SINGULARITIES OF SOLUTIONS OF TIME DEPENDENT HAMILTON-JACOBI EQUATIONS. APPLICATIONS TO RIEMANNIAN GEOMETRY

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ABSTRACT

If $U : [0, +\infty[\times M$ is a uniformly continuous viscosity solution of the evolution Hamilton-Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0,$$

where M is a not necessarily compact manifold, and H is a Tonelli Hamiltonian, we prove the set $\Sigma(U)$, of points in $]0, +\infty[\times M$ where U is not differentiable, is locally contractible. Moreover, we study the homotopy type of $\Sigma(U)$. We also give an application to the singularities of the distance function to a closed subset of a complete Riemannian manifold.

1. Introduction

Let M be a smooth connected but not necessarily compact manifold. We will assume M endowed with a *complete* Riemannian metric g. For $v \in T_x M$, the norm $\|v\|_x$ is $g(v, v)^{1/2}$. We will denote also by $\|\cdot\|_x$ the dual norm on $T_x^* M$. If $\gamma : [a, b] \to M$ is a curve, its length $\ell_g(\gamma)$ (for the metric g) is defined by

$$\ell_g(\gamma) = \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds.$$

The distance d that we will use on M is the Riemannian distance obtained from the Riemannian metric. Namely, if $x, y \in M$ the distance $d(x, y)$ is the infimum of the length of curves joining x to y. Since g is complete, the distance d is complete, and for every pair of points $x, y \in M$, there exists a curve joining x to y whose length is $d(x, y)$. Moreover, for every compact subset $K \subset M$ and every finite $R \geq 0$, the closed R-neighborhood $\bar V_R(K) = \{x \in M \mid d(x, K) \leq R\}$ of K is itself compact.

Before giving our results for the general Hamilton-Jacobi equation, we will give the consequences in Riemannian geometry.

If C is a closed subset of the complete Riemannian manifold (M, g). As usual, the distance function $d_C : M \to [0, +\infty[$ to C is defined by

$$d_C(x) = \inf_{c \in C} d(c, x).$$

We will denote by $\Sigma^*(d_C)$ the set of points in $M \setminus C$ where d_C is not differentiable. Note that $\Sigma^*(d_C)$ has Lebesgue measure 0, since the Lipschitz function d_C on M is differentiable almost everywhere.

Theorem **1.1.** — *Consider the closed subset* C *of the complete Riemannian manifold* (M, g). *Then* $\Sigma^*(d_C)$ *is locally contractible.*

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As a first application, if we take $C = \{p\}$, the set $\Sigma^*(d_p)$ is nothing but the set of $q \in M$ such that there exists two distinct minimizing geodesics from p to q. The closure is known as the cut locus $\text{Cut-locus}_{(M,g)}(p)$ of p for (M, g), see [20, §2.1]. It is well-known that, for M compact, this cut locus $\text{Cut-locus}_{(M,g)}(p)$ is a deformation retract of $M \setminus p$, see [20, Theorem 2.1.8], therefore it is locally contractible. It is also known that the cut locus $\text{Cut-locus}_{(M,g)}(p)$ is, in general, not triangulable, see [18]. However, even if there is an extensive literature on the cut locus, very little was known up to now about the set of $q \in M$ such that there exists two distinct minimizing geodesics from p to q. As Marcel Berger said in [4, Page 284]:

The difficulty for all these studies is an unavoidable dichotomy for cut points: the mixture of points with two different segments and conjugate points.

Our methods permit to separate the study of these two sets.

To state another consequence of Theorem 1.1, we introduce the following definition:

Definition **1.2.** — *If* (M, g) *is a complete Riemannian manifold, we define the subset* $\mathcal{U}(M, g) \subset M \times M$ *as the set of* $(x, y) \in M \times M$ *such that there exists a* unique *minimizing g-geodesic between x and y. This set* $\mathcal{U}(M, g)$ *contains a neighborhood of the diagonal* $\Delta_M \subset M \times M$. *The complement* $\mathcal{NU}(M, g) = M \times M \setminus \mathcal{U}(M, g)$ *is the set of points* $(x, y) \in M$ *such that there exists at least two distinct minimizing g-geodesics between x and y.*

In fact, as we will see in Example 2.24, we have $\mathcal{NU}(M, g) = \Sigma^*(d_{\Delta M})$, the set of singularities in $M \times M \setminus \Delta_M$ of the distance function of points in $M \times M$ to the closed subset Δ_M. Therefore, Theorem 1.1 implies:

Theorem **1.3.** — *For every complete Riemannian manifold* (M, g), *the set* $\mathcal{NU}(M, g) \subset M \times M \setminus \Delta_M$ *is locally contractible. In particular, the set* $\mathcal{NU}(M, g)$ *is locally path connected.*

As above, from $\mathcal{NU}(M, g) = \Sigma^*(d_{\Delta M})$, we recover that $\mathcal{NU}(M, g)$ has Lebesgue measure 0.

Definition **1.4.** — *For a closed subset* $C \subset M$, *we define its Aubry set* $\mathcal{A}^*(C)$ *as the set of points* $x \in M \setminus C$ *such that there exists a curve* $\gamma : [0, +\infty[\to M$ *parameterized by arc-length such that* $d_C(\gamma(t)) = t$, *for all* $t \geq 0$, *and* $x = \gamma(t_0)$ *for some* $t_0 > 0$.

Remark **1.5.** — A curve $\gamma : [0, +\infty[\to M$ parameterized by arc-length such that $d_C(\gamma(t)) = t$ is necessary a g-minimizing geodesic.

We necessarily have $\Sigma^*(d_C) \cap \mathcal{A}^*(C) = \emptyset$ (as follows from Proposition 2.12 and Lemma 2.14).

Theorem **1.6.** — *If* C *is a closed subset of the complete Riemannian manifold* (M, g), *then the inclusion* $\Sigma^*(d_C) \subset M \setminus (C \cup \mathcal{A}^*(C))$ *is a homotopy equivalence.*

If U is a bounded connected component of $M \setminus C$, then $U \cap \mathcal{A}^*(C) = \emptyset$, see §7, and Theorem 1.6 implies that the inclusion $\Sigma^*(d_C) \cap U \subset U$ is a homotopy equivalence. This fact was already known. It is due to Lieutier [21] in the Euclidean case and to Albano, Cannarsa, Nguyen & Sinestrari [1] in the general Riemannian case. The non-compact case is, to our knowledge, new, see however [7] where they study the unbounded components of $\Sigma^*(d_C)$ in the Euclidean case.

A consequence of Theorem 1.6 is

Theorem **1.7.** — *For every compact connected Riemannian manifold* M, *the inclusion* $\mathcal{NU}(M, g) \subset M \times M \setminus \Delta_M$ *is a homotopy equivalence. Therefore the set* $\mathcal{NU}(M, g)$ *is path connected.*

Of course, the homotopy equivalence in this theorem is a consequence of the compact version of Theorem 1.6, which is due, as we said above, to Albano, Cannarsa, Nguyen & Sinestrari [1].

We will give a version for non-compact M in §7.

For sake of completeness, we note that the subset Δ_M is a deformation retract of $\mathcal{U}(M, g)$. In fact, we can get such a retraction using the midpoint in a geodesic segment minimizing the length between the pair of points.

We now state our general results for Tonelli Hamiltonians. The local contractibility Theorem 1.8 is valid under slightly less restrictive conditions on the Hamiltonian H, see §8.

We recall that a Tonelli Hamiltonian $H : T^*M \to \mathbf{R}$ on M (for the complete Riemannian metric g) is a function $H : T^*M \to \mathbf{R}$ that satisfies the following conditions:

(1*) The Hamiltonian H is at least C^2

(2*) (Uniform superlinearity) For every $K \geq 0$, we have

$$C^*(K) = \sup_{(x,p)\in T^*M} K\|p\|_x - H(x, p) < \infty.$$

(3*) (Uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A^*(R) = \sup\{H(x, p) \mid \|p\|_x \leq R\} < +\infty.$$

(4*) (C^2 strict convexity in the fibers) For every $(x, p) \in T^*M$, the second derivative along the fibers $\partial^2 H/\partial p^2(x, p)$ is positive definite.

Note that (2*) implies

$$\forall (x, p) \in T^*M, H(x, p) \geq K\|p\|_x - C^*(K).$$

We will consider viscosity solutions of the Hamilton-Jacobi equation. There are several classical introductions to viscosity solutions [2, 3, 8, 11]. The more recent introductions [14, 15] are well-adapted to our manifold setting.

If $u : \mathrm{N} \to \mathbf{R}$ is a function defined on the manifold N, a singularity of u is a point of N where u is not differentiable. We denote by $\Sigma(u)$ the set of singularities of u.

The goal of this work is to study the topological structure of the set of singularities $\Sigma(\mathrm{U})$, with $\mathrm{U} : \mathrm{O} \to \mathbf{R}$ a continuous viscosity solution of the evolutionary Hamilton-Jacobi equation

$$(\mathbf{1.1}) \qquad \partial_t \mathrm{U} + \mathrm{H}(x, \partial_x \mathrm{U}) = 0,$$

defined on the open subset $\mathrm{O} \subset \mathbf{R} \times \mathrm{M}$.

In [9], we announced the results and sketched the proofs for M compact in the case of the stationary Hamilton-Jacobi equation, i.e. for U of the form $\mathrm{U}(t, x) = u(x) - ct$, with $u : \mathrm{M} \to \mathbf{R}$ and $c \in \mathbf{R}$. We extend the results of [9] to the case of the evolutionary Hamilton-Jacobi equation (1.1) covering also the case when M is non-compact.

Our first result is a local contractibility result.

Theorem **1.8.** — *Let* $\mathrm{H} : \mathrm{T}^*\mathrm{M} \to \mathbf{R}$ *be a Tonelli Hamiltonian. If the function* $\mathrm{U} : \mathrm{O} \to \mathbf{R}$, *defined on the open subset* $\mathrm{O} \subset \mathbf{R} \times \mathrm{M}$ *is a continuous viscosity solution of the evolutionary Hamilton-Jacobi equation* (1.1)*, then the set* $\Sigma(\mathrm{U}) \subset \mathrm{O}$ *of singularities of* U *is locally contractible.*

In fact, as we will see, the above theorem follows from its particular case with $\mathrm{U} :]0, +\infty[\times \mathrm{M} \to \mathbf{R}$.

To give a more global result on the topology of $\Sigma(\mathrm{U})$ we need the Aubry set of a solution of (1.1). For this we first recall that the Lagrangian $\mathrm{L} : \mathrm{TM} \to \mathbf{R}$ (associated to H) is defined by

$$\mathrm{L}(x, v) = \sup_{p \in \mathrm{T}_x^*\mathrm{M}} p(v) - \mathrm{H}(x, p).$$

This Lagrangian L is finite everywhere, and enjoys the same properties as H, namely

(1) The Lagrangian L is at least C^2 (in fact, it is as smooth as H).
(2) (Uniform superlinearity) For every $\mathrm{K} \geq 0$, we have

$$(\mathbf{1.2}) \qquad \mathrm{C}(\mathrm{K}) = \sup_{(x,v) \in \mathrm{TM}} \mathrm{K}\|v\|_x - \mathrm{L}(x, v) < \infty.$$

(3) (Uniform boundedness in the fibers) For every $\mathrm{R} \geq 0$, we have

$$(\mathbf{1.3}) \qquad \mathrm{A}(\mathrm{R}) = \sup\{\mathrm{L}(x, v) \mid \|v\|_x \leq \mathrm{R}\} < +\infty.$$

(4) (C^2 strict convexity in the fibers) for every $(x, v) \in \mathrm{TM}$, the second derivative along the fibers $\partial^2 \mathrm{L}/\partial v^2(x, v)$ is positive definite.

Again (1.2) implies

$$(\mathbf{1.4}) \qquad \forall (x, v) \in \mathrm{TM}, \mathrm{L}(x, v) \geq \mathrm{K}\|v\|_x - \mathrm{C}(\mathrm{K}).$$

A Lagrangian $L : TM \to \mathbf{R}$, on the complete Riemannian manifold (M, g), is said to be Tonelli if it satisfies the conditions (1) to (4) above.

Definition **1.9** (**Aubry set**). — *Let* $U :]0, T[\times M \to \mathbf{R}$, *with* $T \in]0, +\infty]$, *be a viscosity solution, on* $]0, T[\times M$, *of the evolutionary Hamilton-Jacobi equation* (1.1). *The Aubry set* $\mathcal{I}_T(U)$ *of* U *is the set of points* $(t, x) \in]0, T[\times M$ *for which we can find a curve* $\gamma :]0, T[\to M$, *with* $\gamma(t) = x$ *and*

$$U(b, \gamma(b)) - U(a, \gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

for every $a < b \in]0, T[$.

It is well-known that U is differentiable at every point of $\mathcal{I}_T(U)$, see Proposition 2.12. Therefore, we have $\Sigma(U) \cap \mathcal{I}_T(U) = \emptyset$. To avoid further machinery, in this introduction, we will state our results assuming the function $U : [0, t] \times M \to \mathbf{R}$ uniformly continuous.

Theorem **1.10.** — *Let* $H : T^*M \to \mathbf{R}$ *be a Tonelli Hamiltonian. Assume that the uniformly continuous function* $U : [0, t] \times M \to \mathbf{R}$ *is a viscosity solution, on* $]0, t[\times M$, *of the evolutionary Hamilton-Jacobi equation* (1.1). *Then the inclusion* $\Sigma_t(U) = \Sigma(U) \cap]0, t[\times M \subset]0, t[\times M \setminus \mathcal{I}_t(U)$ *is a homotopy equivalence.*

2. Background

We will need to use some of the facts about viscosity solutions and the negative Lax-Oleinik semi-groups. We refer to [14] and [15] for details and proofs.

In the remainder of this work, we will assume that $H : T^*M \to \mathbf{R}$ is a given Tonelli Hamiltonian on the complete Riemannian manifold M. We will denote by $L : TM \to \mathbf{R}$ its associated Lagrangian defined by

$$L(x, v) = \sup_{p \in T_x^*M} p(v) - H(x, p).$$

2.1. *Action and minimizers*

If $\gamma : [a, b] \to M$ is an absolutely continuous curve, its action $\mathbf{L}(\gamma)$ is defined by

$$\mathbf{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Note that since L is bounded from below (by $-C(0)$), we always have $\mathbf{L}(\gamma) > -\infty$, although we may have $\mathbf{L}(\gamma) = +\infty$.

For $x, y \in \mathrm{M}$, and $t > 0$, the minimal action $h_t(x, y)$ to join x to y in time t is

$$h_t(x,y) = \inf_\gamma \mathbf{L}(\gamma) = \inf_\gamma \int_0^t \mathrm{L}(\gamma(s), \dot\gamma(s))\, ds,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to \mathrm{M}$, with $\gamma(0) = x$ and $\gamma(t) = y$.

From the definition of h_t, we obtain the well-known inequality

$$h_{t+s}(x, z) \le h_t(x, y) + h_s(y, z), \text{ for all } t, s > 0 \text{ and } x, y, z \in \mathrm{M}. \tag{2.1}$$

A minimizer (for L) is an absolutely continuous curve $\gamma : [a, b] \to \mathrm{M}$ such that

$$\mathbf{L}(\gamma) = h_{b-a}(\gamma(a), \gamma(b)).$$

Tonelli's theorem [6, 10, 13] states that, for every $a < b \in \mathbf{R}$ and every $x, y \in \mathrm{M}$, there exists a minimizer $\gamma : [a, b] \to \mathrm{M}$, with $\gamma(a) = x$, $\gamma(b) = y$. All minimizers are as smooth as L.

Moreover, all minimizers are extremals, i.e. they satisfy the Euler-Lagrange equation given, in local coordinates, by

$$\frac{d}{dt}\left[\frac{\partial \mathrm{L}}{\partial v}(\gamma(s), \dot\gamma(s))\right] = \frac{\partial \mathrm{L}}{\partial x}(\gamma(s), \dot\gamma(s)). \tag{2.2}$$

As is well-known the 2nd order ODE (2.2) on M yields a 1st order ODE on TM which generates a flow ϕ_t^{L} on TM called the Euler-Lagrange flow. A curve $\gamma : [a, b] \to \mathrm{M}$ is an extremal (i.e. satisfies (2.2)) if and only if its speed curve $s \mapsto (\gamma(s), \dot\gamma(s))$ is (a piece of) an orbit of the Euler-Lagrange flow ϕ_t^{L}.

An absolutely continuous curve $\gamma : [a, b] \to \mathrm{M}$ is called a *local* minimizer (for L) if there exists a neighborhood U of $\gamma([a, b])$ in M such that for any other absolutely continuous curve $\delta : [a, b] \to \mathrm{U}$, with $\delta(a) = \gamma(a)$ and $\delta(b) = \gamma(b)$, we have $\mathbf{L}(\delta) \ge \mathbf{L}(\gamma)$. The regularity part of Tonelli's theorem implies that such a local minimizer $\gamma : [a, b] \to \mathrm{M}$ is as smooth as L and satisfies the Euler-Lagrange equation (2.2). Therefore its speed curve $s \mapsto (\gamma(s), \dot\gamma(s))$ is (a piece of) an orbit of the Euler-Lagrange flow ϕ_t^{L}.

