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A proof of Furstenberg's conjecture on the intersections of $\times p$- and $\times q$-invariant sets

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Abstract

We prove the following conjecture of Furstenberg (1969): if $A, B \subset [0,1]$ are closed and invariant under $\times p \mod 1$ and $\times q \mod 1$, respectively, and if $\log p/\log q \notin \mathbb{Q}$, then for all real numbers u and v,

$$\dim_{\mathrm{H}}(uA+v)\cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}.$$

We obtain this result as a consequence of our study on the intersections of incommensurable self-similar sets on $\mathbb{R}$. Our methods also allow us to give upper bounds for dimensions of arbitrary slices of planar self-similar sets satisfying SSC and certain natural irreducible conditions.

1. Introduction

1.1. *Background and history.* This paper is concerned with Furstenberg's problem [15] about the intersections of Cantor sets. The Cantor sets under consideration are dynamically defined, that is, they are either invariant sets or attractors of certain dynamical systems. Let (X, f) be a dynamical system where $f : X \to X$ is a measurable map on a compact metric space X. Many important dynamical properties of f are displayed by its invariant sets. Supposing that we are given two dynamical systems (X, f) and (X, g), it is reasonable to expect that information about common dynamical features of f and g can be obtained by comparing their respectively invariant sets. We are particularly interested in systems (X, f) and (X, g) that are arisen from two arithmetically or geometrically "independent" origins. In this case, one expects that the two systems should share as few common structures as possible and thus an f-invariant set should intersect a g-invariant set in as small a set as possible.

Keywords: $\times p$-invariant sets, intersections of Cantor sets, rigidity phenomena

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Furstenberg has given in [15] some quantitative formulations of the above philosophy. Let dim denote a dimension function for subsets of X (e.g. Hausdorff dimension). Following Furstenberg, we say that f and g are *transverse* if

$$\dim A \cap B \leq \max\{0, \dim A + \dim B - \dim X\}$$

for all closed sets A and B that are f- and g-invariant, respectively. The present work was motivated by a conjecture of Furstenberg concerning the transversality of two arithmetically "independent" systems.

Two positive real numbers a and b are said to be *multiplicatively independent*, denoted by $a \not\sim b$, if $\log a / \log b \notin \mathbb{Q}$. For a natural number $m \geq 2$, let $T_m : x \mapsto mx \mod 1$ be the m-fold map of the unit interval. We use $\dim_{\mathrm{H}} A$ to denote the Hausdorff dimension of a set A. Furstenberg conjectured that two dynamics T_p and T_q with $p \not\sim q$ are transverse. More precisely,

CONJECTURE 1.1 (Furstenberg, [15]). *Assume that $p \not\sim q$. Let $A_p, B_q \subset [0,1]$ be closed sets that are invariant under T_p and T_q, respectively. Then for all real numbers u and v,*

$$\dim_{\mathrm{H}}(uA_p + v) \cap B_q \leq \max\{0, \dim_{\mathrm{H}} A_p + \dim_{\mathrm{H}} B_q - 1\}.$$

In this paper, we prove Conjecture 1.1. We point out that Conjecture 1.1 is closely related to another conjecture of Furstenberg about expansions of real numbers in different bases, which is stronger and remains open. For $x \in [0,1]$, we denote the orbit of x under the map T_m by $\mathcal{O}_m(x) = \{T_m^k(x) : k \in \mathbb{N}\}$.

CONJECTURE 1.2 (Furstenberg, [15]). *If $p \not\sim q$, then for each $x \in [0,1] \backslash \mathbb{Q}$, we have*

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1. \tag{1.1}$$

Suppose that $p \not\sim q$, A_p is a closed T_p-invariant set and B_q is a closed T_q-invariant set, and $\dim_{\mathrm{H}} A_p + \dim_{\mathrm{H}} B_q < 1$. Then Conjecture 1.1 implies that $\dim_{\mathrm{H}} A_p \cap B_q = 0$, while Conjecture 1.2 predicts that $A_p \cap B_q \subset \mathbb{Q}$. In this respect, Conjecture 1.2 is much stronger than Conjecture 1.1. It seems that Conjecture 1.2 is still far out of reach of current methods. Nevertheless, as observed already by Furstenberg, using Conjecture 1.1 one can obtain some partial results towards Conjecture 1.2: the set of $x \in [0,1]$ that do not satisfy (1.1) has Hausdorff dimension zero. See Theorem 9.4 for a detailed proof.

The aforementioned conjectures belong to the broad category of rigidity problems about $\times p$ and $\times q$ dynamics, where there is a rich literature; see, e.g., the survey paper of Lindenstrauss [25] and the references therein. The study of rigidity properties between $\times p$ and $\times q$ dynamics (when $p \not\sim q$) was initiated by Furstenberg in his landmark paper [14]. In that paper, Furstenberg established the celebrated Diophantine result: if $p \not\sim q$, then the unit interval itself is the only (infinite) closed set that is both T_p and T_q invariant. He has famously conjectured that the measure version of this should be also true: any Borel probability measure on the unit interval invariant under T_p and T_q is a linear combination of Lebesgue measure and an atomic measure supported on finitely many rational points. The best partial result towards this conjecture is due to Rudolph and Johnson [30], [23] who proved the conjecture under the assumption of positive entropy. The research along this line has been fruitful and influential, and it has led to deep advances in Diophantine approximation and homogeneous dynamics (see [25]).

In another direction, Conjecture 1.1 can also be regarded as a problem about slices of fractal sets. Note that the set $(uA_p+v)\cap B_q$ is, up to an affine coordinate change, the intersection of the product set $A_p \times B_q$ with the line $\ell_{u,v} = \{(x,y) : y = ux+v\}$. By a classical result of Marstrand [26], for any Borel set $E \subset \mathbb{R}^2$ and each $u \in \mathbb{R}$, Lebesgue almost every $v \in \mathbb{R}$ satisfies

$$\dim_{\mathrm{H}} E \cap \ell_{u,v} \le \max\{0, \dim_{\mathrm{H}} E - 1\}.$$

In general, this is only an almost every result, and there could be exceptional pairs (u,v) for which the above inequality fails. In most cases, the set of exceptional (u,v) is quite difficult to analyze.

While explicitly determining the exceptional set is in general intractable, for certain fractal sets with regular arithmetical or geometrical structures, it is widely believed that the exceptional set should be very small and could only be caused by some evident algebraic or combinatorial reasons. For A_p, B_q as in Conjecture 1.1, the set $A_p \times B_q$ is such an example, for which it is clear that certain lines parallel to the axes are exceptional for the slice result, and Conjecture 1.1 predicts that these lines are the only exceptions.

There is a rich literature about generic slices of various fractal sets; see, e.g., [26], [18], [24], [27], [6], [7], [2], [34]. However, very little is known about specific slices, and there were few partial results concerning Conjecture 1.1 before the present paper. The first and perhaps also the best one is due to Furstenberg [15, Th. 4]. His result states that under the assumption of the conjecture, if $\overline{\dim}_{\mathrm{B}}(u_0A+v_0)\cap B = \gamma > 0$ for some $u_0 \neq 0, v_0 \in \mathbb{R}$, then for Lebesgue almost every $u \in \mathbb{R}$, there is v such that $\dim_{\mathrm{H}}(uA_p+v)\cap B_q \ge \gamma$. From the last assertion, it is not hard to deduce that in this case, we must have $\dim_{\mathrm{H}} A_p + \dim_{\mathrm{H}} B_q > 1/2$; see [20, Th. 7.9] for the deduction. Thus, under the assumption $\dim_{\mathrm{H}} A_p + \dim_{\mathrm{H}} B_q \le 1/2$, Furstenberg's result confirms Conjecture 1.1. We will return back to [15, Th. 4] in Section 4.2. We would like to mention that the technique (namely, CP-process) Furstenberg introduced and used in [15] is also important for the present work. It will be one of the main ingredients for our proof of Conjecture 1.1.

Recently, Feng, Huang and Rao [12] studied affine embeddings between incommensurable self-similar sets and, as a consequence, they showed that if $p \not\sim q$, then for T_p-invariant self-similar set E and T_q-invariant self-similar set F, there exists a (non-effective) positive constant δ depending on E and F such that the Hausdorff dimension of the intersection of F with each C^1-diffeomorphic image of E does not exceed $\min\{\dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F\} - \delta$. Later, Feng [11] obtained some effective versions of the results of [12], but these effective versions are still far from sufficient for proving Conjecture 1.1. Feng [11] also constructed, for any $s, t \in (0, 1)$ and $\varepsilon > 0$, a T_p-invariant set A of dimension s and a T_q-invariant set B of dimension t that verify Conjecture 1.1 with a loss of ε.

Finally, we note that the slice problem may be considered as "dual" to the projection problem for fractal sets. In that direction, there is a dual version of Conjecture 1.1, also due to Furstenberg and recently settled by Hochman and Shmerkin [21] (some special cases by Peres and Shmerkin[29]), which asserts that under the assumptions of Conjecture 1.1, for each orthogonal projection P_θ from $\mathbb{R}^2$ to $\mathbb{R}$ with direction θ not parallel to the axes, we have

$$\dim_{\mathrm{H}} P_\theta(A_p \times B_q) = \min\{1, \dim_{\mathrm{H}}(A_p \times B_q)\}.$$

Recently, there has been considerable interest in the study of projections of dynamically defined Cantor sets; see, for instance, the survey paper of Shmerkin [32] and the references therein for more details.

1.2. *Statements of general results.* We prove a more general statement about intersections of regular homogeneous self-similar sets on $\mathbb{R}$ (see below for the definition) under natural irreducibility assumptions. Conjecture 1.1 will be a consequence of this general result.

We first recall some relevant definitions. An *iterated function system* (IFS) on $\mathbb{R}^d$ is a finite family $\{f_i\}_{i=1}^m$ of strictly contracting maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$. Its *attractor* is the unique non-empty compact set $X \subset \mathbb{R}^d$ satisfying

$$X = \bigcup_{i=1}^{m} f_i(X).$$

The IFS $\{f_i\}_{i=1}^m$ is called *self-similar* if each map f_i is a similarity transformation. In this case, the attractor is called a self-similar set.

A self-similar IFS $\{f_i\}_{i=1}^m$ defined on the line $\mathbb{R}$ is said to be *regular* and *λ-self-similar* if it satisfies the following conditions:

(1) *regular condition*: there exists an open interval J such that $f_i(J) \subset J$ for each i and $f_i(J) \cap f_j(J) = \emptyset$ for $i \neq j$;
(2) *λ-self-similar condition*: there exists $0 < \lambda < 1$ such that each f_i is of the form $f_i(x) = \lambda x + t_i$.

The attractor of a regular and λ-self-similar IFS will be called a *regular λ-self-similar set.*

We use $\overline{\dim}_{\mathrm{B}}$ to denote upper box-counting dimension.

THEOREM 1.3. *Assume that* $\alpha, \beta \in (0,1)$ *with* $\alpha \nsim \beta$. *Let* $C_\alpha \subset \mathbb{R}$ *be a regular* α*-self-similar set, and let* $C_\beta \subset \mathbb{R}$ *be a regular* β*-self-similar set. Then for all real numbers* u *and* v, *we have*

$$\overline{\dim}_{\mathrm{B}}(uC_\alpha + v) \cap C_\beta \leq \max\{0, \dim_{\mathrm{H}} C_\alpha + \dim_{\mathrm{H}} C_\beta - 1\}.$$

If we compare Theorem 1.3 and Conjecture 1.1, we notice that in Theorem 1.3, α, β are real numbers, and moreover we consider the upper box-counting dimension of intersections.

From Theorem 1.3, we can deduce a slightly stronger result than what is stated in Conjecture 1.1.

THEOREM 1.4. *Under the assumptions of Conjecture* 1.1, *we have for all real numbers* u *and* v,

$$\overline{\dim}_{\mathrm{B}}(uA_p + v) \cap B_q \leq \max\{0, \dim_{\mathrm{H}} A_p + \dim_{\mathrm{H}} B_q - 1\}.$$

Remark 1.5. (1) One deduces Theorem 1.4 from Theorem 1.3 by using the fact that if $A \subset [0,1]$ is a closed T_m-invariant set, then for any $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and a regular $1/m^k$-self-similar set $\widetilde{A}$ such that $A \subset \widetilde{A}$ and $\dim_{\mathrm{H}} A \geq \dim_{\mathrm{H}} \widetilde{A} - \varepsilon$. See Section 9 for the detailed proof.

(2) In Theorem 1.3, we only consider regular λ-self-similar IFSs, but it also works for some other cases. For example, the same proof works if the regular condition is replaced by the *strong separation condition* (SSC).

(3) Our approach is purely ergodic theoretical; it is quite flexible and can be extended to more general settings. A natural generalization of Theorem 1.3 is to consider intersections of linear and non-linear IFS attractors. Under certain natural circumstances, one should expect similar dimension bounds as above for the intersections. We expect that our methods could be developed further to treat these problems.

(4) Theorem 1.3 has consequences on problems of embeddings between self-similar sets as studied in [12]. See Section 9 for details.

Our next result concerns slices of self-similar sets on the plane with irrational rotation.

Let $\{f_i\}_{i=1}^m$ be a homogeneous self-similar IFS on $\mathbb{R}^2$, where for fixed $\lambda \in (0,1)$ and $\xi \in [0,1)$, each $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f_i(x) = \lambda O_\xi x + t_i,$$

with $t_i \in \mathbb{R}^2$ and O_ξ being the rotation matrix of angle $2\pi\xi \in [0, 2\pi)$.

THEOREM 1.6. *Let* X *be a self-similar set corresponding to an* IFS *as above. Suppose that* ξ *is irrational and the* IFS $\{f_i\}_i$ *satisfies the strong separation condition. Then*

$$\overline{\dim}_{\mathrm{B}}(X \cap \ell) \leq \max\{0, \dim_{\mathrm{H}} X - 1\}$$

for any line ℓ *of* $\mathbb{R}^2$.

Remark 1.7. The irrationality condition for ξ is necessary, as we can see from the 4-corner 1/3-Cantor set (i.e., the product of the classical 1/3-Cantor set C with itself): certain lines parallel to the x or y-axes intersect $C \times C$ in a set that is a copy of C.

We note that Theorems 1.4 and 1.6 have been simultaneously and independently proved by P. Shmerkin [33] using completely different (additive combinatorial) methods.

1.3. *Strategy of the proof.* Let us briefly describe our strategy for proving Theorem 1.6. The proof of Theorem 1.3 follows the same strategy, but is a bit more technical. For a set $A \subset \mathbb{R}^2$, we denote by $N_\delta(A)$ the minimal number of balls of diameter δ needed to cover A.

Let X be a self-similar set satisfying the conditions of Theorem 1.6. Our overall strategy is to show that whenever there exists a line ℓ_0 such that $\overline{\dim}_{\mathrm{B}} X \cap \ell_0 =: \gamma > 0$, then we must have $\dim_{\mathrm{H}} X \geq 1 + \gamma$. To prove this, we proceed to show that for any $\varepsilon > 0$ and all large enough n, there exist $E_n^\varepsilon \subset X$ and a set of angles $F_n^\varepsilon \subset [0, 2\pi)$ satisfying the following properties:

(1) $N_{2^{-n}}(E_n^\varepsilon) \leq 2^{n\varepsilon}$;

(2) $N_{2^{-n}}(F_n^\varepsilon) \geq 2^{n(1-\varepsilon)}$;

(3) for each $t \in F_n^\varepsilon$, there exists a line ℓ_t with angle t intersecting E_n^ε such that $\inf_{x\in X} N_{2^{-n}}\left((X \cap \ell_t) \setminus B(x, r_0)\right) \geq 2^{n(\gamma-\varepsilon)}$, where $r_0 = r(\varepsilon) > 0$ is some constant not depending on n.

From these estimates, one can deduce that $\overline{\dim}_{\mathrm{B}} X \geq 1 + \gamma$. Since the self-similar set X has equal Hausdorff and upper box dimensions, we get $\dim_{\mathrm{H}} X \geq 1 + \gamma$; see Section 7.1.

To show the existence of the sets E_n^ε and F_n^ε described above, we use ergodic methods. We consider the dynamical system (X, W) where W is the inverse map of the IFS $\{f_i\}_{i=1}^m$ on X; that is, the restriction of W on $f_i(X)$ is f_i^{-1}. Then W is expanding and rotating, for each $k \geq 1$ the map W^k transforms a slice $\ell \cap X$ into a finite family $L_k(\ell)$ of slices and the angle of each slice in $L_k(\ell)$ is rotated by $-k\xi$ compared to that of ℓ. For $z \in \ell \cap X$, we denote by $S(\ell, z, k)$ the unique slice in $L_k(\ell)$ containing $W^k(z)$.

Now, for any $\varepsilon > 0$, we would like to find a slice $\ell \cap X$ and a point $z \in \ell \cap X$ such that there exists a set $E_n^\varepsilon \subset X$ satisfying the following:

(i) $N_{2^{-n}}(E_n^\varepsilon) \leq 2^{n\varepsilon}$;

(ii) the set $F_n^\varepsilon(z) := \{-k\xi \mod 2\pi : W^k(z) \in E_n^\varepsilon\}$ satisfies $N_{2^{-n}}(F_n^\varepsilon(z)) \geq 2^{n(1-\varepsilon)}$;

(iii) for MOST $k \in \{i \in \mathbb{N} : W^i(z) \in E_n^\varepsilon\}$, we have

$$\inf_{x\in X} N_{2^{-n}}\left(S(\ell, z, k) \setminus B(x, r_0)\right) \geq 2^{n(\gamma-\varepsilon)}, \tag{1.2}$$

where MOST means such k's have relative density $1-\varepsilon$ in $\{i \in \mathbb{N} : W^i(z) \in E_n^\varepsilon\}$.

