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The Kelmans-Seymour conjecture III: 3-vertices in K_4^-

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ABSTRACT

Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct, such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. We show that one of the following holds: $G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2, or G contains a TK_5 in which x_1 is not a branch vertex, or $\{x_2, y_1, y_2\}$ may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$ contains TK_5.

1. Introduction

We use notation and terminology from [2,3]. For a graph K, we use TK to denote a *subdivision* of K. The vertices of TK corresponding to the vertices of K are its *branch vertices*. Kelmans [6] and, independently, Seymour [11] conjectured that every 5-connected nonplanar graph contains TK_5. In [7,8], this conjecture is shown to be true for graphs containing K_4^-.

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In [2] we outline a strategy to prove the Kelmans-Seymour conjecture for graphs not containing K_4^-. Let G be a 5-connected nonplanar graph not containing K_4^-. Then by a result of Kawarabayashi [4], G contains an edge e such that G/e is 5-connected. If G/e is planar, we can apply a discharging argument. So assume G/e is not planar. Let M be a maximal connected subgraph of G such that G/M is 5-connected and nonplanar (so $|V(M)| \geq 2$). Let z denote the vertex representing the contraction of M, and let $H = G/M$. Then one of the following holds:

(a) H contains a subgraph K such that $K \cong K_4^-$ and z has degree 2 in K.
(b) H contains a subgraph K such that $K \cong K_4^-$ and z has degree 3 in K.
(c) H does not contain K_4^-, and there exists $T \subseteq H$, with $z \in V(T)$ and either $T \cong K_2$ or $T \cong K_3$, such that H/T is 5-connected and planar.
(d) H does not contain K_4^-, and for any $T \subseteq H$ with $z \in V(T)$ and either $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

Note that local structure around z (in particular, K_4^- containing z) will help us find TK_5 in G from certain TK_5 in H.

In [2] we deal with certain special separations and the results can be used to take care of (c). In [3] we prove results that can be used to take care of (a). In this paper, we prove the following, which can be used to take care of (b).

Theorem 1.1. *Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Then one of the following holds:*

(i) *G contains a TK_5 in which x_1 is not a branch vertex.*
(ii) *$G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
(iii) *x_2, y_1, y_2 may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$ contains TK_5.*

This paper is organized as follows. In Section 2, we list a number of known results that will be used in the proof of Theorem 1.1. The steps we take to prove Theorem 1.1 is quite similar to the arguments in [3]. First, we find a path in G from x_1 to x_2 such that the graph obtained from G by removing that path satisfies certain connectivity requirements. What is different here is that we need the path to include x_1z_0 or x_1z_1. We find this path in Section 3, see Figs. 1 and 2. In Section 4, we derive further structural information of the graph G. In Section 5, we find a substructure of G consisting of five additional paths, see Fig. 3. In Section 6, we use this substructure to find a TK_5 in $G - \{x_1v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$.

2. Lemmas

For each positive integer m, let $[m] = \{1, \dots, m\}$. For convenience, we recall a technical notion from [2] (originated from [12]). A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \dots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

- for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,
- for $i \in [k]$, $|N(A_i)| \leq 3$, and
- if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each $i \in [k]$) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc in the plane with no edge crossing.

If, in addition, $b_1, \dots, b_n$ are vertices in G such that $b_i \notin A_j$ for all $i \in [n]$ and $j \in [k]$, $p(G, \mathcal{A})$ can be drawn in a closed disc in the plane with no edge crossing, and $b_1, \dots, b_n$ occur on the boundary of the disc in this cyclic order, then we say that $(G, \mathcal{A}, b_1, \dots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_1, \dots, b_n)$ is 3-planar.

We can now state the following result of Seymour [12]; equivalent versions can be found in [1,14,13].

Lemma 2.1. *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then exactly one of the following holds:*

(i) *G contains disjoint paths from s_1, s_2 to t_1, t_2, respectively.*
(ii) *(G, s_1, s_2, t_1, t_2) is 3-planar.*

We also state a generalization of Lemma 2.1, which is a consequence of Theorems 2.3 and 2.4 in [10].

Lemma 2.2. *Let G be a graph, $v_1, \dots, v_n \in V(G)$ be distinct, and $n \geq 4$. Then exactly one of the following holds:*

(i) *There exist $1 \leq i < j < k < l \leq n$ such that G contains disjoint paths from v_i, v_j to v_k, v_l, respectively.*
(ii) *$(G, v_1, v_2, \dots, v_n)$ is 3-planar.*

We will make use of the following result of Perfect [9]. A collection of paths in a graph are said to be *independent* if no internal vertex of any path in this collection belongs to another path in the collection.

Lemma 2.3. *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G - u)$. Suppose there exist k independent paths from u to distinct $a_1, \dots, a_k \in A$, respectively, and otherwise disjoint* *from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \dots, P_n$ in G from u to n distinct vertices in A and otherwise disjoint from A then $P_1, \dots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i \in [k]$.*

We will also use a result of Watkins and Mesner [15] on cycles through three vertices.

Lemma 2.4. *Let G be a 2-connected graph and let y_1, y_2, y_3 be three distinct vertices of G. There is no cycle in G containing $\{y_1, y_2, y_3\}$ if, and only if, one of the following holds:*

- (*i*) *There exists a 2-cut S in G and there exist pairwise disjoint subgraphs D_{y_i} of $G-S$, $i \in [3]$, such that $y_i \in V(D_{y_i})$ and each D_{y_i} is a union of components of $G-S$.*
- (*ii*) *There exist 2-cuts S_{y_i} of G, $i \in [3]$, and pairwise disjoint subgraphs D_{y_i} of G, such that $y_i \in V(D_{y_i})$, each D_{y_i} is a union of components of $G - S_{y_i}$, there exists $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*
- (*iii*) *There exist pairwise disjoint 2-cuts S_{y_i} in G, $i \in [3]$, and pairwise disjoint subgraphs D_{y_i} of $G - S_{y_i}$ such that $y_i \in V(D_{y_i})$, D_{y_i} is a union of components of $G - S_{y_i}$, and $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_{y_i} for $i \in [3]$.*

The next result is Theorem 3.2 from [7].

Lemma 2.5. *Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Suppose $G - x_1x_2$ contains a path X between x_1 and x_2 such that $G - X$ is 2-connected, $X - x_2$ is induced in G, and $y_1, y_2 \notin V(X)$. Let $v \in V(X)$ such that $x_2v \in E(X)$. Then G contains a TK_5 in which x_2v is an edge and x_1, x_2, y_1, y_2 are branch vertices.*

It is easy to see that under the conditions of Lemma 2.5, $G - \{x_2u : u \notin \{v, x_1, y_1, y_2\}\}$ contains TK_5. The next result is Corollary 2.11 in [5]. Let G be a graph and $A \subseteq V(G)$. For any positive integer k, we say that G is (k, A)-connected if, for each $S \subseteq V(G)$ with $|S| < k$, every component of $G - S$ must contain a vertex from A. We say that (G, A) is *plane* if G is drawn in the plane with no edge crossings, and the vertices in A are incident with the outer face of G; and we say that (G, A) is *planar* if G admits a planar drawing such that (G, A) is plane.

Lemma 2.6. *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is plane, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Suppose there exists $w \in N(a)$ such that w is not incident with the outer face of $G - a$. Then*

- (*i*) *the vertices of $G - a$ cofacial with w induce a cycle C_w in $G - a$, and*
- (*ii*) *$G - a$ contains paths P_1, P_2, P_3 from w to $A - \{a\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A| = 1$ for $i \in [3]$.*

The next four results are (essentially) Theorem 1.1, Theorem 1.2, Proposition 2.3 and Proposition 4.2, respectively, in [2]. Note that condition (*iii*) in three of these four results (Theorem 1.1, Theorem 1.2 and Proposition 4.2 in [2]) states that G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$ and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$. This condition implies that G contains a K_4^- in which a is of degree 2. We only need the weaker versions of these results.

Lemma 2.7. *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*
(*ii*) *$G - a$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which a is of degree 2.*

Lemma 2.8. *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*
(*ii*) *$G - a$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which a is of degree 2.*
(*iii*) *For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains TK_5.*

Lemma 2.9. *Let G be a graph, $A \subseteq V(G)$, and $a \in A$ such that $|A| = 6$, $|V(G)| \geq 8$, $(G - a, A - \{a\})$ is planar, and G is $(5, A)$-connected. Then one of the following holds:*

(*i*) *$G - a$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which the degree of a is 2.*
(*ii*) *G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, $|V(G_2)| \geq 7$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar.*

Lemma 2.10. *Let G be a 5-connected nonplanar graph and $a \in V(G)$ such that $G - a$ is planar. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*
(*ii*) *$G - a$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which a is of degree 2.*

We also need Lemma 3.1 in [3]. Let G be a graph and $\{u, v\} \subseteq V(G)$. We say that a sequence of blocks $B_1, \dots, B_k$ in G is *a chain of blocks* from u to v if $|V(B_i) \cap V(B_{i+1})| = 1$ for $i \in [k-1]$, $V(B_i) \cap V(B_j) = \emptyset$ for any $1 \le i < i+1 < j \le k$, $u, v \in V(B_1)$ are distinct when $k = 1$, and $u \in V(B_1) - V(B_2)$ and $v \in V(B_k) - V(B_{k-1})$ when $k \ge 2$. A block is *nontrivial* if it is 2-connected.

Lemma 2.11. *Let G be a graph and $A = \{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that G is $(4, A)$-connected. Suppose there exists a path X in $G - x_1x_2$ from x_1 to x_2 such that $G - X$ contains a chain of blocks B from y_1 to y_2. Then one of the following holds:*

- *(i) There is a 4-separation (G_1, G_2) in G such that $B + \{x_1, x_2\} \subseteq G_1$, $|V(G_2)| \ge 6$, and $(G_2, V(G_1 \cap G_2))$ is planar.*
- *(ii) There exists an induced path X' in $G - x_1x_2$ from x_1 to x_2 such that $G - X'$ is a chain of blocks from y_1 to y_2 and contains B.*

3. Nonseparating paths

Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. To take care of case (b) described in Section 1, we need to find a path in G satisfying certain properties (see *(iv)* of Lemma 3.2). As a first step, we prove the following.

Lemma 3.1. *Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Let $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$ be distinct. Then one of the following holds:*

- *(i) G contains a TK_5 in which x_1 is not a branch vertex.*
- *(ii) $G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
- *(iii) There exist $i \in \{0, 1\}$ and an induced path X in $G - x_1$ from z_i to x_2 such that $(G - x_1) - X$ is a chain of blocks from y_1 to y_2, $z_{1-i} \notin V(X)$, and one of y_1, y_2 is contained in a nontrivial block of $(G - x_1) - X$.*

Proof. We may assume $G - x_1$ contains disjoint paths X, Y from z_1, y_1 to x_2, y_2, respectively. For, otherwise, since G is 5-connected, it follows from Lemma 2.1 that $(G - x_1, z_1, y_1, x_2, y_2)$ is planar; so *(i)* or *(ii)* holds by Lemma 2.10.

Hence $(G - x_1) - X$ contains a chain of blocks from y_1 to y_2, say B. We may assume that $(G - x_1) - X$ is a chain of blocks from y_1 to y_2. For otherwise, we may apply Lemma 2.11 to conclude that G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $B + \{x_1, x_2, z_1\} \subseteq G_1$, $|V(G_2)| \ge 7$, and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar. If $|V(G_1)| \ge 7$ then *(i)* or *(ii)* follows from Lemma 2.7. So assume $|V(G_1)| \le 6$. Since $y_1y_2 \notin E(G)$, $|V(G_1)| = 6$ and $|V(B)| = 3$. Let $V(B) = \{y_1, y_2, v\}$. Since G is 5-connected and

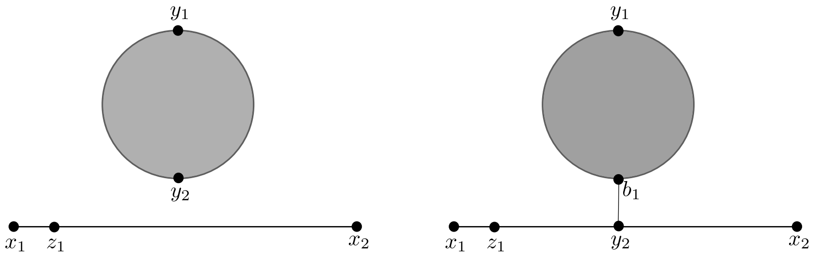

Fig. 1. Structure of G in *(iii)* of Lemma 3.2.

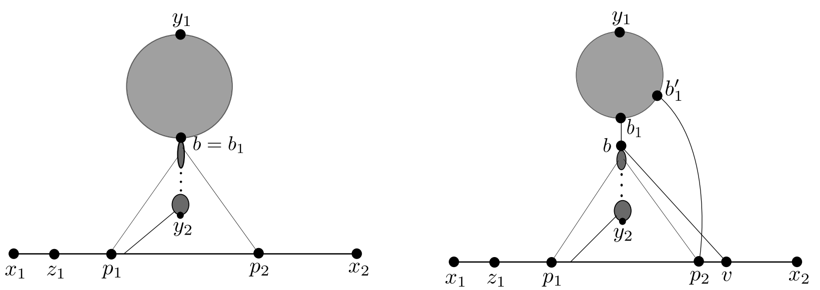

Fig. 2. Structure of G in *(iv)* of Lemma 3.2 (with $j = 1$).

$y_1y_2 \notin E(G)$, $\{x_1, x_2, y_1, y_2, z_1\} = V(G_1 \cap G_2) = N(v)$. Hence, $G[\{v, x_1, x_2, y_1\}] - x_1x_2$ is a K_4^- in which x_1 is of degree 2, and *(ii)* holds.

We may further assume that $z_0 \notin V(X)$. For, suppose $z_0 \in V(X)$. Since G is 5-connected and X is induced in $G - x_1$, every vertex of X has at least two neighbors in $(G - x_1) - X$. Hence, $(G - x_1) - z_0Xx_2$ is also a chain of blocks from y_1 to y_2. So we may use z_0Xx_2 as X.

Let B_1, B_2 be the blocks in $(G-x_1)-X$ containing y_1, y_2, respectively. If one of B_1, B_2 is nontrivial, then *(iii)* holds. So we may assume that $|V(B_1)| = |V(B_2)| = 2$. Since X is induced and G is 5-connected, there exists $z \in N(x_2) - (\{x_1, y_1, y_2\} \cup V(X))$, and y_1 and y_2 each have at least two neighbors on $X - x_2$. Let Z be a path in $(G-x_1)-X-\{y_1, y_2\}$ from z_0 to z. Then $(G - x_1) - Z$ contains a chain of blocks, say B, from y_1 to y_2, and the blocks in $(G - x_1) - Z$ containing y_1 or y_2 are nontrivial. Thus, we may apply Lemma 2.11 to G, Z and B. If *(ii)* of Lemma 2.11 holds, we have *(iii)*. So assume *(i)* of Lemma 2.11 holds. Then, as in the second paragraph of this proof, *(i)* or *(ii)* follows from Lemma 2.7. □

We have results from [2,3,8] that can be used to deal with *(i)* or *(ii)* of Lemma 3.1. In this paper, we deal with *(iii)* of Lemma 3.1. Parts *(iii)* and *(iv)* of the next lemma give more detailed structure of G when *(iii)* of Lemma 3.1 occurs. We refer the reader to Fig. 1 for *(iii)* of Lemma 3.2, and Fig. 2 for *(iv)* of Lemma 3.2.

For a graph H and a subgraph L of H, an L-*bridge* of H is a subgraph of H that is induced by an edge in $E(H) - E(L)$ with both incident vertices in $V(L)$, or is induced by the edges in a component of $H - L$ as well as edges from that component to L.

Lemma 3.2. *Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Let $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$ be distinct and let $G' := G - \{x_1x : x \notin \{x_2, y_1, y_2, z_0, z_1\}\}$. Then one of the following holds:*

(i) *G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.*
(ii) *$G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
(iii) *The notation of y_1, y_2, z_0, z_1 may be chosen so that $(G - x_1) - x_2y_2$ has an induced path X from z_1 to x_2 such that $z_0, y_1 \notin V(X)$, and $(G - x_1) - X$ is 2-connected.*
(iv) *The notation of z_0, z_1 may be chosen so that there exists an induced path X in $G - x_1$ from z_1 to x_2 such that $z_0 \notin V(X)$, $(G - x_1) - X$ is a chain of blocks $B_1, \dots, B_k$ from y_1 to y_2 with B_1 nontrivial, $z_0 \in V(B_1)$ when z_1 has at least two neighbors in B_1, and $(G - x_1) - x_2y_2$ has a 3-separation (Y_1, Y_2) such that $V(Y_1 \cap Y_2) = \{b, p_1, p_2\}$, z_1, p_1, p_2, x_2 occur on X in this order, $Y_1 = G[B_1 \cup z_1Xp_1 \cup p_2Xx_2 + b]$, $p_1Xp_2 + y_2 \subseteq Y_2$, and p_1, p_2 each have at least two neighbors in $Y_2 - B_1$. Moreover, if $b \notin V(B_1)$ then $V(B_2) = \{b_1, b\}$ with $b_1 \in V(B_1)$, and there exists some $j \in [2]$ such that p_{3-j} has a unique neighbor b_1' in B_1, b has a unique neighbor v in $X - p_1Xp_2$ such that $vp_{3-j} \in E(X) - E(p_1Xp_2)$, $vb_1 \notin E(G)$, and $p_jb \notin E(G)$.*

Proof. We begin our proof by applying Lemma 3.1 to G, x_1, x_2, y_1, y_2. If (i) or (ii) of Lemma 3.1 holds then assertion (i) or (ii) of this lemma holds. So we may assume that (iii) of Lemma 3.1 holds. Then we may assume $(G - x_1) - x_2y_2$ has an induced path X from z_1 to x_2 such that $z_0, y_1 \notin V(X)$, $(G - x_1) - X$ has a nontrivial block B_1 containing y_1, and y_1 is not a cut vertex of $(G - x_1) - X$. (Note that we are not requiring the stronger condition that $(G - x_1) - X$ be a chain of blocks from y_1 to y_2.) We choose such a path X that

(1) B_1 is maximal,
(2) subject to (1), whenever possible, $(G - x_1) - X$ has a chain of blocks from y_1 to y_2 and containing B_1, and
(3) subject to (2), the component H of $(G - x_1) - X$ containing B_1 is maximal.

Let $\mathcal{C}$ be the set of all components of $(G - x_1) - X$ different from H. Then

(4) $\mathcal{C} = \emptyset$, $H = (G - x_1) - X$, and if $y_2 \notin V(X)$ then H is a chain of blocks from y_1 to y_2 and containing B_1.

First, suppose $\mathcal{C} = \emptyset$. Then $H = (G - x_1) - X$. Suppose $y_2 \notin V(X)$. Then H has a chain of blocks, say B, from y_1 to y_2 and containing B_1. By applying Lemma 2.11 to $G-x_1, z_1, x_2, y_1, y_2, X$ can be chosen so that $(G-x_1)-X$ is a chain of blocks from y_1 to y_2, or G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $B + \{x_1, x_2, z_1\} \subseteq G_1$, $|V(G_2)| \geq 7$ and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar. We may assume the latter as otherwise (4) holds. Since B_1 is nontrivial and $y_1y_2 \notin E(G)$, $|V(B)| \geq 4$. So $|V(G_1)| \geq 7$; and (i) or (ii) follows from Lemma 2.7.

Now suppose $\mathcal{C} \neq \emptyset$. For each $D \in \mathcal{C}$, let $u_D, v_D \in V(X)$ be the neighbors of D in $G-x_2y_2$ with u_DXv_D maximal, and assume that z_1, u_D, v_D, x_2 occur on X in this order. Define a new graph $G_{\mathcal{C}}$ such that $V(G_{\mathcal{C}}) = \mathcal{C}$, and two components $C, D \in \mathcal{C}$ are adjacent in $G_{\mathcal{C}}$ if $u_CXv_C - \{u_C, v_C\}$ contains a neighbor of D or $u_DXv_D - \{u_D, v_D\}$ contains a neighbor of C.

Note that, for any component $\mathcal{D}$ of $G_{\mathcal{C}}$, $\bigcup_{D\in V(\mathcal{D})} u_DXv_D$ is a subpath of X. Since G is 5-connected, there exist $y \in V(H)$ and $C \in V(\mathcal{D})$ with $N(y) \cap V(u_CXv_C - \{u_C, v_C\}) \neq \emptyset$.

If $y \neq y_1$ then let Q be an induced path in $G[C + \{u_C, v_C\}] - x_2y_2$ from u_C to v_C, and let X' be obtained from X by replacing u_CXv_C with Q. Then B_1 is contained in a block of $(G-x_1)-X'$, and y_1 is not a cut vertex of $(G-x_1)-X'$. Moreover, if $(G-x_1)-X$ has a chain of blocks from y_1 to y_2 then so does $(G-x_1)-X'$. However, the component of $(G-x_1)-X'$ containing B_1 is larger than H, contradicting (3).

So we may assume that $y = y_1$ for all choices of y and C. Let $uXv := \bigcup_{D\in V(\mathcal{D})} u_DXv_D$. Since G is 5-connected, $y_2 \in V(\bigcup_{D\in V(\mathcal{D})} D) \cup V(uXv - \{u,v\})$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{u, v, x_1, x_2, y_1\}$, $G_1 := G[\bigcup_{D\in V(\mathcal{D})} D \cup uXv + \{x_1, x_2, y_1\}]$, and $B_1 \cup z_1Xu \cup vXx_2 \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. Since $G[\{x_1, x_2, y_1\}] \cong K_3$, (i) or (ii) follows from Lemma 2.8. This completes the proof of (4).

Let $\mathcal{B}$ be the set of all B_1-bridges of H. For each $D \in \mathcal{B}$, let $b_D \in V(D) \cap V(B_1)$ and $u_D, v_D \in V(X)$ be the neighbors of $D - b_D$ in $G - x_2y_2$ with u_DXv_D maximal. Define a new graph $G_{\mathcal{B}}$ such that $V(G_{\mathcal{B}}) = \mathcal{B}$, and two B_1-bridges $C, D \in \mathcal{B}$ are adjacent in $G_{\mathcal{B}}$ if $u_CXv_C - \{u_C, v_C\}$ contains a neighbor of $D - b_D$ or $u_DXv_D - \{u_D, v_D\}$ contains a neighbor of $C - b_C$. Note that, for any component $\mathcal{D}$ of $G_{\mathcal{B}}$, $\bigcup_{D\in V(\mathcal{D})} u_DXv_D$ is a subpath of X, whose ends are denoted by $u_{\mathcal{D}}, v_{\mathcal{D}}$. We let $S_{\mathcal{D}} := \{b_D : D \in V(\mathcal{D})\} \cup (N(u_{\mathcal{D}}Xv_{\mathcal{D}} - \{u_{\mathcal{D}}, v_{\mathcal{D}}\}) \cap V(B_1))$. We may assume that

(5) for any component $\mathcal{D}$ of $G_{\mathcal{B}}$, $|S_{\mathcal{D}}| \leq 2$ and $y_2 \in \left(\bigcup_{D\in V(\mathcal{D})} V(D) - S_{\mathcal{D}}\right) \cup V(u_{\mathcal{D}}Xv_{\mathcal{D}} - \{u_{\mathcal{D}}, v_{\mathcal{D}}\})$.

