - V(T)$ contains K_4^-, and hence, G contains K_4^-; so G contains TK_5 by the main result in [7]. Cases (c) and (d) will be treated in two subsequent papers. In this paper, we deal with (a) by taking advantage of the K_4^- containing z. We prove the following result, in which the vertex y_2 plays the role of z above. Note that for positive integer k, we use the notation $[k]$ for $\{1, \dots, k\}$.

Theorem 1.1. *Let G be a 5-connected nonplanar graph and $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Then one of the following holds:*

(i) *G contains a TK_5 in which y_2 is not a branch vertex.*
(ii) *$G - y_2$ contains K_4^-.*
(iii) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.*
(iv) *For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.*

We now briefly outline how Theorem 1.1 can be used to resolve situation (a); a more precise argument will be given in the final paper of the series. Let G be a 5-connected nonplanar graph, let M be a connected subgraph of G, and let z denote the vertex in G/M representing the contraction of M. Suppose G/M contains a subgraph K such that $K \cong K_4^-$ and z has degree 2 in K. We show that G contains TK_5. Let $V(K) = \{x_1, x_2, y_1, y_2\}$ with $y_1y_2 \notin E(G)$, and we may assume $z = y_2$. We now apply Theorem 1.1, with G/M

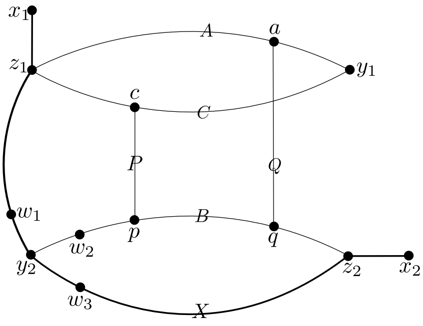

Fig. 1. The paths X, A, B, C, P, Q.

playing the role of G. If (i) of Theorem 1.1 occurs then G/M contains a TK_5 in which y_2 is not a branch vertex; so it is easy to see that G contains TK_5. If (ii) of Theorem 1.1 occurs then $G/M - y_2$ contains K_4^-; so G contains K_4^- and, hence, G contains TK_5 by the main result in [7]. If (iii) of Theorem 1.1 occurs then we may apply Proposition 1.3 in [3] to reduce the problem to a situation similar to the case when (iv) of Theorem 1.1 occurs. Now assume (iv) of Theorem 1.1 occurs. Let M' denote the subgraph of G induced by $M \cup N(M)$, where $N(M)$ is the set of all neighbors of M in G. Since G is 5-connected, there exist $x \in V(M)$ and $w_1, w_2, w_3 \in N(M) - \{x_1, x_2\}$ such that M' has five paths from w to x_1, x_2, w_1, w_2, w_3, with only w in common. Now apply (iv) of Theorem 1.1 with this choice of w_1, w_2, w_3. We can see that a TK_5 in $G/M - \{y_2 v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ can be modified (if necessary) to give a TK_5 in G.

The arguments used in this paper to prove Theorem 1.1 is similar to those used in [6,7]. Namely, we first find a substructure in the graph which consists of eight special paths X, Y, Z, A, B, C, P, Q, see Fig. 1, where X is the path in bold and Y, Z are not shown. We will then attempt to use this intermediate structure to find the desired TK_5 for Theorem 1.1. However, since the TK_5 we are looking for must avoid y_2 as a branch vertex or use certain special edges at y_2, the arguments here are more involved and make heavy use of the option (ii).

We organize this paper as follows. In Section 2, we collect a few known results that will be used in the proof of Theorem 1.1. In Section 3, we find the path X in G between x_1 and x_2 whose deletion results in a graph satisfying certain connectivity requirement. In Section 4, we find the paths Y, Z, A, B, C, P, Q in G and produce the desired intermediate structure in G. In Section 5, we use this structure to find the desired TK_5 for Theorem 1.1.

We end this section with some notation and terminology. Let G be a graph. A *separation* in G consists of a pair of subgraphs G_1, G_2 of G, denoted as (G_1, G_2), such that $E(G_1) \cup E(G_2) = E(G)$, $E(G_1 \cap G_2) = \emptyset$, and neither G_1 nor G_2 is a subgraph of the other. The *order* of this separation is $|V(G_1) \cap V(G_2)|$, and (G_1, G_2) is said to be a *k-separation* if its order is k. When $K \subseteq G$ and $L \subseteq G$, we let $K - L = K - V(K \cap L)$. For $S \subseteq V(G)$, we may view S as a subgraph of G with vertex set S and edge set $\emptyset$. For $H \subseteq G$, $N_G(H)$ denotes the neighborhood of H (not including the vertices in $V(H)$). For any $x \in V(G)$, we use $N_G(x)$ to denote the neighborhood of x in G. When understood, the reference to G may be dropped. We may view paths as sequences of vertices. The *ends* of a path P are the vertices of the minimum degree in P, and all other vertices of P (if any) are its *internal* vertices. A collection of paths is said to be *independent* if no vertex of any path in this collection is an internal vertex of any other path in the collection.

2. Previous results

In this section, we list a few known results that we need. We begin with a technical notion. A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a collection $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

- for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,
- for $i \in [k]$, $|N(A_i)| \leq 3$, and
- if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each $i \in [k]$) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc with no crossing edges.

If, in addition, $b_1, \ldots, b_n$ are vertices in G such that $b_i \notin A_j$ for all $i \in [n]$ and $j \in [k]$, $p(G, A)$ can be drawn in a closed disc in the plane with no crossing edges, and $b_1, \ldots, b_n$ occur on the boundary of the disc in this cyclic order, then we say that $(G, \mathcal{A}, b_1, \ldots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_1, \ldots, b_n)$ is *3-planar*.

Let G be a graph and $A \subseteq V(G)$, and let k be a positive integer. Recall from [3] that G is (k, A)-*connected* if, for any cut T of G with $|T| < k$, every component of $G - T$ contains a vertex from A. Suppose G is $(4, \{b_1, \ldots, b_n\})$-connected. If $(G, \mathcal{A}, b_1, \ldots, b_n)$ is *3-planar* then $\mathcal{A} = \emptyset$. In this case, we say that $(G, b_1, \ldots, b_n)$ is *planar*, and if we do not wish to specify the order of $b_1, \ldots, b_n$ then we say that $(G, \{b_1, \ldots, b_n\})$ is *planar*.

We can now state the following result of Seymour [11]; equivalent versions can be found in [1,13,12].

Lemma 2.1. *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then exactly one of the following holds:*

(i) G contains disjoint paths from s_1, s_2 to t_1, t_2, respectively.
(ii) (G, s_1, s_2, t_1, t_2) is 3-planar.

We also state a generalization of Lemma 2.1, which is a consequence of Theorems 2.3 and 2.4 in [9].

Lemma 2.2. *Let G be a graph, $v_1, \ldots, v_n \in V(G)$ be distinct, and $n \geq 4$. Then exactly one of the following holds:*

(i) There exist $1 \leq i < j < k < l \leq n$ such that G contains disjoint paths from v_i, v_j to v_k, v_l, respectively.
(ii) $(G, v_1, v_2, \ldots, v_n)$ is 3-planar.

Note that the outcomes (i), (ii) and (iii) in the next three lemmas are essentially the same. The first of these three lemmas is Theorem 1.1 in [3].

Lemma 2.3. *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:*

(i) G contains a TK_5 in which a is not a branch vertex.
(ii) $G - a$ contains K_4^-.
(iii) G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$, $G_1 \subseteq G_1'$, and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.

The next result we need is Theorem 1.2 from [3].

Lemma 2.4. *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

(i) G contains a TK_5 in which a is not a branch vertex.
(ii) $G - a$ contains K_4^-.
(iii) G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$ and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.
(iv) For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains TK_5.

We also need Proposition 4.2 from [3].

Lemma 2.5. *Let G be a 5-connected nonplanar graph and $a \in V(G)$ such that $G - a$ is planar. Then one of the following holds:*

(i) G contains a TK_5 in which a is not a branch vertex.
(ii) $G - a$ contains K_4^-.

(*iii*) *G has a* 5*-separation* (G_1, G_2) *such that* $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ *and* G_2 *is the graph obtained from the edge-disjoint union of the 8-cycle* $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ *and the 4-cycle* $b_1b_2b_3b_4b_1$ *by adding* a *and the edges* ab_i *for* $i \in [4]$.

We will make use of the following result of Perfect [8] on independent paths. A collection of paths in a graph are said to be *independent* if no internal vertex of a path in this collection belongs to another path in the collection.

Lemma 2.6. *Let* G *be a graph,* $u \in V(G)$*, and* $A \subseteq V(G-u)$*. Suppose there exist* k *independent paths from* u *to distinct* $a_1, \ldots, a_k \in A$*, respectively, and otherwise disjoint from* A*. Then for any* $n \geq k$*, if there exist* n *independent paths* $P_1, \ldots, P_n$ *in* G *from* u *to* n *distinct vertices in* A *and otherwise disjoint from* A *then* $P_1, \ldots, P_n$ *may be chosen so that* $a_i \in V(P_i)$ *for* $i \in [k]$.

We will also use a result of Watkins and Mesner [14] on cycles through three vertices.

Lemma 2.7. *Let* G *be a* 2*-connected graph and let* y_1, y_2, y_3 *be three distinct vertices of* G*. Then there is no cycle in* G *containing* $\{y_1, y_2, y_3\}$ *if, and only if, one of the following statements holds:*

(*i*) *There exists a 2-cut* S *in* G *and there exist pairwise disjoint subgraphs* D_{y_i} *of* $G-S$*,* $i = 1, 2, 3$*, such that* $y_i \in V(D_{y_i})$ *and each* D_{y_i} *is a union of components of* $G - S$.
(*ii*) *There exist 2-cuts* S_{y_i} *of* G*,* $i = 1, 2, 3$*,* $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$*, and pairwise disjoint subgraphs* D_{y_i} *of* G*, such that* $y_i \in V(D_{y_i})$*, each* D_{y_i} *is a union of components of* $G - S_{y_i}$*, and* $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_v - \{z\}$ *are pairwise disjoint.*
(*iii*) *There exist pairwise disjoint* 2*-cuts* S_{y_i} *in* G*,* $i = 1, 2, 3$*, and pairwise disjoint subgraphs* D_{y_i} *of* $R - S_{y_i}$ *such that* $y_i \in V(D_{y_i})$*, each* D_{y_i} *is a union of components of* $G - S_{y_i}$*, and* $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ *has precisely two components, each containing exactly one vertex from* S_{y_i} *for* $i \in [3]$.

3. Nonseparating paths

Our strategy to prove Theorem 1.1 is to find the paths X, A, B, C, P, Q in G that form the structure shown in Fig. 1. The goal of this section is to complete the first step: find the path X which is shown in bold-face in Fig. 5. A precise statement of the properties of X is given in outcome (*iv*) of Lemma 3.2. But first we need to prove Lemma 3.1, which will be used in the proof of Lemma 3.2 to modify certain paths to obtain the desired X.

We need the concept of chain of blocks. Let G be a graph and $\{u, v\} \subseteq V(G)$. A *block* in G is either a maximal 2-connected subgraph of G or a subgraph of G induced by a cut edge of G. We say that a sequence of blocks $B_1, \ldots, B_k$ in G is a *chain of blocks* in G from u to v if either $k = 1$ and $u, v \in V(B_1)$ are distinct, or $k \geq 2$, $u \in V(B_1) - V(B_2)$, $v \in V(B_k) - V(B_{k-1})$, $|V(B_i) \cap V(B_{i+1})| = 1$ for $i \in [k-1]$, and $V(B_i) \cap V(B_j) = \emptyset$

for any $i, j \in [k]$ with $|i - j| \geq 2$. For convenience, we also view this chain of blocks as $\bigcup_{i=1}^{k} B_i$, a subgraph of G.

The following result was implicit in [2,4]. Since it has not been stated and proved explicitly before, we include a proof. We need the concept of a bridge. Let G be a graph and H a subgraph of G. Then an *H-bridge* of G is a subgraph of G that is either induced by an edge of $G - E(H)$ with both ends in $V(H)$, or induced by the edges in some component of $G - H$ as well as those edges of G from that component to H.

Lemma 3.1. *Let G be a graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that G is $(4, \{x_1, x_2, y_1, y_2\})$-connected. Suppose there exists a path X in $G - x_1x_2$ from x_1 to x_2 such that $G - X$ contains a chain of blocks B from y_1 to y_2. Then one of the following holds:*

- (i) *There is a 4-separation (G_1, G_2) in G such that $B + \{x_1, x_2\} \subseteq G_1$, $|V(G_2)| \geq 6$, and $(G_2, V(G_1 \cap G_2))$ is planar.*
- (ii) *There exists an induced path X' in $G - x_1x_2$ from x_1 to x_2 such that $G - X'$ is a chain of blocks from y_1 to y_2 and contains B.*

Proof. Without loss of generality, we may assume that X is induced in $G - x_1x_2$. We choose such X that

(1) B is maximal (with respect to subgraph containment),
(2) subject to (1), the smallest size of a component of $G - X$ disjoint from B (if exists) is minimal, and
(3) subject to (2), the number of components of $G - X$ is minimal.

We claim that $G - X$ is connected. For, suppose $G - X$ is not connected and let D be a component of $G - X$ other than B such that $|V(D)|$ is minimal. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, there exist distinct $u, v \in N(D) \cap V(X)$. We choose such u, v that uXv is maximal. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, $uXv - \{u, v\}$ contains a neighbor of some component of $G - X$ other than D. Let Q be an induced path in $G[D + \{u, v\}]$ from u to v, and let X' be obtained from X by replacing uXv with Q. Then B is contained in B', the chain of blocks in $G - X'$ from y_1 to y_2. Moreover, either the smallest size of a component of $G - X'$ disjoint from B' is smaller than the smallest size of a component of $G - X$ disjoint from B, or the number of components of $G - X'$ is smaller than the number of components of $G - X$. This gives a contradiction to (1) or (2) or (3). Hence, $G - X$ is connected.

If $G - X = B$, then (ii) holds with $X' := X$. So assume $G - X \neq B$. By (1), each B-bridge of $G - X$ has exactly one vertex in B. Thus, for each B-bridge D of $G - X$, there exist unique $b_D \in V(D) \cap V(B)$. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, there exist distinct $u_D, v_D \in N(D - b_D) \cap V(X)$, and we choose u_D and v_D such that $u_D X v_D$ is maximal.

We now define a new graph $\mathcal{B}$ such that $V(\mathcal{B})$ is the set of all B-bridges of $G - X$, and two B-bridges of $G - X$, C and D, are adjacent if $u_C X v_C - \{u_C, v_C\}$ contains a neighbor of $D - b_D$ or $u_D X v_D - \{u_D, v_D\}$ contains a neighbor of $C - b_C$. Let $\mathcal{D}$ be a connected subgraph of $\mathcal{B}$. Then $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$ is a subpath of X. Let $S_{\mathcal{D}}$ be the union of $\{b_D : D \in V(\mathcal{D})\}$ and the set of those vertices $b \in V(B)$ with the property that b is contained in some $(B \cup X)$-bridge of G that contains internal vertex of $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$. We may assume that

(4) $|S_{\mathcal{D}}| \geq 3$ for any component $\mathcal{D}$ of $\mathcal{B}$.

For, suppose $\mathcal{B}$ has a component $\mathcal{D}$ such that $|S_{\mathcal{D}}| \leq 2$. Let $u, v \in V(X)$ such that $uXv = \bigcup_{D \in V(\mathcal{D})} u_D X v_D$. Then $\{u, v\} \cup S_{\mathcal{D}}$ is a cut in G. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, $|S_{\mathcal{D}}| = 2$. So there is a 4-separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{u, v\} \cup S_{\mathcal{D}}$, $B + \{x_1, x_2\} \subseteq G_1$, and $D \subseteq G_2$ for $D \in V(\mathcal{D})$. Note that $|V(G_2)| \geq 5$.

Suppose $|V(G_2)| = 5$. Then $\mathcal{D}$ consists of a unique B-bridge of $G - X$, say D. Moreover, $|V(D)| = 2$, and the vertex in $V(D) - V(B)$ has all its neighbors contained $\{u, v, b_D\}$, contradicting the assumption that G is $(4, \{x_1, x_2, y_1, y_2\})$-connected.

Therefore, $|V(G_2)| \geq 6$. Let $S_{\mathcal{D}} = \{b_1, b_2\}$. If G_2 has disjoint paths S_1, S_2 from b_1, u to b_2, v, respectively, then choose S_1 to be induced and let $X' = x_1 X u \cup S_1 \cup v X x_2$; now $B \cup S_2$ is contained in the chain of blocks in $G - X'$ from y_1 to y_2, contradicting (1). So no such two paths exist. Hence, by Lemma 2.1, (G_2, u, b_1, v, b_2) is planar; thus (i) holds. □

Now let $\mathcal{D}$ be a component of $\mathcal{B}$. We claim that

(5) there exist $D \in V(\mathcal{D})$, $w_1, w_2 \in V(u_D X v_D) - \{u_D, v_D\}$, and distinct $b_1, b_2 \in S_{\mathcal{D}}$ such that for each $i \in [2]$, $\{b_i, w_i\}$ is contained in a $(B \cup X)$-bridge of G disjoint from $D - b_D$.

To prove (5), let $V(\mathcal{D}) = \{D_1, \dots, D_k\}$ and assume that $u_{D_1} X v_{D_1}$ is maximal among all $u_{D_i} X v_{D_i}$, $i \in [k]$. Moreover, since $\mathcal{D}$ is connected, we may assume that for each $j \in [k]$, $\{D_1, \dots, D_j\}$ induces a connected subgraph $\mathcal{D}_j$ of $\mathcal{D}$.

Suppose (5) fails. Then for each $j \in [k]$, all paths in G that are from $u_{D_j} X v_{D_j} - \{u_{D_j}, v_{D_j}\}$ to B and internally disjoint from $B \cup X \cup D_j$ must end at a common vertex in $S_{\mathcal{D}}$.