To achieve this goal, we first construct an ergodic W-invariant measure ν with positive entropy $h(\nu, W) > 0$ such that for ν-a.e. z, there exists some "good" slice $\ell \cap X$ such that $z \in \ell \cap X$ and the estimate (1.2) holds for most $k \in \mathbb{N}$. Such a measure ν will be constructed in two steps. First, based on the initial slice $\ell_0 \cap X$ with upper box dimension γ, we apply Furstenberg's CP-process machinery to create a rich family of "nice" measures μ that are supported on slices of X, where "nice" roughly means that for μ-a.e. z on the supporting slice $\ell \cap X$ of μ, (1.2) holds for most $k \in \mathbb{N}$. Then a beautiful argument due to Hochman and Shmerkin [22, Th. 2.1], which relates the small-scale structure of a measure to the distribution of W-orbits of its almost every point, will enable us to construct a W-invariant measure ν based on a "nice" measure provided by Furstenberg's CP-process. We show that this W-invariant measure ν admits the desired properties.

After having constructed such a W-invariant (ergodic) measure ν, we apply our third ingredient, which is a general result in ergodic theory and a consequence of Sinai's factor theorem, to show that the space X can be partitioned (up to a part of small ν-measure) into finitely many subsets $\cup_j A_j$ such that for ν-a.e. z and for each j, the set $E_n^\varepsilon := A_j$ satisfies the above properties (i) and (ii).

We would like to mention that if we could prove that the measure ν is weak-mixing (or more precisely, the spectrum of the system (X, W, ν) does not contain ξ), then it is easy to show that for any measurable set $A \subset X$ with $N_{2^{-n}}(A) \leq 2^{n\varepsilon}$ and $\nu(A) > 0$, the set $E_n^\varepsilon := A$ satisfies the required properties (i) and (ii) for ν-a.e. z. But from the construction of ν, it seems difficult to get any information about the mixing or spectral properties of ν. Instead, we have Sinai's factor theorem at our disposal, which provides us a Bernoulli factor system of (X, W, ν) with the same entropy as that of ν, so we can first establish the required properties in the factor system and then "transfer" the results back to the original system (X, W, ν). We note that the application of Sinai's factor theorem in the study of the kind of problems considered in the present paper seems new, and we hope that it may be useful for investigating other related questions.

For proving Theorem 1.3, we follow in principle the same scheme as described above, but instead of considering a single transformation on $K = C_\alpha \times C_\beta$, we consider a skew product U on $K \times [0, 1)$. The component of the map U on K is induced by the inverse maps of the defining IFSs of C_α and C_β and has the effect that it transforms a slice $\ell \cap K$ into finitely many pieces of slices whose slopes are changed in a way similar as the irrational rotation of angle $\theta = \log \alpha / \log \beta$ comparing to that of $\ell \cap K$.

There will be three main steps in the proof of Theorem 1.3, as for the case of Theorem 1.6. First, assuming the existence of a slice $\ell_0 \cap X$ with upper box dimension $\gamma > 0$, we construct a CP-distribution that is supported on "nice" slice measures (with dimension γ) on K. Then based on these "nice" measures, we construct a U-invariant (ergodic) measure ν_∞ whose marginal on K satisfies some similar "nice slice" properties as that of ν; i.e., almost every point with respect to the marginal of ν_∞ lies on a "good" slice of K. After the construction of such a measure ν_∞, we proceed to the last step: apply our ergodic theoretic result to the system $(K \times [0,1), U, \nu_\infty)$ and conclude the proof.

1.4. *Organization of the paper.* In Section 2 we present some general notation and collect some notions and basic properties of symbolic spaces, entropy, dimension and dynamical systems. In Section 3 we recall the CP-process theory. Sections 4–7 are devoted to the proof of Theorem 1.3. In Section 4 we construct an ergodic CP-distribution that is supported on slice measures of $C_\alpha \times C_\beta$. In Section 5 we define the skew-product U and construct the U-invariant measure ν_∞. In Section 6 we state and prove our general ergodic theoretic result. In Section 7 we prove Theorem 1.3. In Section 8 we sketch the proof of Theorem 1.6. In Section 9 we present an application of Theorem 1.3 on embeddings of self-similar sets, and we complete proofs of the remaining statements.

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1.6. *Summary of notation.* For the reader's convenience, we summarize our main notation conventions in Table 1.

2. **Notation and preliminaries**

2.1. *General notation and conventions.*

- We use $\sharp A$ to denote the cardinality of a set A. In a metric space, $B(x,r)$ denotes the closed ball of radius r around x.
- In this paper, a measure is always a Borel probability measure. The set of all Borel probability measures on a metric space X will be denoted by $\mathcal{P}(X)$. Usually, we will not mention the σ-algebra of a measurable space; sets and functions are implicitly assumed to be Borel measurable when it is required.
- If X and Y are metric spaces, and $f : X \to Y$ is any measurable map, then for any $\mu \in \mathcal{P}(X)$, we define $f\mu$ as the push-forward measure $\mu \circ f^{-1}$.
- The topological support of a measure μ is denoted by $\mathrm{supp}(\mu)$; the restriction of μ on a set E is denoted by $\mu|_E$.
- We use δ_x to denote the Dirac measure at a point x.
- We will use standard "big O" and "little o" notation.

$B(x,r)$	The closed ball of radius r around x
$\mathcal{P}(X)$	Space of probability measures on X
$\mu,\nu,\eta,\upsilon,\vartheta$	Measures
P,Q	Probability distributions (elements of $\mathcal{P}(\mathcal{P}(X))$)
$\mu\|_A$	Restriction of μ on A
$\underline{D}(\mu,x),\overline{D}(\mu,x)$	Lower and upper local dimension of μ at x (§2.3)
$\mathcal{D},\mathcal{A},\mathcal{F}$	Partitions
$\mathcal{D}_n(\mathbb{R}^d)$ (or $\mathcal{D}_n$)	Partition of $\mathbb{R}^d$ into n-th level dyadic cubes (§2.3)
$\mathcal{D}_n(x),\mathcal{A}(x)$	The element of $\mathcal{D}_n$ (resp. $\mathcal{A}$) containing x
$N_{2^{-n}}(A)$	The number of elements of $\mathcal{D}_n$ intersecting A (§2.3.2)
Λ	Alphabet set (finite)
σ	Shift map, $\sigma(x)_n = x_{n+1}$
$[a]$	Cylinder set corresponding to $a\in\Lambda^n$
$\mu^{[a]}$	$\sigma^n(\mu\|_{[a]})/\mu([a])$ (§3.2)
$H(\mu,\mathcal{A})$	Shannon entropy (§2.3)
$\mathcal{A}_n^t$	Partition (Definition (5.3))
$\mu^{\mathcal{A}_n^t(z)}$	Definition (5.4)

Table 1.

2.2. *Symbolic space.* In this subsection, we recall some classical notion for symbolic spaces. Let Λ be a finite set that we call an *alphabet set*. Let $\Lambda^{\mathbb{N}}$ be the symbolic space of infinite sequences from the alphabet set. We endow $\Lambda^{\mathbb{N}}$ with the standard metric d_ρ with respect to a number $\rho\in(0,1)$:

$$d_\rho(x,y) = \rho^{\min\{n: x_n\neq y_n\}}. \tag{2.1}$$

Then $(\Lambda^{\mathbb{N}}, d_\rho)$ is a compact totally disconnected metric space.

We denote by $\Lambda^* = \bigcup_{n\geq 0}\Lambda^n$ the set of finite words (with the convention that $\Lambda^0=\{\emptyset\}$). For $n\geq 0$, the *length* of a word $u\in\Lambda^n$, denoted by $|u|$, is defined to be n. For $u\in\Lambda^n$, the n-th level *cylinder* associated to u is the set

$$[u] = \{x\in\Lambda^{\mathbb{N}} : x_1\cdots x_n = u\}.$$

Every cylinder is a closed and open set. For $x\in\Lambda^*\cup\Lambda^{\mathbb{N}}$, we will use

$$x_1^k = x_1\cdots x_k$$

to represent the word consisting of the k first letters of x when $k\leq|x|$. Define the left-shift σ on $\Lambda^{\mathbb{N}}$ by

$$\sigma((x_n)_{n\geq 1}) = (x_{n+1})_{n\geq 1}.$$

2.3. *Dimension and entropy.* In this subsection, we recall some basic notion and facts about dimension and entropy of measures (or sets). We use $\dim_{\mathrm{H}} A$ and $\overline{\dim}_{\mathrm{B}} A$ to denote the Hausdorff dimension and upper box-counting dimension of a set A, respectively.

2.3.1. *Dimension of measures.* Let μ be a Borel measure on a metric space. The *lower* (*Hausdorff*) *dimension* of μ is defined as

$$\dim_*(\mu) = \inf\{\dim_{\mathrm{H}} A : \mu(A) > 0\}.$$

Closely related to the lower dimension of μ is the *lower local dimension*, defined at each $x \in \mathrm{supp}(\mu)$ as

$$\underline{D}(\mu, x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Similarly, we can consider the upper limit and define the upper local dimension $\overline{D}(\mu, x)$ of μ at x. When $\underline{D}(\mu, x) = \overline{D}(\mu, x)$, we say that the local dimension of μ at x exists and denote it by $D(\mu, x)$. If the local dimension of μ exists and is constant μ-almost everywhere, then μ is called *exact dimensional* and the almost sure local dimension is denoted by $\dim(\mu)$. For more details about different definitions of dimensions of measures, we refer the readers to [8], [5], [27], [9].

2.3.2. *Partitions and entropy.* Let μ be a Borel measure on a metric space X. For a finite or countable partition $\mathcal{A}$ of X, the entropy of μ with respect to $\mathcal{A}$ is

$$H(\mu, \mathcal{A}) = -\sum_{A\in\mathcal{A}} \mu(A) \log \mu(A)$$

with the convention that $0 \log 0 = 0$. Here and in what follows, the logarithm is in base e.

Next, we define entropy dimension of measures—first in the symbolic space, then in the Euclidean space. In a symbolic space $(\Lambda^{\mathbb{N}}, d_\rho)$, let $\mathcal{F}_n$ be the partition of $\Lambda^{\mathbb{N}}$ given by the n-th level cylinder sets $\{[u] : u \in \Lambda^n\}$. For a set $A \subset \Lambda^{\mathbb{N}}$, we will use $N_{\rho^n}(A)$ to count the number of elements of $\mathcal{F}_n$ intersecting A. For $\mu \in \mathcal{P}(\Lambda^{\mathbb{N}})$, we define the *entropy dimension* of μ by

$$\dim_e(\mu) = \lim_{n\to\infty} \frac{1}{-n\log\rho} H(\mu, \mathcal{F}_n)$$

if the limit exists; otherwise we consider the upper and lower entropy dimensions $\overline{\dim}_e(\mu)$ and $\underline{\dim}_e(\mu)$ defined by replacing limit, respectively, by lim sup and lim inf.

Now, we define the entropy dimension on Euclidean space. For any $n \geq 0$, let $\mathcal{D}_n(\mathbb{R}^d)$ be the collection of n-th level *dyadic cubes* of $\mathbb{R}^d$, that is,

$$\mathcal{D}_n(\mathbb{R}^d) := \left\{ \prod_{i=1}^{d} [\frac{k_i}{2^n}, \frac{k_i+1}{2^n}) : (k_1, \ldots, k_d) \in \mathbb{Z}^d \right\}.$$

Then $\mathcal{D}_n(\mathbb{R}^d)$ is a partition of $\mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, we will use $N_{2^{-n}}(A)$ to count the number of elements of $\mathcal{D}_n(\mathbb{R}^d)$ intersecting A. For $\mu \in \mathcal{P}(\mathbb{R}^d)$, the entropy dimension of μ is defined as

$$\dim_e(\mu) = \lim_{n\to\infty} \frac{1}{n \log 2} H(\mu, \mathcal{D}_n(\mathbb{R}^d))$$

if the limit exists; otherwise we consider the upper and lower entropy dimensions. We will simply write $\mathcal{D}_n$ for $\mathcal{D}_n(\mathbb{R}^d)$ when no confusion can arise.

The following lemma presents some relationships between different dimensions of a measure.

LEMMA 2.1. *Let μ be a measure on $\mathbb{R}^d$ or $\Lambda^{\mathbb{N}}$. Then*

$$\dim_*(\mu) \leq \underline{\dim}_e(\mu) \leq \overline{\dim}_e(\mu).$$

If μ is exact dimensional, then

$$\dim_*(\mu) = \dim(\mu) = \underline{\dim}_e(\mu) = \overline{\dim}_e(\mu).$$

Proof. Proofs for the Euclidean case can be found in [9]. The symbolic case is analogous. □

2.3.3. *Dimensions of product sets.* We recall the following dimension formula for dimensions of product sets.

LEMMA 2.2 (Theorem 8.10 of [27]). *Let $E, F \subset \mathbb{R}^d$ be non-empty Borel sets. Then*

$$\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F \leq \dim_{\mathrm{H}}(E \times F) \leq \overline{\dim}_{\mathrm{B}}(E \times F) \leq \overline{\dim}_{\mathrm{B}} E + \overline{\dim}_{\mathrm{B}} F.$$

2.4. *Dynamical systems.* In this subsection, we collect some basic notions and properties of dynamical systems. We refer the reader to [38], [3], [4] for more information.

2.4.1. *Measure preserving dynamical system.* By a *Measure preserving dynamical system* (or *dynamical system* for short) we mean a quadruple $(X, \mathcal{B}, T, \mu)$, where X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra on X, $T : X \to X$ is a Borel map and μ is a T-invariant measure. We shall often omit $\mathcal{B}$ in our notation and abbreviate the system to (X, T, μ).

A dynamical system is *ergodic* if the only invariant sets are trivial, i.e., if $\mu(A \Delta T^{-1}A) = 0$, then $\mu(A) = 0$ or $\mu(A) = 1$. By the *ergodic decomposition theorem*, every T-invariant measure μ can be decomposed as mixtures of

T-invariant ergodic measures: $\mu = \int \mu^{(x)} d\mu(x)$, where for μ-a.e. x, $\mu^{(x)}$ is a T-invariant and ergodic measure, called an *ergodic component* of μ. We refer the reader to [4, Ch. 4.2] for more information.

Another important notion in ergodic theory is *weak-mixing*. For the precise definition of weak-mixing and its many equivalent formulations, see [38] and [4]. We will make use the following characterization of weak-mixing (see[4, Th. 2.36]): a dynamical system (X, T, μ) is weakly mixing if and only if, for any ergodic dynamical system (Y, S, ν), the product system $(X \times Y, T \times S, \mu \otimes \nu)$ is also ergodic.

An important class of dynamical systems that we will have occasion to use are *symbolic dynamical systems*, in which X is the symbolic space $\Lambda^{\mathbb{N}}$ and T is the shift transformation σ, and μ is a shift-invariant measure. In the case when μ is a product measure determined by a probability vector $p = (p_i)_{i\in\Lambda}$ on Λ, we call $(\Lambda^{\mathbb{N}}, \sigma, \mu)$ a *Bernoulli shift*.

A dynamical system (Y, S, ν) is a *factor* of (X, T, μ) if there exists a measurable map $\pi : X \to Y$, called the factor map, which is *equivariant*, i.e., $\pi \circ T = S \circ \pi$ and $\pi\mu = \nu$.

Let (X, T, μ) be a dynamical system. A point $x \in X$ is *generic* for μ if

$$\frac{1}{N}\sum_{n=0}^{N-1} \delta_{T^n x} \to \mu \quad \text{as } N \to \infty$$

in the weak-* topology. It follows from the ergodic theorem that if μ is ergodic, then μ-a.e. x is generic for μ.

2.4.2. *Measure-theoretic entropy.* The *measure-theoretic entropy* of a dynamical system (X, T, μ) will be denoted by $h(\mu, T)$. We refer the reader to [38], [3] for precise definition of entropy and related material.

For a finite measurable partition $\mathcal{A}$ of X, we write $\mathcal{A}_n = \bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}$ for the coarsest common refinement of $\mathcal{A}, T^{-1}\mathcal{A}, \ldots, T^{-(n-1)}\mathcal{A}$. We call $\{\mathcal{A}_n\}_{n\geq 1}$ the filtration generated by $\mathcal{A}$ with respect to T. For each $n \geq 1$ and $x \in X$, $\mathcal{A}_n(x)$ is the unique element of $\mathcal{A}_n$ containing x. We use $\mathcal{A}_\infty = \bigvee_{k=0}^{\infty} T^{-k}\mathcal{A}$ to denote the σ-algebra generated by the partitions $\mathcal{A}_n$, $n \geq 1$. We say that $\mathcal{A}$ is a *generator* for T if $\mathcal{A}_\infty$ is the full Borel σ-algebra.

3. CP-processes

3.1. *General theory.* The CP-process theory was pioneered by Furstenberg in [15], initially as a tool to investigate Conjecture 1.1. Recently, a more systematic study of CP-processes was initiated by Furstenberg [16], with further developments by Gavish [17], Hochman [19], Hochman and Shmerkin [21] and others. Let us first recall some basic concepts related to this theory in the symbolic setting.