First, we may assume $|S_{\mathcal{D}}| \leq 2$. For, suppose $|S_{\mathcal{D}}| \geq 3$. Then there exist $D \in V(\mathcal{D})$, $r_1, r_2 \in V(u_DXv_D) - \{u_D, v_D\}$, and distinct $r_1', r_2' \in V(B_1)$ such that for $i \in [2]$, $r_ir_i' \in E(G)$ or $r_i' = b_{D_i}$ for some $D_i \in V(\mathcal{D}) - \{D\}$. (To see this, we choose $D \in V(\mathcal{D})$ such that there is a maximum number of vertices in B_1 from which G has a path to $u_DXv_D - \{u_D, v_D\}$ and internally disjoint from $B_1 \cup D \cup X$. If this number is at most 1, we can show that $|S_{\mathcal{D}}| \leq 2$.) Let $R_i = r_i r_i'$ if $r_i r_i' \in E(G)$; and otherwise let R_i be a path in $G[D_i + r_i]$ from r_i to r_i' and internally disjoint from X. Let Q denote an induced path in $G[D + \{u_D, v_D\}] - b_D - x_2y_2$ between u_D and v_D, and let X' be obtained from X by replacing $u_D X v_D$ with Q. Clearly, the block of $(G - x_1) - X'$ containing y_1 contains B_1 as well as the path $R_1 \cup r_1 X r_2 \cup R_2$. Note that $y_1 \neq b_D$ (as y_1 is not a cut vertex in H). Moreover, if $y_1 = r_i'$ for some $i \in [2]$ then D_i is not defined and $r_i r_i' \in E(G)$. So y_1 is not a cut vertex of $(G - x_1) - X'$. Thus, X' contradicts the choice of X, because of (1).

Now assume $y_2 \notin \bigcup_{D \in V(\mathcal{D})} V(D) \cup V(u_{\mathcal{D}} X v_{\mathcal{D}}) - (\{u_{\mathcal{D}}, v_{\mathcal{D}}\} \cup S_{\mathcal{D}})$. Then $S_{\mathcal{D}} \cup \{u_{\mathcal{D}}, v_{\mathcal{D}}, x_1\}$ is a cut in G; so $|S_{\mathcal{D}}| = 2$ (as G is 5-connected). Let $S_{\mathcal{D}} = \{p, q\}$. Then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{p, q, u_{\mathcal{D}}, v_{\mathcal{D}}, x_1\}$, $B_1 \cup z_1 X u_{\mathcal{D}} \cup v_{\mathcal{D}} X x_2 \subseteq G_1$, and G_2 contains $u_{\mathcal{D}} X v_{\mathcal{D}}$ and the B_1-bridges of H contained in $\mathcal{D}$. If $(G_2 - x_1, u_{\mathcal{D}}, p, v_{\mathcal{D}}, q)$ is planar then, since $|V(G_i)| \geq 7$ for $i \in [2]$, then (i) or (ii) follows from Lemma 2.7. So we may assume that $(G_2 - x_1, u_{\mathcal{D}}, p, v_{\mathcal{D}}, q)$ is not planar. Then by Lemma 2.1, $G_2 - x_1$ contains disjoint paths S, T from $u_{\mathcal{D}}, p$ to $v_{\mathcal{D}}, q$, respectively.

We apply Lemma 2.11 to $G_2 - x_1$ and $\{u_{\mathcal{D}}, v_{\mathcal{D}}, p, q\}$. If (i) of Lemma 2.11 holds then from the separation in $G_2 - x_1$, we derive a 5-separation (G_1', G_2') in G such that $x_1 \in V(G_1' \cap G_2')$, $B_1 \cup T + x_1 \subseteq G_1'$, $|V(G_2')| \geq 7$, and $(G_2' - x_1, V(G_1' \cap G_2') - \{x_1\})$ is planar. So (i) or (ii) follows from Lemma 2.7. We may thus assume that (ii) of Lemma 2.11 holds. Thus, there is an induced path S' in $G_2 - x_1$ from $u_{\mathcal{D}}$ to $v_{\mathcal{D}}$ such that $(G_2 - x_1) - S'$ is a chain of blocks from p to q. Now let X' be obtained from X by replacing $u_{\mathcal{D}} X v_{\mathcal{D}}$ with S'. Then y_1 is not a cut vertex of $(G - x_1) - X'$, and the block of $(G - x_1) - X'$ containing y_1 contains B_1 and $(G_2 - x_1) - S'$, contradicting (1). This completes the proof of (5).

We may also assume that

(6) for any B_1-bridge D of H, $y_2 \notin V(u_D X v_D) - \{u_D, v_D\}$.

For, suppose $y_2 \in V(u_D X v_D) - \{u_D, v_D\}$ for some B_1-bridge D of H. Choose X and D so that, subject to (1)-(3), $u_D X v_D$ is maximal.

We claim that $\{D\}$ is a component of $G_{\mathcal{B}}$. For, otherwise, by the maximality of $u_D X v_D$, there exists a B_1-bridge C of H such that $N(C) \cap V(u_D X v_D - \{u_D, v_D\}) \neq \emptyset$. Let T be an induced path in $G[D + \{u_D, v_D\}] - b_D - x_2y_2$ from u_D to v_D. By replacing $u_D X v_D$ with T we obtain a path X' from X such that y_1 is not a cut vertex in $(G - x_1) - X'$, B_1 is contained in a block of $(G - x_1) - X'$, and $(G - x_1) - X'$ has a chain of blocks from y_1 to y_2 and containing B_1, contradicting the choice of X (in (2) as $y_2 \in V(X)$).

Hence, by (5), $V(G_{\mathcal{B}}) = \{D\}$. If G has an edge from $u_D X v_D - \{u_D, v_D\}$ to $B_1 - y_1$ or if y_1 has two neighbors, one on $u_D X y_2 - u_D$ and one on $v_D X y_2 - v_D$, then let X' be obtained from X by replacing $u_D X v_D$ with an induced path in $G[D + \{u_D, v_D\}] - b_D - x_2y_2$ from u_D to v_D. In the former case, $(G - x_1) - X'$ has a chain of blocks from y_1 to y_2 and containing B_1, contradicting (2). In the latter case, $(G - x_1) - X'$ has a cycle containing

$\{y_1, y_2\}$. So by Lemmas 2.11 and 2.7, (i) or (ii) holds, or there is an induced path X^* in $G - x_1$ from z_1 to x_2 such that $y_1, y_2 \notin V(X^*)$ and $(G - x_1) - X^*$ is 2-connected, and (iii) holds.

Therefore, we may assume $N(u_D X v_D - \{u_D, v_D\}) \cap V(B_1) = \{y_1\}$, and $N(y_1) \cap V(u_D X v_D - \{u_D, v_D\}) \subseteq V(u_D X y_2)$ or $N(y_1) \cap V(u_D X v_D - \{u_D, v_D\}) \subseteq V(v_D X y_2)$. Let $L = G[D \cup u_D X v_D]$ and let $L' = G[L + y_1]$.

Suppose L has disjoint paths from u_D, b_D to v_D, y_2, respectively. We may apply Lemma 2.11 to L and $\{u_D, v_D, b_D, y_2\}$. If L has an induced path S from u_D to v_D such that $L - S$ is a chain of blocks from b_D to y_2 then let X' be obtained from X by replacing $u_D X v_D$ with S; now $(G - x_1) - X'$ is a chain of blocks from y_1 to y_2 and containing B_1, contradicting (2). So we may assume that L has a 4-separation as given in (i) of Lemma 2.11. Thus G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $|V(G_i)| \geq 7$ for $i \in [2]$, and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar. Hence, (i) or (ii) follows from Lemma 2.7.

Thus, we may assume that such disjoint paths do not exist in L. By Lemma 2.1, there exists a collection $\mathcal{A}$ of subsets of $V(L) - \{b_D, u_D, v_D, y_2\}$ such that $(L, \mathcal{A}, u_D, b_D, v_D, y_2)$ is 3-planar.

We now show that $(L' - y_1 v_D, u_D, b_D, v_D, y_2, y_1)$ is planar (when $N(y_1) \cap V(u_D X v_D - \{u_D, v_D\}) \subseteq V(u_D X y_2)$), or $(L' - y_1 u_D, u_D, b_D, v_D, y_1, y_2)$ is planar (when $N(y_1) \cap V(u_D X v_D - \{u_D, v_D\}) \subseteq V(v_D X y_2)$). Since the arguments for these two cases are the same, we consider only the case when $N(y_1) \cap V(u_D X v_D - \{u_D, v_D\}) \subseteq V(u_D X y_2)$. Since G is 5-connected, for each $A \in \mathcal{A}$, $\{x_1, y_1\} \subseteq N(A)$ and $|N_L(A)| = 3$; and since $N(y_1) \cap V(L - b_D) \subseteq V(u_D X y_2)$ and G is 5-connected, $|N_L(A) \cap V(X)| = 2$. For each such A, let $a_1, a_2 \in N_L(A) \cap V(X)$ and let $a \in N_L(A) - V(X)$. If for each $A \in \mathcal{A}$, $(G[A \cup \{a, a_1, a_2, y_1\}], a_1, a, a_2, y_1)$ is planar, then $(L' - y_1 v_D, u_D, b_D, v_D, y_2, y_1)$ is planar. So we may assume that, for some choice of A, $(G[A \cup \{a, a_1, a_2, y_1\}], a_1, a, a_2, y_1)$ is not planar. (Note that $G[A \cup \{a, a_1, a_2, y_1\}]$ is $(4, \{a, a_1, a_2, y_1\})$-connected.) Hence, by Lemma 2.1, $G[A \cup \{a, a_1, a_2, y_1\}]$ contains disjoint paths from a_1, a to a_2, y_1, respectively. So we can apply Lemma 2.11 to $G[A \cup \{a, a_1, a_2, y_1\}]$ and $\{a, a_1, a_2, y_1\}$. If (i) of Lemma 2.11 occurs then G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $|V(G_i)| \geq 7$ for $i \in [2]$, and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar; so (i) or (ii) follows from Lemma 2.7. Hence, we may assume that (ii) of Lemma 2.11 occurs. Then $G[A \cup \{a, a_1, a_2, y_1\}]$ has an induced path S from a_1 to a_2 such that $G[A \cup \{a, a_1, a_2, y_1\}] - S$ is a chain of blocks from y_1 to a. Let X' be obtained from X by replacing $a_1 X a_2$ with S. Then the block of $(G - x_1) - X'$ containing y_1 contains B_1 and $G[A \cup N_L(A) \cup \{y_1\}] - S$, and y_1 is not a cut vertex in $(G - x_1) - X'$, contradicting (1).

Hence, G has a 6-separation (G_1, G_2) with $V(G_1 \cap G_2) = \{b_D, u_D, v_D, x_1, y_1, y_2\}$ and $G_2 - x_1 = L' - y_1 v_D$ (or $G_2 - x_1 = L' - y_1 u_D$). Since $(L' - y_1 v_D, u_D, b_D, v_D, y_2, y_1)$ (or $(L' - y_1 u_D, u_D, b_D, v_D, y_1, y_2)$) is planar and $|V(G_2)| \geq 8$, (i) or (ii) follows from Lemma 2.9 and then Lemma 2.7. This completes the proof of (6).

If $y_2 \in V(X)$ then by (4), (5) and (6), H is 2-connected; so (iii) holds. Thus we may assume $y_2 \notin V(X)$. Then by (4), H is a chain of blocks from y_1 to y_2 and containing

B_1, which we denote as $B_1, \dots, B_k$. We may assume $k \geq 2$; as otherwise, (iii) holds. Let $y_1 \in V(B_1) - V(B_2)$, $y_2 \in V(B_k) - V(B_{k-1})$, and $b_i \in V(B_i) \cap V(B_{i+1})$ for $i \in [k-1]$. Note that

- if z_1 has at least two neighbors in B_1 then $z_0 \in V(B_1)$.

For, suppose z_1 has at least two neighbors in B_1 and $z_0 \notin V(B_1)$. Let $w \in V(X)$ with wXx_2 minimal such that w is a neighbor of $\bigcup_{i=2}^{k} B_i - b_1$ in $G - x_2y_2$. Recall that $z_0 \notin V(X)$. Let W be an induced path in $G[(\bigcup_{i=2}^{k} B_i) + w - b_1] - x_2y_2$ from z_0 to w, and let $X' = W \cup wXx_2$. Then, since y_1 is not a cut vertex of H, y_1 is not a cut vertex of $(G - x_1) - X'$. However, the block of $(G - x_1) - X'$ containing y_1 contains $B_1 + z_1$, contradicting (1).

We further choose X so that, subject to (1), (2) and (3),

(7) B_k is maximal.

Let $q_1, q_2 \in V(X)$ be the neighbors of $\bigcup_{i=2}^{k} B_i - b_1$ in $G - x_2y_2$ with q_1Xq_2 maximal, and assume that z_1, q_1, q_2, x_2 occur on X in this order. We may assume that

(8) there exists $b_1' \in V(B_1 - b_1)$ such that $N(q_1Xq_2 - \{q_1, q_2\}) \cap V(B_1 - b_1) = \{b_1'\}$.

For, otherwise, by (5), $N(q_1Xq_2 - \{q_1, q_2\}) \cap V(B_1 - b_1) = \emptyset$. Hence, (iv) holds with $b = b_1$, $p_1 = q_1$, and $p_2 = q_2$.

Thus G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b_1, b_1', q_1, q_2, x_1, y_2\}$, $G_1 = G[(B_1 \cup z_1Xq_1 \cup q_2Xx_2) + \{x_1, y_2\}]$ and G_2 contains $\bigcup_{i=2}^{k} B_i$ and q_1Xq_2. Note that $xy \notin E(G_2)$ for all $\{x, y\} \subseteq V(G_1 \cap G_2)$. We may assume that

(9) there exists a collection $\mathcal{A}$ of subsets of $V(G_2 - x_1) - \{b_1, b_1', q_1, q_2\}$ such that $(G_2 - x_1, \mathcal{A}, b_1, q_1, b_1', q_2)$ is 3-planar.

For, otherwise, by Lemma 2.1, $G_2 - x_1$ has disjoint paths S, S' from b_1, q_1 to b_1', q_2, respectively. We may choose S' to be induced and let X' be obtained from X by replacing q_1Xq_2 with S'. Then $B_1 \cup S$ is contained in a block of $(G - x_1) - X'$. Thus, by (1), $y_1 = b_1'$ and y_1 is a cut vertex of $(G - x_1) - X'$.

If $G_2 - x_1$ is $(4, \{b_1, b_1', q_1, q_2\})$-connected, let $G_2' = G_2$ and $J = \emptyset = T$. Now suppose $G_2 - x_1$ is not $(4, \{b_1, b_1', q_1, q_2\})$-connected. Since G is 5-connected and y_2 is the only vertex in $V(G_2) - \{b_1, b_1', q_1, q_2, x_1\}$ adjacent to x_2, $G_2 - x_1$ has a 3-cut T separating y_2 from $\{b_1, b_1', q_1, q_2\}$. Choose T so that the component J of $(G_2 - x_1) - T$ containing y_2 is maximal. Let G_2' be obtained from $G_2 - J$ by adding an edge between every pair of vertices in T.

Then $G_2' - x_1$ is $(4, \{b_1, b_1', q_1, q_2\})$-connected, and the paths S, S' also give rise to disjoint paths in $G_2' - x_1$ from b_1, q_1 to b_1', q_2, respectively. Hence by applying Lemma 2.11 (and then Lemma 2.7) to $G_2' - x_1$ and $\{q_1, q_2, b_1, b_1'\}$, we find an induced path S'' in $G_2' - x_1$ from q_1 to q_2 such that $(G_2' - x_1) - S''$ is a chain of blocks from b_1 to b_1'. Note that S'' gives rise to an induced path S^* in G_2 by replacing $S'' \cap G_2'[T]$ with an induced path in $G_2[J + T]$. Let X^* be obtained from X by replacing $q_1 X q_2$ with S^*. Then B_1 is properly contained in a block of $(G - x_1) - X^*$. Since $y_2 \notin V(X)$, $b_1' \notin T \cup V(J)$. Hence, y_1 is not a cut vertex in $(G - x_1) - X^*$. Thus, we have a contradiction to (1) which completes the proof of (9).

We may assume that, for any choice of $\mathcal{A}$ in (9),

(10) $\mathcal{A} \neq \emptyset$.

For, otherwise, $G_2 - x_1$ has no cut of size at most 3 separating y_2 from $\{b_1, b_1', q_1, q_2\}$. Hence, G_2 is $(5, \{b_1, b_1', q_1, q_2, x_1\})$-connected and $(G_2 - x_1, b_1, q_1, b_1', q_2)$ is planar. We may assume that $G_2 - x_1$ is a plane graph with b_1, q_1, b_1', q_2 incident with its outer face.

If y_2 is also incident with the outer face of $G_2 - x_1$ then (i) or (ii) holds by applying Lemma 2.9 to $G_2 - x_1$ and $\{b_1, b_1', q_1, q_2, x_1, y_2\}$ (and then applying Lemma 2.7). So assume that y_2 is not incident with the outer face of $G_2 - x_1$. Then by Lemma 2.6, the vertices of $G_2 - x_1$ cofacial with y_2 induce a cycle C_{y_2} in $G_2 - x_1$, and $G_2 - x_1$ contains paths P_1, P_2, P_3 from y_2 to $\{b_1, b_1', q_1, q_2\}$ such that $V(P_i \cap P_j) = \{y_2\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_{y_2})| = |V(P_i) \cap \{b_1, b_1', q_1, q_2\}| = 1$ for $i \in [3]$. Let $K = C_{y_2} \cup P_1 \cup P_2 \cup P_3$.

If P_1, P_2, P_3 end at q_1, b_1 (respectively, b_1'), q_2, respectively, then let Q be a path in B_1 from y_1 to b_1 (respectively, b_1'); now $K \cup (x_1 z_1 \cup z_1 X q_1) \cup (x_1 x_2 \cup x_2 X q_2) \cup (x_1 y_1 \cup Q) \cup x_1 y_2$ is a TK_5 in G'. For the remaining cases, let Q_1, Q_2 be independent paths in B_1 from y_1 to b_1', b_1, respectively. If P_1, P_2, P_3 end at b_1, q_1, b_1', respectively, then $K \cup Q_1 \cup Q_2 \cup (y_1 x_1 z_1 \cup z_1 X q_1) \cup y_1 x_2 y_2$ is a TK_5 in G'. If P_1, P_2, P_3 end at b_1, q_2, b_1', respectively, then $K \cup Q_1 \cup Q_2 \cup (y_1 x_2 \cup x_2 X q_2) \cup y_1 x_1 y_2$ is a TK_5 in G'. This proves (10). So (i) or (ii) holds.

By (10) and the 5-connectedness of G, we may let $\mathcal{A} = \{A\}$ and $y_2 \in A$. Moreover, $|N(A) - \{x_1, x_2\}| = 3$. Choose $\mathcal{A}$ so that

(11) A is maximal.

Then

(12) $b_1' \notin N(A)$, and we may assume that $N(b') \cap V(B_k - b_{k-1}) = \emptyset$ for all $b' \in N(b_1') \cap V(q_1 X q_2)$, and $|N(A) \cap V(q_1 X q_2)| = 2$.

Suppose $b_1' \in N(A)$. Then $A \cap V(q_1Xq_2 - \{q_1, q_2\}) \neq \emptyset$. Hence, $|N(A) \cap V(q_1Xq_2)| \geq 2$. Since $y_2 \in A$ and $y_2 \notin V(X)$, $|N(A) \cap V(B_i)| \geq 1$ for some $2 \leq i \leq k$, a contradiction as $|N(A) - \{x_1, x_2\}| = 3$.

Now suppose there exist $b' \in N(b_1') \cap V(q_1Xq_2)$ and $b'' \in N(b') \cap V(B_k - b_{k-1})$. Then B_k has independent paths P_2, P_2' from y_2 to b_{k-1}, b'', respectively. Let P_1, P_1' be independent paths in B_1 from y_1 to b_1, b_1', respectively, and let P be a path in $\bigcup_{j=2}^{k-1} B_j$ from b_1 to b_{k-1}. Then $G[\{x_1, x_2, y_1, y_2\}] \cup (b'Xz_1 \cup z_1x_1) \cup b'Xx_2 \cup (b'b_1' \cup P_1') \cup (b'b'' \cup P_2') \cup (P_1 \cup P \cup P_2)$ is a TK_5 in G' with branch vertices b', x_1, x_2, y_1, y_2. Hence, (i) holds.

Finally, assume $|N(A) \cap V(q_1Xq_2)| \leq 1$. Then, since $B_k - b_{k-1}$ has at least two neighbors on q_1Xq_2 (as G is 5-connected), B_k is 2-connected and $V(B_k - b_{k-1}) \not\subseteq A$. Hence, $|N(A) \cap V(B_k)| \geq 2$. Let $q_1', q_2' \in N(B_k - b_{k-1}) \cap V(X)$ such that $q_1'Xq_2'$ is maximal. Then there exists $b' \in N(b_1') \cap V(q_1'Xq_2' - \{q_1', q_2'\})$; otherwise $V(B_k \cup q_1'Xq_2') - \{b_{k-1}, q_1', q_2'\}$ contradicts the choice of A in (11). Since G is 5-connected and $(G_2 - x_1, \mathcal{A}, b_1, q_1, b_1', q_2)$ is 3-planar, b' has a neighbor b'' in $B_k - b_{k-1}$. Now (i) holds by the above paragraph.

So $|N(A) \cap V(q_1Xq_2)| \geq 2$. Indeed, $|N(A) \cap V(q_1Xq_2)| = 2$, since $(G - x_1) - X$ is connected, $y_2 \notin V(X)$, and $|N(A) - \{x_1, x_2\}| = 3$. This concludes the proof of (12).

Since $|N(A) \cap V(q_1Xq_2)| = 2$ (by (12)), there exists $2 \leq l \leq k-1$ such that $b_l \in N(A)$ and $\bigcup_{j=l+1}^{k} V(B_j) \subseteq A$. (Here $l \neq 1$ since $\bigcup_{j=2}^{k} V(B_j) \not\subseteq A$.) Note that $N(A) \cap V(q_1Xq_2) \neq \{q_1, q_2\}$, as b_1' has a neighbor in $q_1Xq_2 - \{q_1, q_2\}$. We may assume that

(13) there exists $i \in [2]$ such that $q_i \in N(A)$ and $N(q_i) \cap V(G_2 - x_1) \subseteq A \cup N(A)$.

