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Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups

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Abstract. Let k be the algebraic closure of a finite field of odd characteristic p, and X a smooth projective scheme over the Witt ring $W(k)$ which is geometrically connected in characteristic zero. We introduce the notion of *Higgs–de Rham flow*[1] and prove that the category of periodic Higgs-de Rham flows over $X/W(k)$ is equivalent to the category of Fontaine modules, hence further equivalent to the category of crystalline representations of the étale fundamental group $\pi_1(X_K)$ of the generic fiber of X, after Fontaine–Laffaille and Faltings, where K is the fraction field of $W(k)$. Moreover, we prove that every semistable Higgs bundle over the special fiber X_k of X of rank $\leq p$ initiates a semistable Higgs–de Rham flow and thus those of rank $\leq p-1$ with trivial Chern classes induce k-representations of $\pi_1(X_K)$. A fundamental construction in this paper is the inverse Cartier transform over a truncated Witt ring. In characteristic p, it was constructed by Ogus–Vologodsky in the nonabelian Hodge theory in positive characteristic; in the affine local case, our construction is related to the local Ogus–Vologodsky correspondence of Shiho.

Keywords. Higgs bundles, p-adic Hodge theory

Contents

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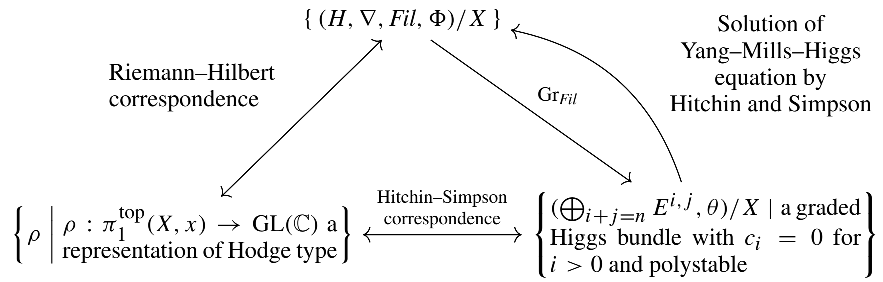
[1] Higgs–de Rham flow was called Higgs–de Rham sequence in [17]. The change has been made because of its analogue to the notion of Yang–Mills–Higgs flow over $\mathbb{C}$.

1. Introduction

Let k be the algebraic closure of a finite field of odd characteristic p, $W := W(k)$ the ring of Witt vectors and K its fraction field. Let X be a smooth projective scheme over W which is geometrically connected in characteristic zero. The paper aims to establish a correspondence between certain Higgs bundles over X with trivial Chern classes which are stable over its special fiber $X_k := X \times_W k$ and certain integral crystalline representations which are absolutely irreducible modulo p of the étale fundamental group $\pi_1(X_K)$ of the generic fiber $X_K := X \times_W K$.

Inspired by the complex analytic theory of Simpson [32], Ogus and Vologodsky [26] established the nonabelian Hodge theorem in positive characteristic, that is, an equivalence of categories between the category of certain nilpotent Higgs modules and the category of certain nilpotent flat modules over a smooth variety over k which is $W_2(k)$-liftable ($\mathrm{char}\, k = 2$ is also allowed). This equivalence generalizes both the classical Cartier descent theorem and the relation between a strict p-torsion Fontaine module and the associated graded Higgs module. Compared with the complex theory, an obvious distinction is that the semistability condition on Higgs modules does not play any role in Ogus–Vologodsky's correspondence. In the meantime, Faltings and others attempted to establish an analogue of Simpson's theory for varieties over p-adic fields. In [6], Faltings established an equivalence of categories between the category of generalized representations of the geometric fundamental group and the category of Higgs bundles over a p-adic curve, which generalizes the earlier work of Deninger–Werner [2] on a partial p-adic analogue of Narasimhan–Seshadri theory. However, the major problem concerning the role of semistability remains open; Faltings asked whether semistable Higgs bundles of degree zero come from genuine representations instead of merely generalized ones. We will address the semistability condition in this paper.

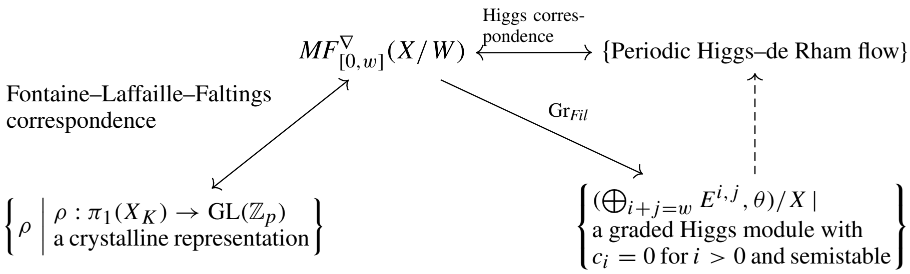
Recall that when X is a complex smooth projective variety, such a correspondence was established first by Hitchin [10] for polystable Higgs bundles over a smooth projective curve and then by Simpson [31] in general. The key step in the Hitchin–Simpson correspondence, as one may see in the following diagram, is to solve the Yang–Mills–Higgs equation and obtain a correspondence from graded Higgs bundles to polarized complex variations of Hodge structure over X.



The notion of polarized $\mathbb{Z}$-variation of Hodge structure $(H, \nabla, \mathit{Fil}, \Phi)$ was introduced by P. Griffiths [9], and later generalized to polarized $\mathbb{C}$-variation of Hodge structure by

P. Deligne [1], where (H, ∇) is a flat bundle with a Hodge filtration, Φ is a polarization which is horizontal with respect to ∇ and satisfies the Riemann–Hodge bilinear relations; $(\bigoplus_{i+j=n} E^{i,j}, \theta)$ denotes a graded Higgs bundle, where θ is a direct sum of $\mathcal{O}_X$-linear morphisms $\theta^{i,j} : E^{i,j} \to E^{i-1,j+1} \otimes \Omega_X$ and $\theta \wedge \theta = 0$.

We return to the p-adic case. A good p-adic analogue of the category of polarized complex variations of Hodge structure is the category $MF^{\nabla}_{[0,w]}(X/W)$ $(w \leq p-1)$ introduced first by Fontaine–Laffaille [7] for $X = \mathrm{Spec}\, W$ and later generalized by Faltings [4, Chapter II] to the general case. An object in that category, called a *Fontaine module*, is also a quadruple $(H, \nabla, \mathit{Fil}, \Phi)$, where (H, ∇) is a flat bundle over X, by which we mean a locally free $\mathcal{O}_X$-module H of finite rank, equipped with an integrable W-connection ∇, Fil is a Hodge filtration, and Φ is a relative Frobenius which is horizontal with respect to ∇ and satisfies some compatibility and strong p-divisibility conditions. The latter condition is a p-adic analogue of the Riemann–Hodge bilinear relations. See §2 for more details. Fontaine–Laffaille in the $X = \mathrm{Spec}\, W$ case and Faltings in the general case proved that there exists a fully faithful functor from the category $MF^{\nabla}_{[0,w]}(X/W)$ $(w \leq p-2)$ to the category of *crystalline representations* of $\pi_1(X_K)$, i.e. a p-adic Riemann–Hilbert correspondence. Our objective is to establish a p-adic analogue of the Higgs correspondence from a certain category of graded Higgs modules with trivial Chern classes to the category of Fontaine modules. Our results are encoded in the following big diagram:



As seen from the above diagram, the central notion in our theory is *Higgs–de Rham flow* (especially the periodic one), which can be viewed as an analogue of Yang–Mills–Higgs flow. To define a Higgs–de Rham flow over $X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$ for all $n \in \mathbb{N}$, the key ingredient is the inverse Cartier transform C_n^{-1} over $W_n := W \otimes \mathbb{Z}/p^n\mathbb{Z}$ (see Definition 5.1). It is built upon the seminal work of Ogus–Vologodsky [26]. They construct the inverse Cartier transform from the category of (suitably nilpotent) Higgs modules over $X'_1 = X_1 \times_{F_k} \mathrm{Spec}\, k$ to the category of (suitably nilpotent) flat modules over X_1. However, over a general smooth scheme X' over W_n, $n \geq 2$, our lifting of inverse Cartier transform operates not on the whole category of (suitably nilpotent) Higgs modules, but on a category which is over a subcategory of graded Higgs modules (over a proper scheme, there is a restriction on the Chern classes of Higgs modules). More details about the category will be given below. In comparison with the construction of Shiho [30], one finds that the existence of a Frobenius lifting over a chosen lifting of X' over W_{n+1} is not assumed in our construction. On the other hand, we do not know whether the functor C_n^{-1} is fully faithful for a proper W_n-scheme when $n \geq 2$.

The Higgs correspondence is established in an inductive way. That is, we shall first define the notion of a Higgs–de Rham flow in characteristic p and then establish the Higgs correspondence for the periodic flows. This is the first step. In this step, we need only assume that the scheme X_1 is smooth over k and W_2-liftable. A choice of such a lifting does matter in the theory. Let us choose and fix a lifting X_2/W_2. Let $S_n :=$ Spec W_n and $F_{S_n} : S_n \to S_n$ be the Frobenius automorphism. Set $X_2' := X_2 \times_{F_{S_2}} S_2$, which is a W_2-lifting of $X_1' = X_1 \times_{F_{S_1}} S_1$. Let $(\mathcal{X}, \mathcal{S}) = (X_1/k, X_2'/W_2)$ and $C^{-1}_{\mathcal{X}/\mathcal{S}}$ the inverse Cartier transform of Ogus–Vologodsky [26] which restricts to an equivalence of categories from the full subcategory $HIG_{p-1}(X_1')$ of nilpotent Higgs modules of exponent $\leq p-1$ to the full subcategory $MIC_{p-1}(X_1)$ of nilpotent flat modules of exponent $\leq p-1$. See also our previous work [16] for an alternative approach via exponential twisting to Ogus–Vologodsky theory over these subcategories. Let $\pi : X_1' \to X_1$ be the base change of F_{S_1} to X_1. From the geometric point of view, it is more natural to make all terms in a flow defined over the same base scheme. So, instead of using $C^{-1}_{\mathcal{X}/\mathcal{S}}$, we take the composite functor $C_1^{-1} := C^{-1}_{\mathcal{X}/\mathcal{S}} \circ \pi^* \circ \iota$ from $HIG_{p-1}(X_1)$ to $MIC_{p-1}(X_1)$, where ι is an automorphism of the category $HIG_{p-1}(X_1)$ defined by sending (E, θ) to $(E, -\theta)$. The reader is advised to take caution concerning this point. For a flat module (H, ∇) over X_1, a *Griffiths transverse filtration Fil* of level $w \geq 0$ on (H, ∇) is defined to be a finite exhaustive decreasing filtration of H by $\mathcal{O}_{X_1}$-submodules,

$$H = Fil^0 \supset Fil^1 \supset \cdots \supset Fil^w \supset Fil^{w+1} = 0,$$

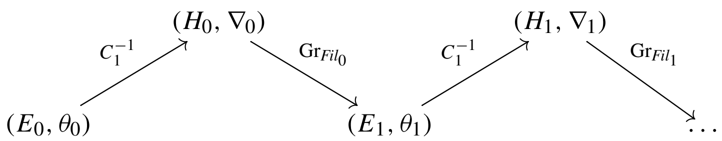
such that its grading $\bigoplus_i Fil^i/Fil^{i+1}$ is torsion-free and Fil obeys Griffiths' transversality

$$\nabla(Fil^i) \subset Fil^{i-1} \otimes \Omega_{X_1/k}, \quad 1 \leq i \leq w.$$

The triple (H, ∇, Fil) is called a *de Rham module*. By taking the grading with respect to the filtration Fil, to every de Rham module (H, ∇, Fil) one can canonically associate a graded Higgs module $(E, \theta) := (\bigoplus_i Fil^i/Fil^{i+1}, \bigoplus_i \bar{\nabla}_i)$, where the $\mathcal{O}_{X_1}$-morphism $\bar{\nabla}_i$ is induced from ∇,

$$\bar{\nabla}_i : Fil^i/Fil^{i+1} \to (Fil^{i-1}/Fil^i) \otimes \Omega_{X_1/k}.$$

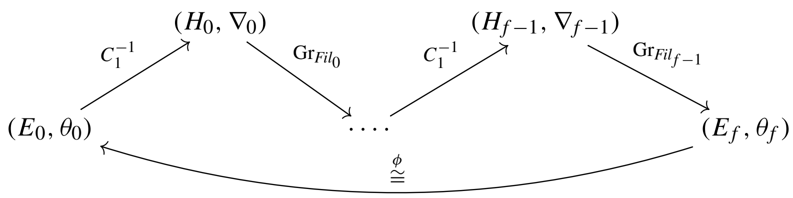
Definition 1.1. A *Higgs–de Rham flow* over X_1 is a diagram of the form



where $(E_0, \theta_0) \in HIG_{p-1}(X_1)$, $Fil_i, i \geq 0$, is a Griffiths transverse filtration on the flat module $(H_i, \nabla_i) := C_1^{-1}(E_i, \theta_i)$ of level $\leq p-1$, and $(E_i, \theta_i), i \geq 1$, is the graded Higgs module associated to the de Rham module $(H_{i-1}, \nabla_{i-1}, Fil_{i-1})$.

If the filtrations $Fil_i, i \geq 0$, in the definition are all of level $\leq w$ ($w \leq p-1$), the flow is said to be a Higgs–de Rham flow *of level* $\leq w$. For a graded torsion-free Higgs module $(E = \bigoplus_i E^i, \theta) \in HIG_{p-1}(X_1)$ (where $\theta : E^i \to E^{i-1} \otimes \Omega_{X_1/k}$), there is a natural Griffiths transverse filtration on $(H, \nabla) := C_1^{-1}(E, \theta)$, which is actually nontrivial in general. Since the filtration $\{E_l := \bigoplus_{i \leq l} E^i\}_l$ of E is θ-invariant, and since C_1^{-1} is an exact functor, $\{(H_l, \nabla_l) := C_1^{-1}(E_l, \theta)\}_l$ is naturally a filtration of (H, ∇) and it is Griffiths transverse. But since $\{H_l\}_l$ is ∇-invariant, the associated graded Higgs module has always the zero Higgs field. It is important to observe that there exist other nontrivial Griffiths transverse filtrations: the Hodge filtration in the geometric case (i.e. the strict p-torsion Fontaine modules) and the Simpson filtration in the ∇-semistable case (see Theorem A.4).

A Higgs–de Rham flow is said to be *periodic* if there exists an isomorphism $\phi : (E_f, \theta_f) \cong (E_0, \theta_0)$ of graded Higgs modules (an explicit ϕ is a part of the definition); it is said to be *preperiodic* if it becomes periodic after removing the first few terms. A Higgs module $(E, \theta) \in HIG_{p-1}(X_1)$ is said to be *(pre)periodic* if it initiates a (pre)periodic Higgs–de Rham flow. The reader is referred to Definition 3.1 for a precise definition. One may visualize a periodic Higgs–de Rham flow of period f via the following diagram:



Theorem 1.2 (Theorem 3.2). *Let X_2 be a smooth scheme over W_2. Let w be an integer between 0 and $p-1$. For each $f \in \mathbb{N}$, there is an equivalence of categories between the full subcategory of strict p-torsion Fontaine modules over X_2/W_2 of Hodge–Tate weight $\leq w$ with endomorphism structure $\mathbb{F}_{p^f}$ and the category of periodic Higgs–de Rham flows over the special fiber X_1 of level $\leq w$ and whose periods are f.*

Next, we construct a lifting to $W_n, n \in \mathbb{N}$, of the inverse Cartier transform of Ogus–Vologodsky [26] restricted to the full subcategory $HIG_{p-2}(X_1')$. Let X_n be a smooth scheme over W_n. We introduce a category $\mathcal{H}(X_n')$, where $X_n' := X_n \times_{F_{S_n}} X_n$, whose object is a tuple $(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\psi})$, where (E, θ) is a nilpotent graded Higgs module over X_n' of exponent $\leq p-2$, $(\bar{H}, \bar{\nabla}, \overline{Fil})$ is a de Rham module over X_{n-1}', and $\bar{\psi}$ is an isomorphism of graded Higgs modules $\mathrm{Gr}_{\overline{Fil}}(\bar{H}, \bar{\nabla}) \cong (E, \theta) \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$. For $n = 1$, such a tuple is reduced to a nilpotent graded Higgs module over X_1' of exponent $\leq p-2$, and therefore $\mathcal{H}(X_1')$ is just the full subcategory $HIG_{p-2}(X_1')$.

Theorem 1.3 (Theorem 4.1). *Assume X_n is W_{n+1}-liftable. There exists a functor $\mathcal{C}_n^{-1}$ from the category $\mathcal{H}(X_n')$ to the category $MIC(X_n)$ of flat modules over X_n such that $\mathcal{C}_n^{-1}$ lifts $\mathcal{C}_{n-1}^{-1}$ and $\mathcal{C}_1^{-1}$ agrees with the inverse Cartier transform $C_{\mathcal{X}/\mathcal{S}}^{-1}$ of Ogus–Vologodsky* [26].

Relying on the existence of an inverse Cartier transform over W_n, we define inductively a periodic Higgs–de Rham flow over X_n/W_n for an arbitrary n (Definition 5.2) and establish the Higgs correspondence over W_n (Theorem 5.3). Passing to the limit, we obtain the p-torsion-free analogue of Theorem 1.2:

Theorem 1.4. *Let X be a smooth scheme over W. For each integer $0 \le w \le p-2$ and each $f \in \mathbb{N}$, there is an equivalence of categories between the category of p-torsion-free Fontaine modules over X/W of Hodge–Tate weight $\le m$ with endomorphism structure $W(\mathbb{F}_{p^f})$ and the category of periodic Higgs–de Rham flows over X of level $\le w$ and whose periods are f.*

The above theorem (and its p-torsion version) and the Fontaine–Laffaille–Faltings correspondence (and its p-torsion version, see Theorem 2.3) together constitute a p-adic (and p-torsion) version of the Hitchin–Simpson correspondence. This correspondence can be regarded as a global and higher Hodge–Tate weight generalization of the Katz correspondence on unit-root F-crystals. Using this correspondence, we construct a p-divisible group with isomorphic Kodaira–Spencer map over W and over the Serre–Tate lifting of an ordinary Abelian variety over k (see Example 5.8).

Now, we shall bring the semistability condition on Higgs modules into consideration, and explain our results along the vertical dotted line in the above big diagram. For a smooth projective variety X_1 over k of positive dimension, we choose (and fix) an ample divisor Z_1 of X_1. Let us consider the (μ-)semistability for a Higgs module (E, θ) over X_1 with respect to the μ_{Z_1}-slope

$$\mu_{Z_1}(E) = c_1(E)Z_1^{\dim X_1 - 1}/\mathrm{rank}(E).$$

The slope of a torsion $\mathcal{O}_{X_1}$-module is set to be infinity. Our first result is

Theorem 1.5 (Theorem 6.6, Theorem A.1). *With notation as above, assume additionally that X_1 is W_2-liftable. Let (E, θ) be a torsion-free nilpotent Higgs module over X_1 of exponent $\le p-1$. Suppose* rank $E \le p$ *and $c_i(E) = 0$, $i > 0$. Then (E, θ) is semistable if and only if it is preperiodic.*

As a consequence, we obtain

Corollary 1.6 (Theorem 6.7). *Let X/W be a smooth projective scheme over W. For each nilpotent semistable Higgs bundle (E, θ) over X_k with* rank $E \le p-1$ *and $c_i(E) = 0$, $i > 0$, there a unique* rank E*-dimensional crystalline k-representation of $\pi_1(X_K)$ up to isomorphism.*

The stronger rank condition in the above corollary results from the application of the Fontaine–Laffaille–Faltings correspondence.

If we try to lift the previous result to the mixed characteristic situation, a nontrivial obstruction occurs, namely the lifting of a Griffiths transverse filtration in positive characteristic to a truncated Witt ring, which prevents us from a direct generalization of Corollary 1.6 to the mixed characteristic case. It turns out that in order to kill the obstruction, one is led to various *ordinary* conditions on the base varieties. By working on this problem for a very simple kind of rank two Higgs bundles of degree zero (the so-called

Higgs bundle with maximal Higgs field) over a curve, we have found a p-adic analogue of the Hitchin–Simpson uniformization of hyperbolic curves which intimately relates the above theory to the theory of ordinary curves due to S. Mochizuki [24]. In particular, the canonical lifting theorem of Mochizuki for ordinary curves has been basically recovered in our recent work [15].

