Quantum Fourier analysis

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Quantum Fourier analysis is a subject that combines an algebraic Fourier transform (pictorial in the case of subfactor theory) with analytic estimates. This provides interesting tools to investigate phenomena such as quantum symmetry. We establish bounds on the quantum Fourier transform $\mathfrak{F}$, as a map between suitably defined L^p spaces, leading to an uncertainty principle for relative entropy. We cite several applications of quantum Fourier analysis in subfactor theory, in category theory, and in quantum information. We suggest a topological inequality, and we outline several open problems.

picture language | uncertainty principles | quantum symmetries | inequalities | quantum entanglement

In this paper, we explore quantum Fourier analysis (QFA), a subject revolving around the study of Fourier analysis of quantum symmetries. The discovery of such symmetries emerged in the 1970s and has flourished ever since. It represents a major advance both in mathematics and in physics, as well as in the relation between these two subjects. Thus, QFA adds an extra dimension to the 200-year-old subject of classical Fourier analysis (CFA), analyzing the Fourier transform F.

CFA led to insights into and to solutions of problems in almost every field of mathematics, including partial differential equations, probability theory, number theory, representation theory, topology, geometry, etc. It ultimately led to the categorization of Fourier duality (1, 2). The Hausdorff–Young inequality is a bound on the norm $M_p = \|F\|_{L^p \to L^q}$, where $q = p/(p-1)$. Hirschman discovered that differentiating M_p gives an uncertainty principle for the Shannon entropy, generalizing the well-known Heisenberg principle. He and Everett conjectured the optimal inequality (3, 4). Deep and beautiful proofs were found (5–7).

Classical hypercontractivity states $\left\|e^{-tH}\right\|_{L^2 \to L^q} \leqslant 1$, where H is a simple harmonic oscillator Hamiltonian with unit angular frequency and $e^{2t} \geqslant q-1 \geqslant 1$ (8–10). The classical Hausdorff–Young inequality is a consequence of $F = e^{i\pi H/2}$ (5). Further inequalities can be found in many papers such as refs. 11–18, suggesting, in retrospect, a bridge from CFA to QFA.

1. QFA

A quantum Fourier transform $\mathfrak{F}$ defines Fourier duality between quantum symmetries, which could be analytic, algebraic, geometric, topological, and categorical. The quantum symmetries could be finite or infinite, discrete or continuous, commutative or noncommutative. In certain contexts $\mathfrak{F}$ can be defined pictorially—as in the picture language program (19). QFA is the study of structures involving $\mathfrak{F}$.

It is possible to estimate various norms $\|\mathfrak{F}\|_{L^p \to L^q}$ as transformations between noncommutative L^p spaces, and results in refs. 20 and 21 represent early breakthroughs in the application and formulation of QFA. As QFA is more sophisticated than CFA, these subjects have differences as well as similarities; we explore them both.

Let us consider an example of similarities and differences. In CFA, the extremizers of the Hausdorff–Young inequality (and many others) are Gaussians. In QFA, on subfactors, the Hausdorff–Young also holds. The extremizers are bishifts of biprojections. A biprojection is a projection whose quantum Fourier transform is a multiple of a projection (22); so the behavior under Fourier transformation of the extremizers in QFA on subfactors are similar to those in CFA, while their algebraic properties differ.

In this paper, we give a unified view of QFA. We establish a "relative" inequality between pictures that yields an uncertainty principle for relative entropy. We propose a universal quantum inequality, namely Eq. **9**, that unifies many other quantum inequalities. This is similar to the way the Brascamp–Lieb inequality unites Young's inequality, Hölder inequality, and others, in CFA. Throughout the paper, we cite applications of QFA. Finally, in section 9, we state some general goals for the future and some open questions.

QFA reveals insight and intrinsic structure, as well as relations between fusion rings, fusion categories, and subfactors. We show how the "Schur product property" provides a powerful obstruction to distinguish mathematical objects. QFA also provides an approach to quantum entanglement, uncertainty relations, and other problems in quantum information. We are certain that QFA will lead to other advances in many different fields.

2. QFA on Fusion Rings

Let us start with fusion rings, as introduced by Lusztig (23); this is an interesting quantum symmetry beyond groups. See ref. 24 for further results and references. A fusion ring $\mathfrak{A}$ is a ring that is free as a $\mathbb{Z}$-module, with a basis $\{x_1, x_2, \ldots, x_m\}$, $m \in \mathbb{N}$, with $x_1 = 1$, and such that

Significance

Classical Fourier analysis, discovered over 200 years ago, remains a cornerstone in understanding almost every field of pure mathematics. Its applications in physics range from classical electromagnetism to the formulation of quantum theory. It gives insights into chemistry, engineering, and information science, and it underlies the theory of communication. Quantum Fourier analysis extends this perspective. It yields insights and inequalities associated with uncertainty principles for quantum symmetries. In this paper, we introduce this mathematical subject, we show how it can solve some theoretical problems, and we give some applications to quantum physics with bounds on entropy and the analysis of quantum entanglement. We believe that quantum Fourier analysis, now in its infancy, will have significant future impact.