For, suppose otherwise. Then for $i \in [2]$, $q_i \notin N(A)$ or $N(q_i) \cap V(G_2 - x_1) \not\subseteq A \cup N(A)$. Hence, $G_2[\bigcup_{j=2}^{l} B_j + \{q_1, q_2\} - b_1]$ contains an induced path P from q_1 to q_2.

We may assume $b_1' \neq y_1$. For, suppose $b_1' = y_1$. Since G is 5-connected, there exists $t \in [2]$ such that $G[\bigcup_{j=l+1}^{k} B_j \cup q_1Xq_2 + y_1] - \{b_l, q_{3-t}\}$ has independent paths P_1, P_2 from y_2 to y_1, q_t, respectively. If q_t has a neighbor $s \in V(B_1)$ then let S be a path in B_1 from s to y_1; now $G[\{x_1, x_2, y_1, y_2\}] \cup (x_1z_1 \cup z_1Xq_1 \cup P \cup q_2Xx_2) \cup (q_ts \cup S) \cup P_2 \cup P_1$ is a TK_5 in G' with branch vertices q_t, x_1, x_2, y_1, y_2, and (i) holds. So assume that q_t has no neighbor in B_1. Then we may assume $q_t \notin \{z_1, x_2\}$ and $q_tx_2 \notin E(X)$; for otherwise, $\{b_1, q_{3-t}, x_1, x_2, y_1\}$ is a 5-cut in G containing the triangle $x_1x_2y_1x_1$, and (i) or (ii) follows from Lemma 2.8. Now let $vq_t \in E(X) - E(q_1Xq_2)$. Then $G[B_1 + v]$ has independent paths R_1, R_2 from v to y_1, b_1, respectively. Let R be a path in $G[\bigcup_{j=2}^{l} B_j + q_{3-t}]$ from b_1 to q_{3-t}. Then $G[\{x_1, x_2, y_1, y_2\}] \cup R_1 \cup (vq_t \cup P_2) \cup (R_2 \cup R \cup (X - (q_1Xq_2 - q_{3-t})) \cup x_1z_1) \cup P_1$ is a TK_5 in G' with branch vertices v, x_1, x_2, y_1, y_2, and (i) holds.

Let $t_1, t_2 \in V(X - x_2) \cap N(B_k - b_{k-1})$ with t_1Xt_2 maximal. Then $t_1 \neq t_2$ as $\{b_{k-1}, t_1, x_1, x_2\}$ is not a cut in G. We claim that $G[B_k \cup t_1Xt_2] - b_{k-1}$ is 2-connected. For, suppose not. Then $G[B_k \cup t_1Xt_2]$ has a 2-separation (L_1, L_2) such that $b_{k-1} \in V(L_1 \cap L_2)$, $t_1Xt_2 \subseteq L_1$, and $y_2 \in L_2$. Now $V(L_1 \cap L_2) \cup \{x_1, x_2\}$ is a cut in G, a contradiction.

Let X' be obtained from X by replacing q_1Xq_2 with P. Then $(G-x_1)-X'$ has a chain of blocks from y_1 to y_2, in which B_1 is a block containing y_1, and the block containing y_2 contains $(B_k - b_{k-1}) \cup t_1Xt_2$ (whose size is larger than B_k as $t_1 \neq t_2$). Since $b'_1 \neq y_1$, y_1 is not a cut vertex in $(G-x_1)-X'$. This contradicts the choice of X for (7) (subject to (1), (2) and (3)). So we have (13).

Then $q_{3-i} \notin N(A)$ (as $N(A) \cap V(q_1Xq_2) \neq \{q_1, q_2\}$), and $x_2 \neq q_i$ (otherwise $N(A) \cup \{x_1\}$ would be a cut in G of size at most 4). Let $a \in N(A) - \{x_1, x_2, q_i, b_l\}$. Then $a \in V(X)$ and $\{a, b_1, b'_1, b_l, q_{3-i}, x_1\}$ is a cut in G. So G has a 6-separation (G'_1, G'_2) such that $V(G'_1 \cap G'_2) = \{a, b_1, b'_1, b_l, q_{3-i}, x_1\}$ and $G'_2 := G_2 - (A \cup \{q_i\})$. Note that $(G'_2 - x_1, b_1, b_l, a, b'_1, q_{3-i})$ is planar.

If $|V(G'_2)| \geq 8$ then we may apply Lemma 2.9 to (G'_1, G'_2) and conclude, with help from Lemma 2.7, that (i) or (ii) holds. So assume $|V(G'_2)| = 6$ or $|V(G'_2)| = 7$. Note that $G - x_1$ has a separation (Y_1, Y_2) such that $V(Y_1 \cap Y_2) = \{a, b_l, q_i\}$, $aXq_i + y_2 \subseteq Y_2$, $Y_1 = G[B_1 \cup G'_2 \cup X - (q_iXa - \{a, q_i\})]$.

Case 1. $|V(G'_2)| = 6$.

Then $l = 2$ and $b_2q_{3-i}, aq_{3-i}, ab'_1 \in E(G)$. We may assume that $b_2q_i \notin E(G)$. For, suppose $b_2q_i \in E(G)$. Let P be a path in $\bigcup_{j=3}^{k-1} B_j$ from b_2 to b_{k-1}. Since G is 5-connected, $B_k - b_{k-1}$ has at least two neighbors on q_iXa. Thus, we may choose $a_1a_2 \in E(G)$ with $a_1 \in V(q_iXa - q_i)$ and $a_2 \in V(B_k - b_{k-1})$. Let Q_1, Q_2 be independent paths in B_k from y_2 to b_{k-1}, a_2, respectively, and P_1, P_2 be independent paths in Y_1 from y_1 to b_1, b'_1, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup (b_2q_1 \cup q_1Xz_1 \cup z_1x_1) \cup (b_2q_2 \cup q_2Xx_2) \cup (P \cup Q_1) \cup (b_2b_1 \cup P_1) \cup (P_2 \cup b'_1a \cup aXa_1 \cup a_1a_2 \cup Q_2)$ is a TK_5 in G' with branch vertices b_2, x_1, x_2, y_1, y_2. So (i) holds.

We claim that $ab_1 \notin E(G)$. For, otherwise, let P be an induced path in $G[\bigcup_{j=3}^{k} B_j + q_i]$ from q_i to b_2. Let X' be obtained from X by replacing q_iXq_{3-i} with $P \cup b_2q_{3-i}$. Then, in $(G-x_1)-X'$, there is a block containing both B_1 and a, and y_1 is not a cut vertex. This contradicts (1).

If $q_{3-i}b_1 \notin E(G)$ then (iv) holds with $b = b_2$, $p_j = q_i$, $p_{3-j} = a$, and $v = q_{3-i}$. So we may assume $q_{3-i}b_1 \in E(G)$. Note that $q_{3-i} \neq x_1$ as $x_1 \notin V(X)$ and $q_{3-i} \in V(X)$. We consider two cases: $x_2 \neq q_{3-i}$ and $x_2 = q_{3-i}$.

First, suppose $x_2 \neq q_{3-i}$. Since G is 5-connected, x_2 has at least one neighbor in $B_1 - b'_1$. Thus, $G[B_1 + x_2]$ has independent paths P_1, P_2 from b_1 to x_2, b'_1, respectively. If $G[Y_2 + x_2]$ contains a path P from q_i to x_2 and containing $\{a, b_2\}$ then $G[\{b_1, b_2, q_{3-i}\}] \cup P \cup P_1 \cup (ab'_1 \cup P_2) \cup aq_{3-i} \cup (x_2x_1z_1 \cup z_1Xq_1) \cup x_2Xq_2$ is a TK_5 in G' with branch vertices $a, b_1, b_2, q_{3-i}, x_2$, and (i) holds. Thus, it remains to prove the existence of P. Note that $G[Y_2 + x_2]$ is $(4, \{a, b_2, q_i, x_2\})$-connected. First, consider the case when $G[Y_2 + x_2]$ has disjoint paths from b_2, x_2 to a, q_i, respectively. Then by Lemma 2.11 and then Lemma 2.7, (i) or (ii) holds, or there is a path S in $G[Y_2 + x_2]$ from a to b_2 such that $G[Y_2 + x_2] - S$ is a chain of blocks from q_i to x_2. Now the existence of P follows from the fact that Y_2 is 2-connected. So assume $G[Y_2 + x_2]$ has no disjoint paths from b_2, x_2 to a, q_i, respectively. Then by Lemma 2.1, $(G[Y_2 + x_2], b_2, x_2, a, q_i)$ is planar. If $|V(G[Y_2 + x_2])| \geq 6$ then (i)

or (ii) follows from Lemma 2.7. So assume $|V(G[Y_2+x_2])| = 5$. If $ab_2 \in E(G)$ then $G[\{q_i, a, b_2, y_2\}] \cong K_4^-$ (as $b_2q_i \notin E(G)$); and if $ab_2 \notin E(G)$ then $ax_1 \in E(G)$ (as G is 5-connected), and $G[\{q_i, a, x_1, y_2\}]$ contains a K_4^- in which x_1 is of degree 2. So (ii) holds.

Now suppose $x_2 = q_{3-i}$. Then we may assume that $b_1' \neq y_1$, for otherwise $G[\{a, x_1, x_2, y_1\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds. Thus B_1 has independent paths P_1, P_2 from b_1 to y_1, b_1', respectively. If Y_2 has a cycle C containing $\{a, b_2, y_2\}$, then $C \cup G[\{a, b_1, b_2, x_2\}] \cup y_2x_2 \cup (P_2 \cup b_1'a) \cup (P_1 \cup y_1x_1y_2)$ is a TK_5 in G' with branch vertices $a, b_1, b_2, q_{3-i}, y_2$ (as we assume $b_1q_{3-i} \in E(G)$). So we may assume that the cycle C in Y_2 does not exist. Since Y_2 is 2-connected, it follows from Lemma 2.4 that Y_2 has 2-cuts S_u, for $u \in \{a, b_2, y_2\}$, separating u from $\{a, b_2, y_2\} - \{u\}$. Since G is 5-connected, we see that S_{y_2} separates $\{q_i, y_2\}$ from $\{a, b_2\}$. Hence, S_{y_2} separates y_2 from $\{a, q_i, y_2\}$; so, since G is 5-connected, $x_1b_2, x_2b_2 \in E(G)$. Now $G[\{b_1, b_2, x_1, x_2\}]$ contains a K_4^- in which x_1 is of degree 2 (as we assume $b_1q_{3-i} \in E(G)$), and (ii) holds.

Case 2. $|V(G_2')| = 7$.

Let $z \in V(G_2') - \{a, b_1', b_1, b_l, q_{3-i}, x_1\}$. Suppose $z \notin V(X)$. Then $b_1'a \in E(G)$. Since G is 5-connected and B_1 is a block of H, $zb_1' \notin E(G)$ and $za, zb_1, zb_l, zq_{3-i}, zx_1 \in E(G)$. We may assume $b_1'q_{3-i} \notin E(G)$, as otherwise, $G[\{a, b_1', q_{3-i}, z\}]$ contains K_4^- and (ii) holds. Thus, $G[B_1 + q_{3-i}]$ has independent paths P_1, P_2 from b_1 to b_1', q_{3-i}, respectively. Note $b_1b_l \in E(G)$ by the maximality of A in (11). In $G[A \cup \{a, b_l, q_i\}]$ we find independent paths Q_1, Q_2 from b_l to q_i, a, respectively. Now $G[\{a, b_1, b_l, q_{3-i}, z\}] \cup (P_1 \cup b_1'a) \cup P_2 \cup Q_2 \cup (q_2Xx_2 \cup x_2x_1z_1 \cup z_1Xq_1 \cup Q_1)$ is a TK_5 in G' with branch vertices a, b_1, b_l, q_{3-i}, z, and (i) holds.

So we may assume $z \in V(X)$. Then $b_1b_l, q_{3-i}b_l \in E(G)$. By (9), $b_1a, b_1z \notin E(G)$. Hence, since G is 5-connected, zb_1', zb_l, $zx_1 \in E(G)$ and $q_{3-i} \neq x_1$. We may assume $x_1q_{3-i} \notin E(G)$; as otherwise, $G[\{b_l, q_{3-i}, x_1, z\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds. Note that $b_1'a \in E(G)$ by the maximality of A in (11). Let $q \in N(q_{3-i}) \cap V(B_1 - b_1)$, and let P_1, P_2 be independent paths in B_1 from b_1' to b_1, q, respectively. Let Q_1, Q_2 be independent paths in $G[Y_2]$ from a to b_l, q_i, respectively. Then $G[\{a, b_l, b_1', q_{3-i}, z\}] \cup (P_1 \cup b_1b_l) \cup (P_2 \cup qq_{3-i}) \cup Q_1 \cup (Q_2 \cup q_1Xz_1 \cup z_1x_1x_2 \cup x_2Xq_2)$ is a TK_5 in G' with branch vertices a, b_1', b_l, q_{3-i}, z, and (i) holds. □

4. Two special cases

To prove Theorem 1.1, we need to take care of the conclusions (iii) and (iv) of Lemma 3.2. Results from [7] can be used to deal with (iii) of Lemma 3.2 when $y_2 \notin V(X)$. So it remains to consider (iii) of Lemma 3.2 with $y_2 \in V(X)$ and (iv) of Lemma 3.2.

We will use the notation in the statement of Lemma 3.2. See Figs. 1 and 2. In particular, X is an induced path in $(G - x_1) - x_2y_2$ from z_1 to x_2 and $G' := G - \{x_1x : x \notin \{x_2, y_1, y_2, z_0, z_1\}\}$. Also recall from (iv) of Lemma 3.2 the separation (Y_1, Y_2) and the vertices $p_j, p_{3-j}, v, b, b_1, b_1'$. Note that $x_2 \neq p_2$; as otherwise, $\{b, p_1, x_1, x_2\}$ would a cut in G. Let z_2 be the neighbor of x_2 on X. For any $x \in V(G)$ and $S \subseteq G$, we use $e(x, S)$ to denote the number of edges in G from x to S.

In this section, we deal with two special cases of Theorem 1.1. First, we need some structural information on Y_2.

Lemma 4.1. *Suppose (iv) of Lemma 3.2 holds. Then Y_2 has independent paths from y_2 to b, p_1, p_2, respectively, and, for $i \in [2]$, Y_2 has a path from b to p_{3-i} and containing $\{y_2, p_i\}$. Moreover, one of the following holds:*

- *(i) G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.*
- *(ii) $G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
- *(iii) If $e(p_i, B_1 - b_1) \geq 1$ for some $i \in [2]$ then Y_2 has a path through b, p_i, y_2, p_{3-i} in order, and $Y_2 - b_1$ has a cycle containing $\{p_1, p_2, y_2\}$. If $b \neq b_1$, $p_2 v \in E(X)$, and $vb, vx_1 \in E(G)$ then Y_2 has a cycle containing $\{b, p_2, y_2\}$.*

Proof. Since G is 5-connected, Y_2 is $(3, \{b, p_1, p_2\})$-connected. So by Menger's theorem, Y_2 has independent paths from y_2 to b, p_1, p_2, respectively.

Next, let $i \in [2]$. We claim that Y_2 has a path Q_i from b to p_{3-i} and containing $\{p_i, y_2\}$. To see this, let $Y_2' := Y_2 + \{t, tb, tp_{3-i}\}$, which is 2-connected. If Y_2' has a cycle C containing $\{b, t, y_2\}$ then $Q_i := C - t$ is as desired. So suppose such a cycle C does not exist. Then by Lemma 2.4, Y_2' has a 2-cut T separating y_2 from $\{p_i, t\}$ and $\{p_i, t\} \cap T = \emptyset$. However, $T \cup \{x_1, x_2\}$ is a cut in G, a contradiction.

We now show that (i) holds or the first part of (iii) holds. Suppose $e(p_i, B_1 - b_1) \geq 1$ for some $i \in [2]$.

First, we may assume that Q_i must go through b, p_i, y_2, p_{3-i} in order. For, suppose Q_i goes through b, y_2, p_i, p_{3-i} in this order. Since $e(p_i, B_1 - b_1) \geq 1$, $G[B_1 + p_i]$ has independent paths P_1, P_2 from y_1 to b_1, p_i, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup Q_i \cup P_2 \cup ((X - (p_1 X p_2 - \{p_1, p_2\})) \cup x_1 z_1) \cup (P_1 \cup b_1 b)$ is a TK_5 in G' with branch vertices p_i, x_1, x_2, y_1, y_2, and (i) holds.

Next, note that $Y_2 - b_1$ is 2-connected. For, suppose not. Then $b = b_1$ and $Y_2 - b_1$ has a 1-separation (Y_{21}, Y_{22}), and we may assume $|V(Y_{21} - Y_{22}) \cap \{p_1, p_2, y_2\}| \leq 1$. Since each of $\{p_1, p_2, y_2\}$ has at least two neighbors in $Y_2 - b_1$, $(V(Y_{21} - Y_{22}) \cap \{p_1, p_2, y_2\}) \cup \{b, x_1\} \cup V(Y_{21} \cap Y_{22})$ is a cut in G of size at most 4, a contradiction.

Now suppose no cycle in $Y_2 - b_1$ contains $\{p_1, p_2, y_2\}$. Then, (i) or (ii) or (iii) of Lemma 2.4 holds. We use the notation in Lemma 2.4 (with p_1, p_2, y_2 playing the roles of y_1, y_2, y_3 there). If (i) of Lemma 2.4 occurs then let $S = \{a_1, a_1'\}$, $a_2 = a_3 = a_1$, and $a_2' = a_3' = a_1'$; if (ii) or (iii) of Lemma 2.4 occurs let $S_{p_j} = \{a_j, a_j'\}$ for $j \in [2]$ and let $S_{y_2} = \{a_3, a_3'\}$. Let A, A' denote the components of $(Y_2 - b_1) - (D_{p_1} \cup D_{p_2} \cup D_{y_2})$ such that $a_j \in V(A)$ and $a_j' \in V(A')$ for $j \in [3]$. Note that if (ii) of Lemma 2.4 occurs and $A \neq A'$, then either $A = a_3$ and $\{a_1', a_2', a_3'\} \subseteq V(A')$, or $A' = a_3'$ and $\{a_1, a_2, a_3\} \subseteq V(A)$.

Since $Y_2 - b_1$ is 2-connected, there exist paths S_1, S_2, S_3 in $D_{p_1}, D_{p_2}, D_{y_2}$, respectively, with S_j from a_j to a_j' for $j \in [3]$, $p_j \in V(S_j)$ for $j \in [2]$, and $y_2 \in V(S_3)$. Since G is 5-connected, $b \in V(D_{y_2})$ or $b = b_1$ has a neighbor in D_{y_2}. Hence, $G[D_{y_2} + b]$ contains a path T_3 from b to some $t \in V(S_3) - \{a_3, a_3'\}$ and internally disjoint from S_3. By symmetry, we may assume $t \in V(y_2 S_3 a_3)$. Let T_1 be a path in A from a_i to a_{3-i}, and T_2 be a path in A' from a_i' to a_3'. Then $T_3 \cup tS_3a_3' \cup T_2 \cup S_i \cup T_1 \cup a_{3-i}S_{3-i}p_{3-i}$ is a path from b to p_{3-i} through y_2, p_i in order. This is a contradiction as we have assumed that such a path Q_i does not exist.

Next, we prove that (i) or (ii) holds or the second part of (iii) holds. Suppose $b \neq b_1$, $p_2v \in E(p_2Xx_2)$, and $vb, vx_1 \in E(G)$. Suppose Y_2 has no cycle containing $\{b, p_2, y_2\}$. Then (i) or (ii) or (iii) of Lemma 2.4 holds. In particular, $\{b, p_2, y_2\}$ is independent in G. We use the notation in Lemma 2.4 (with b, p_2, y_2 playing the roles of y_1, y_2, y_3 there, respectively). So there is a 2-cut $S_{y_2} = \{a_3, a_3'\}$ in Y_2 such that $Y_2 - S_{y_2}$ has a component D_{y_2} with $y_2 \in V(D_{y_2})$ and $b, p_2 \notin V(D_{y_2}) \cup S_{y_2}$. Since G is 5-connected, $p_1 \in V(D_{y_2})$. Note that $Y_2 - D_{y_2}$ is $(4, \{a_3, a_3', b, p_2\})$-connected.

Suppose $(Y_2 - D_{y_2}, a_3, b, a_3', p_2)$ is not planar. Then by Lemma 2.1, $Y_2 - D_{y_2}$ contains disjoint paths from a_3, b to a_3', p_i, respectively. By Lemma 2.11, we may assume that $Y_2 - D_{y_2}$ has an induced path S from b to p_2 such that $(Y_2 - D_{y_2}) - S$ is a chain of blocks from a_3 to a_3'; for otherwise, we may apply Lemma 2.7 to show that (i) or (ii) holds. Thus $Y_2 - D_{y_2}$ has a path S_1 from a_3 to a_3' and containing $\{b, p_2\}$ (as Y_2 is 2-connected). Let S_2 be a path in $G[D_{y_2} + \{a_3, a_3'\}]$ from a_3 to a_3' through y_2. Then $S_1 \cup S_2$ is a cycle in Y_2 containing $\{b, p_2, y_2\}$, a contradiction.

So we may assume $(Y_2 - D_{y_2}, a_3, b, a_3', p_2)$ is planar. If $|V(Y_2 - D_{y_2})| \geq 6$ then (i) or (ii) follows from Lemma 2.7 (by considering the 5-cut $\{a_3, a_3', b, p_i, x_1\}$).

Now suppose $|V(Y_2 - D_{y_2})| = 5$. Let $t \in V(Y_2 - D_{y_2}) - \{a_3, a_3', b, p_2\}$. Since G is 5-connected, $ta_3, ta_3', tb, tp_2, tx_1 \in E(G)$. By symmetry between a_3 and a_3', we may assume $a_3' \in V(X)$. Then $a_3'p_2 \in E(G)$. If $ba_3' \in E(G)$ then $G[\{a_3', b, p_2, t\}] \cong K_4^-$, and (ii) holds. So assume $ba_3' \notin E(G)$. Then, since G is 5-connected, $ba_3, bx_1 \in E(G)$. Now $G[\{a_3, b, t, x_1\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds.