Example **2.1.** — The simplest Tonelli Hamiltonian $\mathrm{H}_g : \mathrm{T}^*\mathrm{M} \to \mathbf{R}$ on the complete Riemannian manifold (M, g) is given by $\mathrm{H}_g(x, p) = \frac{1}{2}\|p\|_x^2$.

Its associated Lagrangian $\mathrm{L}_g : \mathrm{TM} \to \mathbf{R}$ is given by $\mathrm{L}_g(x, v) = \frac{1}{2}\|v\|_x^2$.

If $\gamma : [a, b] \to \mathrm{M}$ is a curve, we will denote by $\mathbf{L}_g(\gamma)$ its action for this Lagrangian L_g, that is

$$\mathbf{L}_g(\gamma) = \frac{1}{2}\int_a^b \|\dot\gamma(s)\|_{\gamma(s)}^2\, ds.$$

The next lemma is well-known and follows from the Cauchy-Schwarz inequality.

Lemma **2.2.** — *For any curve* $\gamma : [a, b] \to M$ *we have*

$$\mathbf{L}_g(\gamma) \geq \frac{\ell_g(\gamma)^2}{2(b-a)} \geq \frac{d^2(\gamma(a), \gamma(b))}{2(b-a)}, \tag{2.3}$$

with equality if and only if γ *is a minimizing g-geodesic.*

Therefore, for every $t > 0$ *and every* $x, y \in M$, *we have*

$$h^g_t(x, y) = \frac{d(x, y)^2}{2t}. \tag{2.4}$$

Moreover, a curve $\gamma : [a, b] \to M$ *is* L_g*-minimizing if and only if it is a minimizing geodesic.*

Example **2.3.** — If (M, g) is a Riemannian manifold, we define the Riemannian manifold $(M \times M, g \times g)$ as $M \times M$ with metric $g \times g$, i.e.

$$(g \times g)_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big) = g_{x_1}(v_1, w_1) + g_{x_2}(v_2, w_2),$$

where we used the identification $T_{(x_1,x_2)}(M \times M) = T_{x_1}M \times T_{x_2}M$. If (M, g) is complete so is $(M \times M, g \times g)$.

Therefore $H_{g\times g} : T^*(M \times M) \to \mathbf{R}$ is given by $H_{g\times g}\big((x_1, x_2), (p_1, p_2)\big) = \frac{1}{2}\|p_1\|^2_{x_1} + \frac{1}{2}\|p_2\|^2_{x_2}$.

Its associated Lagrangian $L_{g\times g} : TM \to \mathbf{R}$ is given by $L_{g\times g}(x_1, x_2, v_1, v_2) = \frac{1}{2}\|v_1\|^2_{x_1} + \frac{1}{2}\|v_2\|^2_{x_2}$.

A curve $\Gamma : [a, b] \to M \times M$ is nothing but a pair of curves $\gamma_1, \gamma_2 : [a, b] \to M$, such that $\Gamma(s) = (\gamma_1(s), \gamma_2(s))$, for all $s \in [a, b]$. We will denote this identification by $\Gamma = (\gamma_1, \gamma_2)$.

Lemma **2.4.** — *If* $\Gamma = (\gamma_1, \gamma_2) : [a, b] \to M \times M$ *is a curve in* $M \times M$, *its length is* $\ell_{g\times g}(\Gamma) = \ell_{g\times g}(\gamma_1, \gamma_2) = \int_a^b \sqrt{\|\dot\gamma_1(s)\|^2_{\gamma_1(s)} + \|\dot\gamma_2(s)\|^2_{\gamma_2(s)}}\, ds$. *The action of* Γ *is given by*

$$\begin{aligned}\mathbf{L}_{g\times g}(\Gamma) = \mathbf{L}_{g\times g}(\gamma_1, \gamma_2) &= \frac{1}{2}\int_a^b \|\dot\gamma_1(s)\|^2_{\gamma_1(s)} + \frac{1}{2}\|\dot\gamma_2(s)\|^2_{\gamma_2(s)}\, ds \\ &= \mathbf{L}_g(\gamma_1) + \mathbf{L}_g(\gamma_2).\end{aligned} \tag{2.5}$$

The $g \times g$*-distance in* $M \times M$ *is given by*

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}.$$

Therefore, since $h^{(g,g)}_t((x_1, x_2), (y_1, y_2)) = d((x_1, x_2), (y_1, y_2))^2/(2t)$, *we get*

$$h^{(g,g)}_t = \frac{d(x_1, y_1)^2 + d(x_2, y_2)^2}{2t}.$$

Most of the proof of Lemma 2.4 is similar to or uses Lemma 2.2.

In order to give the connection between action and viscosity solutions it is convenient to introduce the following (obvious) definition.

Definition **2.5** *(Graph of a curve). — If* $\gamma : [a, b] \to M$, *we define its graph* $\mathrm{Graph}(\gamma) \subset \mathbf{R} \times M$ *as*

$$\mathrm{Graph}(\gamma) = \{(t, \gamma(t) \mid t \in [a, b]\}.$$

We now recall the definition of domination for a function.

Definition **2.6.** — *If* $U : O \to [-\infty, +\infty]$ *is a function defined on the subset* $O \subset \mathbf{R} \times M$, *we say that* U *is dominated by* L *(on* O*) if for every absolutely continuous curve* $\gamma : [a, b] \to M$, *with* $\mathrm{Graph}(\gamma) \subset O$ *and* $\mathbf{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, dt < +\infty$, *we have*

$$U(b, \gamma(b)) \leq U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot\gamma(s))\, dt. \tag{2.6}$$

Note that the right hand side of (2.6) makes always sense since we insisted that $\mathbf{L}(\gamma) < +\infty$.

Remark **2.7.** — 1) We used the inequality (2.6) rather than the usual

$$U(\gamma(b), b) - U(\gamma(a), a) \leq \mathbf{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, dt, \tag{2.7}$$

since it will be convenient to consider real valued functions with possibly infinite values. Of course when U is finite valued (2.6) and (2.7) are equivalent.

2) If $U : I \times M \to \mathbf{R}$, where $I \subset \mathbf{R}$ is an interval, then all curves defined on a subinterval $[a, b] \subset I$ have their graph included in $I \times M$. Therefore, in such a case, the function U is dominated by L if and only if

$$U(t', x') - U(t, x) \leq h_{t'-t}(x, x'), \text{ for every } x, x' \in M, \text{ and every } t < t' \in I. \tag{2.8}$$

A first connection between action and viscosity solutions is given by the following proposition, see [19] or [15] for a proof.

Proposition **2.8.** — *Assume* $V : O \to \mathbf{R}$ *is a continuous function, where* $O \subset \mathbf{R} \times M$ *is an open subset of* $\mathbf{R} \times M$. *Then* V *is a viscosity subsolution of the Hamilton-Jacobi equation* (1.1) *on* O *if and only if it is dominated by* L *on* O, *where* L *is the Lagrangian associated to* H.

2.2. *Calibrated curves, backward characteristics*

Definition **2.9** *(Calibrated curve). — Let* $U : O \to [-\infty, +\infty]$ *be a function. An absolutely continuous curve* $\gamma : [a, b] \to M$ *is said to be* U*-calibrated (for the Lagrangian* L*) if* $\mathrm{Graph}(\gamma) \subset O$, *its action* $\mathbf{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds$ *is finite, and*

$$U(b, \gamma(b)) = U(a, \gamma(a)) + \mathbf{L}(\gamma) = U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot\gamma(s))\, ds. \tag{2.9}$$

Again we used 2.9, rather than the more usual $U(b, \gamma(b)) - U(a, \gamma(a)) = \mathbf{L}(\gamma)$, because we would like to allow possibly infinite values for U.

Proposition **2.10.** — *Suppose* $U : O \to [-\infty, +\infty]$ *is a function defined on the subset* O.
If the absolutely continuous curve $\gamma : [a, b] \to M$ *is piecewise* U*-calibrated, then it is* U*-calibrated.*

Of course, the curve $\gamma : [a, b] \to M$ is said to be piecewise calibrated if we can find a finite sequence $a = t_0 < t_1 < \cdots < t_\ell = b$ such that each restriction $\gamma|[t_i, t_{i+1}]$, $i = 0, \dots, \ell - 1$ is U-calibrated.

The notion of calibrated curve is useful when U is a viscosity subsolution as can be seen from the following well-known proposition.

Proposition **2.11.** — *Suppose the function* $U : O \to [-\infty, +\infty]$ *is dominated by* L. *If* $\gamma : [a, b] \to M$ *is* U*-calibrated, with at least one of the two* $U(a, \gamma(a)), U(b, \gamma(b))$ *finite, we have:*

(1) *For every* $t \in [a, b]$, *the value* $U(t, \gamma(t))$ *is finite.*
(2) *The restriction* $\gamma|[a', b']$ *is also* U*-calibrated, for any subinterval* $[a', b'] \subset [a, b]$.
(3) *the curve* γ *is a local minimizer. Hence it is as smooth as* L *and a solution of the Euler-Lagrange equation* (2.2).

We will need some facts about differentiability and calibrated curves for viscosity subsolutions. We only quote the properties that we will use latter. The reader is referred to [2, 3, 8, 13–16] for context and proofs.

Proposition **2.12.** — *Suppose the function* $U : [a, b] \times V \to \mathbf{R}$, *with* V *an open subset of* M, *is continuous and a viscosity subsolution of* (1.1) *on* $]a, b[\times V$. *Assume that* $\gamma : [a, b] \to M$, *is a* U*-calibrated curve. If* $\partial_x U$ *exists at* $(t, \gamma(t))$ *with* $t \in [a, b]$, *then*

$$\partial_x U(t, \gamma(t)) = \partial_v L(\gamma(t), \dot\gamma(t)).$$

Moreover, the function U *is differentiable at every point* $(t, \gamma(t))$ *with* $t \in]a, b[$.

2.3. *The negative Lax-Oleinik semi-group and the negative Lax-Oleinik evolution*

In fact, viscosity solutions which are continuous are always given by the negative Lax-Oleinik evolution as we now recall.

Once the minimal action is defined, we can introduce the negative Lax-Oleinik semi-group.

If $u : \mathrm{M} \to [-\infty, +\infty]$ is a function and $t > 0$, the function $\mathrm{T}_t^- u : \mathrm{M} \to [-\infty, +\infty]$ is defined by

$$\mathrm{T}_t^- u(x) = \inf_{y \in \mathrm{M}} u(y) + h_t(y, x).$$

We also set $\mathrm{T}_0^- u = u$. The negative Lax-Oleinik semi-group is $\mathrm{T}_t^-, t \geq 0$.

It is convenient to define $\hat{u} : [0, +\infty[\times \mathrm{M} \to [-\infty, +\infty]$ by $\hat{u}(t, x) = \mathrm{T}_t^- u(x)$. This function $\hat{u}$ is called the negative Lax-Oleinik evolution of u.

From the inequality (2.1), we obtain the well-known inequality

$$\hat{u}(t + s, x) \leq \hat{u}(t, y) + h_s(y, x), \text{ for all } t, s > 0 \text{ and } x, y \in \mathrm{M}. \tag{2.10}$$

Example **2.13.** — By (2.4), for the Hamiltonian $\mathrm{H}_g : \mathrm{T}^*\mathrm{M} \to \mathbf{R}$, and Lagrangian $\mathrm{L}_g : \mathrm{TM} \to \mathbf{R}$, defined in Example 2.1, we have $h_t^g(x, y) = \frac{d(x,y)^2}{2t}$. Therefore, the associated negative Lax-Oleinik semi-group T_t^{g-} is defined, when $t > 0$, by

$$\mathrm{T}_t^{g-} u(x) = \inf_{y \in \mathrm{M}} u(y) + \frac{d(y, x)^2}{2t},$$

for $u : \mathrm{M} \to [-\infty, +\infty]$.

If $\mathrm{C} \subset \mathrm{M}$, we define its (modified) characteristic function $\chi_{\mathrm{C}} : \mathrm{M} \to \{0, +\infty\}$ by

$$\chi_{\mathrm{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathrm{C}, \\ +\infty, & \text{if } x \notin \mathrm{C}. \end{cases}$$

Therefore its negative Lax-Oleinik evolution $\hat{\chi}_{\mathrm{C}}$, for the Lagrangian L_g, is defined, for $t > 0$, by

$$\hat{\chi}_{\mathrm{C}}(t, x) = \frac{d_{\mathrm{C}}(x)^2}{2t}, \tag{2.11}$$

since $\hat{\chi}_{\mathrm{C}}(t, x) = \inf_{y \in \mathrm{M}} \chi_{\mathrm{C}}(y) + \frac{d(y,x)^2}{2t} = \inf_{c \in \mathrm{C}} \frac{d(c,x)^2}{2t}$ and $d_{\mathrm{C}}(x) = \inf_{c \in \mathrm{C}} d(c, x)$.

Note that $d_{\mathrm{C}} = d_{\bar{\mathrm{C}}}$, where $\bar{\mathrm{C}}$ is the closure of C in M. Hence, we get $\hat{\chi}_{\mathrm{C}} = \hat{\chi}_{\bar{\mathrm{C}}}$ on $]0, +\infty[\times \mathrm{M}$. Therefore, to study the properties of $\hat{\chi}_{\mathrm{C}}$ on $]0, +\infty[\times \mathrm{M}$, we can always assume that C is a closed subset of M.

Lemma **2.14.** — *Suppose* C *is a closed subset of the complete Riemannian manifold* (M, g)*. A curve* $\gamma : [a, b] \to \mathrm{M}$*, with* $a > 0$*, is* $\hat{\chi}_{\mathrm{C}}$*-calibrated if and only if it is a minimizing g-geodesic and*

$$\frac{d_{\mathrm{C}}^2(\gamma(b))}{2b} - \frac{d_{\mathrm{C}}^2(\gamma(a))}{2a} = \frac{d^2(\gamma(b), \gamma(a))}{2(b - a)}.$$

A curve $\gamma : [0, b] \to \mathrm{M}$ *is* $\hat{\chi}_{\mathrm{C}}$*-calibrated if and only if it is a minimizing g-geodesic, with* $\gamma(0) \in \mathrm{C}$ *and* $d_{\mathrm{C}}(\gamma(b)) = d(\gamma(b), \gamma(0))$.

Proof. — Suppose that the curve $\gamma : [a, b] \to \mathrm{M}$, with $a > 0$, is $\hat{\chi}_{\mathrm{C}}$-calibrated. It must be a minimizing geodesic. We now recall, see Lemma 2.3, that for a minimizing g-geodesic $\gamma : [a, b] \to \mathrm{M}$, we have

$$\mathbf{L}_g(\gamma) = \frac{d^2(\gamma(b), \gamma(a))}{2(b-a)}.$$

Therefore, using (2.11), we see that a minimizing g-geodesic $\gamma : [a, b] \to \mathrm{M}$, with $a > 0$, is calibrated if and only if

$$\frac{d_{\mathrm{C}}^2(\gamma(b))}{2b} = \frac{d_{\mathrm{C}}^2(\gamma(a))}{2a} + \frac{d^2(\gamma(b), \gamma(a))}{2(b-a)}.$$

This finishes to prove the first part with $a > 0$.

The second part can be deduced from the first (using domination) and the fact that $\gamma : [0, b] \to \mathrm{M}$ is $\hat{\chi}_{\mathrm{C}}$-calibrated if and only if the restriction $\gamma|[a, b]$ is $\hat{\chi}_{\mathrm{C}}$-calibrated, for all $a \in]0, b]$. □

We will now give the relationship between the Aubry set $\mathcal{I}_\infty(\hat{\chi}_{\mathrm{C}})$ of $\hat{\chi}_{\mathrm{C}}$, see Definition 1.9, and the Aubry set $\mathcal{A}^*(\mathrm{C})$ of the closed set C, see Definition 1.4.

Proposition **2.15.** — *If* C *is a closed subset of the complete Riemannian manifold* (M, g), *then we have* $\mathcal{I}_\infty(\hat{\chi}_{\mathrm{C}}) =]0, +\infty[\times (\mathrm{C} \cup \mathcal{A}^*(\mathrm{C}))$.

Proof. — From Lemma 2.14, a curve $\gamma : [0, +\infty[\to \mathrm{M}$ is $\hat{\chi}_{\mathrm{C}}$-calibrated for the Lagrangian L_g, if and only if it is a minimizing g-geodesic satisfying

$$\gamma(0) \in \mathrm{C} \text{ and } d_{\mathrm{C}}(\gamma(t)) = d(\gamma(t), \gamma(0)), \text{ for all } t > 0. \tag{2.12}$$

Therefore, a constant curve with value in C is $\hat{\chi}_{\mathrm{C}}$-calibrating. This implies that $]0, +\infty[\times \mathrm{C} \subset \mathcal{I}_\infty(\hat{\chi}_{\mathrm{C}})$.