Stability, rather than merely semistability, on periodic Higgs bundles over k makes the choices involved in Higgs–de Rham flows basically unique. By the unicity of one-periodic Higgs–de Rham flow initializing a stable Higgs bundle (Propositions 7.2 and 7.5), one is able to identify the category of one-periodic stable Higgs modules with a full subcategory of the category of periodic Higgs–de Rham flows. Combined with this unicity, the Higgs correspondence implies the following rigidity result for Fontaine modules.

Theorem 1.7 (Corollary 7.6). *Let X/W be a smooth projective scheme. Moreover, let $(H_i, \nabla_i, Fil_i, \Phi_i)$, $i = 1, 2$, be two p-torsion-free Fontaine modules over X/W, and (E_i, θ_i) the associated graded Higgs modules. If (E_i, θ_i) are isomorphic as graded Higgs modules and additionally are Higgs stable modulo p, then $(H_i, \nabla_i, Fil_i, \Phi_i)$, $i = 1, 2$, are isomorphic.*

Our final result is the following correspondence. An $\mathbb{F}_p$-representation ρ of π_1 is said to be *absolutely irreducible* if $\rho \otimes k$ is irreducible.

Theorem 1.8 (Corollary 7.7). *With notation as above, there is an equivalence of categories between the category of crystalline $\mathbb{Z}_p$- (resp. $\mathbb{Z}/p^n\mathbb{Z}$-) representations of $\pi_1(X_K)$ with Hodge–Tate weight $\leq p-2$ whose mod p reduction is absolutely irreducible and the category of one-periodic Higgs bundles over X/W (resp. X_n/W_n) whose exponent is $\leq p-2$ and whose mod p reduction is stable.*

In the study of Hitchin–Simpson correspondence, one usually concentrates on the subcategory of Higgs bundles with trivial Chern classes. However, the theory developed in this paper turns out to be also useful in the study of Higgs bundles with nontrivial Chern classes. This has been beautifully demonstrated in the recent work [22] of A. Langer on a purely algebraic proof of the Bogomolov–Giesecker inequality for semistable Higgs bundles in the complex case [31, Proposition 3.4] and the Miyaoka–Yau inequality for Chern numbers of complex algebraic surfaces of general type. In his work, the notion of (semistable) Higgs–de Rham flow in characteristic p has played a similar role to the Yang–Mills–Higgs flow over the field of complex numbers.

Our paper basically consists of two parts: Sections 2–5 are devoted to the theory of the Higgs correspondence between the category of p-torsion Fontaine modules with extra endomorphism structures and the category of periodic Higgs–de Rham flows; Sections 6–7 and the appendix aim at applications to produce representations of π_1 from (semi-)stable Higgs bundles. In more detail, in Section 2 we recall the basics of the theory of Fontaine modules; in Section 3, we introduce the notion of a Higgs–de Rham flow in positive characteristic and establish the Higgs correspondence in positive characteristic; in Section 4, we construct a lifting of the inverse Cartier transform of Ogus–Vologodsky over an arbitrary truncated Witt ring; in Section 5, we establish the Higgs correspondence over an arbitrary truncated ring; in Section 6, we introduce the notion of a strongly semistable Higgs module and show that a strongly semistable Higgs module with trivial Chern classes is preperiodic and vice versa, and consequently we produce crystalline representations of the algebraic fundamental groups of the generic fiber with k-coefficients from semistable nilpotent Higgs bundles of small ranks over the closed fiber with trivial Chern classes; in Section 7, we prove a rigidity theorem for Fontaine modules whose associated graded Higgs modules are mod p stable; in the Appendix, we prove (jointly with Y.-H. Yang) that every semistable Higgs module of small rank is strongly semistable, thus partially verifying a conjecture in the first version of the paper [17].

2. Preliminaries on Fontaine modules

The category of Fontaine modules, as introduced by G. Fatlings [4], originates from number theory. In the seminal paper [7], Fontaine and Laffaille introduced the category $MF^{f,q}$ (resp. $MF^{f,q}_{\mathrm{tor}}$) of strongly divisible filtered modules over W (resp. of finite length) and constructed an exact and faithful contravariant functor from the previous category to the category of representations of the Galois group of the local field K (resp. of finite length). A representation lying in the image of that functor is said to be *crystalline*. The significance of the above category, as shown in another seminal paper [8] by Fontaine–Messing, is due to the fact that the crystalline cohomology of many proper algebraic varieties over W lies in that category. In the above cited paper, Faltings generalized both results to a geometric base (see also [5] for the generalization to the semistable reduction case and the case of a very ramified base ring). For us, that category plays the role connecting a certain category of p-adic Higgs modules with a certain category of p-adic representations of algebraic fundamental groups. From the point of view of nonabelian Hodge theory, this category is a nice p-adic analogue of polarized complex variations of Hodge structure, a special but important class of so-called harmonic bundles. One of principal aims of this paper is to establish a correspondence between the category of Fontaine modules and the category of one-periodic Higgs–de Rham flows, in both positive and mixed characteristics. This section is devoted to a brief exposition of the former category and related known results.

Remark 2.1. We remind the reader of the category of F-T-crystals developed in the monograph [25], which is also a p-adic analogue of the category of complex variations of Hodge structure (with no emphasis on polarization). This category of crystals is intimately related to the category of Fontaine modules (see particularly [25, Proposition 5.3.9]). It would be interesting to relate it to a category of Higgs modules. This task has not been touched upon in this paper.

For clarity, we start with p-torsion-free Fontaine modules. Our exposition is based on [4, Ch. II] and [5, §3]. Let X be a smooth W-scheme. For an affine subset $U \subset X$ flat over W, there exists a (nonunique) absolute Frobenius lifting $F_{\hat{U}}$ on its p-adic completion $\hat{U}$. An object in the category $MF^{\nabla}_{[0,w]}(\hat{U})$ is a quadruple $(H, \nabla, Fil, \Phi_{F_{\hat{U}}})$, where

(i) (H, Fil) is a filtered-free $\mathcal{O}_{\hat{U}}$-module with a basis e_i of Fil^i, $0 \le i \le w$.
(ii) ∇ is an integrable connection on H satisfying the Griffiths transversality:

$$\nabla(Fil^i) \subset Fil^{i-1} \otimes \Omega_{\hat{U}}.$$

(iii) The relative Frobenius is an $\mathcal{O}_{\hat{U}}$-linear morphism $\Phi_{F_{\hat{U}}} : F_{\hat{U}}^* H \to H$ with the strong p-divisibility property: $\Phi_{F_{\hat{U}}}(F_{\hat{U}}^* Fil^i) \subset p^i H$ and

$$\sum_{i=0}^{w} \frac{\Phi_{F_{\hat{U}}}(F_{\hat{U}}^* Fil^i)}{p^i} = H. \tag{2.1.1}$$

(iv) The relative Frobenius $\Phi_{F_{\hat{U}}}$ is horizontal with respect to the connection $F_{\hat{U}}^*\nabla$ on $F_{\hat{U}}^* H$ and ∇ on H.

The filtered-freeness in (i) means that the filtration Fil on H has a splitting such that each Fil^i is a direct sum of several copies of $\mathcal{O}_{\hat{U}}$. Equivalently, Fil is a finite exhaustive decreasing filtration of free $\mathcal{O}_{\hat{U}}$-submodules which is split. The pull-back connection $F_{\hat{U}}^*\nabla$ on $F_{\hat{U}}^* H$ is defined by the formula

$$F_{\hat{U}}^*\nabla(f \otimes e) = df \otimes e + f \cdot (dF_{\hat{U}} \otimes 1)(1 \otimes \nabla(e)), \qquad f \in \mathcal{O}_{\hat{U}},\ e \in H|_{\hat{U}}.$$

The horizontality condition (iv) is expressed by the commutativity of the diagram

$$\begin{array}{ccc} F_{\hat{U}}^* H & \xrightarrow{\Phi_{F_{\hat{U}}}} & H \\ {\scriptstyle F_{\hat{U}}^*\nabla}\downarrow & & \downarrow{\scriptstyle \nabla} \\ F_{\hat{U}}^* H \otimes \Omega_{\hat{U}} & \xrightarrow{\Phi_{F_{\hat{U}}} \otimes \mathrm{Id}} & H \otimes \Omega_{\hat{U}} \end{array}$$

As there is no canonical Frobenius liftings on $\hat{U}$, one has to know how the relative Frobenius changes under another Frobenius lifting. This is expressed by a Taylor formula. Let $\hat{U} = \operatorname{Spf} R$ and $F : R \to R$ be an absolute Frobenius lifting. Choose étale local coordinates $\{t_1, \ldots, t_d\}$ of U (that is, fix an étale map $U \to \operatorname{Spec}(W[t_1, \ldots, t_d])$). Let R' be any p-adically complete, p-torsion-free W-algebra, equipped with a Frobenius lifting $F' : R' \to R'$ and a morphism of W-algebras $\iota : R \to R'$. Then the relative Frobenius $\Phi_{F'} : F'^*(\iota^* H) \to \iota^* H$ is the composite

$$F'^*\iota^* H \overset{\alpha}{\cong} \iota^* F^* H \xrightarrow{\iota^*\Phi_F} \iota^* H,$$

where the isomorphism α is given by the formula

$$\alpha(e \otimes 1) = \sum_{\underline{i}} \nabla_{\partial}^{\underline{i}}(e) \otimes \frac{z^{\underline{i}}}{\underline{i}!}.$$

Here $\underline{i} = (i_1, \ldots, i_d)$ is a multi-index, and $z^{\underline{i}} = z_1^{i_1} \cdots z_d^{i_d}$ with $z_i = F' \circ \iota(t_i) - \iota \circ F(t_i)$, $1 \le i \le d$, and $\nabla_{\partial}^{\underline{j}} = \nabla_{\partial_{t_1}}^{i_1} \cdots \nabla_{\partial_{t_d}}^{i_d}$.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine covering of X over W. For each i, let $F_{\hat{U}_i}$ be an absolute Frobenius lifting over $\hat{U}_i$. Then, in view of [4, Theorem 2.3], when $w \le p - 1$, the local categories $\{MF^{\nabla}_{[0,w]}(\hat{U}_i)\}_{i\in I}$ glue into the category $MF^{\nabla}_{[0,w]}(X)$. Its object is called a *Fontaine module* over X/W, and will be denoted again by a quadruple (H, ∇, Fil, Φ), although the relative Frobenius Φ is only *locally* defined and depends on a choice of absolute Frobenius lifting locally. For an open affine U together with an absolute Frobenius lifting $F_{\hat{U}}$ over $\hat{U}$, the symbol $\Phi_{(U,F_{\hat{U}})}$ means the evaluation of Φ at $(U, F_{\hat{U}})$.

Example 2.2. Let $f : Y \to X$ be a proper smooth morphism over W of relative dimension $w \leq p-2$ between smooth W-schemes. Assume that the relative Hodge cohomology $R^i f_*\Omega^j_Y$, $i + j = w$, has no torsion. By [4, Theorem 6.2], the crystalline direct image $R^w f_*(\mathcal{O}_Y, d)$ is an object in $MF^{\nabla}_{[0,w]}(X/W)$.

The fundamental theorem of Fontaine–Laffaille (see [7, Theorem 3.3] for $X = \operatorname{Spec} W$) and Faltings (see [4, Theorem 2.6*]), which is a p-adic analogue of the Riemann–Hilbert correspondence over $\mathbb{C}$, reads:

Theorem 2.3 (Fontaine–Laffaille–Faltings correspondence). *With notation as above, assume furthermore X is proper over W and $w \leq p-2$. There exists a fully faithful contravariant functor* $\mathbf{D}$ *from the category $MF^{\nabla}_{[0,w]}(X/W)$ to the category of étale local systems over X_K.*

The image of the functor $\mathbf{D}$ is called the *crystalline sheaves* of Hodge–Tate weight n over X_K. We remind the reader that the functor $\mathbf{D}$ in [4] is covariant and its image is the category of dual crystalline sheaves. In the above theorem for torsion-free Fontaine modules as well as its p-torsion analogue below, we actually use the dual of the functor $\mathbf{D}$.

Variant 1: p-torsion. A p-torsion Fontaine module is defined in a similar way. In fact, the previous category $MF^{\nabla}_{[0,w]}(X/W)$ is the p-adic limit of its torsion variant (see [4, Ch. II, (c)–(d)]). The major modification in the p-torsion case occurs in the definition of strong p-divisibility. Note that (2.1.1) does not make sense in the p-torsion case. For each natural number n, a strict p^n-torsion Fontaine module (H, ∇, Fil, Φ) means the following: H is a *finitely generated* $\mathcal{O}_{X_n}$-module; Fil is a finite exhaustive decreasing filtration of $\mathcal{O}_{X_n}$-submodules on H satisfying Griffiths' transversality with respect to an integrable connection ∇; Φ is strongly p-divisible, namely, the evaluation of Φ at $(U, F_{\hat{U}})$ is an $\mathcal{O}_{U_n}$-isomorphism ($U_n := U \otimes \mathbb{Z}/p^n\mathbb{Z}$, $F_{U_n} := F_{\hat{U}} \otimes \mathbb{Z}/p^n\mathbb{Z}$)

$$\Phi_{(U_n,F_{U_n})} : F^*_{U_n}\tilde{H}|_{U_n} \cong H|_{U_n},$$

where $\tilde{H}$ is the quotient $\bigoplus_{i=0}^{w} Fil^i/\sim$ with $x \sim py$ for any $x \in Fil^i$ and y the image of x under the natural inclusion $Fil^i \hookrightarrow Fil^{i-1}$; the horizontality for Φ is formulated as in the nontorsion case, which means explicitly that the following diagram is commutative:

$$\begin{array}{ccc}
F^*_{U_n}\tilde{H}|_{U_n} & \xrightarrow{\Phi_{(U_n,F_{U_n})}} & H|_{U_n} \\
{\scriptstyle F^*_{U_n}\tilde{\nabla}}\downarrow & & \downarrow{\scriptstyle \nabla} \\
F^*_{U_n}\tilde{H}|_{U_n} \otimes \Omega_{U_n} & \xrightarrow{\Phi_{(U_n,F_{U_n})}\otimes \mathrm{Id}} & H|_{U_n} \otimes \Omega_{U_n}
\end{array}$$

where $\tilde{\nabla}$ is the integrable p-connection (see Definition 4.4) over $\tilde{H}$ induced by ∇ and $F_{U_n}^*\tilde{\nabla}$ is defined similarly to $F_{\hat{U}}^*\nabla$ in the non-p-torsion case by replacing ∇ resp. $dF_{\hat{U}}$ in the composite therein with $\tilde{\nabla}$ resp. $\frac{dF_{U_n}}{p}$ (see (4.15.1) for a local expression). Let $MF^{\nabla}_{[0,w],n}(X/W)$ denote the category of strict p^m-torsion Fontaine modules. The Fontaine–Laffaille–Faltings correspondence as given above is achieved by taking the p-adic limit of its p-torsion analogue.

Remark 2.4. Using Fitting ideals, Faltings [4, Theorem 2.1] shows that (H, Fil) is indeed locally filtered-free. A slight generalization obtained by A. Ogus using a different method is given in [25, Theorem 5.3.3]. Note also that in the definition of the category $MF^{\nabla}_{[0,w],n}(X/W)$ one actually requires only the existence of a model X_{n+1} over W_{n+1}. Although objects of this category are defined over X_n, the horizontality of the relative Frobenius uses the operator $\frac{dF_{U_{n+1}}}{p}$, where $F_{U_{n+1}}$ is an absolute Frobenius lifting on an open affine $U_{n+1} \subset X_{n+1}$. Also, the existence of X_{n+1} is required for the sake of the transition of two evaluations of the relative Frobenius via the Taylor formula. Therefore, this category requires only the existence of a smooth W_{n+1}-scheme X_{n+1}. In this case, we shall denote it by $MF^{\nabla}_{[0,w]}(X_{n+1}/W_{n+1})$.

Variant 2: extra endomorphism. For our purposes, we need to introduce the category $MF^{\nabla}_{[0,w],f}(X/W)$ of Fontaine modules with endomorphism structure $W(\mathbb{F}_{p^f})$ for each $f \in \mathbb{N}$. It consists of 5-tuples $(H, \nabla, Fil, \Phi, \iota)$, where (H, ∇, Fil, Φ) is a torsion-free Fontaine module and

$$\iota : W(\mathbb{F}_{p^f}) \hookrightarrow \mathrm{End}_{MF}(H, \nabla, Fil, \Phi)$$

is an embedding of $\mathbb{Z}_p$-algebras. A morphism of this category is a morphism in $MF^{\nabla}_{[0,w]}(X/W)$ respecting the endomorphism structure ι. Clearly, the category for $f = 1$ is nothing but the category of Fontaine modules. We introduce similarly its p-torsion counterpart $MF^{\nabla}_{[0,w],n,f}(X/W)$ (and $MF^{\nabla}_{[0,w],f}(X_{n+1}/W_{n+1})$), where the extra endomorphism structure is given by an embedding of $\mathbb{Z}/p^n\mathbb{Z}$-algebras:

$$\iota : W_n(\mathbb{F}_{p^f}) \hookrightarrow \mathrm{End}_{MF}(H, \nabla, Fil, \Phi).$$

3. Higgs correspondence in positive characteristic

Let us begin with the definitions of a (pre)periodic Higgs–de Rham flow and a (pre)periodic Higgs module. Let X_1 be a smooth variety over k and X_2 a W_2-lifting of X_1.

Definition 3.1. A *preperiodic Higgs–de Rham flow* over X_1 (with respect to the given W_2-lifting X_2) is a tuple $(E, \theta, Fil_0, \ldots, Fil_{e+f-1}, \phi)$, where $e \geq 0$ and $f \geq 1$ are integers, (E, θ) is a Higgs module in the category $HIG_{p-1}(X_1)$, Fil_i, $0 \leq i \leq e + f - 1$, is a Griffiths transverse filtration on $C_1^{-1}(E_i, \theta_i)$ where $(E_0, \theta_0) = (E, \theta)$ and

$$(E_i, \theta_i) := \mathrm{Gr}_{Fil_{i-1}}(H_{i-1}, \nabla_{i-1}), \qquad 1 \leq i \leq e + f,$$

is inductively defined, and ϕ is an isomorphism of graded Higgs modules

$$(E_{e+f}, \theta_{e+f}) \cong (E_e, \theta_e).$$

It is said to be *periodic* of period f (or f-periodic) if the integer e above is zero. The (E_i, θ_i)s (resp. (H_i, ∇_i)s) are called the *Higgs* (resp. *de Rham*) *terms* of the flow. A Higgs module (E, θ) over X_1 is said to be (pre)periodic if there exists a (pre)periodic Higgs–de Rham flow with leading Higgs term (E, θ).

One can complete a preperiodic Higgs–de Rham flow over X_1 to a Higgs–de Rham flow in a natural way: note that the isomorphism ϕ induces the isomorphism

$$C_1^{-1}(\phi) : C_1^{-1}(E_{e+f}, \theta_{e+f}) \cong C_1^{-1}(E_e, \theta_e),$$

and therefore one naturally obtains the filtration Fil_{e+f} on $C_1^{-1}(E_{e+f}, \theta_{e+f})$ by pulling back Fil_e via this isomorphism.