So $|V(Y_2 - D_{y_2})| = 4$ and, hence, (i) of Lemma 2.4 occurs, with $V(D_b) = \{b\}$ and $V(D_{p_2}) = \{p_2\}$. We claim that $D := G[D_{y_2} + \{a_3, a_3', x_1\}] + \{c, cx_1, cy_2\}$ has a cycle C containing $\{c, a_3, a_3'\}$; for otherwise, by Lemma 2.4, $D - c$ has a 2-cut either separating a_3 from $\{x_1, y_2, a_3', p_1\}$ or separating a_3' from $\{x_1, y_2, a_3, p_1\}$, contradicting the 5-connectedness of G. Let Q be a path in $G[B_1 + \{b, p_2\}]$ from b to p_2. Now $G[\{a_3, a_3', b, p_2\}] \cup Q \cup (C - c) \cup vx_1 \cup (vXx_2 \cup x_2y_2) \cup vb \cup vp_2$ is a TK_5 in G with branch vertices a_3, a_3', b, p_2, v. □

The next two results provide information on $e(z_i, B_1)$ for $i \in [2]$ in the case when $y_2 \notin V(X)$.

Lemma 4.2. *Suppose (iv) of Lemma 3.2 holds with $b \neq b_1$. Then one of the following holds:*

(*i*) G' *contains* TK_5*, or* G *contains a* TK_5 *in which* x_1 *is not a branch vertex.*
(*ii*) $G - x_1$ *contains* K_4^-*, or* G *contains a subgraph isomorphic to* K_4^- *in which* x_1 *is of degree 2.*
(*iii*) $e(z_i, B_1) \geq 2$ *for* $i \in [2]$.

Proof. Recall the notation from (*iv*) of Lemma 3.2. In particular, $v \in V(X) - V(p_1 X p_2)$. Suppose $e(z_i, B_1) \leq 1$ for some $i \in [2]$.

Case 1. $v \in V(z_1 X p_1 - p_1)$; so $p_1 v \in E(X)$.

In this case, $e(z_1, Y_2) \leq 2$ (with equality only if $z_1 = v$). Hence, $e(z_1, B_1) \geq 2$, since G is 5-connected. Thus, $e(z_2, B_1) \leq 1$. Hence, $z_2 = p_2$ and $e(z_2, B_1) = 1$, since $\{x_1, x_2, p_1, b\}$ cannot be a cut in G. By Lemma 4.1, Y_2 has a path Q from b to p_1 and containing $\{y_2, z_2\}$.

Suppose b, z_2, y_2, p_1 occur on Q in this order. If $b_1' \in N(z_2)$ then let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1' to y_1, x_2, respectively; now $G[\{x_1, x_2, y_2\}] \cup z_2 x_2 \cup (z_2 Q b \cup bv \cup v X z_1 \cup z_1 x_1) \cup z_2 Q y_2 \cup b_1' z_2 \cup (b_1' p_1 \cup p_1 Q y_2) \cup (P_1 \cup y_1 x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1', x_1, x_2, y_2, z_2. So assume $b_1' \notin N(z_2)$. Let P_1, P_2 be independent paths in $G[B_1 + z_2]$ from y_1 to b_1', z_2, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 Q b \cup bv \cup v X z_1 \cup z_1 x_1) \cup z_2 Q y_2 \cup P_2 \cup (y_2 Q p_1 \cup p_1 b_1' \cup P_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume that b, y_2, z_2, p_1 must occur on Q in this order. Then, by Lemma 4.1, we may assume $e(z_2, B_1 - b_1) = 0$; so $b_1 z_2 \in E(G)$. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1 to y_1, x_2, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_2 x_2 \cup (z_2 Q p_1 \cup p_1 X z_1 \cup z_1 x_1) \cup z_2 Q y_2 \cup (b_1 b \cup b Q y_2) \cup b_1 z_2 \cup (P_1 \cup y_1 x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1, x_1, x_2, y_2, z_2.

Case 2. $v \in V(p_2 X x_2 - p_2)$; so $p_2 v \in E(X)$.

Since $\{b, p_2, x_1, x_2\}$ cannot be a cut in G, $e(z_1, B_1) \geq 1$. We consider two cases.

Subcase 2.1. $e(z_1, B_1) = 1$.

Then $z_1 = p_1$. By Lemma 4.1, Y_2 has a path Q from b to p_2 and containing $\{z_1, y_2\}$.

Suppose b, z_1, y_2, p_2 occur on Q in this order. If $b_1' \in N(z_1)$ then $x_2 \neq v$ as $\{x_1, x_2, b_1, b_1'\}$ is not a cut in G; so $e(x_2, B_1 - y_1) \geq 1$. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1' to y_1, x_2, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_1 x_1 \cup (z_1 Q b \cup bv \cup v X x_2) \cup z_1 Q y_2 \cup b_1' z_1 \cup (b_1' p_2 \cup p_2 Q y_2) \cup (P_1 \cup y_1 x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1', x_1, x_2, y_2, z_1. Hence, we may assume $b_1' \notin N(z_1)$. Then let P_1, P_2 be independent paths in $G[B_1 + z_1]$ from y_1 to b_1', z_1, respectively; now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 Q b \cup bv \cup v X x_2) \cup z_1 Q y_2 \cup P_2 \cup (y_2 Q p_2 \cup p_2 b_1' \cup P_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume b, y_2, z_1, p_2 must occur on Q in this order. Hence, by Lemma 4.1, we may assume $e(p_1, B_1 - b_1) = 0$; so $b_1 \in N(z_1)$ as $\{b, p_2, x_1, x_2\}$ is not a cut in G. Then $e(x_2, B_1 - y_1) \geq 1$; otherwise, $x_2 = v$, and $\{b_1, b_1', x_1, x_2\}$ would be a cut in G. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1 to y_1, x_2, respectively. Then

$G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup (z_1Qp_2 \cup p_2Xx_2) \cup z_1Qy_2 \cup b_1z_1 \cup (b_1b \cup bQy_2) \cup (P_1 \cup y_1x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1, z_1, x_1, x_2, y_2.

Subcase 2.2. $e(z_1, B_1) \geq 2$.

Then $e(z_2, B_1) \leq 1$. Hence, $z_2 = p_2$ or $z_2 = v$. Suppose $z_2 = p_2$. Then $x_2 = v$; so $x_1v \in E(G)$. Hence, by (*iii*) of Lemma 4.1, Y_2 has a cycle C containing $\{b, y_2, z_2\}$. Let P_1, P_2 be independent paths in B_1 from y_1 to b_1, b_1', respectively. Now $C \cup x_2y_2 \cup x_2z_2 \cup x_2b \cup y_1x_2 \cup y_1x_1y_2 \cup (P_1 \cup b_1b) \cup (P_2 \cup b_1'z_2)$ is a TK_5 in G' with branch vertices b, x_2, y_1, y_2, z_2.

So we may assume $z_2 = v$. Since $e(z_2, B_1) = 1$, $x_1v \in E(G)$. Hence, by (*iii*) of Lemma 4.1, Y_2 has a cycle C containing $\{b, p_2, y_2\}$. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from x_2 to b_1, b_1', respectively. Note that P_1, P_2 exist since x_2 has at least two neighbors in B_1. Then $C \cup z_2b \cup z_2p_2 \cup z_2x_1y_2 \cup x_2y_2 \cup x_2z_2 \cup (P_1 \cup b_1b) \cup (P_2 \cup b_1'p_2)$ is a TK_5 in G' with branch vertices b, p_2, x_2, y_2, z_2. □

Lemma 4.3. *Suppose $y_2 \notin V(X)$. Then one of the following holds:*

(*i*) *G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.*
(*ii*) *$G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
(*iii*) *There exists $i \in [2]$ such that $e(z_i, B_1 - b_1) \geq 2$ and $e(z_{3-i}, B_1 - b_1) \geq 1$.*

Proof. Suppose (*iii*) fails. First, assume $b \neq b_1$; so (*iv*) of Lemma 3.2 occurs. Then by Lemma 4.2, we have, for $i \in [2]$, $e(z_i, B_1 - b_1) = 1$ and $b_1z_i \in E(G)$. Let P_1, P_2 be independent paths in B_1 from y_1 to b_1, b_1', respectively. Recall, from (*iv*) of Lemma 3.2, the role of $j \in [2]$ and the vertices p_{3-j}, v. Since b_1' is the only neighbor of p_{3-j} in B_1, $p_{3-j} \notin \{z_1, z_2\}$. Let Q be a path in $Y_2 - \{z_1, z_2\}$ from b to p_{3-j} and through y_2. Then $G[\{x_1, x_2, y_1, y_2\}] \cup b_1z_1x_1 \cup b_1z_2x_2 \cup (b_1b \cup bQy_2) \cup P_1 \cup (y_2Qp_{3-j} \cup p_{3-j}b_1' \cup P_2)$ is a TK_5 in G' with branch vertices b_1, x_1, x_2, y_1, y_2.

So we may assume $b = b_1$. Then, for $i \in [2]$, $e(z_i, B_1 - b_1) \geq 1$ as $\{b, p_{3-i}, x_1, x_2\}$ is not a cut in G. Hence, since (*iii*) fails, $e(z_i, B_1 - b_1) = 1$ for $i \in [2]$. For $i \in [2]$, let $z_i' \in N(z_i) \cap V(B_1)$. Since G is 5-connected, $z_1 = p_1$.

Case 1. $z_2 \neq p_2$.

Then, since G is 5-connected, $z_2x_1, z_2b \in E(G)$. First, assume that there is no edge from $p_2Xz_2 - z_2$ to $B_1 - b$. Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, x_1, x_2, z_1, z_2\}$, $B_1 \subseteq G_1$, and $Y_2 \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. Since $x_1x_2z_2x_1$ is a triangle in G, the assertion of the lemma follows from Lemma 2.8.

Hence, we may assume that there exists $uu' \in E(G)$ with $u \in V(p_2Xz_2 - z_2)$ and $u' \in V(B_1 - b)$. Suppose, for some choice of uu', $u' \neq z_1'$ and $B_1 - b$ contains independent paths P_1, P_2 from y_1 to z_1', u', respectively. By Lemma 4.1 (since $e(p_1, B_1 - b_1) = 1$), Y_2 contains a path Q from b to p_2 through p_1, y_2 in order. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup$

$(z_1Qb \cup bz_2x_2) \cup (z_1z_1' \cup P_1) \cup z_1Qy_2 \cup (P_2 \cup u'u \cup uXp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Therefore, we may assume that for any choice of uu', $u' = z_1'$ or the paths P_1, P_2 do not exist. If $u' = z_1'$ for all $u' \in N(p_2Xz_2 - z_2)$ we let $B' = B_1$ and $B'' = \{b, z_1'\}$; otherwise, since B_1 is 2-connected, B_1 has a 2-separation (B', B'') such that $b \in V(B' \cap B'')$, $y_1 \in V(B')$ and $z_1', u' \in V(B'')$ for all $u' \in N(p_2Xz_2 - z_2)$. Thus G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(B' \cap B'') \cup \{x_1, x_2, z_2\}$, $B' \subseteq G_1$ and $B'' \cup Y_2 \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$.

If $|V(G_1)| \geq 7$ then the assertion of the lemma follows from Lemma 2.8 (as $x_1x_2z_2x_1$ is a triangle in G). So assume $|V(G_1)| \leq 6$. Then, since G is 5-connected, $z_2y_1 \in E(G)$. So $G[\{x_1, x_2, y_1, z_2\}] - x_1y_1 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

Case 2. $z_2 = p_2$.

We may assume $z_i' \neq y_1$ for $i \in [2]$. For, otherwise, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, p_{3-i}, x_1, x_2, y_1\}$, $B_1 \subseteq G_1$ and $Y_2 \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$. If $|V(G_1)| \geq 7$ then, since $G[\{x_1, x_2, y_1\}] \cong K_3$, the assertion of the lemma follows from Lemma 2.8. So we may assume $|V(G_1) = 6$. Then $|V(B_1)| = 3$. Let $z \in V(B_1) - \{b_1, y_1\}$. Then, since G is 5-connected, $zx_1, zx_2, zy_1 \in E(G)$; so $G[\{x_1, x_2, y_1, z\}] - x_1x_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

Note that $z_1' \neq z_2'$ as otherwise $\{b_1, x_1, x_2, z_1'\}$ would be a cut in G. Let $K = G[B_1 + \{x_2, z_1, z_2\}]$. Suppose K contains disjoint paths Z_1, Z_2 from z_1, z_2 to x_2, y_1, respectively. By (iii) of Lemma 4.1, let C be a cycle in $Y_2 - b_1$ containing $\{y_2, z_1, z_2\}$. Then $G[\{x_1, x_2, y_2\}] \cup C \cup z_1x_1 \cup z_2x_2 \cup (Z_2 \cup y_1x_1) \cup Z_1$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

So we may assume that such Z_1, Z_2 do not exist. Then by Lemma 2.1, there exists a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(K) - \{x_2, y_1, z_1, z_2\}$ such that $(K, \mathcal{A}, z_1, z_2, x_2, y_1)$ is 3-planar. Since G is 5-connected, either $\mathcal{A} = \emptyset$ or $|\mathcal{A}| = 1$. When $|\mathcal{A}| = 1$ let $\mathcal{A} = \{A\}$; then $b_1 \in A$. We choose $\mathcal{A}$ so that $|\mathcal{A}|$ is minimal and, subject to this, $|A|$ is minimal when $\mathcal{A} = \{A\}$. Note that if A exists then $|A| \geq 2$ (by the minimality of $|\mathcal{A}|$ and $|A|$). Moreover, $|N_K(A)| = 3$ as $N_K(A) \cup \{b_1, x_1\}$ is not a cut in G.

We may assume that if $\mathcal{A} \neq \emptyset$ then $\{x_2, z_1, z_2\} \cap N_K(A) = \emptyset$. For, suppose there exists $u \in \{x_2, z_1, z_2\} \cap N_K(A)$. Let $S := (N_K(A) \cup \{x_1, x_2, z_1, z_2\}) - \{u\}$ if $u \in \{z_1, z_2\}$ and let $S := N_K(A) \cup \{x_1, x_2, z_1, z_2\}$ if $u = x_2$. Then S is a cut in G separating $B_1 - A$ from Y_2. Since G is 5-connected, $|S| = 5$ if $u \in \{z_1, z_2\}$ and $|S| \in \{5, 6\}$ if $u = x_2$. Therefore, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$, $B_1 - A \subseteq G_1$, and $Y_2 \subseteq G_2$. Note that $(G_1 - x_1, S - \{x_1\})$ is planar. Since $|V(Y_2)| \geq 4$, $|V(G_2)| \geq 7$ if $|S| = 5$, and $|V(G_2)| \geq 8$ is $|S| = 6$. Since $b_1, y_1 \notin \{z_1', z_2'\}$, $|V(G_1)| \geq 7$ if $|S| = 5$ and $|V(G_1)| \geq 8$ if $|S| = 6$. Thus, if $|S| = 5$ then the assertion of the lemma follows from Lemma 2.7, and if $|S| = 6$ then the assertion of the lemma follows from Lemma 2.9 and Lemma 2.7.

If $\mathcal{A} = \emptyset$ let $K^* = K$; and if $\mathcal{A} \neq \emptyset$ let K^* be the graph obtained from K by deleting A and adding new edges joining every pair of distinct vertices in $N_K(A)$. Since B_1 is 2-connected and G is 5-connected, $K' := K^* - \{x_2, z_1, z_2\}$ is a 2-connected planar graph.

Take a plane embedding of K' and let D denote its outer cycle. Let $t \in V(D)$ such that $t \in N(x_2)$ and tDz'_2 is minimal.

When $\mathcal{A} \neq \emptyset$, $N_K(A) \nsubseteq V(D)$; as otherwise, if we write $N_K(A) = \{s_1, s_2, s_3\} \subseteq V(D)$ with $s_2 \in V(s_1Ds_3)$, then $\{b_1, s_1, s_3, x_1\}$ is a cut in G, a contradiction. Further, if $\mathcal{A} \neq \emptyset$ let $N_K(A) = \{a, a_1, a_2\}$ with $a \in N_K(A) - V(tDz'_1)$; then, by the minimality of $\mathcal{A}$ and A, $G[A \cup N_K(A)]$ contains disjoint paths P_1, P_2 from a, a_2 to b_1, a_1, respectively. If $\mathcal{A} = \emptyset$ let $Q = tDz'_1$, $P_1 = a = a_1 = a_2 = b_1$ and $P_2 = \emptyset$. If $\mathcal{A} \neq \emptyset$ let $Q = tDz'_1$ if $a_1a_2 \notin E(tDz'_1)$; and otherwise let $Q = (tDz'_1 - a_1a_2) \cup P_2$. Note that Q is a path in B_1.

Suppose $K' - (tDz'_1 - z'_2)$ has independent paths S_1, S_2 from y_1 to $z'_2, \{a, a_1, a_2\}$, respectively, and internally disjoint from $\{a, a_1, a_2\}$. We may assume the notation is chosen so that $a \in V(S_2)$. For $i \in [2]$, let $S'_i = S_i$ if $a_1a_2 \notin E(S_i)$; and otherwise let S'_i be obtained from S_i by replacing a_1a_2 with P_2. By Lemma 4.1, let Q_1, Q_2 be independent paths in Y_2 from y_2 to z_2, b_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup (z'_2Qz'_1 \cup z'_1z_1x_1) \cup (z'_2Qt \cup tx_2) \cup (z'_2z_2 \cup Q_1) \cup S'_1 \cup (S'_2 \cup P_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z'_2.

So we may assume that such S_1, S_2 do not exist. Then by planarity, K' has a cut $\{s_1, s_2, s_3\}$ separating y_1 from $\{a, a_1, a_2, z'_2\}$, with $s_1 \in V(z'_2Dz'_1)$ and $s_3 \in V(tDz'_2)$. Clearly, $\{s_1, s_2, s_3\}$ is also a cut in B_1 separating y_1 from $\{z'_2\} \cup A$. Denote by M the $\{s_1, s_2, s_3\}$-bridge of B_1 containing y_1. If $V(M) - \{s_1, s_2, s_3\} = \{y_1\}$ then $s_1 = z'_1$ and $s_3 = t$; now $G[\{t, x_1, x_2, y_1\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds. So assume $|V(M) - \{s_1, s_2, s_3\}| \geq 2$. Then G has a 6-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, s_2, s_3, x_1, x_2, z_1\}$, $G_2 = G[M + \{x_1, x_2, z_1\}]$, and $(G_2 - x_1, z_1, s_1, s_2, s_3, x_2)$ is planar. It is easy to see that $|V(G_i)| \geq 8$ for $i \in [2]$; so the assertion follows from Lemma 2.9 and then Lemma 2.7. □

5. Substructure

Recall the notation in the statement of Lemma 3.2. See Figs. 1 and 2. In this section, we derive a substructure in G by finding five paths A, B, C, Y, Z in $H := G[B_1 + \{z_1, z_2\}]$. The paths Y, Z are found in the following lemma.

Lemma 5.1. *Suppose $y_2 \in V(X)$ (see (iii) of Lemma 3.2), or $y_2 \notin V(X)$ and $e(z_i, B_1) \geq 2$ for some $i \in [2]$ (see (iv) of Lemma 3.2). Let $b_1 \in N(y_2) \cap V(B_1)$ if $y_2 \in V(X)$, and let $\{b_1\} = V(B_1) \cap V(B_2)$ if $y_2 \notin V(X)$. Then one of the following holds:*

- *(i) G' contains TK_5 or G contains a TK_5 in which x_1 is not a branch vertex.*
- *(ii) $G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
- *(iii) H contains disjoint paths Y, Z from y_1, z_1 to b_1, z_2, respectively.*

Proof. Suppose (iii) fails. Then by Lemma 2.1, there exists a collection $\mathcal{A}$ of subsets of $V(H) - \{b_1, y_1, z_1, z_2\}$ such that $(H, \mathcal{A}, b_1, z_1, y_1, z_2)$ is 3-planar.

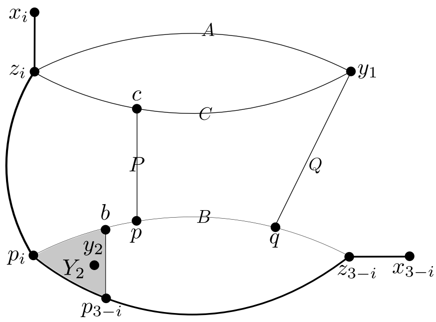

Fig. 3. An intermediate structure.

Since B_1 is 2-connected, $|N_H(A) \cap \{z_1, z_2\}| \leq 1$ for all $A \in \mathcal{A}$. Let $\mathcal{A}' = \{A \in \mathcal{A} : |\{z_1, z_2\} \cap N_H(A)| = 0\}$ and $\mathcal{A}'' = \{A \in \mathcal{A} : |\{z_1, z_2\} \cap N_H(A)| = 1\}$. Let $p(H, \mathcal{A})$ be the graph obtained from H by deleting A (for each $A \in \mathcal{A}$) and adding new edges joining every pair of distinct vertices in $N_H(A)$. Since G is 5-connected and B_1 is 2-connected, $p(H, \mathcal{A}) - \{z_1, z_2\}$ is 2-connected and we may assume that it is drawn in the plane with outer cycle D, such that for each $A \in \mathcal{A}''$, the edges between the vertices in $N_H(A) - \{z_1, z_2\}$ occur on D.

For each $j \in [2]$, let $t_j \in V(D)$ such that H has a path R_j from z_j to t_j and internally disjoint from $p(H, \mathcal{A})$, and, subject to this, t_2, b_1, t_1, y_1 occur on D in clockwise order, and t_2Dt_1 is maximal. When $e(z_1, B_1) \geq 2$, let $t_1' \in V(b_1Dt_1)$ with $t_1'Dt_1$ maximal such that H has independent paths R_1, R_1' from z_1 to t_1, t_1', respectively, and internally disjoint from $p(H, \mathcal{A})$. (Note that, for convenience, we use the same R_j in both cases.) When $e(z_2, B_2) \geq 2$, let $t_2' \in V(t_2Db_1)$ with t_2Dt_2' maximal such that H has independent paths R_2, R_2' from z_2 to t_2, t_2', respectively, and internally disjoint from $p(H, \mathcal{A})$.

Next we define vertices y_{21}, y_{22} and paths Q_1, Q_2, Q_3. If $y_2 \in V(X)$, then let $p_1 = p_2 = b = y_2$, let $Q_j := y_2$ for $j \in [3]$, and let $y_{21}, y_{22} \in N(y_2) \cap V(D)$ such that $t_2', y_{22}, y_{21}, t_1'$ occur on D in clockwise order and $y_{22}Dy_{21}$ is maximal. Suppose $y_2 \notin V(X)$. By Lemma 4.3, we may assume that for some $i \in [2]$, $e(z_i, B_1 - b_1) \geq 2$ and $e(z_{3-i}, B_1 - b_1) \geq 1$. If both $e(z_1, B_1) \geq 2$ and $e(z_2, B_2) \geq 2$, then let $y_{21} = y_{22} = b_1$ and, by Lemma 4.1, let Q_1, Q_2, Q_3 be independent paths in Y_2 from y_2 to p_1, p_2, b, respectively. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, by Lemma 4.1, Y_2 has a path Q_{3-i}^* through b, z_{3-i}, y_2, p_i in order; let $R_{3-i}' := z_{3-i}Q_{3-i}^*b \cup bb_1$, $t_{3-i}' := b_1$, $Q_{3-i} := y_2Q_{3-i}^*z_{3-i}$, and $Q_i := p_iQ_{3-i}^*y_2$. Note that in this final case, R_{3-i} and R_{3-i}' are independent, and Q_3, y_{21} and y_{22} are not defined.