For a curve $\gamma : [0, +\infty[\to \mathrm{M}$ and $\lambda > 0$, we define $\gamma_\lambda : [0, +\infty[\to \mathrm{M}$ by

$$\gamma_\lambda(t) = \gamma(\lambda t).$$

Obviously, the curve $\gamma : [0, +\infty[\to \mathrm{M}$ is a minimizing g-geodesic that satisfies (2.12), if and only γ_λ is also a minimizing g-geodesic satisfying (2.12).

Assume now that $(t, y) \in \mathcal{I}_\infty(\hat{\chi}_{\mathrm{C}})$, with $y \notin \mathrm{C}$. We can find a $\hat{\chi}_{\mathrm{C}}$-calibrated curve $\gamma : [0, +\infty[\to \mathrm{M}$, with $\gamma(t) = y$. Since $\gamma(0) \in \mathrm{C}$, the g-geodesic γ is not constant. Since geodesics are parametrized proportionally to arc-length, we can find λ such that γ_λ is parameterized by arc length. As we saw above, the curve γ_λ is a minimizing g-geodesic satisfying (2.12). Since the g-geodesic γ_λ is minimizing and parametrized by arc-length, we get $d_C(\gamma_\lambda(s)) = d(\gamma_\lambda(s), \gamma_\lambda(0)) = s$. By Definition 1.4, this means that $y \in \mathcal{A}^*(C)$. Hence $\mathcal{I}_\infty(\hat{\chi}_C) \subset]0, +\infty[\times(C \cup \mathcal{A}^*(C))$.

It remains to show that $]0, +\infty[\times\mathcal{A}^*(C) \subset \mathcal{I}_\infty(\hat{\chi}_C)$. Suppose $y \in \mathcal{A}^*(C)$. By Definition 1.4 and Remark 1.5, we can find a minimizing g-geodesic $\gamma : [0, +\infty[\to M$ parameterized by arc-length such that $d_C(\gamma(t)) = d(\gamma(0), \gamma(t))$ and $y = \gamma(t_0)$ for some $t_0 > 0$. Therefore, for every $\lambda > 0$, the curve γ_λ is $\hat{\chi}_C$-calibrated. Therefore, we get $(t/\lambda, \gamma_\lambda(t)) \in \mathcal{I}_\infty(\hat{\chi}_C)$, for every $t > 0$. Since $\gamma_\lambda(t_0/\lambda) = \gamma(t_0) = y$, and $\lambda > 0$ is arbitrary, we conclude that $(t, y) \in \mathcal{I}_\infty(\hat{\chi}_C)$, for every $t > 0$. Hence $]0, +\infty[\times\mathcal{A}^*(C) \subset \mathcal{I}_\infty(\hat{\chi}_C)$. □

Example **2.16.** — We specialize to the diagonal $\Delta_M = \{(x, x) \mid x \in M\} \subset M \times M$ of the Riemannian manifold $(M \times M, g \times g)$. In this case

$$\hat{\chi}_{\Delta_M}(x, y) = \frac{d_{\Delta_M}(x, y)^2}{2t} = \frac{d(x, y)^2}{4t}. \tag{2.13}$$

The left hand side equality follows from (2.4), with $C = \Delta_M$. Moreover, for all $z \in M$ we have

$$\sqrt{2}\sqrt{d^2(x, z) + d^2(z, y)} \geq d(x, z) + d(z, y) \geq d(x, y), \tag{2.14}$$

with equalities if and only if z is the midpoint on a minimal geodesic between x and y. This readily implies $\sqrt{2}d_{\Delta_M}(x, y) = d(x, y)$.

The following lemma, left to the reader, sums up the calibration for the distance function to Δ_M.

Lemma **2.17.** — *Let $\Gamma : [0, t] \to M \times M$ be a curve in $M \times M$, with $\Gamma(s) = (\gamma_1(s), \gamma_2(s))$, with $\gamma_1, \gamma_2 : [0, t] \to M$. The curve Γ is $\hat{\chi}_{\Delta_M}$-calibrated if and only if γ_1 and γ_2 are both minimizing g-geodesics, with $\gamma_1(0) = \gamma_2(0)$ and*

$$d(\gamma_1(0), \gamma_1(t)) = d(\gamma_2(0), \gamma_2(t)) = \frac{d(\gamma_1(t), \gamma_2(t))}{2}.$$

Therefore, if $\Gamma = (\gamma_1, \gamma_2) : [0, t] \to M \times M$ is such a $\hat{\chi}_{\Delta_M}$-calibrated curve, then the curve $\gamma : [-t, t] \to M$ defined by

$$\gamma(s) = \begin{cases} \gamma_1(-s), & \text{if } s \in [-t, 0], \\ \gamma_2(s), & \text{if } s \in [0, t], \end{cases} \tag{2.15}$$

is a minimizing g-geodesic.

Conversely, if $\gamma : [-t,t] \to \mathrm{M}$*, is a minimizing g-geodesic, then* $\Gamma = (\gamma_1, \gamma_2) : [0,t] \to \mathrm{M} \times \mathrm{M}$ *defined, for* $s \in [0,t]$*, by*

$$\gamma_1(s) = \gamma(-s) \qquad \gamma_2(s) = \gamma(s) \tag{2.16}$$

is $\hat{\chi}_{\Delta_{\mathrm{M}}}$*-calibrated.*

The following theorems were obtained in [15].

Theorem **2.18.** — *Let* $u : \mathrm{M} \to [-\infty, +\infty]$ *be a function and denote by* $\hat{u} : [0, +\infty[\times \mathrm{M} \to [-\infty, +\infty]$ *its negative Lax-Oleinik evolution. If* $\hat{u}$ *is finite-valued on* $]0, t_0[\times \mathrm{M}$*, then it is continuous and even locally semiconcave on* $]0, t_0[\times \mathrm{M}$.

Moreover, the negative Lax-Oleinik evolution is a viscosity solution, on $]0, t_0[\times \mathrm{M}$*, of* (1.1).

This is a consequence of [15, Theorem 1.1].

Theorem **2.19.** — *If the continuous function* $\mathrm{U} :]0, t_0[\times \mathrm{M} \to \mathbf{R}$ *is a viscosity solution of* (1.1) *on* $]0, t_0[\times \mathrm{M}$*, then* $\mathrm{U} = \hat{u}$ *on* $]0, t_0[\times \mathrm{M}$ *for some function* $u : \mathrm{M} \to [-\infty, +\infty]$.

This is a consequence of [15, Theorem 1.2].

It follows from Theorem 2.19 that Theorem 1.8 for $\mathrm{O} =]0, t_0[\times \mathrm{M}$ is a particular case of Theorem 3.1 below, which states that $\Sigma_{t_0}(\hat{u})$ is locally contractible for any $u : \mathrm{M} \to [-\infty, +\infty]$ such that $\hat{u}$ is finite on $]0, t_0[\times \mathrm{M}$.

We now recall some more facts on $\hat{u}$ obtained, for example, in [15].

Proposition **2.20.** — *For every function* $u : \mathrm{M} \to [-\infty, +\infty]$*, we have* $\hat{u} = \hat{u}_-$ *on* $]0, +\infty[\times \mathrm{M}$*, with* $u_- : \mathrm{M} \to [-\infty, +\infty]$ *the lower semi-continuous regularization of u given by* $u_-(x) = \liminf_{y \to x} u(y) = \sup_{\mathrm{V}} \inf_{y \in \mathrm{V}} u(y)$*, where the supremum is taken over all neighborhoods* V *of x. This function* u_- *is the largest lower semi-continuous function which is* $\leq u$.

Example **2.21.** — If $\mathrm{C} \subset \mathrm{M}$, it is not difficult to see that the lower semi-continuous regularization of the characteristic function χ_{C} is precisely the characteristic function $\chi_{\bar{\mathrm{C}}}$, where $\bar{\mathrm{C}}$ is the closure of C in M.

As a consequence of Proposition 2.20, without loss of generality, we can assume that the function u is lower semi-continuous when we consider properties of $\hat{u}$ away from $\{0\} \times \mathrm{M}$.

This is quite convenient since, as shown in [15] it allows to have backward characteristics and characterize the points where $\hat{u}$ is differentiable.

Proposition **2.22.** — *If the function* $u : \mathrm{M} \to [-\infty, +\infty]$ *is lower semi-continuous, and its negative Lax-Oleinik evolution* $\hat{u}$ *is finite on* $]0, t_0[\times \mathrm{M}$*, with* $t_0 \in]0, +\infty]$*, then, for every* $(t, x) \in$

$]0,t_0[\times M$, *we can find a backward* $\hat{u}$*-characteristic* $\gamma:[0,t]\to M$ *with* $\gamma(t)=x$. *Moreover, the function* $\hat{u}$ *is differentiable at a point* $(t,x)\in]0,t_0[\times M$, *if and only if there exists a unique backward* $\hat{u}$*-characteristic* $\gamma:[0,t]\to M$ *with* $\gamma(t)=x$.

Example **2.23.** — If C is a closed subset of the complete Riemannian manifold (M,g), it is well-known that the inf in the definition of $d_C(x)$ is always attained and that there exists a g-geodesic $\gamma:[0,t]\to M$ with $\gamma(0)\in C$, $\gamma(t)=x$ and $d_C(x)=d(\gamma(0),x)$. It is also well-known that d_C is differentiable at $x\notin C$ if and only if there is a unique (up to reparametrization) minimizing g-geodesic $\gamma:[0,t]\to M$, with $\gamma(0)\in C$, $\gamma(t)=x$ and $d_C(x)=d(\gamma(0),x)$.

Proposition 2.22 generalizes these facts, on the Lagrangian L_g of Example 2.1, to all Tonelli Lagrangians.

Example **2.24.** — If we apply the previous example to the closed subset Δ_M of the complete Riemannian manifold $(M\times M, g\times g)$ we first recover the well known fact that any pair of points x,y in M can be joined by a minimizing geodesic.

Moreover, the function $d^2_{\Delta_M}$ is differentiable at $(x,y)\in M\times M$ if and only if there is a unique (up to reparametrization) minimizing g-geodesic in M joining x to y.

Therefore, recalling from Definition 1.2 that $\mathcal{U}(M,g)$ is the set of $(x,y)\in M\times M$ such that there exists a *unique* minimizing g-geodesic between x and y, we obtain that the set of points in $M\times M$ where $d^2_{\Delta_M}$ is differentiable is precisely $\mathcal{U}(M,g)$. Hence, its complement $M\times M\setminus\mathcal{U}(M,g)=\mathcal{NU}(M,g)$–the set of points $(x,y)\in M$ such that there exists at least two distinct minimizing g-geodesics between x and y–is the set $\Sigma(d^2_{\Delta_M})$ of singularities of $d^2_{\Delta_M}$. Since, the diagonal Δ_M is contained in $\mathcal{U}(M,g)$, we conclude that

$$\Sigma^*(d_{\Delta_M})=\mathcal{NU}(M,g).$$

2.4. *Cut points and cut time function*

In this subsection, we will consider a lower semi-continuous function $u:M\to[-\infty,+\infty]$ such that $\hat{u}$ is finite on $]0,t_0[\times M$, where $t_0\in]0,+\infty]$. By Theorem 2.18, on $]0,t_0[\times M$ the function $\hat{u}$ is locally semiconcave and a viscosity solution of the evolutionary Hamilton-Jacobi equation (1.1). Moreover, by Proposition 2.22, for every $(t,x)\in]0,t_0[\times M$ we can find a backward $\hat{u}$-characteristic ending at (t,x).

Recall that $\Sigma_{t_0}(\hat{u})$ denotes the set of singularities of $\hat{u}$ contained in $]0,t_0[\times M$.

We now introduce the set $\mathrm{Cut}_{t_0}(\hat{u})\subset]0,t_0[\times M$ of cut points of $\hat{u}$.

Definition **2.25** *(*$\mathrm{Cut}_{t_0}(\hat{u})$*).* — *The set* $\mathrm{Cut}_{t_0}(\hat{u})$ *of cut points of* $\hat{u}$ *is the set of points* $(t,x)\in]0,t_0[\times M$ *where no backward* $\hat{u}$*-characteristic ending at* (t,x) *can be extended to a* $\hat{u}$*-calibrated curve defined on* $[0,t']$, *with* $t'>t$.

Lemma **2.26.** — *The point* $(t,x)\in]0,t_0[\times M$ *is in* $\mathrm{Cut}_{t_0}(\hat{u})$ *if and only if for any* $\hat{u}$*-calibrated curve* $\delta:[a,b]\to M$, *with* $t\in[a,b]$ *and* $\delta(t)=x$, *we have* $t=b$.

Proof. — In fact, if δ is as given above, and $\gamma : [0,t] \to \mathrm{M}$ is a backward $\hat{u}$-characteristic ending at (t,x), then the curve defined on $[0,b] \supset [0,t]$ which is equal to γ on $[0,t]$ and δ on $[t,b]$ is continuous and piecewise calibrated on $[0,b]$. Therefore, by Lemma 2.10, it is a $\hat{u}$-calibrated curve extending γ. If $(t,x) \in \mathrm{Cut}_{t_0}(\hat{u})$, then we must have $t = b$. □

Proposition **2.27.** — *Under the hypothesis above, we have*

$$\Sigma_{t_0}(\hat{u}) \subset \mathrm{Cut}_{t_0}(\hat{u}) \subset]0, t_0[\times \mathrm{M} \setminus \mathcal{I}_{t_0}(\hat{u}).$$

Moreover, the set $\Sigma_{t_0}(\hat{u})$ *is dense in* $\mathrm{Cut}_{t_0}(\hat{u})$. *Hence, we have*

$$\Sigma_{t_0}(\hat{u}) \subset \mathrm{Cut}_{t_0}(\hat{u}) \subset \overline{\Sigma_{t_0}(\hat{u})}.$$

Proof. — The fact that $\mathrm{Cut}_{t_0}(\hat{u}) \cap \mathcal{I}_{t_0}(\hat{u}) = \emptyset$ follows from Lemma 2.26. Note now that for every point $(t,x) \in]0, t_0[\times \mathrm{M} \setminus \mathrm{Cut}_{t_0}(\hat{u})$, we can find a $\hat{u}$-calibrated curve $\gamma : [0,t'] \to \mathrm{M}$ with $t' > t$ and $\gamma(t) = x$. Therefore, by Proposition 2.12, the function $\hat{u}$ is differentiable at (t,x). Hence, we get $\Sigma_{t_0}(\hat{u}) \subset \mathrm{Cut}_{t_0}(\hat{u})$.

We now prove the density of the inclusion $\Sigma_{t_0}(\hat{u}) \subset \mathrm{Cut}_{t_0}(\hat{u})$. Suppose that $\mathrm{V} \subset]0, t_0[\times \mathrm{M}$ is an open neighborhood of $(\bar{t}, \bar{x})$ which contains no point of $\Sigma_{t_0}(\hat{u})$. Since $\hat{u}$ is semiconcave and is differentiable everywhere on the open set V it is $\mathrm{C}^{1,1}$, see [8]. Therefore, if we set

$$\mathrm{X}(t,x) = \partial_p \mathrm{H}\left(x, \partial_x \hat{u}(t,x)\right),$$

we conclude that the vector field $\bar{\mathrm{X}}$ on V defined by

$$\bar{\mathrm{X}}(t,x) = \left(1, \partial_p \mathrm{H}\left(x, \partial_x \hat{u}(t,x)\right)\right)$$

is locally Lipschitz. Therefore, we can find a unique solution $\Gamma : [-\epsilon, \epsilon] \to \mathrm{V}$ of $\dot{\Gamma}(t) = \bar{\mathrm{X}}(t, \Gamma(t))$ with $\Gamma(0) = (\bar{t}, \bar{x})$. Given the form of the vector field $\bar{\mathrm{X}}$, we have $\Gamma(s) = (\bar{t} + s, \bar{\gamma}(s))$ with $\bar{\gamma}$ a curve in M such that $\bar{\gamma}(0) = \bar{x}$. It is well known that $\gamma : [\bar{t} - \epsilon, t + \epsilon] \to \mathrm{M}$, defined by $\gamma(t) = \bar{\gamma}(t - \bar{t})$, is $\hat{u}$-calibrated with $\gamma(\bar{t}) = \bar{x}$, see [12, Proposition 3.4, Page 487]. Lemma 2.26 now implies that $(\bar{t}, \bar{x})$ is not in $\mathrm{Cut}_{t_0}(\hat{u})$. Hence any open subset of $]0, t_0[\times \mathrm{M}$ intersecting $\mathrm{Cut}_{t_0}(\hat{u})$ contains points in $\Sigma_{t_0}(\hat{u})$. □

At this point, it is convenient to introduce the cut time function $\tau :]0, t_0[\times \mathrm{M} \to]0, t_0]$ for $\hat{u}$.

Definition **2.28** *(Cut time function).* — *For* $(t,x) \in]0, t_0[\times \mathrm{M}$, *we define* $\tau(t,x)$ *as the supremum of the* $t' \in [t, t_0[$ *such that there exists a* $\hat{u}$*-calibrating curve* $\gamma : [t, t'] \to \mathrm{M}$, *with* $\gamma(t) = x$. *The function* $\tau :]0, t_0[\times \mathrm{M} \to]0, t_0]$ *for* $\hat{u}$ *is the cut time of* $\hat{u}$.