Recall that $\mathcal{P}(X)$ is the set of all Borel probability measures on a metric space X. A *distribution* is a Borel probability measure on $\mathcal{P}(X)$ (or even larger spaces). Notice that distributions are measures on space of measures.

Fix a finite alphabet Λ. For $0 < \rho < 1$, consider the symbolic space $\Lambda^{\mathbb{N}}$ endowed with the metric defined as (2.1). Let

$$\Omega = \left\{(\mu, x) \in \mathcal{P}(\Lambda^{\mathbb{N}}) \times \Lambda^{\mathbb{N}} : x \in \operatorname{supp}(\mu)\right\}.$$

The CP-process theory studies the dynamical properties under the action of magnification of measures.

Definition 3.1 (Magnification dynamics). We define the *magnification operator* $M : \Omega \to \Omega$ as

$$M(\mu, x) = (\mu^{[x_1]}, \sigma(x)),$$

where $\mu^{[x_1]} = \sigma(\mu|_{[x_1]})/\mu([x_1])$.

It is clear that $M(\Omega) \subset \Omega$ and M is continuous. For any distribution P on Ω (i.e., $P \in \mathcal{P}(\Omega)$), we denote by P_1 its marginal on the measure coordinate.

Definition 3.2 (Adaptedness). A distribution P on Ω is called *adapted* if for every $f \in C(\mathcal{P}(\Lambda^{\mathbb{N}}) \times \Lambda^{\mathbb{N}})$,

$$\int f(\mu, x) dP(\mu, x) = \int \left(\int f(\mu, x) d\mu(x)\right) dP_1(\mu).$$

In other words, P is adapted if, conditioned on the measure component being μ, the point component x is distributed according to μ. In particular, if a property holds for P-a.e. (μ, x) and P is adapted, then this property holds for P_1-a.e. μ and μ-a.e. x.

Definition 3.3 (CP-distribution). A distribution P on Ω is a *CP-distribution* if it is M-invariant and adapted. In this case, we call the system (Ω, P, M) a CP-process.

A CP-distribution P is *ergodic* if the measure preserving system (Ω, P, M) is ergodic in the usual sense. If it is not ergodic, then we can consider its ergodic decomposition.

PROPOSITION 3.4. *The ergodic components of a* CP-*distribution are adapted*; *in particular*, *they are ergodic* CP-*distributions.*

A proof of this result is indicated in the remark following Proposition 5.1 of [16]. See also [31, Prop. 22] and [19, Th. 1.3] for alternative proofs.

3.2. *Dimension and generic properties of* CP-*processes.* In this subsection, we list some useful properties of CP-processes that we will use later. The first one concerns dimension information of typical measures for ergodic CP-distributions.

Proposition 3.5 (Theorem 2.1 of [16]). *Let P be an ergodic* CP*-distribution. Then P_1-almost every measure μ is exact dimensional with dimension*

$$\dim \mu = \frac{1}{\log \rho^{-1}} \int -\log \nu[x_1] dP(\nu, x) = \frac{1}{\log \rho^{-1}} \int \sum_{i\in\Lambda} -\nu[i] \log \nu[i] dP_1(\nu).$$

For an ergodic CP-distribution P, we denote by $\dim P$ the almost sure dimension of μ for a P-typical μ.

Several times we will use the following lemma, which is an immediate consequence of the ergodic theorem and the adaptedness property of CP-processes. We denote

$$\mu^{[x_1^n]} = \sigma^n(\mu|_{[x_1^n]})/\mu([x_1^n]). \tag{3.1}$$

Lemma 3.6. *Let P be an ergodic* CP*-distribution. Then P_1-a.e. μ generates P_1 in the sense that for μ-a.e. x, we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} \delta_{\mu^{[x_1^n]}} \to P_1 \quad \textit{weak-* as } N \to \infty. \tag{3.2}$$

For a measure μ that generates P_1 in the above sense, we say μ is generic for P_1. As a corollary of Proposition 3.5 and Lemma 3.6, we obtain the following easy but useful properties concerning typical measures of CP-distributions with positive dimension. Similar results have appeared in [22].

Proposition 3.7. *Let P be an ergodic* CP*-distribution with $\dim P = h > 0$. For any $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for each μ that is generic for P_1 and for μ-a.e. x,*

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le k \le N : \max_{u \in \Lambda^{n_0(\varepsilon)}} \mu^{[x_1^k]}([u]) \le \varepsilon \right\} > 1 - \varepsilon \tag{3.3}$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le k \le N : H(\mu^{[x_1^k]}, \mathcal{F}_n) \ge n(h \log \rho^{-1} - \varepsilon) \right\} > 1 - \varepsilon \quad \textit{for all } n \ge n_0(\varepsilon). \tag{3.4}$$

In particular, for P_1-a.e. μ and μ-a.e. x, the above properties hold.

Proof. The proof is similar to that of [22, Lemma 4.11]. Fix any $\varepsilon > 0$. By Proposition 3.5, P_1-a.e. ν is exact dimensional with dimension $h > 0$, so ν is non-atomic and using Lemma 2.1, we have

$$\lim_{n\to\infty} \frac{1}{n} H(\nu, \mathcal{F}_n) = h \log \rho^{-1}.$$

Thus for P_1-a.e. ν, there exists a finite integer $n(\nu)$ such that for each $n \ge n(\nu)$,

$$\max_{u\in\Lambda^n} \nu([u]) < \varepsilon \quad \text{and} \quad H(\nu, \mathcal{F}_n) > n(h \log \rho^{-1} - \varepsilon). \tag{3.5}$$

It follows that there exist a set E_ε of measures with $P_1(E_\varepsilon) > 1-\varepsilon$ and a finite $n_0(\varepsilon) \in \mathbb{N}$ such that $n_0(\varepsilon) \geq n(\nu)$ for $\nu \in E_\varepsilon$. For any $n \geq n_0(\varepsilon)$, let E_ε^n be the set of measures ν such that (3.5) holds. Then $E_\varepsilon \subset E_\varepsilon^n$ and E_ε^n is open. Since μ generates P_1, we have, for μ-a.e. x,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\mu^{[x_1^k]}}(E_\varepsilon^n) \geq P_1(E_\varepsilon^n) > 1-\varepsilon.$$

The above statement holds for each $n \geq n_0(\varepsilon)$, which is what we wanted to show. □

Remark 3.8. In the above proof, we saw that properties (3.3) and (3.4) hold for each pair (μ, x) satisfying (3.2).

4. Constructions of CP-distributions based on $K = C_\alpha \times C_\beta$

Let us first recall the sets C_α and C_β and some preliminary results about them. Fix two real numbers $0 < \beta < \alpha < 1$ such that $\theta = \log\alpha/\log\beta$ is irrational. Let $\Phi = \{\phi_i(x) = \alpha x + \lambda_i^\alpha\}_{i=1}^m$ and $\Psi = \{\psi_j(x) = \beta x + \lambda_i^\beta\}_{j=1}^l$ be two regular self-similar IFSs on $\mathbb{R}$. Let C_α be the attractor of Φ and C_β be the attractor of Ψ. Let $K = C_\alpha \times C_\beta$.

In this section, assuming the existence of a slice $\ell_0 \cap K$ with upper box dimension $\gamma > 0$, we construct a family of ergodic CP-distributions having dimensions at least γ and supported on measures that are supported on slices of K. The construction of such CP-distributions is essentially due to Furstenberg [15]; we just reinterpret the material in our setting.

Since the IFSs Φ and Ψ satisfy the convex open set condition, there exist open intervals I_α and I_β with $\phi_i(I_\alpha) \subset I_\alpha$ $(1 \leq i \leq m)$ and $\psi_j(I_\beta) \subset I_\beta$ $(1 \leq j \leq l)$ such that

$$\phi_{i_1}(I_\alpha) \cap \phi_{i_2}(I_\alpha) = \emptyset \text{ for } i_1 \neq i_2 \quad\text{and}\quad \psi_{j_1}(I_\beta) \cap \psi_{j_2}(I_\beta) = \emptyset \text{ for } j_1 \neq j_2.$$

Let $\{I_\alpha^i\}_{i=1}^m$ be a partition of $\bigcup_{i=1}^m \phi_i\left(\overline{I_\alpha}\right)$ such that each I_α^i is an interval that may be open, closed or half open and whose interior is the same as that of $\phi_i\left(\overline{I_\alpha}\right)$. Similarly, we choose such a partition $\{I_\beta^j\}_{j=1}^l$ for $\bigcup_{j=1}^l \psi_j\left(\overline{I_\beta}\right)$. Then we define S_α to be the inverse map of Φ on $\bigcup_i \phi_i(I_\alpha)$, that is, the restriction of S_α on I_α^i is ϕ_i^{-1} for $1 \leq i \leq m$. Let S_β be the inverse map of Ψ on $\bigcup_j \psi_j(I_\beta)$. We define two maps on $(\bigcup_i \phi_i(I_\alpha)) \times (\bigcup_j \psi_j(I_\beta))$ by

$$\varphi_1(x,y) = (S_\alpha(x), y) \quad\text{and}\quad \varphi_2(x,y) = (S_\alpha(x), S_\beta(y)).$$

Then $K = C_\alpha \times C_\beta$ is invariant under both maps φ_1 and φ_2. Given a line ℓ with slope u that intersects K, then φ_1 transforms $\ell \cap [0,1]^2$ into finitely many line segments, each with slope αu and φ_2 transforms $\ell \cap [0,1]^2$ into finitely many line segments, each with slope $\alpha u/\beta$.

Now suppose that there exists a line ℓ that intersects K in a set of upper box dimension $\gamma > 0$. The same will be true for at least one of the lines of $\varphi_1(\ell)$ and for one of the lines of $\varphi_2(\ell)$. We can continue in this way and finally we will find a family L of infinitely many lines such that each line of L intersects K in a set of upper box dimension γ. If the initial line ℓ has slope u with $u \notin \{0, \infty\}$, then for each pair $(n, m) \in \mathbb{N}^2$ with $n \geq m$, there exists a line in L with slope $u\alpha^n/\beta^m$. Since $\log \alpha / \log \beta$ is irrational, the set $\{u\alpha^n/\beta^m : n \geq m\}$ is dense in $(0, +\infty)$ or in $(-\infty, 0)$ depending on whether $u > 0$ or $u < 0$.

In the rest of this paper, we always make the assumption that

$$\text{there exists a line } \ell_0 \text{ with slope } u_0 \in (0, +\infty) \text{ such that } \overline{\dim}_{\mathrm{B}}(\ell_0 \cap K) = \gamma > 0. \tag{4.1}$$

Our ultimate aim is to show that, in this case, we must have $\dim_{\mathrm{H}} K \geq 1 + \gamma$. For the case of negative slope u_0, we apply a reflection to C_α to make the slope positive.

In the rest of this section, we will follow Furstenberg [15] to construct an ergodic CP-distribution (with dimension γ) on the space of measures that are supported on slices of K with slopes in $[1, 1/\beta]$. In the end of Section 4.2, as a direct application of this CP-distribution, we will give the proof of Furstenberg's main result in [15, Th. 4]: under the assumption (4.1), for Lebesgue almost all $u \in (0, +\infty)$, there exists a slice of K with slope u and Hausdorff dimension $\geq \gamma$.

4.1. *Symbolic setting.* Let $\Lambda_\alpha = \{\lambda_i^\alpha\}_{i=1}^m$ and $\Lambda_\beta = \{\lambda_j^\beta\}_{j=1}^l$. Note that C_α can be written as

$$C_\alpha = \left\{ \sum_{n=1}^{\infty} \alpha^{n-1} a_n : (a_n)_{n\geq 1} \in \Lambda_\alpha^{\mathbb{N}} \right\}.$$

A similar representation holds for C_β, replacing α by β and Λ_α by Λ_β.

Write $\Lambda = \Lambda_\alpha \times \Lambda_\beta$. Let $X = \Lambda^{\mathbb{N}}$. Recall that $\theta = \log \alpha / \log \beta$. For each $t \in [0, 1) = \mathbb{R}/\mathbb{Z}$, we construct a tree $X_t \subset X = \Lambda^{\mathbb{N}}$ as follows. For $s \in [0, 1)$, write $L(s) = \Lambda$ if $s \in [0, \theta)$ and $L(s) = \Lambda_\alpha \times \{\lambda_1^\beta\}$ otherwise. We define

$$R_\theta(s) = s - \theta \mod 1 \quad \text{for } s \in [0, 1).$$

In the rest of this paper, we identify $[0, 1)$ with $\mathbb{R}/\mathbb{Z}$; thus $[0, 1)$ is compact and R_θ is continuous on it.

Let

$$X_t = \prod_{n=0}^{\infty} L(R_\theta^n(t)).$$

By definition, for $x \in X_t$, the shifted point $\sigma(x)$ is an element of $X_{R_\theta(t)}$. On each X_t we consider the metric d_α (recall (2.1)).

For $s \in [0,1)$, let $Z(s) = \{n \geq 0 : R_\theta^n(s) \in [0,\theta)\}$. We write the elements of $Z(s)$ in an increasing order as $w_1(s) < w_2(s) < \cdots$. We define a projection map $\pi_t : X_t \to K$ by

$$\pi_t\left((a_n)_n, (b_n)_n\right) = \left(\sum_{n=1}^{\infty} \alpha^{n-1} a_n, \sum_{n=1}^{\infty} \beta^{n-1} b_{w_n(t)}\right).$$

Note that π_t is a surjective map.

Let us record for later use some properties about X_t and π_t in the following lemma. We use $\mathrm{cov}_r(A)$ to denote the minimal number of balls of diameter r needed to cover a set A. Recall also the notation $N_{\alpha^k}(A)$ (see Section 2.3.2).

Lemma 4.1.

(1) *If* $t_k, t \in [0,1)$ *are such that* $t_k \to t$ *and* $R_\theta^n(t) \neq \theta$ *for all* n, *then* $X_{t_k} \to X_t$ (*under the Hausdorff metric*) *and* $\pi_{t_k} \to \pi_t$.

(2) *There exists a constant* $C_1 > 0$ *such that the maps* π_t *are uniformly* C_1*-Lipschitz.*

(3) *There exists a constant* $C_2 > 0$ *such that for all* $t \in [0,1)$ *and all* $A \subset X_t$, $N_{\alpha^k}(A) \leq C_2 \cdot \mathrm{cov}_{\alpha^k}(\pi_t(A))$ *for each* $k \in \mathbb{N}$.

(4) *For all* $t \in [0,1)$ *and all* $A \subset X_t$, *we have* $\dim_{\mathrm{H}} A = \dim_{\mathrm{H}} \pi_t(A)$.

Proof. We give the proof for parts (1) and (2), the other parts are obvious. The first part follows from the fact that, if $R_\theta^n(t) \neq \theta$ for all $n \leq M$, then for all sufficiently large k, we have $w_n(t_k) = w_n(t)$ for all $n \leq M$, and this implies that the first M generations of the trees X_{t_k} and X_t coincide, and that π_{t_k} is uniformly close to π_t.

To prove part (2), it suffices to show that there is C_1 (independent of t) such that

$$C_1^{-1} \alpha^k \leq \beta^{r_k(t)} \leq C_1 \alpha^k,$$

where $r_k(t) = \sharp\{0 \leq i \leq k-1 : R_\theta^i(t) \in [0,\theta)\}$. This is equivalent to saying that $|r_k(t) - k\theta|$ is bounded by some uniform constant. To show this, we only need to observe that $r_k(t)$ is the number of $i \in \{0, \ldots, k-1\}$ such that there exists an integer n with $t - i\theta \geq n$ and $t - (i+1)\theta < n$. Thus $-r_k(t)$ is the largest integer not greater than $t - k\theta$, from which we deduce that $|r_k(t) - k\theta| \leq 2$. $\square$

Let $\ell_{u,z}$ denote the line through z with slope u. We define

$$\mathcal{F} = \left\{(A, x, t) : t \in [0,1), A \subset X_t \text{ is compact}, x \in A, \pi_t(A) \subset K \cap \ell_{\beta^{-t}, \pi_t(x)}\right\}.$$

Note that for any line $\ell_{\beta^{-t},z}$ with $t \in [0,1), z \in K$ and any $x \in \pi_t^{-1}(z)$, we have the set $(\pi_t^{-1}(K \cap \ell_{\beta^{-t},z}), x, t) \in \mathcal{F}$.

Lemma 4.2.

(1) *If* $(A, x, t) \in \mathcal{F}$, *then* $(\sigma(A \cap [x_1]), \sigma(x), R_\theta(t)) \in \mathcal{F}$.

(2) *Suppose that* $(A_k, y_k, t_k) \to (A, x, t)$ *and* $(A_k, y_k, t_k) \in \mathcal{F}$ *for each* k*. If* $R_\theta^n(t) \neq \theta$ *for all* n*, then* $(A, x, t) \in \mathcal{F}$*.*

Proof. Note that for $x' \in X_t$, we have $\pi_{R_\theta(t)}(\sigma(x')) = \Phi_t(\pi_t(x'))$. Thus we have

$$\pi_{R_\theta(t)}(\sigma(A \cap [x_1])) = \Phi_t(\pi_t(A \cap [x_1])).$$

From this we deduce claim (1). Claim (2) is a consequence of part (1) of Lemma 4.1. □

4.2. *Construction of* CP*-distributions.* Consider the space

$$Y = \mathcal{P}(X) \times X \times [0, 1).$$

Note that Y is a compact space. We define a map $\hat{M}$ on Y by

$$\hat{M}(\mu, x, t) = (\mu^{[x_1]}, \sigma(x), R_\theta(t)).$$

The map $\hat{M}$ can be viewed as an "extension" of the magnification operator M in Definition 3.1. It is continuous on Y (where we consider the weak topology on $\mathcal{P}(X)$).