Let $\mathcal{A}_1 = \{A \in \mathcal{A} : z_1 \in N_H(A)$ or $N_H(A) \subseteq V(b_1Dy_1)\}$, $\mathcal{A}_2 = \{A \in \mathcal{A} : z_2 \in N_H(A)$ or $N_H(A) \subseteq V(y_1Db_1)\}$, and $A_j = \bigcup_{A \in \mathcal{A}_j} A$ for $j \in [2]$. Let $F_1 := G'[V(z_1Xp_1) \cup A_1 \cup V(b_1Dy_1)]$ and $F_2 := G'[V(x_2Xp_2) \cup A_2 \cup V(y_1Db_1)]$. Write $b_1Dy_1 = v_1 \dots v_m$ and $z_1Xp_1 = v_{m+1} \dots v_n$ with $v_1 = b_1$, $v_m = y_1$, $v_{m+1} = z_1$, and $v_n = p_1$. Write $y_1Db_1 = u_1 \dots u_k$ and $p_2Xx_2 = u_{k+1} \dots u_l$ such that $u_1 = y_1$, $u_k = b_1$, $u_{k+1} = p_2$ and $u_l = x_2$. We may assume that

(1) $(F_1, v_1, \dots, v_n)$ and $(F_2, u_1, \dots, u_l)$ are planar.

Note that F_1 is $(4, \{v_1, \dots, v_k\})$-connected, and F_2 is $(4, \{u_1, \dots, u_l\})$-connected. We only prove that $(F_1, v_1, \dots, v_n)$ is planar; the argument for $(F_2, u_1, \dots, u_l)$ is similar. Suppose $(F_1, v_1, \dots, v_n)$ is not planar. Then by Lemma 2.2, there exist $1 \le q < r < s < t \le n$ such that F_1 contains disjoint paths S_1, S_2 from v_q, v_r to v_s, v_t, respectively. By the definition of F_1 (and since X is induced), we see that $r \le m$ and $s \ge m+1$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to y_1Dt_2, $t_2'Dv_q$, v_rDy_1, respectively. (By this, we mean that T_1 is between y_1 and t_2, T_2 is between t_2' and v_q, and T_3 is between v_r and y_1.) Hence, $G[\{x_1, x_2, y_1, y_2\}] \cup z_2x_2 \cup (z_2Xp_2 \cup Q_2) \cup (R_2 \cup T_1) \cup (R_2' \cup T_2 \cup S_1 \cup v_sXz_1 \cup z_1x_1) \cup (T_3 \cup S_2 \cup v_tXp_1 \cup Q_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (1).

We may also assume that

(2) $N_H(x_2) \subseteq V(F_2 + x_1)$.

For, suppose there exists $a \in N_H(x_2) - V(F_2 + x_1)$. If $a \notin A$ for all $A \in \mathcal{A}$ let $a' = a$ and $S = a$; and if $a \in A$ for some $A \in \mathcal{A}$ then let $a' \in N_H(A) - V(F_2)$ and S be a path in $G[A + a']$ from a to a'.

First, we may choose a and a' so that $a' \notin V(t_1Dy_1 - y_1)$ and no 2-cut of B_1 separating a from y_1Dt_2 is contained in t_1Dy_1. For, otherwise, we may assume $a' \in V(t_1Dy_1 - y_1)$ (by modifying A if necessary). Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $t_2'Dt_1', t_1Da', y_1Dt_2$, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup z_2x_2 \cup (R_1' \cup T_1 \cup R_2') \cup (z_1Xp_1 \cup Q_1) \cup (z_2Xp_2 \cup Q_2) \cup (R_1 \cup T_2 \cup S \cup ax_2) \cup (R_2 \cup T_3 \cup y_1x_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

Suppose that $p(H, \mathcal{A}) - \{z_1, z_2\} - t_1Dt_2$ has a path T from a' to t_1'. Let T_1, T_2 be independent paths in B_1 corresponding to T, t_1Dt_2, respectively. So $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (R_1 \cup t_1T_2y_1) \cup (R_1' \cup T_1 \cup S \cup ax_2) \cup (y_1T_2t_2 \cup R_2 \cup z_2Xp_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume that such T does not exist. By planarity, there is a cut $\{s_1, s_2\}$ in B_1 separating t_1' from $N_H(x_2) - V(F_2 + x_1)$, with $s_1, s_2 \in V(t_1Dt_2)$. Since $\{s_1, s_2\} \not\subseteq V(t_1Dy_1)$ and $a \notin V(F_2+x_1)$, we may let $s_1 \in V(t_1Dy_1 - y_1)$ and $s_2 \in V(y_1Dt_2 - y_1)$. Let M be the $\{s_1, s_2\}$-bridge of B_1 containing y_1. We choose $\{s_1, s_2\}$ so that M is minimal (subject to the property that $s_1 \in V(t_1Dy_1 - y_1)$ and $s_2 \in V(y_1Dt_2 - y_1)$).

Since $\{s_1, s_2, x_1, x_2\}$ cannot be a cut in G, there exists $vv' \in E(G)$ with $v' \in V(M) - \{s_1, s_2\}$ and $v \in V(z_jXp_j - z_j)$ for some $j \in [2]$. By minimality, $M - S_j$ has independent paths P_1, P_2 from y_1 to s_{3-j}, v', respectively. Let T_1 be a path in $B_1 - (M - s_j)$ corresponding to $t_2'Dt_1'$, and T_2 be a path in $B_1 - (M - s_j)$ corresponding to t_1Ds_1 (when $j = 2$) or s_2Dt_2 (when $j = 1$). Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-j}x_{3-j} \cup (z_{3-j}Xp_{3-j} \cup Q_{3-j}) \cup (R_{3-j}' \cup T_1 \cup R_j' \cup z_jx_j) \cup (R_{3-j} \cup T_2 \cup P_1) \cup (P_2 \cup v'v \cup vXp_j \cup Q_j)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

We may assume

(3) $N(z_1Xp_1 - z_1) \cap V(B_1) \nsubseteq V(F_1)$ or $N(z_2Xp_2 - z_2) \cap V(B_1) \nsubseteq V(F_2)$.

For, suppose $N(z_jXp_j - z_j) \cap V(B_1) \subseteq V(F_j)$ for $j \in [2]$. If $y_2 \in V(X)$ then by (1) and (2), $G - x_1$ is planar; so the assertion of this lemma follows from Lemma 2.10. Hence, we may assume $y_2 \notin V(X)$. By (1) and (2), $b = b_1$, and $(G[B_1 \cup z_1Xp_1 \cup p_2Xx_2], p_1, b, p_2, x_2)$ is planar. So G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, p_1, p_2, x_1, x_2\}$ and $G_2 = G[(B_1 \cup z_1Xp_1 \cup x_2Xp_2) + x_1]$. Clearly, $|V(G_j)| \geq 7$ for $j \in [2]$. Hence, the assertion of this lemma follows from Lemma 2.7.

Since the rest of the argument is the same for the two cases in (3), we will assume

(4) $N(z_2Xp_2 - z_2) \cap V(B_1) \nsubseteq V(F_2)$ (and, hence, $e(z_2, B_1) \geq 2$).

Let $vv' \in E(G)$ with $v \in V(B_1 - F_2)$ and $v' \in V(z_2Xp_2 - z_2)$. Let $v'' = v$ and $S = v$ if $v \notin A$ for all $A \in \mathcal{A}$; otherwise, let $v \in A \in \mathcal{A}$ and $v'' \in N_H(A)$ such that $v'' \notin V(F_2)$, and let S be a path in $G[A + v'']$ from v to v''.

Suppose $(p(H, \mathcal{A}) - \{z_1, z_2\}) - t_2'Dt_1'$ has independent paths P_1, P_2 from y_1 to t_1, v'', respectively. Let P_1', P_2', T be independent paths in B_1 corresponding to $P_1, P_2, t_2'Dt_1'$, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (R_1 \cup P_1') \cup (z_1Xp_1 \cup Q_1) \cup (R_1' \cup T \cup R_2' \cup z_2x_2) \cup (P_2' \cup S \cup vv' \cup v'Xp_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume that such P_1, P_2 do not exist in $p(H, \mathcal{A})$. Then by planarity and the existence of t_1Dy_1, $p(H, \mathcal{A}) - \{z_1, z_2\}$ has a cut $\{s_1, s_2\}$, with $s_1 \in V(t_2'Dt_1')$ and $s_2 \in V(t_1Dy_1)$, separating y_1 from $\{v'', t_1\}$. Clearly, $\{s_1, s_2\}$ is also a cut in B_1. Denote by M_v, M_y the $\{s_1, s_2\}$-bridges of B_1 containing $\{v'', t_1\}, y_1$, respectively. We choose $\{s_1, s_2\}$ so that M_y is minimal. Since v is arbitrary, we have $N(z_2Xp_2 - z_2) \cap V(B_1 - F_2) \subseteq V(M_v)$. We further choose vv' with $v'Xx_2$ minimal.

Recall that y_{22} is defined only when $y_2 \in V(X)$, or when $y_2 \notin V(X)$ and both $e(z_1, B_1) \geq 2$ and $e(z_2, B_2) \geq 2$. We may assume

(5) $y_{22} \in V(M_v)$ (when defined) and, for any $q \in V(p_2Xv' - v')$, $N(q) \cap V(M_y - \{s_1, s_2\}) = \emptyset$.

Suppose (5) fails. If y_{22} is defined and $y_{22} \notin V(M_v)$ let $q = b$, $q' = y_{22}$, and $Q' = q'q \cup Q_3$; and if y_{22} is defined, $y_{22} \in V(M_v)$, and there exist $q \in V(p_2Xv' - v')$ and $q' \in N(q) \cap V(M_y - \{s_1, s_2\})$, then let $Q' = q'q \cup qXp_2 \cup Q_2$.

Since B_1 is 2-connected, there exists $j \in [2]$ such that $M_v - s_{3-j}$ contains disjoint paths T_1, T_2 from $\{t_1, t_1'\}$ to $\{v'', s_j\}$. Note that $R_1 \cup R_1' \cup T_1 \cup T_2$ contains independent paths T_1', T_2' from z_1 to v'', s_j, respectively. If M_y contains independent paths S_1, S_2 from y_1 to q', s_j, then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (T_1' \cup S \cup vv' \cup v'Xx_2) \cup (T_2' \cup$

$S_2) \cup (Q' \cup S_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we may assume S_1, S_2 do not exist in M_y; hence M_y has a cut vertex c that separates y_1 from $\{q', s_j\}$.

By the minimality of M_y and the existence of $y_1 D s_1$, $c \in V(y_1 D t_2' - t_2')$; so we must have $j = 1$. Denote by C_q, C_y the c-bridges of M_y containing $\{q', s_1\}$, y_1, respectively, and choose c with C_y minimal. Then $N(p_2 X v' - v') \cap V(C_y - \{c, s_2\}) = \emptyset$.

We may assume that there exist $uu' \in E(G)$ with $u' \in V(z_1 X p_1 - z_1)$ and $u \in V(C_y) - \{c, s_2\}$. For, otherwise, by (1) and (2), there exists $z \in V(v' X x_2)$ such that $\{c, s_2, x_1, x_2, z\}$ is a cut in G, and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{c, s_2, x_1, x_2, z\}$, $M_v \cup z_1 X z \cup Y_2 \subseteq G_1$, $M_y \subseteq G_2$, and $(G_2 - x_1, \{c, s_2, x_2, z\})$ is planar. Clearly, $|V(G_1)| \geq 7$ and $|V(G_2)| \geq 6$. If $|V(G_2)| \geq 7$ then the assertion of the lemma follows from Lemma 2.7. So assume $|V(G_2)| = 6$. Then $z = z_2$ and $y_1 z_2 \in E(G)$; now $G[\{x_1, x_2, y_1, z_2\}] - x_1 z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

By the minimality of M_y and C_y, $C_y - s_2$ has independent paths U_1, U_2 from y_1 to c, u, respectively. In $M_v - s_1$, we find a path T from t_1 to v''. Let X^* be an induced path in $G - x_1$ from z_1 to x_2 such that $V(X') \subseteq V(R_1 \cup T \cup S \cup vv' \cup v' X x_2)$. Now $U_1 \cup U_2 \cup (C_q - s_1) \cup uu' \cup u' X p_1 \cup Q_1 \cup Q_2 \cup p_2 X q \cup qq'$ is a subgraph of $(G - x_1) - X^*$ and has a cycle containing $\{y_1, y_2\}$. Hence by Lemma 2.11 and Lemma 2.7, we may assume that $G - x_1$ contains an induced path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. So the assertion of this lemma follows from Lemma 2.5. This proves (5).

We may assume $N(z_1 X p_1 - z_1) \cap V(M_y - \{s_1, s_2\}) \neq \emptyset$. For, otherwise, by (5), G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, s_2, v', x_1, x_2\}$, $G_2 := G[v' X x_2 \cup M_y + x_1]$ and $(G_2 - x_1, s_1, s_2, x_2, v')$ is planar (by (1) and (2)). Clearly, $|V(G_1)| \geq 7$ and $|V(G_2)| \geq 6$. If $|V(G_2)| \geq 7$ then the assertion of this lemma follows from Lemma 2.7. So assume $|V(G_2)| = 6$. Then $v' = z_2$ and $y_1 z_2 \in E(G)$. So $G[\{x_1, x_2, y_1, z_2\}] - x_1 z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

So there exists $uu' \in E(G)$ with $u' \in V(z_1 X p_1 - z_1)$ and $u \in V(M_y) - \{s_1, s_2\}$. Hence, $e(z_1, B_1) \geq 2$; so y_{21}, y_{22}, Q_3 are defined. Let P_u be a path in M_y from u to some $u_D \in V(s_2 D s_1) - \{s_1, s_2\}$ and internally disjoint from $V(D)$ (which exists by minimality of M_y), and P_v be a path in M_v from v'' to some $v_D \in V(s_1 D s_2)$ and internally disjoint from $V(D)$. By the definition of F_2, we may choose v_D so that $v_D \notin V(s_1 D y_{22})$.

We may assume $v_D \in V(t_1' D y_1 - t_1')$. For, suppose $v_D \in V(y_{22} D t_1' - y_{22})$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $t_1 D y_1$, $v_D D t_1'$, $y_1 D y_{22}$, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup (R_1 \cup T_1) \cup (R_1' \cup T_2 \cup P_v \cup S \cup vv' \cup v' X x_2) \cup (T_3 \cup y_{22} b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Next, we consider the location of u_D. Suppose $u_D \in V(t_2' D s_1 - s_1)$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $y_1 D t_2$, $t_2' D u_D$, $y_{21} D y_1$, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 X p_2 \cup Q_2) \cup (R_2 \cup T_1) \cup (R_2' \cup T_2 \cup P_u \cup uu' \cup u' X z_1 \cup z_1 x_1) \cup (T_3 \cup y_{21} b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now suppose $u_D \in V(s_2 D y_1)$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $y_1 D t_2$, $t_2' D t_1'$, $u_D D y_1$, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 X p_2 \cup Q_2) \cup$

$(R_2 \cup T_1) \cup (R_2' \cup T_2 \cup R_1' \cup z_1x_1) \cup (T_3 \cup P_u \cup uu' \cup u'Xp_1 \cup Q_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So we may assume $u_D \in V(y_1Dt_2' - t_2')$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $y_1Du_D, t_2'Dt_1', v_DDy_1$, respectively. Thus, $(G-x_1)-(R_1'\cup T_2\cup R_2'\cup z_2x_2)$ contains the cycle $T_1 \cup P_u \cup uu' \cup u'Xp_1 \cup Q_1 \cup Q_2 \cup p_2Xv' \cup v'v \cup S \cup P_v \cup T_3$. Hence, by Lemma 2.11 and Lemma 2.7, we may assume that $G - x_1$ contains a path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. So the assertion of this lemma follows from Lemma 2.5. □

We now prove the existence of three paths A, B, C in $H := G[B_1 + \{z_1, z_2\}]$.

Lemma 5.2. *Let $b_1 \in N(y_2) \cap V(B_1)$ when $y_2 \in V(X)$, and let $\{b_1\} = V(B_1) \cap V(B_2)$ when $y_2 \notin V(X)$. Then one of the following holds:*

- *(i) G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.*
- *(ii) $G - x_1$ contains K_4^-, or G contains a subgraph isomorphic to K_4^- in which x_1 is of degree 2.*
- *(iii) There exists $i \in [2]$ such that H contains independent paths A, B, C, with A and C from z_i to y_1 and B from b_1 to z_{3-i}.*

Proof. If $y_2 \notin V(X)$ then by Lemma 4.1, let Q_1, Q_2, Q_3 be independent paths in Y_2 from y_2 to p_1, p_2, b, respectively. When $y_2 \in V(X)$ let $Q_1 = Q_2 = Q_3 = y_2$. We may assume that

(1) for $i \in [2]$, H has no path through z_{3-i}, z_i, y_1, b_1 in order.

For, if H has a path S through z_{3-i}, z_i, y_1, b_1 in order. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup z_iSy_1 \cup (z_iSz_{3-i} \cup z_{3-i}x_{3-i}) \cup (y_1Sb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

We may also assume that

(2) for $i \in [2]$ with $e(z_i, B_1 - b_1) \geq 2$, H has a 2-separation (F_i', F_i'') such that $b_1 \in V(F_i')$, $z_i \in V(F_i' - F_i'')$ and $\{y_1, z_{3-i}\} \subseteq V(F_i'' - F_i')$.

Suppose $i \in [2]$ and $e(z_i, B_1 - b_1) \geq 2$. Let K be obtained from H by duplicating z_i and y_1 with copies z_i' and y_1', respectively. So in K, y_1 and y_1' are not adjacent, but have the same set of neighbors, namely $N_H(y_1)$; and the same holds for z_i and z_i'.

Suppose K contains disjoint paths A', B', C' from $\{z_i, z_i', b_1\}$ to $\{y_1, y_1', z_{3-i}\}$, with $z_i \in V(A'), z_i' \in V(C')$ and $b_1 \in V(B')$. If $z_{3-i} \notin V(B')$ then, after identifying y_1 with y_1' and z_i with z_i', we obtain from $A' \cup B' \cup C'$ a path in H from z_{3-i} to b_1 through z_i, y_1 in order, contradicting (1). Hence $z_{3-i} \in V(B')$, and we get the desired paths for *(iii)* from $A' \cup B' \cup C'$, by identifying y_1 with y_1' and z_i with z_i'.

So we may assume that such A', B', C' do not exist. Then K has a separation (K', K'') such that $|V(K' \cap K'')| \leq 2$, $\{b_1, z_i, z_i'\} \subseteq V(K')$ and $\{y_1, y_1', z_{3-i}\} \subseteq V(K'')$. Since $H - z_{3-i}$ is 2-connected, $z_{3-i} \notin V(K' \cap K'')$.

We claim that $z_i, z_i' \notin V(K' \cap K'')$. For, if exactly one of z_i, z_i' is in $V(K' \cap K'')$ then, since z_i, z_i' have the same set of neighbors in K, $V(K' \cap K'') - \{z_i, z_i'\}$ is a cut in H separating $\{z_{3-i}, y_1\}$ from $\{z_i, b_1\}$, a contradiction. Now assume $\{z_i, z_i'\} = V(K' \cap K'')$. Then z_i is a cut vertex in H separating b_1 from $\{y_1, z_{3-i}\}$, a contradiction.

We may assume that $y_1, y_1' \notin V(K' \cap K'')$. First, suppose exactly one of y_1, y_1' is in $V(K' \cap K'')$. Then, since y_1, y_1' have the same set of neighbors in K, $V(K' \cap K'') - \{y_1, y_1'\}$ is a cut in H separating $\{z_{3-i}, y_1\}$ from $\{z_i, b_1\}$, a contradiction. Now assume $\{y_1, y_1'\} = V(K' \cap K'')$. Then y_1 is a cut vertex in H separating z_{3-i} from $\{b_1, z_i\}$. This implies that $N(z_{3-i}) \cap V(B_1) = \{y_1\}$; so $y_2 \notin V(X)$ and $z_{3-i} = p_{3-i}$. We may assume $i = 2$; for otherwise, $G[\{x_1, x_2, y_1, z_2\}] - x_1 z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. Then $z_1 = p_1$, and G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, p_2, x_1, x_2, y_1\}$ and $G_2 = G[B_1 \cup x_2 X p_2 + \{b, x_1\}]$. Note that $x_1 x_2 y_1 x_1$ is a triangle and $|V(G_j)| \geq 7$ for $j \in [2]$. So the assertion of this lemma follows from Lemma 2.8.

Thus, since B_1 is 2-connected, $|V(K' \cap K'')| = 2$. Let $V(K' \cap K'') = \{s, t\}$, and let F_i' (respectively, F_i'') be obtained from K' (respectively, K'') by identifying z_i' with z_i (respectively, y_1' with y_1). Then (F_i', F_i'') gives the desired 2-separation in H, completing the proof of (2).

By Lemma 5.1, we may assume that

(3) H has disjoint paths Y, Z from y_1, z_1 to b_1, z_2, respectively.

We now consider three cases.

Case 1. $e(z_i, B_1 - b_1) \geq 2$ for $i \in [2]$.

For $i \in [2]$, let $V(F_i' \cap F_i'') = \{s_i, t_i\}$ as in (2). Let Z_1, B_1' denote the $\{s_1, t_1\}$-bridges of F_1' containing z_1, b_1, respectively, and let Y_1, Z_2 denote the $\{s_1, t_1\}$-bridges of F_1'' containing y_1, z_2, respectively.

Suppose $Y_1 \neq Z_2$, and suppose $Z_1 \neq B_1'$ or $b_1 \in \{s_1, t_1\}$. Let $b_1 = s_1$ if $b_1 \in \{s_1, t_1\}$. Then Z_1 has independent paths S_1, T_1 from z_1 to s_1, t_1, respectively. Moreover, Z_2 has independent paths S_2, T_2 from z_2 to s_1, t_1, respectively, $B_1' - t_1$ has a path P from s_1 to b_1, and Y_1 has independent paths S_3, T_3 from y_1 to s_1, t_1, respectively. So $x_1 z_1 \cup (z_1 X p_1 \cup Q_1) \cup x_1 y_2 \cup (z_2 X p_2 \cup Q_2) \cup z_2 x_2 x_1 \cup (T_2 \cup T_1) \cup S_1 \cup S_2 \cup (S_3 \cup y_1 x_1) \cup (P \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Thus, we may assume that $Y_1 = Z_2$, or $Z_1 = B_1'$ and $b_1 \notin \{s_1, t_1\}$. First, suppose $Y_1 \neq Z_2$. Then $Z_1 = B_1'$ and $b_1 \notin \{s_1, t_1\}$, and hence $B_1' - \{s_1, t_1\}$ has a path from z_1 to b_1. Since H is 2-connected, $Y_1 \cup Z_2$ has two independent paths from y_1 to z_2. However, this contradicts the existence of the separation (F_2', F_2'').