We give a characterization of the cut-time function.

Proposition **2.29.** — *Suppose* $(t, x) \in]0, t_0[\times M$. *Choose a* $\hat{u}$*-calibrated curve* $\gamma : [0, t] \to M$ *with* $\gamma(t) = x$. *Extending* γ *to an extremal* $\gamma : [0, +\infty[\to M$ *of* L, *we have* $\tau(t, x) = \sup\{t' \in]0, t_0[\mid \gamma|[0, t'] \text{ is } \hat{u}\text{-calibrated}\} \geq t$.

Proof. — Set $S = \sup\{t' \in]0, t_0[\mid \gamma|[0, t'] \text{ is } \hat{u}\text{-calibrated}\}$. Since $\gamma : [0, t] \to M$ is $\hat{u}$-calibrated, we indeed have $S \geq t$. Moreover, if $\gamma|[0, t']$ is $\hat{u}$-calibrated, with $t' > t$, then $\gamma|[t, t']$ is also $\hat{u}$-calibrated. Therefore, we get $S \leq \tau(t, x)$.

On the other hand if $\delta : [t, s] \to M$ is $\hat{u}$-calibrated, with $\delta(t) = x$, then the concatenation $\delta * (\gamma|[0, t])$ of $\gamma|[0, t]$ and δ is also $\hat{u}$-calibrated, therefore $\delta * (\gamma|[0, t])$ is a minimizing extremal. Since the concatenation $\delta * (\gamma|[0, t])$ coincides with γ on $[0, t]$, we must have $\delta * (\gamma|[0, t]) = \gamma$ on $[0, s]$. In particular, we get that $\gamma|[0, s]$ is $\hat{u}$-calibrated. Hence, wet get $\tau(t, x) \leq S$. □

Proposition **2.30.** — *The properties of the cut time function* τ *are:*

(i) $\tau(t, x) \in [t, t_0]$;
(ii) $\tau(t, x) = t$ *if and only if* $(t, x) \in \mathrm{Cut}_{t_0}(\hat{u})$;
(iii) $\tau(t, x) = t_0$ *if and only if* $(t, x) \in \mathcal{I}_{t_0}(\hat{u})$;
(iv) *the function* τ *is upper semi-continuous.*

Proof. — Property (i) is obvious. Property (ii) follows from Proposition 2.26.

Property (iii) follows from Proposition 2.29 and the fact that calibrated curves are minimizers (hence extremals).

For part (iv) Assume that $(t_n, x_n) \to (t, x)$ with $\tau(t_n, x_n) \to t'$. We must show that $t' \leq \tau(t, x)$. By contradiction, assume $t' > \tau(t, x) \geq t$. Fix t'' such that $t' > t'' > \tau(t, x) \geq t$. For n large, we have $\tau(t_n, x_n) > t'' > t_n$. Therefore for such an n we can find a $\hat{u}$-calibrated curve $\gamma_n : [0, t''] \to M$, with $\gamma_n(t_n) = x_n$. Extracting if necessary, we can obtain a curve $\gamma : [0, t''] \to M$ which is a C^1 limit of the minimizers γ_n. This curve γ is also $\hat{u}$-calibrated and satisfies $\gamma(t) = x$, this a contradiction since $t'' > \tau(t, x)$. □

3. Local contractibility

Theorem 1.8 is a consequence of the following more general one.

Theorem **3.1.** — *Assume that the function* $u : M \to [-\infty, +\infty]$ *is such that its negative Lax-Oleinik evolution* $\hat{u}$ *is finite at every point of* $]0, t_0[\times M$. *Then the sets* $\Sigma_{t_0}(\hat{u})$ *and* $\mathrm{Cut}_{t_0}(\hat{u})$ *are locally contractible. In particular, they are locally path connected.*

At this point it is useful to introduce the concept of U-adapted homotopy.

Definition **3.2** *(*U*-adapted homotopy).* — *Suppose* $U : [0, t_0] \times M \to \mathbf{R}$ *is a viscosity solution of the evolution Hamilton-Jacobi equation* (1.1). *A continuous homotopy* $F : S \times [0, \delta] \to M$, *with* $\delta > 0$ *and* $S \subset]0, t_0[\times M$, *is said to be* U*-adapted if it satisfies*

(1) *for all* $(t, x) \in \mathrm{S}$, *we have* $t + \delta < t_0$;
(2) *for all* $(t, x) \in \mathrm{S}$, *we have* $\mathrm{F}[(t, x), 0] = x$;
(3) *if* $(t + s, \mathrm{F}[(t, x), s]) \notin \Sigma_{t_0}(\mathrm{U})$, *for some* $(t, x) \in \mathrm{S}$ *and some* $s \in]0, \delta]$, *then the curve* $\sigma \mapsto \mathrm{F}[(x, t), \sigma - t]$, $\sigma \in [t, t + s]$, *is* U*-calibrated.*

Notation **3.3.** — For such a U-adapted homotopy $\mathrm{F} : \mathrm{S} \times [0, \delta] \to \mathrm{M}$, with $\mathrm{S} \subset]0, t_0[\times \mathrm{M}$, we define $\bar{\mathrm{F}} : \mathrm{S} \times [0, \delta] \to]0, t_0[\times \mathrm{M}$ by

$$\bar{\mathrm{F}}[(t, x), s] = (t + s, \mathrm{F}[(t, x), s]) .$$

Of course, the important property of an adapted homotopy is the last one, as can be seen from the proof of the next Proposition.

Proposition **3.4.** — *Assume that the function* $u : \mathrm{M} \to [-\infty, +\infty]$ *is such that its negative Lax-Oleinik evolution* $\hat{u}$ *is finite at every point of* $]0, t_0[\times \mathrm{M}$. *If* $\mathrm{F} : \mathrm{S} \times [0, \delta] \to \mathrm{M}$, *with* $\mathrm{S} \subset]0, t_0[\times \mathrm{M}$, *is a* $\hat{u}$*-adapted homotopy, then*

$$\bar{\mathrm{F}}[(t, x), s] = (t + s, \mathrm{F}[(t, x), s]) \in \Sigma_{t_0}(\hat{u}),$$

for all s *such that* $\tau(t, x) - t < s \leq \delta$, *where* τ *is the cut time function for* $\hat{u}$.

Proof. — Assume that $\bar{\mathrm{F}}[(t, x), s] = (t + s, \mathrm{F}[(t, x), s]) \notin \Sigma_{t_0}(\hat{u})$, then, by part (3) of Definition 3.2, the curve $\sigma \mapsto \mathrm{F}[(x, t), \sigma - t]$, $\sigma \in [t, t + s]$, is $\hat{u}$-calibrated which implies that $t + s \leq \tau(t, x)$. □

Other nice features of U-adapted homotopies are the stability by restriction and composition, given in the following lemma whose proof is immediate.

Lemma **3.5.** — *Suppose* $\mathrm{U} :]0, t_0[\times \mathrm{M} \to \mathbf{R}$ *is a viscosity solution of the evolutionary Hamilton-Jacobi equation* (1.1). *If* $\mathrm{F}_1 : \mathrm{S}_1 \times [0, \delta_1] \to \mathrm{M}$, *and* $\mathrm{F}_2 : \mathrm{S}_2 \times [0, \delta_2] \to \mathrm{M}$ *are two continuous* U*-adapted homotopies, with* $\mathrm{S}_1, \mathrm{S}_2 \subset]0, t_0[\times \mathrm{M}$ *such that* $\bar{\mathrm{F}}_1[(t, x), \delta_1] = (t + \delta_1, \mathrm{F}_1[(t, x), \delta_1]) \in \mathrm{S}_2$, *for all* $(t, x) \in \mathrm{S}_1$, *then the homotopy* $\mathrm{F} : \mathrm{S}_1 \times [0, \delta_1 + \delta_2] \to \mathrm{M}$ *defined by*

$$\mathrm{F}[(t, x), s] = \begin{cases} \mathrm{F}_1[(t, x), s], & \textit{for } s \in [0, \delta_1], \\ \mathrm{F}_2\left[\bar{\mathrm{F}}_1[(t, x), \delta_1], s - \delta_1\right], & \textit{for } s \in [\delta_1, \delta_1 + \delta_2], \end{cases}$$

is itself U*-adapted.*

We will deduce Theorem 3.1 from the lemma below, whose proof will be postponed to section §4.

Lemma **3.6.** — *Assume that the function* $u : \mathrm{M} \to [-\infty, +\infty]$ *is such that its negative Lax-Oleinik evolution* $\hat{u}$ *is finite at every point of* $]0, t_0[\times \mathrm{M}$.

Then, for every compact subset $\mathrm{C} \subset]0, t_0[\times \mathrm{M}$, *we can find* $\delta > 0$ *and a* $\hat{u}$*-adapted homotopy* $\mathrm{F} : \mathrm{C} \times [0, \delta] \to \mathrm{M}$ *such that for every* $(t, x) \in \mathrm{C}$ *and every* $s \in]0, \delta]$, *we have*

$$\begin{aligned}\hat{u}[t+s, \mathrm{F}((t,x),s)] - h_s[x, \mathrm{F}((t,x),s)] &= \max_{z\in \mathrm{M}} \hat{u}(t+s, z) - h_s(x, z) \\ &\geq \hat{u}(t+s, x) - h_s(x, x).\end{aligned} \tag{3.1}$$

Remark **3.7.** — In the proof of Theorem 3.1, we only use the existence of a $\hat{u}$-adapted homotopy. The inequality (3.1) will be used for the application to Riemannian manifolds. More precisely, we will use inequality (3.1) to obtain Proposition 7.3, which is used in the proof of Theorem 1.6.

Proof of Theorem 3.1. — Fix $(\bar{t}, \bar{x}) \in \mathrm{Cut}_{t_0}(\hat{u})$, and a neighborhood of $(\bar{t}, \bar{x})$ of the form $[a, b] \times \mathrm{K}$, with $[a, b] \subset]0, t_0[$ a compact interval, and $\mathrm{K} \subset \mathrm{M}$ a compact subset. Hence $a < \bar{t} < b$ and $\bar{x} \in \mathring{\mathrm{K}}$, where $\mathring{\mathrm{K}}$ is the interior of K.

We now remark that, to prove the theorem, it suffices to find a neighborhood $[\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$ of $(\bar{t}, \bar{x})$ contained in $[a, b] \times \mathrm{K}$ and a homotopy $\mathrm{H} : [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V} \times [0, 1] \to [a, b] \times \mathrm{K}$ such that

(C1) $\mathrm{H}[(t, x), 0] = (t, x)$, for all $(t, x) \in [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$;
(C2) $\mathrm{H}[(t, x), s] \in \Sigma_{t_0}(\hat{u})$ for all $(t, x) \in \mathrm{Cut}_{t_0}(\hat{u}) \cap [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$, and all $s > 0$;
(C3) $\mathrm{H}[(t, x), 1] \in \Sigma_{t_0}(\hat{u})$, for all $(t, x) \in [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$.

In fact, properties (C1) and (C2) show that the inclusion $\mathrm{Cut}_{t_0}(\hat{u}) \cap [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V} \subset \mathrm{Cut}_{t_0}(\hat{u}) \cap [a, b] \times \mathrm{K}$ (resp. $\Sigma_{t_0}(\hat{u}) \cap [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V} \subset \Sigma_{t_0}(\hat{u}) \cap [a, b] \times \mathrm{K}$) is homotopic to $\mathrm{H}(\cdot, 1)$ as maps with values in $\mathrm{Cut}_{t_0}(\hat{u}) \cap [a, b] \times \mathrm{K}$ (resp. $\Sigma_{t_0}(\hat{u}) \cap [a, b] \times \mathrm{K}$). We now observe that, cutting down the neighborhood V of $\bar{x}$ on the manifold M, we can assume that V is contractible in itself. Hence, so is $[\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$. Therefore, by (C3), we obtain that $\mathrm{H}(\cdot, 1)$ on $[\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$ is homotopic to a constant as maps with values in $\Sigma_{t_0}(\hat{u}) \cap [a, b] \times \mathrm{K}$. This clearly finishes the proof of local contractibility for both $\Sigma_{t_0}(\hat{u})$ and $\mathrm{Cut}_{t_0}(\hat{u})$.

It remains to construct H. We first use Lemma 3.6 to find a $\hat{u}$-adapted homotopy $\mathrm{F} : [a, b] \times \mathrm{K} \times [0, \bar{\delta}] \to \mathrm{M}$ for some $\bar{\delta} > 0$. Since $\mathrm{F}\big[(\bar{t}, \bar{x}), 0\big] = \bar{x} \in \mathring{\mathrm{K}}$, we can find $\delta, \epsilon > 0$, with $\delta < \bar{\delta}$, and a neighborhood $\mathrm{V} \subset \mathring{\mathrm{K}}$ of x such that $[\bar{t} - \epsilon, \bar{t} + \epsilon + \delta] \subset]a, b[$, and $\mathrm{F}\big([\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V} \times [0, \delta]\big) \subset \mathring{\mathrm{K}}$. Since the point $(\bar{t}, \bar{x})$ is in $\mathrm{Cut}_{t_0}(\hat{u})$, we have $\tau(\bar{t}, \bar{x}) = \bar{t}$, where τ is the cut time function of $\hat{u}$. Using that τ is upper semi-continuous, we conclude that $\tau(t, x) - t < \delta$ in a neighborhood of $(\bar{t}, \bar{x})$ in $]0, t_0[\times \mathrm{M}$. Hence, cutting down ϵ and V if necessary, we can assume that $\tau(t, x) - t < \delta$, for $(t, x) \in [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}$. Therefore, Proposition (3.4) applied to the $\hat{u}$-adapted homotopy F implies

$$\bar{\mathrm{F}}\,[(t, x), \delta] = (t + \delta, \mathrm{F}\,[(t, x), \delta]) \in \Sigma_{t_0}(\hat{u}),$$
$$\text{for all } (t, x) \in [\bar{t} - \epsilon, \bar{t} + \epsilon] \times \mathrm{V}.$$

We now define the continuous homotopy $H : [\bar{t}-\epsilon, \bar{t}+\epsilon] \times V \times [0,1] \to [a,b] \times K$ by

$$H[(t,x),s] = \bar{F}[(t,x),s\delta] = (t+s\delta, F[(t,x),s\delta])\,. \tag{3.2}$$

Obviously, the homotopy H satisfies the required property (C1) given above. It also satisfies (C3), since $H[(t,x),1] = \bar{F}[(t,x),\delta] \in \Sigma_{t_0}(\hat{u})$, for $(t,x) \in [\bar{t}-\epsilon, \bar{t}+\epsilon] \times V$. To prove (C2), we remark that $\tau(t,x) = t$, for $(t,x) \in \mathrm{Cut}_{t_0}(\hat{u})$, which, again by Proposition (3.4), implies $H[(t,x),s] = \bar{F}[(t,x),s\delta] \in \Sigma_{t_0}(\hat{u})$, for every $s > 0$. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. — If C is a closed subset of the complete Riemannian manifold M, from Example 2.13, the negative Lax-Oleinik evolution $\hat{\chi}_C$ is given, for $t > 0$, by

$$\hat{\chi}_C(x) = \frac{d_C(x)^2}{2t}.$$

The partial derivative $\partial_t \hat{\chi}_C$ is given by

$$\partial_t \hat{\chi}_C(t,x) = -\frac{d_C(x)^2}{2t^2}.$$

Hence, it is defined and continuous everywhere. This implies that

$$\Sigma(\hat{\chi}_C) =]0,+\infty[\times \Sigma(d_C^2), \tag{3.3}$$

where $\Sigma(d_C^2)$, as usual, is the set of points in M where d_C^2 is not differentiable.

From Theorem 3.1, we obtain that $\Sigma(\hat{\chi}_C) =]0,+\infty[\times \Sigma(d_C^2)$ is locally contractible, which implies that $\Sigma(d_C^2)$ is also locally contractible.

We now observe that d_C^2 is differentiable at every point $c \in C$, since $0 \le d_C^2(x) \le d^2(c,x)$. Therefore, from $d_C > 0$ on $M \setminus C$, we get

$$\Sigma(d_C^2) = \Sigma^*(d_C). \tag{3.4}$$

This finishes the proof Theorem 1.1. □

4. Proof of Lemma 3.6

In this section, the function $u : M \to [-\infty,+\infty]$ is lower semi-continuous function and such that $\hat{u}$ is finite on $]0,t_0[\times M$, where $t_0 \in]0,+\infty]$.

Lemma 3.6 will follow from the lemma below.

Lemma **4.1.** — *Assume that the function* $u : \mathrm{M} \to [-\infty, +\infty]$ *is such that its negative Lax-Oleinik evolution* $\hat{u}$ *is finite at every point of* $]0, t_0[\times \mathrm{M}$.