This section aims to establish the Higgs correspondence between the category of periodic Higgs–de Rham flows over X_1 and the category of strict p-torsion Fontaine modules with extra endomorphism structure. Let us introduce the category $HDF_{w,f}(X_2/W_2)$ as follows: its object is given by a periodic Higgs–de Rham flow $(E, \theta, Fil_0, Fil_1, \dots, Fil_{f-1}, \phi)$ over X_1 such that each filtration Fil_i is of level $\leq w$. Note that (E, θ) in a periodic Higgs–de Rham must be a graded Higgs module. A morphism between two objects is a morphism of graded Higgs modules respecting the additional structures. As an illustration, we explain a morphism in the category $HDF_{w,1}(X_2/W_2)$ of one-periodic Higgs–de Rham flows in detail. Let $(E_i, \theta_i, Fil_i, \phi_i)$, $i = 1, 2$, be two objects in the category. Then a morphism

$$f : (E_1, \theta_1, Fil_1, \phi_1) \to (E_2, \theta_2, Fil_2, \phi_2)$$

is given by a morphism of graded Higgs modules

$$f : (E_1, \theta_1) \to (E_2, \theta_2)$$

such that the induced morphism of flat modules (by the functoriality of C_1^{-1})

$$C_1^{-1}(f) : C_1^{-1}(E_1, \theta_1) \to C_1^{-1}(E_2, \theta_2)$$

preserves the filtrations, and moreover the induced morphism of graded Higgs modules is compatible with ϕs, that is, the following diagram of natural morphisms commutes:

$$\begin{array}{ccc} \mathrm{Gr}_{Fil_1} C_1^{-1}(E_1, \theta_1) & \xrightarrow{\phi_1} & (E_1, \theta_1) \\ {\scriptstyle \mathrm{Gr}\, C_1^{-1}(f)} \downarrow & & \downarrow {\scriptstyle f} \\ \mathrm{Gr}_{Fil_2} C_1^{-1}(E_2, \theta_2) & \xrightarrow{\phi_2} & (E_2, \theta_2) \end{array}$$

Recall that $MF^{\nabla}_{[0,w],f}(X_2/W_2)$ is the category of strict p-torsion Fontaine modules with extra endomorphism $\mathbb{F}_{p^f}$. The Higgs correspondence in positive characteristic is the following

Theorem 3.2. *With notation as above, let $w \leq p-1$ and f be a natural number. Then there is an equivalence of categories between $MF^{\nabla}_{[0,w],f}(X_2/W_2)$ and $HDF_{w,f}(X_2/W_2)$.*

We take an open covering $\{U_i\}$ of X_2/W_2 consisting of open affine subsets which are smooth over W_2, together with an absolute Frobenius lifting F_{U_i} on each U_i. Reducing modulo p, one obtains an open affine covering $\{U_{i,1}\}$ for X_1. We show first a special case of the theorem, namely the case $f=1$.

Proposition 3.3. *There is an equivalence of categories between the category of strict p-torsion Fontaine modules and the category of one-periodic Higgs–de Rham flows over X_1.*

For simplicity, we write MF for $MF^{\nabla}_{[0,w],1}(X_2/W_2)$ and HDF for $HDF_{w,1}(X_2/W_2)$. In the following paragraph, we shall construct two functors

$$\mathcal{GR}: MF \to HDF, \quad \mathcal{IC}: HDF \to MF,$$

and then show they are quasi-inverse to each other. For an $(H, \nabla, Fil, \Phi) \in MF$, let $(E,\theta) := \mathrm{Gr}_{Fil}(H,\nabla)$ be the associated graded Higgs module. The following lemma gives the first functor.

Lemma 3.4. *There is a filtration $Fil_{\exp}$ on $C_1^{-1}(E,\theta)$ together with an isomorphism of graded Higgs modules*

$$\phi_{\exp}: \mathrm{Gr}_{Fil_{\exp}}(C_1^{-1}(E,\theta)) \cong (E,\theta),$$

which is induced by the filtration Fil and the relative Frobenius Φ.

Proof. By [16, Proposition 1.4], the relative Frobenius induces an isomorphism of flat modules

$$\tilde{\Phi}: C_1^{-1}(E,\theta) \cong (H,\nabla).$$

So we define $Fil_{\exp}$ on $C_1^{-1}(E,\theta)$ to be the inverse image of Fil on H by $\tilde{\Phi}$. It induces tautologically an isomorphism of graded Higgs modules

$$\phi_{\exp} = \mathrm{Gr}(\tilde{\Phi}): \mathrm{Gr}_{Fil_{\exp}}(C_1^{-1}(E,\theta)) \cong (E,\theta). \qquad \square$$

Next, the functor C_1^{-1} induces the second functor $\mathcal{IC}$ as follows. Given an object $(E,\theta,Fil,\phi) \in HDF$, we define the triple

$$(H,\nabla,Fil) = (C_1^{-1}(E,\theta), Fil).$$

What remains is to produce a relative Frobenius Φ from the ϕ. This is the most technical point of the whole proof. Following Faltings [4, Ch. II, (d)], it suffices to give for each pair (U_i, F_{U_i}) an $\mathcal{O}_{U_{i,1}}$-morphism

$$\Phi_{(U_i,F_{U_i})}: F^*_{U_{i,1}}\mathrm{Gr}_{Fil} H|_{U_{i,1}} \to H|_{U_{i,1}},$$

where $F_{U_{i,1}}$ is the absolute Frobenius of $U_{i,1}$, satisfying the following conditions:

(1) strong p-divisibility, that is, $\Phi_{(U_i,F_{U_i})}$ is an isomorphism,
(2) horizontality,

(3) over each $U_{ij,1} := U_{i,1} \cap U_{j,1}$, $\Phi_{(U_i, F_{U_i})}$ and $\Phi_{(U_j, F_{U_j})}$ are related via the Taylor formula. Precisely, if the gluing map for $H_i := H|_{U_{i,1}}$ and $H_j := H|_{U_{j,1}}$ is G_{ij}, then we shall have the commutative diagram

$$\begin{array}{ccc} F^*_{U_{i,1}} \mathrm{Gr}_{Fil}\, H_i|_{U_{ij,1}} & \xrightarrow{\Phi_{(U_i,F_{U_i})}|_{U_{ij,1}}} & H_i|_{U_{ij,1}} \\ \downarrow{\scriptstyle \widetilde{G}_{ij}} & & \downarrow{\scriptstyle G_{ij}} \\ F^*_{U_{i,1}} \mathrm{Gr}_{Fil}\, H_j|_{U_{ij,1}} & & H_j|_{U_{ij,1}} \\ \downarrow{\scriptstyle \varepsilon_{ij}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ F^*_{U_{j,1}} \mathrm{Gr}_{Fil}\, H_j|_{U_{ij,1}} & \xrightarrow{\Phi_{(U_j,F_{U_j})}|_{U_{ij,1}}} & H_j|_{U_{ij,1}} \end{array}$$

where $\widetilde{G}_{ij}$ denotes the obvious map induced by G_{ij}, and ε_{ij} is defined by the Taylor formula which is given by the following expression:

$$e \otimes 1 \to e \otimes 1 + \sum_{|\underline{k}|=1}^{w} (\theta'_\partial)^{\underline{k}}(e) \otimes \frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!}, \tag{3.4.1}$$

where θ' denotes the Higgs field of $\mathrm{Gr}_{Fil}(H_j, \nabla)$. Here we take a system of étale local coordinates $\{\tilde{t}_1, \ldots, \tilde{t}_d\}$ of $U_{ij,2}$, which induces a system of étale local coordinates $\{t_1, \ldots, t_d\}$ on $U_{ij,1}$, and $\underline{k} = (k_1, \ldots, k_d)$ is a multi-index, $z^{\underline{k}} = z_1^{k_1} \cdots z_d^{k_d}$ with

$$z_k = F^*_{U_{i,2}}(\tilde{t}_k) - F^*_{U_{j,2}}(\tilde{t}_k),$$

and $(\theta')^{\underline{k}}_\partial = (\theta'_{\partial_{t_1}})^{k_1} \cdots (\theta'_{\partial_{t_d}})^{k_d}$.

Recall that $H = \{H_i := F^*_{U_{i,1}} E|_{U_{i,1}}, G_{ij}\}_{i \in I}$, where G_{ij} has an expression of ε_{ij} similar to (3.4.1) (see [16, proof of Proposition 1.4]):

$$G_{ij}(e \otimes 1) = e \otimes 1 + \sum_{|\underline{k}|=1}^{w} \theta^{\underline{k}}_\partial(e) \otimes \frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!}.$$

We define

$$\Phi_{(U_i, F_{U_i})} = F^*_{U_{i,1}} \phi : F^*_{U_{i,1}} \mathrm{Gr}_{Fil}\, H|_{U_{i,1}} \to F^*_{U_{i,1}} E|_{U_{i,1}}.$$

By construction, $\Phi_{(U_i, F_{U_i})}$ is strongly p-divisible (this is condition (1)). As ϕ is globally defined, we have the diagram

$$\begin{array}{ccc} \mathrm{Gr}_{Fil}\, H_i|_{U_{ij,1}} & \xrightarrow{\phi_i|_{U_{ij,1}}} & E|_{U_{ij,1}} \\ \downarrow{\scriptstyle \mathrm{Gr}(G_{ij})} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathrm{Gr}_{Fil}\, H_j|_{U_{ij,1}} & \xrightarrow{\phi_j|_{U_{ij,1}}} & E|_{U_{ij,1}} \end{array}$$

Pulling back the above diagram via $F^*_{U_{i,1}}$, we get

$$\begin{array}{ccc} F^*_{U_{i,1}}\,\mathrm{Gr}_{Fil}\,H_i|_{U_{ij,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi_i)|_{U_{ij,1}}} & F^*_{U_{i,1}}E|_{U_{ij,1}} \\ \widetilde{G_{ij}}\downarrow & & \downarrow \mathrm{Id} \\ F^*_{U_{i,1}}\,\mathrm{Gr}_{Fil}\,H_j|_{U_{ij,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi_j)|_{U_{ij,1}}} & F^*_{U_{i,1}}E|_{U_{ij,1}} \end{array}$$

Then we extend it to the diagram

$$\begin{array}{ccc} F^*_{U_{i,1}}\,\mathrm{Gr}_{Fil}\,H_i|_{U_{ij,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi_i)|_{U_{ij,1}}} & F^*_{U_{i,1}}E|_{U_{ij,1}} \\ \widetilde{G_{ij}}\downarrow & & \downarrow \mathrm{Id} \\ F^*_{U_{i,1}}\,\mathrm{Gr}_{Fil}\,H_j|_{U_{ij,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi_j)|_{U_{ij,1}}} & F^*_{U_{i,1}}E|_{U_{ij,1}} \\ \varepsilon_{ij}\downarrow & & \downarrow G_{ij} \\ F^*_{U_{j,1}}\,\mathrm{Gr}_{Fil}\,H_j|_{U_{ij,1}} & \xrightarrow{F^*_{U_{j,1}}(\phi_j)|_{U_{ij,1}}} & F^*_{U_{j,1}}E|_{U_{ij,1}} \end{array}$$

As $\phi\circ\theta'=\theta\circ\phi$, for any local section e of $\mathrm{Gr}_{Fil}\,H_j|_{U_{ij,1}}$ we have

$$\begin{aligned} G_{ij}\circ F^*_{U_{i,1}}(\phi_j)(e\otimes 1) &= \phi_j(e)\otimes 1+\sum_{|\underline{k}|=1}^{w}\theta_\partial^{\underline{k}}(\phi_j(e))\otimes\frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!} \\ &= \phi_j(e)\otimes 1+\sum_{|\underline{k}|=1}^{w}\phi_j((\theta')_\partial^{\underline{k}}(e))\otimes\frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!} \\ &= F^*_{U_{j,1}}(\phi_j)\circ\varepsilon_{ij}(e\otimes 1). \end{aligned}$$

So the lower square of the last diagram is commutative (this is condition (3)). What remains to show is condition (2).

Lemma 3.5. *Each $\Phi_{(U_i,F_{U_i})}$ is horizontal with respect to ∇.*

Proof. Put $\tilde H=\mathrm{Gr}_{Fil}\,H$, $\theta'=\mathrm{Gr}_{Fil}\,\nabla$, $\Phi_i=\Phi_{(U_i,F_{U_i})}$ and let $F_{U_{i,1}}$ be the absolute Frobenius over $U_{i,1}$. Following Faltings [4, Ch. II, (d)], it suffices to show the commutativity of the diagram

$$\begin{array}{ccc} F^*_{U_{i,1}}\tilde H|_{U_{i,1}} & \xrightarrow{\Phi_i} & H|_{U_{i,1}} \\ F^*_{U_i}\nabla\downarrow & & \downarrow\nabla \\ F^*_{U_{i,1}}\tilde H|_{U_{i,1}}\otimes\Omega_{U_{i,1}} & \xrightarrow{\Phi_i\otimes\mathrm{Id}} & H|_{U_{i,1}}\otimes\Omega_{U_{i,1}} \end{array}$$

where $F^*_{U_i}\nabla$ is a connection induced by $\frac{dF_{U_i}}{p}(F^*_{U_{i,1}}\theta')$, i.e. the composite of

$$F^*_{U_{i,1}}\tilde H|_{U_{i,1}}\xrightarrow{F^*_{U_{i,1}}(\theta')}F^*_{U_{i,1}}\tilde H|_{U_{i,1}}\otimes F^*_{U_{i,1}}\Omega_{U_{i,1}}\xrightarrow{\mathrm{Id}\otimes\frac{dF_{U_i}}{p}}F^*_{U_{i,1}}\tilde H|_{U_{i,1}}\otimes\Omega_{U_{i,1}}.$$

Thus we have to show the commutativity of the diagram

$$\begin{array}{ccc} F^*_{U_{i,1}}\tilde{H}|_{U_{i,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi)} & F^*_{U_{i,1}}E|_{U_{i,1}} \\ {\scriptstyle \frac{dF_{U_i}}{p}(F^*_{U_{i,1}}(\theta'))}\downarrow & & \downarrow{\scriptstyle \frac{dF_{U_i}}{p}(F^*_{U_{i,1}}(\theta))} \\ F^*_{U_{i,1}}\tilde{H}|_{U_{i,1}}\otimes\Omega_{U_{i,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi)\otimes\mathrm{Id}} & F^*_{U_{i,1}}E|_{U_{i,1}}\otimes\Omega_{U_{i,1}}. \end{array}$$

As ϕ is a morphism of graded Higgs modules, one has the commutative diagram

$$\begin{array}{ccc} \tilde{H}|_{U_i} & \xrightarrow{\phi} & E|_{U_{i,1}} \\ {\scriptstyle \theta'}\downarrow & & \downarrow{\scriptstyle \theta} \\ \tilde{H}|_{U_{i,1}}\otimes\Omega_{U_{i,1}} & \xrightarrow{\phi\otimes\mathrm{Id}} & E|_{U_{i,1}}\otimes\Omega_{U_{i,1}}. \end{array}$$

The pull-back via $F^*_{U_{i,1}}$ of the above diagram yields commutative diagrams

$$\begin{array}{ccccc} F^*_{U_{i,1}}\tilde{H}|_{U_{i,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi)} & F^*_{U_{i,1}}E|_{U_{i,1}} & \xrightarrow{\frac{dF_{U_i}}{p}(F^*_{U_{i,1}}(\theta))} & \\ {\scriptstyle F^*_{U_{i,1}}(\theta')}\downarrow & & \downarrow{\scriptstyle F^*_{U_{i,1}}(\theta)} & & \\ F^*_{U_{i,1}}\tilde{H}|_{U_{i,1}}\otimes F^*_{U_{i,1}}\Omega_{U_{i,1}} & \xrightarrow{F^*_{U_{i,1}}(\phi)\otimes\mathrm{Id}} & F^*_{U_{i,1}}E|_{U_{i,1}}\otimes F^*_{U_{i,1}}\Omega_{U_{i,1}} & \xrightarrow{\mathrm{Id}\otimes\frac{dF_{U_i}}{p}} & F^*_{U_{i,1}}E|_{U_{i,1}}\otimes\Omega_{U_{i,1}} \\ & \searrow{\scriptstyle F^*_{U_{i,1}}(\phi)\otimes\frac{dF_{U_i}}{p}} & & & \end{array}$$

Chasing the outside of the above diagram gives the required commutativity. □

Proof of Proposition 3.3. The equivalence of categories follows by providing natural isomorphisms of functors:

$$\mathcal{GR}\circ\mathcal{IC}\cong\mathrm{Id},\quad \mathcal{IC}\circ\mathcal{GR}\cong\mathrm{Id}.$$

We first define a natural isomorphism $\mathcal{A}$ from $\mathcal{IC}\circ\mathcal{GR}$ to Id: for $(H,\nabla,Fil,\Phi)\in MF$, put

$$(E,\theta,Fil,\phi)=\mathcal{GR}(H,\nabla,Fil,\Phi),\quad (H',\nabla',Fil',\Phi')=\mathcal{IC}(E,\theta,Fil,\phi).$$

Then one verifies that the map

$$\tilde{\Phi}:(H',\nabla')=C_1^{-1}\circ\mathrm{Gr}_{Fil}(H,\nabla)\cong(H,\nabla)$$

gives an isomorphism from (H',∇',Fil',Φ') to (H,∇,Fil,Φ) in the category MF. We call it $\mathcal{A}(H,\nabla,Fil,\Phi)$. It is straightforward to verify that $\mathcal{A}$ is indeed a natural transformation. Conversely, a natural isomorphism $\mathcal{B}$ from $\mathcal{GR}\circ\mathcal{IC}$ to Id is given as follows: for

(E, θ, Fil, ϕ), put

$$(H, \nabla, Fil, \Phi) = \mathcal{IC}(E, \theta, Fil, \phi), \quad (E', \theta', Fil', \phi') = \mathcal{GR}(H, \nabla, Fil, \Phi).$$

Then $\phi : \mathrm{Gr}_{Fil} \circ C_1^{-1}(E, \theta) \cong (E, \theta)$ induces an isomorphism from $(E', \theta', Fil', \phi')$ to (E, θ, Fil, ϕ) in *HDF*, which we define to be $\mathcal{B}(E, \theta, Fil, \phi)$. It is direct to check that $\mathcal{B}$ is a natural isomorphism. □

Before moving to the proof of Theorem 3.2 in general, we shall introduce an intermediate category, the category of one-periodic Higgs–de Rham flows with endomorphism structure $\mathbb{F}_{p^f}$: an object is a 5-tuple $(E, \theta, Fil, \phi, \iota)$, where (E, θ, Fil, ϕ) is an object in *HDF* and $\iota : \mathbb{F}_{p^f} \hookrightarrow \mathrm{End}_{HDF}(E, \theta, Fil, \phi)$ is an embedding of $\mathbb{F}_p$-algebras. As an immediate consequence of Proposition 3.3, we have

Corollary 3.6. *There is an equivalence of categories between the category of strict p-torsion Fontaine modules with endomorphism structure $\mathbb{F}_{p^f}$ and the category of one-periodic Higgs–de Rham flows over X_1 with endomorphism structure $\mathbb{F}_{p^f}$.*

Obviously, Corollary 3.6 and the next proposition will complete the proof of Theorem 3.2.

Proposition 3.7. *There is an equivalence of categories between the category of one-periodic Higgs–de Rham flows of level $\leq w$ over X_1 with endomorphism structure $\mathbb{F}_{p^f}$ and the category $HDF_{w,f}(X_2/W_2)$.*

Start off with an object $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$ in $HDF_{w,f}(X_2/W_2)$. Put

$$(G, \eta) := \bigoplus_{i=0}^{f-1} (E_i, \theta_i)$$

with $(E_0, \theta_0) = (E, \theta)$. As the functor C_1^{-1} is compatible with direct sums, one has the identification

$$C_1^{-1}(G, \eta) = \bigoplus_{i=0}^{f-1} C_1^{-1}(E_i, \theta_i).$$

We equip $C_1^{-1}(G, \eta)$ with the filtration $Fil = \bigoplus_{i=0}^{f-1} Fil_i$ by the above identification. Also ϕ induces a natural isomorphism of graded Higgs modules

$$\tilde{\phi} : \mathrm{Gr}_{Fil}\, C_1^{-1}(G, \eta) \cong (G, \eta)$$

as follows: as

$$\mathrm{Gr}_{Fil}\, C_1^{-1}(G, \eta) = \bigoplus_{i=0}^{r-1} \mathrm{Gr}_{Fil_i}\, C_1^{-1}(E_i, \theta_i),$$

we require that $\tilde{\phi}$ maps the factor $\mathrm{Gr}_{Fil_i}\, C_1^{-1}(E_i, \theta_i)$ identically to the factor (E_{i+1}, θ_{i+1}) for $0 \leq i \leq f-2$ (assume $f \geq 2$ to avoid the trivial case), and the last factor $\mathrm{Gr}_{Fil_{f-1}}(E_{f-1}, \theta_{f-1})$ isomorphically to (E_0, θ_0) via ϕ. Thus the constructed quadruple $(G, \eta, Fil, \tilde{\phi})$ is a periodic Higgs–de Rham flow of period one.