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[$\mathcal{B}1$] $x_j x_k = \sum_{s=1}^{m} N_{j,k}^{s} x_s$, with $N_{j,k}^{s} \in \mathbb{N}$, and

[$\mathcal{B}2$] there exists an involution $*$ on $\{1, 2, \ldots, m\}$ such that $N_{j,k}^{1} = \delta_{j,k^*}$, inducing an antiisomorphism of $\mathfrak{A}$, given by $x_k^* := x_{k^*}$ and $x_k^* x_j^* = (x_j x_k)^*$.

From a fusion ring, one can construct two C^*-algebras with faithful tracial states and a unitary Fourier transform between them. We recall this construction and the QFA on fusion rings as studied in ref. 25. Define a unital *-algebra over $\mathbb{C}$ with basis $\{x_j\}$, with multiplication given by property [$\mathcal{B}1$], and with adjoint given by * in [$\mathcal{B}2$]. Define a linear functional τ given by the Dirac measure on the basis, $\tau(x_j) = \delta_{j,1}$. From [$\mathcal{B}2$] we infer that τ is a strictly positive trace, and therefore it defines an inner product. This gives the C^*-algebra $\mathcal{B}$ by the Gelfand–Naimark–Segal (GNS) construction.

One can define a second (in this case abelian) C^*-algebra $\mathcal{A}$ from $\mathfrak{A}$ with multiplication $\diamond$, adjoint $\#$, given by another strictly positive linear functional d on $\mathcal{A}$. Let

[$\mathcal{A}1$] $x_j \diamond x_k = \delta_{j,k}\, d(x_j)^{-1} x_j$,

[$\mathcal{A}2$] $x_j^{\#} = x_j$.

Here, $d(\cdot)$ is defined to be linear, by sending a basis element x_j to $d(x_j)$, the operator norm of the fusion matrix M_j, with entries $(M_j)_{s,t} = N_{j,s}^{t}$. This is called the Perron–Frobenius dimension of x_j. The trace d (resp. τ) on the finite-dimensional C^*-algebra $\mathcal{A}$ (resp. $\mathcal{B}$) defines L^p norms on $\mathcal{A}$ (resp. L^q norms on $\mathcal{B}$) by $\|a\|_{\mathcal{A}_p} = d(|a|^p)^{1/p}$ and $\|b\|_{\mathcal{B}_q} = \tau(|b|^q)^{1/q}$. Then $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras with the same basis. We use the classical notation $\mathfrak{F}: x \mapsto \widehat{x}$ for the *Fourier transform* as the linear map from $\mathcal{A}$ to $\mathcal{B}$ defined by $\mathfrak{F}(x_j) = x_j$. The Fourier transform can be extended to a map from $L^p(\mathcal{A}, d)$ to $L^q(\mathcal{B}, \tau)$ for $1/p + 1/q = 1$. Plancherel's formula follows as: $\|\mathfrak{F}(x)\|_{\mathcal{B}_2} = \|x\|_{\mathcal{A}_2}$. Similarly, one has $\tau((\mathfrak{F}(x))^* \mathfrak{F}(y)) = d(x^{\#} \circ y)$.

We summarize several results in ref. 25 about QFA on fusion rings, including the quantum Schur product theorem (QSP), the quantum Hausdorff–Young inequality (QHY) with $1/p + 1/q = 1$, the quantum Young inequality (QY) with $1/p + 1/q = 1 + 1/r$, and the basic quantum uncertainty principles (QUP)—defined in terms of the von Neumann entropy, $H_{\mathcal{A}}(|x|^2) = -d((x^{\#} \diamond x) \diamond \log(x^{\#} \diamond x))$, and $H_{\mathcal{B}}(|\widehat{x}|^2) = -\tau((\widehat{x}^* \widehat{x}) \log(\widehat{x}^* \widehat{x}))$, and in terms of the support $\mathcal{S}_{\mathcal{A}}(x) = d(R(x))$, $\mathcal{S}_{\mathcal{B}}(\widehat{x}) = \tau(R(\widehat{x}))$, where $R(\cdot)$ is the range projection.

Theorem 2.1. *For nonzero* $x, y \in \mathcal{A}$,

[*QSP*]: $\mathfrak{F}^{-1}(\widehat{x}\widehat{y}) \geqslant_{\mathcal{A}} 0$, *whenever both* $x, y \geqslant_{\mathcal{A}} 0$.

[*QHY*]: $\|\widehat{x}\|_{\mathcal{B}_q} \leqslant \|x\|_{\mathcal{A}_p}$, *for* $1 \leqslant p \leqslant 2$.

[*QY*]: $\|\mathfrak{F}^{-1}(\widehat{x}\widehat{y})\|_{\mathcal{A}_r} \leqslant \|x\|_{\mathcal{A}_p} \|y\|_{\mathcal{A}_q}$, *for* $1 \leqslant p, q, r \leqslant \infty$.

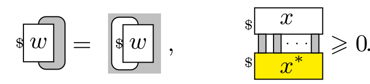
[*QUP*−1]: $H_{\mathcal{A}}(|x|^2) + H_{\mathcal{B}}(|\widehat{x}|^2) + 2\|x\|_{\mathcal{A}_2}^2 \log \|x\|_{\mathcal{A}_2}^2 \geqslant 0$.