By the assumption (4.1) and the discussion preceding it, there exist some $t_0 \in [0, 1)$ and a line ℓ with slope β^{-t_0} such that $\overline{\dim}_{\mathrm{B}} K \cap \ell = \gamma > 0$. Let $E = \pi_{t_0}^{-1}(K \cap \ell)$. Then by parts (2) and (3) of Lemma 4.1, we have $\overline{\dim}_{\mathrm{B}} E = \gamma$ (in the space X_{t_0}). Thus there exists a sequence $n_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \frac{\log N_{\alpha^{n_k}}(E)}{-n_k \log \alpha} = \gamma. \tag{4.2}$$

We define a sequence of measures $\{\mu_k\}_k$ on E by setting

$$\mu_k = \frac{1}{N_{\alpha^{n_k}}(E)} \sum_{u \in \Lambda^{n_k} : [u] \cap E \neq \emptyset} \delta_{x_u},$$

where x_u is some point in $[u] \cap E$. Finally, let

$$P_k = \frac{1}{N_{\alpha^{n_k}}(E)} \sum_{u \in \Lambda^{n_k} : [u] \cap E \neq \emptyset} \delta_{(\mu_k, x_u, t_0)}$$

and

$$Q_k = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \hat{M}^i P_k.$$

By the construction of P_k, it is clear that for any $f \in C(Y)$, we have

$$\int f(\mu, x, t) dP_k(\mu, x, t) = \int \Big(\int f(\mu, x, t) d\mu(x) \Big) d(P_k)_{1,3}(\mu, t),$$

where we use $(P_k)_{1,3}$ to denote the marginal of P_k on the first and third coordinates. The same is true for Q_k. Let us call a distribution $P \in \mathcal{P}(Y)$ *globally adapted* if it satisfies the above identity. It follows from the definition that if a property holds for P-a.e. (μ, x, t) and P is globally adapted, then this property holds for $P_{1,3}$-a.e. (μ, t) and μ-a.e. x. Clearly, for a globally adapted distribution, its marginal on the first two coordinates (μ, x) is adapted in the sense of Definition 3.2. For each $P \in \mathcal{P}(Y)$, we define

$$H(P) = \int \frac{1}{\log \alpha} \log \mu[x_1] dP_{1,2}(\mu, x),$$

where $P_{1,2}$ is the marginal of P on (μ, x). Let us calculate

$$\begin{aligned} H(Q_k) &= \frac{1}{n_k} \frac{1}{N_{\alpha^{n_k}}(E)} \sum_{u \in \Lambda^{n_k} : [u] \cap E \neq \emptyset} \sum_{i=1}^{n_k} \frac{1}{\log \alpha} \log \frac{\mu_k[u_1^i]}{\mu_k[u_1^{i-1}]} \\ &= \frac{1}{n_k} \frac{1}{N_{\alpha^{n_k}}(E)} \sum_{u \in \Lambda^{n_k} : [u] \cap E \neq \emptyset} \frac{1}{\log \alpha} \log \mu_k[u] = \frac{\log N_{\alpha^{n_k}}(E)}{-n_k \log \alpha}. \end{aligned}$$

It follows from (4.2) that

$$H(Q_k) \to \gamma \quad \text{as } k \to \infty.$$

Passing to a further subsequence we can assume that $Q_k \to Q$ in $\mathcal{P}(Y)$. Now by continuity of $\hat{M}$, Q is $\hat{M}$-invariant; and since each Q_k is globally adapted, we deduce that Q is also globally adapted. Thus the marginal of Q on (μ, x) is a CP-distribution. Since the map H is continuous on $\mathcal{P}(Y)$, we have

$$H(Q) = \lim_{k \to \infty} H(Q_k) = \gamma.$$

Let

$$Q = \int Q^{(\mu,x,t)} dQ(\mu, x, t)$$

be the ergodic decomposition of Q. We define

$$\mathcal{E}_\gamma = \left\{ (\mu, x, t) \in Y : H(Q^{(\mu,x,t)}) \geq \gamma \right\}. \tag{4.3}$$

Then we have $Q(\mathcal{E}_\gamma) > 0$, and for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$, the marginal of $Q^{(\mu,x,t)}$ on the first two coordinates, denoted by $Q_{1,2}^{(\mu,x,t)}$, is an ergodic CP-distribution with dimension $H(Q^{(\mu,x,t)}) \geq \gamma$. Note that for the adaptedness of $Q_{1,2}^{(\mu,x,t)}$, we have used Proposition 3.4.

Let

$$\Xi_{\mathcal{F}} = \bigcup_{(A,x,t) \in \mathcal{F}} \mathcal{P}(A) \times \{x\} \times \{t\}.$$

LEMMA 4.3. *The distribution Q is supported on $\Xi_{\mathcal{F}}$. In particular, this holds for Q-a.e. ergodic component of Q.*

Proof. We need to prove that for Q-a.e. (μ, x, t), we have $(\mathrm{supp}(\mu), x, t) \in \mathcal{F}$. Since Q is a weak limit of Q_k and each Q_k is supported on $\Xi_{\mathcal{F}}$, it follows that Q is supported on triples of the form

$$(\mu, x, t) = \lim_{k \to \infty} (\mu_k, x_k, t_k)$$

with $(\mathrm{supp}(\mu_k), x_k, t_k) \in \mathcal{F}$. Now, since the marginal of Q on the third coordinate is an R_θ-invariant measure on $[0, 1)$, it must be the Lebesgue measure.

Thus for Q-a.e. (μ, x, t), we have $R_\theta^n(t) \neq \theta$ for all n. From this, part (2) of Lemma 4.2 and the fact that $\mathrm{supp}(\mu) \subset \liminf_{k\to\infty} \mathrm{supp}(\mu_k)$, we deduce that $(\mathrm{supp}(\mu), x, t) \in \mathcal{F}$. □

We finish this subsection by giving the proof of the following result of Furstenberg [15, Th. 4] by using the CP-distributions $\{Q_{1,2}^{(\mu,x,t)}\}_{(\mu,x,t)}$ we constructed above.

THEOREM 4.4 (Furstenberg, [15]). *Assume that* (4.1) *holds. Then for Lebesgue almost every* $u \in (0, +\infty)$, *there exists a line* ℓ *with slope* u *such that* $\dim_{\mathrm{H}} \ell \cap K \geq \gamma$.

Proof. By the discussion preceding assumption (4.1), we only need to show that for Lebesgue almost every $u \in [1, \beta^{-1}]$, there exists a line ℓ with slope u such that $\dim_{\mathrm{H}} \ell \cap K \geq \gamma$. Let $Q, \mathcal{E}_\gamma, \Xi_{\mathcal{F}}$ be as above. We choose an element $(\mu, x, t) \in \mathcal{E}_\gamma$ such that the ergodic component $Q^{(\mu,x,t)}$ is supported on $\Xi_{\mathcal{F}}$ and its marginal $Q_{1,2}^{(\mu,x,t)}$ is an ergodic CP-distribution (with dimension at least γ). Thus for $Q^{(\mu,x,t)}$-a.e. (ϑ, y, s), ϑ is a measure with dimension at least γ. Again, since the marginal of $Q^{(\mu,x,t)}$ on the third coordinate is an R_θ-invariant measure on $[0,1)$, it must be the Lebesgue measure. Hence for Lebesgue almost every $s \in [0,1)$, there exists (ϑ, y) such that $(\vartheta, y, s) \in \Xi_{\mathcal{F}}$ and $\dim \vartheta \geq \gamma$. From the definition of $\Xi_{\mathcal{F}}$ and part (4) of Lemma 4.1, we deduce that there exists a line ℓ with slope β^{-s} such that $\dim_{\mathrm{H}} \ell \cap K \geq \gamma$. □

5. A skew product U on $K \times [0,1)$ and a class of U-invariant measures

In the previous section, we constructed a family of ergodic $\hat{M}$-invariant distributions $\{Q^{(\mu,x,t)}\}_{(\mu,x,t)\in\mathcal{E}_\gamma}$ whose marginals on the first two coordinates are ergodic CP-distributions having dimensions at least γ and supported on measures that are supported on slices of K. In Section 5.1, we will define a skew product on $K \times [0,1)$, which can be regarded as the geometric version of the shift map σ on X_t ($t \in [0,1)$), and we study some partitions generated by U. In Section 5.2, we will construct a family of U-invariant measures such that each of them is a certain form of superposition of measures distributed according to $Q^{(\mu,x,t)}$ with some $(\mu, x, t) \in \mathcal{E}_\gamma$. In Section 5.3, we will study some further properties of such a U-invariant measure.

5.1. *The transformation U and some basic properties.* For each $t \in [0,1)$, we define a map $\Phi_t : K \to K$ by

$$\Phi_t(x,y) = \begin{cases} (S_\alpha(x), S_\beta(y)) & \text{if } t \in [0,\theta), \\ (S_\alpha(x), y) & \text{otherwise.} \end{cases} \tag{5.1}$$

Note that, by the discussion about S_α and S_β at the beginning of Section 4, we have the following result.

LEMMA 5.1. *If ℓ is a line with slope β^{-t} ($t \in [0,1)$) that intersects K, then $\Phi_t(\ell)$ consists of a finite number of lines, each of which has slope $\beta^{-R_\theta(t)}$.*

We consider the following transformation $U : K \times [0,1) \to K \times [0,1)$ defined as a skew product:

$$U(z,t) = (\Phi_t(z), R_\theta(t)).$$

Recall that Φ_t is defined by (5.1) and R_θ is the irrational rotation map defined by $R_\theta(t) = t - \theta \mod 1$.

Let us write $U_t^n(z)$ for the first component of $U^n(z,t)$. Then it follows from the definition of U that we have

$$U_t^n(z) = \Phi_{R_\theta^{n-1}(t)} \circ \cdots \circ \Phi_{R_\theta(t)} \circ \Phi_t(z) = (S_\alpha^n(z_1), S_\beta^{r_n(t)}(z_2)) \text{ for } z = (z_1, z_2),$$

where $r_n(t) := \sharp\{0 \le k \le n-1 : R_\theta^k(t) \in [0,\theta)\}$.

In the following, we define a sequence of refining partitions of $K \times [0,1)$, which is generated by U. First, recall that $\{I_\alpha^i\}_{i=1}^m$ and $\{I_\beta^j\}_{j=1}^l$ are, respectively, partitions of $\bigcup_{i=1}^m \phi_i(\overline{I_\alpha})$ and $\bigcup_{j=1}^l \psi_j(\overline{I_\beta})$; see the beginning of Section 4. We take $\mathcal{C} = \{[0,\theta), [0,1) \setminus [0,\theta)\}$ as a partition of $[0,1)$. Let

$$\mathcal{B}_1 = \{I_\alpha^i \cap C_\alpha\}_{i=1}^m \times \{I_\beta^j \cap C_\beta\}_{j=1}^l \times \mathcal{C} \tag{5.2}$$

be our first level partition of $K \times [0,1)$. Then for $n \ge 2$, let

$$\mathcal{B}_n = \bigvee_{k=0}^{n-1} U^{-k}(\mathcal{B}_1).$$

For later use, let us give some more details about the partitions $\{\mathcal{B}_n\}_n$. For $n \ge 1$, let

$$\mathcal{C}_n = \bigvee_{k=0}^{n-1} R_\theta^{-k}(\mathcal{C}).$$

Recall that the map U_t^k is defined via the relation $U^k(z,t) = (U_t^k(z), R_\theta^k(t))$. For $n \ge 1$ and $t \in [0,1)$, let

$$\mathcal{A}_n^t = \bigvee_{k=0}^{n-1} (U_t^k)^{-1} \left(\{I_\alpha^i \cap C_\alpha\}_{i=1}^m \times \{I_\beta^j \cap C_\beta\}_{j=1}^l \right).$$

Note that by the fact $U_t^n(z) = (S_\alpha^n(z_1), S_\beta^{r_n(t)}(z_2))$, we have

$$\mathcal{A}_n^t = \left(\bigvee_{k=0}^{n-1} S_\alpha^{-k} \left(\{I_\alpha^i \cap C_\alpha\}_{i=1}^m \right) \right) \times \left(\bigvee_{k=0}^{n-1} S_\beta^{-r_k(t)} \left(\{I_\beta^j \cap C_\beta\}_{j=1}^l \right) \right). \tag{5.3}$$

Thus by the definition of $\{r_k(t)\}_k$, we have $\mathcal{A}_n^t = \mathcal{A}_n^{t'}$ if t and t' both belong to a same element of $\mathcal{C}_n$. By the definition of U, it is not hard to check that each element of $\mathcal{B}_n$ has the form $A \times C$ with some $C \in \mathcal{C}_n$ and $A \in \mathcal{A}_n^t$ for some $t \in C$.

As usual, for all $z \in K$, we write $\mathcal{A}_n^t(z)$ for the unique element of $\mathcal{A}_n^t$ containing z. For any measure $\nu \in \mathcal{P}(K)$ and $z \in \operatorname{supp}(\nu)$, we write

$$\nu^{\mathcal{A}_n^t(z)} = U_t^n\left(\frac{\nu|_{\mathcal{A}_n^t(z)}}{\nu(\mathcal{A}_n^t(z))}\right). \tag{5.4}$$

Note that if $\nu \in \mathcal{P}(\ell \cap K)$ for some line ℓ with slope β^{-t}, then $\nu^{\mathcal{A}_n^t(z)} \in \mathcal{P}(\ell' \cap K)$ for some line ℓ' with slope $\beta^{-R_\theta^n(t)}$. Recall also the notation $\mu^{[x_1^n]}$ (see (3.1)). In what follows, the boundary of $\mathcal{A}_n^t$ should be understood as relative to the space K.

Lemma 5.2.

(1) *Let $t \in [0,1)$ and $x \in X_t$. If $\pi_t(x)$ is not at the boundary of $\mathcal{A}_n^t(\pi_t(x))$, then the set $\pi_t([x_1^n])$ coincides with $\mathcal{A}_n^t(\pi_t(x)) \cap K$ except possibly at the boundary points of $\mathcal{A}_n^t(\pi_t(x))$.*

(2) *Let $(\mu, x, t) \in \Xi_{\mathcal{F}}$. If μ is non-atomic, then for μ-a.e. x and $n \geq 1$, we have*

$$\pi_{R_\theta^n(t)}\left(\mu^{[x_1^n]}\right) = (\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))}. \tag{5.5}$$

Proof. The part (1) is clear; we only need to prove (2). By definition, $\pi_t\mu$ is a measure supported on some slice of K with the form $K \cap \ell_{\beta^{-t},z}$ for some $z \in K$. It is clear that, for all $n \geq 1$ and each element A of $\mathcal{A}_n^t$, the support of $\pi_t\mu$ intersects the boundary of A in at most two points. Since μ is non-atomic, it follows that $\pi_t\mu$ gives zero measure to the boundary of A. Thus for μ-a.e. x and $n \geq 1$,

$$\pi_t(\mu|_{[x_1^n]}) = \pi_t\mu|_{\mathcal{A}_n^t(\pi_t(x))}.$$

Note that for $t \in [0,1)$ and $x \in X_t$, we have

$$U^n(\pi_t(x), t) = (U_t^n(\pi_t(x)), R_\theta^n(t)) = (\pi_{R_\theta^n(t)}(\sigma^n(x)), R_\theta^n(t)).$$

Combining the above conclusions, we obtain (5.5). □

5.2. *Construction of a class of U-invariant measures.* This subsection is devoted to the construction of a class of U-invariant measures. We will first define these measures and then show that they are U-invariant.

Let Q be the $\hat{M}$-invariant distribution constructed in Section 4.2. Recall that $Q = \int Q^{(\mu,x,t)} dQ(\mu, x, t)$ is the ergodic decomposition of Q. By the ergodic theorem, for Q-a.e. (μ, x, t), the triple (μ, x, t) generates $Q^{(\mu,x,t)}$ in the sense that

$$\frac{1}{N}\sum_{n=0}^{N-1} \delta_{\hat{M}^n(\mu,x,t)} \to Q^{(\mu,x,t)} \quad \text{as } N \to \infty \tag{5.6}$$

in the weak-* topology. Consider the map $G : \Xi_{\mathcal{F}} \to \mathcal{P}(K \times [0,1))$ defined by

$$G(\mu, x, t) = \pi_t\mu \times \delta_t.$$

Then G is continuous. It follows from (5.6) that for Q-a.e. (μ, x, t),

$$\frac{1}{N}\sum_{n=0}^{N-1} G(\hat{M}^n(\mu, x, t)) \to \int G dQ^{(\mu,x,t)} \text{ as } N \to \infty. \tag{5.7}$$

Recall that by the definition of $\hat{M}$, we have

$$\hat{M}^n(\mu, x, t) = \left(\mu^{[x_1^n]}, \sigma^n(x), R_\theta^n(t)\right). \tag{5.8}$$

We use $Q_{1,3}$ and $Q_{1,3}^{(\mu,x,t)}$ to denote, respectively, the marginals of Q and $Q^{(\mu,x,t)}$ on the first and third coordinates. Recall the definition of $\mathcal{E}_\gamma$ (see (4.3)).