Lemma 3.8. *With notation as above, there is a natural embedding of $\mathbb{F}_p$-algebras*

$$\iota : \mathbb{F}_{p^r} \to \mathrm{End}_{HDF}(G, \eta, Fil, \tilde{\phi}).$$

Thus the extended tuple $(G, \eta, Fil, \tilde{\phi}, \iota)$ is a one-periodic Higgs–de Rham flow with endomorphism structure $\mathbb{F}_{p^f}$.

Proof. Choose a primitive element ξ_1 in $\mathbb{F}_{p^f}|\mathbb{F}_p$ once and for all. To define the embedding ι, it suffices to specify the image $s := \iota(\xi_1)$, which is defined as follows: Write

$$(G, \eta) = (E_0, \theta_0) \oplus (E_1, \theta_1) \oplus \cdots \oplus (E_{f-1}, \theta_{f-1}).$$

Then $s = m_{\xi_1} \oplus m_{\xi_1^p} \oplus \cdots \oplus m_{\xi_1^{p^{f-1}}}$, where $m_{\xi_1^{p^i}}$, $i = 0, \ldots, f-1$, is multiplication by $\xi_1^{p^i}$. It defines an endomorphism of (G, η) and preserves Fil on $C_1^{-1}(G, \eta)$. Write $(\mathrm{Gr}_{Fil} \circ C_1^{-1})(s)$ for the induced endomorphism of $\mathrm{Gr}_{Fil}\, C_1^{-1}(G, \eta)$. It remains to verify the commutativity

$$\tilde{\phi} \circ s = (\mathrm{Gr}_{Fil} \circ C_1^{-1})(s) \circ \tilde{\phi}.$$

In terms of a local basis, it boils down to the following obvious equality:

$$\begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ \phi & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 & 0 & \ldots & 0 \\ 0 & \xi_1^p & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \xi_1^{p^{f-1}} \end{pmatrix} = \begin{pmatrix} \xi_1^p & 0 & \ldots & 0 \\ 0 & \xi_1^{p^2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \xi_1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ \phi & 0 & \ldots & 0 \end{pmatrix}.$$

□

Conversely, given a one-periodic Higgs–de Rham flow with endomorphism structure $\mathbb{F}_{p^f}$, say $(G, \eta, Fil, \phi, \iota)$, we can associate to it an object in $HDF_{w,f}(X_2/W_2)$ as follows. The endomorphism $\iota(\xi_1)$ decomposes (G, η) into eigenspaces:

$$(G, \eta) = \bigoplus_{i=0}^{f-1} (G_i, \eta_i),$$

where (G_i, η_i) is the eigenspace corresponding to the eigenvalue $\xi_1^{p^i}$. The isomorphism $C_1^{-1}(\iota(\xi_1))$ induces the eigen-decomposition of the de Rham module as well:

$$(C_1^{-1}(G, \eta), Fil) = \bigoplus_{i=0}^{f-1} (C_1^{-1}(G_i, \eta_i), Fil_i).$$

Under this decomposition, the isomorphism $\phi : \mathrm{Gr}_{Fil}\, C_1^{-1}(G, \eta) \cong (G, \eta)$ decomposes into $\bigoplus_{i=0}^{f-1} \phi_i$ such that for $i \leq f-2$,

$$\phi_i : \mathrm{Gr}_{Fil_i}\, C_1^{-1}(G_i, \eta_i) \cong (G_{i+1}, \eta_{i+1}),$$

and $\phi_{f-1} : \mathrm{Gr}_{Fil_{f-1}}\, C_1^{-1}(G_{f-1}, \eta_{f-1}) \cong (G_0, \eta_0)$. Set $(E, \theta) = (G_0, \eta_0)$.

Lemma 3.9. *Let $(E_0, \theta_0) = (E, \theta)$. Then the filtrations $\{Fil_i\}$ and the isomorphisms $\{\phi_i\}$ of graded Higgs modules induce inductively a filtration $\widetilde{Fil}_i$ on $C_1^{-1}(E_i, \theta_i)$, $i = 0, \ldots, f-1$, and an isomorphism of graded Higgs modules*

$$\tilde{\phi} : \mathrm{Gr}_{\widetilde{Fil}_{f-1}} C_1^{-1}(E_{f-1}, \theta_{f-1}) \cong (E, \theta).$$

Thus the extended tuple $(E, \theta, \widetilde{Fil}_0, \ldots, \widetilde{Fil}_{f-1}, \tilde{\phi})$ is an object in $HDF_{w,f}(X_2/W_2)$.

Proof. The filtration $\widetilde{Fil}_0$ on $C_1^{-1}(E_0, \theta_0)$ is just Fil_0. Set

$$(E_1, \theta_1) = \mathrm{Gr}_{Fil_0} C_1^{-1}(E_0, \theta_0).$$

Via the isomorphism

$$C_1^{-1}(\phi_0) : C_1^{-1} \mathrm{Gr}_{Fil_0} C_1^{-1}(G_0, \eta_0) \cong C_1^{-1}(G_1, \eta_1),$$

we obtain a filtration $\widetilde{Fil}_1$ on $C_1^{-1}(E_1, \theta_1)$ from Fil_1 on $C_1^{-1}(G_1, \eta_1)$ by pull-back. By construction, one has the isomorphism

$$\mathrm{Gr}\, C_1^{-1}(\phi_0) : \mathrm{Gr}_{\widetilde{Fil}_1} C_1^{-1}(E_1, \theta_1) \cong \mathrm{Gr}_{Fil_1} C_1^{-1}(G_1, \eta_1).$$

Repeating the same procedure for (E_2, θ_2) and so on, we shall inductively obtain a filtration $\widetilde{Fil}_i$ on $C_1^{-1}(E_i, \theta_i)$ for $i = 1, \ldots, f-1$. Finally, we define

$$\tilde{\phi} : \mathrm{Gr}_{\widetilde{Fil}_{f-1}} C_1^{-1}(E_{f-1}, \theta_{f-1}) = (\mathrm{Gr}_{\widetilde{Fil}_{f-1}} C_1^{-1}) \circ \cdots \circ (\mathrm{Gr}_{\widetilde{Fil}_0} C_1^{-1})(E, \theta) \to (E, \theta)$$

to be the composite $(\mathrm{Gr}\, C_1^{-1})^{f-1}(\phi_0) \circ \cdots \circ (\mathrm{Gr}\, C_1^{-1})(\phi_{f-2}) \circ \phi_{f-1}$. □

Proof of Proposition 3.7. For $f = 1$, there is nothing to prove. Suppose $f \geq 2$. Note that Lemma 3.8 gives a functor $\mathcal{E}$ from $HDF_{w,f}(X_2/W_2)$ to the category of one-periodic Higgs–de Rham flows with endomorphism structure $\mathbb{F}_{p^f}$, while Lemma 3.9 gives a functor $\mathcal{D}$ in the opposite direction. We show that they give an equivalence of categories. It is direct to see that

$$\mathcal{D} \circ \mathcal{E} = \mathrm{Id}.$$

So it remains to give a natural isomorphism τ between $\mathcal{E} \circ \mathcal{D}$ and Id. For $(E, \theta, Fil, \phi, \iota)$, let

$$\mathcal{D}(E, \theta, Fil, \phi, \iota) = (G, \eta, Fil_0, \ldots, Fil_{f-1}, \tilde{\phi}),$$

and

$$\mathcal{E}(G, \eta, Fil_0, \ldots, Fil_{f-1}, \tilde{\phi}) = (E', \theta', Fil', \phi', \iota').$$

By the construction, (E', θ') is equal to

$$(G, \eta) \oplus \mathrm{Gr}_{Fil_0} C_1^{-1}(G, \eta) \oplus \cdots \oplus (\mathrm{Gr}_{Fil_{f-2}} C_1^{-1}) \circ \cdots \circ (\mathrm{Gr}_{Fil_0} C_1^{-1})(G, \eta).$$

Let $(E, \theta) = (E_0, \theta_0) \oplus (E_1, \theta_1) \oplus \cdots \oplus (E_{f-1}, \theta_{f-1})$ be the eigen-decomposition of (E, θ) under $\iota(\xi_1)$. For $1 \leq i \leq f-1$, there is a natural isomorphism $\phi_{i-1} \circ (\mathrm{Gr}\, C^{-1})\phi_{i-2} \circ \cdots \circ (\mathrm{Gr}\, C^{-1})^{i-1}\phi_0$ of graded Higgs modules between the factors:

$$(\mathrm{Gr}_{Fil_{i-1}} C_1^{-1}) \circ (\mathrm{Gr}_{Fil_{i-2}} C_1^{-1}) \cdots \circ (\mathrm{Gr}_{Fil_0} C_1^{-1})(G, \eta) \cong (E_i, \theta_i).$$

Thus $\mathrm{Id} \oplus \bigoplus_{i=1}^{f-1} \phi_{i-1} \circ (\mathrm{Gr}\, C^{-1})\phi_{i-2} \circ \cdots \circ (\mathrm{Gr}\, C^{-1})^{i-1}\phi_0$ provides an isomorphism of graded Higgs modules from (E', θ') to (E, θ). It is easy to check that it yields an isomorphism of $\tau(E, \theta, Fil, \phi, \iota)$ in the latter category. The functoriality of τ is easily verified. □

This completes the Higgs correspondence in positive characteristic. In the following we deduce from it some direct consequences.

Crystalline $\mathbb{F}_{p^f}$-representations. Let X/W be a smooth proper scheme over W. An $\mathbb{F}_{p^f}$-representation of $\pi_1(X_K)$ is said to be crystalline if it is crystalline as an $\mathbb{F}_p$-representation by restriction of scalars. In other words, a crystalline $\mathbb{F}_{p^f}$-representation is a crystalline $\mathbb{F}_p$-representation $\mathbb{V}$ together with an embedding of $\mathbb{F}_p$-algebras $\mathbb{F}_{p^f} \hookrightarrow \mathrm{End}_{\pi_1(X_K)}(\mathbb{V})$. Similarly, one has the notion of crystalline $W_n(\mathbb{F}_{p^f})$-representation for $n \in \mathbb{N} \cup \{\infty\}$. The following corollary is immediate from Theorems 2.3 and 3.2.

Corollary 3.10. *Let X/W be a smooth proper scheme. Assume $w \leq p-2$. There is an equivalence of categories between the category of crystalline $\mathbb{F}_{p^f}$-representations of $\pi_1(X_K)$ with Hodge–Tate weight $\leq w$ and the category of f-periodic Higgs–de Rham flows of level $\leq w$ over X_1.*

For an object $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi) \in HDF_{w,f}(X_2/W_2)$, we define its shift and lengthening as follows: for $(E_f, \theta_f) = \mathrm{Gr}_{Fil_{f-1}}(H_{f-1}, \nabla_{f-1})$, $C_1^{-1}(\phi)$ induces the pull-back filtration $(C_1^{-1}(\phi))^* Fil_0$ on $C_1^{-1}(E_f, \theta_f)$ and an isomorphism of graded Higgs modules $\mathrm{Gr}\, C_1^{-1}(\phi)$ on the gradings. Then it is easy to check that the tuple

$$(E_1, \theta_1, Fil_1, \ldots, Fil_{f-1}, C_1^{-1}(\phi)^* Fil_0, \mathrm{Gr}\, C_1^{-1}(\phi))$$

is an object in $HDF_{w,f}(X_2/W_2)$, which we call the *shift* of $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$. For any multiple lf, $l \geq 1$, we can lengthen $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$ to an object of $HDF_{w,lf}(X_2/W_2)$: similar to the above, we can inductively define the induced filtration on (H_j, ∇_j), $f \leq j \leq lf-1$, from Fil_is via ϕ. One has the induced isomorphism of graded Higgs modules

$$(\mathrm{Gr}\, C_1^{-1})^{l'f}(\phi) : (E_{(l'+1)f}, \theta_{(l'+1)f}) \cong (E_{l'f}, \theta_{l'f}), \quad 0 \leq l' \leq l-1.$$

The isomorphism $\phi_l : (E_{lf}, \theta_{lf}) \cong (E_0, \theta_0)$ is defined to be their composite. The resulting object $(E, \theta, Fil_0, \ldots, Fil_{lf-1}, \phi_l)$ is called the $(l-1)$-fold *lengthening* of $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$. The following result will be obvious from the proof of Theorem 3.2.

Corollary 3.11. *With notation as in Corollary 3.10, let ρ be the corresponding crystalline $\mathbb{F}_{p^f}$-representation to $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$. Then:*

(i) *The shift of $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$ corresponds to $\rho^\sigma = \rho \otimes_{\mathbb{F}_{p^f}, \sigma} \mathbb{F}_{p^f}$, the σ-conjugation of ρ. Here $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^f}|\mathbb{F}_p)$ is the Frobenius element.*
(ii) *For $l \in \mathbb{N}$, the $(l-1)$-fold lengthening of $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi)$ corresponds to the extension of scalars $\rho \otimes_{\mathbb{F}_{p^f}} \mathbb{F}_{p^{lf}}$.*

Local freeness of preperiodic Higgs modules. In the following, we explain that a preperiodic Higgs module is locally free, an a posteriori property.

Proposition 3.12. *Every periodic Higgs module is locally free.*

Proof. Let (E,θ) be a periodic Higgs module. Then a periodic Higgs–de Rham flow with leading term (E,θ) gives an object in the category $HDF_{w,f}(X_2/W_2)$ for some f. Let $(H,\nabla,Fil,\Phi,\iota)$ be the corresponding object in $MF^{\nabla}_{[0,w],f}(X_2/W_2)$ according to Theorem 3.2. The proof of [4, Theorem 2.1] (cf. page 32 of loc. cit.) asserts that Fil is a filtration of locally free subsheaves of H and locally split, which implies that $\mathrm{Gr}_{Fil}\,H$ is locally free. It follows immediately that (E,θ) is also locally free. □

A. Langer [22, §5.3, Proposition 1(1)] has obtained the following enhancement of the previous result (notice however that the assumption that $\operatorname{rank} E \leq p$ in the statement on the Higgs module is in our case unnecessary due to a slightly different definition of Higgs–de Rham flow).

Corollary 3.13 (Langer). *Every preperiodic Higgs module is locally free.*

Proof. This follows from Proposition 3.12 and [22, Lemma 3]. □

Corollary 3.14. *Griffiths transverse filtrations in a preperiodic Higgs–de Rham flow are filtrations by locally free subsheaves and locally split.*

Proof. The periodic case has been explained in the proof of Proposition 3.12. But it also follows from Corollary 3.13, by induction on the level of filtrations. So does the preperiodic case. □

4. Inverse Cartier transform over a truncated Witt ring

The inverse Cartier transform of Ogus–Vologodsky [26] has played a pivotal role in the notion of a (periodic) Higgs–de Rham flow in characteristic p. In order to obtain the analogous notion of (periodic) Higgs–de Rham flow over a truncated Witt ring, we need to construct a lifting of the inverse Cartier transform. In this section, X_n is a smooth scheme over W_n and X_{n+1} is a W_{n+1}-lifting of X_n.

The anonymous referee has kindly pointed out to us that the work of A. Shiho [30] is related to our construction below. Recall, for each $n\in\mathbb{N}$, $S_n=\operatorname{Spec} W_n$ and $F_{S_n}: S_n\to S_n$ the Frobenius automorphism. Shiho [30] constructs a functor from the category of quasi-nilpotent Higgs modules on $X_n^{(n)}$ to the category of quasi-nilpotent flat modules on X_n, where X_n is a smooth scheme over W_n and $X_n^{(n)}=X_n\times_{F^n_{S_n}} S_n$, under the assumption that $X_n^{(m)}=X_n\times_{F^m_{S_n}} S_n$, $0\leq m\leq n$, admits a smooth lifting $X_{n+1}^{(m)}$ to S_{n+1} and the Frobenius liftings $F^m_{n+1}: X_{n+1}^{(m-1)}\to X_{n+1}^{(m)}$ over S_{n+1} exist. The functor is a nice p-adic reincarnation of the notion of λ-connection in complex differential geometry. However, the assumption on the Frobenius lifting is very restrictive for a projective W_n-scheme, which is however the basic assumption to formulate the semistability for Higgs modules. For example, a smooth projective curve over W_2 admits no Frobenius lifting once its genus is greater than one. Our construction was inspired by the fact that Ogus–Vologodsky's construction extends the theory of strict p-torsion Fontaine modules (see [26, §4], and also [27], [16]). We have worked out a generalization for strict p^n-torsion Fontaine submodules in [29] and the current construction is a further generalization (without assuming the existence of an ambient strict p^n-torsion Fontaine module).

Let $X'_n = X_n \times_{F_{S_n}} S_n$. Then X'_{n+1} is a smooth lifting of X'_n over W_{n+1}. The mod p^{n-1} reduction $X_n \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$ of X_n is denoted by X_{n-1}. Similarly for X'_{n-1}. Let us introduce a category $\mathcal{H}(X'_n)$ of Higgs modules over X'_n as follows: an object is given by a tuple

$$(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\psi}),$$

where (E, θ) is a graded Higgs module over X'_n of exponent $\leq p-2$, $(\bar{H}, \bar{\nabla}, \overline{Fil})$ a de Rham module over X'_{n-1} with the level of Hodge filtration $\leq p-2$ and

$$\bar{\psi} : \mathrm{Gr}_{\overline{Fil}}(\bar{H}, \bar{\nabla}) \cong (\bar{E}, \bar{\theta}) := (E, \theta) \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$$

an isomorphism of graded Higgs modules over X'_{n-1}. A morphism in the category is defined in the obvious way. For $n = 1$, the above tuple reduces to a nilpotent graded Higgs module over X'_1 of exponent $\leq p-2$. So $\mathcal{H}(X'_1)$ is a full subcategory of $HIG_{p-1}(X'_1)$.

Theorem 4.1. *With notation as above, there exists a functor $\mathcal{C}_n^{-1}$ from the category $\mathcal{H}(X'_n)$ to the category $MIC(X_n)$ of flat modules over X_n such that $\mathcal{C}_n^{-1}$ lifts $\mathcal{C}_{n-1}^{-1}$ and $\mathcal{C}_1^{-1}$ agrees with the inverse Cartier transform $C^{-1}_{\mathcal{X}/\mathcal{S}}$ of Ogus–Vologodsky* [26] *with $(\mathcal{X}, \mathcal{S}) = (X_1/k, X'_2/W_2)$.*

We shall also introduce an intermediate category $\widetilde{MIC}(X'_n)$, which we call the category of twisted flat modules over X'_n. The construction of the functor $\mathcal{C}_n^{-1}$ consists in constructing a functor $\mathcal{T}_n : \mathcal{H}(X'_n) \to \widetilde{MIC}(X'_n)$ and then a functor $\mathcal{F}_n : \widetilde{MIC}(X'_n) \to MIC(X_n)$. The category $\widetilde{MIC}(X'_n)$ is closely related to the category of quasi-nilpotent $\mathcal{O}_{X'_n}$-modules with integrable p-connections of Shiho [30] which we shall explain later. The motivation to introduce this new category is mainly the necessity to make sense of p-powers in the denominators appearing in the Taylor formula (4.16.1).

Let X be a smooth scheme over S_n. First recall that a *Lie algebroid* on X is a locally free $\mathcal{O}_X$-module $\mathcal{A}$ equipped with a skew-symmetric $\mathcal{O}_X$-bilinear pairing

$$[\cdot,\cdot]_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

satisfying the Jacobi identity and an action of $\mathcal{A}$ on $\mathcal{O}_X$ by derivations (so-called *anchor map*) which is given by an $\mathcal{O}_X$-linear morphism of Lie algebras

$$\alpha : \mathcal{A} \to T_X$$

subject to the compatibility condition (Leibniz rule)

$$[x, fy]_{\mathcal{A}} = \alpha(x)(f)y + f[x, y]_{\mathcal{A}}.$$

Let us consider the Lie algebroid $(T_X, \alpha, \{\cdot,\cdot\} := p[\cdot,\cdot])$, with the anchor $\alpha = p \cdot \mathrm{Id} : T_X \to T_X$ and $\{D_1, D_2\} = p[D_1, D_2]$ for any local sections $D_i, i = 1, 2$ of T_X, where $[\cdot,\cdot]$ is the usual Lie bracket for the tangent sheaf. Let $\mathcal{D}_X^{(-1)}$ denote the sheaf of enveloping algebras of the Lie algebroid $(T_X, \alpha, \{\cdot,\cdot\})$. Thus $\mathcal{D}_X^{(-1)}$ is generated by the algebra of functions $\mathcal{O}_X$ and the $\mathcal{O}_X$-module of derivations T_X, subject to the module and commutator relations

$$f \cdot D = fD, \quad D \cdot f - f \cdot D = pD(f), \quad D \in T_X,\ f \in \mathcal{O}_X,$$

and the Lie algebroid relation

$$D_1 \cdot D_2 - D_2 \cdot D_1 = \{D_1, D_2\}, \quad D_1, D_2 \in T_X.$$

Next we introduce a sheaf of twisted differential operators on X.