[*QUP*-2] : $\mathcal{S}_{\mathcal{A}}(x) \mathcal{S}_{\mathcal{B}}(\widehat{x}) \geqslant 1$.

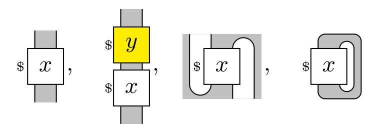
3. QFA on Subfactors

Modern subfactor theory was initiated by Jones in 1983 by his remarkable index theorem (26). A subfactor of type II_1 is an inclusion of II_1 factors $\mathcal{N} \subset \mathcal{M}$, and its index describes the relative size of these two factors. Jones' index theorem asserts that the index δ^2 of a subfactor belongs to the set $\{4\cos^2(\frac{\pi}{n}) : n = 3, 4, \cdots\} \cup [4, \infty]$, and every possible value can be realized as the index of a subfactor. Subfactor theory turns out to be a natural framework to study quantum symmetry in statistical physics and quantum field theory (e.g., ref. 27). Subfactor planar algebras (28) provide a pictorial tool to study subfactor theory. A planar algebra $\mathscr{P}_\bullet = \{\mathscr{P}_{n,\pm} : n \geqslant 0\}$ is a family of finite-dimensional vector spaces with an action of the operad of *planar tangles*, similar to *topological quantum field theory* (29). One represents an element in $\mathscr{P}_{n,\pm}$ (called an n-box) by a labeled rectangle with $2n$ strings attached to its boundary. Each vector space $\mathscr{P}_{n,\pm}$ is equipped with an involution *, which is compatible with the involution of planar tangles. This involution * is called the adjoint and given pictorially by vertical reflection.

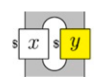
Any $w \in \mathscr{P}_{1,+}$ satisfies the spherical condition, and any $x \in \mathscr{P}_{n,+}$ satisfies the "reflection-positivity" condition,


[1]

The action of planar tangles turns $\mathscr{P}_{n,\pm}$ into C^*-algebras, and the trace gives a Hilbert-space representation by the GNS construction. We set $\mathcal{A} = \mathscr{P}_{2,+}$ and $\mathcal{B} = \mathscr{P}_{2,-}$. For elements in $\mathcal{A}$, one has pictorial representations for x, multiplication xy, the Fourier transform $\mathfrak{F}(x)$, and the trace $tr(x)$ as follows:



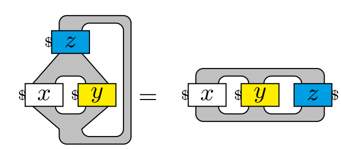
Define the convolution product $*$ on $\mathcal{A}$ by

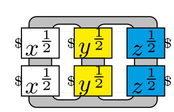

$$= x * y = \mathfrak{F}^{-1}(\mathfrak{F}(y)\mathfrak{F}(x))\,. \qquad [2]$$

The C^*-algebra $\mathcal{B}$ has a similar pictorial representation. These pictures not only make QFA transparent; they also provide a precise framework for proofs.

Theorem 3.1 (Schur product theorem; theorem 4.1 in ref. 20). *For any* $0 \leqslant x, y \in \mathcal{A}$ *(or* $0 \leqslant x, y \in \mathcal{B}$*), one has* $0 \leqslant x * y$.

Proof. Since tr is faithful, it suffices to show that $tr\,((x * y)z) \geqslant 0$ for any $0 \leqslant z \in \mathcal{A}$. Since $\mathcal{A}$ is a C^*-algebra, there exist elements $x^{\frac{1}{2}}, y^{\frac{1}{2}}, z^{\frac{1}{2}} \in \mathcal{A}$, such that

$tr((x * y)z) =$

[3]

$=$

$\geqslant 0.$ **[4]**

Here, we infer positivity from the reflection positivity in Eq. **1**. This proof works on $\mathcal{B}$ by switching the shading, since the dual of a subfactor is also a subfactor.

The Schur product property holds on both $\mathcal{A}$ and $\mathcal{B}$ for subfactors. On fusion rings, it holds on $\mathcal{A}$ but not necessarily on $\mathcal{B}$. We discuss this essential difference revealed by QFA in more detail in section 4.

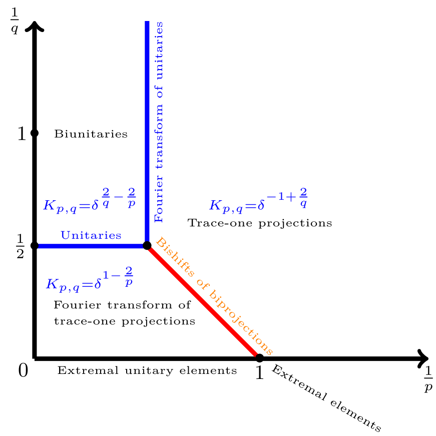
Applications. An important application of the Schur product theorem comes in the classification of *abelian* subfactor planar algebras (20). One can regard this as the fundamental theorem for abelian subfactors, extending the fundamental theorem for finite abelian groups. This was the first classification that requires a bound neither on the Jones index nor on the dimension.