So $Y_1 = Z_2$. Thus, by symmetry, we may assume $t_2 \in V(Y_1) - \{s_1, t_1\}$. Suppose $b_1 \notin \{s_1, t_1\}$ and $B_1' = Z_1$. Then $s_2 \in V(B_1') - \{s_1, t_1\}$. Moreover, $\{s_2, t_2\}$ separates s_1

from t_1 in H; for otherwise, either t_2 separates z_2 from $\{b_1, y_1, z_1\}$ in H, or t_2 separates y_1 from $\{b_1, z_1, z_2\}$ in H, a contradiction. Thus, we may assume that in H, $\{s_2, t_2\}$ separates $\{b_1, s_1, z_2\}$ from $\{t_1, y_1, z_1\}$. However, this contradicts (3).

Therefore, $B_1' \neq Z_1$ or $b_1 \in \{s_1, t_1\}$. If $b_1 \notin \{s_1, t_1\}$ then $B_1' \neq Z_1$; so $s_2 \in \{s_1, t_1\}$ (because of (F_2', F_2'')), and we may assume $s_2 = s_1$. If $b_1 \in \{s_1, t_1\}$ then we may assume that $b_1 = s_1$; so $s_2 = s_1$ or, in Z_1, s_2 separates s_1 from $\{t_1, z_1\}$. Let Y_1', Z_2' be the t_2-bridges of $Y_1 - \{s_1, t_1\}$ containing y_1, z_2, respectively. Again, because of the existence of (F_2', F_2''), t_1 has no neighbor in $Z_2' - t_2$. Hence, by (3), s_1 has a neighbor in $Y_1' - t_2$; and, thus, $s_2 = s_1$ and $G[Y_1' + \{s_1, t_1\}]$ has disjoint paths S_1, T_1 from s_1, t_1 to y_1, t_2, respectively. Let S_2, T_2 be independent paths in $G[Z_2' + s_1]$ from z_2 to s_1, t_2, respectively, and S, T be independent paths in Z_1 from z_1 to s_1, t_1, respectively. Let P be a path in $B_1' - t_1$ from s_1 to b_1. Then $x_1z_1 \cup (z_1Xp_1 \cup Q_1) \cup x_1y_2 \cup (z_2Xp_2 \cup Q_2) \cup z_2x_2x_1 \cup (T_2 \cup T_1 \cup T) \cup S \cup (S_1 \cup y_1x_1) \cup S_2 \cup (P \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Case 2. $e(z_2, B_1 - b_1) \geq 2$.

If $y_2 \in V(X)$ then $e(z_1, B_1 - b_1) \geq 2$, and if $y_2 \notin V(X)$ then, by Lemma 4.3, $e(z_1, B_1 - b_1) \geq 1$. In view of Case 1, we may assume $e(z_1, B_1 - b_1) = 1$; so $z_1 = p_1$ and $y_2 \notin V(X)$. Note that if $b \neq b_1$ then, by Lemma 4.2, we may assume $z_1b_1 \in E(G)$; so $b_1 \in V(F_2' \cap F_2'')$. By Lemma 4.1, we may assume that Y_2 has a path Q from p_2 to b through y_2, z_1 in this order.

For convenience, let $F' := F_2'$, $F'' := F_2''$, $s := s_2$ and $t := t_2$. So $b_1, z_2 \in V(F')$ and $y_1, z_1 \in V(F'')$. We choose (F', F'') so that F'' is minimal. Let z_1' denote the unique neighbor of z_1 in $B_1 - b_1$.

Subcase 2.1. $N(z_2Xp_2 - z_2) \cap V(F'' - \{z_1, s, t\}) \not\subseteq \{z_1'\}$.

Let $uu' \in E(G)$, with $u \in V(F'') - \{z_1, z_1', s, t\}$ and $u' \in V(z_2Xp_2 - z_2)$. Note that F' contains a path S from z_2 to b_1 such that $|V(S) \cap \{s, t\}| \leq 1$. Moreover, if there exists $r \in \{s, t\}$ such that $r \in V(S)$ for all such path S, then $b_1 = r$.

If $(F'' - z_1) - S$ contains independent paths T_1, T_2 from y_1 to z_1', u, respectively, then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup S \cup z_2x_2) \cup (z_1z_1' \cup T_1) \cup (T_2 \cup uu' \cup u'Xp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume that such T_1, T_2 do not exist. Hence, there is a cut vertex c in $(F'' - z_1) - S$ separating y_1 from $\{u, z_1'\}$. Denote by M_1, M_2 the $(\{c\} \cup (V(S) \cap \{s, t\}))$-bridges of $F'' - z_1$ containing y_1, $\{u, z_1'\}$, respectively. We may choose c so that M_1 is minimal. Then $N(z_2Xp_2 - z_2) \cap V(F'') \subseteq V(M_2)$ (as uu' was chosen arbitrarily).

Since G is 5-connected, $\{s, t\} \subseteq V(M_1)$ (as otherwise $\{c, x_1, x_2\} \cup (\{s, t\} \cap V(M_1))$ would be a cut in G), and M_1 contains independent paths R_1, R_2, R_3 from y_1 to c, s, t, respectively. Since B_1 is 2-connected, $\{s, t\} \cap V(M_2) \neq \emptyset$ and there exist choices of u and $r \in \{s, t\} \cap V(M_2)$ such that M_2 contains disjoint paths R_4, R_5 from $\{z_1', u\}$ to $\{c, r\}$ and avoiding $\{s, t\} \cap V(M_2) - \{r\}$. Thus, $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$ contains independent paths from y_1 to z_1', u, respectively, and avoiding $\{s, t\} \cap V(M_2) - \{r\}$. By the non-existence of T_1 and T_2, $r \in V(S)$ for every choice of S. Hence, $b_1 = r$, $\{s, t\} \cap V(M_2) = \{r\}$, and

$V(S) \cap \{s,t\} = \{r\}$ for every choice of S. Without loss of generality, we may assume that $r = t$.

We further choose uu' so that $u'Xp_2$ is maximal. Suppose $N(u'Xp_2 - u') \cap V(F' - \{s,t\}) = \emptyset$. Then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s,t,u',x_1,x_2\}$ and $G_2 = G[F' \cup x_2Xu' + x_1]$. Clearly, $|V(G_1)| \geq 7$. Since $e(z_2, B_1 - b_1) \geq 2$, $|V(G_2)| \geq 7$. If $(G_2 - x_1, x_2, s, t, u')$ is planar then the assertion of this lemma follows from Lemma 2.7. Hence, we may assume, by Lemma 2.1, that $G_2 - x_1$ contains disjoint paths X_1, X_2 from u', x_2 to s, t, respectively. Let X_3 be a path in $M_2 - t$ from z_1' to c. Then $G[\{x_1,x_2,y_1,y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup X_2) \cup (z_1z_1' \cup X_3 \cup R_1) \cup (R_2 \cup X_1 \cup u'Xp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So assume that there exists $ww' \in E(G)$ with $w' \in V(u'Xp_2 - u')$ and $w \in V(F' - \{s,t\})$. Let S_1 be a path in $F' - t$ from w to s and S_2 be a path in $M_2 - t$ from z_1' to u. Then $G[\{x_1,x_2,y_1,y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup R_3) \cup (z_1z_1' \cup S_2 \cup uu' \cup u'Xx_2) \cup (R_2 \cup S_1 \cup ww' \cup w'Xp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Subcase 2.2. $N(z_2Xp_2 - z_2) \cap V(F'' - \{z_1, s, t\}) \subseteq \{z_1'\}$.

Then $\{s, t, x_1, x_2, z_1'\}$ is a 5-cut in G separating $F'' - z_1$ from $F' \cup Y_2$. Since G is 5-connected, $F'' - z_1$ has independent paths T_1, T_2, T_3 from y_1 to s, t, z_1', respectively. Next, we find a path R in $F'' - z_1$ from s to t and containing $\{y_1, z_1'\}$. For this, let $F_g := (F'' - z_1) + \{g, gs, gt\}$, where g is a new vertex. Since G is 5-connected and we are in Subcase 2.2, F_g has no 2-cut separating y_1 from $\{g, z_1'\}$. Hence, by Lemma 2.4, there is a cycle in F_g containing $\{g, y_1, z_1'\}$ and, after removing g from this cycle, we get the desired R.

Let $x = p_2$ if $N(z_2Xp_2 - z_2) \cap V(F'' - \{z_1, s, t\}) = \emptyset$ and, otherwise, let $x \in N(z_1') \cap N(z_2Xp_2 - z_2)$ with xXz_2 minimal.

We may assume that $N(xXp_2 - x) \cap V(B_1 - \{b_1, z_1'\}) = \emptyset$. For, otherwise, there exists $rr' \in E(G)$ such that $r \in V(B_1) - \{b_1, z_1'\}$ and $r' \in V(xXp_2 - x)$. Then $r \in V(F')$ and $x \neq p_2$; so $xz_1' \in E(G)$. Note that F' has disjoint paths from $\{s,t\}$ to $\{b_1, r\}$, which, combined with T_1, T_2, gives independent paths P_1, P_2 in $B_1 - z_1'$ from y_1 to b_1, r, respectively. Hence, in $(G - x_1) - (z_1z_1'x \cup xXx_2)$, $\{y_1, y_2\}$ is contained in the cycle $P_1 \cup P_2 \cup rr' \cup r'Xp_2 \cup Q_2 \cup Q_3 \cup bb_1$. Hence, by Lemma 2.11 and Lemma 2.7, we may assume that $G - x_1$ has a path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X)$, and $(G - x_1) - X'$ is 2-connected. Thus, the assertion of this lemma follows from Lemma 2.5.

We may assume $b = b_1$. For, suppose $b \neq b_1$. Then, using the notation from *(iv)* of Lemma 3.2, $v \in V(p_2Xx_2 - p_2)$ and $b_1' \in V(B_1 - b_1)$. Let P_1, P_2 be independent paths in B_1 from y_1 to b_1, b_1', respectively. Then $G[\{x_1,x_2,y_1,y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup P_1) \cup (z_1Qb \cup bv \cup vXx_2) \cup (P_2 \cup b_1'p_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Therefore, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b_1, s, t, x, x_1, x_2\}$ and $G_2 = G[F' \cup xXx_2 + x_1]$. Let $G_2' = G_2 + \{r, rs, rt\}$, where r is a new vertex.

We may assume that $(G_2' - x_1, \mathcal{A}, b_1, x, x_2, r)$ is 3-planar for some collection $\mathcal{A}$ of subsets of $V(G_2' - x_1) - \{b_1, x, x_2, r\}$. For, otherwise, by Lemma 2.1, $G_2' - x_1$ contains disjoint paths R_1, R_2 from b_1, x to x_2, r, respectively. Let $R = T_2 \cup (R_2 - r)$ if $R_2 - r$

ends at t, and $R = T_1 \cup (R - r)$ otherwise. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb_1 \cup R_1) \cup (z_1z_1' \cup T_3) \cup (R \cup xXp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

We choose $\mathcal{A}$ to be minimal and define J, s', t' as follows. If $\mathcal{A} = \emptyset$ then after relabeling of s, t (if necessary), we may assume $(G_2' - x_1, b_1, x, x_2, s, t)$ is planar and let $J = G_2$, $s' = s$ and $t' = t$. Now assume $\mathcal{A} \neq \emptyset$. Then, by the minimality of $\mathcal{A}$ and 5-connectedness of G, $\mathcal{A}$ has a unique member, say A, such that $r \in N(A)$ and $\{s, t\} \subseteq A$ and, moreover, $G'[A \cup \{s', t'\}]$ is connected, where $N(A) \cap V(F') = \{r, s', t'\}$. Let J denote the $\{s', t', x_1\}$-bridge of G_2' containing $\{b_1, x, x_2\}$. We may assume, after suitable labeling of s', t', $(J - x_1, b_1, x, x_2, s', t')$ is planar.

Suppose $b_1 \in \{s', t'\}$. Then G has a 5-separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{s', t', x, x_1, x_2\}$ and $L_2 = J$. If $|V(J)| \geq 7$ then the assertion of this lemma follows from Lemma 2.7. So assume $|V(J)| \leq 6$. Since $e(z_2, B_1 - b_1) \geq 2$, there exists $v \in N(z_2) \cap V(F' - \{s', t', z_2\})$. Since G is 5-connected, $vx_1, vx_2 \in E(G)$. Hence, $G[\{v, x_1, x_2, z_2\}]$ contains a K_4^- in which x_1 is of degree 2.

Thus, we may assume that $b_1 \notin \{s', t'\}$. Then G has a 6-separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{b_1, s', t', x, x_1, x_2\}$ and $L_2 = J$. If $|V(J)| \geq 8$ then the assertion of this lemma follows from Lemmas 2.9 and 2.7.

So assume $|V(J)| \leq 7$. By planarity of J and 2-connectedness of B_1, $z_2t' \notin E(G)$. Thus, since $e(z_2, B_1 - b_1) \geq 2$, $z_2s' \in E(G)$ and there exists $v \in V(J) - \{b_1, s', t', x, x_2, z_2\}$ such that $z_2v \in E(G)$. So $|V(J)| = 7$ and $z_2 = x$. By the minimality of F', $vt' \in E(G)$; and by the 2-connectedness of B_1, $\{vs', vb_1\} \subseteq E(G)$. By planarity of J, $x_2v \notin E(G)$. Thus, $vx_1 \in E(G)$ as G is 5-connected. Then we may assume $x_1b_1 \notin E(G)$; for otherwise $G[\{b_1, t', v, x_1\}] - x_1t' \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. We may also assume $x_1z_2 \notin E(G)$; for otherwise $G[\{s', v, x_1, z_2\}] - x_1s' \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. So $z_2 = p_2$ as G is 5-connected.

If $L := G[(F'' - z_1) + A \cup \{s', t'\}]$ has independent paths P_1, P_2 from t' to s', z_1', respectively, and if Y_2 has a cycle C containing $\{b, z_1, z_2\}$, then $G[\{b_1, t', v\}] \cup z_2v \cup (z_2s' \cup P_1) \cup C \cup (z_1z_1' \cup P_2) \cup z_1x_1v$ is a TK_5 in G with branch vertices b_1, t', v, z_1, z_2. So we may assume P_1, P_2 do not exist, or C does not exist.

Suppose P_1, P_2 do not exist in L. Then L has 1-separation (L_1, L_2) such that $t' \in V(L_1 - L_2)$ and $\{s', z_1'\} \subseteq V(L_2)$. Since G is 5-connected, $|V(L_1)| = 2$ and $x_1t' \in E(G)$. Now $G[\{b_1, t', v, x_1\}] - x_1b_1 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

Now assume C does not exist. Then by Lemma 2.4, Y_2 has 2-cuts S_b, S_z such that b_1 is a in component D_b of $Y_2 - S_b$, $p_1 = z_1$ is in a component D_z of $Y_2 - S_z$, and $V(D_b) \cap (V(D_z) \cup S_z \cup \{p_2\}) = \emptyset = V(D_z) \cap (V(D_b) \cup S_b \cup \{p_2\})$. If $y_2 \notin V(D_b)$ then $S_b \cup \{b, x_1\}$ is a cut in G, a contradiction. So $y_2 \in V(D_b)$. Then $y_2 \notin V(D_z)$. Then $S_z \cup \{x_1, z_1'\}$ is a cut in G, a contradiction.

Case 3. $e(z_2, B_1 - b_1) \leq 1$.

If $y_2 \in V(X)$ then, since G is 5-connected, $e(z_1, B_1 - b_1) \geq 2$ and $e(z_2, B_1 - b_1) = 1$. If $y_2 \notin V(X)$ then, by (iii) of Lemma 4.3, $e(z_2, B_1 - b_1) = 1$ and $e(z_1, B_1 - b_1) \geq 2$.

For convenience, let $F' := F_1'$, $F'' := F_1''$, $s := s_1$ and $t := t_1$. Then $b_1, z_1 \in V(F')$ and $y_1, z_2 \in V(F'') - V(F')$. We choose (F', F'') so that F'' is minimal. Let z_2' denote the unique neighbor of z_2 in $B_1 - b_1$. We may assume $z_2' \neq y_1$; for, otherwise, $G[\{x_1, x_2, y_1, z_1'\}] - x_1 z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. Note that if $z_2 \neq p_2$ then $z_2 b_1, z_2 x_1 \in E(G)$. By (iii) of Lemma 4.1, $G[Y_2 + b_1 + p_2 X z_2]$ contains a path Q from p_1 to b_1 through y_2, p_2 in order.

Subcase 3.1. $N(z_1 X p_1 - z_1) \cap V(F'' - \{s, t, z_2\}) \not\subseteq \{z_2'\}$.

Let $uu' \in E(G)$ with $u' \in V(z_1 X p_1 - z_1)$ and $u \in V(F'') - \{s, t, z_2, z_2'\}$. Since B_1 is 2-connected, F' contains a path S from z_1 to b_1 such that $|V(S) \cap \{s, t\}| \leq 1$.

Suppose $(F'' - z_2) - S$ contains independent paths S_1, S_2 from y_1 to z_2', u, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup z_2 Q y_2 \cup (z_2 Q b_1 \cup S \cup z_1 x_1) \cup (z_2 z_2' \cup S_1) \cup (S_2 \cup uu' \cup u' X p_1 \cup p_1 Q y_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So we may assume that such S_1, S_2 do not exist in $(F'' - z_2) - S$ for any choice of S and any choice of u. Hence, $(F'' - z_2) - S$ has a cut vertex c which separates y_1 from $N(z_1 X p_1 - z_1) \cup \{z_2'\}$. Denote by M_1, M_2 the $(\{c\} \cup (\{s, t\} \cap V(S)))$-bridges of $F'' - z_2$ containing y_1, $(N(z_1 X p_1 - z_1) \cap V(F'' - \{s, t, z_2\})) \cup \{z_2'\}$, respectively. Since G is 5-connected, $\{s, t\} \subseteq V(M_1)$ (to avoid the cut $\{c, x_1, x_2\} \cup (V(S) \cap \{s, t\})$) and M_1 contains independent paths R_1, R_2, R_3 from y_1 to c, s, t, respectively. Since B_1 is 2-connected, $\{s, t\} \cap V(M_2) \neq \emptyset$. Note that there exists $r \in \{s, t\} \cap V(M_2)$ such that M_2 contains disjoint paths T_1, T_2 from $\{z_2', u\}$ to $\{c, r\}$ and avoiding $\{s, t\} \cap V(M_2) - \{r\}$. Now $R_1 \cup R_3 \cup T_1 \cup T_2$ contains independent paths from y_1 to z_2', u, respectively, and avoiding $\{s, t\} \cap V(M_2) - \{r\}$. So by the nonexistence of S_1, S_2, $r \in V(S)$ for every choice of S, which implies $b_1 = r$. So we may assume $b_1 = t$.

Choose uu' so that $u' X p_1$ is maximal. Since $\{s, t, u', x_1\}$ cannot be a cut in G separating F' from $F'' \cup Y_2 \cup p_2 X x_2$, there exists $ww' \in E(G)$ such that $w \in V(F' - \{s, t, z_1\})$ and $w' \in V(u' X p_1 - u') \cup V(p_2 X x_2)$.

Suppose $w' \in V(u' X p_1 - u')$. Let P_1 be a path in $F' - \{z_1, t\}$ from w to s and P_2 be a path in $M_2 - t$ from z_2' to u. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup z_2 Q y_2 \cup (z_2 Q b_1 \cup R_3) \cup (z_2 z_2' \cup P_2 \cup uu' \cup u' X z_1 \cup z_1 x_1) \cup (R_2 \cup P_1 \cup ww' \cup w' X p_1 \cup p_1 Q y_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume $w' \in V(p_2 X x_2)$. Let W be a path in $F' - t$ from z_1 to w. Then $X' := W \cup ww' \cup w' X x_2$ is a path in $G - x_1$ from z_1 to x_2 such that in $(G - x_1) - X'$, $\{y_1, y_2\}$ is contained in a cycle (which is contained in $(Y_2 - p_2) \cup p_1 X u' \cup u' u \cup M_2 \cup (M_1 - s)$). Hence by Lemma 2.11 and Lemma 2.7, we may assume that X' is induced, $y_1, y_2 \notin V(X)$, and $(G - x_1) - X'$ is 2-connected. Thus, the assertion of this lemma follows from Lemma 2.5.

Subcase 3.2. $N(z_1 X p_1 - z_1) \cap V(F'' - \{s, t, z_2\}) \subseteq \{z_2'\}$.

First, we show that $\{s, t, x_1, x_2, z_2'\}$ is a 5-cut in G separating $F'' - z_2$ from $F' \cup Y_2 \cup X$. For, otherwise, there exists $ww' \in E(G)$ with $w \in V(F'') - \{s, t, z_2'\}$ and $w' \in V(p_2 X z_2 - z_2)$. Let P_1, P_2 be independent paths in F' from z_1 to r, b_1, respectively, with $r \in \{s, t\}$. Without loss of generality, we may assume $r = s$. By the minimality of F'', $F'' - t$ has independent paths R_1, R_2 from y_1 to s, w, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (P_1 \cup R_1) \cup (P_2 \cup b_1z_2x_2) \cup (R_2 \cup ww' \cup w'Xp_2 \cup Q_2)$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1.

Hence, since G is 5-connected, $F'' - z_2$ contains independent paths T_1, T_2, T_3 from y_1 to s, t, z_2', respectively. Let $F_g := (F'' - z_2) + \{g, gs, gt\}$, where g is a new vertex; then by Lemma 2.4, F_g has a cycle containing $\{g, y_1, z_2'\}$. Thus, we may assume by symmetry that $F'' - z_2$ has a path S from s to t and through y_1, z_2' in order.

We may assume $N(x_2) \cap V(F' - \{s,t\}) = \emptyset$. For, suppose there exists $x_2^* \in N(x_2) \cap V(F' - \{s,t\})$. Since B_1 is 2-connected, F' contains independent paths R_1, R_2 from z_1 to x_2^*, r, respectively, for some $r \in \{s,t\}$. (This can be done by considering whether or not z_1 and x_2^* are contained in the same $\{s,t\}$-bridge of F'.) Let $T = T_1$ if $r = s$, and $T = T_2$ if $r = t$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (R_1 \cup x_2^*x_2) \cup (R_2 \cup T) \cup (Q_2 \cup p_2Xz_2 \cup z_2z_2' \cup T_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Let $x = p_1$ if $N(z_2') \cap V(z_1Xp_1 - z_1) = \emptyset$, and otherwise let $x \in N(z_2') \cap V(z_1Xp_1 - z_1)$ with z_1Xx minimal.