Definition 4.2. With notation as above, let $U \subset X$ be an open affine subset over S_n. Define $\widetilde{\mathcal{D}}_X(U)$ to be the algebra generated over $\mathcal{D}_X^{(-1)}(U)$ by the symbols

$$\{\gamma_m(D_1, \ldots, D_{p-1+m}) \mid m \in \mathbb{N} \text{ and } D_1, \ldots, D_{p-1+m} \in T_X(U)\},$$

subject to the following six relations:

(1) $p^m \cdot \gamma_m(D_1, \ldots, D_{p-1+m}) = D_1 \cdots D_{p-1+m}$;
(2) $\gamma_m(D_1, \ldots, \alpha D_i + \alpha' D_i', \ldots D_{p-1+m}) = \alpha \cdot \gamma_m(D_1, \ldots, D_i, \ldots, D_{p-1+m})$ $+ \alpha' \cdot \gamma_m(D_1, \ldots, D_i', \ldots, D_{p-1+m}), \alpha, \alpha' \in W_n, D_i, D_i' \in T_X(U)$;
(3) for $m \geq 1$, $\gamma_m(D_1, \ldots, D_{p-1+m}) \cdot f = f \cdot \gamma_m(D_1, \ldots, D_{p-1+m})$ $+ \sum_{i=1}^{p-2+m} \gamma_{m-1}(D_1, \ldots, D_{i-1}, D_i(f)D_{i+1}, \ldots, D_{p-1+m})$ $+ \gamma_{m-1}(D_1, \ldots, D_{p-2+m}) \cdot D_{p-1+m}(f)$;
(4) $\gamma_m(D_1, \ldots, D_i, D_{i+1}, \ldots, D_{p-1+m}) = \gamma_m(D_1, \ldots, D_{i+1}, D_i, \ldots, D_{p-1+m})$ $+ \gamma_{m-1}(D_1, \ldots, D_{i-1}, [D_i, D_{i+1}], D_{i+2}, \ldots, D_{p-1+m})$;
(5) $\gamma_{m_1}(D_1, \ldots, D_{p-1+m_1}) \cdot \gamma_{m_2}(D_{p+m_1}, \ldots, D_{2p-2+m_1+m_2})$ $= p^{p-1}\gamma_{p-1+m_1+m_2}(D_1, \ldots, D_{2p-2+m_1+m_2})$;
(6) $\gamma_m(D_1, \ldots, D_{p-1+m}) \cdot D_{p+m} = D_1 \cdot \gamma_m(D_2, \ldots, D_{p+m})$ $= p \cdot \gamma_{m+1}(D_1, \ldots, D_{p-1+m}, D_{p+m})$.

The sheaf $\widetilde{\mathcal{D}}_X$ of twisted differential operators is associated to the presheaf $U \mapsto \widetilde{\mathcal{D}}_X(U)$.

The above definition is to justify the p-powers appearing in the denominators of differential operators. In particular, one may regard $\gamma_m(D_1, \ldots, D_{p-1+m})$ as a symbol for $\frac{D_1 \cdots D_{p-1+m}}{p^m}$. Since $\widetilde{\mathcal{D}}_X$ contains $\mathcal{O}_X$ as a subsheaf of algebras, it has a natural left $\mathcal{O}_X$-module structure. Thus it contains T_X as a left $\mathcal{O}_X$-submodule.

Definition 4.3. A *twisted connection* on an $\mathcal{O}_X$-module H is a W_n-morphism

$$\tilde{\nabla} : \widetilde{\mathcal{D}}_X \to \mathrm{End}_{W_n}(H)$$

between sheaves of W_n-algebras extending the structural morphism $\mathcal{O}_X \to \mathrm{End}_{W_n}(H)$.

A coherent $\mathcal{O}_X$-module equipped with a twisted connection is said to a *twisted flat module*. Let $\widetilde{MIC}(X)$ be the category of twisted flat modules over X. An explanation of the relation to the notion of an $\mathcal{O}_X$-module with integrable p-connection is in order. We cite the following definition from Shiho [30]:

Definition 4.4 ([30, Definitions 1.1–1.2,1.5]). With notation as above, a *p-connection* ∇ on an $\mathcal{O}_X$-module H is a W_n-linear map $\nabla : H \to H \otimes \Omega_X$ such that

$$\nabla(fh) = p \cdot df \otimes h + f\nabla(h), \qquad f \in \mathcal{O}_X, h \in H.$$

It is said to be *integrable* if $\nabla_1 \circ \nabla = 0$, where

$$\nabla_1 : H \otimes \Omega_X \to H \otimes \Omega_X^2$$

is the W_n-linear map defined via the formula

$$\nabla_1(h \otimes \omega) = \nabla(h) \wedge \omega + ph \otimes d\omega.$$

Moreover, an integrable p-connection (H, ∇) is *quasi-nilpotent* if locally $\theta^a = 0$ once $|a| \geq N$ for some natural number N where θ is the connection one-form $\nabla(e) = \sum_i \theta_i(e)dt_i$ with respect to étale local coordinates $\{t_1, \dots, t_d\}$ of X and $\theta^a = \prod_i \theta_i^{a_i}$ for a multi-index $a = (a_1, \dots, a_d) \in \mathbb{N}^d$ and $|a| = \sum_i a_i$.

In the above definition, the natural number N depends on the local expression of a local section e. However, in [30, Lemma 1.6], an integrable p-connection ∇ on H being quasi-nilpotent is shown to be independent of the choices of local coordinates. Let us denote the category of $\mathcal{O}_X$-modules with p-connections by $MC^{(-1)}(X)$ and with integrable p-connections by $MIC^{(-1)}(X)$. We also need to introduce a level structure on the quasi-nilpotency of an integrable p-connection.

Definition 4.5. With notation as above, let (H, ∇) be a quasi-nilpotent $\mathcal{O}_X$-module with an integrable p-connection. It is said to be *quasi-nilpotent of level* $\leq m$ if

$$\nabla_{D_1} \circ \cdots \circ \nabla_{D_{m+1}} = 0 \qquad \text{for any } D_1, \dots D_{m+1} \in T_X.$$

Denote the category of quasi-nilpotent $\mathcal{O}_X$-modules with an integrable p-connection of level $\leq m$ by $MIC_m^{(-1)}(X)$.

We point out that it is rather subtle to express the level structure in terms of local coordinates. Now there is a functor from $\widetilde{MIC}(X)$ to $MIC_{p-2+n}^{(-1)}(X)$: given a twisted connection $\tilde{\nabla}$ on H, one considers its restriction to $T_X \subset \tilde{\mathcal{D}}_X$ which yields a W_n-linear morphism

$$\nabla := \tilde{\nabla}|_{T_X} : H \to H \otimes \Omega_X.$$

It follows directly from the definition that this is indeed an integrable p-connection and quasi-nilpotent of level $\leq p - 2 + n$. Notice that the integrability amounts to the relation

$$\nabla(D_1) \circ \nabla(D_2) - \nabla(D_2) \circ \nabla(D_1) = \nabla(\{D_1, D_2\}), \qquad D_1, D_2 \in T_X.$$

For $m \geq p$, we may write

$$\frac{\nabla(D_1)\circ\cdots\circ\nabla(D_m)}{m!} = \frac{p^{m+1-p}}{m!}\cdot\widetilde{\nabla}(\gamma_{m+1-p}(D_1,\ldots,D_m)),$$

and as the factor $\frac{p^{m+1-p}}{m!}$ converges to zero p-adically as $m \to \infty$, we have $\frac{\nabla(D_1)\circ\cdots\circ\nabla(D_m)}{m!} \to 0$ as $m \to \infty$. This convergence property is crucial in the construction of our second functor $\mathcal{F}_n$. As a side remark, we do not know whether the above functor from $\widetilde{MIC}(X)$ to $MIC^{(-1)}_{p-2+n}(X)$ is essentially surjective (but it is not faithful). Later, we shall come back to this point again.

Now we can proceed to the construction of our first functor $\mathcal{T}_n : \mathcal{H}(X'_n) \to \widetilde{MIC}(X'_n)$. We shall present two approaches. The first one, which is our original approach, is a method of "local lifting and global gluing" which may be more familiar to a reader whose background is in complex algebraic geometry. The second approach is based on a beautiful construction suggested by the referee who kindly allows us to reproduce his/her idea here. This direct approach is much more transparent.

First approach. Let X be a smooth scheme over S_n. Let $\bar{X} := X \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$ be its mod p^{n-1} reduction. Since our arguments rely heavily on local calculations using a basis of local sections of an $\mathcal{O}_X$-module, we will restrict ourselves in this approach to the full subcategory $\mathcal{H}_{\mathrm{lf}}(X) \subset \mathcal{H}(X)$ consisting of *locally free* objects. However, this restriction is mainly for simplicity in the local arguments. For a general coherent object, we may use a set of minimal generators with possible relations for a coherent object locally. Also, we need to introduce some other categories which will be used only in this approach. The category $\mathcal{H}^{\mathrm{ni}}(X)$ (resp. $\mathcal{H}^{\mathrm{ni}}_{\mathrm{lf}}(X)$) is a variant of $\mathcal{H}(X)$ (resp. $\mathcal{H}_{\mathrm{lf}}(X)$): its object is also a tuple $(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\psi})$; but integrability of θ or $\bar{\nabla}$ is not required. For brevity, an object in $\mathcal{H}(X)$ or $\mathcal{H}^{\mathrm{ni}}(X)$ is written as $(E, \bar{H})$. Second, let $MCF_{p-2}(X)$ be a category of filtered $\mathcal{O}_X$-modules equipped with (not necessarily integrable) connections: its object is a triple (H, ∇, Fil), where H is a locally free $\mathcal{O}_X$-module, ∇ is a W_n-linear connection on H, and Fil a finite exhaustive decreasing filtration of locally free $\mathcal{O}_X$-submodules on H of level $\leq p-2$ which is locally split and satisfies Griffiths' transversality. There is a diagram of these categories connected by natural functors:

$$\begin{array}{ccccc}
 & MCF_{p-2}(X) & & & \\
 & \downarrow{\scriptstyle R_n} & \searrow^{G_n} & & \\
\mathcal{H}(X) \hookrightarrow & \mathcal{H}^{\mathrm{ni}}(X) & & MC^{(-1)}(X) & \hookleftarrow MIC^{(-1)}_{p-3+n}(X)
\end{array}$$

The functor R_n is the obvious one: to (H, ∇, Fil), one associates the graded Higgs module $(E, \theta) = \mathrm{Gr}_{Fil}(H, \nabla)$, which is locally free by the assumption on the filtration, and also $(\bar{H}, \bar{\nabla}, \overline{Fil})$, its mod p^{n-1} reduction. One notices that there is a natural isomorphism of graded Higgs modules

$$\bar{\psi} : \mathrm{Gr}_{\overline{Fil}}(\bar{H}, \bar{\nabla}) \cong \mathrm{Gr}_{Fil}(H, \nabla) \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}.$$

The functor G_n is a variant of a construction due to Faltings [4, Ch. II], which originates from [7] in the case $X = S_n$. Given an object $(H, \nabla, Fil) \in MCF_{p-2}(X)$, the object $(\tilde{H}, \tilde{\nabla}) = G_n(H, \nabla, Fil)$ is defined as follows: $\tilde{H}$ is the cokernel of the first map of the exact sequence

$$\bigoplus_i Fil^i \xrightarrow{[-1]-p\cdot \mathrm{Id}} \bigoplus_i Fil^i \xrightarrow{\rho} \tilde{H} \to 0, \tag{4.5.1}$$

where $[-1] := \bigoplus_i (Fil^i \hookrightarrow Fil^{i-1})$, Id denotes the identity map, and ρ is the natural projection map. Here we have extended the filtration so that

$$Fil^i = Fil^0,\ i \leq -1, \qquad Fil^j = 0,\ j \geq p.$$

Consider the W_n-linear map

$$\nabla' := \bigoplus_i \nabla|_{Fil^i}, \qquad \nabla|_{Fil^i} : Fil^i \to Fil^{i-1} \otimes \Omega_X.$$

The image $([-1]-p\cdot\mathrm{Id})(\bigoplus_i Fil^i) \subset \bigoplus_i \nabla|_{Fil^i}$ being preserved, ∇' induces a W_n-linear map

$$\tilde{\nabla} : \tilde{H} \to \tilde{H} \otimes \Omega_X.$$

One checks immediately that $\tilde{\nabla}$ is indeed a p-connection.

First, we show that the functor R_n is locally essentially surjective.

Lemma 4.6. *Let X be a smooth affine scheme over S_n. Let $(E, \bar{H})$ be an object in $\mathcal{H}^{\mathrm{ni}}_{\mathrm{lf}}(X)$. Suppose each component of $E = \bigoplus_k E_k$ is a free $\mathcal{O}_X$-module. Then there exists an object $(H, \nabla, Fil) \in MCF_{p-2}(X)$ such that $R_n(H, \nabla, Fil) = (E, \bar{H})$.*

Proof. Write $E = \bigoplus_{k=0}^{w} E^{w-k,k}$, $\theta = \bigoplus_{k=0}^{w-1} \theta_w$, and for $0 \leq k \leq w$ take a basis $\{e_k\}$ for the $\mathcal{O}_X$-module $E^{w-k,k}$. Relative to this basis, θ_k is expressed in terms of a matrix of differential one-forms which we denote again by θ_k. Put the bar over θ_k as well as $\{e_k\}$ to mean their mod p^{n-1} reduction. Then take the basis $\{f'_k\}_{0\leq k\leq w}$ of $\mathrm{Gr}_{\overline{Fil}}(\bar{H})$ such that

$$\bar{\psi}(f'_k) = \bar{e}_k.$$

Also choose a basis $\{\bar{f}_k\}_{0\leq k\leq w}$ of $\bar{H}$ whose image in $\mathrm{Gr}_{\overline{Fil}}(\bar{H})$ is $\{f'_k\}_{0\leq k\leq w}$. By Griffiths' transversality, the connection matrix $(\bar{a}_{ij})$ representing $\bar{\nabla}$ in the basis $\{\bar{f}_k\}$, i.e.,

$$\bar{\nabla}(\bar{f}_i) = \sum_j \bar{a}_{ij} \bar{f}_j,$$

has the property $\bar{a}_{ij} = 0,\ j > i+1$. Now we take a matrix (a_{ij}) of differential one-forms over X as follows: for $i \geq j$, take any lift a_{ij} of $\bar{a}_{ij}$; for $j = i+1$, take $a_{ij} = \theta_i$; for $j > i+1$, take $a_{ij} = 0$. Now let H be the free $\mathcal{O}_X$-module generated by elements $\{f_k\}_{0\leq k\leq w}$ whose mod p^{n-1} reductions are $\{\bar{f}_k\}_{0\leq k\leq w}$ and let Fil^{w-k} be the submodule freely generated by the elements $\{f_j\}_{0\leq j\leq k}$, which gives a filtration Fil on H. Then a connection ∇ on H is determined by the formula $\nabla(f_i) = \sum_j a_{ij} f_j$. The triple (H, ∇, Fil) so constructed is indeed an object in $MCF_{p-2}(X)$ lifting $(E, \bar{H})$, i.e., $R_n(H, \nabla, Fil) = (E, \bar{H})$ as required. □

Next we show the following

Lemma 4.7. *If $R_n(H, \nabla, Fil)$ lies in the full subcategory $\mathcal{H}_{\mathrm{lf}}(X) \subset \mathcal{H}^{\mathrm{ni}}_{\mathrm{lf}}(X)$, then $G_n(H, \nabla, Fil)$ lies in the full subcategory $MIC^{(-1)}_{p-3+n}(X)$.*

Proof. Set $(\tilde{H}, \tilde{\nabla}) := G_n(H, \nabla, Fil)$. Once the integrability of $\tilde{\nabla}$ is verified, the statement on quasi-nilpotency of level $\leq p-3+n$ follows directly from the definition. As integrability is a local property, we take étale local coordinates $\{t_1, \dots, t_d\}$ on X and a filtered basis $\{f_k\}_{0\leq k\leq w}$ of H. It suffices to check that for $1 \leq i, j \leq d$, $\tilde{\nabla}(\partial t_i)$ commutes with $\tilde{\nabla}(\partial t_j)$.

Let $\tilde{f}_k$ be the image of f_k (elements of Fil^k) in $\tilde{H}$. Then $\{\tilde{f}_k\}_{0\leq k\leq w}$ is a basis of $\tilde{H}$. As in Lemma 4.6, we may assume the matrix $A := (a_{ij})$ of ∇ in the basis $\{f_k\}_{0\leq k\leq w}$ has the property that $a_{ij} = 0$ for $j > i+1$. Then the matrix of $\tilde{\nabla}$ in the basis $\{\tilde{f}_k\}_{0\leq k\leq m}$ is $\tilde{A} = (\tilde{a}_{ij})$, with $\tilde{a}_{ij} = p^{i+1-j}a_{ij}$ for $j \leq i+1$, and the other entries zero. Then $\tilde{\nabla}(\partial t_i) \circ \tilde{\nabla}(\partial t_j)$ in the basis $\{\tilde{f}_k\}$ is represented by the matrix

$$p\partial t_i(\partial t_j \tilde{A}) + (\partial t_j \tilde{A}) \cdot (\partial t_i \tilde{A}).$$

It remains to show the following equality for all i, j:

$$p\partial t_i(\partial t_j \tilde{A}) + (\partial t_j \tilde{A}) \cdot (\partial t_i \tilde{A}) - p\partial t_j(\partial t_i \tilde{A}) - (\partial t_i \tilde{A}) \cdot (\partial t_j \tilde{A}) = 0.$$

For $0 \leq r \leq w-2$ with $t = r+1$ and $s = r+2$, the corresponding entry in the above equality means

$$(\partial t_i a_{rt})(\partial t_j a_{ts}) - (\partial t_j a_{rt})(\partial t_i a_{ts}) = 0,$$

which is actually equivalent to the integrability of the Higgs field θ; for $0 \leq r, s \leq w$ with $s \leq r+1$, this means that

$$p^{r+2-s}(\partial t_i \wedge \partial t_j)(da_{rs}) = p^{r+2-s}\sum_{t=0}^{w}\left[(\partial t_i a_{rt})(\partial t_j a_{ts}) - (\partial t_j a_{rt})(\partial t_i a_{ts})\right].$$

As $r+2-s \geq 1$, the above equation is implied by the following equation, which is equivalent to the integrability of $\bar{\nabla}$:

$$(\partial \bar{t}_i \wedge \partial \bar{t}_j)(d\bar{a}_{rs}) = \sum_{t=0}^{w}\left[(\partial \bar{t}_i \bar{a}_{rt})(\partial \bar{t}_j \bar{a}_{ts}) - (\partial \bar{t}_j \bar{a}_{rt})(\partial \bar{t}_i \bar{a}_{ts})\right].$$

Here $\{\bar{t}_1, \dots, \bar{t}_d\}$ are the induced étale local coordinates on $\bar{X}$. □

We shall take a step further to land the object $G_n(H, \nabla, Fil)$ in the last lemma in the category of twisted connections.

