The Modular Chromatic Number of Trees

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Abstract: The concept of modular coloring and modular chromatic number was proposed by *F. Okamoto*, *E. Salehi* and *P. Zhang* in 2010. They also proved that the modular chromatic numbers of trees are either 2 or 3. In this paper, we give a necessary and sufficient condition for a tree to have modular chromatic number 3.

Keywords: trees, modular coloring, modular chromatic number

1. Introduction

We follow [3] for graph-theoretical terms and notations not defined in this paper.

Coloring is a very active research area of graph theory. The famous Four Color Conjecture was one of the problems that make significance contribution to the development of graph theory. The Four Color Conjecture was proved in 1977 by *Appel*, *Haken* and *Koch* with the help of computers. In 1996, the proof was simplified by *Robertson*, *Sanders*, *Seymour* and *Thomas*. But computers are still needed in the new proof. Besides the concept of the classic coloring, scholars also propose many new concepts of coloring on graphs such as List Coloring (see 8.4 of [3]) and Fractional Coloring (see Chapter 7 of [4]).

In 2010, *F. Okamoto*, *E. Salehi* and *P. Zhang* gave the definition of modular coloring in their paper [1]. Let G=(V,E) be a simple graph. For integer k, let c: V → {0,1, … ,k-1} be a mapping on the set of vertices of G. Given a mapping c, let $\sigma(v)=\sum_{u\in N(v)} c(u) \pmod k$ for every vertex c. If σ is a proper coloring for G, in other words, $\sigma(x)\neq\sigma(y)$ for any pair of adjacent vertices x and y, then c is called a **modular k-coloring** of G. An integer k is said to be the **modular chromatic number** of G, denoted by mc(G)=k, if k is the smallest integer such that there exists a modular k-coloring for G.

In [1] the modular chromatic numbers of a few families of graphs were determined. For path of order n P_n, mc(P_n)=2.Let C_n be a cycle of order n, then mc(C_n)=2; and mc(C_n)=3 otherwise. For grid graph M (the Cartesian product of two paths), [2] proved that mc(M)=2.

It was proved in [1] that the modular chromatic number of trees are either 2 or 3 by giving a modular 3-coloring to an arbitrary tree constructively. In other words, the result and its proof do not give a characterization of trees with modular chromatic number 3.

In this paper, we give a necessary and sufficient condition for trees with modular chromatic number of 3. Before stating our main result, we need a few definitions.

The path of order 5 is called the **W-graph**. Let (X, Y) be a bipartition of the W-graph with |X| = 3. Vertices in X are called **big** vertices, and vertices in Y are **little** vertices.

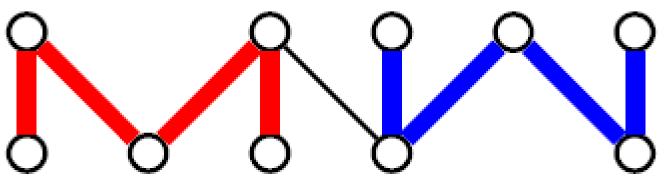
W-trees are defined recursively from the W-graph. The W-graph is a W-tree. The graph obtained from two W-trees by pasting along a big vertex is also a W-tree; a big (little) vertex of the two smaller W-trees is still a big(little) vertex of the larger W-tree.

Obviously, a W-tree has an odd number of big vertices, and the degree of a little vertex is always 2.

Let N[X] be the closed neighborhood of the vertex set X, namely N[X] = X ∪ N(X).

Let T be a tree with bipartition (A, B). If there exists a vertex subset X A such that the subgraph induced by N[X] is a W-tree and X is the set of big vertices, then we say T has an **induced W-tree with respect to A**. A tree is said to be a **WM-tree** if it has induced W-trees with respect to both A and B.

The smallest order of a WM-tree is 10, and there is only one WM-tree of order 10 up to isomorphism, as shown in the following figure.



The main result of our paper is the following theorem.

Theorem 1. *Let T be a tree with bipartition (A, B), then mc(T) = 3 if and only if T is a WM-tree.*

2. Proof of Theorem 1

(**Sufficiency**) Suppose to the contrary that mc(T) = 2, and c is a modular 2-coloring of T . First assume σ(a) = 1 for all a ∈ A. Since T is a WM-tree, T has an induced W-tree with bipartition (X, Y), where X A is the set of big vertices. Then we have

$$1=|X| = \sum_{x\in X} \sigma(x)=\sum_{y\in Y} 2c(y)=0(\mathrm{mod}2),$$

a contradiction. If σ(a) = 0 for all a ∈ A, we may apply a similar argument to an induced W-tree with respect to B to conclude a contradiction.

(**Necessity**) It suffices to show if T is not a WM-tree, then mc(T) = 2. Without loss of generality, we may assume T has no induced W-tree with respect to A. We will prove by induction on the order of T the following slightly stronger result.

Claim 1. *If T has no induced W-tree with respect to A, then there exists a modular 2-coloring c of T , such that σ(a) = 1 for a ∈ A.*

Without loss of generality, we assume every modular 2-coloring c (or c_i , c') in the rest of the proof has the property that c(a) = 0 for a ∈ A (or a ∈ A_i , a ∈ A').

It is not hard to verify that Claim 1 is true for all the trees with order at most 4. So we assume T is a not a WM-tree, and has at least 5 vertices in the rest of the proof.