Theorem 3.2 (theorem 6.7 in ref. 20). *An irreducible subfactor planar algebra is abelian if and only if it is a free product of the simplest (Temperley–Lieb–Jones) planar algebras and finite abelian groups.*

Also one obtains a geometric proof of the (mathematical-physics version of) reflection-positivity condition.

Theorem 3.3. *Reflection positivity holds for Hamiltonians on planar para algebras (theorem 7.1 in ref. 30) and on Levin–Wen models (theorem 3.2 in ref. 31).*

Quantum Hausdorff–Young Inequalities. Many estimates for the norm $K_{p,q} = \|\mathfrak{F}\|_{L^p \to L^q}$ have been established in the quantum case, including when p, q are not dual (21, 32). Instead of synthesizing these estimates into one theorem, we give a picture (5) illustrating the known bounds for $K_{p,q}$, with the extremizers for various regions and their boundaries labeled by text.



[5]

The constant δ is the square root of the Jones index. The extremizers of these inequalities have nine different characterizations. In particular, the red line $1/p+1/q=1$, for $1/2 \leqslant 1/p \leqslant 1$ corresponds to the quantum Hausdorff–Young inequality. Moreover, all of the other quantum inequalities, such as quantum Young's inequality, in Theorem 2.1 have been proved for subfactors planar algebras in ref. 21.

4. QFA on Unitary Fusion Categories

The QFA on subfactors also applies for unitary fusion categories through the quantum double construction, see e.g., ref. 33. Let $\mathcal{C}$ be a unitary fusion category and $I=\{X_1, X_2, \ldots, X_m\}$ be the set of simple objects. There is a Frobenius algebra γ in $\mathcal{C}\boxtimes\overline{\mathcal{C}}$ whose object is $\bigoplus_{i=1}^{m} X_i \boxtimes \overline{X_i}$. Following the quantum double construction, we obtain an irreducible subfactor planar algebra, such that $\mathcal{A}=\mathscr{P}_{2,+}=\hom_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(\gamma)$ and $\mathcal{B}=\mathscr{P}_{2,-}=\hom_{\gamma-\gamma}(\gamma\otimes\gamma)$. Applying QFA to $\mathcal{A}$ on this subfactor, we obtain inequalities on the Grothendieck ring of unitary fusion categories as stated in Theorem 2.1. Applying QFA to $\mathcal{B}$, we obtain inequalities on the dual of the Grothendieck ring, which turn out to be highly nontrivial.

Application: Analytic Obstructions. It is important to determine whether a fusion ring can be the Grothendieck ring of a unitary fusion category. QFA provides powerful analytic obstructions to the unitary categorification of fusion rings. The quantum inequalities in Theorem 2.1 holds on the dual of Grothendieck rings. However, they may not necessarily hold on the dual of fusion rings, thereby providing analytic obstructions of the unitary categorification of fusion rings. Even the Schur product property on the dual of a fusion ring, $0 \leqslant x * y$ if $0 \leqslant x, y \in \mathcal{B}$, gives by itself a surprisingly efficient analytic obstruction:

Theorem 4.1 (25). *If a fusion ring can be unitarily categorified, then the Schur product property holds on its dual.*

There are 34 examples in the classification of simple integral fusion rings up to rank 8 and Frobenius–Perron dimension less than 3,780. Four of them are group-like. Methods based on previously known analytic, algebraic, and number theoretic obstructions did not determine whether the remaining 30 could be unitarily categorified. As a consequence of the Schur-product obstruction, 28 out of the 30 have no unitary categorification, as shown in ref. 25.

Example: Let us recall one example from ref. 25 to illustrate this obstruction. Let $\mathfrak{A}$ be the rank-7 simple integral fusion ring with the following seven fusion matrices, equal to

$$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&1&0&1&1\\ 0&0&1&0&1&1&1\\ 0&1&0&0&1&1&1\\ 0&0&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&1&0&1&1&1\\ 1&1&1&0&0&1&1\\ 0&0&0&1&1&1&1\\ 0&1&0&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0&0\\ 0&1&0&0&1&1&1\\ 0&0&0&1&1&1&1\\ 1&0&1&1&0&1&1\\ 0&1&1&0&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&1&1&1&1&1\\ 0&1&0&1&1&1&1\\ 0&1&1&0&1&1&1\\ 1&1&1&1&1&1&1\\ 0&1&1&1&1&2&1\\ 0&1&1&1&1&1&2 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&2&1\\ 1&1&1&1&2&0&3\\ 0&1&1&1&1&3&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&2\\ 0&1&1&1&1&3&1\\ 1&1&1&1&2&1&2 \end{pmatrix}.$$

The eigenvalue table of these matrices (where $\zeta^7=1$) is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 5 & -1 & -\zeta-\zeta^6 & -\zeta^5-\zeta^2 & -\zeta^4-\zeta^3 & 0 & 0\\ 5 & -1 & -\zeta^5-\zeta^2 & -\zeta^4-\zeta^3 & -\zeta-\zeta^6 & 0 & 0\\ 5 & -1 & -\zeta^4-\zeta^3 & -\zeta-\zeta^6 & -\zeta^5-\zeta^2 & 0 & 0\\ 6 & 0 & -1 & -1 & -1 & 1 & 1\\ 7 & 1 & 0 & 0 & 0 & 0 & -3\\ 7 & 1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \qquad [6]$$

The first column is the Perron–Frobenius dimension of the seven simple objects. Take $X=x_1+x_5-3x_6+2x_7$, then $X=X^*=X^2/15$.