We now apply induction on j to show that $|S_{\mathcal{D}_j}| \leq 2$ for $j \in [k]$. This is clear if $j = 1$; as otherwise we would have (5). Now assume that $|S_{\mathcal{D}_j}| \leq 2$ for some $j \in [k-1]$. By definition, $\mathcal{D}_j$ is connected. Therefore, since $u_{D_1} X v_{D_1}$ is maximal and G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, D_{j+1} must have a neighbor in $u_{D_i} X v_{D_i} - \{u_{D_i}, v_{D_i}\}$ for some $i \in [j]$. Hence, $b_{D_{j+1}} \in S_{\mathcal{D}_j}$. To show that $|S_{\mathcal{D}_{j+1}}| \leq 2$, we also need to consider those vertices $b \in V(B)$ with the property that there exists $w \in V(u_{D_{j+1}} X v_{D_{j+1}}) - \{u_{D_{j+1}}, v_{D_{j+1}}\}$ such that $\{b, w\}$ is contained in some $(B \cup X)$-bridge of G disjoint from

D_{j+1}. Suppose $b \notin S_{\mathcal{D}_j}$ for some choice of b. Then $w \notin V(u_{D_i} X v_{D_i}) - \{u_{D_i}, v_{D_i}\}$ and, hence, D_i has a neighbor in $u_{D_{j+1}} X v_{D_{j+1}} - \{u_{D_{j+1}}, v_{D_{j+1}}\}$; which implies that $b = b_{D_i} \in S_{\mathcal{D}_j}$, a contradiction. Therefore, $b \in S_{\mathcal{D}_j}$ for all choices of such b; so $|S_{\mathcal{D}_{j+1}}| \leq 2$.

Thus, $|S_{\mathcal{D}}| \leq 2$, contradicting (4). So (5) holds. □

Let P denote an induced path in $G[D + \{u_D, v_D\}]$ between u_D and v_D, and let X' be obtained from X by replacing $u_D X v_D$ with P. Clearly, the chain of blocks in $G - X'$ from y_1 to y_2 contains B as well as a path from b_1 to b_2 that is internally disjoint from $D \cup B$. This is a contradiction to (1). ∎

We now show that the conclusion of Theorem 1.1 holds or we can find a path X in G such that $y_1, y_2 \notin V(X)$ and $(G - y_2) - X$ is 2-connected. Note that the outcomes (i), (ii) and (iii) in the next lemma are the same as those in Lemmas 2.3, 2.4 and 2.5.

Lemma 3.2. *Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1 y_2 \notin E(G)$. Then one of the following holds:*

- (i) *G contains a TK_5 in which y_2 is not a branch vertex.*
- (ii) *$G - y_2$ contains K_4^-.*
- (iii) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$ and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_1$ and the 4-cycle $b_1 b_2 b_3 b_4 b_1$ by adding y_2 and the edges $y_2 b_i$ for $i \in [4]$.*
- (iv) *For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2 v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5, or $G - x_1 x_2$ has an induced path X from x_1 to x_2 such that $y_1, y_2 \notin V(X)$, $w_1, w_2, w_3 \in V(X)$, and $(G - y_2) - X$ is 2-connected.*

Proof. First, we may assume that

(1) $G - x_1 x_2$ has an induced path X from x_1 to x_2 such that $y_1, y_2 \notin V(X)$ and $(G - y_2) - X$ is 2-connected.

To see this, we first show that there exists $z \in N(y_1) - \{x_1, x_2\}$ such that $K := (G - x_1 x_2) - y_2 - y_1 z$ has disjoint paths from x_1, y_1 to x_2, z, respectively. For, suppose $z \in N(y_1) - \{x_1, x_2\}$ and K has no disjoint paths from x_1, y_1 to x_2, z, respectively. Then by Lemma 2.1, (K, x_1, y_1, x_2, z) is planar. Let $z_1, z_2 \in N(y_1) - \{x_1, x_2, z\}$. Then, since G is 5-connected and (K, x_1, y_1, x_2, z) is planar, $K - y_1 z_1$ has disjoint paths from x_1, y_1 to x_2, z_2, respective. Hence, z_2 gives the desired choice for z.

Since G is 5-connected, $(G - x_1 x_2) - \{y_1, y_2, z\}$ has a path X from x_1 to x_2. Thus, we may apply Lemma 3.1 to $G - y_2$, X and $B = y_1 z$.

Suppose (i) of Lemma 3.1 holds. Then G has a 5-separation (G_1, G_2) such that $y_2 \in V(G_1 \cap G_2)$, $\{x_1, x_2, y_1, z\} \subseteq V(G_1)$ and $y_1 z \in E(G_1)$, $|V(G_2)| \geq 7$, and $(G_2 - y_2, V(G_1 \cap$

$G_2) - \{y_2\})$ is planar. If $|V(G_1)| \geq 7$ then, by Lemma 2.3, (i) or (ii) or (iii) of this lemma holds. Now assume $|V(G_1)| = 5$. Then $V(G_1 \cap G_2) = \{x_1, x_2, y_1, y_2, z\}$. Moreover, $(G_2 - y_2, x_1, x_2, y_1, z)$ is planar, since $(G - x_1x_2) - y_2 - y_1z$ has disjoint paths from x_1, y_1 to x_2, z, respectively. Hence, $G_1 - y_2$ has a K_4^- or $G - y_2$ is planar; so (ii) of this lemma holds in the former case, and (i) or (ii) or (iii) of this lemma holds in the latter case by Lemma 2.5. Thus we may assume that $|V(G_1)| = 6$. Let $v \in V(G_1 - G_2)$. Then $v \neq y_2$. Since G is 5-connected, v must be adjacent to all vertices in $V(G_1 \cap G_2)$. Thus, $v \neq y_1$ as $y_1y_2 \notin E(G)$. Now $|V(G_1 \cap G_2) \cap \{x_1, x_2, z\}| \geq 2$. Therefore, $G[\{v, y_1\} \cup (V(G_1 \cap G_2) \cap \{x_1, x_2, z\})]$ contains K_4^-; so (ii) holds.

So we may assume that (ii) of Lemma 3.1 holds. Then $(G - y_2) - x_1x_2$ has an induced path, also denoted by X, from x_1 to x_2 such that $(G - y_2) - X$ is a chain of blocks from y_1 to z. Since $zy_1 \in E(G)$, $(G - y_2) - X$ is in fact a block. If $V((G - y_2) - X) = \{y_1, z\}$ then, since G is 5-connected and X is induced in $(G - y_2) - x_1x_2$, $G[\{x_1, x_2, z, y_1\}] \cong K_4$; so (ii) holds. Hence, $V((G - y_2) - X) \neq \{y_1, z\}$ and $(G - y_2) - X$ is 2-connected; thus (1) holds. □

We wish to prove (iv). So let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ and assume that

$$G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$$

does not contain TK_5. We may assume that

(2) $w_1, w_2, w_3 \notin V(X)$.

For, suppose not. If $w_1, w_2, w_3 \in V(X)$ then (iv) holds. So, without loss of generality, we may assume $w_1 \in V(X) - \{x_1, x_2\}$ and $w_2 \in V(G - X)$. Since X is induced in $G - x_1x_2$ and G is 5-connected, w_1 has at least two neighbors in $(G - y_2) - X$; so $(G - y_2) - (X - w_1)$ is 2-connected and, hence, contains independent paths P_1, P_2 from y_1 to w_1, w_2, respectively. Then $w_1Xx_1 \cup w_1Xx_2 \cup w_1y_2 \cup P_1 \cup (y_2w_2 \cup P_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2, a contradiction. □

(3) For any $u \in V(x_1Xx_2) - \{x_1, x_2\}$, $\{u, y_1, y_2\}$ is not contained in any cycle in $G' - (X - u)$.

For, suppose there exists $u \in V(x_1Xx_2) - \{x_1, x_2\}$ such that $\{u, y_1, y_2\}$ is contained in a cycle C in $G' - (X - u)$. Then $uXx_1 \cup uXx_2 \cup C \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices u, x_1, x_2, y_1, y_2, a contradiction. So we have (3). □

Let $y_3 \in V(X)$ such that $y_3x_2 \in E(X)$, and let $H := G' - (X - y_3)$. By (1) and (2), $G' - X$ is 2-connected. Hence, since X is induced in $G - x_1x_2$ and G is 5-connected, y_3 has at least two neighbors in $G' - X$; so H is 2-connected. By (3), no cycle in H contains $\{y_1, y_2, y_3\}$. Thus, we apply Lemma 2.7 to H. In order to treat simultaneously the three outcomes of Lemma 2.7, we introduce some notation (based on those in Lemma 2.7). Let $S_{y_i} = \{a_i, b_i\}$ for $i \in [3]$, such that: if Lemma 2.7(i) occurs we let $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$, and $S_{y_i} = S$ for $i \in [3]$; if Lemma 2.7(ii) occurs then let $a_1 = a_2 = a_3$; and if Lemma 2.7(iii) then $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ belong to different components of $H - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$. Let B_a, B_b denote the components of $H - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ such that for $i \in [3]$, $a_i \in V(B_a)$ and $b_i \in V(B_b)$. Note that $B_a = B_b$ is possible, but only if Lemma 2.7(i) or Lemma 2.7(ii) occurs.

For convenience, let $D_i' := G'[D_{y_i} + \{a_i, b_i\}]$ for $i \in [3]$. We choose the cuts S_{y_i} so that

(4) $D_1' \cup D_2' \cup D_3'$ is maximal.

By (4), if Lemma 2.7(i) occurs then we have $a_1 = a_2 = a_3$ and $V(B_a) = \{a_1\}$, and $b_1 = b_2 = b_3$ and $V(B_b) = \{b_1\}$. Since H is 2-connected, D_i', for each $i \in [3]$, contains a path Y_i from a_i to b_i and through y_i. In addition, since $(G - y_2) - X$ is 2-connected, for any $v \in V(D_3') - \{a_3, b_3, y_3\}$, $D_3' - y_3$ contains a path from a_3 to b_3 through v.

(5) If $B_a \cap B_b = \emptyset$ then $|V(B_a)| = 1$ or B_a is 2-connected, and $|V(B_b)| = 1$ or B_b is 2-connected. If $B_a \cap B_b \neq \emptyset$ then $B_a = B_b$ and $B_a - a_3$ is 2-connected.

First, suppose $B_a \cap B_b = \emptyset$. By symmetry, we only prove the claim for B_a. If $|V(B_a)| = 1$ or B_a is 2-connected then we have (5). So assume $|V(B_a)| > 1$ (hence $|V(B_a)| \geq 3$) and B_a is not 2-connected. Then B_a has a separation (B_1, B_2) such that $|V(B_1 \cap B_2)| \leq 1$. Since B_a is connected and H is 2-connected, $|V(B_1 \cap B_2)| = 1$ and we may assume that, for some permutation ijk of [3], $a_i \in V(B_1) - V(B_2)$ and $a_j, a_k \in V(B_2)$ (or $b_i \in V(B_1) - V(B_2)$ and $b_j, b_k \in V(B_2)$ when Lemma 2.7(ii) occurs). Replacing S_{y_i}, D_i' by $V(B_1 \cap B_2) \cup \{b_i\}, D_i' \cup B_1$, respectively, while keeping $S_{y_j}, D_j', S_{y_k}, D_k'$ unchanged, we derive a contradiction to (4).

Now assume $B_a \cap B_b \neq \emptyset$. Then $B_a = B_b$ by definition, and $a_1 = a_2 = a_3$ by our assumption above for the case when Lemma 2.7(ii) occurs. Note that $B_a - a_3$ is connected (since H is 2-connected). Suppose $B_a - a_3$ is not 2-connected. Then B_a has a 2-separation (B_1, B_2) with $a_3 \in V(B_1 \cap B_2)$. First, suppose for some permutation ijk of [3], $b_i \in V(B_1) - V(B_2)$ and $b_j, b_k \in V(B_2)$. Then replacing S_{y_i}, D_i' by $V(B_1 \cap B_2), D_i' \cup B_1$, respectively, while keeping $S_{y_j}, D_j', S_{y_k}, D_k'$ unchanged, we derive a contradiction to (4). Therefore, by the symmetry between B_1 and B_2, we may assume $\{b_1, b_2, b_3\} \subseteq V(B_1)$. Since G is 5-connected, there exists $rr' \in E(G)$ such that $r \in V(X) - \{y_3, x_2\}$ and $r' \in V(B_2 - B_1)$. Let R be a path in $B_2 - (B_1 - a_3)$ from a_3 to r', and R' a path in $B_1 - a_3$ from b_1 to b_2. Then $(R \cup r'r \cup rXx_1) \cup (a_3Y_3y_3 \cup y_3x_2) \cup a_3Y_1y_1 \cup a_3Y_2y_2 \cup (y_1Y_1b_1 \cup R' \cup b_2Y_2y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices a_3, x_1, x_2, y_1, y_2, a contradiction. □

(6) D_{y_i} is connected for $i \in [3]$.

Suppose D_{y_i} is not connected for some $i \in [3]$, and let D be a component of D_{y_i} not containing y_i. Since G is 5-connected, there exists $rr' \in E(G)$ such that $r \in V(X) - \{x_2, y_3\}$ and $r' \in V(D)$.

Let R be a path in $G[D+a_i]$ from a_i to r', and R' a path from b_1 to b_2 in $B_b - a_3$. Note that R' exists since $|V(B_b)| = 1$ or B_b is 2-connected (by (5)), and R' is independent of R (by the choice of R). By (5), let A_1, A_2, A_3 be independent paths in B_a from a_i to a_1, a_2, a_3, respectively. Note that R' is independent of each A_i, since if $B_a = B_b$ then A_i is trivial. Then $(R \cup r'r \cup rXx_1) \cup (A_1 \cup a_1Y_1y_1) \cup (A_2 \cup a_2Y_2y_2) \cup (A_3 \cup a_3Y_3y_3 \cup y_3x_2) \cup (y_1Y_1b_1 \cup R' \cup b_2Y_2y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices a_i, x_1, x_2, y_1, y_2, a contradiction. □

(7) If $a_1 = a_2 = a_3$ then $N(a_3) \cap V(X - \{x_2, y_3\}) = \emptyset$.

For, suppose $a_1 = a_2 = a_3$ and there exists $u \in N(a_3) \cap V(X - \{x_2, y_3\})$. Let Q be a path in $B_b - a_3$ between b_1 and b_2, and let P be a path in $D'_3 - b_3$ from a_3 to y_3. Note that Q exists since $|V(B_b)| = 1$ or B_b is 2-connected (by (5)). Then $(a_3u \cup uXx_1) \cup (P \cup y_3x_2) \cup a_3Y_1y_1 \cup a_3Y_2y_2 \cup (y_1Y_1b_1 \cup Q \cup b_2Y_2y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices a_3, x_1, x_2, y_1, y_2, a contradiction. □

We may assume that

(8) there exists $u \in V(X) - \{x_1, x_2, y_3\}$ such that $N(u) - \{y_2\} \not\subseteq V(X \cup D'_3)$.

For, suppose no such vertex exists. Then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_3, b_3, x_1, x_2, y_2\}$, $X \cup D'_3 \subseteq G_1$, and $D'_1 \cup D'_2 \cup B_a \cup B_b \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$ since $|N(y_1)| \geq 5$ and $y_1y_2 \notin E(G)$. If $|V(G_1)| \geq 7$ then, by Lemma 2.4 (with y_2, x_1, x_2 as a, a_1, a_2 there, respectively), *(i)* or *(ii)* or *(iii)* or *(iv)* of this lemma holds. So we may assume $|V(G_1)| = 6$. Then $X = x_1y_3x_2$ and $V(D_{y_3}) = \{y_3\}$. Hence, $G[\{x_1, x_2, y_1, y_3\}] \cong K_4^-$; so *(ii)* holds. □

(9) For all $u \in V(X) - \{x_1, x_2, y_3\}$ with $N(u) - \{y_2\} \not\subseteq V(X \cup D'_3)$, $N(u) \cap V(D'_3 - y_3) = \emptyset$.

For, suppose there exist $u \in V(X) - \{x_1, x_2, y_3\}$, $u_1 \in (N(u) - \{y_2\}) - V(X \cup D'_3)$, and $u_2 \in N(u) \cap V(D'_3 - y_3)$. Recall (see before (5)) that $y_i \in V(Y_i)$ for $i \in [3]$ and that there is a path Y'_3 in $D'_3 - y_3$ from a_3 to b_3 through u_2.