Lemma 5.3. *For $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x with $(\mu, x, t) \in \mathcal{E}_\gamma$, we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} (\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))} \times \delta_{R_\theta^n(t)} \to \nu^{(\mu,x,t)} := \int \pi_s\vartheta \times \delta_s \, dQ_{1,3}^{(\mu,x,t)}(\vartheta, s) \text{ as } N \to \infty \tag{5.9}$$

in the weak- topology.*

Proof. First, we claim that for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$, the measure μ is non-atomic. To see this, recall that for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$, the triple (μ, x, t) generates $Q^{(\mu,x,t)}$, and the marginal $Q_{1,2}^{(\mu,x,t)}$ is an ergodic CP-distribution with positive dimension. Let us fix any such $(\mu, x, t) \in \mathcal{E}_\gamma$. Then (μ, x) generates the marginal $Q_{1,2}^{(\mu,x,t)}$, and it follows from Proposition 3.7, (3.3) (and Remark 3.8) that μ is non-atomic.

Now, combining (5.7), (5.8), (5.4) and part (2) of Lemma 5.2, we get (5.9) for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$. Since Q is globally adapted, we deduce that (5.9) holds for $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x such that $(\mu, x, t) \in \mathcal{E}_\gamma$. □

We saw in the above proof that formula (5.9) actually holds for $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x with $H(Q^{(\mu,x,t)}) > 0$, but we will not use this fact.

The rest of this subsection is devoted to the proof of the following.

Proposition 5.4. *For $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x with $(\mu, x, t) \in \mathcal{E}_\gamma$, the measure $\nu^{(\mu,x,t)}$ is U-invariant.*

Our idea for the proof of Proposition 5.4 is inspired by [22, Th. 2.1] where it is shown, for a Borel map T of a compact metric space X, how to relate the small-scale structure of a measure $\upsilon \in \mathcal{P}(X)$ to the distribution of T-orbits of υ-typical points.

The proof of Proposition 5.4 relies on three lemmas. For any $(z,t) \in K \times [0,1)$, we define a sequence of measures

$$\eta_N(z,t) = \frac{1}{N}\sum_{n=0}^{N-1} \delta_{U^n(z,t)}, \quad N \geq 1.$$

The first lemma shows that, for a given measure $\upsilon \in \mathcal{P}(K)$, when restricted on the elements of $\mathcal{B}_k, k \geq 1$, the measures $\eta_N(z,t)$ and the Cesàro averages of $\upsilon^{\mathcal{A}_n^t(z)} \times \delta_{R_\theta^n(t)}$ are asymptotically the same for υ-a.e. z.

Lemma 5.5. *Let $\upsilon \in \mathcal{P}(K)$. For any $t \in [0,1)$, $k \geq 1$ and each $B \in \mathcal{B}_k$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \left(1_B(U^n(z,t)) - \upsilon^{\mathcal{A}_n^t(z)} \times \delta_{R_\theta^n(t)}(B)\right) = 0 \ \textit{ for } \upsilon\textit{-a.e. } z.$$

Proof. Fix $k \geq 1$. Let $B \in \mathcal{B}_k$. Recall that we can write $B = A \times C$ with some $C \in \mathcal{C}_k$ and $A \in \mathcal{A}_k^{t'}$ for some $t' \in C$. Then $1_B(U^n(z,t)) = 1_A(U_t^n(z))1_C(R_\theta^n(t))$ and $\upsilon^{\mathcal{A}_n^t(z)} \times \delta_{R_\theta^n(t)}(B) = \upsilon^{\mathcal{A}_n^t(z)}(A)1_C(R_\theta^n(t))$. Observe that by the definition of $\upsilon^{\mathcal{A}_n^t(z)}$, we have

$$\upsilon^{\mathcal{A}_n^t(z)}(A) = \mathbb{E}_\upsilon(1_A \circ U_t^n | \mathcal{A}_n^t)(z).$$

Let $f_n(z) = \mathbb{E}_\upsilon(1_A \circ U_t^n|\mathcal{A}_n^t)(z)1_C(R_\theta^n(t)) - 1_A(U_t^n(z))1_C(R_\theta^n(t))$. Note that f_n is bounded uniformly in n. We only need to prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f_n(z) = 0$$

for υ-a.e. z. For this, it is sufficient to show that for certain $k' \geq 1$ and each $p = 0, \ldots, k'-1$, we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f_{nk'+p}(z) = 0$ for υ-a.e. z.

Now, for each $n \geq 1$, we have $\mathbb{E}_\upsilon(f_n|\mathcal{A}_n^t) = 0$. By the definition of the partitions $\{\mathcal{A}_n^t\}_n$ (see (5.3)), it is clear that there exists $k'' \in \mathbb{N}$ such that for all $s, s' \in [0,1)$ and all $n \geq 1$, $\mathcal{A}_{n+k''}^s$ refines $\mathcal{A}_n^{s'}$. Because of this and since $A \in \mathcal{A}_k^{t'}$, the map $1_A \circ U_t^n$ is $\mathcal{A}_{n+k'}^t$-measurable for $k' = k + k''$. Thus $\{f_{nk'+p}\}_n$ is a sequence of bounded martingale differences for the filtration $\{\mathcal{A}_{nk'+p}^t\}_n$, from which we deduce that their Cesàro averages converge to 0 for υ-a.e. z; see [10, Th. 3 in Ch. VII.9]. □

Lemma 5.6. *For $Q_{1,3}$-a.e. (μ,t) and μ-a.e. x with $(\mu,x,t) \in \mathcal{E}_\gamma$, we have that for any $k \geq 1$ and each $B \in \mathcal{B}_k$,*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} (\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))} \times \delta_{R_\theta^n(t)}((\partial B)^{(\varepsilon)}) = o(1) \quad \textit{as } \varepsilon \to 0, \tag{5.10}$$

where $E^{(\varepsilon)}$ denotes the ε-neighborhood of a set E.

Proof. By the global adaptedness of Q, we only need to show (5.10) for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$.

Fix $k \geq 1$, and let $B \in \mathcal{B}_k$. Recall that $B = A \times C$ with $C \in \mathcal{C}_k$ and $A \in \mathcal{A}_k^t$ for some $t \in C$. Observe that we have

$$(\partial B)^{(\varepsilon)} \subset \Big(K \times (\partial C)^{(\varepsilon)}\Big) \bigcup \Big((\partial A)^{(\varepsilon)} \times [0,1)\Big).$$

Thus it is sufficient to show that for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{R_\theta^n(t)}((\partial C)^{(\varepsilon)}) = o(1) \text{ as } \varepsilon \to 0 \tag{5.11}$$

and

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\pi_t\mu)^{\mathcal{A}_n^t(z)}((\partial A)^{(\varepsilon)}) = o(1) \text{ as } \varepsilon \to 0. \tag{5.12}$$

The statement (5.11) is clearly true. Actually, since θ is irrational, for any $t \in [0,1)$, the limsup in (5.11) is a limit and it is bounded by the Lebesgue measure of $(\partial C)^{(\varepsilon)}$ that is $o(1)$ when $\varepsilon \to 0$.

Now, let us prove (5.12). In the proof of Lemma 5.3, we have seen that for Q-a.e. $(\mu, x, t) \in \mathcal{E}_\gamma$, (μ, x) generates an ergodic CP-distribution $Q_{1,2}^{(\mu,x,t)}$ with positive dimension. It follows from Proposition 3.7, (3.3) (and Remark 3.8) that for any $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{1 \leq k \leq N : \max_{u \in \Lambda^{n_0(\varepsilon)}} \mu^{[x_1^k]}([u]) \leq \varepsilon \right\} > 1 - \varepsilon. \tag{5.13}$$

Now, recalling $\pi_{R_\theta^k(t)}(\mu^{[x_1^k]}) = (\pi_t\mu)^{\mathcal{A}_k^t(\pi_t(x))}$ and using part (3) of Lemma 4.1, we deduce that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{1 \leq k \leq N : \sup_{y \in K} (\pi_t\mu)^{\mathcal{A}_k^t(\pi_t(x))}(B(y, \delta(\varepsilon))) \leq \varepsilon \right\} > 1 - \varepsilon. \tag{5.14}$$

By definition, all elements in $\mathcal{A}_n^t$ have uniformly bounded eccentricities[1] (less than $1/\beta$). On the other hand, the measure $(\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))}$ is supported on some slice of K with slope between 1 and $1/\beta$. Hence there exists an absolute constant depending only on β such that for any $A \in \mathcal{A}_n^t$, the intersection of the support of $(\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))}$ with $(\partial A)^{(\varepsilon)}$ is included in two balls of diameter less than ε times this constant. Combining this fact with (5.14), we get (5.12). □

[1]The eccentricity of a rectangle is the ratio of the lengths of the longest and shortest side. Here we are actually referring to the eccentricity of the convex hull of $\mathcal{A}_n^t$ but not itself, since $\mathcal{A}_n^t$ is in general a Cantor set.

The following lemma says that the measures $\eta_N(z,t)$ and the Cesàro averages of $(\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))} \times \delta_{R_\theta^n(t)}$ are asymptotically the same for typical (μ, x, t) in $\mathcal{E}_\gamma$.

LEMMA 5.7. *For $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x with $(\mu, x, t) \in \mathcal{E}_\gamma$, we have*

$$\eta_N(\pi_t(x), t) \to \nu^{(\mu,x,t)} \quad \text{as } N \to \infty$$

in the weak- topology.*

Proof. By the definition of $\{\mathcal{B}_n\}_n$, it is clear that $\max_{B\in\mathcal{B}_n} \operatorname{diam}(B) \to 0$ as $n \to \infty$. So the partitions $\{\mathcal{B}_n\}_n$ generate the Borel σ-algebra of $K \times [0, 1)$. Now by this fact and Lemma 5.5, it is well known that for proving Lemma 5.7 we only need to show the following: for $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x with $(\mu, x, t) \in \mathcal{E}_\gamma$, whenever $\eta_{N_k}(\pi_t(x), t) \to \upsilon$ along some $N_k \to \infty$, then $\upsilon(\partial B) = 0$ for each $B \in \mathcal{B}_n$ and all $n \geq 1$. For this, we use Lemma 5.6. Fix any $n_0 \geq 1$ and $B \in \mathcal{B}_{n_0}$. For any $\varepsilon > 0$, let $f_\varepsilon \in C(K \times [0, 1))$ be such that $1_{\partial B} \leq f_\varepsilon \leq 1_{(\partial B)^{(\varepsilon)}}$. Since $\max_{B\in\mathcal{B}_k} \operatorname{diam}(B) \to 0$ as $k \to \infty$, for n large enough we can find a finite family $\{B_i\} \subset \mathcal{B}_n$ such that $(\partial B)^{(\varepsilon)} \subset \cup_i B_i \subset (\partial B)^{(2\varepsilon)}$. Now if $\eta_{N_k}(\pi_t(x), t) \to \upsilon$, then by Lemma 5.5 and Lemma 5.6, we have

$$\begin{aligned}
\int f_\varepsilon d\upsilon &= \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} f_\varepsilon(U^n(\pi_t(x), t)) \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))} \times \delta_{R_\theta^n(t)}(\cup_i B_i) \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\pi_t\mu)^{\mathcal{A}_n^t(\pi_t(x))} \times \delta_{R_\theta^n(t)}((\partial B)^{(2\varepsilon)}) \\
&= o(1) \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

This implies that $\upsilon(\partial B) = 0$. □

We are now ready to prove Proposition 5.4.

Proof of Proposition 5.4. By Lemma 5.7, for $Q_{1,3}$-a.e. (μ, t) and μ-a.e. x with $(\mu, x, t) \in \mathcal{E}_\gamma$, $\nu^{(\mu,x,t)}$ is a measure according to which certain orbit $\{U^n(z, t)\}_n$ equidistributes. Thus for proving the U-invariance of $\nu^{(\mu,x,t)}$, we only need to show that it gives zero measure to the set of discontinuities of U. This is an immediate consequence of the fact that the discontinuities of U are contained in the set $\bigcup_{B\in\mathcal{B}_1} \partial B$, since in the proof of Lemma 5.7 we have shown that $\nu^{(\mu,x,t)}$ gives zero measure to this set. □

5.3. *Further properties of a U-invariant measure ν_∞.* From now on, let us fix an element $(\mu_0, x_0, t_0) \in \mathcal{E}_\gamma$ such that $Q_{1,2}^{(\mu_0,x_0,t_0)}$ is an ergodic CP-distribution with dimension $\geq \gamma$ and the measure

$$\nu_\infty := \nu^{(\mu_0,x_0,t_0)} = \int \pi_s\mu \times \delta_t \, dQ_{1,3}^{(\mu_0,x_0,t_0)}(\mu, t)$$

is U-invariant.

Applying Proposition 3.7 to the ergodic CP-distribution $Q_{1,2}^{(\mu_0,x_0,t_0)}$ we get that for any $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for $Q_1^{(\mu_0,x_0,t_0)}$-a.e. μ and μ-a.e. x,

(5.15)

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \Big\{ 1 \leq k \leq N : \max_{u\in\Lambda^{n_0(\varepsilon)}} \mu^{[x_1^k]}([u]) \leq \varepsilon \text{ and } H(\mu^{[x_1^k]}, \mathcal{F}_n) \geq n(\gamma \log \alpha^{-1} - \varepsilon) \Big\} > 1 - 2\varepsilon \quad \text{for all } n \geq n_0(\varepsilon).$$

Here we use $Q_1^{(\mu_0,x_0,t_0)}$ to denote the measure marginal of $Q_{1,2}^{(\mu_0,x_0,t_0)}$. Now, using part (3) of Lemma 4.1, we deduce that for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $n_1(\varepsilon) \in \mathbb{N}$ such that for $Q_1^{(\mu_0,x_0,t_0)}$-a.e. μ and $\pi_t\mu$-a.e. z,

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \Big\{ 1 \leq k \leq N : \sup_{y\in K} (\pi_t\mu)^{\mathcal{A}_k^t(z)}(B(y, \delta(\varepsilon))) \leq \varepsilon \text{ and } H((\pi_t\mu)^{\mathcal{A}_k^t(z)}, \mathcal{D}_n) \geq n(\gamma \log 2 - 2\varepsilon) \Big\} > 1 - 2\varepsilon \quad \text{for all } n \geq n_1(\varepsilon). \tag{5.16}$$

In particular, the above property also holds for $Q_{1,3}^{(\mu_0,x_0,t_0)}$-a.e. (μ, t) and $\pi_t\mu$-a.e. z. On the other hand, since the measure ν_∞ has the form $\int \pi_t\mu \times \delta_t \, dQ_{1,3}^{(\mu_0,x_0,t_0)}(\mu, t)$, selecting a pair (z, t) according to ν_∞ can be done by first selecting a pair (μ, t) according to $Q_{1,3}^{(\mu_0,x_0,t_0)}$ and then selecting a point z according to $\pi_t\mu$.

It follows from the above discussions that we have

Proposition 5.8. *The measure ν_∞ satisfies the following property*:

(5.17) *For any $\varepsilon > 0$, there are $\delta(\varepsilon) > 0$ and $n_1(\varepsilon)$ such that for ν_∞-a.e. (z, t), we can find $\mu \in \mathcal{P}(X)$ such that $\pi_t\mu \in \mathcal{P}(\ell \cap K)$ for some line ℓ with slope β^{-t} and (5.16) holds for $\pi_t\mu$ and z.*

In particular, almost every ergodic component of ν_∞ still satisfies this property.

In the rest of this paper, we choose an ergodic component of ν_∞ that satisfies property (5.17) and still denote it by ν_∞. We have thus proved the following:

THEOREM 5.9. *There exists a U-invariant ergodic measure ν_∞ that satisfies property (5.17).*

6. An ergodic theoretic result

This section is devoted to the proof of the following theorem in ergodic theory. Recall that a sequence $\{x_k\}_{k\in\mathbb{N}} \in [0,1)$ is called *uniformly distributed* (UD) if for any sub-interval J of $[0,1)$, we have

$$\lim_{N\to\infty} N^{-1}\sharp\{0 \leq k \leq N-1 : x_k \in J\} = \mathcal{L}(J).$$

THEOREM 6.1. *Let (X,T,μ) be an ergodic dynamical system. Let $\mathcal{A}$ be a generator with finite cardinality, and let $\{\mathcal{A}_n\}_n$ be the filtration generated by $\mathcal{A}$ with respect to T (see Section 2.4.2). Suppose that $\mu(\partial A) = 0$ for each $A \in \mathcal{A}_n$ and all $n \geq 1$. Let ξ be an irrational number. For any $\varepsilon > 0$, there exists $n_2 = n_2(\varepsilon) \in \mathbb{N}$ such that for each $n \geq n_2$, we can find a disjoint family $\{C_i\}_{i=1}^{N(n,\varepsilon)}$ of measurable subsets $C_i \subset X$ satisfying the following properties:*

(1) *We have $\mu\left(\bigcup_i C_i\right) \geq 1-\varepsilon$.*
(2) *For each $1 \leq i \leq N(n,\varepsilon)$, we have $\sharp\{A \in \mathcal{A}_n : C_i \bigcap A \neq \emptyset\} \leq e^{n\varepsilon}$.*
(3) *There exists another disjoint family $\{\widetilde{C}_i\}_{i=1}^{N(n,\varepsilon)}$ of measurable subsets $\widetilde{C}_i \subset X$ such that for each $1 \leq i \leq N(n,\varepsilon)$, we have $C_i \subset \widetilde{C}_i$ and $\mu(C_i) \geq (1-\varepsilon)\mu(\widetilde{C}_i)$, and moreover, for μ-a.e. x the sequence*

$$\left\{R_\xi^k(0) \in [0,1) : k \in \mathbb{N} \text{ and } T^k(x) \in \widetilde{C}_i\right\}$$

is UD. *Here R_ξ is the irrational rotation map defined by $R_\xi(t) = t - \xi \mod 1$.*

Remark 6.2. The conclusion of the theorem holds without the condition that the generator $\mathcal{A}$ has finite cardinality, but we will not use this fact. Assuming the condition on $\mathcal{A}$ will make the proof shorter.