The Schur product property on $\mathcal{B}$, equivalent to a dual version of Eq. **4**, yields (with $x=y=z=X$) that

$$d((\widehat{XX^*})\diamond(\widehat{XX^*})\diamond(\widehat{XX^*}))\geqslant 0. \qquad [7]$$

However, it follows directly that Eq. **7** is false in this case, as

$$\frac{1^3}{1}+\frac{0^3}{5}+\frac{0^3}{5}+\frac{0^3}{5}+\frac{1^3}{6}+\frac{(-3)^3}{7}+\frac{2^3}{7}=-\frac{65}{42}<0.$$

Therefore, the fusion ring $\mathfrak{A}$ cannot be unitarily categorified.

5. QFA on Locally Compact Quantum Groups

The previous results focus on finite quantum symmetry, such as fusion rings and finite-index subfactors. One might ask whether QFA can be established for infinite quantum symmetry. The answer is "yes"; there are results on infinite-dimensional Kac algebras and locally compact quantum groups. This relies on the theory of noncommutative L^p spaces (e.g., ref. 34).

We recall the definition of the Fourier transform on locally compact quantum groups, of which the Fourier transform on Kac algebra is a special case (35). Let $\mathbb{G}$ be a locally compact quantum group and φ the left Haar weight. Suppose W is the multiplicative unitary, ϕ is a normal semifinite faithful weight on the commutant of $L^\infty(\mathbb{G})$, and $\hat{d}$ is a normal semifinite faithful weight on the commutant of $L^\infty(\widehat{\mathbb{G}})$. Let $d=\frac{d\varphi}{d\phi}$, $\hat{d}=\frac{d\hat{\varphi}}{d\hat{\phi}}$ be the Connes' spatial derivatives, and let $L^p(\phi)$, $L^p(\hat{\phi})$ be Hilsum's space for any $1\leqslant p\leqslant\infty$. The Fourier transform $\mathcal{F}_p: L^p(\phi)\to L^q(\hat{\phi})$, for $1\leqslant p\leqslant 2$, and $1/p+1/q=1$, is defined by

$$\mathcal{F}_p(xd^{1/p})=(\varphi\otimes\iota)(W(x\otimes 1))\hat{d}^{1/q}, \forall x\in\mathcal{T}_\varphi^2.$$

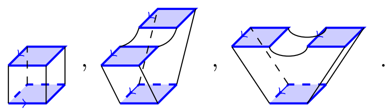
Here, $\mathcal{T}_\varphi\subset\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$ is the space of elements analytic with respect to φ. Even the definition of the convolution on locally compact quantum groups is nontrivial.

The quantum inequalities in Theorem 2.1 on these infinite quantum symmetries have been partially studied in refs. 36–40. The quantum uncertainty principle QUP-2 in Theorem 2.1 becomes a continuous family of inequalities on locally compact quantum groups (40).

6. Surface Algebras and A Universal Inequality

Surface Algebras. Many inequalities in CFA have not been axiomatized in a pictorial framework. Z.L. introduced surface algebras in ref. 41, formalizing the extension of planar algebras from two-dimensional (2D) to three-dimensional (3D) space, outlined in ref. 42. Surface algebras are an extensive framework to capture additional pictorial features of Fourier analysis.

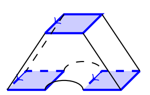
For any subfactor planar algebra, the actions of planar tangles can be further extended to the actions of surface tangles. (The arrow in planar diagrams corresponds to the \$ sign in planar algebras. The clockwise/anticlockwise orientation of the arrow indicates the input/output disc in surface algebras.) One can represent Fourier transform, multiplication, and convolution as the action of the following surface tangles in the 3D space:



[8]

Using 3D pictures, one can consider the Fourier duality for surface tangles with multiple inputs and outputs. The 3D formalism has provided deep insights into quantum information, algebraic identities, and various other connections with physics.

One can consider a finite-dimensional Kac algebra K as $\mathcal{A}$ and its dual $\widehat{K}$ as $\mathcal{B}$, with a Fourier transform from K to $\widehat{K}$ defined analogously to section 5. The pair of Kac algebras K and $\widehat{K}$ can be understood as $\mathcal{A}$ and $\mathcal{B}$ for the surface algebra. The comultiplication is given by the picture:



The Hopf-axiom that the comultiplication is an algebraic homomorphism reduces to the *string–genus relation* of surface tangles given in equation 17 of ref. 42.