Suppose $z_2'x_2 \in E(G)$. Then we may assume $x_1z_2 \notin E(G)$; for otherwise, $G[\{x_1, x_2, z_2, z_2'\}] - x_1z_2' \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. Hence, $z_2 = p_2$, and $\{b_1, s, t, x, x_1\}$ is a 5-cut in G separating $F' \cup z_1Xx$ from $F'' \cup Y_2$. Since G is 5-connected, $b_1 \notin \{s,t\}$. Let (G_1, G_2) be a 5-separation in G such that $V(G_1 \cap G_2) = \{b_1, s, t, x, x_1\}$ and $G_2 = G[F' \cup z_1Xx + x_1]$. Clearly, $|V(G_2)| \geq 7$. We may assume $|V(G_1)| \geq 7$; for, if not, $|V(G_1)| = 6$, and $G[\{b_1, s, t, z_1\}] - st \cong K_4^-$ and (ii) holds. If $(G_2 - x_1, b_1, x, s, t)$ is planar then the assertion of this lemma follows from Lemma 2.7. So we may assume that this is not the case. Then by Lemma 2.1, $G_2 - x_1$ has disjoint paths S_1, S_2 from s, t to b_1, x, respectively. Now $z_2z_2'x_2z_2 \cup y_1x_2 \cup y_1Sz_2' \cup (y_1Ss \cup S_1 \cup b_1Qz_2) \cup y_2Qz_2 \cup (y_2Qp_1 \cup p_1Xx \cup S_2 \cup tSz_2') \cup y_2x_2 \cup y_2x_1y_1$ is a TK_5 in G' with branch vertices x_2, y_1, y_2, z_2, z_2'.

Now assume $z_2'x_2 \notin E(G)$. Then x_2 has a neighbor in $F'' - \{y_1, z_2'\}$ (as $N(x_2) \cap V(F' - \{s,t\}) \neq \emptyset$). Let r be a new vertex. We may assume that $(F'' + \{r, rs, rt\}) - z_2$ has disjoint paths S_1, S_2 from r, z_2' to x_2, y_1, respectively. For, suppose such paths S_1, S_2 do not exist. Then by Lemma 2.1, there exists a collection $\mathcal{A}$ of disjoint subsets of $F'' - \{x_2, y_1, z_2\}$ such that $(F'' + \{r, rs, rt\}) - z_2, r, y_1, x_2, z_2')$ is 3-planar. Since G is 5-connected and F'' is minimal, we may assume $(F'' - z_2, s, t, y_1, x_2, z_2')$ is planar. Thus, since z_2' is the only neighbor of z_2 in $F'' - F'$, G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{s, t, x_1, x_2, z_2\}$, $G_2' - x_1 = F''$, and $(G_2' - x_1, s, t, x_2, z_2)$ is planar. Since $|V(G_j')| \geq 7$ for $j \in [2]$, the assertion of this lemma follows from Lemma 2.7.

Without loss of generality, let $rs \in S_1$. If $F' - t$ has independent paths P_1, P_2 from z_1 to s, b_1, respectively, then $G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup (P_1 \cup (S_1 - r)) \cup (z_1Xp_1 \cup p_1Qy_2) \cup (z_2z_2' \cup S_2 \cup y_1x_1) \cup z_2x_2 \cup z_2Qy_2 \cup (z_2Qb_1 \cup P_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2. So we may assume that such P_1, P_2 do not exist in $F' - t$.

Thus F' has a 2-separation (F_1, F_2) such that $t \in V(F_1 \cap F_2)$, $z_1 \in V(F_1 - F_2)$ and $\{b_1, s\} \subseteq V(F_2 - F_1)$. Choose this separation so that F_1 is minimal. Let $s' \in$

$V(F_1 \cap F_2) - \{t\}$. Since $\{s', t, z_1, x_1\}$ cannot be a cut in G, $V(F_1) = \{s', t, z_1\}$ or there exists $zz' \in E(G)$ such that $z \in V(z_1 X p_1 - z_1) \cup V(p_2 X z_2 - z_2)$ and $z' \in V(F_1) - \{s', t, z_1\}$.

First, assume $V(F_1) = \{s', t, z_1\}$. Then $z_1 = p_1$ as G is 5-connected. By (iii) of Lemma 4.1, let Q' be a path in Y_2 from p_2 to b_1 and through y_2, p_1 in order, and let C be a cycle in $Y_2 - b_1$ containing $\{p_1, p_2, y_2\}$. Let $C' := Q' \cup p_2 X z_2 \cup z_2 b_1$ if $z_2 \neq p_2$; and let $C' := C$ if $z_2 = p_2$. If $F' - \{b_1, t, z_1\}$ has a path S from s' to s then $x_1 x_2 y_2 x_1 \cup z_1 x_1 \cup z_2 x_2 \cup C' \cup (z_1 s' \cup S \cup S_1) \cup (z_2 z_2' \cup S_2 \cup y_1 x_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2. So we may assume such S does not exist. Then F' has a separation (L', L'') such that $V(L' \cap L'') = \{b_1, t\}$, $\{s', z_1\} \subseteq V(L')$ and $s \in V(L'') - \{b_1, t\}$. Since G is 5-connected, $\{b_1, t, x_1, z_1\}$ is not a cut in G, and $L' - \{b_1, t, z_1\}$ has a path S' from s' to some $z \in N(p_2 X z_2 - z_2)$. Let $z' \in N(z) \cap V(p_2 X z_2 - z_2)$. Let S be a path in $L'' - t$ from s to b_1. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup Q_1 \cup (z_1 s' \cup S' \cup zz' \cup z' X x_2) \cup (z_1 t \cup T_2) \cup (T_1 \cup S \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Thus, we may assume that $zz' \in E(G)$ such that $z \in V(z_1 X p_1 - z_1) \cup V(p_2 X z_2 - z_2)$ and $z' \in V(F_1) - \{z_1, s', t\}$.

Suppose $z \in V(x X p_1 - x)$. Let $X^* = z_1 X x \cup x z_2' z_2 x_2$. Then, $T_1 \cup T_2 \cup (F' - z_1) \cup zz' \cup z X p_1 \cup Y_2$ is contained in $G - X^*$ and has a cycle containing $\{y_1, y_2\}$. Hence, by Lemma 2.11 and then Lemma 2.7, we may assume that $G - x_1$ has an induced path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X')$ and $G - X'$ is 2-connected. Then the assertion of this lemma follows from Lemma 2.5.

Now suppose $z \in V(p_2 X z_2 - z_2)$. By the minimality of F_1, $F_1 - t$ has independent paths L_1, L_2 from z_1 to s', z', respectively. In $F_2 \cup (F'' - z_2)$, we find independent paths L_1', L_2' from y_1 to s', b_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup (L_1 \cup L_1') \cup (L_2 \cup z' z \cup z X x_2) \cup (L_2' \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Hence, we may assume $z \in V(z_1 X x - z_1)$ for all such zz'. Choose z with $z_1 X z$ is maximal. Since $\{s', t, x_1, z\}$ cannot be a cut in G, there exists $uu' \in E(G)$ such that $u \in V(z_1 X z) - \{z_1, z\}$ and $u' \in V(F_2) - \{s', t\}$. Let P_1 be a path in $F_1 - \{s', z_1\}$ from z' to t, and P_2 be a path in $F_2 - t$ from u' to b_1. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 z_2' \cup T_3) \cup (z_2 X p_2 \cup p_2 Q y_2) \cup (z_2 Q b_1 \cup P_2 \cup u' u \cup u X z_1 \cup z_1 x_1) \cup (T_2 \cup P_1 \cup z' z \cup z X p_1 \cup p_1 Q y_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. □

6. Finding TK_5

Recall the notation from Lemma 3.2 and the previous section. In particular, $H := G[B_1 + \{z_1, z_2\}]$, $G' := G - \{x_1 x : x \notin \{x_2, y_1, y_2, z_0, z_1\}\}$, $b_1 \in N(y_2) \cap V(B_1)$ and $p_1 = p_2 = b = y_2$ if $y_2 \in V(X)$, and $b_1 \in V(B_1 \cap B_2)$ and $V(Y_1 \cap Y_2) = \{b, p_1, p_2\}$ if $y_2 \notin V(X)$. Our objective is to find TK_5 in G' using the structural information on H produced in the previous sections. By Lemma 4.1,

(A1) Y_2 has independent paths Q_1, Q_2, Q_3 from y_2 to p_1, p_2, b, respectively.

Note that if $y_2 \in V(X)$ then $e(z_1, B_1 - b_1) \geq 2$ and $e(z_2, B_1 - b_1) \geq 1$. Thus, by Lemma 4.3, we may assume that there exists $i \in [2]$ such that $e(z_i, B_1 - b_1) \geq 2$ and $e(z_{3-i}, B_1 - b_1) \geq 1$. (Hence, by Lemma 4.2, $e(z_{3-i}, B_1) = 1$ only if $b = b_1$ and, therefore, $z_{3-i} = p_{3-i}$.) Then by Lemma 4.1,

(A2) Y_2 has a path T from b to p_i through p_{3-i}, y_2 in order, respectively.

By Lemma 5.1, we may assume that

(A3) H has disjoint paths Y, Z from y_1, z_1 to b_1, z_2, respectively.

By Lemma 5.2, we may assume that

(A4) H has independent paths A, B, C, with A, C from z_i to y_1, and B from b_1 to z_{3-i}.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of all $(A \cup C)$-bridges of H with attachments on both A and C. We may choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,
(b) whenever possible, $J(A, C) \subseteq L(A, C)$,
(c) $J(A, C)$ is maximal, and
(d) $L(A, C)$ is maximal.

We refer the reader to Fig. 3 for an illustration. We may assume that

(A5) for any $j \in [2]$, H contains no path from z_j to b_1 and through z_{3-j}, y_1 in order.

For, suppose H does contain a path R from z_j to b_1 and through z_{3-j}, y_1 in order. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-j}x_{3-j} \cup (z_{3-j}Xp_{3-j} \cup Q_{3-j}) \cup (z_{3-j}Rz_j \cup z_jx_j) \cup z_{3-j}Ry_1 \cup (y_1Rb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-j}$. Thus, we may assume (A5).

Since B_1 is 2-connected and $e(z_{3-i}, B_1 - b_1) \geq 1$, H has disjoint paths P, Q from $p, q \in V(B)$ to $c, a \in V(A \cup C) - \{z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$. By symmetry between A and C, we may assume that b_1, p, q, z_{3-i} occur on B in order. By (A5), $c \neq y_1$. We choose such P, Q that the following are satisfied in order listed:

(A6) qBz_{3-i} is minimal, pBz_{3-i} is maximal, the subpath of $(A \cup C) - z_i$ between a and y_1 is minimal, and the subpath of $(A \cup C) - z_i$ between c and y_1 is maximal.

Let B' denote the union of B and the B-bridges of H not containing $A \cup C$. Note that all paths in H from $A \cup C$ to B' and internally disjoint from B' must have an end in B. We may assume that

(A7) if $e(z_{3-i}, B_1) \geq 2$ then, for any $q^* \in V(B' - q)$, B' has independent paths from z_{3-i} to q, q^*, respectively.

For, suppose $e(z_{3-i}, B_1) \geq 2$ and for some $q^* \in V(B' - q)$, B' has no independent paths from z_{3-i} to q, q^*, respectively. Then $q \neq z_{3-i}$, and B' has a 1-separation (B'_1, B'_2) such that $q, q^* \in V(B'_2)$ and $z_{3-i} \in V(B'_1) - V(B'_2)$. Note that $b_1 \in V(B'_2)$. Choose (B'_1, B'_2) with B'_1 minimal, and let $z \in V(B'_1 \cap B'_2)$.

Since $e(z_{3-i}, B_1) \geq 2$, $|V(B'_1)| \geq 3$; so H has a path R from some $s \in V(B'_1 - z)$ to some $t \in V(A \cup C \cup P \cup Q)$ and internally disjoint from $A \cup B \cup C \cup P \cup Q$. By the choice of P, Q in (A6), we see that $t = z_i$. Let S be a path in B'_1 from z_{3-i} to s, respectively. Let $R = A \cup y_1Cc$ if $c \in V(C)$, and $R = C \cup y_1Ac$ if $c \in V(A)$. Then Now $S \cup R \cup P \cup pBb_1$ is a path contradicting (A5).

We will show that we may assume $a = y_1$ (see (3)), derive structural information about G' and H (see (4)–(7)), and consider whether or not $z_i \in V(J(A, C))$ (see Case 1 and Case 2). First, we may assume that

(1) $N(y_1) \cap V(z_jXp_j - z_j) = \emptyset$ for $j \in [2]$.

For, suppose there exists $s \in N(y_1) \cap V(z_jXp_j - z_j)$ for some $j \in [2]$. By symmetry, assume $c \in V(C)$. If $j = 3 - i$ then, using Q_1, Q_2, Q_3 from (A1), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup A \cup (z_iCc \cup P \cup pBz_{3-i} \cup z_{3-i}x_{3-i}) \cup (y_1s \cup sXp_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

So assume $j = i$. Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$. Recall the path T from (A2). Note that $z_{3-i}Tb \cup bb_1 \cup A \cup B \cup C \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to z_i, y_1, respectively. Hence $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (S_1 \cup z_ix_i) \cup S_2 \cup (y_1s \cup sXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Now assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $P_1 \cup P_2 \cup A \cup B \cup C \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to z_i, y_1, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup z_ix_i) \cup S_2 \cup (y_1s \cup sXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. This proves (1).

We may also assume

(2) $y_1 \in V(J(A, C))$.

For, suppose $y_1 \notin V(J(A, C))$. By (1) and 5-connectedness of G, $y_1 \in V(D_1)$ for some $(A \cup C)$-bridge D_1 of H with $D_1 \neq J(A, C)$. Thus, let $D_1, \ldots, D_k$ be a maximal sequence of $(A \cup C)$-bridges of H with $D_j \neq J(A, C)$ for $j \in [k]$, such that, for each $l \in [k-1]$,

$$D_{l+1} \text{ has a vertex not in } \bigcup_{j\in[l]}(c_jCy_1 \cup a_jAy_1) \text{ and a vertex not in } \bigcap_{j\in[l]}(z_iCc_j \cup z_iAa_j),$$

where for each $j \in [k]$, $a_j \in V(D_j \cap A)$ and $c_j \in V(D_j \cap C)$ such that a_jAy_1 and c_jCy_1 are maximal. Let $S_l := \bigcup_{j\in[l]}(D_j \cup a_jAy_1 \cup c_jCy_1)$.

We claim that for any $l \in [k]$ and for any $r_l \in V(S_l)-\{a_l, c_l\}$, S_l has three independent paths A_l, C_l, R_l from y_1 to a_l, c_l, r_l, respectively. This is obvious for $l = 1$ (if $a_l = y_1$, or $c_l = y_1$, or $r_l = y_1$ then A_l, or C_l, or R_l is a trivial path). Now assume $k \geq 2$ and the claim holds for some $l \in [k-1]$. Let $r_{l+1} \in V(S_{l+1})-\{a_{l+1}, c_{l+1}\}$. When $r_{l+1} \in V(S_l)-\{a_l, c_l\}$ let $r_l := r_{l+1}$; otherwise, let $r_l \in V(a_lAy_1 - a_l) \cup V(c_lCy_1 - c_l)$ with $r_l \in V(D_{l+1})$. By assumption, S_l has independent paths A_l, C_l, R_l from y_1 to a_l, c_l, r_l, respectively. If $r_{l+1} \in V(S_l)-\{a_l, c_l\}$ then $A_{l+1} := A_l \cup a_lAa_{l+1}, C_{l+1} := C_l \cup c_lCc_{l+1}, R_{l+1} := R_l$ are the desired paths in S_{l+1}. If $r_{l+1} \in V(D_{l+1})-V(A\cup C)$ then let P_{l+1} be a path in D_{l+1} from r_l to r_{l+1} internally disjoint from $A \cup C$; we see that $A_{l+1} := A_l \cup a_lAa_{l+1}, C_{l+1} := C_l \cup c_lCc_{l+1}, R_{l+1} := R_l \cup P_{l+1}$ are the desired paths in S_{l+1}. So we may assume by symmetry that $r_{l+1} \in V(a_{l+1}Aa_l - a_{l+1})$. Let Q_{l+1} be a path in D_{l+1} from r_l to a_{l+1} internally disjoint from $A \cup C$. Now $R_{l+1} := A_l \cup a_lAr_{l+1}, C_{l+1} := C_l \cup c_lCc_{l+1}, A_{l+1} := R_l \cup Q_{l+1}$ are the desired paths in S_{l+1}.

Hence, by (c), $J(A, C)$ does not intersect $(a_kAy_1 \cup c_kCy_1) - \{a_k, c_k\}$. In particular, $a, c \notin (a_kAy_1 \cup c_kCy_1) - \{a_k, c_k\}$. Since G is 5-connected, $\{a_k, c_k, x_1, x_2\}$ cannot be a cut in G separating S_k from $X \cup J(A, C)$. So there exists $ss' \in E(G)$ such that $s \in V(S_k) - \{a_k, c_k\}$ and $s' \in V(z_1Xp_1 \cup z_2Xp_2)$. By the above claim, let A_k, C_k, R_k be independent paths in S_k from y_1 to a_k, c_k, s, respectively; so $s' \notin \{z_1, z_2\}$ by (c).

Suppose $s' \in V(z_{3-i}Xp_{3-i} - z_{3-i})$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup (z_iCc \cup P \cup pBz_{3-i} \cup z_{3-i}x_{3-i}) \cup (z_iAa_k \cup A_k) \cup (R_k \cup ss' \cup s'Xp_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

So we may assume $s' \in V(z_iXp_i - z_i)$. Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$. Recall the path T from (A2). Note that $z_{3-i}Tb \cup bb_1 \cup z_iAa_k \cup z_iCc_k \cup P \cup Q \cup B$ contains independent paths S_1, S_2 from z_{3-i} to z_i, v, respectively, for some $v \in \{a_k, c_k\}$. Let $S = A_k$ if $v = a_k$, and $S = C_k$ if $v = c_k$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (S_1 \cup z_ix_i) \cup (S_2 \cup S) \cup (R_k \cup ss' \cup s'Xp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Hence, we may assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then, $P_1 \cup P_2 \cup z_iAa_k \cup z_iCc_k \cup P \cup Q \cup B$ contains independent paths S_1, S_2 from z_{3-i} to z_i, v, respectively, for some $v \in \{a_k, c_k\}$. Let $S = A_k$ if $v = a_k$, and $S = C_k$ if $v = c_k$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup z_ix_i) \cup (S_2 \cup S) \cup (R_k \cup ss' \cup s'Xp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. This completes the proof of (2).

For convenience, we let $K := A \cup B \cup C \cup P \cup Q$. We claim that

(3) $a = y_1$

Suppose $a \neq y_1$. By (2), $J(A,C)$ has a path S from y_1 to some vertex $s \in V(P \cup Q \cup B) - \{c, a\}$ and internally disjoint from K. By (A6), $s \notin V(Q \cup qBz_{3-i})$. So $s \in V(P \cup b_1Bq) - \{a, q\}$. If $a \in V(A)$ let $R = aAz_i$ and $R' = C$; and if $a \in V(C)$ let $R = aCz_i$ and $R' = A$. Also, let $S' = S \cup sBb_1$ if $s \in V(B)$, and $S' = S \cup sPp \cup pBb_1$ if $s \in V(P)$. Then $z_{3-i}Bq \cup Q \cup R \cup R' \cup S'$ is a path contradicting (A5).

By symmetry between A and C, we may assume $c \in V(C)$. Before we distinguish cases according to whether or not $z_i \in V(J(A,C))$, we derive further information about G'. We may assume that

(4) for any path W in G' from x_i to some $w \in V(K) - \{z_i, y_1\}$ and internally disjoint from K, we have $w \in V(A) - \{z_i, y_1\}$.

To see this, suppose $w \notin V(A) - \{z_i, y_1\}$. First, assume $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$. Recall the path T from (A2), and note that $z_{3-i}Tb_1 \cup B \cup (C - z_i) \cup P \cup Q \cup W$ contains independent paths S_1, S_2 from z_{3-i} to x_i, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup S_1 \cup S_2 \cup (A \cup z_iXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Thus, we may assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths in B' from (A7) with $q^* = p$. So $P_1 \cup P_2 \cup B \cup (C - z_i) \cup W \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to x_i, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup S_1 \cup S_2 \cup (A \cup z_iXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. This completes the proof of (4).

Since G is 5-connected and $z_0 \in V(B_1)$ when $e(z_1, B_1) \geq 2$ (see (*iv*) of Lemma 3.2), it follows from (4) that

G' has a path W from x_i to $w \in V(A) - \{y_1, z_i\}$ and internally disjoint from K.

Hence, $|V(A)| \geq 3$. Also, $|V(C)| \geq 3$ as $c \in V(C) - \{y_1, z_1\}$. Since A and C are induced paths in H,

$$y_1z_i \notin E(G).$$

We may assume that

(5) G' has no path from $z_{3-i}Xp_{3-i} - y_2$ to $(A \cup C) - y_1$ and internally disjoint from K, G' has no path from $z_iXp_i - z_i$ to $(A \cup cCy_1) - \{z_i, c\}$ and internally disjoint from K, and if $i = 1$ then G' has no path from x_2 to $(A \cup C) - y_1$ and internally disjoint from K.

First, suppose S is a path in G' from some $s \in V(z_{3-i}Xp_{3-i} - y_2)$ to some $s' \in V(A \cup C) - \{y_1\}$. Then $A \cup C \cup S$ contains independent paths S_1, S_2 from z_i to y_1, s, respectively.

Hence, $G[\{x_1, x_2, y_1, y_2\}] \cup z_i x_i \cup (z_i X p_i \cup Q_i) \cup S_1 \cup (S_2 \cup sXz_{3-i} \cup z_{3-i}x_{3-i}) \cup (Q \cup qBb_1 \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

Now assume that S is a path in G' from some $s \in V(z_i X p_i - z_i)$ to some $s' \in V(A \cup cCy_1) - \{z_i, c\}$ and internally disjoint from K. Let $S' = y_1 A s'$ if $s' \in V(A)$, and $S' = y_1 C s'$ if $s' \in V(cCy_1)$. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Bq \cup Q) \cup (z_{3-i}Tb_1 \cup b_1 Bp \cup P \cup cCz_i \cup z_i x_i) \cup (S' \cup S \cup sXp_i \cup p_i T y_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup cCz_i \cup z_i x_i) \cup (S' \cup S \cup sXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Now suppose $i = 1$ and S is a path in G' from x_2 to some $s \in V(A \cup C) - \{y_1\}$ and internally disjoint from K. If $s \in V(A - y_1)$, then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup C \cup (z_1 A s \cup S) \cup (Q \cup qBb_1 \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So assume $s \in V(C - y_1)$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup A \cup (z_1 C s \cup S) \cup (Q \cup qBb_1 \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. This completes the proof of (5).