We will use Sinai's factor theorem in the proof of Theorem 6.1.

THEOREM 6.3 (Sinai's factor theorem). *Let (X,T,μ) be an ergodic dynamical system. Then any Bernoulli shift $(\Sigma^{\mathbb{N}},\sigma,\nu)$ with $h(\nu,\sigma) \leq h(\mu,T)$ is a factor of (X,T,μ).*

For definitions of Bernoulli shift and factor, see Section 2.4. In the above version of Sinai's factor theorem, we include the case when $h(\mu,T) = 0$ — in this case the theorem is obviously true since every Bernoulli shift with zero entropy is a trivial one-point system (the product measure ν is a Dirac measure at a fixed point) that is trivially a factor of (X,T,μ). The original version of Sinai's factor theorem [36], [37] was stated for invertible systems, but it also implicitly applies to non-invertible ones. (For the proof, see also [28].)

For the rest of this section, we fix an ergodic dynamical system (X, T, μ) satisfying the hypothesis of Theorem 6.1 and let $(\Sigma^{\mathbb{N}}, \sigma, \nu)$ be a Bernoulli shift with $h(\nu, \sigma) = h(\mu, T)$.

It follows from Sinai's factor theorem that there exists a factor map $\pi : X \to \Sigma^{\mathbb{N}}$ such that

$$\pi \circ T = \sigma \circ \pi \quad \text{and} \quad \nu = \pi\mu.$$

By Rohlin's disintegration theorem, there exists a system of conditional measures $(\mu_y)_{y\in\Sigma^{\mathbb{N}}}$ of μ with respect to π satisfying the following properties:

(1) for ν-a.e. y, μ_y is a Borel probability measure supported on $\pi^{-1}(y)$;
(2) for every μ-measurable $B \subset X$, the map $y \mapsto \mu_y(B)$ is ν-measurable and

$$\mu(B) = \int_{\Sigma^{\mathbb{N}}} \mu_y(B) d\nu(y);$$

(3) moreover for ν-a.e. y, the measure μ_y can be obtained as the weak-* limit of $\lim_{r\to 0} \mu_{\pi^{-1}(B(y,r))}$, where $\mu_{\pi^{-1}(B(y,r))}$ is defined by

$$\mu_{\pi^{-1}(B(y,r))}(A) = \frac{\mu\left(\pi^{-1}(B(y,r)) \bigcap A\right)}{\mu\left(\pi^{-1}(B(y,r))\right)}.$$

For a proof of the above version of Rohlin's disintegration theorem, see [35].

The proof of Theorem 6.1 relies on two lemmas. Recall that $\{\mathcal{A}_n\}_n$ is the filtration associated to the generator $\mathcal{A}$ and, for $x \in X$, $\mathcal{A}_n(x)$ is the unique element of $\mathcal{A}_n$ containing x.

LEMMA 6.4. *Suppose that μ satisfies the hypothesis of Theorem 6.1. Let ν and $(\mu_y)_{y\in\Sigma^{\mathbb{N}}}$ be as above. For any $\delta > 0$, we have*

(i) *there exist a measurable set $A_\delta \subset X$ with $\mu(A_\delta) > 1 - \delta$ and $n' \in \mathbb{N}$ such that for each $x \in A_\delta$,*

$$\mu_{\pi(x)}(\mathcal{A}_n(x)) \geq e^{-n\delta} \quad \text{for all } n \geq n'; \tag{6.1}$$

(ii) *for any $n \geq 1$, there exist a measurable set $B_\delta^n \subset \Sigma^{\mathbb{N}}$ with $\nu(B_\delta^n) > 1 - \delta$ and $r = r(\delta, n) > 0$ such that for each $y \in B_\delta^n$ and each $A \in \mathcal{A}_n$, we have*

$$\frac{\mu\left(\pi^{-1}(B(y,r)) \bigcap A\right)}{\mu\left(\pi^{-1}(B(y,r))\right)} \geq (1-\delta)\mu_y(A). \tag{6.2}$$

Proof. (i) Since $(\Sigma^{\mathbb{N}}, \sigma, \nu)$ is a factor of (X, T, μ) with $h(\nu, \sigma) = h(\mu, T)$, it follows from the conditional Shannon-McMillan-Breiman Theorem [3, Th. 3.3.7] that for μ-a.e. x,

$$\lim_{n\to\infty} \frac{\log \mu_{\pi(x)}(\mathcal{A}_n(x))}{-n} = 0.$$

By Egorov's theorem, there exist a measurable set $A_\delta \subset X$ with $\mu(A_\delta) > 1-\delta$ and $n' \in \mathbb{N}$ such that for each $x \in A_\delta$,

$$\frac{\log \mu_{\pi(x)}(\mathcal{A}_n(x))}{-n} \leq \delta \quad \text{for all } n \geq n'.$$

This is exactly (6.1).

(ii) Fix any $n \geq 1$. By hypothesis, $\mu(\partial A) = 0$ for all $A \in \mathcal{A}_n$. The same holds for μ_y for ν-a.e. y. Recall that by Rohlin's disintegration theorem, for ν-a.e. y, μ_y is the weak-* limit of $\mu_{\pi^{-1}(B(y,r))}$ as $r \to 0$. Thus, by Portmanteau's theorem, we deduce that for ν-a.e. y and for all $A \in \mathcal{A}_n$,

$$\lim_{r\to\infty} \frac{\mu\left(\pi^{-1}(B(y,r)) \bigcap A\right)}{\mu\left(\pi^{-1}(B(y,r))\right)} = \mu_y(A).$$

Then we can again apply Egorov's theorem to obtain a measurable set $B_\delta^n \subset \Sigma^{\mathbb{N}}$ with $\nu(B_\delta^n) > 1-\delta$ and $r = r(\delta, n) > 0$ such that for each $y \in B_\delta^n$ and each $A \in \mathcal{A}_n$, we have (6.2). □

The following result is an easy consequence of the mixing property of the Bernoulli shift $(\Sigma^{\mathbb{N}}, \sigma, \nu)$.

LEMMA 6.5. *For any measurable set $B \subset \Sigma^{\mathbb{N}}$ with $\nu(B) > 0$, the sequence*

$$\left\{R_\xi^k(0) : k \in \mathbb{N} \quad \text{and } T^k(x) \in \pi^{-1}(B)\right\}$$

is UD *for μ-a.e. $x \in X$.*

Proof. Since the Bernoulli shift $(\Sigma^{\mathbb{N}}, \sigma, \nu)$ is weak-mixing, for any irrational rotation system $([0,1), R_\xi, \mathcal{L})$, the product system

$$(\Sigma^{\mathbb{N}} \times [0,1), \sigma \times R_\xi, \nu \times \mathcal{L})$$

is ergodic (see Section 2.4.1). We claim that if $B \subset \Sigma^{\mathbb{N}}$ is measurable with $\nu(B) > 0$, then the set

$$\left\{R_\xi^k(0) : k \in \mathbb{N} \quad \text{and } \sigma^k(y) \in B\right\}$$

is UD for ν-a.e. $y \in \Sigma^{\mathbb{N}}$. To see this, note that by the ergodic theorem, for ν-a.e. y and $\mathcal{L}$-a.e. t, the sequence $\{x_n(y,t)\}_n := \{R_\xi^k(t) : k \in \mathbb{N} \text{ and } \sigma^k(y) \in B\}$ satisfies $\lim_{N\to\infty} N^{-1}\sharp\{1 \leq n \leq N : x_n(y,t) \in J\} = \mathcal{L}(J)$ for each dyadic interval $J \in \mathcal{D}_k([0,1))$, $k \geq 1$. This clearly implies that the sequence $\{x_n(y,t)\}_n$ is UD. Since $R_\xi^k(t) = R_\xi^k(0) + t$ in $[0,1)$, we deduce that $\{x_n(y,0)\}_n$ is UD for ν-a.e. y, as claimed.

On the other hand, since $(\Sigma^{\mathbb{N}}, \sigma, \nu)$ is a factor of (X, T, μ) with factor map π, we have for μ-a.e. $x \in X$,

$$\{k \in \mathbb{N} : T^k(x) \in \pi^{-1}(B)\} = \{k \in \mathbb{N} : \sigma^k(\pi(x)) \in B\}.$$

Combining this with the above claim, we get the desired result. □

Proof of Theorem 6.1. Fix $\varepsilon > 0$. Let $\delta > 0$ be a small constant which we will choose later. Let A_δ and $n' =: n_2$ be the set and the number provided by Lemma 6.4(i). Then we have

$$\int_{\Sigma^{\mathbb{N}}} \mu_y(A_\delta) d\nu(y) = \mu(A_\delta) > 1 - \delta.$$

From this, we deduce that there exists $\delta_1 > 0$, with $\delta_1 = o(1)$ when $\delta \to 0$, so that the following holds: we can find a measurable set $B_1 \subset \Sigma^{\mathbb{N}}$ with $\nu(B_1) > 1 - \delta_1$ such that for each $y \in B_1$, we have $\mu_y(A_\delta) > 1 - \delta_1$. For instance, we can take $\delta_1 = \sqrt{\delta}$.

Fix any $n \geq n_2$. Let B_δ^n and r be the measurable set and the number provided by Lemma 6.4(ii). Note that we have $\nu(B_\delta^n) > 1 - \delta$. Let $B_2 = B_1 \cap B_\delta^n$. Then we have $\nu(B_2) > 1 - \delta - \delta_1$. For each $y \in B_2$, let

$$E(y, n) = \left\{ A \in \mathcal{A}_n : \pi^{-1}(y) \cap A_\delta \cap A \neq \emptyset \right\}.$$

By the definition of A_δ, if $x \in A_\delta$, then $\mu_{\pi(x)}(\mathcal{A}_n(x)) \geq e^{-n\delta}$. It follows that for each $A \in E(y, n)$, we have $\mu_y(A) \geq e^{-n\delta}$. Since μ_y is a probability measure, we deduce that $\sharp(E(y, n)) \leq e^{n\delta}$ for each $y \in B_2$.

Now, let us consider the following collection of balls of $\Sigma^{\mathbb{N}}$:

$$\left\{ B(y, r) \subset \Sigma^{\mathbb{N}} : y \in B_2 \text{ and } \nu(B(y, r)) > 0 \right\}.$$

Since we use an ultra-metric in $\Sigma^{\mathbb{N}}$, the above collection is actually finite. Let us enumerate its elements by $\{B_i\}_{i=1}^{N(n)}$. Note that B_i's are disjoint balls. For each $1 \leq i \leq N(n)$, let us define

$$\widetilde{C}_i = \pi^{-1}(B_i) \quad \text{and} \quad C_i = \pi^{-1}(B_i) \bigcap \left(\bigcup_{A \in E(y,n)} A \right),$$

where y is some point in B_2 such that $B(y, r) = B_i$. Now we can make our choice of δ. In the following we fix δ small enough such that

$$\delta \leq \varepsilon \text{ and } (1 - \delta - \delta_1)(1 - \delta)(1 - \delta_1) \geq 1 - \varepsilon.$$

Let $N(n, \varepsilon) := N(n)$. We claim that the families $\{C_i\}_{i=1}^{N(n,\varepsilon)}$ and $\{\widetilde{C}_i\}_{i=1}^{N(n,\varepsilon)}$ satisfy properties (1), (2) and (3) in Theorem 6.1.

We first verify property (2). We have seen that $\sharp(E(y, n)) \leq e^{n\delta}$ for each $y \in B_2$. By the definition of C_i and the assumption $\delta \leq \varepsilon$, this clearly implies property (2).

Now, we verify properties (1) and (3). Observe that $\mathcal{A}_n$ is a partition of X, thus by definition of $E(y, n)$ we have for $y \in B_2$,

$$\pi^{-1}(y) \cap A_\delta \subset \bigcup_{A \in E(y,n)} A.$$

Note that by the choice of B_1, we have

$$\mu_y\left(\pi^{-1}(y)\cap A_\delta\right)=\mu_y(A_\delta)>1-\delta_1$$

for each $y\in B_1$. From these two facts, we deduce that if $y\in B_2\subset B_1$, then

$$\mu_y\left(\bigcup_{A\in E(y,n)} A\right)\geq 1-\delta_1. \tag{6.3}$$

On the other hand, recall that each $y\in B_\delta^n$ satisfies (6.2) for all $A\in\mathcal{A}_n$. Using this and the fact $B_2\subset B_\delta^n$, we deduce from from (6.3) that for each $y\in B_2$, we have

$$\mu\left(\pi^{-1}(B(y,r))\bigcap\left(\bigcup_{A\in E(y,n)} A\right)\right)\geq(1-\delta)(1-\delta_1)\mu\left(\pi^{-1}(B(y,r))\right).$$

Combining this with the definitions of C_i and $\widetilde{C}_i$ and the choice of δ, we get

$$\mu(C_i)\geq(1-\delta)(1-\delta_1)\mu(\widetilde{C}_i)\geq(1-\varepsilon)\mu(\widetilde{C}_i)$$

for each $1\leq i\leq N(n,\varepsilon)$. Note also that

$$\mu\left(\cup_i\widetilde{C}_i\right)=\mu\left(\cup_i\pi^{-1}(B_i)\right)=\nu\left(\cup_i B_i\right)\geq\nu(B_2)\geq 1-\delta-\delta_1.$$

Thus again by the choice of δ, we obtain

$$\mu\left(\cup_i C_i\right)\geq(1-\delta)(1-\delta_1)\mu\left(\cup_i\widetilde{C}_i\right)\geq(1-\delta-\delta_1)(1-\delta)(1-\delta_1)\geq 1-\varepsilon.$$

It remains to show that the sequence

$$\left\{R_\xi^k(0)\in[0,1):k\in\mathbb{N}\text{ and }T^k(x)\in\widetilde{C}_i\right\}$$

is UD on $[0,1)$. This is implied by Lemma 6.5. □

7. Proof of Theorem 1.3

The following result is essential for proving Theorem 1.3. It is a consequence of property (5.17) of ν_∞ and an application of Theorem 6.1 to the system $(K\times[0,1),U,\nu_\infty)$. Recall that Π_1 is the projection from $K\times[0,1)$ to K and $N_{2^{-n}}(A)$ denotes the number of n-level dyadic cubes intersecting a set A.

Proposition 7.1. *For any $\varepsilon>0$, there exist $r_0=r_0(\varepsilon)>0$ and $n_3=n_3(\varepsilon)\in\mathbb{N}$ such that for each $n\geq n_3$, the following is true: for ν_∞-a.e. (z,t), we can find a measure $\nu\in\mathcal{P}(K)$, a measurable set $D\subset K\times[0,1)$ and a subset $\mathcal{N}\subset\mathbb{N}$ satisfying the following properties:*

(1) *the measure $\nu\in\mathcal{P}(\ell\cap K)$ for some line ℓ with slope β^{-t};*
(2) *$n^{-1}\log N_{2^{-n}}(\Pi_1(D))\leq\varepsilon$;*
(3) *for each $k\in\mathcal{N}$, $U^k(z,t)\in D$;*
(4) *$\mathcal{L}\left(\overline{\{R_\theta^k(t):k\in\mathcal{N}\}}\right)\geq 1-\varepsilon$, where $\mathcal{L}$ denotes the normalized Lebesgue measure on $[0,1)$ (i.e., $\mathcal{L}([0,1))=1$);*

(5) *for each* $k \in \mathcal{N}$,

$$\inf_{y\in K} \frac{1}{n\log 2} H\left(\nu^{\mathcal{A}_k^t(z)}|_{B(y,r_0)^c}, \mathcal{D}_n\right) \geq \gamma - \varepsilon^{\frac{1}{2}}.$$

Recall that $\nu^{\mathcal{A}_k^t(z)}$ is defined as (5.4), and it is supported on some slice $\ell' \cap K$ with slope $\beta^{-R_\theta^k(t)}$. Recall also that $\eta|_E$ denotes the restriction of a measure η on E; see Section 2.3.2 for the definition of entropy.

For the proof of Proposition 7.1, we need two elementary lemmas. For $F_1 \subset F_2 \subset \mathbb{N}$, we define the *upper density* of F_1 in F_2, denoted $\overline{d}(F_1, F_2)$, as

$$\overline{d}(F_1, F_2) = \limsup_{N\to\infty} \frac{\sharp\{F_1 \cap [0, N-1]\}}{\sharp\{F_2 \cap [0, N-1]\}}.$$

Similarly, we define the *lower density* $\underline{d}(F_1, F_2)$ of F_1 in F_2. If $\overline{d}(F_1, F_2) = \underline{d}(F_1, F_2)$, then we say the density of F_1 in F_2 exists and denote it by $d(F_1, F_2)$.