Then, for every compact subset of the form $[a, b] \times \mathrm{K} \subset]0, t_0[\times \mathrm{M}$, *we can find* $\delta > 0$, *with* $[a-\delta, b+\delta] \times \mathrm{K} \subset]0, t_0[\times \mathrm{M}$, *and such that for every* $(t, x) \in [a, b] \times \mathrm{K}$ *and every* $s \in]0, \delta]$, *we can find a* unique $y = y(s, t, x) \in \mathrm{M}$ *such that*

$$\hat{u}(t+s, y) - h_s(x, y) = \max_{z \in \mathrm{M}} \hat{u}(t+s, z) - h_s(x, z).$$

Moreover, for every $r > 0$, *we can find* $\delta(r) \in]0, \delta]$ *such that* $d(x, y(s, t, x)) \le r$, *for all* $(t, x) \in [a, b] \times \mathrm{K}$ *and all* $s \in]0, \delta(r)]$.

Proof. — We first show the second part of this Lemma. Since we will prove the uniqueness by the end of the proof, for now, we will show that for a given $r > 0$, the maximum $\max_{z \in \mathrm{M}} \hat{u}(t+s, z) - h_s(x, z)$, for s small, can only be achieved in $\bar{\mathrm{B}}(x, r)$.

We start with defining by defining the function $\tilde{u} :]0, +\infty[\times[0, +\infty[\times \mathrm{M} \to [-\infty, +\infty]$ by

$$\tilde{u}(t, s, x) = \begin{cases} \sup_{z \in \mathrm{M}} \hat{u}(t+s, z) - h_s(x, z), & \text{for } s > 0, \\ \hat{u}(t, x), & \text{for } s = 0. \quad \square \end{cases} \tag{4.1}$$

Claim **4.2.** — *For all* $(t, s, x) \in]0, +\infty[\times[0, +\infty[\times \mathrm{M} \to [-\infty, +\infty]$, *we have*

$$\hat{u}(t, x) \ge \tilde{u}(t, s, x) \ge \hat{u}(t+s, x) - |\mathrm{A}(0)|s, \tag{4.2}$$

where $\mathrm{A}(0) = \sup\{\mathrm{L}(z, 0) \mid z \in \mathrm{M}\}$.

In particular $\tilde{u}(t, s, x)$ *is finite when* $t + s < t_0$ *and continuous at every* $(t, 0, x)$, *with* $t < t_0$.

Proof. — By definition (4.1) of $\tilde{u}$, the inequalities (4.2) are true for $s = 0$.

For $s > 0$, we have $\tilde{u}(t, s, x) = \sup_{z \in \mathrm{M}} \hat{u}(t+s, z) - h_s(x, z)$. Therefore, the left hand side inequality of (4.2) follows for $\hat{u}(t+s, z) \le \hat{u}(t, x) + h_s(x, z)$, which is (2.10).

For the right hand side, we have $\tilde{u}(t, s, x) = \max_{z \in \mathrm{M}} \hat{u}(t+s, z) - h_s(x, z) \ge \hat{u}(t+s, x) - h_s(x, x)$. But $h_s(x, x)$ is less than the action of the constant path at x, hence

$$h_s(x, x) \le \int_0^s \mathrm{L}(x, 0)\, dt = \mathrm{L}(x, 0)s \le |\mathrm{A}(0)|s,$$

which implies $\tilde{u}(t, s, x) \ge \hat{u}(t+s, x) - h_s(x, x) \ge \hat{u}(t+s, x) - |\mathrm{A}(0)|s$. $\square$

We fix a positive number $\Delta > 0$ such that $[a-\Delta, b+\Delta] \subset]0, t_0[$.

By Theorem 2.18, the function $\hat{u}$ is continuous (and even locally semiconcave) on $]0, t_0[\times \mathrm{M}$. Therefore it is bounded on the compact subset $[a-\Delta, b+\Delta] \times \bar{\mathrm{V}}_2(\mathrm{K})$, where $\bar{\mathrm{V}}_2(\mathrm{K}) = \{y \in \mathrm{M} \mid d_{\mathrm{K}}(y) \le 2\}$. Set

$$\mathrm{B} = \sup\{|\hat{u}(t, x)| \mid (t, x) \in [a-\Delta, b+\Delta] \times \bar{\mathrm{V}}_2(\mathrm{K})\} < +\infty. \tag{4.3}$$

Claim **4.3.** — *For all $s, t > 0$, with $t, t+s \in [a-\Delta, b+\Delta]$ and all $x \in \bar{V}_2(K)$, we have*

$$B \geq \tilde{u}(t, s, x) \geq -B - |A(0)|s \geq -B - |A(0)|t_0. \tag{4.4}$$

Proof. — This follows from (4.2), since (t, x) and $(t+s, x)$ are both in $[a-\Delta, b+\Delta] \times \bar{V}_2(K)$ and $t, t+s \in [a-\Delta, b+\Delta] \subset]0, t_0[$. □

Claim **4.4.** — *For every $r \in]0, 1]$, we can find $\delta(r) \in]0, \Delta]$ such that*

$$\hat{u}(t+s, y) - h_s(x, y) \leq -B - |A(0)|t_0 - 1,$$

$(t, x) \in [a, b] \times \bar{V}_1(K)$, *all $s \in]0, \delta(r)]$, and all $y \in M$ such that $d(x, y) > r$.*

Proof. — To prove this, we take a minimizer $\gamma : [0, s] \to M$ such that $\gamma(0) = x, \gamma(s) = y$. Since $d(x, y) > r$, we can find $s' \in]0, s[$ such that $d(x, \gamma(s')) = r$. Since γ is a minimizer joining x to y, we have $h_s(x, y) = h_{s'}(x, \gamma(s')) + h_{s-s'}(\gamma(s'), y)$. Moreover, the function $\hat{u}$ satisfies $\hat{u}(t+s, y) \leq \hat{u}(t+s', \gamma(s')) + h_{s-s'}(\gamma(s'), y)$. Therefore

$$\hat{u}(t+s, y) - h_s(x, y) \leq \hat{u}(t+s', \gamma(s')) - h_{s'}(x, \gamma(s')) \leq B - h_{s'}(x, \gamma(s')),$$

where the last equality comes from $(t, \gamma(s')) \in [a, b] \times \bar{V}_2(K)$ and $s' \in]0, \Delta]$, since $d(x, \gamma(s')) = r \leq 1$ and $0 < s' < s \leq \Delta$. Integrating (1.4) along the speed curve of $\gamma|[0, s']$, we obtain

$$h_{s'}(x, \gamma(s')) = \int_0^{s'} L(\gamma(t), \dot{\gamma}(t))\, dt \geq K\ell_g(\gamma[[0, s']) - C(K)s',$$

for all $K \geq 0$. Since $\ell_g(\gamma[[0, s']) \geq d(x, \gamma(s')) = r$, we conclude that

$$\hat{u}(t+s, y) - h_s(x, y) \leq B - Kr + C(K)s'.$$

Since $r > 0$, we can then choose K such that $B - Kr \leq -B - |A(0)|t_0 - 2$. Now that K is chosen we can take $\delta(r) = \min(1/C(K), \Delta)$. Hence, if $s \leq \delta(r)$, since $s' \leq s$, we get

$$\begin{aligned}\hat{u}(t+s, y) - h_s(x, y) &\leq B - Kr + C(K)s' \leq -B - |A(0)|t_0 - 2 + C(K)s' \\ &\leq -B - |A(0)|t_0 - 1,\end{aligned}$$

as desired. This finishes the proof of 4.4. □

Claim 4.4 implies $\{y \in M \mid \hat{u}(t+s, z) - h_s(x, z) \geq -B - |A(0)|\Delta\} \subset \bar{B}(x, r)$, for all $(t, x) \in [a, b] \times \bar{V}_1(K)$, all $r \in]0, 1]$ and all $s \in]0, \delta(r)]$.

Therefore from (4.4), for $(t,x) \in [a,b] \times \bar{V}_1(K)$ and $s \in]0, \delta(r)]$, where $r \leq 1$, we have

$$\tag{4.5} \begin{aligned} \tilde{u}(t,s,x) = \max_{y \in M} u(t+s,y) - h_s(x,y) &= \max_{y \in \bar{B}(x,r)} u(t+s,y) - h_s(x,y) \\ &= \max_{y \in \bar{V}_2(K)} u(t+s,y) - h_s(x,y). \end{aligned}$$

Hence, this maximum is achieved at a point in the compact subset $\bar{B}(x,r)$ (and only at points in $\bar{B}(x,r)$).

Since $y \in \bar{V}_2(K)$ is compact, the equality 4.5 (together with Claim 4.2 to cover the case $s = 0$) implies that the function $\tilde{u}$ is continuous on $[a,b] \times [0, \delta(1)] \times \bar{V}_1(K)$.

To prove the uniqueness part of Lemma 4.1, we need to introduce the positive Lax-Oleinik semi-group T_t^+, $t \geq 0$, whose definition we now recall.

If $u : M \to [-\infty, +\infty]$ is a function and $t > 0$, the function $T_t^+ u : M \to [-\infty, +\infty]$ is defined by

$$\tag{4.6} T_t^+ u(x) = \sup_{y \in M} u(y) - h_t(x,y).$$

We also set $T_0^+ u = u$.

Remark **4.5.** — As is well-known, the semi-group $\check{T}_t^-$ defined by $\check{T}_t^-(u) = -T_t^+(-u)$ is in fact the negative Lax-Oleinik semi-group associated with the Lagrangian $\check{L} : TM \to \mathbf{R}$ defined by $\check{L}(x,v) = L(x,-v)$.

For $u : M \to [-\infty, +\infty]$, it is convenient to define $\check{u} : [0, +\infty[\times M \to [-\infty, +\infty]$ by

$$\check{u}(t,x) = T_t^+ u(x).$$

We will call $\check{u}$ the *positive* Lax-Oleinik evolution.

Recalling that $\hat{u}(t+s,y) = T_{t+s}^- u(y)$, for t, s, we see that

$$\tag{4.7} T_s^+ T_{t+s}^- u(x) = \max_{y \in M} \hat{u}(t+s,y) - h_s(y,x) = \tilde{u}(t,s,x).$$

This equality is also valid for $s = 0$, from the definition of T_0^+ and $\tilde{u}$.

Claim **4.6.** — *For $t, s > 0$, with $t, t+s \in]0, t_0[$, the function $T_s^+ T_{t+s}^- u$ is locally semi-convex.*

Proof. — Fix $t, s > 0$, with $t, t+s \in]0, t_0[$, set $v = T_{t+s}^- u$. Then $T_s^+ T_{t+s}^- u = \check{v}(s, \cdot)$ and $s \in]0, t+s[$. We first show that $\check{v}$ is finite everywhere on $]0, t+s[\times M$. We have $\check{v}(s,x) = \sup_{z \in M} \hat{u}(t+s,z) - h_s(x,z)$. Hence $\check{v}(s,x) \geq u(t+s,x) - h_s(x,x) > -\infty$ since $t+s \in]0, t_0[$.

From (2.10), we obtain $\hat{u}(t+s, z) \leq \hat{u}(t, x) + h_s(x, z)$. Hence $\check{v}(s, x) \leq \hat{u}(t, x) < +\infty$, where the last inequality follows from $t \in]0, t_0[$.

Since $\check{v}$ is finite everywhere on $]0, t+s[\times M$, using Remark 4.5 above, we obtain from Theorem 2.18 that the positive Lax-Oleinik evolution $\check{v}$ is locally semi-convex on $]0, t+s[\times M$. Therefore $T_s^+ T_{s+t}^- u = \check{v}(s, \cdot)$ is locally semi-convex on M. □

The main point to prove the uniqueness statement in Lemma 4.1 is the following Lemma.

Claim **4.7.** — *There exists* $\delta \in]0, \delta(1)]$, *such that* $T_s^+ T_{s+t}^- u$ *is* C^1 *(and even* $C^{1,1}$*) on* $\mathring{V}_{1/2}(K) = \{y \in | \, d(y, K) < 1/2\}$, *for* $t \in [a, b]$, *and* $s \in]0, \delta]$.

Proof. — As we know already from Claim 4.6 that $T_s^+ T_{s+t}^- u$ is locally semi-convex, it suffices to prove that $T_s^+ T_{s+t}^- u$ is locally semi-concave on a neighborhood of $\bar{V}_{1/2}(K)$.

Since $\hat{u}$ is locally semi-concave on $]0, t_0[\times M$, we obtain that the family of functions $(T_t^- u)_{t \in [a-\Delta, b+\Delta]}$ is equicontinuous and equi-semiconcave on the compact neighborhood $\bar{V}_1(K)$ of K.

If M is compact, we could take $K = M$, and the C^1 property above follows from [5].

For the noncompact case, we will use [17, Appendix B], which adapts some of the results of [5] to the noncompact setting.

To be able to use [17, Appendix B], it is useful to introduce, for $r > 0$ a modified family $T_t^{+,r}$ of positive Lax-Oleinik operators defined by

$$T_t^{+,r} u(x) = \sup\{u(y) - h_t(x, y) \mid d(y, x) \leq r\},$$

for $u : M \to [-\infty, +\infty]$.

Using a covering of a compact set by a finite number of domains of charts of the manifold M, it is not difficult to see that the following lemma is a consequence of the positive Lax-Oleinik version of [17, Lemma B.7]. □

Lemma **4.8.** — *Let* K_1 *and* K_2 *be two compact subsets of* M *with* $K_1 \subset \mathring{K}_2$.

Assume that $\mathcal{F}^+$ *is a family of functions from* M *to* **R**, *which is equi-continuous and equi-semiconcave on* K_2. *Then there exits* $r_0 \in]0, 1]$, *and* $\delta_0 > 0$, *such that the family* $\{T_t^{+,r_0} w \mid w \in \mathcal{F}^+, t \in [0, \delta_0]\}$ *is equi-semiconcave on* K_1.

End of proof of Claim 4.7. — We apply Lemma 4.8 above with the family of functions $\mathcal{F}^+ = (T_t^- u)_{t \in [a-\Delta, b+\Delta]}$ and the compact sets $K_2 = \bar{V}_1(K)$, $K_1 = \bar{V}_{1/2}(K)$. Note that by (4.5), we have

$$T_s^+ T_{s+t}^- u(x) = T_s^{+,r} T_{s+t}^- u(x), \text{ for } t \in [a, b], s \in]0, \delta(r)] \text{ and } x \in \bar{V}_1(K).$$

Therefore by Lemma 4.8, we get that, for $s \leq \bar{\delta} = \min(\delta_0, \delta(r_0), \delta(1))$ and $t \in [a, b]$, the function $T_s^+ T_{s+t}^- u$ is semiconcave on $\bar{V}_{1/2}(K)$. We conclude that, for $s \in]0, \delta]$, $t \in [a, b]$, the function $T_s^+ T_{s+t}^- u$ is indeed $C^{1,1}$ on $\mathring{V}_{1/2}(K)$. This finishes the proof of Claim 4.7. □

As we will show below, the construction of the homotopy in Lemma 3.6 will follow easily from the uniqueness given in Lemma 4.1.

Claim **4.9.** — *Fix* δ *given by Claim 4.7. Assume* $(t, s, x) \in [a, b] \times]0, \delta[\times K$. *Then there exists a unique* $y = y(t, s, x) \in M$ *such that*

$$\tilde{u}(t, s, x) = T_{s+t}^- u(y) - h_s(x, y). \tag{4.8}$$

Moreover, if $s \leq \min(\delta, \delta(r))$, *where* $\delta(r)$ *is given by Lemma 4.1, we have* $d(y, x) \leq r$, *for the* y *given by* (4.8).

Proof. — We have

$$\tilde{u}(t, s, x) = \max_{y \in M} u(t + s, y) - h_s(x, y).$$

Therefore, the existence of $y = y(t, s, x)$, as we saw, follows from (4.5).

We now show the uniqueness. Suppose that $y = y(t, s, x)$ is a point where $u(t + s, y) - h_s(x, y)$ achieves its maximum $\tilde{u}(t, s, x)$. Calling $\gamma : [0, s] \to M$ the minimizer such that $\gamma(0) = x$ and $\gamma(s) = y$, we have

$$T_s^+ T_{s+t}^- u(\gamma(0)) = \tilde{u}(t, s, \gamma(0)) = T_{s+t}^- u(\gamma(s)) - h(\gamma(0), \gamma(s)).$$

Since $T_s^+ T_{s+t}^- u$ is differentiable at $x = \gamma(0)$, we use the T_t^+ version of Proposition 2.12 (again this version follows from Remark 4.5) to obtain that

$$d_x T_s^+ T_{s+t}^- u = \partial_v L(\gamma(0), \dot{\gamma}(0)).$$

Hence the curve $\gamma : [0, s] \to M$ is unique, since it is (as all minimizers) a solution of the Euler-Lagrange equation. This implies the uniqueness of $y = \gamma(s)$. □

By Claim 4.9, we can define the function $F : ([a, b] \times K) \times]0, \delta[\to M$ by taking as $F[(t, x), s]$ the unique $y = y(t, s, x) \in M$ such that

$$\tilde{u}(t, s, x) = T_{s+t}^- u(y) - h_s(x, y). \tag{4.9}$$

In other words $F[(t, x), s]$ is the unique y, where $T_{s+t}^- u(y) - h_s(x, y)$ achieves its maximum. We extend this function F to $([a, b] \times K) \times [0, \eta[$ by $F[(t, x), 0] = x$. Therefore F satisfies part (2) of the Definition 3.2 of $\hat{u}$-adapted.