Case I. There exists a leaf b_0 ∈ B. Let a_0 be the unique neighbor of b_0 , and let T_1 , T_2 , . . . , T_k (k ≥ 1) be all the components of T − {a_0 , b_0 }. The trees T_i have bipartition (A_i , B_i) where A_i = A ∩ V (T_i). Then none of the trees T_i (1 ≤ i ≤ k) has an induced W-tree with respect to A_i . Otherwise, the induced W-tree of some T_i is also an induced W-tree of T with respect to A. By induction hypothesis, there exist modular 2-colorings c_i for T_i such that σ_i (a) = 1 for a ∈ A_i . Let b_i ∈ V (T_i) be the unique neighbor of a_0 in T , define a coloring for T as below:

$$c(v)=\begin{cases} c_i(v), v\in V(T_i) \\ 0, v=a_0 \\ 1-\sum_{1}^{k} c(b_i) (\mathrm{mod}2), v=b_0 \end{cases}$$

It is easy to check c is the coloring we desire.

Case II. All the leaves are in A.

Case IIa. There exists a leaf a_0 ∈ A whose unique neighbor b_0 has degree bigger than 2. Let T_1' , T_2' , . . . , T_k' (k ≥ 2) be all the components of T − {a_0 , b_0 }. Let T_i be the subgraph of T induced by V (T'_i) ∪ {a_0 , b_0 }.Every subtree T_i has a natural bipartition (A_i , B_i) where A_i = A ∩ V (T_i). Then none of T_i has an induced W-tree with respect to A_i . So by induction hypothesis, we have modular 2-colorings c_i for T_i such that σ_i (a) = 1 for a ∈ A_i . Since a_0 is a leaf for all T_i , we have σ_i (a_0) = c_i (b_0) = 1 for 1 ≤ i ≤ k. Thus, the following coloring is well-defined, and is a modular 2-coloring for T with the desired property:

$$c(v)=\begin{cases} c_i(v), v\in B \\ 0, v\in A \end{cases}$$

Now assume every leaf of T has a neighbor of degree exactly 2. We make the following claim.

Claim 2. *Let T be a tree with bipartition (A, B), all the leaves of T are in A, and every leaf has a neighbor of degree 2. Then at least one of the following two conditions is satisfied.*

(1)There exist two leaves a_1 and a_2 whose distance $d_T(a_1, a_2)$ in T is 4,

(2)There exists a leaf a_1 whose second neighbor also has degree 2.

Proof of Claim 2. Let a_0 and a_1 be two leaves with maximum distance among all pairs of leaves in T . Consider the second neighbor of a_1 , which is denoted by a_3 . If $d(a_3) = 2$, then condition (2) is satisfied. Suppose $d(a_3) \geq 3$. Let T_0 be a component of $T - a_3$ which contains neither a_0 nor a_1 . Let $a_2 \in A$ be a leaf of T_0 such that it is also a leaf of T . Then the distance between a_2 and a_3 in T must be 2. Otherwise a_0 and a_2 will be a pair of leaves with longer distance in T , which contradicts with our selection of a_0 and a_1 . Therefore a_1 and a_2 are the two vertices satisfy condition (1).

Case IIb. There exist two leaves a_1 and a_2 in A with distance 4. Let b_1 and b_2 be the unique neighbor of a_1 and a_2 , respectively. And let a_3 be the unique common second neighbor of a_1 and a_2 . Let $T' = T - \{a_1, a_2, b_1, b_2\}$ with bipartition (A' , B'), where $A' = A \cap V(T)$. Then T' has no induced W-tree with respect to A . Suppose to the contrary that T' has an induced W-tree W with respect to A'. If $a_3 \in W$, then W is also an induced W-tree of T with respect to A. If $a_3 \in W$, then the subtree induced by $V(W) \cup \{a_1, a_2, b_1, b_2\}$ is an induced W-tree of T with respect to A. In either case, there is a contradiction.

Hence, by induction hypothesis, T' has a modular 2-coloring c' such that $\sigma'(a) = 1$ for $a \in A'$. Let

$$c(v)=\begin{cases} c'(v), v\in T' \\ 0, v\in \{a_1, a_2\} \\ 1, v\in \{b_1, b_2\} \end{cases}$$

It is easy to check that c is a modular 2-coloring of T such that $\sigma(a) = 1$ for $a \in A$.

Case IIc. There exists a leaf a_1 whose second neighbor has degree 2. Let b_1 and a_2 be the neighbor and second neighbor of a_1 , respectively. Since the degree of a_2 is 2, let b_2 be the other neighbor of a_2 .

Let $T_1, T_2, \ldots, T_k$ ($k \geq 1$) be all the components of $T - \{a_1, a_2, b_1, b_2\}$. Let (A_i, B_i) be the bipartition of T_i where $A_i = A \cap V(T_i)$. By similar argument as in the proof of Case IIb, T_i has no induced W-tree with respect to A_i. Let c_i be a modular 2-coloring of T_i with $\sigma_i(a) = 1$ for $a \in A_i$. Then

$$c(v) = \begin{cases} c_i(v), v \in T_i \\ 0, v \in \{a_1, a_2, b_2\} \\ 1, v = b_1 \end{cases}$$

is the desired modular 2-coloring for T .

By summing up all the above cases, we have proved Claim 1.

References

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