Suppose $u_1 \in V(D_{y_i})$ for some $i \in [2]$. Then $D'_i - b_i$ (or $D'_i - a_i$) has a path Y'_i from u_1 to a_i (or b_i) through y_i. If Y'_i ends at a_i then let P_a, P_b be disjoint paths in $B_a \cup B_b$ from a_1, b_3 to a_2, b_{3-i}, respectively; now $Y'_i \cup P_a \cup Y_{3-i} \cup P_b \cup b_3Y'_3u_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So Y'_i ends at b_i. Let P_b, P_a be disjoint paths in $B_a \cup B_b$ from b_1, a_{3-i} to b_2, a_3, respectively. Then $Y'_i \cup P_b \cup Y_{3-i} \cup P_a \cup a_3Y'_3u_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Thus, since $u_1 \in (N(u) - \{y_2\}) - V(X \cup D_3')$, we have $u_1 \in V(B_a \cup B_b)$. Note that when Lemma 2.7(*ii*) occurs, we have $V(B_a) = \{a_1 = a_2 = a_3\}$ and by (7), $u_1 \neq a_3$; so $u_1 \in V(B_b)$. When Lemma 2.7(*ii*) does not occur, we have symmetry between B_a and B_b; so we may assume $u_1 \in V(B_b)$. Note that $u_1 \notin \{a_3, b_3\}$ (as $u_1 \notin V(D_3')$ by definition) and $B_b - a_3$ is 2-connected (by (5) and the fact that $B_b - a_3 = B_b$ if $B_a \cap B_b = \emptyset$). Hence, $B_b - a_3$ has disjoint paths Q_1, Q_2 from $\{u_1, b_3\}$ to $\{b_1, b_2\}$. By symmetry between b_1 and b_2, we may assume Q_1 is between u_1 and b_1 and Q_2 is between b_3 and b_2. Let P be a path in B_a from a_1 to a_2. Note that if $|V(B_a)| = 1$ then $V(P) = \{a_3\}$, and if $|V(B_a)| > 1$ then $B_a \cap B_b = \emptyset$. Hence P is disjoint from Q_1 and Q_2. Then $Q_1 \cup u_1uu_2 \cup u_2Y_3'b_3 \cup Q_2 \cup Y_2 \cup P \cup Y_1$ is a cycle in $G' - (X - u)$ containing $\{y_1, y_2, u\}$, contradicting (3). □

(10) For any $u \in V(X) - \{x_1, x_2, y_3\}$ with $N(u) - \{y_2\} \not\subseteq V(X \cup D_3')$, $\{a_j, b_j\} \not\subseteq N(u)$ for each $j \in [2]$ and there exists $i \in [2]$ such that $(N(u) - \{y_2\}) - V(X) \subseteq V(D_i')$.

First, we show that $\{a_j, b_j\} \not\subseteq N(u)$ for each $j \in [2]$. For, suppose $u_1 = a_j$ and $u_2 = b_j$ for some $j \in [2]$. Then, since $\{u_1, u_2\} \cap \{a_3, b_3\} = \emptyset$, we have $B_a \cap B_b = \emptyset$, $|V(B_a)| \geq 2$, and $|V(B_b)| \geq 2$. Thus by (5), B_a and B_b are 2-connected. So let P_1, P_2 be independent paths in B_a from a_j to a_{3-j}, a_3, respectively, and Q_1, Q_2 be independent paths in B_b from b_i to b_{3-j}, b_3, respectively. Now $ua_j \cup ub_j \cup a_jY_jy_j \cup b_jY_jy_j \cup (y_jx_1 \cup x_1Xu) \cup (P_1 \cup Y_{3-j} \cup Q_1) \cup (P_2 \cup a_3Y_3y_3) \cup (Q_2 \cup b_3Y_3y_3) \cup uXy_3 \cup y_jx_2y_3$ is a TK_5 in G' with branch vertices a_j, b_j, u, y_j, y_3, a contradiction. Thus, $\{a_j, b_j\} \not\subseteq N(u)$ for each $j \in [2]$.

Now let $u_1, u_2 \in (N(u) - \{y_2\}) - V(X \cup D_3')$ be distinct. Note that such u_1, u_2 exist by (9) and the fact that X is induced in $G' - x_1x_2$.

We claim that $\{u_1, u_2\} \not\subseteq V(B_a)$ and $\{u_1, u_2\} \not\subseteq V(B_b)$. Recall that if $B_a \cap B_b \neq \emptyset$ then $B_a = B_b$, and note that if $B_a \cap B_b = \emptyset$ then there is symmetry between B_a and B_b. So if the claim fails we may assume that $u_1, u_2 \in V(B_b)$. Then by (5), $B_b - a_3$ is 2-connected; so $B_b - a_3$ contains disjoint paths Q_1, Q_2 from $\{u_1, u_2\}$ to $\{b_1, b_2\}$. If $B_a = B_b$ then let $P = a_3$; and if $B_a \cap B_b = \emptyset$ then let P be a path in B_a from a_1 to a_2. Now $Q_1 \cup u_1uu_2 \cup Q_2 \cup Y_1 \cup P \cup Y_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $\{u_1, u_2\} \not\subseteq V(B_a)$ and $\{u_1, u_2\} \not\subseteq V(B_b)$.

Now suppose $u_1 \in V(B_a - a_3)$ and $u_2 \in V(B_b - b_3)$. Then $|V(B_a)| \geq 2$ and $|V(B_b)| \geq 2$, and, by the above claim, $B_a \cap B_b = \emptyset$; so by (5), B_a and B_b are 2-connected. Let Y_3' be a path in $D_3' - y_3$ from a_3 to b_3. First, assume that $u_1 \in \{a_1, a_2\}$ or $u_2 \in \{b_1, b_2\}$. By symmetry, we may assume $u_1 = a_1$. So $u_2 \neq b_1$. Since B_a and B_b are 2-connected, $B_a - a_1$ contains a path P from a_2 to a_3, and B_b contains disjoint paths Q_1, Q_2 from $\{b_2, b_3\}$ to b_1, u_2, respectively. Then $Y_1 \cup Q_1 \cup Y_2 \cup P \cup Y_3' \cup Q_2 \cup u_1uu_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $u_1 \notin \{a_1, a_2\}$ and $u_2 \notin \{b_1, b_2\}$. Since B_a is 2-connected, we may assume by symmetry that B_a contains disjoint paths P_1, P_2 from u_1, a_3 to a_1, a_2, respectively. Since B_b is 2-connected, B_b contains disjoint paths Q_1, Q_2 from b_1, u_2, respectively to $\{b_2, b_3\}$. Now $P_1 \cup Y_1 \cup Q_1 \cup Y_2 \cup P_2 \cup Y_3' \cup Q_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Therefore, we may assume $u_1 \in V(D_{y_i})$ for some $i \in [2]$. By symmetry, we may assume that $u_1 \in V(D_{y_1})$ and $D'_1 - a_1$ contains a path R_1 from u_1 to b_1 and through y_1. If $(N(u) - \{y_2\}) - V(X) \subseteq V(D'_1)$ then we have (10). So we may assume $u_2 \notin V(D'_1)$.

Suppose $u_2 \in V(D_{y_2})$. If $D'_2 - a_2$ contains a path R_2 from u_2 to b_2 through y_2 then let Q be a path in B_b from b_1 to b_2; now $R_1 \cup Q \cup R_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $D'_2 - b_2$ contains a path R_2 from u_2 to a_2 and through y_2. Now let P be a path in B_a from a_2 to a_3, Q be a path in $B_b - a_3$ from b_1 to b_3. Let Y'_3 be a path in $D'_3 - y_3$ from a_3 to b_3. Then $R_1 \cup Q \cup Y'_3 \cup P \cup R_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Hence, $u_2 \notin V(D_{y_2})$ and, hence, $u_2 \in V(B_a \cup B_b) - \{a_1, b_1\}$. Suppose $u_2 \in V(B_b)$. Then, by (5), $B_b - a_3$ is 2-connected; so let Q_1, Q_2 be disjoint paths in $B_b - a_3$ from b_1, u_2, respectively, to $\{b_2, b_3\}$. Let P be a path in B_a from a_2 to a_3. Now $u_2uu_1 \cup R_1 \cup Q_1 \cup Q_2 \cup Y_2 \cup P \cup Y'_3$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $u_2 \notin V(B_b)$ and $u_2 \in V(B_a - a_1)$; hence $B_a \cap B_b = \emptyset$. Let P be a path in B_a from u_2 to a_2 and Q be a path in B_b from b_1 to b_2. Then $u_2uu_1 \cup R_1 \cup Q \cup Y_2 \cup P$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). This completes the proof of (10). □

By (10) and by symmetry, let $u \in V(X) - \{x_1, x_2, y_3\}$ and $u_1, u_2 \in N(u)$ such that $u_1 \in V(D_{y_1})$ and $u_2 \in V(D'_1)$. If $G[D'_1 + u]$ contains independent paths R_1, R_2 from u to a_1, b_1, respectively, such that $y_1 \in V(R_1 \cup R_2)$, then let P be a path in B_a between a_1 and a_2 and Q be a path in $B_b - a_3$ between b_1 and b_2; now $R_1 \cup P \cup Y_2 \cup Q \cup R_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So such paths do not exist. Then in the 2-connected graph $D^*_1 := G[D'_1 + u] + \{c, ca_1, cb_1\}$ (by adding a new vertex c), there is no cycle containing $\{c, u, y_1\}$. Hence, by Lemma 2.7, D^*_1 has a 2-cut T separating y_1 from $\{u, c\}$, and $T \cap \{u, c\} = \emptyset$.

We choose u, u_1, u_2 and T so that the T-bridge of D^*_1 containing y_1, denoted as B, is minimal. Then $B - T$ contains no neighbor of $X - \{x_1, x_2\}$. Hence, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, x_2, y_2\} \cup V(T)$, $B \subseteq G_1$, and $X \cup D'_2 \cup D'_3 \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$. Since $y_1y_2 \notin E(G)$ and G is 5-connected, $|V(G_1)| \geq 7$. So (i) or (ii) or (iii) or (iv) of this lemma follows from Lemma 2.4. ∎

4. An intermediate substructure

The objective of this section is to derive the structure in G', as described in Fig. 1, which includes paths X, Y, Z, A, B, C, P, Q. (In the next section, we will use this substructure to find the desired TK_5 in G or G'.) First, we use the path in outcome (iv) of Lemma 3.2 to get the desired path X. We then prove Lemma 4.1 which can be used to find the paths Y, Z. After that we prove Lemma 4.2 which will be used to find the paths A, B, C. The final step in this section is to prove Lemma 4.3 which can be used to find the paths P and Q.

By Lemma 3.2, to prove Theorem 1.1 it suffices to deal with the second part of (iv) of Lemma 3.2. Thus, let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$, let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct, and let P be an induced path in $G - x_1x_2$ from x_1 to x_2 such that $y_1, y_2 \notin V(P)$, $w_1, w_2, w_3 \in V(P)$, and $(G - y_2) - P$ is 2-connected.

Without loss of generality, assume x_1, w_1, w_2, w_3, x_2 occur on P in order. Let

$$X := x_1Pw_1 \cup w_1y_2w_3 \cup w_3Px_2,$$

and let

$$G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}.$$

Then X is an induced path in $G' - x_1x_2$, $y_1 \notin V(X)$, and x_1, w_1, y_2, w_3, x_2 occur on X in order. Moreover, $G' - X$ is 2-connected, since w_2 has at least two neighbors in $(G - y_2) - P$ (as G is 5-connected and P is induced in $G - x_1x_2$). For convenience, we record this situation by calling $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ a 9-*tuple.*

We prove the following result, to be used to find special paths Y, Z in G'.

Lemma 4.1. *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ be a 9-tuple. Then one of the following holds:*

- (i) *G contains a TK_5 in which y_2 is not a branch vertex, or G' contains TK_5.*
- (ii) *$G - y_2$ contains K_4^-.*
- (iii) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.*
- (iv) *There exist $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$, $z_2 \in V(x_2Xy_2) - \{x_2, y_2\}$ such that $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ has disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively.*

Proof. Let K be the graph obtained from $G - \{x_1, x_2, y_2\}$ by contracting $x_iXy_2 - \{x_i, y_2\}$ to the new vertex u_i, for $i \in [2]$. Note that K is 2-connected; since G is 5-connected, X is induced in $G' - x_1x_2$, and $G - X$ is 2-connected. We may assume that

(1) there exists a collection $\mathcal{A}$ of subsets of $V(K) - \{u_1, u_2, w_2, y_1\}$ such that $(K, \mathcal{A}, u_1, y_1, u_2, w_2)$ is 3-planar.

For, suppose this is not the case. Then by Lemma 2.1, K contains disjoint paths, say Y, U, from y_1, u_1 to w_2, u_2, respectively. Let v_i denote the neighbor of u_i in the path U, and let $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ be a neighbor of v_i in G. Then $Z := (U - \{u_1, u_2\}) + \{z_1, z_2, z_1v_1, z_2v_2\}$ is a path between z_1 and z_2. Now $Y + \{y_2, y_2w_2\}, Z$ are the desired paths for (iv). So we may assume (1). □

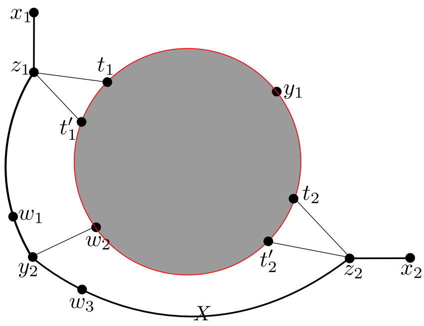

Fig. 2. Structures in a 9-tuple.

Since $G - X$ is 2-connected, $|N_K(A) \cap \{u_1, u_2, w_2\}| \leq 1$ for all $A \in \mathcal{A}$ (by the second property in the definition of 3-planar graphs). Let $p(K, \mathcal{A})$ be the graph obtained from K by (for each $A \in \mathcal{A}$) deleting A and adding new edges joining every pair of distinct vertices in $N_K(A)$. Since G is 5-connected and $G-X$ is 2-connected, we may assume (by the third property in the definition of 3-planar graphs) that $p(K, \mathcal{A}) - \{u_1, u_2\}$ is a 2-connected plane graph, and by (1) (and the final part of the definition of 3-planar graphs for $(K, \mathcal{A}, u_1, y_1, u_2, w_2)$ to be 3-planar), for each $A \in \mathcal{A}$ with $N_K(A) \cap \{u_1, u_2\} \neq \emptyset$ the edge joining the vertices in $N_K(A) - \{u_1, u_2\}$ occur on D, the outer cycle of $p(K, \mathcal{A}) - \{u_1, u_2\}$. Note that $y_1, w_2 \in V(D)$.

Let $t_1 \in V(D)$ with $t_1 D y_1$ minimal such that $u_1 t_1 \in E(p(K, \mathcal{A}))$; and let $t_2 \in V(D)$ with $y_1 D t_2$ minimal such that $u_2 t_2 \in E(p(K, \mathcal{A}))$. (So t_1, y_1, t_2, w_2 occur on D in clockwise order and possibly $t_1 = y_1$ and/or $t_2 = y_1$.) Since K is 2-connected and X is induced in $G' - x_1 x_2$, there exist $z_1 \in V(x_1 X y_2) - \{x_1, y_2\}$ and independent paths R_1, R_1' in G from z_1 to D and internally disjoint from $V(p(K, \mathcal{A})) \cup V(X)$, such that R_1 ends at t_1 and R_1' ends at some vertex $t_1' \neq t_1$, and w_2, t_1', t_1, y_1 occur on D in clockwise order. Similarly, there exist $z_2 \in V(x_2 X y_2) - \{x_2, y_2\}$ and independent paths R_2, R_2' in G from z_2 to D and internally disjoint from $V(p(K, \mathcal{A})) \cup V(X)$, such that R_2 ends at t_2, R_2' ends at some vertex $t_2' \neq t_2$, and y_1, t_2, t_2', w_2 occur on D in clockwise order. (See Fig. 2.)

We may assume that

(2) $K - \{u_1, u_2\}$ has no 2-separation (K', K'') such that $V(K' \cap K'') \subseteq V(t_1 D t_2)$, $|V(K')| \geq 3$, and $V(t_2 D t_1) \subseteq V(K'')$.

For, suppose such a separation (K', K'') does exist in $K - \{u_1, u_2\}$. Then by the definition of t_1 and t_2, we see that G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(K' \cap K'') \cup \{x_1, x_2, y_2\}$, $K' \subseteq V(G_1)$ and $K'' \cup X \subseteq G_2$. Note that $G[\{x_1, x_2, y_2\}]$ is a triangle in G, $|V(G_2)| \geq 7$, and $|V(G_1)| \geq 6$ (as $|V(K')| \geq 3$). If $|V(G_1)| \geq 7$ then by Lemma 2.4, (i) or (ii) or (iii) of this lemma holds. (Note that if (iv) of Lemma 2.4 holds then G' has a TK_5; so (i) of this lemma holds.) So assume $|V(G_1)| = 6$, and let $v \in V(G_1 - G_2)$.

Since G is 5-connected, $N(v) = V(G_1 \cap G_2)$. In particular, $v \neq y_1$ as $y_1y_2 \notin E(G)$. Then $G[\{v, x_1, x_2, y_1\}]$ contains K_4^-, and (ii) holds. So we may assume (2). $\square$

Next we may assume that

(3) each neighbor of x_1 is contained in $V(X)$, or $V(t_1Dy_1)$, or some $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, and each neighbor of x_2 is contained in $V(X)$, or $V(y_1Dt_2)$, or some $A \in \mathcal{A}$ with $u_2 \in N_K(A)$.

For, otherwise, we may assume by symmetry that there exists $a \in N(x_1) - V(X)$ such that $a \notin V(t_1Dy_1)$ and $a \notin A$ for each $A \in \mathcal{A}$ with $u_1 \in N_K(A)$.

First, we define vertex a' and a path S. Let $a' = a$ and $S = a$ if $a \notin A$ for all $A \in \mathcal{A}$. Now suppose $a \in A$ for some $A \in \mathcal{A}$. Then there exists $a' \in N_K(A) - V(t_1Dt_2)$; for otherwise $N_K(A) \subseteq V(t_1Dt_2)$ and two vertices in $N_K(A)$ would form a cut in $K - \{u_1, u_2\}$ separating a from $V(t_2Dt_1)$, contradicting (2). So $G[A + a']$ contains a path S from a to a'.