Claim **4.10.** — *The homotopy* $F : ([a, b] \times K) \times [0, \delta] \to M$, *defined above, is continuous and* $\hat{u}$*-adapted.*

Proof. — We first show that F is continuous on $([a, b] \times K)\times]0, \delta]$. For this, we remark that, since $\delta \leq \delta(1)$, the map F takes values in the compact set $\bar{V}_2(K)$. Therefore, to show that F is continuous on $([a, b] \times K)\times]0, \delta]$, it suffices to show that its graph is closed in $([a, b] \times K)\times]0, \delta] \times \bar{V}_2(K)$. By the definition of F, its graph Graph(F) is given by

$$\begin{aligned}&\mathrm{Graph}(F) = \{\big((t, x), s, y\big) \in ([a, b] \times K)\times]0, \delta] \times \bar{V}_2(K) \mid \\ &\tilde{u}(t, s, x) = T^-_{s+t}u(y) - h_s(x, y)\}.\end{aligned}$$

Since, as we observed above, the three functions $(t, s, x) \mapsto \tilde{u}(t, s, x)$, $(t, s, y) \mapsto T^-_{s+t}u(y)$ and $(s, x, y) \mapsto h_s(x, y)$ are all continuous for $(t, s, x, y) \in [a, b] \times]0, \delta] \times K \times M$, we conclude that Graph(F) is closed.

To show that F is continuous at a point in $([a, b] \times K) \times \{0\}$, using Lemma 4.1, we note that $d(x, F[(t, x), s]) \leq r$, for $0 < s \leq \delta(r)$.

We now check condition (3) of Definition 3.2 for F. Fix $(t, x) \in [a, b] \times K$. Assume that $\hat{u}$ is differentiable at $(t + s, F[(t, x), s])$, for some $s > 0$. Choose a minimizer $\gamma : [t, t+s] \to M$, with $\gamma(t) = x$ and $\gamma(t + s) = F[(t, x), s]$. Since

$$T^+_s T^-_{s+t}u(\gamma(t)) = \tilde{u}(t, s, \gamma(t)) = T^-_{s+t}u(\gamma(t + s)) - h(\gamma(t), \gamma(t + s)),$$

by the T^+_τ version of Proposition 2.12, we obtain

$$d_{\gamma(t+s)}(T^-_{s+t}u) = \partial_v L(\gamma(t + s), \dot{\gamma}(t + s)).$$

We now observe that, by the same Proposition 2.12, the backward $\hat{u}$-characteristic $\bar{\gamma} : [0, t + s] \to M$ ending at $\gamma(t + s)$ satisfies

$$d_{\bar{\gamma}(t+s)}(T^-_{s+t}u) = \partial_v L(\bar{\gamma}(t + s), \dot{\bar{\gamma}}(t + s)).$$

Since $\gamma(t + s) = \bar{\gamma}(t + s)$, we conclude that the two extremals γ and $\bar{\gamma}$ have the same position and speed at time $t + s$, therefore $\gamma = \bar{\gamma}|[t, t + s]$. In particular, the curve γ is $\hat{u}$-calibrated.

To finish the proof that F satisfies part (3) of Definition 3.2, since $\gamma : [t, t+s] \to M$ satisfies with $\gamma(t) = x$, $\gamma(t + s) = F[(t, x), s]$ and is $\hat{u}$-calibrated, it suffices to apply the next lemma which shows that $\gamma(t + s') = F[(t, x), s']$, for all $s' \in [0, s]$. □

Lemma **4.11.** — *If* $\gamma : [t, t + s] \to M$ *is* $\hat{u}$*-calibrated then, for all* $s' \in [0, s]$, *we have*

$$\tilde{u}(t, s', \gamma(t)) = \hat{u}(t, \gamma(t)) = \hat{u}(s' + t, \gamma(t + s')) - h_{s'}(\gamma(t), \gamma(t + s')).$$

Therefore, the function F *defined above, see* (4.9), *satisfies*

$$\gamma(t + s') = F[(t, x), s'], \ \textit{for all } s' \in [0, \min(s, \delta)].$$

Proof. — Note that the equality $\hat{u}(t,\gamma(t)) = \hat{u}(s'+t,\gamma(t+s')) - h_{s'}(\gamma(t),\gamma(t+s'))$ is equivalent to $\hat{u}(s'+t,\gamma(t+s')) = \hat{u}(t,\gamma(t)) + h_{s'}(\gamma(t),\gamma(t+s'))$, which is a consequence of the $\hat{u}$-calibration of γ. Therefore, it remains to show that $\tilde{u}(t,s',\gamma(t)) = \hat{u}(t,\gamma(t))$. Note that the inequality $\tilde{u}(t,s',\gamma(t)) \leq \hat{u}(t,\gamma(t))$ follows from (4.2). On the other hand, by definition (4.1) of $\tilde{u}$, we have $\tilde{u}(t,s',\gamma(t)) \geq \hat{u}(s'+t,\gamma(t+s')) - h_{s'}(\gamma(t),\gamma(t+s')) = \hat{u}(t,\gamma(t))$. □

To finish the proof of Lemma 3.6, it remains to observe that (3.1) follows from the definition of F, and that any compact subset C of $]0,t_0[\times M$ is contained in a subset of the form $[a,b]\times K$, where K is compact subset of M and $[a,b] \subset]0,t_0[$.

Remark **4.12.** — It follows from Lemma 4.11, the homotopy $F : C\times[0,\delta]\to M$ constructed in Lemma 3.6 satisfies the following condition:

(CC) If $\gamma : [t,t+s]\to M$ is $\hat{u}$-calibrated, with $\gamma(t) = x$, then $F((t,x),s') = \gamma(t+s')$, for all $s' \in [0,\min(\delta,s)]$.

Note also that, if the adapted homotopies F_1, F_2 of Lemma 3.5 satisfy condition (CC), so does the homotopy F obtained from them in that Lemma 3.5.

Therefore, as the reader, will realise, all adapted homotopies that we construct will also satisfy condition (CC).

5. Constructions of global homotopy equivalences

The goal of this section is to establish some results that will help to construct homotopies of the type mentioned in the proof of Theorem 1.10.

Proposition **5.1.** — *Assume* $u : M \to [-\infty,+\infty]$ *is such that* $\hat{u}$ *is finite on* $]0,+\infty[\times M$. *If there exists* $\delta > 0$ *such that, for every compact subset* $C \subset]0,+\infty[\times M$, *we can find a* $\hat{u}$*-adapted homotopy* $F : C\times[0,\delta]\to M$, *then, for every* $T \in]0,+\infty]$, *the inclusions*

$$\Sigma_T(\hat{u}) \subset \mathrm{Cut}_T(\hat{u}) \subset]0,+\infty[\times M \setminus \mathcal{I}_T(\hat{u})$$

are all homotopy equivalences.

We reduce the proof of Proposition 5.1 to several lemmas.

Lemma **5.2.** — *Under the hypothesis of Proposition 5.1, for every compact subset* $C \subset]0,+\infty[\times M$, *there exists a* $\hat{u}$*-adapted homotopy* $F : C\times[0,+\infty[\to M$.

Proof. — We show, by induction on the integer $n\geq 1$, how to extend the $\hat{u}$-adapted homotopy $F : C\times[0,\delta]\to M$ to a $\hat{u}$-adapted homotopy $F : C\times[0,n\delta]\to M$. Assume $F : C\times[0,n\delta]\to M$ is constructed for some $n\geq 1$. As introduced in Notation 3.3, we define $\bar{F} : C\times[0,n]\to]0,+\infty[\times M$ by

$$\bar{F}[(t,x),s] = (t+s, F[(t,x),s])\,.$$

Since $\bar{F}$ is continuous, the subset $C_n = \bar{F}[C \times \{n\delta\}]$ is a compact subset of $]0, +\infty[\times M$. Therefore, by the hypothesis of Proposition 5.1 applied to C_n instead of C, we can find a $\hat{u}$-adapted homotopy $F_n : C_n \times [0, \delta] \to M$. By Lemma 3.5, if we extend F to $C \times [n\delta, (n+1)\delta]$, by $F[(t, x), s] = F_n\left[\bar{F}[(t, x), n], s - n\right]$, it will be $\hat{u}$-adapted on $C \times [0, (n+1)\delta]$. □

Lemma **5.3.** — *Under the hypothesis of Proposition 5.1, for every compact subset* C *of* $]0, T[\times M \setminus \mathcal{I}_T(\hat{u})$, *where* $T \in]0, +\infty]$, *we can find a continuous homotopy*

$$G : \big(]0, T[\times M \setminus \mathcal{I}_T(\hat{u})\big) \times [0, 1] \to]0, T[\times M \setminus \mathcal{I}_T(\hat{u})$$

such that:

$$G[(t, x), 1] \in \Sigma_T(\hat{u}), \text{ for all } (t, x) \in C, \tag{5.1}$$

$$G[(t, x), 0) = (t, x), \text{ for all } (t, x) \in]0, T[\times M \setminus \mathcal{I}_T(\hat{u}), \tag{5.2}$$

and

$$G(\Sigma_T(\hat{u}) \times [0, 1]) \subset \Sigma_T(\hat{u}), \tag{5.3}$$

$$G(\mathrm{Cut}_T(\hat{u}) \times [0, 1]) \subset \mathrm{Cut}_T(\hat{u}). \tag{5.4}$$

Proof. — We choose C′ a compact neighborhood of C in $]0, T[\times M \setminus \mathcal{I}_T(\hat{u})$. We apply Lemma 5.2 for C′ (instead of C) to obtain the homotopy $F : C' \times [0, +\infty[\to M$. As introduced in Notation 3.3, we define $\bar{F} : C' \times [0, +\infty[\to]0, +\infty[\times M$ by

$$\bar{F}[(t, x), s] = (t + s, F[(t, x), s])\,.$$

We first observe that the image of $\bar{F}$ avoids $\mathcal{I}_T(\hat{u})$. In fact, if $(t + s, y) = \bar{F}[(t, x), s] = (t + s, F[(t, x), s])$ is in $\mathcal{I}_T(\hat{u})$, then $0 < t + s < T$, and there exists a $\hat{u}$-calibrated curve $\gamma : [0, T[\to M$ such that $\gamma(t + s) = y$. By Proposition 2.12, this implies that $\hat{u}$ is differentiable at $(t + s, y)$ and any $\hat{u}$-calibrated curve $\delta : [a, t + s] \to M$ ending at y must coincide with $\gamma|[a, t + s]$. Using again that $\hat{u}$ is differentiable at $(t + s, y) = \bar{F}[(t, x), s] = (t + s, F[(t, x), s])$, and that F is $\hat{u}$-adapted, we obtain that the curve $\sigma \to F[(t, x), \sigma - t], \sigma \in [t, t + s]$, which ends at y, is $\hat{u}$-calibrated. Therefore $F[(t, x), \sigma - t] = \gamma(\sigma)$, and $\bar{F}[(t, x), \sigma - t] \in \mathcal{I}_T(\hat{u})$, for $\sigma \in [t, t + s]$. In particular, for $\sigma = t$, this would imply $(t, x) \in \mathcal{I}_T(\hat{u})$, which is impossible, since (t, x) is in C′ which is disjoint from $\mathcal{I}_T(\hat{u})$. Thus we obtained

$$F : C' \times]0, +\infty[\to M \setminus \mathcal{I}_T(\hat{u}). \tag{5.5}$$

Since F is $\hat{u}$-adapted, we must have

$$\bar{F}\left((C' \cap \Sigma_T(\hat{u})) \times [0, +\infty[\right) \subset \Sigma_T(\hat{u}) \tag{5.6}$$

$$\bar{F}\left((C' \cap \mathrm{Cut}_T(\hat{u})) \times [0, +\infty[\right) \subset \mathrm{Cut}_T(\hat{u}). \tag{5.7}$$

Since C' is a compact subset of $]0,T[\times M\setminus\mathcal{I}_T(\hat u)$, and the upper semi-continuous τ is $<$ T everywhere in $]0,T[\times M\setminus\mathcal{I}_T(\hat u)$, we can choose a finite $T_0<T$ such that the cut time function τ is $<T_0$ on C'. Since C' is a compact neighborhood of C in $]0,T[\times M\setminus\mathcal{I}_T(\hat u)$, we can find a continuous function $\alpha: C'\to[0,1]$ such that α is identically equal to 1 on C, and α is identically 0 on $\partial C'$, the boundary of C' in $]0,T[\times M\setminus\mathcal{I}_T(\hat u)$. We define $G: C'\times[0,1]\to]0,+\infty[\times M$ by

$$G[(t,x),s]=\bar F[(t,x),s\alpha(t,x)(T_0-t)].$$

In particular, we get

$$G[(t,x),s]=(t+s\alpha(t,x)(T_0-t),F[(t,x),s\alpha(t,x)(T_0-t)])\,.$$

Since $t+s\alpha(t,x)(T_0-t)\le T_0$, the image $G(C'\times[0,1])$ of G is, in fact, contained in $]0,T_0]\times M$. Taken together with (5.5), it implies

$$G(C'\times[0,1])\subset]0,T[\times M\setminus\mathcal{I}_T(\hat u).$$

Properties (5.2), (5.3) and (5.4) of G on $C'\times[0,1]$ follow from properties (5.6) and (5.7) of $\bar F$. Since for $(t,x)\in C$, we have $\alpha(t,x)=1$, we get

$$G[(t,x),1]=(t+(T_0-t),F[(t,x),(T_0-t)])=(T_0,F[(t,x),(T_0-t)])\,.$$

Since F is $\hat u$-adapted and $\tau<T_0$ on $C'\supset C$, we obtain property (5.1), from Proposition 3.4.

Since $G[(t,x),s]=(t,x)$, for $(t,x)\in\partial C'$, we can extend G continuously to $(]0,T_0[\times M\setminus\mathcal{I}_T(\hat u))\times[0,1]$ by $G[(t,x),s]=(t,x)$, for $(t,x)\notin C'$.

It is not difficult to check that this extension G still has the required properties (5.1) to (5.4). □

Proof of Proposition 5.1. — We can find a sequence of compact subsets C_n, $n\ge 1$ of $]0,T[\times M\setminus\mathcal{I}_T(\hat u)$, such that $C_n\subset\mathring{C}_{n+1}$, and $]0,T[\times M\setminus\mathcal{I}_T(\hat u)=\cup_{n\ge0}C_n$. We construct a homotopy $H:\big(]0,T[\times M\setminus\mathcal{I}_T(\hat u)\big)\times[0,+\infty[\to]0,T[\times M\setminus\mathcal{I}_T(\hat u)$ such that $H(\Sigma_T(\hat u)\times[0,+\infty[)\subset\Sigma_T(\hat u)$, $H(\mathrm{Cut}_T(\hat u)\times[0,+\infty[)\subset\mathrm{Cut}_T(\hat u)$, $H((t,x,0)=(t,x)$, for all $(t,x)\in\,]0,T[\times M\setminus\mathcal{I}_T(\hat u)$, and

$$H(C_n\times[n,+\infty[)\subset\Sigma_T(\hat u),\ \text{ for all } n\ge1.$$

We will construct H on $]0,T[\times M\setminus\mathcal{I}_T(\hat u)\times[0,n]$ by induction on $n\ge1$.

We start by applying Lemma 5.3 to the compact set C_1 to obtain the homotopy

$$G_1:\big(]0,T[\times M\setminus\mathcal{I}_T(\hat u)\big)\times[0,1]\to]0,T[\times M\setminus\mathcal{I}_T(\hat u),$$

with

$$G_1[(t,x),0]=(t,x),\ \text{ for all } (t,x)\in\big(]0,T[\times M\setminus\mathcal{I}_T(\hat u)\big),$$

and

$$G_1(C_1 \times \{1\} \subset \Sigma_T(\hat{u}),$$
$$G_1(\Sigma_T(\hat{u}) \times [0,1]) \subset \Sigma_T(\hat{u}),$$
$$G_1(\mathrm{Cut}_T(\hat{u}) \times [0,1]) \subset \mathrm{Cut}_T(\hat{u}).$$

We then set $H[(t,x),s] = G_1[(t,x),s]$, for $(t,x) \in 0, T[\times M \setminus \mathcal{I}_T(\hat{u}), s \in [0,1]$. Assuming that H has been constructed on $\big(]0,T[\times M \setminus \mathcal{I}_T(\hat{u})\big) \times [0,n]$, we apply Lemma 5.3 to the compact set $H(C_{n+1} \times \{n\})$ to obtain the homotopy

$$G_n : \big(]0,T[\times M \setminus \mathcal{I}_T(\hat{u})\big) \times [0,1] \to]0,T[\times M \setminus \mathcal{I}_T(\hat{u}),$$

with

$$G_n[(t,x),0] = (t,x), \text{ for all } (t,x) \in \big(]0,T[\times M \setminus \mathcal{I}_T(\hat{u})\big),$$

and

$$G_n[H(C_{n+1} \times \{n\}) \times \{1\}] \subset \Sigma_T(\hat{u}),$$
$$G_n(\Sigma_T(\hat{u}) \times [0,1]) \subset \Sigma_T(\hat{u}),$$
$$G_n(\mathrm{Cut}_T(\hat{u}) \times [0,1]) \subset \mathrm{Cut}_T(\hat{u}).$$

We then define H on $\big(]0,T[\times M \setminus \mathcal{I}_T(\hat{u})\big) \times [n, n+1]$ by

$$H[(t,x),s] = G_n(H[(t,x),n], t-n).$$

It is not difficult to check that H satisfies the required properties.