A Universal Inequality. In a subfactor planar/surface algebra $\mathscr{P}$, the Fourier transform, the multiplication, and the convolution can be realized by planar/surface tangles. In general, a surface tangle is a multilinear map on $\bigoplus_{n\in\mathbb{N}}\mathscr{P}_{n,\pm}$. Now, we give a pictorial inequality in the quantum case, motivated by the classical Brascamp–Lieb inequality. We replace the dual of the linear map $B_j:\mathbb{R}^n\to\mathbb{R}^{k_j}$ by a surface tangle T_j with k_j input discs and n output discs; moreover, the n-output discs are identical for different j:

$$\left\|\prod_{j=1}^{m} T_j(x_j)\right\|_1 \leqslant C\prod_{j=1}^{m}\|x_j\|_{p_j}, \qquad [9]$$

and C is the best constant.

This topological inequality includes the quantum Hausdorff–Young inequality, quantum Hölder inequality, and quantum version of Young's inequality. The best constants of these three inequalities are achieved at biprojections.

For those familiar with the Quon language (42), we can consider the pictorial inequalities whose T_js are surface tangles with braided charged strings. In particular, if all of the inputs and outputs are 2-boxes, corresponding to qudits, then T_j can be any Clifford transformation on qudits. These Clifford transformations can be considered as a quantum analog of the dual of a linear transformation $B_j:(\mathbb{Z}_d)^n\to(\mathbb{Z}_d)^{k_j}$. Considering the action on density matrices, the n-qudit Clifford gates on Pauli matrices are symplectic transformations on $2n$-dimensional symplectic spaces over $\mathbb{Z}_d$.

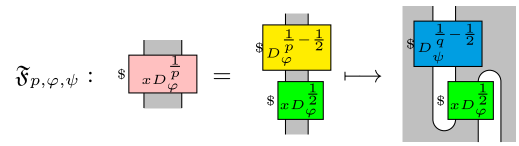
7. Relative Inequalities, Entropy, and Uncertainty

Here, we present a relative, quantum, Hausdorff–Young inequality. This leads to a relative, quantum, entropic uncertainty principle.

Let $\mathscr{P}_\bullet$ be an irreducible subfactor planar algebra with the Markov trace tr. Let φ (resp. ψ) be a faithful state on $\mathscr{P}_{2,+}$ (resp. $\mathscr{P}_{2,-}$). Let D_φ (resp. D_ψ) be the density operator of φ (resp. ψ), namely, $\varphi(\cdot)=tr(D_\varphi\,\cdot)$. Now, we define a Fourier transform $\mathfrak{F}_{p,\varphi,\psi}:L^p(\mathscr{P}_{2,+},tr)\to L^q(\mathscr{P}_{2,-},tr)$ for $1\leqslant p\leqslant 2$, $q=p/(p-1)$ as

$$\mathfrak{F}_{p,\varphi,\psi}(xD_\varphi^{1/p})=\mathfrak{F}(xD_\varphi^{1/2})D_\psi^{1/q-1/2}. \qquad [10]$$

This Fourier transform is represented pictorially as follows,



[11]

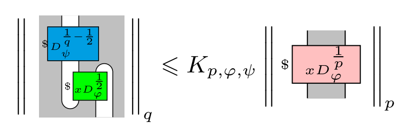
From Plancherel's theorem for $\mathfrak{F}$, we infer Plancherel's theorem for $\mathfrak{F}_{p,\varphi,\psi}$,

$$\|\mathfrak{F}_{2,\varphi,\psi}(xD_\varphi^{1/2})\|_2=\|\mathfrak{F}(xD_\varphi^{1/2})\|_2=\|xD_\varphi^{1/2}\|_2. \qquad [12]$$

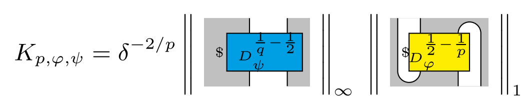
Theorem 7.1 (Relative, quantum Hausdorff–Young inequality). *Let $\mathscr{P}_\bullet$ be an irreducible subfactor planar algebra and let φ,ψ be faithful states on $\mathscr{P}_{2,\pm}$. Then for any $x\in\mathscr{P}_{2,+}$, $1\leqslant p\leqslant 2$, and dual $2\leqslant q=p/(p-1)$, we have*

$$\|\mathfrak{F}_{p,\varphi,\psi}(xD_\varphi^{1/p})\|_q\leqslant K_{p,\varphi,\psi}\|xD_\varphi^{1/p}\|_p. \qquad [13]$$

Here, $K_{p,\varphi,\psi}=\delta^{-2/p}\|D_\psi^{1/q-1/2}\|_\infty\|\mathfrak{F}(D_\varphi^{1/2-1/p})\|_1$. Pictorially,

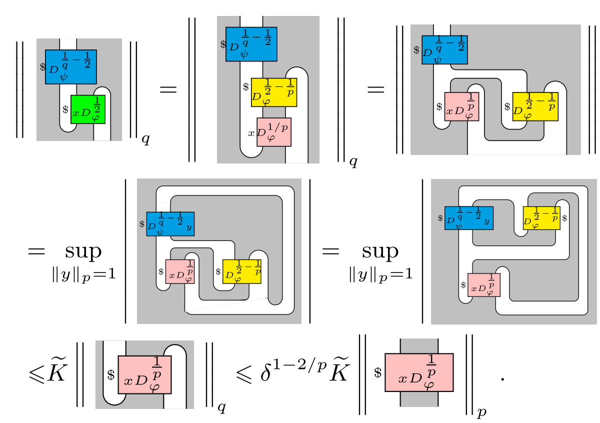


where

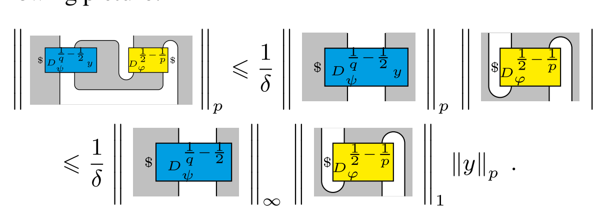


[14]