Note that, by (2), there is a path T from a' to some $u \in V(t_2Dt_1) - \{t_1, t_2\}$ in $p(K, \mathcal{A}) - \{u_1, u_2, y_2\} - t_1Dt_2$. Recall the paths R_i, R_i', $i \in [2]$, defined before (2). Then $t_1Dt_2 \cup R_1 \cup R_2$ and $R_2' \cup t_2'Du \cup T$ give independent paths T_1, T_2, T_3 in $G - (X - \{z_1, z_2\})$ with T_1, T_2 (coming from $t_1Dt_2 \cup R_1 \cup R_2$) from y_1 to z_1, z_2, respectively, and T_3 from a' to z_2. Moreover, T_1, T_2, T_3 can be chosen to be independent of both S and X. Hence, $z_2Xx_2 \cup z_2Xy_2 \cup T_2 \cup (T_3 \cup S \cup ax_1) \cup (T_1 \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2; so (i) holds. $\square$

Label the vertices of w_2Dy_1 and x_1Xy_2 such that $w_2Dy_1 = v_1 \dots v_k$ and $x_1Xy_2 = v_{k+1} \dots v_n$, with $v_1 = w_2$, $v_k = y_1$, $v_{k+1} = x_1$ and $v_n = y_2$. Let G_1 denote the union of x_1Xy_2, $\{v_1, \dots, v_k\}$, $G[A \cup (N_K(A) - u_1)]$ for each $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, all edges of G' from x_1Xy_2 to $\{v_1, \dots, v_k\}$, and all edges of G' from x_1Xy_2 to A for each $A \in \mathcal{A}$ with $u_1 \in N_K(A)$. Note that G_1 is $(4, \{v_1, \dots, v_n\})$-connected. (For, if S is a cut in G_1 with $|S| \leq 3$ separating some vertex of G_1 from $\{v_1, \dots, v_n\}$, then $S \cup \{y_2\}$ would be a cut in G of size at most 4, contradicting the assumption that G is 5-connected.) Similarly, let $y_1Dw_2 = z_1 \dots z_l$ and $x_2Xy_2 = z_{l+1} \dots z_m$, with $z_1 = w_2$, $z_l = y_1$, $z_{l+1} = x_2$ and $z_m = y_2$. Let G_2 denote the union of y_2Xx_2, $\{z_1, \dots, z_l\}$, $G[A \cup (N_K(A) - u_2)]$ for each $A \in \mathcal{A}$ with $u_2 \in N_K(A)$, all edges of G' from y_2Xx_2 to $\{z_1, \dots, z_l\}$, and all edges of G' from y_2Xx_2 to A for each $A \in \mathcal{A}$ with $u_2 \in N_K(A)$. Note that G_2 is $(4, \{z_1, \dots, z_m\})$-connected.

If both $(G_1, v_1, \dots, v_n)$ and $(G_2, z_1, \dots, z_m)$ are planar then it follows from (1) and (3) that $G - y_2$ is planar; so (i) or (ii) or (iii) of this lemma holds by Lemma 2.5. Hence, we may assume by symmetry that $(G_1, v_1, \dots, v_n)$ is not planar. Then by Lemma 2.2, there exist $1 \leq q < r < s < t \leq n$ such that G_1 has disjoint paths Q_1, Q_2 from v_q, v_r to v_s, v_t, respectively, and internally disjoint from $\{v_1, \dots, v_n\}$.

Since (K, u_1, y_1, u_2, w_2) is 3-planar (by (1)) and G_1 consists of x_1Xy_2, $\{v_1, \dots, v_k\}$, $G'[N_K(A) - \{u_1\}]$ for each $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, and edges of G' between $V(x_1Xy_2)$

and $\{v_1, \ldots, v_k\}$, we have $q, r \le k$ and $s, t \ge k+1$. Note that the paths $y_1 D t_2$, $t_2' D v_q$, $v_r D y_1$ give rise to independent paths P_1, P_2, P_3 in $K - \{u_1, u_2\}$, with P_1 from y_1 to t_2, P_2 from t_2' to v_q, and P_3 from v_r to y_1. Therefore, $z_2 X x_2 \cup z_2 X y_2 \cup (R_2 \cup P_1) \cup (R_2' \cup P_2 \cup Q_1 \cup v_s X x_1) \cup (P_3 \cup Q_2 \cup v_t X y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So (i) of this lemma holds. ∎

Outcome (iv) of Lemma 4.1 motivates the concept of 11-tuple. We say that $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ is an 11-*tuple* (where it is possible that $z_1 = w_1$ and/or $z_2 = w_3$) if

- $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ is a 9-tuple, and $z_i \in V(x_i X y_2) - \{x_i, y_2\}$ for $i \in [2]$,
- $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ contains disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively, and
- subject to the above conditions, $z_1 X z_2$ is maximal.

Since G is 5-connected and X is induced in $G' - x_1x_2$, each z_i $(i \in [2])$ has at least two neighbors in $H - \{y_2, z_1, z_2\}$ (which is 2-connected). So $H - y_2, H - \{y_2, z_1\}$ and $H - \{y_2, z_2\}$ are all 2-connected. Note that y_2 has exactly one neighbor in $H - \{y_2, z_1, z_2\}$, namely, w_2.

Lemma 4.2. *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ be an 11-tuple and Y, Z be disjoint paths in $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ from y_1, z_1 to y_2, z_2, respectively. Then G contains a TK_5 in which y_2 is not a branch vertex, or G' contains TK_5, or the following holds:*

- (i) *for $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1 z_i \notin E(G)$), and thus*
- (ii) *there exists $i \in [2]$ such that H contains independent paths A, B, C, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.*

Proof. First, suppose, for some $i \in [2]$, there is a path P in H from z_i to y_2 such that z_i, z_{3-i}, y_1, y_2 occur on P in order. Then $z_{3-i} X x_{3-i} \cup z_{3-i} X y_2 \cup (z_{3-i} P z_i \cup z_i X x_i) \cup z_{3-i} P y_1 \cup y_1 P y_2 \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So we may assume that such P does not exist. Hence by the existence of Y, Z in H, we have $y_1 z_1, y_1 z_2 \notin E(G)$, and (i) holds.

So from now on we may assume that (i) holds. For each $i \in [2]$, let H_i denote the graph obtained from H by duplicating z_i and y_1, and let z_i' and y_1' denote the duplicates of z_i and y_1, respectively. So in H_i, y_1 and y_1' are not adjacent, and have the same set of neighbors, namely $N_H(y_1)$; and the same holds for z_i and z_i'.

First, suppose for some $i \in [2]$, H_i contains pairwise disjoint paths A', B', C' from $\{z_i, z_i', y_2\}$ to $\{y_1, y_1', z_{3-i}\}$, with $z_i \in V(A')$, $z_i' \in V(C')$ and $y_2 \in V(B')$. If $z_{3-i} \notin V(B')$, then after identifying y_1 with y_1' and z_i with z_i', we obtain from $A' \cup B' \cup C'$ a path in

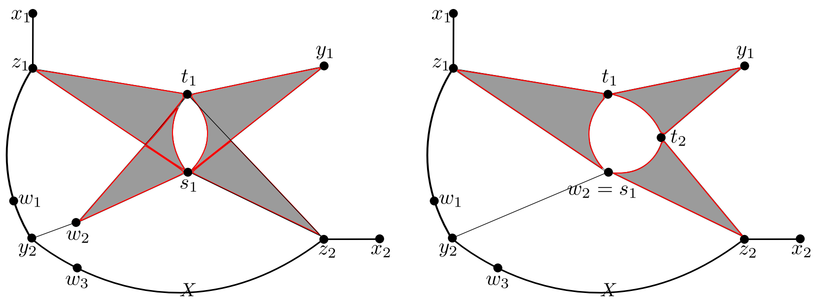

Fig. 3. The bridges Y_1, Y_2, Z_1 and Z_2.

H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting our assumption that (i) holds. Hence $z_{3-i} \in V(B')$. Then we get the desired paths for (ii) from $A' \cup B' \cup C'$ by identifying y_1 with y_1' and z_i with z_i'.

So we may assume that for each $i \in [2]$, H_i does not contain three pairwise disjoint paths from $\{y_2, z_i, z_i'\}$ to $\{y_1, y_1', z_{3-i}\}$. Then H_i has a separation (H_i', H_i'') such that $|V(H_i' \cap H_i'')| = 2$, $\{y_2, z_i, z_i'\} \subseteq V(H_i')$ and $\{y_1, y_1', z_{3-i}\} \subseteq V(H_i'')$.

We claim that $y_1, y_2, y_1', z_i', z_1, z_2 \notin V(H_i' \cap H_i'')$ for $i \in [2]$. Since $H - y_2$ is 2-connected, $y_2 \notin V(H_i' \cap H_i'')$. Since $H - \{z_{3-i}, y_2\}$ is 2-connected, $z_{3-i} \notin V(H_i' \cap H_i'')$. Also, $\{y_1, y_1'\} \neq V(H_i' \cap H_i'')$, since otherwise y_1 would be a cut vertex in $H - y_2$ separating z_{3-i} from z_i. Now suppose one of y_1, y_1' is in $V(H_i' \cap H_i'')$; then since y_1, y_1' are duplicates, the vertex in $V(H_i' \cap H_i'') - \{y_1, y_1'\}$ is a cut vertex in $H - y_2$ separating $\{y_1, z_{3-i}\}$ from z_i, a contradiction. So $y_1, y_1' \notin V(H_i' \cap H_i'')$. Similar argument shows that $z_i, z_i' \notin V(H_i' \cap H_i'')$.

For $i \in [2]$, let $V(H_i' \cap H_i'') = \{s_i, t_i\}$, and let F_i' (respectively, F_i'') be obtained from H_i' (respectively, H_i'') by identifying z_i' with z_i (respectively, y_1' with y_1). Then (F_i', F_i'') is a 2-separation in H such that $V(F_i' \cap F_i'') = \{s_i, t_i\}$, $\{y_2, z_i\} \subseteq V(F_i') - \{s_i, t_i\}$, and $\{y_1, z_{3-i}\} \subseteq V(F_i'') - \{s_i, t_i\}$. Let Z_1, Y_2 denote the $\{s_1, t_1\}$-bridges of F_1' containing z_1, y_2, respectively; and let Z_2, Y_1 denote the $\{s_1, t_1\}$-bridges of F_1'' containing z_2, y_1, respectively.

Suppose $Y_1 \neq Z_2$ and $Y_2 \neq Z_1$. See the left picture in Fig. 3. Since $H - y_2$ is 2-connected, there exist independent P_1, Q_1 in Z_1 from z_1 to s_1, t_1, respectively, independent paths P_2, Q_2 in Z_2 from z_2 to s_1, t_1, respectively, independent paths P_3, Q_3 in Y_1 from y_1 to s_1, t_1, respectively, and a path S in Y_2 from y_2 to one of $\{s_1, t_1\}$ and avoiding the other, say avoiding t_1. Then $z_1 X x_1 \cup z_1 X y_2 \cup y_2 x_1 \cup P_1 \cup S \cup (P_3 \cup y_1 x_1) \cup (Q_2 \cup Q_1) \cup P_2 \cup z_2 X y_2 \cup (z_2 X x_2 \cup x_2 x_1)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Hence, we may assume $Y_1 = Z_2$ or $Y_2 = Z_1$. Indeed, $Y_1 = Z_2$. For, if $Y_1 \neq Z_2$ then $Y_2 = Z_1$, $Y_2 - \{s_1, t_1\}$ has a path from y_2 to z_1, and $Y_1 \cup Z_2$ has two independent paths from y_1 to z_2 (since $H - y_2$ is 2-connected). Now these three paths contradict the existence of the cut $\{s_2, t_2\}$ in H.

Then $\{s_2, t_2\} \cap V(Y_1 - \{s_1, t_1\}) \neq \emptyset$. Without loss of generality, we may assume that $t_2 \in V(Y_1) - \{s_1, t_1\}$. See the right picture in Fig. 3.

Suppose $Y_2 = Z_1$. Then $s_2 \in V(Y_2) - \{s_1, t_1\}$ and we may assume that in H, $\{s_2, t_2\}$ separates $\{t_1, y_1, z_1\}$ from $\{s_1, y_2, z_2\}$. Hence, in Y_1, t_2 separates $\{y_1, t_1\}$ from $\{z_2, s_1\}$, and in Y_2, s_2 separates $\{z_1, t_1\}$ from $\{y_2, s_1\}$. But this contradicts the existence of the paths Y and Z in H.

So $Y_2 \neq Z_1$. Since $H - y_2$ is 2-connected and $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$, we must have $V(Y_2) = \{y_2, w_2\}$; so $s_2 = w_2 \in \{s_1, t_1\}$. By symmetry, we may assume that $s_2 = w_2 = s_1$. Let Y_1', Z_2' be the $\{s_2, t_2\}$-bridge of Y_1 containing y_1, z_2, respectively. Then $t_1 \notin V(Z_2')$; for, otherwise, $H - \{s_2, t_2\}$ would contain a path from z_2 to z_1, a contradiction (as (H_2', H_2') is a separation in H_2 with $\{y_2, z_2, z_2'\} \subseteq V(H_2')$ and $\{y_1, y_1', z_1\} \subseteq V(H_2'')$, and $z_1, z_1', z_2 \notin V(H_2' \cap H_2'') = \{s_2, t_2\}$). Therefore, $t_1 \in V(Y_1')$ and, because of the paths Y and Z, Y_1' contains disjoint paths R_1, R_2 from $s_2 = s_1, t_1$ to y_1, t_2, respectively. Since $H - y_2$ is 2-connected, Z_1 has independent paths P_1, Q_1 from z_1 to $s_2 = s_1, t_1$, respectively, and Z_2' has independent paths P_2, Q_2 from z_2 to $s_2 = s_1, t_2$, respectively. Now $z_1Xx_1 \cup z_1Xy_2 \cup y_2x_1 \cup P_1 \cup s_1y_2 \cup (R_1 \cup y_1x_1) \cup P_2 \cup (Q_2 \cup R_2 \cup Q_1) \cup z_2Xy_2 \cup (z_2Xx_2 \cup x_2x_1)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2. ∎

We now show that if we cannot find the desired TK_5 in G or G' then H contains two more paths P, Q from $A \cup C$ to B (see outcome *(iii)* of the lemma below), competing the structure in Fig. 1.

Lemma 4.3. *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ be an 11-tuple and Y, Z be disjoint paths in $H := G' - V(X - \{y_2, z_1, z_2\} \cup E(X))$ from y_1, z_1 to y_2, z_2, respectively. Then G contains a TK_5 in which y_2 is not a branch vertex, or G' contains TK_5, or the following holds:*

- *(i) there exist $i \in [2]$ and independent paths A, B, C in H, with A and C from z_i to y_1, and B from y_2 to z_{3-i},*
- *(ii) for each $i \in [2]$ satisfying (i), $z_{3-i}x_{3-i} \in E(X)$, and*
- *(iii) H contains two disjoint paths from $V(B - y_2)$ to $V(A \cup C) - \{y_1, z_i\}$ and internally disjoint from $A \cup B \cup C$, with one ending in A and the other ending in C.*

Proof. By Lemma 4.2, we may assume that

(1) for each $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1z_i \notin E(G)$), and
(2) there exist $i \in [2]$ and independent paths A, B, C in H, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of $(A \cup C)$-bridges of H each of which intersects both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$. We choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H with A, C from z_i to y_1 and B from y_2 to z_{3-i},

(b) whenever possible, $J(A, C) \subseteq L(A, C)$,
(c) $J(A, C)$ is maximal, and
(d) $L(A, C)$ is maximal.

We now show that *(ii)* and *(iii)* hold **even** with the restrictions (a), (b), (c) and (d) above. Let B' denote the union of B and the B-bridges of H not containing $A \cup C$.

(3) For each $i \in [2]$ for which (2) holds, if *(iii)* holds for i then *(ii)* holds for i.

Suppose we have $i \in [2]$ for which (2) holds, with the paths A, B, C given as above, and suppose *(iii)* holds for i with disjoint paths P, Q in H from $V(B-y_2)$ to $V(A \cup C) - \{y_1, z_i\}$ and internally disjoint from $A \cup B \cup C$, with one ending in A and the other ending in C. So let $V(P \cap B) = \{p\}$, $V(Q \cap B) = \{q\}$, $V(P \cap C) = \{c\}$ and $V(Q \cap A) = \{a\}$. By the symmetry between A and C, we may assume that y_2, p, q, z_{3-i} occur on B in order. We may further choose P, Q so that pBz_{3-i} is maximal.

To prove *(ii)*, suppose there exists $x \in V(z_{3-i}Xx_{3-i}) - \{x_{3-i}, z_{3-i}\}$. We claim that $N(x) \cap V(H) - \{y_1\} \subseteq V(B')$. For suppose $N(x) \cap V(H) - \{y_1\} \nsubseteq V(B')$. Then x has a neighbor, say x', that is contained in a B-bridge of H containing $A \cup C \cup (P-p) \cup (Q-q)$. Thus, since $H - \{y_2, z_1, z_2\}$ is 2-connected, G' has a path T from x to $(A - y_1) \cup (C - y_1) \cup (P - p) \cup (Q - q)$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q$. By considering whether $x' \in V(C - y_1) \cup V(P - p)$ or $x' \in V(A - y_1) \cup V(Q - q))$, it is straightforward to check that $A \cup B \cup C \cup P \cup Q \cup T$ contains disjoint paths from y_1, z_i to y_2, x, respectively. But this contradicts the choice of Y and Z in the 11-tuple (that z_1Xz_2 is maximal).

So $N(x) \cap V(H) - \{y_1\} \subseteq V(B')$. Consider $B'' := G[(B' - z_{3-i}) + x]$. If B'' contains disjoint paths P', Q' from y_2, x to p, q, respectively, then $Q' \cup Q \cup aAz_i$ and $P' \cup P \cup cCy_1$ contradict the choice of Y, Z. If B'' contains disjoint paths P'', Q'' from x, y_2 to p, q, respectively, then $Q'' \cup Q \cup aAy_1$ and $P'' \cup P \cup cCz_i$ contradict the choice of Y, Z. So we may assume that there is a cut vertex z in B'' separating $\{x, y_2\}$ from $\{p, q\}$. Note that $z \in V(y_2Bp)$.