Lemma 4.8. *With notation as above, the integrable p-connection of $G_n(H, \nabla, Fil)$ can be extended to a twisted connection in a functorial way.*

Proof. If (H, ∇, Fil) is the mod p^n reduction of an object over W, then it is clear how to extend $\tilde{\nabla}$ to a twisted connection: for $\gamma_m(D_1, \dots, D_{p-1+m})$ with $D_i \in T_X$, one defines $\tilde{\nabla}(\gamma_m(D_1, \dots, D_{p-1+m}))$ to be $\frac{\tilde{\nabla}_{D_1} \circ \cdots \circ \tilde{\nabla}_{D_{p-1+m}}}{p^m}$, which is defined by first lifting to W, then dividing by p^m and finally reducing modulo p^n. The reason that $\tilde{\nabla}_{D_1} \circ \cdots \circ \tilde{\nabla}_{D_{p-1+m}}$ is divisible by p^m is as follows. By Griffiths' transversality,

$$\nabla'_{D_{m+1}} \circ \cdots \circ \nabla'_{D_{p-1+m}} : Fil^i \to Fil^{i+1-p}.$$

As $Fil^j = 0$ for $j \geq w+1$, and $w \leq p-2$, one sees that the image of $\nabla'_{D_{m+1}} \circ \cdots \circ \nabla'_{D_{p-1+m}}$ lies in Fil^i with $i \leq -1$. In the quotient $\tilde{H}$, for $i \leq -1$, the image of each element in Fil^{i-1} is equal to p times the same element in Fil^i. Thus, since ∇' always shifts the indices of direct factors of $\bigoplus_i Fil^i$ by minus one, the restriction of $\tilde{\nabla}$ to the image $\overline{Fil}^{-1}$ (which is isomorphic to H) of Fil^{-1} in $\tilde{H}$ is divisible by p. On the other hand, this also gives us a way to extend $\tilde{\nabla}$ to a twisted connection in the general case. Indeed, we simply define $\tilde{\nabla}(\gamma_m(D_1, \dots, D_{p-1+m}))$ to be the composite $\frac{\nabla'_{D_1}}{p} \circ \cdots \circ \frac{\nabla'_{D_m}}{p} \circ \nabla'_{D_{m+1}} \circ \cdots \circ \nabla'_{D_{p-1+m}}$, where $\frac{\nabla'_{D_i}}{p}$ is defined to be the map ∇_{D_i} on $\overline{Fil}^{-1} \subset \tilde{H}$. One verifies directly that this defines a twisted connection which extends $\tilde{\nabla}$. ◻

We believe the answer to the following questions is affirmative:

Question 4.9. Can $MIC^{(-1)}_{p-3+n}(X)$ be realized as a full subcategory of $\widetilde{MIC}(X)$?

We shall also provide the gluing morphisms.

Lemma 4.10. *Let* $(H_i, \nabla_i, Fil_i) \in MCF_{p-2}(X)$, $i = 1, 2, 3$. *Suppose there are isomorphisms for* $1 \leq i < j \leq 3$,

$$\bar{f}_{ij} : (\bar{H}_i, \bar{\nabla}_i, \overline{Fil}_i) \cong (\bar{H}_j, \bar{\nabla}_j, \overline{Fil}_j), \qquad f^G_{ij} : \mathrm{Gr}_{Fil_i}(H_i, \nabla_i) \cong \mathrm{Gr}_{Fil_j}(H_j, \nabla_j),$$

satisfying

$$\mathrm{Gr}(\bar{f}_{ij}) = f^G_{ij} \bmod p^{n-1}.$$

Then there are isomorphisms in the category $\widetilde{MIC}(X)$ *between the objects* $(\tilde{H}_i, \tilde{\nabla}_i) := G_n(H_i, \nabla_i, Fil_i)$,

$$\tilde{f}_{ij} : (\tilde{H}_i, \tilde{\nabla}_i) \cong (\tilde{H}_j, \tilde{\nabla}_j).$$

Moreover, if the cocycle conditions

$$\bar{f}_{13} = \bar{f}_{23} \circ \bar{f}_{12}, \qquad f^G_{13} = f^G_{23} \circ f^G_{12}$$

are satisfied then also

$$\tilde{f}_{13} = \tilde{f}_{23} \circ \tilde{f}_{12}.$$

Proof. For an $s \in Fil_i^k \backslash Fil_i^{k+1}$, we denote by $\tilde{s}$ (resp. $\hat{s}, \bar{s}$) its image under the natural map $Fil_i^k \to \tilde{H}_i$ (resp. $Fil_i^k \to Fil_i^k/Fil_i^{k+1}$, $H_i \to \bar{H}_i$). Consider sections of Fil_j^k whose images under the map $H_j \to \bar{H}_j$ are equal to $\bar{f}_{ij}(\bar{s})$ and under the map $Fil_j^k \to Fil_j^k/Fil_j^{k+1}$ equal to $f_{ij}^G(\hat{s})$ at the same time. As the difference of any two such sections lies in $p^{n-1}Fil_j^{k+1}$, they give rise to a unique section $\tilde{f}_{ij}(\tilde{s}) \in \tilde{H}_j$. So we define $\tilde{f}_{ij} : \tilde{H}_i \to \tilde{H}_j$ by sending $\tilde{s}$ to $\tilde{f}_{ij}(\tilde{s})$. It is straightforward to verify the well-definedness of $\tilde{f}_{ij}$ as well as the compatibility with twisted connections. The cocycle condition for $\{\tilde{f}_{ij}\}$ follows directly from the definition. □

Now we can give the construction of the first functor. Notice that it does not require X to be W_{n+1}-liftable.

Proposition 4.11. *Let X be a smooth scheme over S_n. Then there is a functor $\mathcal{T}_n$ from $\mathcal{H}_{\mathrm{lf}}(X)$ to $\widetilde{MIC}(X)$.*

Proof. Let $(E, \bar{H})$ be an object in $\mathcal{H}_{\mathrm{lf}}(X)$. Take an open affine covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that E and $\bar{H}$ are free modules over each U_i. By Lemma 4.6, we can take an object $(H_i, \nabla_i, Fil_i) \in MCF_{p-2}(U_i)$ for $U_i \in \mathcal{U}$ such that $R_n(H_i, \nabla_i, Fil_i) = (E, \bar{H})|_{U_i}$ (we call (H_i, ∇_i, Fil_i) a local lifting). Since over $U_i \cap U_j$, the restrictions of (H_i, ∇_i, Fil_i) and (H_j, ∇_j, Fil_j) are both local liftings of $(E, \bar{H})|_{U_i \cap U_j}$, by Lemma 4.10 there are isomorphisms

$$\tilde{f}_{ij} : G_n(H_i, \nabla_i, Fil_i)|_{U_i \cap U_j} \cong G_n(H_j, \nabla_j, Fil_j)|_{U_i \cap U_j}$$

satisfying the cocycle condition on $U_i \cap U_j \cap U_k$. Thus gluing the local objects $\{G_n(H_i, \nabla_i, Fil_i)\}_{U_i \in \mathcal{U}}$ in $\widetilde{MIC}(U_i)$ via the isomorphisms $\{\tilde{f}_{ij}\}$ yields an object in $\widetilde{MIC}(X)$. We denote the object so constructed by $(\tilde{H}, \tilde{\nabla})_{\mathcal{U}, \mathcal{L}_{\mathcal{U}}}$, where $\mathcal{U}$ denotes an open affine covering of X and $\mathcal{L}_{\mathcal{U}}$ consists of a local lifting of $(E, \bar{H})$ restricted to each $U \in \mathcal{U}$. We need to show that this object is independent of the choice of local liftings and the choice of open affine coverings up to canonical isomorphism. For two sets of local liftings $\mathcal{L}_{\mathcal{U}}^1 = \{(H_i^1, \nabla_i^1, Fil_i^1)\}_{i \in I}$ and $\mathcal{L}_{\mathcal{U}}^2 = \{(H_i^2, \nabla_i^2, Fil_i^2)\}_{i \in I}$, Lemma 4.10 provides an isomorphism over each $U_i \in \mathcal{U}$:

$$\mu_i : G_n(H_i^1, \nabla_i^1, Fil_i^{*1}) \cong G_n(H_i^2, \nabla_i^2, Fil_i^{*2})$$

and over $U_i \cap U_j$, the equality $\tilde{f}_{ij}^2 \circ \mu_i = \mu_j \circ \tilde{f}_{ij}^1$ holds. So the local isomorphisms $\{\mu_i\}_{i \in I}$ glue together to a global isomorphism

$$\mu : (\tilde{H}, \tilde{\nabla})_{\mathcal{U}, \mathcal{L}_{\mathcal{U}}} \cong (\tilde{H}, \tilde{\nabla})_{\mathcal{U}, \mathcal{L}'_{\mathcal{U}}}.$$

One can further show that for any three choices of local liftings, the resulting isomorphisms satisfy the cocycle relation. So the object $(\tilde{H}, \tilde{\nabla})_{\mathcal{U}, \mathcal{L}_{\mathcal{U}}}$ is independent of the choice of local liftings up to canonical isomorphism and therefore can be denoted by $(\tilde{H}, \tilde{\nabla})_{\mathcal{U}}$. Next for any two open affine coverings $\mathcal{U}, \mathcal{U}'$ of X, we find a common refinement $\mathcal{U}''$ of both. Again, by Lemma 4.10, one constructs an isomorphism

$$\nu_{\mathcal{U}, \mathcal{U}''} : (\tilde{H}, \tilde{\nabla})_{\mathcal{U}} \cong (\tilde{H}, \tilde{\nabla})_{\mathcal{U}''}$$

and similarly $\nu_{\mathcal{U}',\mathcal{U}''}$. Then one defines the isomorphism

$$\nu_{\mathcal{U},\mathcal{U}'} = \nu^{-1}_{\mathcal{U}',\mathcal{U}''} \circ \nu_{\mathcal{U},\mathcal{U}''} : (\tilde{H}, \tilde{\nabla})_{\mathcal{U}} \cong (\tilde{H}, \tilde{\nabla})_{\mathcal{U}'}$$

which also satisfies the cocycle relation for any three choices of open affine coverings (another way to remove this independence is to take the direct limit with respect to the directed set of all open affine coverings with the partial order given by refinement). Thus, we get our functor $\mathcal{T}_n$ by associating $(\tilde{H}, \tilde{\nabla})$ to $(E, \bar{H})$ as above which is defined up to canonical isomorphism. □

Second approach. This approach is due to the referee, who showed us a global construction of the functor from the whole category $\mathcal{H}(X)$ to $MIC^{(-1)}(X)$ in the weight one case. We shall generalize his/her method in the following.

With notation as above, for an object $(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\psi}) \in \mathcal{H}(X)$, the isomorphism of graded Higgs modules $\bar{\psi} : \mathrm{Gr}_{\overline{Fil}}(\bar{H}, \bar{\nabla}) \cong (\bar{E}, \bar{\theta})$ allows us to define the following composite morphism:

$$j : \overline{Fil}^i \to \overline{Fil}^i / \overline{Fil}^{i+1} \overset{\bar{\psi}}{\cong} \bar{E}^i.$$

Let $\eta : E^i \to \bar{E}^i$ be the natural projection by mod p^{n-1} reduction. Let $\overline{Fil}^i \times_{\bar{E}^i} E^i$ denote the kernel of the morphism of $\mathcal{O}_X$-modules

$$\overline{Fil}^i \oplus E^i \xrightarrow{-j+\eta} \bar{E}^i.$$

We are going to associate to $(E, \bar{H})$ a twisted flat module $(H^\sharp, \nabla^\sharp)$. The $\mathcal{O}_X$-module $H^\sharp$ is defined to be the cokernel of the morphism

$$\bigoplus_i (\overline{Fil}^i \times_{\bar{E}^i} E^i) \xrightarrow{\epsilon} \bigoplus_i (\overline{Fil}^i \times_{\bar{E}^i} E^i),$$

where for $x \times y \in \overline{Fil}^{i+1} \times_{\bar{E}^{i+1}} E^{i+1}$,

$$\epsilon(x \times y) = (x \times 0) + (-px \times -py) \in (\overline{Fil}^i \times_{\bar{E}^i} E^i) \oplus (\overline{Fil}^{i+1} \times_{\bar{E}^{i+1}} E^{i+1}).$$

Let $c : \bigoplus_i (\overline{Fil}^i \times_{\bar{E}^i} E^i) \to H^\sharp$ be the natural morphism. We define a W_n-linear additive map (not a connection) as follows:

$$\nabla'' : \bigoplus_i (\overline{Fil}^i \times_{\bar{E}^i} E^i) \to \bigoplus_i (\overline{Fil}^{i-1} \times_{\bar{E}^{i-1}} E^{i-1}) \otimes \Omega_X, \tag{4.11.1}$$

which takes $x \times y \in \overline{Fil}^i \times_{\bar{E}^i} E^i$ to $\bar{\nabla}(x) \times \theta(y) \in (\overline{Fil}^{i-1} \times_{\bar{E}^{i-1}} E^{i-1}) \otimes \Omega_X$. Here we have used the natural identification

$$(\overline{Fil}^{i-1} \otimes \Omega_{\bar{X}}) \times_{\bar{E}^{i-1} \otimes \Omega_{\bar{X}}} (E^{i-1} \otimes \Omega_X) \cong (\overline{Fil}^{i-1} \times_{\bar{E}^{i-1}} E^{i-1}) \otimes \Omega_X.$$

We shall show that this map descends to a p-connection over $H^\sharp$.

Lemma 4.12. *Let A_i be the image of $\overline{Fil}^i \times_{\bar{E}^i} E^i$ under the map ϵ. Then*

$$\nabla''(A_i) \subset A_{i-1} \otimes \Omega_X$$

and the induced W_n-linear additive map $\nabla^\sharp : H^\sharp \to H^\sharp \otimes \Omega_X$ is an integrable p-connection.

Proof. For $x \times y \in \overline{Fil}^i \times_{\bar{E}^i} E^i$, one computes

$$\nabla''(\epsilon(x \times y)) = \bar{\nabla}(x) \times 0 + \bar{\nabla}(-px) \times \theta(-py) = \epsilon(\bar{\nabla}(x) \times \theta(y)),$$

which lies in $A_{i-1} \otimes \Omega_X$. Therefore, ∇'' induces an additive W_n-linear map $\nabla^\sharp$ on $H^\sharp$. It suffices to show $\nabla^\sharp$ is a p-connection, as integrability follows directly from the integrability of $\bar{\nabla}$ and θ. For $f \in \mathcal{O}_X$ and $x \times y \in \overline{Fil}^i \times_{\bar{E}^i} E^i$, it suffices to prove

$$\nabla^\sharp(fc(x \times y)) = f\nabla^\sharp(c(x \times y)) + pc(x \times y) \otimes df.$$

Because

$$\begin{aligned}\nabla''(f(x \times y)) &= \nabla''(\bar{f} \cdot x \times f \cdot y) = \bar{\nabla}(\bar{f} \cdot x) \times \theta(fy)\\ &= (d\bar{f} \cdot x + \bar{f}\bar{\nabla}(x)) \times f\theta(y) = d\bar{f} \cdot x \times 0 + \bar{f}\bar{\nabla}(x) \times f\theta(y)\\ &= (x \times 0) \otimes df + f\nabla''(x \times y),\end{aligned}$$

where $\bar{f} \in \mathcal{O}_{\bar{X}}$ is the mod p^{n-1} reduction of f, and because $c(x \times 0) = c(p(x \times y))$, the required equality follows. □

Lemma 4.13. *With notation as above, the p-connection $\nabla^\sharp$ extends to a twisted connection.*

Proof. This step is similar to Lemma 4.8. Let $D_1, \ldots, D_{p-1+m} \in T_X$ for $m \geq 0$. Note that

$$\nabla''_{D_{m+1}} \circ \cdots \circ \nabla''_{D_{p-1+m}} : \overline{Fil}^i \times_{\bar{E}^i} E^i \to \overline{Fil}^{i+1-p} \times_{\bar{E}^{i+1-p}} E^{i+1-p},$$

and as $\overline{Fil}^i = 0$ and $E^i = 0$ for $i \geq w+1$ and $w \leq p-2$, the image of $\nabla''_{D_{m+1}} \circ \cdots \circ \nabla''_{D_{p-1+m}}$ lies in $\overline{Fil}^i \times_{\bar{E}^i} E^i$ for $i \leq -1$. As $\overline{Fil}^i = \bar{H}$ and $\bar{E}^i = E^i = 0$ for $i \leq -1$, one defines the map

$$\frac{\nabla''}{p} := \bar{\nabla} \times 0 : \overline{Fil}^i \times_{\bar{E}^i} E^i \to \overline{Fil}^i \times_{\bar{E}^i} E^i \otimes \Omega_X, \qquad i \leq -1.$$

Then the map

$$\nabla^\sharp(\gamma_m(D_1, \ldots, D_{p-1+m})) := \frac{\nabla''_{D_1}}{p} \circ \cdots \circ \frac{\nabla''_{D_m}}{p} \circ \nabla''_{D_{m+1}} \circ \cdots \circ \nabla''_{D_{p-1+m}}$$

gives rise to a twisted connection extending the p-connection $\nabla^\sharp$. □

Equivalence

Proposition 4.14. *The functor $\mathcal{T}_n^\sharp$, restricted to the subcategory $\mathcal{H}_{\mathrm{lf}}(X)$, is naturally equivalent to $\mathcal{T}_n$.*

Proof. Given an object $(E, \bar{H}) \in \mathcal{H}_{\mathrm{lf}}(X)$, we set

$$(\tilde{H}, \tilde{\nabla}) := \mathcal{T}_n(E, \bar{H}), \qquad (H^{\sharp}, \nabla^{\sharp}) := \mathcal{T}_n^{\sharp}(E, \bar{H}),$$

defined as above. We are going to exhibit a natural isomorphism

$$\lambda : (H^{\sharp}, \nabla^{\sharp}) \cong (\tilde{H}, \tilde{\nabla}).$$

Without loss of generality, we may assume that there is an object $(H, \nabla, Fil) \in MCF_{p-2}$ such that $R_n(H, \nabla, Fil) = (E, \bar{H})$ and $G_n(H, \nabla, Fil) = (\tilde{H}, \widetilde{\nabla})$. Consider the commutative diagram

$$\begin{array}{ccccc} & & \bigoplus_i Fil^i/pFil^i & & \\ & \swarrow^{p^{n-1}} & \downarrow^{p^{n-1}\cdot[-1]} & \searrow^{0} & \\ \bigoplus_i Fil^i & \xrightarrow{[-1]-p\cdot \mathrm{Id}} & \bigoplus_i Fil^i & \xrightarrow{\rho} & \tilde{H} \\ \downarrow^{\eta\times j} & \searrow^{\kappa} & \downarrow^{\eta\times j} & \nearrow_{\bar{c}} & \\ \bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i & \xrightarrow[{[-1]\times 0-p\cdot \mathrm{Id}}]{} & \bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i & & \end{array} \tag{4.14.1}$$

The middle row is (4.5.1), the exact sequence defining $\tilde{H}$. The middle column is also an exact sequence because of the local freeness assumption, where η denotes mod p^{n-1} reduction and j the natural projection $Fil^i \to Fil^i/Fil^{i+1} = E^i$. By the commutativity of the upper left triangle, ρ factors through the surjective map $\bar{c}$, so that there is an exact sequence

$$\bigoplus_i Fil^i \xrightarrow{\kappa} \bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i \xrightarrow{\bar{c}} \tilde{H} \to 0.$$

By the commutativity of the lower left triangle and the surjectivity of the left vertical map $\eta \times j$, the above exact sequence yields the exact sequence

$$\bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i \xrightarrow{[-1]\times 0-p\cdot \mathrm{Id}} \bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i \xrightarrow{\bar{c}} \tilde{H} \to 0.$$

Thus there is a unique isomorphism $\lambda : H^{\sharp} \to \tilde{H}$ rendering the following diagram commutative:

$$\begin{array}{ccccc} \bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i & \xrightarrow{[-1]\times 0-p\cdot \mathrm{Id}} & \bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i & \xrightarrow{\bar{c}} & \tilde{H} \\ & & & \searrow^{c} & \uparrow \wr\, \lambda \\ & & & & H^{\sharp} \end{array}$$

The isomorphism λ is compatible with twisted connections. Indeed, the twisted connection $\tilde{\nabla}$ on $\tilde{H}$ is induced by the maps

$$\nabla' : Fil^i \to Fil^{i-1} \otimes \Omega_{X_n}, \quad i \geq 0,$$
$$\frac{\nabla'}{p} : Fil^j \to Fil^j \otimes \Omega_{X_n}, \quad j \leq -1,$$

while $\nabla^\sharp$ on $H^\sharp$ by the formula (4.11.1) and $\frac{\nabla''}{p}$ in Lemma 4.13. Since the vertical map $\eta \times j$ in the middle column of diagram (4.14.1) is compatible with ∇' on $\bigoplus_i Fil^i$ and with ∇'' on $\bigoplus_i \overline{Fil}^i \times_{\bar{E}^i} E^i$, and also with $\frac{\nabla'}{p}$ on $\bigoplus_{j\leq -1} Fil^j$ and $\frac{\nabla''}{p}$ on $\bigoplus_{j\leq -1} \overline{Fil}^j \times_{\bar{E}^j} E^j$, it follows that λ is compatible with $\widetilde{\nabla}$ on $\tilde{H}$ and $\nabla^\sharp$ on $H^\sharp$. □

This completes the construction of the first functor from $\mathcal{H}(X'_n)$ to $\widetilde{MIC}(X'_n)$, by setting $X = X'_n$ in the above two approaches.