Proof. We give the elementary and insightful picture proof:



The first inequality is a consequence of the quantum Hölder inequality. The second inequality is a consequence of the quantum Hausdorff–Young inequality (theorem 4.8 in ref. 21). Here, the constant is $\widetilde{K}=\sup_{\|y\|_p=1}\widetilde{K}(y)$, with $\widetilde{K}(y)$ equal to the following picture:



To obtain the first inequality, we use the quantum Young inequality (theorem 4.13 in ref. 21). To obtain the second inequality, we use the quantum Hölder inequality.

Relative Entropy. We formulate relative entropy (RE) and the corresponding relative entropic quantum (REQ) uncertainty principle. For two positive functionals ω, φ on $\mathscr{P}_{2,+}$, recall that the relative entropy (43) is

$$S(\omega\|\varphi) = tr(D_\omega(\log D_\omega - \log D_\varphi)).$$

The Relative Entropic Quantum Uncertainty Principle. For a positive functional ω on $\mathscr{P}_{2,+}$, define $\widehat{\omega}$ as the positive functional on $\mathscr{P}_{2,-}$ given by the density matrix

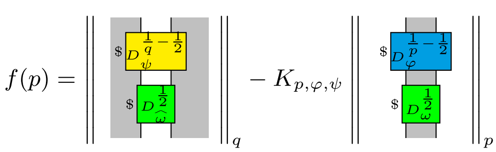
$$D_{\widehat{\omega}} = |\mathfrak{F}(D_\omega^{1/2})|^2. \quad [15]$$

It follows that $\widehat{\omega}(1) = \omega(1)$. If ω is a state, so is $\widehat{\omega}$.

Theorem 7.2 (REQ Uncertainty Principle). *Let $\mathscr{P}_\bullet$ be an irreducible subfactor planar algebra and φ, ψ be faithful positive functionals on $\mathscr{P}_{2,\pm}$. Then, for any state ω on $\mathscr{P}_{2,+}$,*

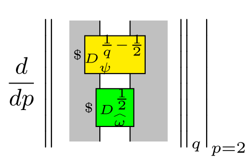
$$S(\omega\|\varphi) + S(\widehat{\omega}\|\psi) \leqslant \log\|D_\psi^{-1}\|_\infty - \frac{1}{\delta^2} tr(\log D_\varphi) - 2\log\delta.$$

Proof. Note that $\mathfrak{F}_{p,\varphi,\psi}(D_\omega^{1/2} D_\varphi^{1/p-1/2}) = \mathfrak{F}(D_\omega^{1/2}) D_\psi^{1/q-1/2}$. As $\|AB\|_p = \||A|\,B\|_p$, using Eq. **15**, we infer that $\mathfrak{F}_{p,\varphi,\psi}(D_\omega^{1/2} D_\varphi^{1/p-1/2})$ and $D_{\widehat{\omega}}^{1/2} D_\psi^{1/q-1/2}$ have the same q norms. Define the function $f(p)$ as a picture, where $q = p/(p-1)$, and where $K_{p,\varphi,\psi}$ is defined in Eq. **14**,

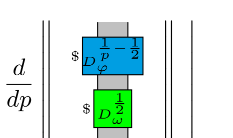


The picture $f(p)$ is negative for $1 \leqslant p \leqslant 2$, by Theorem 7.1. Also $f(2) = 0$ by Plancherel's theorem, so the left derivative $f'_-(2) \geqslant 0$. Then, Theorem 7.2 is a consequence of the expressions for the derivatives in the following lemma.

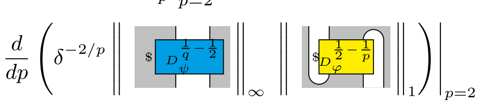
Lemma 7.3. *For any positive functional ω on $\mathscr{P}_{2,+}$, we have*



$$= \begin{array}{l} -\dfrac{1}{4} tr(D_{\widehat{\omega}})^{1/2} \log tr(D_{\widehat{\omega}}) \\ -\dfrac{1}{4\, tr(D_{\widehat{\omega}})^{1/2}} S(\widehat{\omega}\|\psi) \end{array},$$



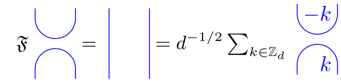
$$= \begin{array}{l} -\dfrac{1}{4} tr(D_\omega)^{1/2} \log tr(D_\omega) \\ +\dfrac{1}{4\, tr(D_\omega)^{1/2}} S(\omega\|\varphi) \end{array},$$