Since x has at least two neighbors in $B'' - y_2$ (because G is 5-connected and X is induced in $G' - x_1x_2$), the z-bridge of B'' containing $\{x, y_2\}$ has at least three vertices. Therefore, from the maximality of pBz_{3-i} and 2-connectedness of $H - \{y_2, z_1, z_2\}$, there is a path in H from y_1 to $y_2Bz - \{y_2, z\}$ and internally disjoint from $P \cup Q \cup A \cup C \cup B'$. So there is a path Y' in H from y_1 to y_2 and disjoint from $P \cup Q \cup A \cup C \cup pBz_{3-i}$. Now $z_{3-i}Bp \cup P \cup cCz_i \cup A \cup Y'$ is a path in H through z_{3-i}, z_i, y_1, y_2 in order, contradicting (1). □

By (2) and (3), it suffices to prove *(iii)*. Since $H - \{y_2, z_i\}$ is 2-connected, it contains disjoint paths P, Q from $B - y_2$ to some distinct vertices $s, t \in V(A \cup C) - \{z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$.

(4) We may choose P, Q so that $s \neq y_1$ and $t \neq y_1$.

For, otherwise, $H-\{y_2, z_i\}$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{v, y_1\}$ for some $v \in V(H)$, $(A \cup C) - z_i \subseteq H_1$ and $B - y_2 \subseteq H_2$. Recall the disjoint paths Y, Z in H from y_1, z_1 to y_2, z_2, respectively. Suppose $v \notin V(Z)$. Then $Z - z_i \subseteq H_2 - \{y_1, v\}$. Hence we may choose Y (by modifying $Y \cap H_1$) so that $V(Y \cap A) = \{y_1\}$ or $V(Y \cap C) = \{y_1\}$. (To see how to modify $Y \cap H_1$, let y denote the vertex of Y in $A \cup C$ with $y_1 Y y$ maximal. Replace $y_1 Y y$ with $y_1 A y$ (when $y \in V(A)$) or $y_1 C y$ (when $y \in V(C)$).) Now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting (1). So $v \in V(Z)$. Hence $Y \subseteq H_2 - v$, and we may choose Z (by modifying $Z \cap H_1$) so that $V(Z \cap A) = \{z_i\}$ or $V(Z \cap C) = \{z_i\}$. Now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting (1) and completing the proof of (4). □

If $s \in V(A - y_1)$ and $t \in V(C - y_1)$ or $s \in V(C - y_1)$ and $t \in V(A - y_1)$, then P, Q are the desired paths for (iii). So we may assume by symmetry that $s, t \in V(C)$. Let $V(P \cap B) = \{p\}$ and $V(Q \cap B) = \{q\}$ such that y_2, p, q, z_{3-i} occur on B in this order. Then z_i, s, t, y_1 must occur on C in order; for otherwise, $z_{3-i} B q \cup Q \cup t C z_i \cup A \cup y_1 C s \cup P \cup p B y_2$ is a path in H through z_{3-i}, z_i, y_1, y_2 in order, contradicting (1). We choose P, Q so that

(∗) subject to (4), the following conditions hold in order: sCt is maximal, pBz_{3-i} is maximal, and qBz_{3-i} is minimal.

Now consider B', the union of B and the B-bridges of H not containing $A \cup C$. Note from definition that

(∗∗) $(P - p) \cup (Q - q)$ is disjoint from B', and every path in H from $A \cup C$ to B' and internally disjoint from $A \cup B' \cup C$ must end in B.

For convenience, let $K = P \cup Q \cup A \cup B' \cup C$.

(5) $B' - y_2$ contains independent paths P', Q' from z_{3-i} to p, q, respectively.

For, otherwise, $B' - y_2$ has a cut vertex z separating z_{3-i} from $\{p, q\}$. Clearly, $z \in V(qBz_{3-i} - z_{3-i})$, and we choose z so that zBz_{3-i} is minimal. (See Fig. 4.)

Let B'' denote the z-bridge of $B' - y_2$ containing z_{3-i}; then $zBz_{3-i} \subseteq B''$. Since $H - \{y_2, z_i\}$ is 2-connected, it contains a path W from some $w' \in V(B'' - z)$ to some $w \in V(P \cup Q \cup A \cup C) - \{z_i\}$ and internally disjoint from K. By (∗∗), $w' \in V(zBz_{3-i}) - \{z, z_{3-i}\}$. By (∗), $w \notin V(Q) \cup V(tCy_1 - y_1)$. Moreover, $w \notin V(P) \cup V(z_i Ct - t)$; for otherwise, $(P - p) \cup (z_i Ct - t) \cup W \cup w' B z_{3-i}$ contains a path from z_{3-i} to z_i which, together with $A \cup y_1 Ct \cup Q \cup qBy_2$, forms a path in H through z_{3-i}, z_i, y_1, y_2 in order, contradicting (1).

If $w \in V(A) - \{z_i, y_1\}$ then P, W give the desired paths for (iii). So we may assume $w = y_1$ for every choice of W. Then $\{y_1, z\}$ is a cut in H separating B'' from $A \cup y_2 Bz \cup C \cup P \cup Q$. Since Y and Z are disjoint paths in H, we have $z \in V(Z)$, $z_i Zz \cap (B'' - z) = \emptyset$,

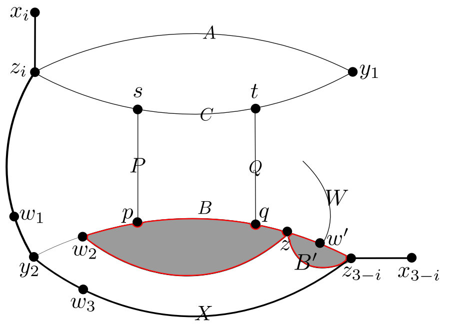

Fig. 4. Structure in the proof of (5).

and $Y \cap (B'' \cup (W - y_1)) = \emptyset$. By the minimality of zBz_{3-i}, B'' has independent paths P'', Q'' from z_{3-i} to z, w', respectively. Now $z_iZz \cup P'' \cup Q'' \cup W \cup Y$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (1). □

(6) We may assume that $J(A, C) \not\subseteq L(A, C)$.

For, otherwise, there is a path R from B to some $r \in V(A) - \{y_1, z_i\}$ and internally disjoint from $A \cup B' \cup C$. If $R \cap (P \cup Q) \neq \emptyset$, then it is easy to check that $P \cup Q \cup R$ contains the desired paths for (iii). So we may assume $R \cap (P \cup Q) = \emptyset$. If $y_2 \notin V(R)$, then P, R are the desired paths for (iii). So assume $y_2 \in V(R)$. Recall the paths P', Q' from (5). Then $z_iCs \cup P \cup P' \cup Q' \cup Q \cup tCy_1 \cup y_1Ar \cup R$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (1). □

Let $J = J(A, C) \cup C$. Then by (1), J does not contain disjoint paths from y_2, z_i to y_1, z_{3-i}, respectively. So by Lemma 2.1, there exists a collection $\mathcal{A}$ of subsets of $V(J) - \{y_1, y_2, z_1, z_2\}$ such that $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar. We choose $\mathcal{A}$ so that every member of $\mathcal{A}$ is minimal and, subject to this, $|\mathcal{A}|$ is minimum. Then

(7) for any $D \in \mathcal{A}$ and any $v \in V(D)$, $(J[D + N_J(D)], N_J(D) \cup \{v\})$ is not 3-planar.

For, suppose there exist $D \in \mathcal{A}$, $v \in D$, and a collection of subsets $\mathcal{A}'$ of $V(D - v)$ such that $(J[D + N_J(D)], \mathcal{A}', N_J(D) \cup \{v\})$ is 3-planar. Then, with $\mathcal{A}'' = (\mathcal{A} - \{D\}) \cup \mathcal{A}'$, $(J, \mathcal{A}'', z_i, y_1, z_{3-i}, y_2)$ is 3-planar. So $\mathcal{A}''$ contradicts the choice of $\mathcal{A}$. □

Let $v_1, \ldots, v_k$ be the vertices of $L(A, C) \cap (C - \{y_1, z_i\})$ such that $z_i, v_1, \ldots, v_k, y_1$ occur on C in the order listed. We claim that

(8) $(J, z_i, v_1, \ldots, v_k, y_1, z_{3-i}, y_2)$ is 3-planar.

For, suppose otherwise. Let $w_1, \ldots, w_l$ denote the vertices of $C - \{y_1, z_i\}$ not contained in any $D \in \mathcal{A}$ and assume that $z_i, w_1, \ldots, w_l, y_1$ occur on C in order. Then, since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, $(J, \mathcal{A}, z_i, w_1, \dots, w_l, y_1, z_{3-i}, y_2)$ is 3-planar. Thus, if $\{v_1, \dots, v_k\} \subseteq \{w_1, \dots, w_l\}$ then (8) holds. So we may assume that there exist $j \in [k]$ and $D \in \mathcal{A}$ such that $v_j \in D$. Since $H - y_2$ is 2-connected, there exist distinct $c_1, c_2 \in V(C) \cap N_J(D)$ and we choose such c_1, c_2 with c_1Cc_2 maximal.

Suppose $N_J(D) \subseteq V(C)$. Then, since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, J has a separation (J_1, J_2) such that $V(J_1 \cap J_2) = \{c_1, c_2\}$, $D \cup V(c_1Cc_2) \subseteq V(J_1)$, and $B \subseteq J_2$. Since J has only one C-bridge and C is induced in H, we have $J_1 = c_1Cc_2$. Now let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by removing all members of $\mathcal{A}$ contained in $V(J_1)$. Then $(J, \mathcal{A}', z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting the choice of $\mathcal{A}$.

Thus, let $c \in N_J(D) - V(C)$. So $c \in V(J(A, C))$. Let $D' = J[D + \{c_1, c_2, c\}]$. By (7) and Lemma 2.1, D' contains disjoint paths R, T, with R from v_j to c and T from c_1 to c_2. We may assume T is induced D'. Let C' be obtained from C by replacing c_1Cc_2 with T. Then A, B, C' are induced paths in H, with A, C' from z_i to y_1 and B from y_2 to z_{3-i}; so (a) holds for A, B, C'. Because of P and Q, $J(A, C')$ intersects $C' - \{y_1, z_i\}$. Since $c \in V(J(A, C)) - V(C))$, we see that $c \in V(J(A, C'))$; so $v_j \in V(J(A, C'))$. Since $v_j \in V(L(A, C))$, $J(A, C')$ intersects $A - \{y_1, z_i\}$. Thus, $J(A, C') \subseteq L(A, C')$, contradicting (b) (via (6)). $\square$

(9) There exist disjoint paths R_1, R_2 in $L(A, C)$ from some $r_1, r_2 \in V(C)$ to some $r_1', r_2' \in V(A)$, respectively, and internally disjoint from $A \cup C$, such that z_i, r_1, r_2, y_1 occur on C in this order and z_i, r_2', r_1', y_1 occur on A in this order.

We prove (9) by studying the $(A \cup C)$-bridges of H other than $J(A, C)$. For any $(A \cup C)$-bridge T of H with $T \neq J(A, C)$, if T intersects A let $a_1(T), a_2(T) \in V(T \cap A)$ with $a_1(T)Aa_2(T)$ maximal, and if T intersects C let $c_1(T), c_2(T) \in V(T \cap C)$ with $c_1(T)Cc_2(T)$ maximal. Possibly $a_1(T) = a_2(T)$ and/or $c_1(T) = c_2(T)$, but they are distinct if $T \not\subseteq L(A, C)$. We choose the notation so that $z_i, a_1(T), a_2(T), y_1$ occur on A in order, and $z_i, c_1(T), c_2(T), y_1$ occur on C in order.

If T_1, T_2 are $(A \cup C)$-bridges of H such that $T_2 \subseteq L(A, C)$, $T_1 \neq J(A, C)$, and $T_1 \cap (A \cup C) \subseteq C$ (or $T_1 \cap (A \cup C) \subseteq A$), then $c_1(T_1)Cc_2(T_1) - \{c_1(T_1), c_2(T_1)\}$ (or $a_1(T_1)Aa_2(T_1) - \{a_1(T_1), a_2(T_1)\}$) does not intersect T_2. For, otherwise, we may modify C (or A) by replacing $c_1(T_1)Cc_2(T_1)$ (or $a_1(T_1)Aa_2(T_1)$) with an induced path in T_1 from $c_1(T_1)$ to $c_2(T_1)$ (or from $a_1(T_1)$ to $a_2(T_1)$). The new A and C satisfy (a) and (b). Thus, the new A and C do not enlarge $J(A, C)$, as otherwise we have a contradiction to (c). However, the new A and C do enlarge $L(A, C)$, contradicting (d).

Because of the disjoint paths Y and Z in H, $(H, z_i, y_1, z_{3-i}, y_2)$ is not 3-planar. By (1) $A - \{y_1, z_i\} \neq \emptyset$. Hence, since $H - \{y_2, z_1, z_2\}$ is 2-connected, $L(A, C) \neq \emptyset$. Thus, since $(J, z_i, v_1, \dots, v_k, y_1, z_{3-i}, y_2)$ is 3-planar (by (8)) and $J(A, C)$ does not intersect $A - \{y_1, z_i\}$ (by (6)), one of the following holds: There exist $(A \cup C)$-bridges T_1, T_2 of H such that $T_1 \cup T_2 \subseteq L(A, C)$, $z_iAa_2(T_1)$ properly contains $z_iAa_1(T_2)$, and $c_1(T_1)Cy_1$

properly contains $c_2(T_2)Cy_1$; or there exists an $(A \cup C)$-bridge T of H such that $T \subseteq L(A, C)$ and $T \cup a_1(T)Aa_2(T) \cup c_1(T)Cc_2(T)$ has disjoint paths from $a_1(T), a_2(T)$ to $c_2(T), c_1(T)$, respectively. In either case, we have (9). □

(10) $r_1, r_2 \in V(tCy_1)$ for all choices of R_1, R_2 in (9), or $r_1, r_2 \in V(z_iCs)$ for all choices of R_1, R_2 in (9).

For, otherwise, there exist R_1, R_2 as in (9) such that $r_1 \in V(z_iCs)$ and $r_2 \in V(tCy_1)$, or $r_1 \in V(sCt) - \{s, t\}$, or $r_2 \in V(sCt) - \{s, t\}$. Let $A' := z_iAr_2' \cup R_2 \cup r_2Cy_1$ and $C' := z_iCr_1 \cup R_1 \cup r_1'Ay_1$. We may assume A', C' are induced paths in H (by taking induced paths in $H[A']$ and $H[C']$). Note that A', B, C' satisfy (a), and $J(A, C) \subseteq J(A', C')$. However, because of P and Q, $J(A', C')$ intersects both $A' - \{z_i, y_1\}$ and $C' - \{z_i, y_1\}$, contradicting (b) (via (6)) in the choice of A, B, C. □

If $r_1, r_2 \in V(z_iCs)$ for all choices of R_1, R_2 in (9) then we choose such R_1, R_2 that z_iAr_1' and z_iCr_2 are maximal, and let $z' := r_1'$ and $z'' = r_2$; otherwise, define $z' = z'' = z_i$. Similarly, if $r_1, r_2 \in V(tCy_1)$ for all choices of R_1, R_2 in (9), then we choose such R_1, R_2 that y_1Ar_2' and y_1Cr_1 are maximal, and let $y' := r_2'$ and $y'' = r_1$; otherwise, define $y' = y'' = y_1$. By (10), z_i, z', y', y_1 occur on A in order, and z_i, z'', s, t, y'', y_1 occur on C in order.

Note that H has a path W from some $y \in V(B) \cup V(P - s) \cup V(Q - t)$ to some $w \in V(z_iAz' - \{z', z_i\}) \cup V(z_iCz'' - \{z'', z_i\}) \cup V(y'Ay_1 - \{y', y_1\}) \cup V(y''Cy_1 - \{y'', y_1\})$ such that W is internally disjoint from K. For, otherwise, $(H, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting the existence of the disjoint paths Y and Z. By (6), $w \notin V(A)$. If $w \in V(z_iAz' - \{z', z_i\}) \cup V(y'Ay_1 - \{y', y_1\})$ then we can find the desired P, Q. So assume $w \in V(z_iCz'' - \{z'', z_i\}) \cup V(y''Cy_1 - \{y'', y_1\})$. By $(*)$ and (1), $y \notin V(B - y_2)$ and $y \notin V(P \cup Q)$. This forces $y = y_2$, which is impossible as $N_H(y_2) = \{w_2\}$. ∎

Remark. Note from the proof of Lemma 4.3 (see the statement below (d)) that the outcomes *(ii)* and *(iii)* of Lemma 4.3 hold for those paths A, B, C that satisfy (a), (b), (c) and (d).

5. Finding TK_5

In this section, we prove Theorem 1.1 by finding the desired TK_5 using the structure derived in Section 4. We will first give a precise description of this intermediate structure and list certain properties related to this structure. This is done in (1), (2), (3) and (a), (b), (c), (d) below. We then prove Claims (4)–(12), whose main purpose is to obtain (12) which gives an additional path Q' that is independent of the intermediate structure. See Fig. 5. Using the path Q', we prove the next group of claims (13)–(17) to force a local structure around a special vertex z_1. The proof of Theorem 1.1 concludes with claims (18)–(21).

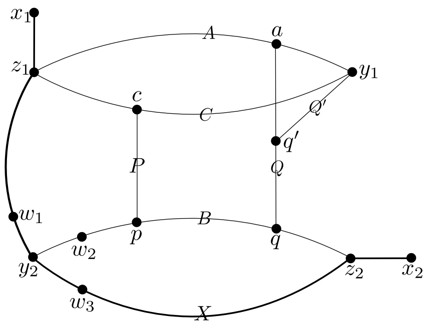

Fig. 5. Intermediate structure with additional path Q'.

Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct and let $G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$.

We may assume that $G' - x_1x_2$ has an induced path L from x_1 to x_2 such that $y_1, y_2 \notin V(L)$, $(G - y_2) - L$ is 2-connected, and w_1, w_2, w_3 occur on L in that order; for otherwise, the conclusion of Theorem 1.1 follows from Lemma 3.2. Hence, $G' - x_1x_2$ has an induced path X from x_1 to x_2 such that $w_2, y_1 \notin V(X)$, $w_1y_2, w_3y_2 \in E(X)$, x_1, w_1, y_2, w_3, x_2 occur on X in order, and $G' - X = G - X$ is 2-connected. Hence, $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ is a 9-tuple.