Since $C_n \subset \mathring{C}_{n+1}$, we can define a continuous function $\alpha :]0,T[\times M \setminus \mathcal{I}_T(\hat{u}) \to \mathbf{R}$ such that $\alpha = n+1$ on ∂C_n, $\alpha(C_0) \subset [0,1]$, and $\alpha(C_n \setminus \mathring{C}_{n-1}) \subset [n, n+1]$. Therefore $H[(t,x), \alpha(t,x)] \in \Sigma_T(\hat{u})$, for all $(t,x) \in]0,T[\times M \setminus \mathcal{I}_T(\hat{u})$.

We now define the homotopy $\tilde{H} : M \setminus \mathcal{I}(u) \times [0,1] \to M \setminus \mathcal{I}(u)$ by

$$\tilde{H}[(x,t),s] = H[(t,x), s\alpha(t,x)].$$

It is not difficult to check that $\tilde{H}[(x,t),0] = (x,t), \tilde{H}[(x,t),1] \in \Sigma_T(\hat{u})$, for all $(t,x) \in]0,T[\times M \setminus \mathcal{I}_T(\hat{u})$, $\tilde{H}(\Sigma_T(\hat{u}) \times [0,1]) \subset \Sigma_T(\hat{u})$, and $\tilde{H}(\mathrm{Cut}_T(\hat{u}) \times [0,1]) \subset \mathrm{Cut}_T(\hat{u})$. This finishes the proof of Proposition 5.1. □

Here is a useful criterion that allows to show that the hypothesis of Proposition 5.1 holds.

Lemma **5.4.** — *Let* $u : \mathrm{M} \to [-\infty, +\infty]$ *be such that* $\hat{u}$ *is finite on* $]0, +\infty[\times \mathrm{M}$. *Assume for every* $t_0 > 0$, *we can find a constant* κ_{t_0}, *such that, for every compact subset* $\mathrm{C} \subset [t_0, +\infty[\times \mathrm{M}$, *we can find* $\delta > 0$ *and a* $\hat{u}$*-adapted homotopy* $\mathrm{F} : \mathrm{C} \times [0, \delta] \to \mathrm{M}$, *such that*

$$d(\mathrm{F}[(t, x), s], x) \leq \kappa_{t_0} s, \text{ for all } (t, x) \in \mathrm{C} \text{ and all } s \in [0, \delta].$$

Then for every compact subset $\mathrm{C} \subset]0, +\infty[\times \mathrm{M}$, *we can find a* $\hat{u}$*-adapted homotopy* $\tilde{\mathrm{F}} : \mathrm{C} \times [0, +\infty[\to \mathrm{M}$.

Proof. — By Proposition 5.1, it suffices to show that for such a compact subset of $\mathrm{C} \subset]0, +\infty[\times \mathrm{M}$, we can find a $\hat{u}$-adapted homotopy $\tilde{\mathrm{F}} : \mathrm{C} \times [0, 1] \to \mathrm{M}$.

We first find $a, b \in]0, +\infty[$, with $a < b$, and a compact subset $\mathrm{K} \subset \mathrm{M}$ such that $\mathrm{C} \subset [a, b] \times \mathrm{K}$.

We now use the hypothesis of the Lemma, applied to the compact set $[a, b+1] \times \bar{\mathrm{V}}_{\kappa_a}(\mathrm{K}) \subset [a, +\infty[\times \mathrm{M}$, to find a $\hat{u}$-adapted homotopy

$$\mathrm{F} : \big([a, b+1] \times \bar{\mathrm{V}}_{\kappa_a}(\mathrm{K})\big) \times [0, \delta] \to \mathrm{M},$$

such that, for all $(t, x) \in [a, b+1] \times \bar{\mathrm{V}}_{\kappa_a}(\mathrm{K})$ and all $s \in [0, \delta]$, we have

$$d\left(\mathrm{F}[(t, x), s], x\right) \leq \kappa_a s. \tag{5.8}$$

If $\delta \geq 1$ we have finished. If $\delta < 1$, we will use an argument analogous to the one in the proof of Proposition 5.1. Choose $n_0 \geq 2$ such that $(n_0 - 1)\delta < 1 \leq n_0\delta$. Note that $n_0\delta < 1 + \delta \leq 2$. By induction on $n = 1, \ldots, n_0$, we will construct an extension of F on $\mathrm{C} \times [0, \delta]$ to a $\hat{u}$-adapted homotopy $\tilde{\mathrm{F}} : \mathrm{C} \times [0, n\delta] \to \mathrm{M}$ which also satisfies

$$d\left(\tilde{\mathrm{F}}[(t, x), s], x\right) \leq \kappa_a s, \text{ for all } (t, x) \in \mathrm{C} \text{ and all } s \in [0, n\delta]. \tag{5.9}$$

For $n = 1$, we just take as $\tilde{\mathrm{F}}$ the restriction of F to $\mathrm{C} \times [0, \delta]$. Assuming that $\tilde{\mathrm{F}} : \mathrm{C} \times [0, n\delta] \to \mathrm{M}$ has been constructed for $n < n_0$, we note that for every $(t, x) \in \mathrm{C} \subset [a, b] \times \mathrm{K}$ and every $s \in [0, n\delta] \subset [0, 1]$, we have $t + s \in [a, b+1]$ and $d(\tilde{\mathrm{F}}[(t, x), s], x) \leq \kappa_a s \leq \kappa_a$. Hence $\left(t + n\delta, \tilde{\mathrm{F}}[(t, x), n\delta]\right) \in [a, b+1] \times \bar{\mathrm{V}}_{\kappa_a}(\mathrm{K})$. Therefore, for $s \in [n\delta, (n+1)\delta]$, we can set

$$\tilde{\mathrm{F}}[(t, x), s] = \mathrm{F}\Big[(t + n\delta, \tilde{\mathrm{F}}[(t, x), n\delta]), s - n\delta\Big].$$

By Lemma 3.5, this extension $\tilde{\mathrm{F}}$ of F is still $\hat{u}$-adapted. It remains to check (5.9) for $s \in [n\delta, (n+1)\delta]$. Noting that

$$\tilde{\mathrm{F}}[(t, x), s] = \mathrm{F}\Big[(t + n\delta, \tilde{\mathrm{F}}[(t, x), n\delta]), s - n\delta\Big],$$

from (5.8), we obtain $d(\tilde{F}[(t,x),s],\tilde{F}[(t,x),n\delta]) \leq \kappa_a(s-n\delta)$, hence

$$\begin{aligned} d(\tilde{F}[(t,x),s],x) &\leq d(\tilde{F}[(t,x),s],\tilde{F}[(t,x),n\delta]) + d(\tilde{F}[(t,x),n\delta],x) \\ &\leq \kappa_a(s-n\delta) + \kappa_a n\delta = \kappa_a s. \end{aligned}$$

Since $n_0\delta > 1$, this finishes the proof of the Lemma. □

6. Functions Lipschitz in the large

To state the generalization we have in mind, we recall the definition of Lipschitz in the large for a function, see [22, Definition A.5] or [15].

Definition **6.1.** — *Let* X *be a metric space whose distance is denoted by* d. *A function* $u : X \to \mathbf{R}$ *is said to be Lipschitz in the large if there exists a constant* $K < +\infty$ *such that*

$$|u(y) - u(x)| \leq K + Kd(x,y), \text{ for every } x, y \in X.$$

When the inequality above is satisfied, we will say that u *is Lipschitz in the large with constant* K.

Note that we do not assume in the definition above that u is continuous.

Obviously, when X is compact $u : X \to \mathbf{R}$ is Lipschitz in the large if and only if u is bounded.

As is shown in [15, Proposition 10.3], the function $u : X \to \mathbf{R}$ is Lipschitz in the large if and only if there exits a (globally) Lipschitz function $\varphi : X \to \mathbf{R}$ such that

$$\|u - \varphi\|_\infty = \sup_{x\in X} |u(x) - \varphi(x)| < +\infty.$$

In particular, a Lipschitz in the large function $u : M \to \mathbf{R}$ is bounded from below by a Lipschitz function and therefore $\hat{u}$ is finite everywhere on $[0,+\infty[\times M$.

As we will see below the next theorem generalizes Theorem 1.10 stated in the introduction.

Theorem **6.2.** — *Assume* $u : M \to \mathbf{R}$ *is a Lipschitz in the large function. For every* $T \in]0,+\infty]$ *the inclusions* $\Sigma_T(\hat{u}) \subset \mathrm{Cut}_T(\hat{u}) \subset]0,T[\times M \setminus \mathcal{I}_T(\hat{u})$ *are homotopy equivalences.*

It is not difficult to see that, for a function $u : M \to \mathbf{R}$ Lipschitz in the large with constant K, its lower semi-continuous regularization u_- is itself Lipschitz in the large with constant K. Therefore, by Proposition 2.20, without loss of generality we can prove Theorem 6.2 adding the assumption that u is lower semi-continuous.

To prove Theorem 6.2, we need a preliminary result, namely Corollary 6.4 below. To obtain this Corollary, we need following result, whose proof is standard and can be found in [15, Theorem 10.4].

Proposition **6.3.** — *If u is Lipschitz in the large, then for every $t_0 > 0$, the restriction of its negative Lax-Oleinik evolution $\hat{u}$ to $[t_0, +\infty[\times M$ is (globally) Lipschitz.*

Corollary **6.4.** — *If u is Lipschitz in the large, then for every $t_0 > 0$, we can find a constant κ_{t_0} such that for every $t \geq t_0$, $s > 0$, $x, y \in M$ with $\hat{u}(t+s, y) - h_s(x,y) \geq \hat{u}(t+s, x) - h_s(x,x)$, we have $d(x,y) \leq \kappa_{t_0} s$.*

Proof. — By Proposition 6.3 above we can find a global Lipschitz constant $\lambda < +\infty$ for $\hat{u}$ on $[t_0, +\infty[\times M$. The inequality $\hat{u}(t+s, y) - h_s(x,y) \geq \hat{u}(t+s, x) - h_s(x,x)$ yields

$$h_s(x,y) \leq \hat{u}(t+s, y) - \hat{u}(t+s, x) + h_s(x,x).$$

Since $h_s(x,x) \leq sA(0)$ (where $A(0) = \sup_{x\in M} L(x,0)$ was defined before, see (1.3)), we obtain

$$\begin{aligned} h_s(x,y) &\leq \hat{u}(s+t, y) - \hat{u}(s+t, x) + h_s(x,x) \\ &\leq \lambda d(x,y) + A(0)s. \end{aligned}$$

But by the uniform superlinearity of L, see (1.4), we know that $h_s(x,y) \geq (\lambda+1)d(x,y) - C(\lambda+1)s$. Hence, we have $(\lambda+1)d(x,y) - C(\lambda+1)s \leq \lambda d(x,y) + A(0)s$, from which the inequality $d(x,y) \leq [C(\lambda+1) + A(0)]s$ follows. □

Proof of Theorem 6.2. — Suppose $C \subset]0, +\infty[\times M$ is compact. We show that there exists a $\hat{u}$-adapted homotopy $F : C \times [0, +\infty[\to M$. This will imply Theorem 6.2 by Proposition 5.1.

Choose $t_0 > 0$ such that $C \subset]t_0, +\infty[\times M$. By Lemma 3.6, we can find a $\hat{u}$-adapted homotopy $F_0 : C \times [0, \delta] \to M$ with

$$\hat{u}[t+s, F_0((t,x),s)] - h_s[x, F_0((t,x),s)] \geq \hat{u}(t+s, x) - h_s(x,x).$$

By Corollary 6.4, we obtain $d(F_0((t,x),s), x) \leq \kappa_{t_0} s$. The existence of a $\hat{u}$-adapted homotopy $F : C \times [0, +\infty[\to M$ now follows from Lemma 5.1 and Lemma 5.2. □

Proof of Theorem 1.10. — Under the hypothesis of Theorem 1.10, by Theorem 2.19, we know that $U = \hat{u}$, with $u : M \to \mathbf{R}$ defined by $u(x) = U(0,x)$. Note that the function u itself is uniformly continuous.

Any uniformly continuous function $u : M \to \mathbf{R}$ is Lipschitz in the large, since it is a uniform limit of Lipschitz functions, see for example [15, Lemma A.1]. Therefore, Theorem 1.10 is a consequence of the more general Theorem 6.2. □

Corollary **6.5.** — *Assume that the function $u : M \to [-\infty, +\infty]$, with $\hat{u}$ finite everywhere on $]0, +\infty[$, is such that $T_t^- u = \hat{u}(t, \cdot)$ is Lipschitz in the large, for every $t > 0$. Then, for every $T \in]0, +\infty]$, the inclusions $\Sigma_T(\hat{u}) \subset \mathrm{Cut}_T(\hat{u}) \subset]0, T[\times M \setminus \mathcal{I}_T(\hat{u})$ are homotopy equivalences.*

Proof. — Again, like in the proof of Theorem 6.2, it suffices to show that, for every compact subset $C \subset]0,+\infty[\times M$, there exists a $\hat{u}$-adapted homotopy $F : C \times [0,+\infty[\to M$.

By compactness of C, there exists $t_0 > 0$ such that $C \subset]t_0,+\infty[\times M$. Set $u_{t_0} = T^-_{t_0} u = \hat{u}(t_0,\cdot)$. We have $\hat{u}_{t_0}(t,x) = \hat{u}(t_0+t,x)$. Setting $C_{t_0} = \{(t-t_0,x) \mid (t,x) \in C\} \subset]0,+\infty[$, since u_{t_0} is Lipschitz in the large, by the beginning of the proof of Theorem 6.2, we know that we can find a $\hat{u}_{t_0}$-adapted homotopy $F_0 : C_{t_0} \times [0,+\infty[\to M$. It is not difficult to see that $F : C \times [0,+\infty[\to M$, defined by $F(t,x) = F_0(t-t_0,x)$, is a $\hat{u}$-adapted homotopy. □

Corollary **6.6.** — *Suppose that the manifold* M *is compact. If the function* $u : M \to [-\infty,+\infty]$ *is such that* $\hat{u}$ *is finite everywhere on* $]0,+\infty[\times M$*, then, for every* $T \in]0,+\infty]$*, the inclusions* $\Sigma_T(\hat{u}) \subset \mathrm{Cut}_T(\hat{u}) \subset]0,T[\times M \setminus \mathcal{I}_T(\hat{u})$ *are homotopy equivalences.*

Proof. — By Theorem 2.18, we know that $T^-_t u$ is continuous on M, for every $t > 0$, therefore Lipschitz in the large, since M is compact. It suffices now to apply Corollary 6.5. □

Lemma **6.7.** — *Assume that* $C \subset M$ *is a closed subset of the complete Riemannian manifold* (M,g)*. If the curve* $\gamma : [0,+\infty[\to M$ *is* $\hat{\chi}_C$*-calibrated, then either* γ *is a constant curve with value in* C *or* $d_C(\gamma(t)) \to +\infty$ *when* $t \to +\infty$*. Therefore, if* U *is a relatively compact component of* $M \setminus C$*, then* $]0,+\infty[\times U$ *is disjoint from* $\mathcal{I}_{+\infty}(\hat{\chi}_C)$*.*

In particular, if M *is compact, then for any closed subset* C *we have* $\mathcal{I}_{+\infty}(\hat{\chi}_C) =]0,+\infty[\times C$*.*

Proof. — Assume (t_0,x_0), with $x_0 \in U$ and $t_0 > 0$, is in the Aubry set $\mathcal{I}_{+\infty}(\hat{\chi}_C)$. We can find a $\hat{\chi}_C$-calibrated curve $\gamma : [0,+\infty[\to M$, with $\gamma(t_0) = x_0$. By Lemma 2.14, the curve γ is a g-geodesic satisfying $\gamma(0) \in C$ and $d_C(\gamma(t)) = d(\gamma(0),\gamma(t)) = vt$, for all $t \geq 0$, where $v = \|\dot{\gamma}(s)\|_{\gamma(s)}$ is its constant norm of speed, from which the first part and the last part of the lemma follow.

For the second part, we note that, for a non-constant $\hat{\chi}_C$-calibrated curve $\gamma : [0,+\infty[\to M$, the image $\gamma(]0,+\infty[)$ is contained in a unique connected component of $M \setminus C$ on which d_C is not bounded. □

Corollary **6.8.** — *Assume that* $C \subset M$ *is a closed subset of the* compact *Riemannian manifold* M*. Then the inclusion* $\Sigma^*(d_C) \subset M \setminus C$ *is a homotopy equivalence.*

Proof. — Since M is compact $\mathcal{I}_{+\infty}(\hat{\chi}_C) =]0,+\infty[\times C$, by the previous Lemma 6.7.