LEMMA 7.2. *Let* $\{x_k\}_{k\in\mathbb{N}} \subset [0, 1)$ *be a sequence that is* UD. *Suppose that* $F \subset \mathbb{N}$. *Then*

$$\mathcal{L}\left(\overline{\{x_k : k \in F\}}\right) \geq \overline{d}(F, \mathbb{N}).$$

Proof. Let $E = \overline{\{x_k : k \in F\}}$. If $\mathcal{L}(E^c) > 0$, then for any $\varepsilon > 0$, we can find finitely many intervals $\{J_i\}_i \subset E^c$ such that $\mathcal{L}(\cup_i J_i) > \mathcal{L}(E^c) - \varepsilon$. Now since $\{x_k\}_{k\in\mathbb{N}}$ is UD, we have

$$\begin{aligned}\mathcal{L}(\cup_i J_i) &= \lim_{N\to\infty} N^{-1}\sharp\{1 \leq k \leq N : x_k \in \cup_i J_i\} \\ &= 1 - \lim_{N\to\infty} N^{-1}\sharp\{1 \leq k \leq N : x_k \notin \cup_i J_i\} \leq 1 - \overline{d}(F).\end{aligned}$$ □

LEMMA 7.3. *Let* $\eta \in \mathcal{P}(\mathbb{R}^d)$ *and* $0 < \delta < 1$. *If* $\sup_{y\in\mathbb{R}^d} \eta(B(y, \delta)) \leq \varepsilon$, *then for* $n \in \mathbb{N}$ *with* $2^{-n} \leq \delta$, *we have*

$$\inf_{y\in\mathbb{R}^d} H(\eta|_{B(y,\delta)^c}, \mathcal{D}_n) \geq H(\eta, \mathcal{D}_n) - C_1 n\varepsilon^{\frac{1}{2}}$$

for some constant C_1 *depending only on* d.

Proof. We will use the elementary fact that if μ is a finite (not necessarily probability) measure on a metric space X, then for any finite partition $\mathcal{A} = \{A_i\}_{i=1}^k$ of X, we have

$$H(\mu, \mathcal{A}) \leq \sum_i \frac{\mu(X)}{k} \log \frac{k}{\mu(X)} = \mu(X)\log k + \mu(X)\log \frac{1}{\mu(X)}, \tag{7.1}$$

with equality only if $\mu(A_i) = \mu(X)/k$ for each i.

Recall that $\mathcal{D}_n$ is the collection of n-th level dyadic cubes of $\mathbb{R}^d$. Fix any $y_0 \in \mathbb{R}^d$. Let $\mathcal{A} = \{w \in \mathcal{D}_n : w \cap B(y_0, \delta) \neq \emptyset\}$ and $E = \cup_{w\in\mathcal{A}} w$. Note that since $2^{-n} \leq \delta$, for some constant $C' = C'(d)$, we have $\operatorname{diam}(E) \leq C'\delta$ and E

can be covered by less than C' balls of diameter δ, thus $\eta(E) \le C'\varepsilon$. Now to conclude the proof we only need to notice that

$$H(\eta|_{B(y_0,\delta)^c}, \mathcal{D}_n) \ge H(\eta, \mathcal{D}_n) - H(\eta|_E, \mathcal{A})$$

and by (7.1),

$$H(\eta|_E, \mathcal{A}) \le \eta(E) \log \sharp \mathcal{A} + \eta(E) \log \frac{1}{\eta(E)} \le C_1 n \varepsilon^{\frac{1}{2}}$$

for some constant $C_1 = C_1(d)$. □

Now we are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Fix any $\varepsilon > 0$. Recall that by Theorem 5.9, the measure ν_∞ is ergodic and satisfies property (5.17). Let $r_0(\varepsilon) := \delta(\varepsilon)$, where $\delta(\varepsilon)$ is the constant appearing in property (5.17).

Recall that $\mathcal{B}_1$ is the partition of $K \times [0, 1)$ defined in (5.2). Since $\mathcal{B}_1$ is a generator with finite cardinality and $\nu_\infty(\partial B) = 0$ for each $B \in \mathcal{B}_n$, $n \ge 1$ (see the proof of Lemma 5.7), we can apply Theorem 6.1 to the system $(K \times [0, 1), U, \nu_\infty)$. Let $n_2(\varepsilon)$ be the integer provided by Theorem 6.1. Let

$$n_3(\varepsilon) := \max\{n_2(\varepsilon), n_2(\varepsilon)\frac{\log \alpha^{-1}}{\log 2}, n_1(\varepsilon)\},$$

where $n_1(\varepsilon)$ is the integer appearing in (5.17).

We fix any $n \ge n_3(\varepsilon)$. Let $\widetilde{n} = \lfloor n \frac{\log 2}{\log \alpha^{-1}} \rfloor + 1$, where $\lfloor x \rfloor$ denotes the integer part of x. By the choice of $n_3(\varepsilon)$, we have $\widetilde{n} \ge n_2(\varepsilon)$. Then by Theorem 6.1, we can find a disjoint family $\{C_i\}_{i=1}^{N(\widetilde{n},\varepsilon)}$ of measurable subsets $C_i \subset K \times [0, 1)$ satisfying the following properties:

(i) we have $\nu_\infty(\bigcup_i C_i) \ge 1 - \varepsilon$;
(ii) for $1 \le i \le N(\widetilde{n}, \varepsilon)$, we have $\sharp \left\{E \in \mathcal{B}_{\widetilde{n}} : C_i \cap E \ne \emptyset\right\} \le e^{\varepsilon \widetilde{n}}$;
(iii) there exists another disjoint family $\{\widetilde{C}_i\}_{i=1}^{N(\widetilde{n},\varepsilon)}$ of measurable subsets $\widetilde{C}_i \subset K \times [0, 1)$ such that for each $1 \le i \le N(\widetilde{n}, \varepsilon)$, we have $C_i \subset \widetilde{C}_i$, $\nu_\infty(C_i) \ge (1 - \varepsilon)\nu_\infty(\widetilde{C}_i)$ and for ν_∞-a.e. (z, t), the sequence

$$\left\{R_\theta^k(t) \in [0, 1) : k \in \mathbb{N} \text{ and } U^k(z, t) \in \widetilde{C}_i\right\} \tag{7.2}$$

is UD.

Now, it follows from the above property (iii) and property (5.17) that the following set

$$A' := \begin{cases} (z,t) : & \text{the sequence (7.2) is UD for each } 1 \le i \le N(\widetilde{n}, \varepsilon) \text{ and there} \\ & \text{exists } \mu = \mu_{z,t} \text{ such that } \pi_t\mu \in \mathcal{P}(l \cap K) \text{ for some line } l \text{ with} \\ & \text{slope } \beta^{-t} \text{ and (5.16) holds for } \pi_t\mu \text{ and } z. \end{cases}$$

has full ν_∞-measure. For $1 \le i \le N(\widetilde{n}, \varepsilon)$, let

$$H(C_i, z, t) = \left\{k \in \mathbb{N} : U^k(z,t) \in C_i\right\}$$

and

$$H(\widetilde{C}_i, z, t) = \left\{k \in \mathbb{N} \text{ and } U^k(z,t) \in \widetilde{C}_i\right\}.$$

Let A'' be the set of (z,t) such that for each i,

$$d(H(C_i,z,t),\mathbb{N}) = \nu_\infty(C_i) \quad \text{and} \quad d(H(\widetilde{C}_i,z,t),\mathbb{N}) = \nu_\infty(\widetilde{C}_i).$$

Recall that for a subset F of $\mathbb{N}$, $d(F,\mathbb{N})$ denotes the density of F in $\mathbb{N}$. By ergodicity of ν_∞, A'' also has full ν_∞-measure. Let $A = A' \cap A''$. Then we still have $\nu_\infty(A) = 1$.

Now, let us pick any $(z,t) \in A$. In the following, we will find a measure $\nu \in \mathcal{P}(K)$, a measurable set $D \subset K \times [0,1)$ and a subset $\mathcal{N} \subset \mathbb{N}$ satisfying properties (1)–(5) in the statement of Proposition 7.1.

Note that since $A \subset A'$, $(z,t) \in A'$. It follows that there exists $\mu = \mu_{z,t}$ such that $\pi_t\mu \in \mathcal{P}(\ell \cap K)$ for some line ℓ with slope β^{-t} and (5.16) holds for $\pi_t\mu$ and z. Let

$$\nu = \pi_t \mu_{z,t}.$$

Recall that $r_0(\varepsilon) = \delta(\varepsilon)$ and $n \ge n_3(\varepsilon) \ge n_1(\varepsilon)$, where $\delta(\varepsilon)$ and $n_1(\varepsilon)$ are the constant and the integer appearing in property (5.16). Thus by (5.16), the set

$$A(\nu,z,t) := \left\{k \in \mathbb{N} : \sup_{y \in K} \nu^{\mathcal{A}_k^t(z)}(B(y,\delta(\varepsilon))) \le \varepsilon \right.$$
$$\left. \text{and } H(\nu^{\mathcal{A}_k^t(z)}, \mathcal{D}_n) \ge n(\gamma \log 2 - 2\varepsilon)\right\}$$

has lower density at least $1 - 2\varepsilon$ in $\mathbb{N}$. On the other hand, by the above property (i), the density of $\bigcup_{i=1}^{N(\widetilde{n},\varepsilon)} H(C_i,z,t)$ in $\mathbb{N}$ is at least $1-\varepsilon$. Note also that the $H(C_i,z,t)$'s are disjoint. It follows that there exists at least one $1 \le i_0 \le N(\widetilde{n},\varepsilon)$ such that the lower density of $A(\nu,z,t) \cap H(C_{i_0},z,t)$ in $H(C_{i_0},z,t)$ is at least $1-3\varepsilon$. Let

$$D = C_{i_0} \quad \text{and} \quad \mathcal{N} = A(\nu,z,t) \cap H(C_{i_0},z,t).$$

Since $H(C_{i_0},z,t)$ has density at least $(1-\varepsilon)$ in $H(\widetilde{C}_{i_0},z,t)$, we deduce that the lower density of $\mathcal{N}$ in $H(\widetilde{C}_{i_0},z,t)$ is at least $(1-3\varepsilon)(1-\varepsilon)$. Now, since $(z,t) \in A'$, the sequence

$$\left\{R_\theta^k(t) \in [0,1) : k \in H(\widetilde{C}_{i_0},z,t)\right\}$$

is UD in $[0,1)$. From Lemma 7.2, we obtain

$$\mathcal{L}\left(\overline{\{R_\theta^k(t) : k \in \mathcal{N}\}}\right) \ge (1-3\varepsilon)(1-\varepsilon) \ge 1-4\varepsilon.$$

Let us now consider the projection $\Pi_1(D)$. By the above property (ii), we have

$$\sharp\{E \in \mathcal{B}_{\widetilde{n}} : D \cap E \neq \emptyset\} \leq e^{\varepsilon \widetilde{n}}.$$

It follows that

$$\sharp\{A \in \Pi_1(\mathcal{B}_{\widetilde{n}}) : \Pi_1(D) \cap A \neq \emptyset\} \leq e^{\varepsilon \widetilde{n}}.$$

Recall that each element of $\Pi_1(\mathcal{B}_{\widetilde{n}})$ is in $\mathcal{A}^t_{\widetilde{n}}$ for some $t \in [0,1)$. By definition, it is clear that each element in $\mathcal{A}^t_{\widetilde{n}}$ can be covered by C_2 balls of diameter $\alpha^{\widetilde{n}}$, where C_2 is a constant depending only on the geometry of $\mathbb{R}^2$, α and β. By the choice of $\widetilde{n}$, we have $\alpha^{\widetilde{n}} \leq 2^{-n}$. Thus we get

$$n^{-1} \log N_{2^{-n}}(\Pi_1(D)) \leq C_3 \varepsilon$$

for some constant C_3 depending only on $\mathbb{R}^2$, α and β. It remains to show property (5) of Proposition 7.1. For this, we use the fact that for each $k \in \mathcal{N}$, the measure $\nu^{\mathcal{A}^t_k(z)}$ satisfies the inequalities in the definition of $A(\nu, z, t)$ and apply Lemma 7.3 to $\nu^{\mathcal{A}^t_k(z)}$ to get

$$\inf_{y \in K} \frac{1}{n \log 2} H\left(\nu^{\mathcal{A}^t_k(z)}|_{B(y, r_0(\varepsilon))^c}, \mathcal{D}_n\right) \geq \gamma - C_4 \varepsilon^{\frac{1}{2}}$$

for some constant C_4 depending only on $\mathbb{R}^2$, α and β. Note that to effectively apply Lemma 7.3 we need to assume that $n \geq n_3(\varepsilon)$ was chosen large enough so that $2^{-n} \leq r_0(\varepsilon)$. For this we may replace $n_3(\varepsilon)$, if necessary, by a larger number, which we continue to denote by $n_3(\varepsilon)$, such that $2^{-n_3(\varepsilon)} \leq r_0(\varepsilon)$. Letting $C = \max\{C_3, 4, C_4^2\}$, we get that the chosen ν, D and $\mathcal{N}$ satisfy properties (1)–(5) of Proposition 7.1 provided that in (1)–(5) we replace ε by $C\varepsilon$. To complete the proof, we only need to replace $r_0(\varepsilon)$ and $n_3(\varepsilon)$ by $r_0(\varepsilon/C)$ and $n_3(\varepsilon/C)$, respectively. □

7.1. *Proof of Theorem* 1.3. Recall that we initially assumed (4.1) and we need to prove $\dim_H K \geq 1 + \gamma$. Since $K = C_\alpha \times C_\beta$, $\dim_H C_\alpha = \overline{\dim}_B C_\alpha$ and $\dim_H C_\beta = \overline{\dim}_B C_\beta$, by Lemma 2.2, $\dim_H K = \overline{\dim}_B K$. Thus it suffices to show that $\overline{\dim}_B K \geq 1 + \gamma$.

Fix a small $\varepsilon > 0$. Let $r_0 = r_0(\varepsilon)$ and $n_3 = n_3(\varepsilon)$ be as in Proposition 7.1. Fix any large $n \geq n_3$. Choose a point $(z,t) \in K \times [0,1)$, a measure $\nu \in \mathcal{P}(K)$, a measurable set $D \subset K \times [0,1)$ and a subset $\mathcal{N} \subset \mathbb{N}$ satisfying properties (1)–(5) of Proposition 7.1.

We claim that for any $k \in \mathcal{N}$,

$$\inf_{y \in K} \frac{1}{n \log 2} \log N_{2^{-n}}\left(\operatorname{supp}\left(\nu^{\mathcal{A}^t_k(z)}\right) \setminus B(y, r_0)\right) \geq \gamma - o(1) \tag{7.3}$$
$$\text{as } \varepsilon \to 0 \text{ and } n \to \infty.$$

The claim is just a consequence of property (5) and the elementary formula (7.1).

Note that since $\nu \in \mathcal{P}(\ell \cap K)$ for some line ℓ with slope β^{-t}, $\nu^{\mathcal{A}_k^t(z)}$ is a measure supported on some other slice $\ell' \cap K$ with slope $\beta^{-R_\theta^k(t)}$. Note also that for each $k \in \mathcal{N}$, we have $\Pi_1(U^k(z,t)) \in \Pi_1(D)$ and the support of $\nu^{\mathcal{A}_k^t(z)}$ intersects $\Pi_1(D)$.

Let us summarize the consequences of properties (1)–(5): For any $\varepsilon > 0$, there exist a set $F = \{R_\theta^k(t) : k \in \mathcal{N}\} \subset [0,1)$ with $\mathcal{L}\left(\overline{F}\right) \geq 1 - C\varepsilon$ and a set $D_1 = \Pi_1(D) \subset K$ with $n^{-1} \log N_{2^{-n}}(D_1) \leq C\varepsilon$ such that for each $s \in F$, there exists a line $\ell = \ell_s$ with slope β^{-s} intersecting D_1 and satisfying

(7.4)
$$\inf_{y \in K} \frac{1}{n \log 2} \log N_{2^{-n}}\left(\ell \cap K \setminus B(y, r_0)\right) \geq \gamma - o(1) \quad \text{as } \varepsilon \to 0 \text{ and } n \to \infty.$$

Now, let us consider the set $\widetilde{K} := K - D_1 = \{w - v : w \in K, v \in D_1\}$. It follows from the above summarized property that for any $t \in F$, we can find some line $\ell = \ell'_t$ with slope β^{-t} satisfying (7.4) and passing through an n-th level dyadic cube containing the origin. From this, it is easy to check that we have

$$\frac{\log N_{2^{-n}}(\widetilde{K})}{n \log 2} \geq 1 + \gamma - o(1) \quad \text{as } \varepsilon \to 0 \text{ and } n \to \infty.$$

It is a well-known fact that for each $d \geq 1$, there exists a constant $C(d)$ such that $N_{2^{-n}}(A+B) \leq C(d) N_{2^{-n}}(A) N_{2^{-n}}(B)$ for any $A, B \subset \mathbb{R}^d$. Since $n^{-1} \log N_{2^{-n}}(D_1) = o(1)$, it follows that

$$\frac{\log N_{2^{-n}}(K)}{n \log 2} \geq 1 + \gamma - o(1) \quad \text{as } \varepsilon \to 0 \text{ and } n \to \infty.$$

This implies $\overline{\dim}_{\mathrm{B}}(K) \geq 1 + \gamma$.