We may assume that there exist $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ for $i \in [2]$ such that $H := G' - (X - \{y_2, z_1, z_2\})$ has disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively; for, otherwise, the conclusion of Theorem 1.1 follows from Lemma 4.1. We choose such Y, Z that z_1Xz_2 is maximal. Then $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ is an 11-tuple.

By Lemma 4.2 and by symmetry, we may assume that

(1) for $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1z_i \notin E(G)$),

and that there exist independent paths A, B, C in H with A and C from z_1 to y_1, and B from y_2 to z_2.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of $(A \cup C)$-bridges of H intersecting both $A - \{y_1, z_1\}$ and $C - \{y_1, z_1\}$. We may choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,
(b) whenever possible $J(A, C) \subseteq L(A, C)$,
(c) $J(A, C)$ is maximal, and
(d) $L(A, C)$ is maximal.

By Lemma 4.3 and its proof (see the remark at the end of Section 4), we may assume that

$$z_2x_2 \in E(X)$$

and that there exist disjoint paths P, Q in H from $p, q \in V(B - y_2)$ to $c \in V(C) - \{y_1, z_1\}, a \in V(A) - \{y_1, z_1\}$, respectively, and internally disjoint from $A \cup B \cup C$. By symmetry between A and C, we assume that y_2, p, q, z_2 occur on B in order. We further choose A, B, C, P, Q so that

(2) the following conditions hold in order: qBz_2 is minimal, pBz_2 is maximal, and $aAy_1 \cup cCz_1$ is minimal.

Let B' denote the union of B and the B-bridges of H not containing $A \cup C$. Note that all paths in H from $A \cup C$ to B' and internally disjoint from B' must have an end in B. For convenience, let

$$K := A \cup B' \cup C \cup P \cup Q.$$

Then

(3) H has no path from $aAy_1 - a$ to $z_1Cc - c$ and internally disjoint from K.

For, suppose S is a path in H from some vertex $s \in V(aAy_1 - a)$ to some vertex $s' \in V(z_1Cc - c)$ and internally disjoint from K. Then $z_2Bq \cup Q \cup aAz_1 \cup z_1Cs' \cup S \cup sAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). □

We proceed by proving a number of claims from which Theorem 1.1 will follow. Our intermediate goal is to prove (12) that H contains a path from y_1 to $Q - a$ and internally disjoint from K. However, the claims leading to (12) will also be useful when we later consider structure of G near z_1.

(4) $B' - y_2$ has no cut vertex contained in $qBz_2 - z_2$ and, hence, for any $q^* \in V(B') - \{y_2, q\}$, $B' - y_2$ has independent paths P_1, P_2 from z_2 to q, q^*, respectively.

For, suppose $B' - y_2$ contains a cut vertex u with $u \in V(qBz_2 - z_2)$. Choose u so that uBz_2 is minimal. Since $H - \{y_2, z_1\}$ is 2-connected, there is a path S in H from some $s' \in V(uBz_2 - u)$ to some $s \in V(A \cup C \cup P \cup Q) - \{p, q\}$ and internally disjoint from K. By the minimality of uBz_2, the u-bridge of $B' - y_2$ containing uBz_2 has independent paths R_1, R_2 from z_2 to s', u, respectively. By the minimality of qBz_2 in (2), S is disjoint from $(P \cup Q \cup A \cup C) - \{z_1, y_1\}$. If $s = z_1$ then $(R_1 \cup S) \cup A \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). So $s = y_1$. Then $(z_1Aa \cup Q \cup qBu \cup R_2) \cup (R_1 \cup S) \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_1, z_2, y_1, y_2 in order, contradicting (1).

Hence, $B' - y_2$ has no cut vertex contained in $qBz_2 - z_2$. Thus, the second half of (4) follows from Menger's theorem. □

(5) We may assume that G' has no path from $aAy_1 - a$ to z_1Xz_2 and internally disjoint from $K \cup X$, and no path from $cCy_1 - c$ to $z_1Xz_2 - z_1$ and internally disjoint from $K \cup X$.

For, suppose S is a path in G' from some $s \in V(aAy_1 - a) \cup V(cCy_1 - c)$ to some $s' \in V(z_1Xz_2)$ and internally disjoint from $K \cup X$, such that $s' \neq z_1$ if $s \in V(cCy_1 - c)$. Note that $s' \neq z_1$, as otherwise $s \in V(aAy_1 - a)$ which contradicts (3). If $s' = z_2$ then $s = y_1$ by (2); so $(z_1Aa \cup Q \cup qBz_2) \cup (S \cup y_1Cc \cup P \cup pBy_2)$ is a path in H through z_1, z_2, y_1, y_2 in order, contradicting (1). Hence, $s' \in V(z_1Xz_2) - \{z_1, z_2\}$.

Suppose $s' \in V(z_1Xy_2 - z_1)$. Let P_1, P_2 be the paths in (4) with $q^* = p$. If $s \in V(aAy_1 - a)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_2 \cup P \cup cCy_1) \cup (P_1 \cup Q \cup aAz_1 \cup z_1Xx_1) \cup (y_1As \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $s \in V(cAy_1 - c)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_2 \cup P \cup cCz_1 \cup z_1Xx_1) \cup (P_1 \cup Q \cup aAy_1) \cup (y_1Cs \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume $s' \in V(z_2Xy_2 - z_2)$. If $s \in V(aAy_1 - a)$, then $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup Q \cup qBz_2 \cup z_2x_2) \cup (y_1As \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s \in V(cCy_1 - c)$, then $z_1Xx_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (y_1Cs \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. This completes the proof of (5). $\square$

Denote by $L(A)$ (respectively, $L(C)$) the union of $(A \cup C)$-bridges of H not intersecting $C - \{y_1, z_1\}$ (respectively, $A - \{y_1, z_1\}$). Let $C' = C \cup L(C)$. The next four claims concern paths from $x_1Xz_1 - z_1$ to other parts of G'. We may assume that

(6) $N(x_1Xz_1 - \{x_1, z_1\}) \subseteq V(C') \cup \{x_1, z_1\}$, and that G' has no disjoint paths from $s_1, s_2 \in V(x_1Xz_1 - z_1)$ to $s'_1, s'_2 \in V(C)$, respectively, which are internally disjoint from $K \cup X$ with $s'_2 \in V(cCy_1 - c)$, x_1, s_1, s_2, z_1 on X in order, and z_1, s'_1, s'_2, y_1 on C in order.

To prove (6), suppose first $N(x_1Xz_1 - \{x_1, z_1\}) \not\subseteq V(C') \cup \{x_1, z_1\}$. Then there exists a path S in G' from some $s \in V(x_1Xz_1) - \{x_1, z_1\}$ to some $s' \in V(A \cup B' \cup P \cup Q) - \{c, y_1, y_2, z_1, z_2\}$ and internally disjoint from $K \cup X$. If $s' \in V(A) - \{z_1, y_1\}$ then $y_1Cc \cup P \cup pBy_2$, $S \cup s'Aa \cup Q \cup qBz_2$ contradict the choice of Y, Z. If $s' \in V(Q - a)$ then $y_1Cc \cup P \cup pBy_2$, $S \cup s'Qq \cup qBz_2$ contradict the choice of Y, Z. If $s' \in V(P - c)$ then let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup pPs' \cup S \cup sXx_1) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $s' \in V(B') - \{y_2, p, q\}$ then let P_1, P_2 be the paths in (4) with $q^* = s'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup S \cup sXx_1) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Next, assume G' has disjoint paths S_1, S_2 from $s_1, s_2 \in V(x_1Xz_1 - z_1)$ to $s'_1, s'_2 \in V(C)$, respectively, and internally disjoint from $K \cup X$ such that $s'_2 \in V(cCy_1 - c)$, x_1, s_1, s_2, z_1 occur on X in order, and z_1, s'_1, s'_2, y_1 occur on C in order. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCs'_1 \cup S_1 \cup s_1Xx_1) \cup (y_1Cs'_2 \cup S_2 \cup s_2Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. □

(7) For any path W in G' from x_1 to some $w \in V(K) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$, we may assume $w \in V(A \cup C) - \{y_1, z_1\}$. (Note that such W exists as G is 5-connected and $G' - X$ is 2-connected.)

For, let W be a path in G' from x_1 to $w \in V(K) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$, such that $w \notin V(A \cup C) - \{z_1, y_1\}$. Then $w \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$.

Suppose $w \in V(B' - q)$. Let P_1, P_2 be the paths in (4) with $q^* = w$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume $w \notin V(B' - q)$. Let P_1, P_2 be the paths in (4) with $q^* = p$. If $w \in V(P - c)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup pPw \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $w \in V(Q - a)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQw \cup W) \cup (P_2 \cup P \cup cCy_1) \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. □

(8) We may assume that G' has no path from $x_1Xz_1 - x_1$ to y_1 and internally disjoint from $K \cup X$.

For, suppose that R is a path in G' from some $x \in V(x_1Xz_1 - x_1)$ to y_1 and internally disjoint from $K \cup X$. Then $x \neq z_1$; as otherwise $z_2Bq \cup Q \cup aAz_1 \cup R \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). Let P_1, P_2 be the paths in (4) with $q^* = p$. We use W from (7).

If $w \in V(A) - \{z_1, y_1\}$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAw \cup W) \cup (P_2 \cup P \cup cCy_1) \cup (R \cup xXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $w \in V(C) - \{z_1, y_1\}$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCw \cup W) \cup (R \cup xXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. □

(9) If G' has a path from $x_1Xz_1 - \{x_1, z_1\}$ to $cCy_1 - c$ and internally disjoint from $K \cup X$, then we may assume that
- $w \in V(C) - \{y_1, z_1\}$ for every choice of W in (7), and
- G' has no path from x_2 to $C - \{y_1, z_1\}$ and internally disjoint from $K \cup X$.

For, let S be a path in G' from some $s \in V(x_1Xz_1) - \{x_1, z_1\}$ to $V(cCy_1 - c)$ and internally disjoint from $K \cup X$. Since X is induced in $G' - x_1x_2$, $G'[H - \{y_2, z_1, z_2\} + s]$ is 2-connected. Hence, since $N(x_1Xz_1 - \{x_1, z_1\}) \subseteq V(C') \cup \{x_1, z_1\}$ (by (6)), G' has independent paths S_1, S_2 from s to distinct $s_1, s_2 \in V(C) - \{z_1, y_1\}$ and internally disjoint from $K \cup X$. We may assume that z_1, s_1, s_2, y_1 occur on C in this order and choose S_1, S_2 so that s_2Cy_1 is minimal.

We claim that $s_2 \in V(cCy_1 - c)$. For, suppose $s_2 \in V(z_1Cc)$. Then let $s' \in V(S_1 \cup S_2) \cap V(S)$ and $s'' \in V(cCy_1 - c) \cap V(S)$ such that $s'Ss'' - s'$ is disjoint from $S_1 \cup S_2$. If $s' \in V(S_1)$ then $S_2, sS_1s' \cup s'Ss''$ contradict the choice of S_1, S_2 (that s_2Cy_1 is minimal). So $s' \in V(S_2)$. Then $S_1, sS_2s' \cup s'Ss''$ contradict the choice of S_1, S_2 (that s_2Cy_1 is minimal).

Suppose we may choose the W in (7) with $w \in V(A) - \{z_1, y_1\}$. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAw \cup W) \cup sXx_1 \cup sXy_2 \cup (P_2 \cup P \cup cCs_1 \cup S_1) \cup (S_2 \cup s_2Cy_1 \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_2, z_2.

Now assume that S' is a path in G' from x_2 to some $s' \in V(C) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$. Then $S_1 \cup S_2 \cup S' \cup (C - z_1)$ contains independent paths S_1', S_2' which are from s to y_1, x_2, respectively (when $s' \in V(z_1Cs_2) - \{s_2, z_1\}$), or from s to c, x_2, respectively (when $s' \in V(s_2Cy_1 - y_1)$). If S_1', S_2' end at y_1, x_2, respectively, then $sXx_1 \cup sXy_2 \cup S_1' \cup S_2' \cup (y_1Aa \cup Q \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_1, y_2. So assume that S_1', S_2' end at c, x_2, respectively. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $sXx_1 \cup sXy_2 \cup S_2' \cup z_2x_2 \cup z_2Xy_2 \cup (S_1' \cup P \cup P_2) \cup (P_1 \cup Q \cup aAy_1 \cup y_1x_1) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_2, z_2. □

The next two claims deal with $L(A)$ and $L(C)$. First, we may assume that

(10) $L(A) \cap A \subseteq z_1Aa$.

In order to see this, for any $(A \cup C)$-bridge R of H contained in $L(A)$, let $z(R), y(R) \in V(R \cap A)$ such that $z(R)Ay(R)$ is maximal. Suppose for some $(A \cup C)$-bridge R_1 of H contained in $L(A)$, we have $y(R_1)Az(R_1) \not\subseteq z_1Aa$. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(A)$, such that for each $i \in \{2, \ldots, m\}$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1} z(R_j)Ay(R_j)$ (which is a path). Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} z(R_j)Ay(R_j) = a_1Aa_2$. Then $J(A, C)$ does not intersect $a_1Aa_2 - \{a_1, a_2\}$; as otherwise, we could modify A inside some R_i to make $J(A, C)$ larger, contradicting (c). So $a_1, a_2 \in V(aAy_1)$. Moreover, G' has no path from $a_1Aa_2 - \{a_1, a_2\}$ to C and internally disjoint from $K \cup X$; for otherwise, we could modify A inside some R_i to make $L(A, C)$ larger, contradicting (d). Hence by (5) and the choice of $R_1, \ldots, R_m$, $\{a_1, a_2, x_1, x_2, y_2\}$ is a cut in G. Thus, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_1, a_2, x_1, x_2, y_2\}$, $P \cup Q \cup B' \cup C \cup X \subseteq G_1$, and $a_1Aa_2 \cup \left(\bigcup_{j=1}^{m} R_j\right) \subseteq G_2$. (In particular, $X \subseteq G_1$ follows from the first part of (5) and (6) and the fact $z_2x_2 \in E(X)$.)

Let $z \in V(G_2) - \{a_1, a_2, x_1, x_2, y_2\}$ and assume z_1, a_1, a_2, y_1 occur on A in order. Since G is 5-connected, $G_2 - y_2$ contains four independent paths R_1, R_2, R_3, R_4 from z to x_1, x_2, a_1, a_2, respectively. Now $R_1 \cup R_2 \cup (R_3 \cup a_1Az_1 \cup z_1Xy_2) \cup (R_4 \cup a_2Ay_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z. □

(11) We may assume that if R is an $(A \cup C)$-bridge of H contained in $L(C)$ and $R \cap (cCy_1 - c) \neq \emptyset$ then $|V(R) - V(C)| = 1$ and $N(R - C) = \{c_1, c_2, s_1, s_2, y_2\}$, with $c_1Cc_2 = c_1c_2$ and $s_1s_2 = s_1Xs_2 \subseteq z_1Xx_1$.

For any $(A \cup C)$-bridge R in $L(C)$, let $z(R), y(R) \in V(C \cap R)$ such that $z(R)Cy(R)$ is maximal. Let R_1 be an $(A \cup C)$-bridge of H contained in $L(C)$ such that $R_1 \cap (cCy_1 - c) \neq \emptyset$.

Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(C)$, such that for each $i \in \{2, \ldots, m\}$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1} z(R_j)Cy(R_j)$ (which is a path). Let $c_1, c_2 \in V(C)$ such that $c_1Cc_2 = \bigcup_{j=1}^{m} z(R_j)Cy(R_j)$, with z_1, c_1, c_2, y_1 on C in order. So $c_2 \in V(cCy_1 - c)$ and, hence, $c_1 \in V(cCy_1 - y_1)$ by (c) and the existence of P. Let $R' = \bigcup_{j=1}^{m} R_j \cup c_1Cc_2$.

By (c), G' has no path from $c_1Cc_2 - \{c_1, c_2\}$ to $V(B' \cup P \cup Q) \cup \{z_1\}$ and internally disjoint from $K \cup X$. By (d), G' has no path from $c_1Cc_2 - \{c_1, c_2\}$ to $A - \{y_1, z_1\}$ and internally disjoint from $K \cup X$.

If $N(x_2) \cap V(R' - \{c_1, c_2\}) \neq \emptyset$ then by the second part of (5) and the second part of (9), $N(R' - \{c_1, c_2\}) = \{x_1, x_2, y_2, c_1, c_2\}$. Let $z \in V(R') - \{c_1, c_2\}$. Since G is 5-connected, R' has independent paths W_1, W_2, W_3, W_4 from z to x_1, x_2, c_2, c_1, respectively. Now $W_1 \cup W_2 \cup (W_3 \cup c_2Cy_1) \cup (W_4 \cup c_1Cz_1 \cup z_1Xy_2) \cup (y_1Aa \cup Q \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z.

So we may assume $N(x_2) \cap V(R' - \{c_1, c_2\}) = \emptyset$. Since G is 5-connected, it follows from (5) that there exist distinct $s_1, s_2 \in V(x_1Xz_1 - z_1) \cap N(R' - \{c_1, c_2\})$. Choose s_1, s_2 such that s_1Xs_2 is maximal and assume that x_1, s_1, s_2, z_1 occur on X in this order. By (6) (in particular, the second part of (6)), $\{c_1, c_2, s_1, s_2, y_2\}$ is a 5-cut in G; so G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{c_1, c_2, s_1, s_2, y_2\}$, $A \cup z_1Xx_2 \cup P \cup Q \subseteq G_1$, and $R' \cup c_1Cc_2 \cup s_1Xs_2 \subseteq G_2$. By the second part of (6) again, $(G_2 - y_2, c_1, c_2, s_1, s_2)$ is planar (since G is 5-connected). Clearly, $|V(G_1)| \geq 7$. If $|V(G_2)| \geq 7$ then by Lemma 2.3, (i) or (ii) or (iii) of this theorem holds. So we may assume that $|V(G_2)| = 6$, and we have the assertion of (11). $\square$

We may assume that

(12) H has a path Q' from y_1 to some $q' \in V(Q - a)$ and internally disjoint from K.