By Corollary 6.6, we obtain that the inclusion $\Sigma^*(\hat{\chi}_C) \subset]0,+\infty[\times(M \setminus C)$ is a homotopy equivalence. But, by (3.3), we have $\Sigma^*(\hat{\chi}_C) =]0,+\infty[\times \Sigma(d_C^2)$ and by (3.4), we have $\Sigma(d_C^2) = \Sigma^*(d_C)$. This finishes the proof of the Corollary. □

7. More applications to complete non-compact Riemannian manifold

Assume that C is a closed subset of the complete Riemannian manifold (M, g). If M is not compact, then neither χ_C nor $\hat{\chi}_C(t_0, \cdot) = d_C(\cdot)^2/2t$ are necessarily Lipschitz in the large. So, we cannot apply Corollary 6.5 to obtain the global homotopy type of $\Sigma^*(d_C) = \Sigma(d_C^2)$. Instead, we will show that Proposition 5.1 directly applies.

For $y \in M$ and $t, s > 0$, we introduce the function $\varphi_{t,s,y} : M \to \mathbf{R}$ defined by

$$\varphi_{t,s,y}(x) = \hat{\chi}_C(t+s, x) - h_s^g(x, y) = \frac{d_C(x)^2}{2(t+s)} - \frac{d(y,x)^2}{2s}. \tag{7.1}$$

Lemma **7.1.** — *Suppose that $x, y \in M$ are such that $\varphi_{t,s,y}(x) \geq \varphi_{t,s,y}(y)$. Then we have*

$$d(x,y) \leq \frac{2s}{t} d_C(y) \text{ and } d_C(x) \leq \left(1 + \frac{2s}{t}\right) d_C(y).$$

Proof. — From the definition (7.1), the inequality $\varphi_{t,s,y}(x) \geq \varphi_{t,s,y}(y)$ translates to

$$\frac{d_C(x)^2}{2(t+s)} - \frac{d(y,x)^2}{2s} \geq \frac{d_C(y)^2}{2(t+s)}.$$

Using $d_C(x) \leq d_C(y) + d(x,y)$, we obtain

$$\begin{aligned}\frac{d_C(y)^2}{2(t+s)} &\leq \frac{[d_C(y) + d(x,y)]^2}{2(t+s)} - \frac{d(y,x)^2}{2s} \\ &= \frac{d_C(y)^2}{2(t+s)} + \frac{2d_C(y)d(x,y)}{2(t+s)} + \left(\frac{1}{2(t+s)} - \frac{1}{2s}\right) d(x,y)^2 \\ &= \frac{d_C(y)^2}{2(t+s)} + \frac{2d_C(y)d(x,y)}{2(t+s)} - \frac{td(x,y)^2}{2s(t+s)},\end{aligned}$$

from which the inequality $d(x,y) \leq 2sd_C(y)/t$ follows. To prove the last inequality of the lemma, we again use $d_C(x) \leq d_C(y) + d(x,y)$. □

Lemma **7.2.** — *Assume that $(t_0, x_0), \ldots, (t_n, x_n), \ldots$ is a (finite or infinite) sequence in $]0, +\infty[\times M$, with $s_n = t_{n+1} - t_n \geq 0$, and such that $\varphi_{t_n,s_n,x_n}(x_{n+1}) \geq \varphi_{t_n,s_n,x_n}(x_n)$. Then*

$$d_C(x_n) \leq e^{2t_n/t_0} d_C(x_0) \text{ and } d(x_n, x_0) \leq 2e^{3t_n/t_0} d_C(x_0).$$

Proof. — From Lemma 7.1, we get

$$d(x_{n+1}, x_n) \leq \frac{2s_n}{t_n} d_C(x_n) \text{ and } d_C(x_{n+1}) \leq \left(1 + \frac{2s_n}{t_n}\right) d_C(x_n). \tag{7.2}$$

Therefore $d_C(x_n) \leq \prod_{i=0}^{n-1}\left(1+\frac{2s_i}{t_i}\right) d_C(x_0)$ from the second inequality above. The first inequality in the lemma follows from $\prod_{i=0}^{n-1}\left(1+\frac{2s_i}{t_i}\right) \leq e^{2t_n/t_0}$, which we now establish by taking log's. Using that $\log(1+t) \leq t$, for $t \in]0,+\infty[$, that t_i is non-decreasing and $\sum_{i=0}^{n-1} s_i = t_n - t_0 \leq t_n$, we get

$$\log \prod_{i=0}^{n-1}\left(1+\frac{2s_i}{t_i}\right) = \sum_{i=0}^{n-1} \log\left(1+\frac{2s_i}{t_i}\right) \leq 2\sum_{i=0}^{n-1}\frac{s_i}{t_i} \leq 2\frac{\sum_{i=0}^{n-1} s_i}{t_0} \leq \frac{2t_n}{t_0}.$$

We now combine the already established first inequality of the lemma with the inequality (7.2) to obtain

$$d(x_n, x_0) \leq \sum_{i=0}^{n-1} d(x_{i+1}, x_i) \leq \sum_{i=0}^{n-1} \frac{2s_i}{t_i} d_C(x_i) \leq 2\left(\sum_{i=0}^{n-1}\frac{s_i}{t_i} e^{2t_i/t_0}\right) d_C(x_0).$$

Combining this last inequality with t_i non-decreasing, we obtain

$$d(x_n, x_0) \leq 2\left(\sum_{i=0}^{n-1}\frac{s_i}{t_i} e^{2t_i/t_0}\right) d_C(x_0) \leq 2\left(\frac{\sum_{i=0}^{n-1} s_i}{t_0} e^{2t_n/t_0}\right) d_C(x_0).$$

Therefore, since $\sum_{i=0}^{n-1} s_i \leq t_n$ and $t \leq \exp(t)$, we get

$$d(x_n, x_0) \leq 2\left(\frac{\sum_{i=0}^{n-1} s_i}{t_0} e^{2t_n/t_0}\right) d_C(x_0) \leq 2\frac{t_n}{t_0} e^{2t_n/t_0} d_C(x_0) \leq 2e^{3t_n/t_0} d_C(x_0). \qquad \square$$

We now prove that χ_C satisfies the hypothesis of Proposition 5.1.

Proposition **7.3.** — *Suppose* C *is a closed subset of the complete Riemannian manifold* (M, g). *Then for every compact subset* $K \subset]0,+\infty[\times M$, *we can find a* $\hat{\chi}_C$*-adapted homotopy* $F : K \times [0,1] \to M$.

Proof. — The compact subset $K \subset]0,+\infty[\times M$ is contained in a set of the form $[a,b] \times A$, where A is a compact subset of M and $0 < a < b < +\infty$. We then define κ by

$$\kappa = e^{3(b+1)/a} \max_A d_C < +\infty.$$

Since A is compact, its neighborhood $\bar{V}_\kappa(A) = \{x \in M \mid d_A(x) \leq \kappa\}$ is also compact. By Lemma 3.6, we can find $\delta > 0$ and a $\hat{\chi}_C$-adapted homotopy $F : \big([a, b+1] \times \bar{V}_\kappa(A)\big) \times [0,\delta] \to M$.

If $\delta \geq 1$, then the restriction of F to $K \times [0,1]$ does the job. If not, cutting down on δ we assume $\delta = 1/n$, with n an integer ≥ 2.

We will show that we can extend F by induction on $i = 1, \dots, n-1$ to $F : K \times [0, (i+1)/n] \to M$ by

$$F((t, x), s) = F[(t + i/n, F(t + i/n, x)), s - i/n], \text{ for } s \in [i/n, (i+1)/n].$$

This homotopy is well-defined if we show by induction that, for $(t, x) \in K \subset [a, b] \times A$, the sequence $(x_i, t + i/n)$, with $x_0 = x$, defined by induction for $i = 1, \dots n-1$ as

$$x_i = F[(t + (i-1)/n, x_{i-1}), 1/n],$$

is such that $x_i \in \bar{V}_\kappa(A)$, for $i = 1, \dots, n-1$.

Since $F : ([a, b+1] \times \bar{V}_\kappa(A)) \times [0, 1/n] \to M$ is defined by Lemma 3.6, the inequality (3.1) implies that the sequence x_i, while it make sense for $i = 0, 1, \dots, n-1$, satisfies the hypothesis of Lemma 7.2 with $t_i = t + i/n \leq t + 1 \leq b + 1$. Therefore

$$d(x_i, x_0) \leq e^{3t_i/t} d_C(x_0) \leq e^{3(b+1)/a} \max_A d_C = \kappa,$$

since $x_0 = x \in A$, $t \in [a, b]$ and $t_i \leq b + 1$. This implies that $x_i \in \bar{V}_\kappa(A)$. □

Proof of Theorem 1.6. — Since we know by Proposition 7.3 that $\hat{\chi}_C$ satisfies the hypothesis of Proposition 5.1, we obtain that the inclusion $\Sigma_\infty(\hat{\chi}_C) \subset]0, +\infty[\times M \setminus \mathcal{I}_\infty(\hat{\chi}_C)$ is a homotopy equivalence. But, by (3.3) and (3.4), we have $\Sigma_\infty(\hat{\chi}_C) =]0, +\infty[\times \Sigma(d_C^2) =]0, +\infty[\times \Sigma^*(d_C)$ and, by Proposition 2.15, we have $\mathcal{I}_\infty(\hat{\chi}_C) =]0, +\infty[\times (C \cup \mathcal{A}^*(C))$. Therefore, the inclusion $]0, +\infty[\times \Sigma^*(d_C) \subset]0, +\infty[\times (M \setminus (C \cup \mathcal{A}^*(C)))$ is a homotopy equivalence, which implies that $\Sigma^*(d_C) \subset (M \setminus (C \cup \mathcal{A}^*(C))$ is itself a homotopy equivalence. □

We now explain the generalization of Theorem 1.7 to non-compact complete Riemannian manifold.

We first introduce the subset $\mathcal{AU}(M, g) \subset M \times M$.

Definition **7.4.** — *For a complete connected Riemannian manifold* (M, g)*, the subset* $\mathcal{AU}(M, g) \subset M \times M$ *is the set of points* $(x, y) \in M \times M$ *such that there exists a* minimizing *g-geodesic* $\gamma :]-\infty, +\infty[\to M$*, with* $\gamma(a) = x$, $\gamma(b) = y$*, for some* $a < b \in \mathbf{R}$.

Lemma **7.5.** — *We have* $\mathcal{AU}(M, g) = \Delta_M \cup \mathcal{A}^*(d_{\Delta_M})$.

The proof is left to the reader. The following generalization of Theorem 1.7 now follows from the lemma above and Theorem 1.6.

Theorem **7.6.** — *For every connected complete Riemannian manifold* M*, the inclusion* $\mathcal{NU}(M, g) \subset M \times M \setminus \mathcal{AU}(M, g)$ *is a homotopy equivalence.*

8. More results on local contractibility

In fact, our local contractibility result can be applied for viscosity solution defined only on an open subset, and also for Hamiltonians which are not necessarily uniformly superlinear. More precisely, we have.

Theorem **8.1.** — *Suppose* $\mathrm{H} : \mathrm{T}^*\mathrm{M} \to \mathbf{R}$ *is a* C^2 *Hamiltonian such that:*

(a) *(Superlinearity on compact subsets) For every compact subset* C *and every real number* $\mathrm{K} \geq 0$*, we have* $\sup\{\mathrm{K}\|p\|_x - \mathrm{H}(x, p) \mid x \in \mathrm{C}, p \in \mathrm{T}^*_x\mathrm{M}\} < \infty$.

(b) *(*C^2 *strict convexity in the fibers) For every* $(x, p) \in \mathrm{T}^*\mathrm{M}$*, the second derivative along the fibers* $\partial^2\mathrm{H}/\partial p^2(x, p)$ *is positive definite.*

If the continuous function $\mathrm{U} : \mathrm{O} \to \mathbf{R}$ *defined on the open subset* $\mathrm{O} \subset \mathbf{R} \times \mathrm{M}$ *is a viscosity solution of the Hamilton-Jacobi equation* (1.1)*, then its set of singularities* $\Sigma(\mathrm{U}) \subset \mathrm{O}$ *is locally contractible.*

Proof. — Since the result is local, we can assume that $\mathrm{O} =]a, b[\times \mathrm{W}$, with $\mathrm{W} \subset \mathrm{M}$ open with $\bar{\mathrm{W}}$ compact. We will first modify H outside of $\mathrm{T}^*\mathrm{W}$ to reduce to the case where the Hamiltonian is Tonelli. Choose a C^∞ function $\varphi : \mathrm{M} \to [0, 1]$ with compact support such that φ is identically 1 on the compact subset $\bar{\mathrm{W}}$. Define $\tilde{\mathrm{H}} : \mathrm{T}^*\mathrm{M} \to \mathbf{R}$ by

$$\tilde{\mathrm{H}}(x, p) = (1 - \varphi(x))\|p\|_x^2 + \varphi(x)\mathrm{H}(x, p).$$

Since the support of φ is compact, using the properties of H, it is not difficult to check that $\tilde{\mathrm{H}}$ is Tonelli and coincides with H on $\mathrm{T}^*\mathrm{W}$. This last fact implies that $\mathrm{U} :]a, b[\times \mathrm{W} \to \mathbf{R}$ is also a viscosity solution of (1.1) for $\tilde{\mathrm{H}}$.

Therefore without loss of generality, we can assume that H is Tonelli.

Fix $(t_0, x_0) \in]a, b[\times \mathrm{W}$. Pick $\eta > 0$ such that $[t_0 - \eta, t_0 + \eta] \times \bar{\mathrm{B}}(x_0, 2\eta) \subset]a, b[\times \mathrm{W}$.

The viscosity solution U is Lipschitz on the compact set $[t_0 - \eta, t_0 + \eta] \times \bar{\mathrm{B}}(x_0, 2\eta)$. Therefore, since speeds of U-calibrating curves are related to upper differentials of U by the Legendre transform, we can find a finite constant K such that for every $[\alpha, \beta] \subset [t_0 - \eta, t_0 + \eta]$ and every curve $\gamma : [\alpha, \beta] \to \bar{\mathrm{B}}(x_0, 2\eta)$, which is U-calibrating, we have $\|\dot{\gamma}(s)\|_{\gamma(s)} \leq \mathrm{K}$. Choose now $\epsilon > 0$ such that $\epsilon < \eta$ and $2\mathrm{K}\epsilon < \eta$. Since $\mathrm{U} : \mathrm{O} \to \mathbf{R}$, as any viscosity solution, has backward characteristics ending at any given point (see [15, Theorem 7.6]), from the definition of K and the choice of ϵ, for every $(t, x) \in]t_0 - \epsilon, t_0 + \epsilon] \times \bar{\mathrm{B}}(x_0, \eta)$, we can find a U-calibrated curve $\gamma_{t,x} : [t_0 - \epsilon, t] \to \bar{\mathrm{B}}(x_0, 2\eta)$. It follows that $\mathrm{U}(t, x) = \inf_{y \in \bar{\mathrm{B}}(x_0, 2\eta)} \mathrm{U}(t_0 - \epsilon, y) + h_{t - t_0 + \epsilon}(y, x)$, for all $(t, x) \in]t_0 - \epsilon, t_0 + \epsilon] \times \bar{\mathrm{B}}(x_0, \eta)$. Therefore, if we define $u : \mathrm{M} \to \mathbf{R}$ by

$$u(y) = \begin{cases} \mathrm{U}(t_0 - \epsilon, y), & \text{if } y \in \bar{\mathrm{B}}(x_0, 2\eta), \\ +\infty, & \text{if } y \in \bar{\mathrm{B}}(x_0, 2\eta), \end{cases}$$

we obtain

$$\mathrm{U}(t, x) = \hat{u}(t - t_0 + \epsilon, x) \text{ for all } (t, x) \in]t_0 - \epsilon, t_0 + \epsilon] \times \bar{\mathrm{B}}(x_0, \eta). \quad (\mathbf{8.1})$$

Since U is continuous and $\bar{\mathrm{B}}(x_0, 2\eta)$ compact and not empty, the function $\hat{u}$ is bounded below and not identically $+\infty$. Therefore $\hat{u}$ is finite everywhere and $\Sigma^*(\hat{u})$ is locally contractible. By (8.1), we have $\Sigma(\mathrm{U}) \cap]t_0 - \epsilon, t_0 + \epsilon[\times \mathring{\mathrm{B}}(x_0, \eta) = \{(s + t_0 - \epsilon, x) \mid (s, x) \in \Sigma^*(\hat{u}) \cap]0, 2\epsilon[\times \mathring{\mathrm{B}}(x_0, \eta)$. Hence $\Sigma(\mathrm{U}) \cap]t_0 - \epsilon, t_0 + \epsilon[\times \mathring{\mathrm{B}}(x_0, \eta)$ is itself locally contractible. □

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