8. **Proof of Theorem 1.6**

For proving Theorem 1.6, we follow the same scheme as in the proof of Theorem 1.3. We only give a sketch of the proof.

Let X be a self-similar set satisfying the conditions of Theorem 1.6. Suppose that there exists a slice $\ell_0 \cap X$ with upper box dimension $\gamma > 0$. Our aim is to show that we must have $\dim_{\mathrm{H}} X \geq 1 + \gamma$.

Construction of CP-*distributions based on* $\ell_0 \cap X$. We will first construct an ergodic CP-distribution Q with dimension at least γ such that Q_1-almost every measure is supported on a slice of X.

We first recall some notation. Let $\mathcal{F} = \{f_i(x) = \lambda O_\xi x + t_i\}_{i=1}^m$ be the IFS generating X. Recall that $\lambda \in (0,1), t_i \in \mathbb{R}^2$ and O_ξ is the rotation matrix of angle $2\pi\xi \in [0, 2\pi)$ with ξ irrational.

Write $\Lambda = \{t_i\}_{i=1}^m$. Consider the symbolic space $\Lambda^{\mathbb{N}}$ endowed with the metric d_λ (recall (2.1)). Let $\Pi : \Lambda^{\mathbb{N}} \to X$ be the projection map defined as

$$\Pi((x_n)_n) = \sum_{n=1}^{\infty} \lambda^{n-1} O_\xi^{n-1} x_n.$$

Then $X = \Pi(\Lambda^{\mathbb{N}})$. Note that since $\mathcal{F}$ satisfies the strong separation condition, the map Π is bi-Lipschitz. Let $M : \mathcal{P}(\Lambda^{\mathbb{N}}) \times \Lambda^{\mathbb{N}}$ be the magnification operator defined as

$$M(\mu, x) = (\mu^{[x_1]}, \sigma(x)).$$

Recall that for some line ℓ_0, we have $\overline{\dim}_{\mathrm{B}} X \cap \ell_0 = \gamma$. Let $A = \Pi^{-1}(X \cap \ell_0)$. Since Π is bi-Lipschitz, the upper box dimension of A is also γ. Thus there exists a sequence $n_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \frac{N_{\lambda^{n_k}}(A)}{-n_k \log \lambda} = \gamma.$$

Similarly as in Section 4.2, we define a sequence of measures $\{\mu_k\}_k$ on A:

$$\mu_k = \frac{1}{N_{\lambda^{n_k}}(A)} \sum_{u \in \Lambda^{n_k} : [u] \cap A \neq \emptyset} \delta_{x_u},$$

where x_u is some point in $[u] \cap A$. Then we set

$$P_k = \frac{1}{N_{\lambda^{n_k}}(A)} \sum_{u \in \Lambda^{n_k} : [u] \cap A \neq \emptyset} \delta_{(\mu_k, x_u)} \quad \text{and} \quad Q_k = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} M^i P_k.$$

Let Q be an accumulation point of $\{Q_k\}_k$. Then Q is M-invariant and adapted, and thus it is a CP-distribution. Moreover, it has dimension

$$H(Q) = \int \frac{1}{\log \alpha} \log \mu[x_1] dQ(\mu, x) = \gamma.$$

One can also show that the measure component of Q is supported on measures that are supported on slices of X. Up to replacing Q by one of its ergodic components with dimension $\geq \gamma$, we may assume that Q is an ergodic CP-distribution with dimension at least γ and that Q is supported on measures that are supported on slices of X.

The transformation W on X and a W-invariant measure ν. Let W be the inverse map of the IFS $\mathcal{F}$ on X; that is, the restriction of W on $f_i(X)$ is f_i^{-1}. Then W is expanding and rotating, and it transforms a slice $l \cap X$ into finitely many pieces of slices with the angle of each of the transformed slices being rotated by $-\xi$ comparing to that of the initial slice l.

We use $\mathcal{A}_n$ to denote the partition of X given by

$$\{\Pi([u]) : u \in \Lambda^n\}.$$

For any measure $\eta \in \mathcal{P}(X)$ and $x \in \text{supp}(\eta)$, we write

$$\eta^{\mathcal{A}_n(x)} = W^n \left(\frac{\eta|_{\mathcal{A}_n(x)}}{\eta(\mathcal{A}_n(x))} \right).$$

Consider the map $G : \mathcal{P}(\Lambda^{\mathbb{N}}) \times \Lambda^{\mathbb{N}} \to \mathcal{P}(X)$ defined by

$$G(\mu, x) = \Pi\mu.$$

Then G is continuous. Applying the ergodic theorem to the CP-distribution Q, we get for Q-a.e. (μ, x),

$$\frac{1}{N} \sum_{n=0}^{N-1} G(M^n(\mu, x)) \to \int G dQ \quad \text{as } N \to \infty.$$

By the definition of M, we have $G(M^n(\mu, x)) = (\Pi\mu)^{\mathcal{A}_n(x)}$. Thus for Q-a.e. (μ, x),

$$\frac{1}{N} \sum_{n=0}^{N-1} (\Pi\mu)^{\mathcal{A}_n(x)} \to \int \Pi\mu dQ \quad \text{as } N \to \infty.$$

Now, with similar arguments as in the proof of Proposition 5.4, we can prove that the measure $\nu := \int \Pi\mu dQ$ is actually W-invariant. Furthermore, by proceeding analogously as in Section 5.3, we can show that ν satisfy a similar property as (5.17): for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for ν-a.e. $z \in X$, there exists $\mu \in \mathcal{P}(\Lambda^{\mathbb{N}})$ with $\Pi\mu \in \mathcal{P}(l \cap X)$ for some line l and

(8.1)

$$\liminf_{N \to \infty} \frac{1}{N} \sharp \Big\{ 1 \leq k \leq N : \sup_{y \in K} (\Pi\mu)^{\mathcal{A}_k(z)}(B(y, \delta)) \leq \varepsilon \text{ and } H((\Pi\mu)^{\mathcal{A}_k(z)}, \mathcal{D}_n) \geq n(\gamma \log 2 - \varepsilon) \Big\} > 1 - \varepsilon \quad \text{for all } n \geq n_0.$$

Up to taking an ergodic component, we may also assume that ν is ergodic.

Applying the ergodic theoretical result to the system (X, W, ν), *and conclusion.* Now, we apply Theorem 6.1 to the system (X, W, ν) and proceed as in Section 7 to finally conclude that $\overline{\dim}_{\mathrm{B}}(X) \geq 1 + \gamma$. Since X has equal Hausdorff and upper box dimensions, we get $\dim_{\mathrm{H}} X \geq 1 + \gamma$.

9. **Embeddings of self-similar sets and proofs of the remaining statements**

In this section, we first present and prove an application of Theorem 1.3 in the study of affine embeddings of self-similar sets, and then we complete the proofs of the remaining statements: Theorem 1.4 and the claim that Conjecture 1.2 holds outside a set of Hausdorff dimension zero.

9.1. *Embeddings of self-similar sets.* Let $\Phi = \{\phi_i(x) = \alpha_i x + a_i\}_{i=1}^m$ and $\Psi = \{\psi_i(x) = \beta_j x + b_j\}_{j=1}^l$ be two self-similar IFSs on $\mathbb{R}$. We denote their attractors by X_Φ and X_Ψ, respectively. The problem of affine embeddings of self-similar sets was studied in [12]. The following conjecture is a special case of [12, Conj. 1.2].

Conjecture 9.1. *Let Φ, Ψ be the self-similar* IFS*s defined above. Assume that X_Ψ is not a singleton and Φ satisfies the* SSC *and* $\dim_{\mathrm{H}} X_\Phi < 1$. *If there exist real numbers $v, u \neq 0$ such that $uX_\Psi + v \subset X_\Phi$, then for each $1 \leq j \leq l$, there exist rational numbers $r_{i,j} \geq 0$ such that $\beta_j = \prod_{i=1}^m \alpha_i^{r_{i,j}}$.*

Some special cases of Conjecture 9.1 have been proved in [12], and more recently in [1], [13]. As a corollary of Theorem 1.3, we show that Conjecture 9.1 holds under the assumption that Φ is homogeneous.

Corollary 9.2. *Under the assumptions of Conjecture 9.1, suppose further that Φ is homogeneous: there exists $0 < \alpha < 1$ such that $\alpha_i = \alpha$ for each $1 \leq i \leq m$. Then the conclusion of Conjecture 9.1 holds, i.e.,* $\log \beta_j / \log \alpha \in \mathbb{Q}$ *for each $1 \leq j \leq l$.*

Proof of Corollary 9.2. We first prove the conclusion under the assumption that X_Ψ satisfies the SSC. Fix any $j_0 \in \{1, \dots, l\}$. We will show that $\log \beta_{j_0} / \log \alpha \in \mathbb{Q}$. Choose any $j \in \{1, \dots, l\} \setminus \{j_0\}$, and let X_1 be the attractor of the homogeneous self-similar IFS $\{\psi_{j_0} \circ \psi_j, \psi_j \circ \psi_{j_0}\}$. Since X_Ψ satisfies the SSC, the same holds for X_1. Note that $X_1 \subset X_\Psi$, and thus by hypothesis we have $uX_1 + v \subset X_\Phi$. We claim that $\log(\beta_{j_0}\beta_j)/\log \alpha \in \mathbb{Q}$. Otherwise, by Theorem 1.3 (and part (2) of Remark 1.5), we would have

$$\dim_{\mathrm{H}}(uX_1 + v) \cap X_\Phi \leq \max\{0, \dim_{\mathrm{H}} X_1 + \dim_{\mathrm{H}} X_\Phi - 1\} < \dim_{\mathrm{H}} X_1,$$

which contradicts the fact $(uX_1+v) \cap X_\Phi = uX_1+v$. Similarly, we can consider the IFS $\{\psi_{j_0} \circ \psi_j^2, \psi_j^2 \circ \psi_{j_0}\}$ and deduce that $\log(\beta_{j_0}\beta_j^2)/\log \alpha \in \mathbb{Q}$. Then we get $\log \beta_{j_0}/\log \alpha \in \mathbb{Q}$.

Now we consider general X_Ψ. Fix any $j_1 \in \{1, \dots, l\}$. We will show that $\log \beta_{j_1}/\log \alpha \in \mathbb{Q}$. Since X_Ψ is not a singleton, there exists $j \in \{1, \dots, l\}$ such that ψ_{j_1} and ψ_j have different fixed points. From this we deduce that for large enough n, the IFS $\{\psi_{j_1}^n, \psi_j^n\}$ satisfies the SSC. Let X_2 be the attractor of this IFS. Then we have $uX_2 + v \subset X_\Phi$. From this and what we have just proved, we deduce that $\log \beta_{j_1}^n / \log \alpha \in \mathbb{Q}$, which in turn implies that $\log \beta_{j_1}/\log \alpha \in \mathbb{Q}$. □

9.2. *Proofs of the remaining statements.* We first complete the proof of Theorem 1.4. Following Furstenberg, we call $C \subset \mathbb{R}$ a p-Cantor set if it is the attractor of a certain IFS $\mathcal{F} = \{x/p + i/p\}_{i \in \Lambda}$ for some $\Lambda \subset \{0, \dots, p-1\}$. Clearly, each p-Cantor set is a regular $1/p$-self-similar set.

PROPOSITION 9.3. *Let $A \subset \mathbb{T} = [0,1)$ be a T_m-invariant closed set. Then for any $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and an m^k-Cantor set $\widetilde{A}$ such that $A \subset \widetilde{A}$ and* $\dim_{\mathrm{H}} A \geq \dim_{\mathrm{H}} \widetilde{A} - \varepsilon$.

Proof. Let us denote by $\mathcal{D}_k^m$ the set of k-th level m-adic intervals of $\mathbb{T} = [0,1)$, i.e., $\mathcal{D}_k^m = \big\{[i/m^k, (i+1)/m^k) : 0 \leq i \leq m^k - 1\big\}$. Let $N_{m^{-k}}(A)$ be the number of elements in $\mathcal{D}_k^m$ intersecting A. It is a classical result, due to Furstenberg [14], that any T_m-invariant closed set has equal Hausdorff and box dimensions. Thus we have

$$\dim_{\mathrm{H}} A = \lim_{k \to \infty} \frac{\log N_{m^{-k}}(A)}{k \log m}.$$

Let us fix a large enough k such that $\frac{\log N_{m^{-k}}(A)}{k \log m} \leq \dim_{\mathrm{H}} A + \varepsilon$. We consider the IFS

$$\mathcal{F} = \left\{ \frac{1}{m^k} x + \frac{i}{m^k} : 0 \leq i \leq m^k - 1 \ \text{ and } [i/m^k, (i+1)/m^k) \cap A \neq \emptyset \right\}.$$

Since A is T_m-invariant, it is also T_m^k-invariant, from which we deduce that A is a sub-attractor of $\mathcal{F}$, i.e., $A \subset \bigcup_{f \in \mathcal{F}} f(A)$. Let $\widetilde{A}$ be the attractor of $\mathcal{F}$. Then $\widetilde{A}$ is a m^k-Cantor set and $A \subset \widetilde{A}$. Now, it remains to show $\dim_{\mathrm{H}} A \geq \dim_{\mathrm{H}} \widetilde{A} - \varepsilon$. For this, we only need to notice that $\widetilde{A}$ satisfies the open set condition, and it is well known that its Hausdorff dimension is $\frac{\log N_{m^{-k}}(A)}{k \log m}$. By the choice of k, we get the desired result. □

Proof of Theorem 1.4. Let $A \subset \mathbb{T}$ be closed and T_p-invariant, and let $B \subset \mathbb{T}$ be closed and T_q-invariant, with $p \not\sim q$. Fix any $\varepsilon > 0$. By Proposition 9.3, for some large k and l, there exist a p^k-Cantor set $\widetilde{A}$ and a q^l-Cantor set $\widetilde{B}$ such that $A \subset \widetilde{A}$, $\dim_{\mathrm{H}} A \geq \dim_{\mathrm{H}} \widetilde{A} - \varepsilon$, $B \subset \widetilde{B}$ and $\dim_{\mathrm{H}} B \geq \dim_{\mathrm{H}} \widetilde{B} - \varepsilon$. Now, from the hypothesis $p \not\sim q$ we deduce that $p^k \not\sim q^l$, thus we can apply Theorem 1.3 to the sets $\widetilde{A}$ and $\widetilde{B}$ to get

$$\overline{\dim}_{\mathrm{B}}(u\widetilde{A} + v) \cap \widetilde{B} \leq \max\{0, \dim_{\mathrm{H}} \widetilde{A} + \dim_{\mathrm{H}} \widetilde{B} - 1\}.$$

From this we deduce that

$$\overline{\dim}_{\mathrm{B}}(uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\} + 2\varepsilon.$$

Since ε is arbitrary, we get the desired result. □

We now show that Conjecture 1.2 holds outside a set of Hausdorff dimension zero.

THEOREM 9.4. *If $p \not\sim q$, then the set of $x \in [0,1]$ that do not satisfy*

$$\dim_{\mathrm{H}} \overline{O_p(x)} + \dim_{\mathrm{H}} \overline{O_q(x)} \geq 1$$

has Hausdorff dimension zero; *in fact it is a countable union of sets with upper box dimension zero.*

Proof. Let $E = \left\{x \in [0,1] : \dim_{\mathrm{H}} \overline{O_p(x)} + \dim_{\mathrm{H}} \overline{O_q(x)} < 1\right\}$. We need to show that the set E is a countable union of sets with upper box dimension zero.

In the following, by a T_m-invariant set we always mean a T_m-invariant and closed set of $[0,1]$. Let

$$F_1 = \Big\{(A,B) : A \text{ is a } T_p\text{-invariant set}, B \text{ is a } T_q\text{-invariant set} \\ \text{and } \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B < 1\Big\}$$

and

$$F_2 = \Big\{(\widetilde{A},\widetilde{B}) : \widetilde{A} \text{ is a } p^k\text{-Cantor set}, \widetilde{B} \text{ is a } q^l\text{-Cantor set} \\ \text{and } \dim_{\mathrm{H}} \widetilde{A} + \dim_{\mathrm{H}} \widetilde{B} < 1, k,l \in \mathbb{N}\Big\}.$$

By Proposition 9.3, for each pair $(A,B) \in F_1$, there exists $(\widetilde{A},\widetilde{B}) \in F_2$ such that $A \subset \widetilde{A}$ and $B \subset \widetilde{B}$. Thus we have

$$E \subset \bigcup_{(A,B)\in F_1} A \cap B \subset \bigcup_{(\widetilde{A},\widetilde{B})\in F_2} \widetilde{A} \cap \widetilde{B}.$$

Now, note that for each $k \in \mathbb{N}$, there are only finitely many p^k-Cantor sets and finitely many q^k-Cantor sets. Thus the cardinality of F_2 is at most countable. Since $p \not\sim q$, we have $p^k \not\sim q^l$ for any $k,l \in \mathbb{N}$. Thus by Theorem 1.3, for each $(\widetilde{A},\widetilde{B}) \in F_2$, we have

$$\overline{\dim}_{\mathrm{B}}(\widetilde{A} \cap \widetilde{B}) \le \max\{0, \dim_{\mathrm{H}} \widetilde{A} + \dim_{\mathrm{H}} \widetilde{B} - 1\} = 0.$$

Hence E is contained in a countable union of sets with upper box dimension zero. □

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