First, suppose that $y_1 \in V(J(A, C))$. Then, H has a path Q' from y_1 to some $q' \in V(P - c) \cup V(Q - a) \cup V(B)$ and internally disjoint from K. We may assume $q' \in V(P - c) \cup V(B - q)$; for otherwise, $q' \in V(Q - a)$ and (12) holds. If $q' \in V(P - c) \cup V(y_2Bq - q)$ then $(P - c) \cup (y_2Bq - q) \cup Q'$ contains a path Q'' from y_1 to y_2; so $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup Q \cup qBz_2 \cup z_2x_2) \cup Q'' \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. Hence, we may assume $q' \in V(qBz_2 - q)$. Let W, w be given as in (7). If $w \in V(A) - \{y_1, z_1\}$ then let P_1, P_2 be the paths in (4) with $q^* = q'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAw \cup W) \cup (P_2 \cup Q') \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So assume $w \in V(C) - \{y_1, z_1\}$. Then by the first part of (4), B' contains independent paths P_1, P_2 from z_2 to p, q', respectively. Now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup P \cup cCw \cup W) \cup (P_2 \cup Q') \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, we may assume that $y_1 \notin V(J(A,C))$. Note that $y_1 \notin V(L(A))$ (by (10)) and $y_1 \notin V(L(C))$ (by (8) and (11)). Hence, since $y_1y_2 \notin E(G)$ and G is 5-connected, y_1 is contained in some $(A \cup C)$-bridge of H, say D_1, with $D_1 \subseteq L(A,C)$ and $D_1 \neq J(A,C)$. Note that $|V(D_1)| \geq 3$ as A and C are induced paths. For any $(A \cup C)$-bridge D of H with $D \subseteq L(A,C)$ and $D \neq J(A,C)$, let $a(D) \in V(A) \cap V(D)$ and $c(D) \in V(C) \cap V(D)$ such that $z_1Aa(D)$ and $z_1Cc(D)$ are minimal.

Let $D_1, \ldots, D_k$ be a maximal sequence of $(A \cup C)$-bridges of H with $D_i \subseteq L(A,C)$ (so $D_i \neq J(A,C)$ by (6)) for $i \in [k]$, such that, for each $i \in [k-1]$, $D_{i+1} \cap (A \cup C)$ is not contained in $\bigcup_{j=1}^{i}(c(D_j)Cy_1 \cup a(D_j)Ay_1)$, and $D_{i+1} \cap (A \cup C)$ is not contained in $\bigcap_{j=1}^{i}(z_1Cc(D_j) \cup z_1Aa(D_j))$. Note that for any $i \in [k]$, $\bigcup_{j=1}^{i} a(D_j)Ay_1$ and $\bigcup_{j=1}^{i} c(D_j)Cy_1$ are paths. So let $a_i \in V(A)$ and $c_i \in V(C)$ such that $\bigcup_{j=1}^{i} a(D_j)Ay_1 = a_iAy_1$ and $\bigcup_{j=1}^{i} c(D_j)Cy_1 = c_iCy_1$. Let $S_i = a_iAy_1 \cup c_iCy_1 \cup \left(\bigcup_{j=1}^{i} D_j\right)$.

Next, we claim that for any $l \in [k]$ and for any $r_l \in V(S_l) - \{a_l, c_l\}$ there exist three independent paths A_l, C_l, R_l in S_l from y_1 to a_l, c_l, r_l, respectively. This is clear when $l = 1$; note that if $a_l = y_1$, or $c_l = y_1$, or $r_l = y_1$ then A_l, or C_l, or R_l is a trivial path. Now assume that the assertion is true for some $l \in [k-1]$. Let $r_{l+1} \in V(S_{l+1}) - \{a_{l+1}, c_{l+1}\}$. When $r_{l+1} \in V(S_l) - \{a_l, c_l\}$ let $r_l := r_{l+1}$; otherwise, let $r_l \in V(D_{l+1})$ with $r_l \in V(a_lAy_1 - a_l) \cup V(c_lCy_1 - c_l)$ (which exists as $D_{l+1} \cap (A \cup C)$ is not contained in $\bigcap_{j=1}^{l}(z_1Cc(D_j) \cup z_1Aa(D_j)) = z_1Aa_l \cup z_1Cc_l$). By induction hypothesis, there are three independent paths A_l, C_l, R_l in S_l from y_1 to a_l, c_l, r_l, respectively. If $r_{l+1} \in V(S_l) - \{a_l, c_l\}$ then $A_{l+1} := A_l \cup a_lAa_{l+1}, C_{l+1} := C_l \cup c_lCc_{l+1}, R_{l+1} := R_l$ are the desired paths in S_{l+1}. If $r_{l+1} \in V(D_{l+1}) - V(A \cup C)$ then, by the choice of r_l, let P_{l+1} be a path in D_{l+1} from r_l to r_{l+1} and internally disjoint from $A \cup C$; we see that $A_{l+1} := A_l \cup a_lAa_{l+1}, C_{l+1} := C_l \cup c_lCc_{l+1}, R_{l+1} := R_l \cup P_{l+1}$ are the desired paths in S_{l+1}. So we may assume by symmetry that $r_{l+1} \in V(a_{l+1}Aa_l - a_{l+1})$. Let Q_{l+1} be a path in D_{l+1} from r_l to a_{l+1} and internally disjoint from $A \cup C$. Now $R_{l+1} := A_l \cup a_lAr_{l+1}, C_{l+1} := C_l \cup c_lCc_{l+1}, A_{l+1} := R_l \cup Q_{l+1}$ are the desired paths in S_{l+1}.

We claim that $J(A,C)$ has no vertex in $(a_kAy_1 \cup c_kCy_1) - \{a_k, c_k\}$. For, suppose there exists $r \in V(J(A,C))$ such that $r \in V(a_kAy_1 - a_k) \cup V(c_kCy_1 - c_k)$. Then let A_k, C_k, R_k be independent (induced) paths in S_k from y_1 to a_k, c_k, r, respectively. Let A', C' be obtained from A, C by replacing a_kAy_1, c_kCy_1 with A_k, C_k, respectively. We see that $J(A', C')$ contains $J(A,C)$ and R_k, contradicting (c).

Therefore, $a \in V(z_1Aa_k)$ and $c \in V(z_1Cc_k)$. Moreover, no $(A \cup C)$-bridge of H in $L(A)$ intersects $a_kAy_1 - a_k$ (by (10)). Let S'_k be the union of S_k and those $(A \cup C)$-bridges of H that are contained in $L(C)$ and intersect $c_kCy_1 - c_k$. By (11), if R is an $(A \cup C)$-bridge of H contained in S'_k then $N(R - C) - V(C)$ consists of two adjacent vertices of x_1Xz_1. So by (5), $N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_2, y_2\} \subseteq V(x_1Xz_1)$. Since G is 5-connected, $N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_2, y_2\} \neq \emptyset$.

We may assume that $N(S'_k - \{a_k, c_k\}) - \{y_2, x_2, a_k, c_k\} \neq \{x_1\}$. For, otherwise, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_k, c_k, x_1, x_2, y_2\}$, $X \cup P \cup Q \subseteq G_1$, and

$S'_k \subseteq G_2$. Clearly, $|V(G_1)| \geq 7$. Since G is 5-connected and $y_1y_2 \notin E(G)$, $|V(G_2)| \geq 7$. Hence, the assertion of Theorem 1.1 follows from Lemma 2.4.

Thus, we may let $z \in N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_1, x_2, y_2\}$ such that x_1Xz is maximal. Then $z \neq z_1$. For, suppose $z = z_1$. Let $r \in V(S'_k) - \{a_k, c_k\}$ such that $rz_1 \in E(G)$. Let $r' = r$ if $r \in V(S_k)$ and, otherwise, let $r' \in V(c_kCy_1 - c_k)$ with $r'r \in E(G)$ (which exists by (11)). Let A_k, C_k, R_k be independent (induced) paths in S_k from y_1 to a_k, c_k, r', respectively. Now $z_2Bq \cup Q \cup aAz_1 \cup (z_1rr' \cup R_k) \cup C_k \cup c_kCc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

Let C^* be the subgraph of G induced by the union of $x_1Xz - x_1$ and the vertices of $L(C) - C$ adjacent to $c_kCy_1 - c_k$ (each of which, by (11), has exactly two neighbors on C and exactly two on x_1Xz_1 that are actually on x_1Xz). Clearly, C^* is connected. Let $G_z = G[x_1Xz \cup S'_k + x_2]$ and let G'_z be the graph obtained from $G_z - \{x_1, x_2\}$ by contracting C^* to a new vertex c^*.

Note that G'_z has no disjoint paths from a_k, c_k to c^*, y_1, respectively; as otherwise, such paths, $c_kCc \cup P \cup pBy_2$, and $a_kAa \cup Q \cup qBz_2$ give two disjoint paths in H which would contradict the choice of Y, Z. Hence, by Lemma 2.1, there exists a collection $\mathcal{A}$ of subsets of $V(G'_z) - \{a_k, c_k, c^*, y_1\}$ such that $(G'_z, \mathcal{A}, a_k, c_k, c^*, y_1)$ is 3-planar. We choose $\mathcal{A}$ so that each member of $\mathcal{A}$ is minimal and, subject to this, $|\mathcal{A}|$ is minimal.

We claim that $\mathcal{A} = \emptyset$. For, let $T \in \mathcal{A}$. By (10), $T \cap V(L(A)) = \emptyset$. Moreover, $T \cap V(L(C)) = \emptyset$; for otherwise, by (11), $c^* \in N(T)$ and $|N(T) \cap V(C)| = 2$; so by (11) again (and since C is induced in H), $(G'_z, \mathcal{A} - \{T\}, a_k, c_k, c^*, y_1)$ is 3-planar, contradicting the choice of $\mathcal{A}$. Thus, $G[T]$ has a component, say T', such that $T' \subseteq L(A, C)$. Hence, for any $t \in V(T')$, $L(A, C)$ has a path from t to $aAy_1 - y_1$ (respectively, $cCy_1 - y_1$) and internally disjoint from $A \cup C$. Since G is 5-connected, $\{x_1, x_2\} \cap N(T') \neq \emptyset$. Therefore, for some $i \in [2]$, G' contains a path from x_i to $aAy_1 - y_1$ as well as a path from x_i to $cCy_1 - y_1$, both internally disjoint from $K \cup X$. However, this contradicts (9).

Hence, $(G'_z, a_k, c_k, c^*, y_1)$ is planar. So by (6) and (11), $(G_z - x_2, a_k, c_k, z, x_1, y_1)$ is planar. By (9) and (10), $N(x_2) \cap V(S_k) \subseteq V(a_kAy_1)$. Therefore, since $(G_z - x_2) - a_kAy_1$ is connected (by (10)), (G_z, a_k, c_k, z, x_2) is planar.

We claim that $\{a_k, c_k, z, x_2, y_2\}$ is a 5-cut in G. For, otherwise, by (7) and (9), G' has a path S_1 from x_1 to $z_1Cc_k - \{z_1, c_k\}$ and internally disjoint from $K \cup X$. However, G' has a path S_2 from z to $c_kXy_1 - c_k$ and internally disjoint from $K \cup X$. Now S_1, S_2 contradict the second part of (6).

Hence, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_k, c_k, z, x_2, y_2\}$, $B' \cup P \cup Q \cup zXx_2 \subseteq G_1$, and $G_z \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. So (i) or (ii) or (iii) of Theorem 1.1 follows from Lemma 2.3. $\square$

Now that we have established (12), the remainder of this proof will make heavy use of Q'. Our next goal is to obtain structure around z_1, which is done using claims (13)–(17). We may assume that

(13) $x_1z_1 \in E(X)$, $w \in V(A) - \{y_1, z_1\}$ for every choice of the path W in (7), and G' has no path that is from x_2 to $(A \cup C) - y_1$ and internally disjoint from $K \cup Q' \cup X$.

To prove (13), let P_1, P_2 be the paths in (4) with $q^* = p$. Suppose $x_1z_1 \notin E(X)$. Let $s \in V(x_1Xz_1 - x_1)$ such that $x_1s \in E(X)$. By the first part of (6), G has a path S from s to some $s' \in V(C) - \{y_1, z_1\}$ and internally disjoint from $K \cup Q' \cup X$ (as $Q' \subseteq J(A, C)$). Hence, $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCs' \cup S \cup sx_1) \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now suppose the path W in (7) ends at $w \in V(C) - \{y_1, z_1\}$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCw \cup W) \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Finally, suppose G' has a path S from x_2 to some $s \in V(A \cup C) - \{y_1\}$ and internally disjoint from $K \cup Q' \cup X$. If $s \in V(A - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1As \cup S) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s \in V(C - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cs \cup S) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. □

(14) We may assume that G' has no path that is from y_2Xz_2 to $(A \cup C) - y_1$ and internally disjoint from $K \cup Q' \cup X$, and no path that is from $y_2Xz_1 - z_1$ to $A - z_1$ and internally disjoint from $K \cup Q' \cup X$.

Suppose (14) fails. First, suppose S is a path in G' from some $s \in V(y_2Xz_2)$ to some $s' \in V(A \cup C) - \{y_1\}$ and internally disjoint from $K \cup Q' \cup X$. Then $s \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$. Recall from (13) that $z_1x_1 \in E(G)$. If $s' \in V(C - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cs' \cup S \cup sXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s' \in V(A - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1As' \cup S \cup sXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Now suppose S is a path in G' from $s \in V(y_2Xz_1 - z_1)$ to $s' \in V(A - z_1)$ and internally disjoint from $K \cup Q' \cup X$. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCz_1 \cup z_1x_1) \cup (y_1As' \cup S \cup sXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. □

(15) We may assume that
- $J(A, C) \cap (z_1Cc - c) = \emptyset$,
- any path in $J(A, C)$ that is from $A - \{y_1, z_1\}$ to $(P - c) \cup (Q - \{a, q\}) \cup (Q' - y_1) \cup (B - q)$ and internally disjoint from $K \cup Q'$ must end on $(Q \cup Q') - q$, and
- for any $(A \cup C)$-bridge D of H with $D \neq J(A, C)$, if $V(D) \cap V(z_1Cc - c) \neq \emptyset$ and $u \in V(D) \cap V(z_1Ay_1 - z_1)$ then $J(A, C) \cap (z_1Au - \{z_1, u\}) = \emptyset$.

First, suppose there exists $s \in V(J(A, C)) \cap V(z_1Cc - c)$. Then H has a path S from s to some $s' \in V(P - c) \cup V(Q - a) \cup V(Q' - y_1) \cup V(B - y_2)$ and internally disjoint from

$K \cup Q'$. If $s' \in V(Q'-y_1) \cup V(Q-a) \cup V(z_2Bp-p)$ then $S \cup (Q'-y_1) \cup (Q-a) \cup (z_2Bp-p)$ contains a path S' from s to z_2; so $S' \cup sCz_1 \cup A \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). Hence, $s' \in V(P-c) \cup V(y_2Bp-y_2)$ and, by (2), $s = z_1$. Let P_1, P_2 be the paths in (4) with $q^* = p$ (if $s' \in V(P-c)$) or $q^* = s'$ (if $s' \in V(y_2Bp) - \{p, y_2\}$). Then $S \cup (P-c) \cup P_2$ contains a path S' from z_1 to z_2. Let W, w be given as in (7). By (13), $w \in V(A) - \{y_1, z_1\}$. Now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAw \cup W) \cup z_1x_1 \cup z_1Xy_2 \cup S' \cup (C \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

Now suppose S is a path in $J(A,C)$ from $s \in V(A-\{y_1, z_1\})$ to $s' \in V(P-c) \cup V(B-q)$ and internally disjoint from $K \cup Q'$. Since $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$, $s' \neq y_2$. Let P_1, P_2 be the paths in (4) with $q^* = p$ (if $s' \in V(P-c)$) or $q^* = s'$ (if $s' \in V(B-q)$). Let S' be a path in $P_2 \cup S \cup (P-c)$ from s to z_2. Let W, w be given as in (7). By (13), $w \in V(A) - \{y_1, z_1\}$. Hence, $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (S' \cup sAw \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Finally, suppose D is some $(A \cup C)$-bridge of H with $D \neq J(A,C)$, and assume that there exist $v \in V(D) \cap V(z_1Cc - c)$ and $u \in V(D) \cap V(z_1Ay_1 - z_1)$. Then D has a path T from v to u and internally disjoint from $K \cup Q'$. If there exists $s \in V(J(A,C)) \cap V(z_1Au - \{z_1, u\})$ then, by the second item of (15), $J(A,C)$ has a path S from s to some $s' \in V(Q-a)$ and internally disjoint from K. Now $z_2Bq \cup qQs' \cup S \cup sAz_1 \cup z_1Cv \cup T \cup uAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). □

(16) We may assume $L(A) = \emptyset$.

For, suppose $L(A) \neq \emptyset$. For each $(A \cup C)$-bridge R of H contained in $L(A)$, let $a_1(R), a_2(R) \in V(R \cap A)$ with $a_1(R)Aa_2(R)$ maximal. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(A)$, such that for $i = 2, \ldots, m$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1}(a_1(R_j)Aa_2(R_j))$ (which is a path). Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} a_1(R_j)Aa_2(R_j) = a_1Aa_2$. Let $M = \bigcup_{j=1}^{m} R_j$.