Next, we are going to construct the second functor $\mathcal{F}_n : \widetilde{MIC}(X'_n) \to MIC(X_n)$. In contrast to the first functor, it requires the W_{n+1}-liftability of X'_n and depends on the choice of W_{n+1}-liftings. Again, the method of "local lifting and global gluing" in the first approach of the former functor will be applied. Here "local lifting" means a local lifting of the relative Frobenius and the gluing morphisms are provided by the difference of two relative Frobenius liftings via the Taylor formula. We recall that A. Shiho [30] has obtained a functor from the category of quasi-nilpotent modules with integrable p-connections over X'_n to the category of flat modules over X_n under the assumptions both on the existence of W_{n+1}-liftings of X'_n, X_n and on the existence of the relative Frobenius lifting over W_{n+1}. His construction is closely related to ours in the local case, but they were independently obtained.

Proposition 4.15. *Let X be a smooth scheme over S_n and $X' = X \times_{F_{S_n}} S_n$. Assume X' admits a smooth lifting $\tilde{X}'$ over S_{n+1}. Then there is a functor $\mathcal{F}_n$ from the category $\widetilde{MIC}(X')$ to $MIC(X)$.*

Proof. We divide the whole construction into several small steps.

Step 0. For convenience, let $\tilde{X} = \tilde{X}' \times_{F^{-1}_{S_{n+1}}} S_{n+1}$, which lifts X and will be used in Step 2. Let $\tilde{\mathcal{U}}' = \{\tilde{U}'_i\}_{i\in I}$ be an open affine covering of $\tilde{X}'$ (assume each member is flat over S_{n+1}). Then by the obvious base change, it induces an open affine covering $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ of $\tilde{X}$, and by mod p^n reduction, open affine coverings $\mathcal{U}'$ (resp. $\mathcal{U}$) of X' (resp. X). For each $i \in I$, we take a morphism $\tilde{F}_i : \tilde{U}_i \to \tilde{U}'_i$ over S_{n+1} lifting the relative Frobenius $\tilde{U} \otimes \mathbb{Z}/p\mathbb{Z} \to \tilde{U}' \otimes \mathbb{Z}/p\mathbb{Z}$ over k and denote $\tilde{F}_i \otimes \mathbb{Z}/p^n\mathbb{Z}$ by F_i. Given a twisted flat module $(\tilde{H}, \tilde{\nabla})$ over X', its restriction to U'_i is denoted by $(\tilde{H}_i, \tilde{\nabla}_i)$.

Step 1. Each $\tilde{F}_i$ defines a morphism over S_n:

$$\frac{d\tilde{F}_i}{p} : F_i^*\Omega_{U'_i} \to \Omega_{U_i}.$$

Put $H_i := F_i^*\tilde{H}_i$. As discussed before (see the paragraph after Definition 4.5), one can naturally regard $\tilde{\nabla}_i$ as an integrable p-connection on $\tilde{H}_i$. By doing so, we can consider the following formula:

$$\nabla_i(f \otimes e) := df \otimes e + f \cdot \left(\frac{d\tilde{F}_i}{p} \otimes 1\right)(1 \otimes \tilde{\nabla}_i(e)), \qquad f \in \mathcal{O}_{U_i},\ e \in \tilde{H}_i. \quad (4.15.1)$$

Claim 4.16. *∇_i is well-defined and it defines an integrable connection on H_i.*

Proof. One computes that

$$\nabla_i(1\otimes fe)=\left(\frac{d\tilde{F}_i}{p}\otimes 1\right)(1\otimes\tilde{\nabla}_i(fe))=\left(\frac{d\tilde{F}_i}{p}\otimes 1\right)(1\otimes pdf\cdot e+1\otimes f\cdot\tilde{\nabla}_i(e))$$
$$=d(F_i^*f)\otimes e+F_i^*f\cdot\left(\frac{d\tilde{F}_i}{p}\otimes 1\right)(1\otimes\tilde{\nabla}_i(e))=\nabla_i(F_i^*f\otimes e),$$

which shows the well-definedness. Let $\{t_1,\dots,t_d\}$ be étale local coordinates on U_i and $\{t'_\alpha\}_{1\le\alpha\le d}$ the corresponding ones on U'_i. Then for $1\le j,k\le d$ and any $e\in\tilde{H}_i$,

$$\nabla_i(\partial t_j)(1\otimes e)=\sum_{\alpha=1}^d\left(\partial t_j\frac{d\tilde{F}_i}{p}(1\otimes dt'_\alpha)\right)\otimes\tilde{\nabla}(\partial t'_\alpha)(e).$$

So one computes

$$\begin{aligned}\nabla_i(\partial t_k)\circ\nabla_i(\partial t_j)(1\otimes e)&=\sum_{\alpha=1}^d\partial_{t_k}\left(\partial t_j\frac{d\tilde{F}_i}{p}(1\otimes dt'_\alpha)\right)\otimes\tilde{\nabla}(\partial t'_\alpha)(e)\\&\quad+\sum_{1\le\alpha,\beta\le d}\left(\partial t_j\frac{d\tilde{F}_i}{p}(1\otimes dt'_\alpha)\right)\cdot\left(\partial t_k\frac{d\tilde{F}_i}{p}(1\otimes dt'_\beta)\right)\otimes\tilde{\nabla}(\partial t'_\beta)\circ\tilde{\nabla}(\partial t'_\alpha)(e)\\&=\sum_{\alpha=1}^d\partial_{t_j}\left(\partial t_k\frac{d\tilde{F}_i}{p}(1\otimes dt'_\alpha)\right)\otimes\tilde{\nabla}(\partial t'_\alpha)(e)\\&\quad+\sum_{1\le\alpha,\beta\le d}\left(\partial t_j\frac{d\tilde{F}_i}{p}(1\otimes dt'_\alpha)\right)\cdot\left(\partial t_k\frac{d\tilde{F}_i}{p}(1\otimes dt'_\beta)\right)\otimes\tilde{\nabla}(\partial t'_\alpha)\circ\tilde{\nabla}(\partial t'_\beta)(e)\\&=\nabla_i(\partial t_j)\circ\nabla_i(\partial t_k)(1\otimes e).\end{aligned}$$

In the second equality above we have used the integrability of $\tilde{\nabla}_i$, i.e., $\tilde{\nabla}_i(\partial t'_\alpha)$ commutes with $\tilde{\nabla}_i(\partial t'_\beta)$. Then for any $f\in\mathcal{O}_{U_i}$ and any $e\in\tilde{H}_i$, one has

$$\begin{aligned}&\nabla_i(\partial t_k)\circ\nabla_i(\partial t_j)(f\otimes e)\\&=\frac{\partial^2 f}{\partial t_k\partial t_j}\otimes e+\frac{\partial f}{\partial t_j}\cdot\nabla_i(\partial t_k)(1\otimes e)+\frac{\partial f}{\partial t_k}\cdot\nabla_i(\partial t_j)(1\otimes e)+f\cdot\nabla_i(\partial t_k)\nabla_i(\partial t_j)(1\otimes e)\\&=\frac{\partial^2 f}{\partial t_j\partial t_k}\otimes e+\frac{\partial f}{\partial t_j}\cdot\nabla_i(\partial t_k)(1\otimes e)+\frac{\partial f}{\partial t_k}\cdot\nabla_i(\partial t_j)(1\otimes e)+f\cdot\nabla_i(\partial t_j)\nabla_i(\partial t_k)(1\otimes e)\\&=\nabla_i(\partial t_j)\circ\nabla_i(\partial t_k)(f\otimes e).\end{aligned}$$

So $\nabla_i(\partial t_k)$ commutes with $\nabla_i(\partial t_j)$ for $1\le k,j\le d$, that is, ∇_i is integrable as claimed. □

Step 2. The previous step provides a set $\{(H_i,\nabla_i)\}_{i\in I}$ of local flat modules and we want to glue them into one global flat module. The point is to use the Taylor formula involving the difference of two relative Frobenius liftings to construct the gluing morphisms. This is well-known if p is nonnilpotent in the base ring, and we use the formalism of a twisted connection to make sense of the p-powers in the denominators of the Taylor formula.

Let $\tilde{U}_i = \operatorname{Spec} \tilde{R}_i$, $i = 1, 2$, be two smooth schemes over S_{n+1}, equipped with a lifting of relative Frobenius $\tilde{F}_i : \tilde{U}_i \to \tilde{U}'_i$. Assume there are étale local coordinates $\{\tilde{t}_i\}$ for $\tilde{U}_1$ and suppose there is a morphism $\iota : \tilde{U}_2 \to \tilde{U}_1$ over S_{n+1} which induces the morphism $\iota' : \tilde{U}'_2 \to \tilde{U}'_1$. Let $R_i = \tilde{R}_i \otimes \mathbb{Z}/p^n\mathbb{Z}$, $U_i = \operatorname{Spec} R_i$ and $F_i = \tilde{F}_i \otimes \mathbb{Z}/p^n\mathbb{Z}$. Let $\{\tilde{t}'_i\}$ (resp. $\{t_i\}$) be the induced coordinate functions on $\tilde{U}'_1$ (resp. U_1). Given a twisted flat module $(\tilde{H}, \tilde{\nabla})$ over U'_1, there is an isomorphism of R_2-modules $G_{21} : F_2^*\iota'^*\tilde{H} \cong \iota^* F_1^* \tilde{H}$ given by the Taylor formula

$$G_{21}(e \otimes 1) = \sum_J \frac{\tilde{\nabla}(\partial)^J}{J!}(e) \otimes z^J, \qquad e \in \tilde{H}, \tag{4.16.1}$$

where $J := (j_1, \ldots, j_d)$ with each component $j_l \geq 0$, $J! := \prod_{l=1}^d j_l!$, and

$$\frac{\tilde{\nabla}(\partial)^J}{J!} := \frac{(\tilde{\nabla}(\partial t'_1))^{j_1}}{j_1!} \circ \cdots \circ \frac{(\tilde{\nabla}(\partial t'_d))^{j_d}}{j_d!}$$

with

$$z^J := \prod_{l=1}^d z_l^{j_l}, \qquad z_l := \frac{(\tilde{F}_2^*\iota'^*)(\tilde{t}'_l) - (\iota^*\tilde{F}_1^*)(\tilde{t}'_l)}{p}.$$

As already explained right after Definition 4.5, $\frac{\tilde{\nabla}(\partial)^J}{J!}$ converges to zero p-adically as $|J| \to \infty$, and the summation in (4.16.1) is actually finite. There is the cocycle relation between the isomorphisms for three objects: $G_{31} = G_{21} \circ G_{32}$. The proof is mostly formal: suppressing ιs in (4.16.1) and writing $\hat{z}$ (resp. $\tilde{z}$) for the z-function appearing in G_{32} (resp. G_{31}), as $\tilde{z} = z + \hat{z}$, one calculates

$$\begin{aligned} G_{21} \circ G_{32}(e \otimes 1) &= \sum_I \sum_J \frac{\tilde{\nabla}(\partial)^{I+J}}{I!J!}(e) \otimes z^J \cdot \hat{z}^I \\ &= \sum_K \frac{\tilde{\nabla}(\partial)^K}{K!}(e) \otimes \left(\sum_{I+J=K} \frac{K!}{I!J!} \cdot z^J \cdot \hat{z}^I \right) \\ &= \sum_K \frac{\tilde{\nabla}(\partial)^K}{K!}(e) \otimes (z + \hat{z})^K = \sum_K \frac{\tilde{\nabla}(\partial)^K}{K!}(e) \otimes \tilde{z}^K = G_{31}(e \otimes 1), \end{aligned}$$

and the cocycle condition follows. Returning to our case, we shall apply the Taylor formula in the following way: write $\tilde{U}_{ij} = \tilde{U}_i \cap \tilde{U}_j$ and take a relative Frobenius lifting $\tilde{F}_{ij} : \tilde{U}_{ij} \to \tilde{U}'_{ij}$. Let $\iota_1 : \tilde{U}_{ij} \to \tilde{U}_i$ and $\iota_2 : \tilde{U}_{ij} \to \tilde{U}_j$ be the natural inclusions. Then we obtain the isomorphisms α_i and α_j in the following diagram:

$$\begin{array}{ccccc} & & F_{ij}^*\tilde{H}|_{U'_{ij}} & & \\ & \overset{\alpha_i}{\underset{\cong}{\swarrow}} & & \overset{\alpha_j}{\underset{\cong}{\searrow}} & \\ H_i|_{U_{ij}} = F_i^*\tilde{H}|_{U'_{ij}} & & & & F_j^*\tilde{H}|_{U'_{ij}} = H_j|_{U_{ij}} \end{array}$$

Then we define $G_{ij} := \alpha_j \circ \alpha_i^{-1} : H_i|_{U_{ij}} \to H_j|_{U_{ij}}$ and the set of isomorphisms $\{G_{ij}\}$ satisfies the cocycle condition. There is one more property of G_{ij}, namely, it is compatible with connections. For $1 \le l \le d$, let us identify l with the multiple index $(0, \dots, 1, \dots, 0)$ with 1 at the l-th position. Then one calculates

$$(\mathrm{Id} \otimes G_{ij})(\nabla_i(e \otimes 1)) = \sum_J \sum_{l=1}^{d} \frac{\tilde{\nabla}(\partial)^{J+l}}{J!}(e) \otimes z^J \cdot \frac{d\tilde{F}_i}{p}(dt'_l),$$

and

$$\nabla_j(G_{ij}(e \otimes 1)) = \sum_J \sum_{l=1}^{d} \frac{\tilde{\nabla}(\partial)^{J+l}}{J!}(e) \otimes z^J \cdot \left(\frac{d\tilde{F}_j}{p}(dt'_l) + dz_l\right).$$

As

$$dz_l = \frac{d\tilde{F}_i}{p}(dt'_l) - \frac{d\tilde{F}_j}{p}(dt'_l),$$

it follows that

$$(\mathrm{Id} \otimes G_{ij})(\nabla_i(e \otimes 1)) = \nabla_j(G_{ij}(e \otimes 1)).$$

Thus, we use the set $\{G_{ij}\}$ to glue the local flat modules $\{(H_i, \nabla_i)\}$ together and obtain an object (H, ∇) in $MIC(X)$.

Step 3. We need to show that (H, ∇) is independent of the choice of relative Frobenius liftings and of open affine coverings. But, after our argument on a similar independence-type statement in Proposition 4.11, this step becomes entirely formal. Therefore, to summarize, we have constructed a flat module (H, ∇) over X up to canonical isomorphism from any twisted flat module $(\tilde{H}, \tilde{\nabla})$ over X', and this gives our second functor $\mathcal{F}_n$. □

Remark 4.17. The above can be simplified when a global lifting $F_{\tilde{X}}$ of the relative Frobenius over $\tilde{X}$ (a W_{n+1}-lifting of X) exists. Set $F_X = F_{\tilde{X}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. In this case, one has a globally defined morphism $\frac{dF_{\tilde{X}}}{p} : F_X^*\Omega_{X'} \to \Omega_X$. Then $H = F_X^*\tilde{H}$ and ∇ is defined by (4.15.1) with various local $\frac{d\tilde{F}_i}{p}$s replaced simply by $\frac{dF_{\tilde{X}}}{p}$. The reason is as follows: in Step 0, we may take an open affine covering $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ of $\tilde{X}$ (whose elements are flat over W_{n+1}) such that $F_{\tilde{X}} : \tilde{U}_i \to \tilde{U}'_i$ for all i and then take $\tilde{F}_i$ to be the restriction to $F_{\tilde{X}}$ to $\tilde{U}_i$. Then it follows that the gluing functions G_{ij} are all identity.

Proof of Theorem 4.1. We define $\mathcal{C}_n^{-1} : \mathcal{H}(X'_n) \to MIC(X_n)$ to be the composite of $\mathcal{T}_n : \mathcal{H}(X'_n) \to \widetilde{MIC}(X'_n)$ in Proposition 4.11 and $\mathcal{F}_n : \widetilde{MIC}(X'_n) \to MIC(X_n)$ in Proposition 4.15. From their very constructions, the two functors $\mathcal{T}_n$ and $\mathcal{F}_n$ are compatible with mod p^{n-1} reduction. So is their composite. The equivalence of the functor $\mathcal{C}_1^{-1}$ over $W_1 = k$ with the inverse Cartier transform of Ogus–Vologodsky [26] has been verified in [16]. □

5. Higgs correspondence in mixed characteristic

This section aims to establish the Higgs correspondence between the category of strict p^n-torsion Fontaine modules with endomorphism $W_n(\mathbb{F}_{p^f})$ (Variant 2, §2) and the category of periodic Higgs–de Rham flows over X_n, for $n \geq 2$, so that it lifts the one in positive characteristic and its limit, as $n \to \infty$, yields the Higgs correspondence in mixed characteristic.

We proceed first to the definition of a Higgs–de Rham flow over X_n. We use the notation from §4. Let $\pi_n : X'_n \to X_n$ be the natural morphism by base change, which induces the obvious equivalence of categories $\pi_n^* : \mathcal{H}(X_n) \cong \mathcal{H}(X'_n)$. We define the functor

$$C_n^{-1} := \mathcal{C}_n^{-1} \circ \pi_n^* : \mathcal{H}(X_n) \to MIC(X_n).$$

Definition 5.1. A *Higgs–de Rham flow* over X_n is a diagram of the form

$$\begin{array}{ccccccccc}
 & & & & (H_0,\nabla_0) & & & (H_1,\nabla_1) & \\
 & & & \nearrow^{C_n^{-1}} & & \searrow^{\mathrm{Gr}_{Fil_0}} & \nearrow^{C_n^{-1}} & & \searrow^{\mathrm{Gr}_{Fil_1}} \\
(\bar{H}_{-1},\bar{\nabla}_{-1}) & & & (E_0,\theta_0) & & & (E_1,\theta_1) & & \ldots \\
 & \searrow^{\mathrm{Gr}_{\overline{Fil}_{-1}}} & & \downarrow & & & & & \\
 & & (\bar{E}_{-1},\bar{\theta}_{-1}) \xrightarrow[\cong]{\bar{\phi}} & (\bar{E}_0,\bar{\theta}_0) & & & & &
\end{array}$$

Here in the upper line, $(H_i,\nabla_i) := C_n^{-1}(E_i,\theta_i)$ and Fil_i is a finite exhaustive decreasing filtration of $\mathcal{O}_{X_n}$-submodules on H_i satisfying Griffiths' transversality with respect to ∇_i; in the middle line each term (E_i,θ_i) is a graded Higgs module over X_n and $(\bar{H}_{-1},\bar{\nabla}_{-1},\overline{Fil}_{-1})$ is a de Rham module over X_{n-1}; in the bottom line, $\bar{\phi}$ is an isomorphism of graded Higgs modules from $(\bar{E}_{-1},\bar{\theta}_{-1}) = \mathrm{Gr}_{\overline{Fil}_{-1}}(\bar{H}_{-1},\bar{\nabla}_{-1})$ to $(\bar{E}_0,\bar{\theta}_0)$, the mod p^{n-1} reduction of (E_0,θ_0).

We emphasize that, in the above definition, the filtrations Fil_i, $i \geq 0$, and the isomorphism $\bar{\phi}$ are part of the defining data of the flow. Also, an explanation of the various inverse Cartier transforms in the flow is in order: $C_n^{-1}(E_0,\theta_0)$ is an abbreviation for

$$C_n^{-1}(E_0,\theta_0,\bar{H}_{-1},\bar{\nabla}_{-1},\overline{Fil}_{-1},\bar{\phi}),$$

and $C_n^{-1}(E_i,\theta_i)$ for $i \geq 1$ is an abbreviation for

$$C_n^{-1}(E_i,\theta_i,(C_n^{-1}(E_{i-1},\theta_{i-1}),Fil_{i-1}) \otimes \mathbb{Z}/p^{n-1}\mathbb{Z},\mathrm{Id}).$$

Comparing this with the notion of a periodic Higgs–de Rham flow over X_1 (Definition 3.1), one finds that the extra data in the lower left corner in Definition 5.1 puts an additional condition on (E_0,θ_0) when $n \geq 2$. This is caused by the construction of the functor C_n^{-1}, $n \geq 2$. It is of interest to characterize those graded Higgs modules satisfying this condition, but more importantly, to know whether this assumption on C_n^{-1}, $n \geq 2$, could be relaxed.