(6) We may assume that

- (6.1) any path in $J(A, C)$ from $A - \{z_i, y_1\}$ to $(P \cup Q \cup B) - \{c, y_1\}$ and internally disjoint from K must end on Q,
- (6.2) if an $(A \cup C)$-bridge of H contained in $L(A, C)$ intersects $z_i C c - c$ and contains a vertex $z \in V(A - z_i)$ then $J(A, C) \cap (z_i A z - \{z_i, z\}) = \emptyset$, and
- (6.3) $J(A, C) \cap (z_i C c - \{z_i, c\}) = \emptyset$, and any path in $J(A, C)$ from z_i to $(P \cup Q \cup B) - \{c, y_1\}$ and internally disjoint from K must end on $(P - c) \cup b_1 B p$.

To prove (6.1), let S be a path in $J(A, C)$ from $s \in V(A) - \{z_i, y_1\}$ to $s' \in V(P \cup B) - \{c, q, y_1\}$ and internally disjoint from K. Note that $s' \notin V(qBz_{3-i} - q)$ by (A6). Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and we use the path T from (A2). Let S' be a path in $(P - c) \cup (b_1 B q - q)$ from b_1 to s'. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Tb_1 \cup S' \cup S \cup sAw \cup W) \cup (z_{3-i}Bq \cup Q) \cup (C \cup z_i X p_i \cup p_i T y_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So we may assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be the paths from (A7), with $q^* = p$ when $s' \in V(P)$ and $q^* = s'$ when $s' \in V(B)$. So $P_1 \cup P_2 \cup B \cup S \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to s, y_1, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup sAw \cup W) \cup S_2 \cup (C \cup z_i X p_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

To prove (6.2), let D be a path contained in $L(A, C)$ from $z' \in V(z_i C c - c)$ to $z \in V(A - z_i)$ and internally disjoint from K. Suppose there exists $s \in V(J(A, C)) \cap V(z_i A z - \{z_i, z\})$. By (6.1), $J(A, C)$ has a path S from s to some $s' \in V(Q - y_1)$ and internally disjoint from K. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_i x_i \cup (z_i X p_i \cup Q_i) \cup (z_i A s \cup S \cup$

$s'Qq \cup qBz_{3-i} \cup z_{3-i}x_{3-i}) \cup (z_iCz' \cup D \cup zAy_1) \cup (y_1Cc \cup P \cup pBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

To prove (6.3), let S be a path in $J(A,C)$ from $s \in V(z_iCc - c)$ to $s' \in V(P \cup Q \cup B) - \{c, y_1\}$ and internally disjoint from K. Suppose $s' \in V(Q \cup z_{3-i}Bp) - \{p, y_1\}$. Then $(S \cup Q \cup pBz_{3-i}) - \{p, y_1\}$ contains a path S' from s to z_{3-i}. So $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup (z_iCs \cup S' \cup z_{3-i}x_{3-i}) \cup A \cup (y_1Cc \cup P \cup pBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i. Thus, we may assume $s' \in V(P - c) \cup V(b_1Bp)$. By (A6), $s = z_i$. This proves (6).

Denote by $L(A)$ (respectively, $L(C)$) the union of all $(A \cup C)$-bridges of H whose intersection with $A \cup C$ is contained in A (respectively, C).

(7) $L(A) = \emptyset$, and $L(C) \cap C \subseteq z_iCc$.

Suppose $L(A) \neq \emptyset$, and let R_1 be an $(A \cup C)$-bridge of H contained in $L(A)$. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(A)$, such that for $2 \le i \le m$, R_i has a vertex internal to $\bigcup_{j=1}^{i-1} l_jAr_j$ (which is a path), where $l_j, r_j \in V(R_j \cap A)$ with l_jAr_j maximal. Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} l_jAr_j = a_1Aa_2$. By (c), $J(A,C) \cap (a_1Aa_2 - \{a_1, a_2\}) = \emptyset$; by (d) and the maximality of $R_1, \ldots, R_m$, $L(A,C)$ has no path from $a_1Aa_2 - \{a_1, a_2\}$ to $(A - a_1Aa_2) \cup (C - \{y_1, z_i\})$; and by (5), $(z_1Xp_1 \cup z_2Xp_2) - \{a_1, a_2, z_i\}$ contains no neighbor of $(\bigcup_{j=1}^{m} R_j \cup a_1Aa_2) - \{a_1, a_2\}$. Hence, $\{a_1, a_2, x_1, x_2\}$ is a cut in G, a contradiction. Therefore, $L(A) = \emptyset$.

Now assume $L(C) \cap C \not\subseteq z_iCc$, and let R_1 be an $(A \cup C)$-bridge of H contained in $L(C)$ such that $R_1 \cap (cCy_1 - c) \neq \emptyset$. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(C)$ such that for $2 \le i \le m$, R_i has a vertex internal to $\bigcup_{j=1}^{i-1} l_jCr_j$ (which is a path), where $l_j, r_j \in V(R_j \cap C)$ with l_jCr_j maximal. Let $c_1, c_2 \in V(C)$ such that $\bigcup_{j=1}^{m} l_jCr_j = c_1Cc_2$. By the existence of P and (c), $c_1, c_2 \in V(cCy_1)$; by (c), $J(A,C) \cap (c_1Cc_2 - \{c_1, c_2\}) = \emptyset$; by (d), $L(A,C) \cap (c_1Cc_2 - \{c_1, c_2\}) = \emptyset$; and by (5) and the maximality of $R_1, \ldots, R_m$, $z_1Xp_1 \cup z_2Xp_2$ contains no neighbor of $(\bigcup_{j=1}^{m} R_j \cup c_1Cc_2) - \{c_1, c_2\}$. Hence, $\{c_1, c_2, x_1, x_2\}$ is a cut in G, a contradiction. Therefore, $L(C) \cap C \subseteq z_iCc$. This proves (7).

Let F be the union of all $(A \cup C)$-bridges of H different from $J(A,C)$ and intersecting $z_iCc - c$. When $F \neq \emptyset$, let $a^* \in V(F \cap A)$ with a^*Ay_1 minimal, and let r be the neighbor of $(F \cup z_iAa^* \cup z_iCc) - \{a^*, c\}$ on z_iXp_i with rXp_i minimal.

Case 1. $z_i \in V(J(A,C))$.

By (6.3), $J(A,C)$ contains a path S from z_i to some $s \in V(P - c) \cup V(b_1Bp)$ and internally disjoint from K.

Subcase 1.1. $F \neq \emptyset$.

Suppose $r \neq z_i$. Then by (5) and the definition of r, G' has a path R from r to $r' \in V(z_iCc) - \{z_i, c\}$ and internally disjoint from $K \cup X$, and by (6.3), R is disjoint from

$J(A, C)$. First, assume $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and we use the path T from (A2). Note that $S \cup P \cup b_1 Bp$ contains a path S' from z_i to b_1. Hence, $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Tb \cup bb_1 \cup S' \cup z_i x_i) \cup (z_{3-i}Bq \cup Q) \cup (y_1 Cr' \cup R \cup rXp_i \cup p_i Ty_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. So $P_1 \cup P_2 \cup B \cup S \cup (P - c) \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to z_i, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup z_i x_i) \cup S_2 \cup (y_1 Cr' \cup R \cup rXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

So $r = z_i$. By (c) and (d), G' has no path from $z_i Aa^* - \{a^*, z_i\}$ to $(cCy_1 - c) \cup (a^* Ay_1 - a^*)$ and internally disjoint from K. Hence, by (5), $\{a^*, c, x_1, x_2, z_i\}$ is a cut in G, and $i = 2$. Let $F^* := G[F \cup z_i Aa^* \cup z_i Cc + \{x_1, x_2\}]$

Suppose $F^* - x_1$ has disjoint paths S_1, S_2 from x_i, z_i to c, a^*, respectively. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Tb \cup bb_1 \cup b_1 Bp \cup P \cup S_1) \cup (z_{3-i}Bq \cup Q) \cup (y_1 Aa^* \cup S_2 \cup z_i Xp_i \cup p_i Ty_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. Now assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup S_1) \cup (y_1 Aa^* \cup S_2 \cup z_i Xp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Thus, we may assume that such S_1, S_2 do not exist. Then by Lemma 2.1, $(F^* - x_1, x_i, z_i, c, a^*)$ is planar. If $|V(F^*)| \geq 7$, then the assertion of Theorem 1.1 follows from Lemma 2.7. So assume $|V(F^*)| = 6$. Let $z \in V(F^* - x_1) - \{x_i, z_i, c, a^*\}$. Then $G[\{x_i, z_i, z, c\}] \cong K_4^-$, and (ii) of Theorem 1.1 holds (as $i = 2$ in this case).

Subcase 1.2. $F = \emptyset$.

Then $L(C) = \emptyset$ by (7). Also, $L(A) = \emptyset$ by (7); hence, by (4) and the comment preceding (5), $W = x_i w$ with $w \in V(A) - \{z_i, y_1\}$.

We may assume that $J(A, C) \cap (A - \{z_i, y_1\}) = \emptyset$. For, otherwise, let $t \in V(J(A, C)) \cap V(A - \{z_i, y_1\})$. By (6.1), $J(A, C)$ contains a path T from t to $t' \in V(Q - y_1)$ and internally disjoint from K, and T must be internally disjoint from S. Note that $(S \cup P \cup b_1 Bp) - c$ contains a path S' from z_i to b_1 and internally disjoint from $T \cup Q \cup z_{3-i}Bq$. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup z_i x_i \cup (z_i Xp_i \cup p_i Ty_2) \cup (z_{3-i}Tb \cup bb_1 \cup S') \cup (C \cup y_1 x_{3-i}) \cup (z_{3-i}Bq \cup qQt' \cup T \cup tAw \cup wx_i)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $P_1 \cup P_2 \cup B \cup S \cup (P - c) \cup (Q - y_1) \cup T$ contains independent paths S_1, S_2 from z_{3-i} to z_i, t, respectively. Now $G[\{x_1, x_2, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup z_i x_i \cup (z_i Xp_i \cup Q_i) \cup S_1 \cup (C \cup y_1 x_{3-i}) \cup (S_2 \cup tAw \cup wx_i)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

By (A5), $J := J(A, C) \cup C$ contains no disjoint paths from z_i, y_1 to z_{3-i}, b_1, respectively. Hence by Lemma 2.1, there exists a collection $\mathcal{L}$ of subsets of $V(J) - \{b_1, y_1, z_1, z_2\}$ such that $(J, \mathcal{L}, z_i, y_1, z_{3-i}, b_1)$ is 3-planar. We choose $\mathcal{L}$ so that each $L \in \mathcal{L}$ is minimal and, subject to this, $|\mathcal{L}|$ is minimal.

We claim that for each $L \in \mathcal{L}$, $L \cap V(L(A,C)) = \emptyset$. For suppose there exists $L \in \mathcal{L}$ such that $L \cap V(L(A,C)) \neq \emptyset$. Then $|N_J(L) \cap V(C)| \geq 2$. Assume for the moment that $N_J(L) \subseteq V(C)$. Then, since $L(C) = \emptyset$ and $J(A,C) \cap (A - \{z_i, y_1\}) = \emptyset$, $L \subseteq V(C)$. However, since C is an induced path in G, we see that $(J, \mathcal{L} - \{L\}, z_i, y_1, z_{3-i}, b_1)$ is 3-planar, contradicting the choice of $\mathcal{L}$. Thus, let $N_J(L) = \{t_1, t_2, t_3\}$ such that $t_1, t_2 \in V(C)$ and $t_3 \notin V(C)$. Then $J(A,C)$ contains a path R from t_3 to B and internally disjoint from $B \cup C$. Let $t \in L \cap V(L(A,C))$. By the minimality of L, $G[L + \{t_1, t_2, t_3\}]$ contains disjoint paths T_1, T_2 from t_1, t to t_2, t_3, respectively. We may choose T_1 to be induced, and let $C' := z_iCt_1 \cup T_1 \cup t_2Cy_1$. Then A, B, C' satisfy (a), but $J(A, C') \subseteq L(A, C')$ (because of T_2), contradicting (2) (as $J(A,C) \cap (A - \{z_i, y_1\}) = \emptyset$).

Because of the existence of Y, Z in (A3), there are disjoint paths R_1, R_2 in $L(A,C)$ from $r_1, r_2 \in V(A)$ to $r'_1, r'_2 \in V(C)$ such that z_i, r_1, r_2, y_1 occur on A in order and z_i, r'_2, r'_1, y_1 occur on C in order. Let $A' = z_iAr_1 \cup R_1 \cup r'_1Cy_1$ and $C' = z_iCr'_2 \cup R_2 \cup r_2Ay_1$. Let $t_1, t_2 \in V(C - \{z_i, y_1\}) \cap V(J(A,C))$ with t_1Ct_2 maximal, and assume that z_i, t_1, t_2, y_1 occur on C in this order. By the planarity of $(J, z_i, y_1, z_{3-i}, b_1)$ and by (6.3), $t_1 = c$.

Then either $t_1Ct_2 \subseteq z_iCr'_2$ for all choices of R_1 and R_2, or $t_1Ct_2 \subseteq r'_1Cy_1$ for all choices of R_1 and R_2; for otherwise, $J(A', C') \subseteq L(A', C')$, and A', B, C' contradict the choice of A, B, C in (b). Moreover, since $F = \emptyset$, $t_1Ct_2 \subseteq z_iCr'_2$ for all choices of R_1 and R_2. Choose R_1, R_2 so that z_iAr_1 and $z_iCr'_2$ are minimal. Since G is 5-connected, $\{r_1, r'_2, x_1, y_1\}$ cannot be a cut in G. So by (5), G' has a path R from x_2 to some $v \in V(r_1Ay_1 - \{r_1, y_1\}) \cup V(r'_2Cy_1 - \{r'_2, y_1\})$ and internally disjoint from K.

First, assume $i = 1$. If $v \in V(r_1Ay_1) - \{r_1, y_1\}$ then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup C \cup (z_iXp_i \cup Q_i) \cup (zAv \cup R) \cup (Q \cup qBz_{3-i} \cup z_{3-i}Xp_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i. If $v \in V(r'_2Cy_1) - \{r'_2, y_1\}$ then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup A \cup (z_iXp_i \cup Q_i) \cup (z_iCv \cup R) \cup (Q \cup qBz_{3-i} \cup z_{3-i}Xp_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

Hence, we may assume $i = 2$. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Bq \cup Q) \cup (z_{3-i}Tb_1 \cup b_1Bp \cup P \cup cCr'_2 \cup R_2 \cup r_2Av \cup R) \cup (y_1Cr'_1 \cup R_1 \cup r_1Az_i \cup z_iXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup cCr'_2 \cup R_2 \cup r_2Av \cup R) \cup (y_1Cr'_1 \cup R_1 \cup r_1Az_i \cup z_iXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Case 2. $z_i \notin V(J(A,C))$.

Then $F \neq \emptyset$ as the degree of z_i in G' is at least 5. So a^* and r are defined.

Subcase 2.1. $r \neq z_i$, and G' contains a path S from some $s \in V(z_iXr) - \{z_i, r\}$ to some $s' \in V(P \cup Q \cup B') - \{y_1, c\}$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q \cup X$.

First, assume $s' \in V(Q - y_1) \cup V(pBz_{3-i} - p)$. Then $S \cup (Q - y_1) \cup (pBz_{3-i} - p)$ has a path S' from s to z_{3-i}. By (5), let R be a path in G' from r to some $r' \in V(z_iCc) - \{z_i, c\}$ and internally disjoint from $A \cup C \cup J(A,C) \cup X$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXs \cup$

$S' \cup z_{3-i}x_{3-i}) \cup A \cup (z_iCr' \cup R \cup rXp_i \cup Q_i) \cup (y_1Cc \cup P \cup pBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

Hence, we may assume $s' \in V(P-c) \cup V(B'-(pBz_{3-i}-p)))$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$ if $s' \in P$ and $q^* = s'$ if $s' \in V(B' - (pBz_{3-i} - p))$. Since $F \neq \emptyset$ and $B_1 := H - \{z_1, z_2\}$ is 2-connected, $a^* \neq z_i$; so G' has a path R' from r to some $r' \in V(z_iAa^* - z_i)$ and internally disjoint from $A \cup cCy_1 \cup J(A,C) \cup X$.

Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and we use the path T from (A2). Note that $P_1 \cup P_2 \cup S \cup Q \cup B \cup z_{3-i}Tb \cup bb_1$ contains independent paths S_1, S_2 from z_{3-i} to s', y_1, respectively. So $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (S_1 \cup S \cup sXz_i \cup z_ix_i) \cup S_2 \cup (y_1Ar' \cup R' \cup rXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Now assume $e(z_{3-i}, B_1) \geq 2$. Note that $P_1 \cup P_2 \cup B \cup S \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to s, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup S_2 \cup (S_1 \cup sXz_i \cup z_ix_i) \cup (y_1Ar' \cup R' \cup rXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Subcase 2.2. $r = z_i$, or G' contains no path from $z_iXr - \{z_i, r\}$ to $(P \cup Q \cup B') - \{y_1, c\}$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q \cup X$.

By (c) and (d), G' has no path from $z_iAa^* - \{a*, z_i\}$ to $(aCy_1 - a) \cup (a^*Ay_1 - a^*)$ and internally disjoint from K. Then by (5), (6.2) and (6.3), $\{a^*, c, r, x_1, x_2\}$ is a cut in G. Hence, since G is 5-connected, $i = 2$ by (5). Therefore, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a^*, c, r, x_1, x_2\}$ and $G_2 = G[F \cup z_2Cc \cup z_2Aa^* \cup x_2Xr + x_1]$.

Suppose $G_2 - x_1$ contains disjoint paths S_1, S_2 from r, x_2 to a^*, c, respectively. If $e(z_1, B_1) = 1$ then $z_1 = p_1$ and, using the path T from (A2) with $i = 2$, we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Ty_2 \cup (z_1Bq \cup Q) \cup (z_1Tb \cup bb_1 \cup b_1Bp \cup P \cup S_2) \cup (y_1Aa^* \cup S_1 \cup rXp_2 \cup p_2Ty_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So assume $e(z_1, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup S_2) \cup (y_1Aa^* \cup S_1 \cup rXp_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Thus, we may assume that such S_1, S_2 do not exist in $G_2 - x_1$. Then by Lemma 2.1, $(G_2 - x_1, r, x_2, a^*, c)$ is planar. If $|V(G_2)| \geq 7$ then the assertion of Theorem 1.1 follows from Lemma 2.7. So assume $|V(G_2)| \leq 6$. If $r = z_2$ and there exists $z \in V(G_2) - \{a^*, c, x_1, x_2, z_2\}$ then $za^*, zc, zx_1, zx_2, zz_2 \in E(G)$ (as G is 5-connected); so $G[\{c, x_2, z, z_2\}]$ contains K_4^- and (*ii*) of Theorem 1.1 holds. Hence, we may assume that $r \neq z_2$ or $V(G_2) = \{a^*, c, x_1, x_2, z_2\}$. Then, $z_2x_1, z_2c \in E(G)$ and $L(C) = \emptyset$ (by (7)).

Recall that $y_1z_2 \notin E(G)$; so $G[\{x_1, x_2, y_1, z_2\}] \cong K_4^-$. We complete the proof of Theorem 1.1 by proving (*iv*) for this new K_4^-. Let $z_0', z_1' \in N(x_1) - \{x_2, y_1, z_2\}$ be distinct and let $G'' := G - \{x_1v : v \notin \{x_2, y_1, z_0', z_1', z_2\}\}$.

Suppose $z_1' \in V(J(A,C)) - V(A \cup C)$ or $z_1' \in V(Y_2)$ or $z_1' \in V(X)$. Then $(J(A,C) \cup Y_2 \cup X \cup x_2y_2 \cup bb_1) - (A \cup C)$ contains a path from z_1' to x_2. Hence, $G - x_1$ contains an induced path X' from z_1' to x_2 such that $A \cup C$ is a cycle in $(G - x_1) - X'$ and $\{y_1, z_2\} \subseteq V(A \cup C)$. So by Lemma 2.11, we may assume that X' is chosen so that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. Then by Lemma 2.5, G'' contains TK_5 (which uses $G[\{x_1, x_2, z_2, y_1\}]$ and x_1z_1').

So assume $z_1' \in V(L(A,C) - J(A,C)) \cup V(A \cup C)$ (as $L(A) = L(C) = \emptyset$). In fact, $z_1' \in V(C) - \{z_2, y_1\}$. For otherwise, $(W \cup L(A,C) \cup A) - C$ contains an induced path X' from z_1' to x_2, where W comes from (4) and the remark preceding (5). Then $(G-x_1)-X'$ contains $C \cup Q \cup qBb_1 \cup (X - \{x_1, x_2\}) \cup Y_2$, which has a cycle containing $\{y_1, z_2\}$. By Lemma 2.11, we may assume that X' is chosen so that $y_1, y_2 \notin V(X')$ and $(G-x_1)-X'$ is 2-connected. Now the assertion of Theorem 1.1 follows from Lemma 2.5.

If $z_1' \in V(J(A,C))$, then there is a path P' in $J(A,C)$ from z_1' to some $p' \in V(B)$ and internally disjoint from $A \cup B \cup C$. So $G[\{x_1, x_2, y_1, z_2\}] \cup z_1'x_1 \cup z_1'Cz_2 \cup z_1'Cy_1 \cup (P' \cup p'Bb_1 \cup b_1b \cup Q_3 \cup y_2x_2) \cup A$ is a TK_5 in G'' with branch vertices x_1, x_2, y_1, z_2, z_1'.

Thus, we may assume that $z_1' \notin V(J(A,C))$. So there is a path A' in $L(A,C)$ from z_1' to some $a' \in V(A)$ and internally disjoint from $J(A,C) \cup A \cup C$. Recall the path W from (4) and the remark preceding (5). Now $G[\{x_1, x_2, y_1, z_2\}] \cup z_1'x_1 \cup z_1'Cz_2 \cup z_1'Cy_1 \cup (A' \cup a'Aw \cup W) \cup (Q \cup qBb_1 \cup b_1b \cup Q_3 \cup Q_2 \cup p_2Xz_2)$ is a TK_5 in G'' with branch vertices x_1, x_2, y_1, z_2, z_1'. □

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