By (c), $J(A,C) \cap (a_1Aa_2 - \{a_1, a_2\}) = \emptyset$. By (d), $L(A,C) \cap (a_1Aa_2 - \{a_1, a_2\}) = \emptyset$. By (10), $a_1, a_2 \in V(z_1Aa)$. So $z_1 \in \{a_1, a_2\}$ or $z_1 \notin N(M \cup a_1Aa_2 - \{a_1, a_2\})$. By (14), $V(z_1Xz_2 - \{y_2, z_1\}) \cap N(M \cup a_1Aa_2 - \{a_1, a_2\}) = \emptyset$. By (13), $x_2 \notin N(M \cup a_1Aa_2 - \{a_1, a_2\})$. Thus, $\{a_1, a_2, x_1, y_2\}$ is a cut in G separating M from X, which is a contradiction (since G is 5-connected). □

(17) $z_1c \in E(C)$, $z_1y_2 \in E(G)$, and z_1 has degree 5 in G.

To prove (17), let C^* be the union of z_1Cc and all $(A \cup C)$-bridges of H intersecting $z_1Cc - c$. By (15), $V(C^* \cap J(A,C)) = \{c\}$.

Suppose $C^* = z_1Cc$. Then, since A, C are induced paths in H and $L(A) = \emptyset$ (by (16)) and $z_1y_2 \in E(G)$ (as z_1 has degree at least 5 in G). Moreover, $z_1Cc = z_1c$; as otherwise any vertex of $z_1Cc - \{c, z_1\}$ would have degree 2 in G) (as $V(C^* \cap J(A,C)) = \{c\}$), a contradiction. Hence, we have (17).

So we may assume $C^* - z_1Cc \neq \emptyset$. Since $G' - X$ is 2-connected and $J(A,C) \cap (zCc - c) = \emptyset$ (by (15)), $(C^* - z_1Cc) \cap (A - z_1) \neq \emptyset$. Moreover, if $|V(z_1Cc)| \geq 3$ then there is a path in C^* from $z_1Cc - \{c, z_1\}$ to $A - z_1$ and internally disjoint from $A \cup C \cup J(A,C)$. Let $a^* \in V(A \cap C^*)$ with a^*Ay_1 minimal, and let $u \in V(z_1Xy_2)$ with uXy_2 minimal such that u is a neighbor of $(C^* - c) \cup (z_1Aa^* - a^*)$.

We may assume that $\{a^*, c, u, x_1, y_2\}$ is a 5-cut in G. First, note, by the first and third item of (15), that $J(A,C) \cap ((z_1Aa^* - a^*) \cup (z_1Cc - c)) = \emptyset$ (in particular, $a^* \in V(z_1Aa)$). Hence, if $u = z_1$ then it follows from (d), (13) and (14) that $\{a^*, c, u, x_1, y_2\}$ is a 5-cut in G. So we may assume $u \neq z_1$. Then G' contains a path T that is from u to some $u' \in V(A - z_1)$ and internally disjoint from $A \cup cCy_1 \cup P \cup Q \cup Q' \cup B'$. Suppose $\{a^*, c, u, x_1, y_2\}$ is not a 5-cut in G. Then by (d), (13) and (14), G' has a path R from $r \in V(z_1Xu - u)$ to $r' \in V(P - c) \cup V(Q - a) \cup V(Q' - y_1) \cup V(B')$ and internally disjoint from $K \cup X$. Note that $r' \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$. If $r' \in V(B' - q)$ then let P_1, P_2 be the paths in (4) with $q^* = r'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup R \cup rXx_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. If $r' \in V(P - c)$ then let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup pPr' \cup R \cup rXx_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. Now assume $r' \in V(Q - a) \cup V(Q' - y_1)$. Then $(Q - a) \cup (Q' - y_1) \cup R$ contains a path R' from r to q. Let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup R' \cup rXx_1) \cup (P_2 \cup P \cup cCy_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a^*, c, u, x_1, y_2\}$, $uXx_2 \cup P \cup Q \subseteq G_1$, and $C^* \cup z_1Cc \cup z_1Aa^* \subseteq G_2$. Clearly, $|V(G_1)| \geq 7$. Suppose $G_2 - y_2$ contains disjoint paths T_1, T_2 from u, x_1 to a^*, c, respectively. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup T_2) \cup (y_1Aa^* \cup T_1 \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume that such T_1, T_2 do not exist. Then by Lemma 2.1, $(G_2 - y_2, u, x_1, a^*, c)$ is planar (as G is 5-connected). If $|V(G_2)| \geq 7$ then, by Lemma 2.3, (i) or (ii) or (iii) of Theorem 1.1 holds. Hence, we may assume that $|V(G_2)| = 6$. Note that $u \neq z_1$; for otherwise, since G is 5-connected, a^* is adjacent to $x_1Xz_1 - \{x_1, z_1\}$, contradicting the first part of (6). Thus, $V(G_2 - G_1) = \{z_1\}$. Since G is 5-connected, we have (17). □

We have now forced a structure around z_1. Next, we study the structure of $G'[B' \cup y_2Xz_2]$ to complete the proof of Theorem 1.1. Recall that B' is the union of B and all B-bridges of H not containing $A \cup C$. We may assume that

(18) $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar.

For, otherwise, by Lemma 2.1, $G'[B' \cup y_2Xz_2]$ has disjoint paths R_1, R_2 from q, p to y_2, z_2, respectively. Now $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup R_2 \cup z_2x_2) \cup (R_1 \cup qQq' \cup Q') \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we may assume (18). □

Since G is 5-connected, G is $(5, V(K \cup Q' \cup y_2Xx_2 \cup z_1x_1))$-connected. Recall that $w_1y_2 \in E(x_1Xy_2)$ by the definition of w_1. Note that w_1y_2 and w_1Xz_1 are independent paths in G from w_1 to y_2, z_1, respectively. So by Lemma 2.6, G has five independent paths Z_1, Z_2, Z_3, Z_4, Z_5 from w_1 to z_1, y_2, z_3, z_4, z_5, respectively, and internally disjoint from $K \cup Q' \cup y_2Xx_2 \cup z_1x_1$, where $z_3, z_4, z_5 \in V(K \cup Q' \cup y_2Xx_2 \cup z_1x_1)$. Note that we may assume $Z_2 = w_1y_2$. Hence, Z_1, Z_2, Z_3, Z_4, Z_5 are paths in G'. By the fact that X is induced in $G' - x_1x_2$ and by the first part of (14), we have $z_3, z_4, z_5 \notin V(y_2Xz_2 \cup z_1x_1)$. By the second part of (14), we have $z_3, z_4, z_5 \notin V(A - z_1)$. By the second part of (5), $z_3, z_4, z_5 \notin V(cCy_1 - c)$. So by (17), $z_3, z_4, z_5 \in V(P) \cup V(Q - a) \cup V(Q') \cup V(B' - y_2)$. Recall that $L(A) = \emptyset$ from (16), and recall W and w from (7) and (13). We may assume that

(19) at least two of Z_3, Z_4, Z_5 end in $B' - y_2$.

First, suppose at least two of Z_3, Z_4, Z_5 end on P. Without loss of generality, let c, z_3, z_4, p occur on P in this order. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $(P_1 \cup Q \cup aAw \cup W) \cup z_2x_2 \cup z_2Xy_2 \cup (Z_1 \cup z_1x_1) \cup Z_2 \cup (Z_4 \cup z_4Pp \cup P_2) \cup (Z_3 \cup z_3Pc \cup cCy_1 \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Now assume at least two of Z_3, Z_4, Z_5 are on $Q \cup Q'$, say Z_3 and Z_4. Then $Z_3 \cup Z_4 \cup Q \cup Q'$ contains two independent paths Z_3', Z_4' from w_1 to z', q, respectively, where $z' \in \{a, y_1\}$. Hence $(Z_1 \cup z_1x_1) \cup Z_2 \cup (Z_3' \cup z'Ay_1) \cup (Z_4' \cup qBz_2 \cup z_2x_2) \cup (y_2Bp \cup P \cup cCy_1) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2.

So we may assume that $z_3 \in V(B') - \{p, q\}$, and hence $Z_3 = w_1z_3$. Suppose none of Z_4, Z_5 ends in $B' - y_2$. Then we may assume $z_4 \in V(P - p)$. Let P_1, P_2 be the paths in (4) with $q^* = z_3$. Then $(Z_1 \cup z_1x_1) \cup Z_2 \cup (Z_4 \cup z_4Pc \cup cCy_1 \cup y_1x_2) \cup z_2x_2 \cup z_2Xy_2 \cup (Z_3 \cup P_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2. □

(20) We may assume that
- w_1 has at most one neighbor in B' that is in qBz_2 or separated from y_2Bp in $G'[B' \cup y_2Xz_2]$ by a 2-cut contained in qBz_2, and
- w_1 has at most one neighbor in B' that is in $y_2Bp - y_2$ or separated from qBz_2 in $G'[B' \cup y_2Xz_2]$ by a 2-cut contained in y_2Bp.

For the first item, suppose there exist distinct $v_1, v_2 \in N(w_1) \cap V(B')$ such that for $i \in [2]$, $v_i \in V(qBz_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating v_i from y_2Bp. Then, since $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $H - y_2$

is 2-connected, $G'[B'+w_1]-y_2Bp$ contains independent paths S_1,S_2 from w_1 to q,z_2, respectively. So $w_1Xx_1 \cup w_1y_2 \cup (S_1 \cup qQq' \cup Q') \cup (S_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices w_1,x_1,x_2,y_1,y_2.

Now suppose there exist distinct $v_1,v_2 \in N(w_1) \cap V(B')$ such that for $i \in [2]$, $v_i \in V(y_2Bp)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in y_2Bp and separating v_i from qBz_2. Then, since $(G'[B' \cup y_2Xz_2],p,q,z_2,y_2)$ is 3-planar (by (18)) and $H-y_2$ is 2-connected, $G'[B'+w_1]-(qBz_2-z_2)$ has independent paths S_1,S_2 from w_1 to p,z_2, respectively. Recall W,w from (7) and (13). Now $w_1Xx_1 \cup w_1y_2 \cup (S_1 \cup P \cup cCy_1 \cup y_1x_2) \cup z_2x_2 \cup z_2Xy_2 \cup S_2 \cup (z_2Bq \cup Q \cup aAw \cup W) \cup G[\{x_1,x_2,y_2\}]$ is a TK_5 in G' with branch vertices w_1,x_1,x_2,y_2,z_2. □

(21) $G'[B' \cup y_2Xz_2]$ has a 2-separation (B_1,B_2) such that $N(w_1) \cap V(B'-y_2) \subseteq V(B_1)$, $pBq \subseteq B_1$, and $y_2Xz_2 \subseteq B_2$.

To prove (21), let $z \in N(w_1) \cap V(B')$ be arbitrary. If there exists a path S in $B'-(pBy_2 \cup (qBz_2-z_2))$ from z_2 to z, then $z_2x_2 \cup z_2Xy_2 \cup (z_2Bq \cup qQq' \cup Q') \cup (S \cup zw_1 \cup w_1Xx_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices x_1,x_2,y_1,y_2,z_2. So we may assume that such path S does not exist. Then, since $(G'[B' \cup y_2Xz_2],p,q,z_2,y_2)$ is 3-planar (by (18)) and $G'-X$ is 2-connected, $z \in V(y_2Bp \cup qBz_2)$ (in which case let $B'_z=z$ and $B''_z=G'[B' \cup y_2Xz_2]$), or $G'[B' \cup y_2Xz_2]$ has a 2-separation (B'_z,B''_z) such that $B'_z \cap B''_z \subseteq y_2Bp \cup qBz_2 \cup y_2Xz_2$, $z \in V(B'_z-B''_z)$ and $z_2 \in V(B''_z-B'_z)$.

We claim that we may assume that w_1 has exactly two neighbors in B', say v_1,v_2, such that $v_1 \in V(y_2Bp-y_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in y_2Bp and separating v_1 from qBz_2, and $v_2 \in V(qBz_2-z_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating v_2 from y_2Bp. This follows from (20) if for every choice of z, $B'_z \cap B''_z \subseteq y_2Bp$ or $B'_z \cap B''_z \subseteq qBz_2$. So we may assume that there exists $v \in N(w_1) \cap V(B')$ such that $pBq \subseteq B'_v$ and we choose v and (B'_v,B''_v) with B'_v maximal. If, for each choice of z, $pBq \subseteq B'_z$ for some choice of (B'_z,B''_z), then, by (18), we have (21). Thus, we may assume that there exists $z \in N(w_1) \cap V(B')$ such that $pBq \not\subseteq B'_z$ for any choice of (B'_z,B''_z). Then $B'_z \cap B''_z \subseteq y_2Bp$ or $B'_z \cap B''_z \subseteq qBz_2$. First, assume $B'_z \cap B''_z \subseteq qBz_2$. Then by the maximality of B'_v, $B'-y_2Bp$ has independent paths T_1,T_2 from z_2 to q,z, respectively. Hence, $z_2x_2 \cup z_2Xy_2 \cup (T_1 \cup qQq' \cup Q') \cup (T_2 \cup zw_1 \cup w_1Xx_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices x_1,x_2,y_1,y_2,z_2. Now assume $B'_z \cap B''_z \subseteq y_2Bp$. Then by (20), for any $t \in N(w_1) \cap V(B'_v)$, $t \notin V(y_2Bp-y_2)$ and $G'[B' \cup y_2Xz_2]$ has no 2-cut contained in y_2Bp and separating t from qBz_2. If for every choice of $t \in N(w_1) \cap V(B'_v)$, we have $t \in V(qBz_2-z_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating t from y_2Bp then the claim follows from (20). Hence, we may assume that t can be chosen so that $t \notin V(qBz_2-z_2)$ and $G'[B' \cup y_2Xz_2]$ has no 2-cut contained in qBz_2 and separating t from y_2Bp. Then, by (18) and 2-connectedness of $G'-X$, $G[B'+w_1]-(qBz_2-z_2)$ has independent paths S_1,S_2 from w_1 to p,z_2, respectively. Now $w_1Xx_1 \cup w_1y_2 \cup (S_1 \cup P \cup cCy_1 \cup y_1x_2) \cup z_2x_2 \cup z_2Xy_2 \cup S_2 \cup (z_2Bq \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Thus, we may assume that the paths Z_3, Z_4, Z_5 (see the paragraph preceding (19)) may be chosen so that $Z_3 = w_1v_1$, $Z_4 = w_1v_2$, and Z_5 ends at some $v_3 \in V(P \cup Q \cup Q') - \{a, p, q\}$. Suppose $v_3 \in V(P - p)$. Let P_1, P_2 be the paths in (4) with $q^* = v_1$. Recall W, w from (7) and (13). Then $w_1Xx_1 \cup w_1y_2 \cup (Z_5 \cup v_3Pc \cup cCy_1 \cup y_1x_2) \cup z_2x_2 \cup z_2Xy_2 \cup (w_1v_1 \cup P_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Now assume $v_3 \in V(Q \cup Q') - \{a, q\}$. Then $(B' - y_2Bp) \cup Z_5 \cup Q \cup Q' \cup (A - z_1) \cup w_1v_2$ has independent paths R_1, R_2 from w_1 to y_1, z_2, respectively. So $w_1Xx_1 \cup w_1y_2 \cup R_1 \cup (R_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2. This completes the proof of (21). □

By (21), let $V(B_1 \cap B_2) = \{t_1, t_2\}$ with $t_1 \in V(y_2Bp)$ and $t_2 \in V(qBz_2)$. Choose $\{t_1, t_2\}$ so that B_2 is minimal. Then we may assume that $(G'[B_2 + x_2], t_1, t_2, x_2, y_2)$ is 3-planar. For, otherwise, by Lemma 2.1, $G'[B_2 + x_2]$ contains disjoint paths T_1, T_2 from t_1, t_2 to x_2, y_2, respectively. Then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBt_1 \cup T_1) \cup (Q' \cup q'Qq \cup qBt_2 \cup T_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Suppose there exists $ss' \in E(G)$ such that $s \in V(z_1Xw_1 - w_1)$ and $s' \in V(B_2) - \{t_1, t_2\}$. Then $s' \notin V(X)$, as X is induced in $G' - x_1x_2$. By (19) and (21), at least two of Z_3, Z_4, Z_5 end in B_1 (and these paths have length 1). So by (20), we may assume that Z_3 ends at $z_3 \in V(B_1)$ and $B_1 - qBt_2$ contains a path R from z_3 to p. Note that $Z_3 = w_1z_3$. By the minimality of B_2 and 2-connectedness of $H - y_2$, $(B_2 - t_1) - (y_2Xz_2 - z_2)$ contains independent paths R_1, R_2 from z_2 to s', t_2, respectively. Now $z_2x_2 \cup z_2Xy_2 \cup (R_1 \cup s's \cup sXx_1) \cup (R_2 \cup t_2Bq \cup qQq' \cup Q') \cup (y_1Cc \cup P \cup R \cup z_3w_1y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, we may assume that ss' does not exist. Since G is 5-connected, $\{t_1, t_2, y_2, x_2\}$ is not a cut in G. So H has a path T from some $t \in V(y_2Xx_2) - \{y_2, x_2\}$ to some $t' \in V(P \cup Q \cup Q' \cup A \cup C) - \{p, q\}$ and internally disjoint from $K \cup Q'$. By (14), $t' \notin V(A \cup C) - \{y_1\}$.

If $t' \in V(P - p)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup cPt' \cup T \cup tXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we assume $t' \in V(Q \cup Q') - \{a, q\}$.

If $q \neq q'$ or $t' \in V(Q)$ then $(T \cup Q \cup Q') - q$ has a path Q^* from t to y_1; now $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (Q^* \cup sXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So assume $q = q'$ and $t' \in V(Q) - \{a, q\}$. Then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup aQt' \cup T \cup tXx_2) \cup (Q' \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. ∎

Acknowledgment

We thank the anonymous referees for their detailed comments and helpful suggestions.

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