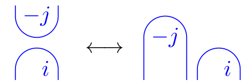
$$= \frac{1}{2}\log\delta - \frac{1}{4}\log\|D_\psi^{-1}\|_\infty + \frac{1}{4\delta^2} tr(\log D_\varphi).$$

8. QFA and Quantum Entanglement

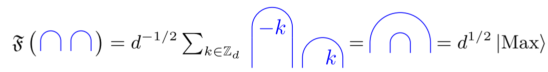
Here, we use pictures as in refs. 30 and 44, which do not require shading. The Fourier transform of a multiple of the projection onto the zero-vector for the group $\mathbb{Z}_d$, namely $d^{1/2}|0\rangle\langle 0|$, is the identity,

 [16]

One can identify a linear transformation T on $\mathbb{C}^d$ as a vector $\widehat{T}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$, namely



Identifying Eq. **16** in this way gives the illustration of how the Fourier transform $\mathfrak{F}$ acts on product states. In particular,



If $d = 2$, then $k = -k \in \mathbb{Z}_d$, and $|\mathrm{Max}\rangle$ is the usual Bell state.

A similar picture and vector $|\mathrm{Max}\rangle_n = \mathfrak{F}|\vec{0}\rangle_n$ exists for n-qudits (44). This vector $|\mathrm{Max}\rangle_n$ generalizes the classical Bell state for $d = 2$ for qubits to a maximally entangled state for n-qudits of order d. Furthermore, one can apply $\mathfrak{F}$ to any product basis state; one thereby obtains a Max basis (of maximally entangled states). These are related to a corresponding n-qudit Greenberger–Horne–Zeilinger (GHZ) basis (45). In section 4 in ref. 19, one finds expressions for the various $|\mathrm{Max}\rangle_n$ basis states, as well as their relation to, and expressions for, the $|\mathrm{GHZ}\rangle_n$ basis states. Also see ref. 42 for a Quon interpretation.

The quantum uncertainty principle QUP-1 in Theorem 2.1 for subfactors gives a lower bound for the entanglement entropy. The product ground state has minimal entanglement entropy. Each Max state has maximal entanglement entropy. In addition, we obtain a relative entropic uncertainty principle for quantum entanglement by applying Theorem 7.2.

9. Some Future Directions and Goals

We propose a few specific questions but first cite some general directions that appear ripe for the development of QFA:

- Establish additive combinatorics for quantum symmetries, such as for unitary modular tensor categories.
- Establish a general theory for $\mathfrak{F}(\mathscr{P}_{n,\pm})$, where $\mathfrak{F}^{2n} = 1$.
- Understand Fourier analysis for infinite quantum symmetries within a pictorial framework, such as surface algebras.
- Seek further applications of QFA to quantum information.

Questions on the Universal Inequality. There are three central problems for the classical Brascamp–Lieb inequality on $\mathbb{R}$:

1) Can one find for which tuples of linear maps the best constant is finite?
2) Can the best constant be achieved? If so, it is proved in ref. 46 that there exist Gaussian extremizers.
3) Are all extremizers Gaussian?

Since all $\mathscr{P}_{n,\pm}$, $n \in \mathbb{N}$, are finite-dimensional, the best constant C of the universal inequality is finite and the extremizer exists by the compactness. We ask the following questions for the universal inequality:

Question 9.1. *In which case is the best constant achieved by tensor product of n-projections (the natural generalization of biprojections)?*

Question 9.2. *If all of the inputs belong to $\mathscr{P}_{2,\pm}$, are all of the extremizers bishifts of biprojections?*

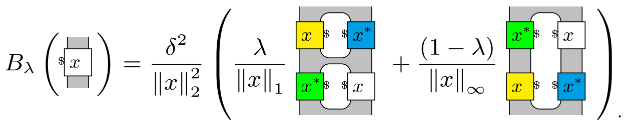
Question 9.3 (Finite abelian groups). *What are the best constants of the universal inequality on finite abelian groups?*

Questions on Subfactor Planar Algebras. Suppose $\mathscr{P}_{\bullet,\pm}$ is an irreducible subfactor planar algebra.

Conjecture 9.4. *For any $\varepsilon > 0$, there exists ε' such that if $x \in \mathscr{P}_{2,\pm}$, $\|x - P\|_2 < \varepsilon'$, and $\|\mathfrak{F}(x) - \lambda Q\|_2 < \varepsilon'$, for some* *projections P, Q and constant λ, then there is a biprojection B, such that $\|x - B\| < \varepsilon$.*

Question 9.5. *Can one characterize the extremizers for the uncertainty principles on n-boxes, for $n \geqslant 3$?*

Block Renormalization Map and Quantum Central Limit Theorem. The block map B_λ is a composition of convolution and multiplication,



The limit points of the iteration of the block map are all biprojections for finite-index, irreducible subfactors (47). We regard this result as a quantum 2D central limit theorem.

Conjecture 9.6. *For any $f \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, f converges either to 0 or to a Gaussian function, under the action of the iteration of the block map $2^n B_\lambda$.*

Conjecture 9.7. *For any $f \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the Hirschman–Beckner entropy decreases under the action of the block map $2^n B_\lambda$. The same question remains for finite cyclic groups.*

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