Next, we turn to the central notion of a periodic Higgs–de Rham flow over X_n which is defined in an inductive way. For a flat bundle (H, ∇) over $X_n, n \geq 1$, a finite exhaustive decreasing filtration on H is said to be a *Hodge filtration* if it consists of locally free subsheaves of H, is locally split, and obeys Griffiths' transversality with respect to ∇.

Definition 5.2. A *periodic Higgs–de Rham flow* over $X_n, n \geq 2$, of period $f \in \mathbb{N}$ consists of the following data:

(1) a periodic Higgs–de Rham flow $(\bar{E}, \bar{\theta}, \overline{Fil}_0, \ldots, \overline{Fil}_{f-1}, \bar{\phi}) \in HDF_{p-2,f}(X_n/W_n)$;
(2) a graded Higgs bundle $(E, \theta) \in HIG_{p-2}(X_n)$ lifting $(\bar{E}, \bar{\theta})$;
(3) a Hodge filtration Fil_i of level $\leq p-2$ on $C_n^{-1}(E_i, \theta_i)$ lifting the Hodge filtration $\overline{Fil}_i$ on $C_{n-1}^{-1}(\bar{E}_i, \bar{\theta}_i)$;
(4) an isomorphism of graded Higgs modules over X_n

$$\phi : \mathrm{Gr}_{Fil_{f-1}}(C_n^{-1}(E_{f-1}, \theta_{f-1})) \cong (E, \theta)$$

lifting $\bar{\phi}$.

In the above definition, $(E_0, \theta_0) := (E, \theta)$ and $(C_{n-1}^{-1}(\bar{E}_{f-1}, \bar{\theta}_{f-1}), \overline{Fil}_{f-1}, \bar{\phi})$ together make up an object in the category $\mathcal{H}(X_n)$ so that $C_n^{-1}(E_0, \theta_0)$ is naturally defined. Also, $C_n^{-1}(E_i, \theta_i)$ for $i \geq 1$ is naturally defined, which is simply

$$C_n^{-1}(E_i, \theta_i, C_{n-1}^{-1}(E_{i-1}, \theta_{i-1}), \overline{Fil}_{i-1}).$$

By Corollary 3.14, the filtrations on a periodic Higgs–de Rham flow over X_1 are indeed Hodge filtrations.

Thus the data of a periodic Higgs–de Rham flow of period f over X_n, $n \geq 2$, is encoded in the tuple

$$(E, \theta, Fil_0, \ldots, Fil_{f-1}, \phi, \bar{E}, \bar{\theta}, \overline{Fil}_0, \ldots, \overline{Fil}_{f-1}, \bar{\phi}).$$

A morphism between two tuples is given by a pair $(f, \bar{f})$, where $\bar{f}$ is a morphism in the category $HDF_{p-2,f}(X_n/W_n)$ and f is a morphism of graded Higgs modules over X_n lifting $\bar{f}$ and satisfying natural properties. Let us explain this in the one-periodic case in detail. Let $(E_i, \theta_i, Fil_i, \phi_i, \bar{E}_i, \bar{\theta}_i, \overline{Fil}_i, \bar{\phi}_i)$, $i = 1, 2$, be two one-periodic flows over X_n. Then a morphism

$$(f, \bar{f}) : (E_1, \theta_1, Fil_1, \phi_1, \bar{E}_1, \bar{\theta}_1, \overline{Fil}_1, \bar{\phi}_1) \to (E_2, \theta_2, Fil_2, \phi_2, \bar{E}_2, \bar{\theta}_2, \overline{Fil}_2, \bar{\phi}_2)$$

means the following: first,

$$\bar{f} : (\bar{E}_1, \bar{\theta}_1, \overline{Fil}_1, \bar{\phi}_1) \to (\bar{E}_2, \bar{\theta}_2, \overline{Fil}_2, \bar{\phi}_2)$$

is a morphism in $HDF_{p-2,1}(X_n/W_n)$; second, $f : (E_1, \theta_1) \to (E_1, \theta_2)$ is a morphism of graded Higgs modules over X_n lifting

$$\bar{f} : (\bar{E}_1, \bar{\theta}_1) \to (\bar{E}_2, \bar{\theta}_2);$$

third, the morphism

$$C_n^{-1}(f): C_n^{-1}(E_1, \theta_1) \to C_n^{-1}(E_2, \theta_2),$$

which is naturally defined by the previous two properties, is compatible with the Hodge filtrations and the induced morphism on graded Higgs modules is compatible with ϕs, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Gr}_{Fil_1} C_n^{-1}(E_1, \theta_1) & \xrightarrow{\phi_1} & (E_1, \theta_1) \\ {\scriptstyle \mathrm{Gr}\, C_n^{-1}(f)} \downarrow & & \downarrow {\scriptstyle f} \\ \mathrm{Gr}_{Fil_2} C_n^{-1}(E_2, \theta_2) & \xrightarrow{\phi_2} & (E_2, \theta_2) \end{array}$$

Thus, the category $HDF_{p-2,f}(X_{n+1}/W_{n+1})$ of periodic Higgs–de Rham flows of period f over X_n is indeed inductively defined. Also, relying upon the previous two definitions, it is straightforward to define a preperiodic Higgs–de Rham flow over X_n (the key is to ensure the well-definedness of the inverse Cartier transform of each Higgs term). Since this will not be used later, we leave this task to the reader. Recall that $MF^{\nabla}_{[0,p-2],f}(X_{n+1}/W_{n+1})$ is the category of strict p^n-torsion Fontaine modules with extra endomorphism $W_n(\mathbb{F}_{p^f})$. The following theorem lifts Theorem 3.2 in characteristic p (but notice the stronger restriction on the Hodge–Tate weight).

Theorem 5.3. *Let X_{n+1} be a smooth scheme over W_{n+1}. For $f \in \mathbb{N}$, there is an equivalence of categories between the category $MF^{\nabla}_{[0,p-2],f}(X_{n+1}/W_{n+1})$ and the category $HDF_{p-2,f}(X_{n+1}/W_{n+1})$.*

Similar to the characteristic p case, the theorem will be reduced to the one-periodic case. Let us introduce the category of one-periodic Higgs–de Rham flows over X_n with endomorphism structure $W_n(\mathbb{F}_{p^f})$. Its object is a tuple $(E, \theta, Fil, \phi, \iota, \bar{E}, \bar{\theta}, \overline{Fil}, \bar{\phi})$, where $(E, \theta, Fil, \phi, \bar{E}, \bar{\theta}, \overline{Fil}, \bar{\phi})$ is an object in the category $HDF_n := HDF_{p-2,f}(X_{n+1}/W_{n+1})$ and

$$\iota: W_n(\mathbb{F}_{p^f}) \hookrightarrow \mathrm{End}_{HDF_n}(E, \theta, Fil, \phi, \bar{E}, \bar{\theta}, \overline{Fil}, \bar{\phi})$$

is an embedding of $W_n(\mathbb{F}_p)$-algebras. A morphism of this category is a morphism of one-periodic Higgs–de Rham flows compatible with endomorphism structures. Thus Theorem 5.3 follows from the next two propositions.

Proposition 5.4. *There is an equivalence of categories between the category $MF^{\nabla}_{[0,p-2],1}(X_{n+1}/W_{n+1})$ and $HDF_{p-2,1}(X_{n+1}/W_{n+1})$.*

This proposition is just the one-periodic case of Theorem 5.3, which implies immediately that $MF^{\nabla}_{[0,p-2],f}(X_{n+1}/W_{n+1})$ is equivalent to the category of one-periodic Higgs–de Rham flows over X_n with endomorphism structure $W_n(\mathbb{F}_{p^f})$. We postpone its proof and first show

Proposition 5.5. *There is an equivalence of categories between the category of one-periodic Higgs–de Rham flows over X_n with endomorphism structure $W_n(\mathbb{F}_{p^f})$ and the category $HDF_{p-2,f}(X_{n+1}/W_{n+1})$.*

Proof. Recall that we have chosen a primitive element $\xi_1 \in \mathbb{F}_{p^f}$ in the proof of Lemma 3.8. Let ξ be the Teichmüller lift of ξ_1 in $W(\mathbb{F}_{p^f})$ and let ξ_n be the mod p^n reduction of ξ. Thus ξ_n is a generator of $W_n(\mathbb{F}_{p^f})$ as a $W_n(\mathbb{F}_p)$-algebra. The Frobenius automorphism of $W_n = W_n(k)$ acts on ξ_n by the power p map. We argue by induction on n. The $n = 1$ case is Proposition 3.7, where we have constructed the functor $\mathcal{E}$ and its quasi-inverse $\mathcal{D}$. Relabel them as $\mathcal{E}_1$ and $\mathcal{D}_1$. As the induction hypothesis, we assume that we have constructed sequences $\{\mathcal{E}_i\}_{1\leq i\leq n-1}$ and $\{\mathcal{D}_i\}_{1\leq i\leq n-1}$ of functors such that (i) $\mathcal{E}_i$ and $\mathcal{D}_i$ are quasi-inverse to each other and therefore give an equivalence of categories between the category of one-periodic Higgs–de Rham flows over X_i with endomorphism structure $W_i(\mathbb{F}_{p^f})$ and the category $HDF_{p-2,f}(X_i)$; (ii) $\mathcal{E}_i$ (resp. $\mathcal{D}_i$) lifts $\mathcal{E}_{i-1}$ (resp. $\mathcal{D}_{i-1}$).

Now we proceed to the case of n. Let us start with an object of $HDF_{p-2,f}(X_n)$:

$$(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi, \bar{E}, \bar{\theta}, \overline{Fil}_0, \dots, \overline{Fil}_{f-1}, \bar{\phi}).$$

Following the characteristic p case, we put

$$(G, \eta) := \bigoplus_{i=0}^{f-1}(E_i, \theta_i),$$

where $(E_0, \theta_0) = (E, \theta)$ and $(E_i, \theta_i) = C_n^{-1}(E_{i-1}, \theta_{i-1})$, $i \geq 1$, are the remaining Higgs terms defined inductively in the flow. The Hodge filtration Fil on $C_n^{-1}(G, \eta)$ as well as the isomorphism $\tilde{\phi}$ of graded Higgs modules over X_n are defined exactly in the same way as in the characteristic p case (see the paragraph before Lemma 3.8). Clearly, (G, θ) (resp. Fil and $\tilde{\phi}$) lifts $(\bar{G}, \bar{\theta})$ (resp. $\overline{Fil}$ and $\bar{\tilde{\phi}}$) and therefore $(G, \eta, Fil, \tilde{\phi}, \bar{G}, \bar{\theta}, \overline{Fil}, \bar{\tilde{\phi}})$ is a one-periodic flow over X_n. Replacing ξ_1 in the proof of Lemma 3.8 with ξ_n, one equips this one-periodic flow with an endomorphism structure $W_n(\mathbb{F}_{p^f})$. This gives us a functor $\mathcal{E}_n$. Conversely, to a one-periodic Higgs–de Rham flow $(G, \eta, Fil, \phi, \iota, \bar{G}, \bar{\eta}, \overline{Fil}, \bar{\phi})$ over X_n with endomorphism structure $W_n(\mathbb{F}_{p^f})$, we associate an f-periodic Higgs–de Rham flow over X_n as follows: first, the induction hypothesis gives us an f-periodic flow over X_{n-1} (by abuse of notation we have omitted the part over X_{n-1} in the following expression):

$$(\bar{E}, \bar{\theta}, \overline{\widetilde{Fil}}_0, \dots, \overline{\widetilde{Fil}}_{f-1}, \bar{\tilde{\phi}}).$$

Second, let $(G, \eta) = \bigoplus_{i=0}^{f-1}(G_i, \eta_i)$ be the eigen-decomposition under the endomorphism $\iota(\xi_n)$. Then $C_n^{-1}(G_i, \eta_i)$ is naturally defined and one has the eigen-decomposition under $C_n^{-1}(\xi_n)$:

$$(C_n^{-1}(G, \eta), Fil) = \bigoplus_{i=0}^{f-1}(C_n^{-1}(G_i, \eta_i), Fil_i).$$

So we put $(E, \theta) = (G_0, \eta_0)$, and closely following the constructions in the characteristic p case (see Lemma 3.9), one obtains the filtrations $\widetilde{Fil}_i$, $0 \leq i \leq f-1$, and the isomorphism $\tilde{\phi}$ of graded Higgs modules over X_n, so that the extended tuple

$$(E, \theta, \widetilde{Fil}_0, \dots, \widetilde{Fil}_{f-1}, \tilde{\phi}, \bar{E}, \bar{\theta}, \overline{\widetilde{Fil}}_0, \dots, \overline{\widetilde{Fil}}_{f-1}, \bar{\tilde{\phi}})$$

is an object in $HDF_{p-2,f}(X_n)$. This gives us the functor $\mathcal{D}_n$ in the reverse direction. Given the proof of Proposition 3.7, the proof that $\mathcal{E}_n$ and $\mathcal{D}_n$ give an equivalence of categories becomes completely formal and is therefore omitted. Finally, the lifting properties of $\mathcal{E}_n$ and $\mathcal{D}_n$ are direct consequences of our choice of ξ_n at the beginning. □

We now turn to the proof of Proposition 5.4. In the following, we choose and fix an open affine covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X_{n+1}/W_{n+1} whose elements are flat over W_{n+1}, and for each i, an absolute Frobenius lifting F_i over U_i. First a lemma:

Lemma 5.6. *Let (H, ∇, Fil, Φ) be a strict p^n-torsion Fontaine module. Let $(\bar{H}, \bar{\nabla}, \overline{Fil})$ be the mod p^{n-1} reduction of (H, ∇, Fil) and (E, θ) the associated graded Higgs bundle. Then the relative Frobenius Φ naturally induces an isomorphism of flat bundles over X_n:*

$$\tilde{\Phi} : C_n^{-1}(E, \theta) \cong (H, \nabla).$$

Proof. The lemma follows from the strong p-divisibility and horizontality of Φ (see §2) and the very construction of C_n^{-1}. For $n = 1$, this is [16, Proposition 1.4]. For simplicity, let us ignore the issue of the obvious base change caused by the Frobenius automorphism of the base ring W_n in the argument. By the first approach for the functor $\mathcal{T}_n$, it follows that

$$\mathcal{T}_n(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}) = G_n(H, \nabla, Fil).$$

Write it $(\tilde{H}, \tilde{\nabla})$. Let $\mathcal{U}_n = \{U_{i,n}\}_{i\in I}$ be the induced open affine covering of X_n, and let $F_{i,n}$ be the induced absolute Frobenius lifting over $U_{i,n}$. Then the evaluation $\Phi_i := \Phi_{(U_{i,n},F_{i,n})}$ of Φ is an isomorphism of local flat bundles:

$$\Phi_i : (F_{i,n}^* \tilde{H}|_{U_{i,n}}, F_{i,n}^*(\tilde{\nabla}|_{U_{i,n}})) \cong (H, \nabla)|_{U_{i,n}},$$

where the connection $F_i^*(\tilde{\nabla}|_{U_{i,n}})$ is defined via (4.15.1). The fact that different evaluations of Φ are related via the Taylor formula means

$$\Phi_i = \Phi_j \circ \varepsilon_{ij}, \tag{5.6.1}$$

where the ε_{ij} are given by (4.16.1) (replacing the indices 21 in that formula with ij). Thus, the very construction of C_n^{-1} means exactly that the local isomorphisms Φ_i glue together to a global one $\tilde{\Phi}$ from $C_n^{-1}(E, \theta)$ to (H, ∇). □

Proof of Proposition 5.4. We divide the whole proof into three steps, following the one in the characteristic p case. The proof is by induction on n, where the $n = 1$ case is Proposition 3.3. Rewrite the functors in the proof of Proposition 3.3 as $\mathcal{GR}_1 := \mathcal{GR}$ and $\mathcal{IC}_1 := \mathcal{IC}$. As the induction hypothesis, we assume the existence of functors $\mathcal{GR}_i : MF^{\nabla}_{[0,p-2],1}(X_{i+1}/W_{i+1}) \to HDF_{p-2,1}(X_{i+1}/W_{i+1})$ and $\mathcal{IC}_i$ in the opposite direction for $1 \le i \le n-1$ such that (i) $\mathcal{GR}_i$ and $\mathcal{IC}_i$ are quasi-inverse to each other, and (2) $\mathcal{GR}_i$ (resp. $\mathcal{IC}_i$) lifts $\mathcal{GR}_{i-1}$ (resp. $\mathcal{IC}_{i-1}$). In the following, we construct a lifting $\mathcal{GR}_n$ (resp. $\mathcal{IC}_n$) of $\mathcal{GR}_{n-1}$ (resp. $\mathcal{IC}_{n-1}$) such that $\mathcal{GR}_n$ and $\mathcal{IC}_n$ are quasi-inverse to each other.

From Fontaine module to one-periodic Higgs–de Rham flow: Let (H, ∇, Fil, Φ) be a strict p^n-torsion Fontaine module over X_{n+1}/W_{n+1}. Its mod p^{n-1} reduction $(\bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\Phi})$ is a strict p^{n-1}-torsion Fontaine module. Let

$$(\bar{E}, \bar{\theta}, \overline{Fil}_{\exp}, \bar{\phi}, \bar{\bar{E}}, \bar{\bar{\theta}}, \overline{\overline{Fil}}_{\exp}, \bar{\bar{\phi}})$$

be the corresponding one-periodic flow over X_{n-1} to $(\bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\Phi})$ via the functor $\mathcal{GR}_{n-1}$. Set $(E, \theta) = \mathrm{Gr}_{Fil}(H, \nabla)$. By Lemma 5.6, we define $Fil_{\exp}$ on $C_n^{-1}(E, \theta)$ to be the pull-back of Fil on H via the isomorphism $\tilde{\Phi}$. Thus, we also obtain an isomorphism of graded Higgs modules over X_n:

$$\phi := \mathrm{Gr}(\tilde{\Phi}) : \mathrm{Gr}_{Fil_{\exp}}(C_n^{-1}(E, \theta)) \cong \mathrm{Gr}_{Fil}(H, \nabla) = (E, \theta).$$

The lifting property of the inverse Cartier transform in Theorem 4.1 implies that the resulting tuple $(E, \theta, Fil_{\exp}, \phi)$ lifts $(\bar{E}, \bar{\theta}, \overline{Fil}_{\exp}, \bar{\phi})$, so that

$$\mathcal{GR}_n(H, \nabla, Fil, \Phi) := (E, \theta, Fil_{\exp}, \phi, \bar{E}, \bar{\theta}, \overline{Fil}_{\exp}, \bar{\phi})$$

is a one-periodic Higgs–de Rham flow over X_n. Clearly, the functor $\mathcal{GR}_n$ lifts $\mathcal{GR}_{n-1}$.

From one-periodic Higgs–de Rham flow to Fontaine module: From a given object $(E, \theta, Fil, \phi, \bar{E}, \bar{\theta}, \overline{Fil}, \bar{\phi}) \in HDF_{p-2,1}(X_n)$, one immediately derives the de Rham module

$$(H, \nabla, Fil) := (C_n^{-1}(E, \theta), Fil).$$

In order to complete it to a Fontaine module, it remains to put a relative Frobenius Φ on it. Let $(\bar{H}, \bar{\nabla})$ be the mod p^{n-1} reduction of (H, ∇) which is equal to $C_{n-1}^{-1}(\bar{E}, \bar{\theta})$. Set $(E_1, \theta_1) = \mathrm{Gr}_{Fil}(H, \nabla)$. Then we have two objects in the category $\mathcal{H}(X_n)$: $(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\phi})$ and $(E_1, \theta_1, \bar{H}, \bar{\nabla}, \overline{Fil}, \mathrm{Id})$. Then $\phi : (E_1, \theta_1) \cong (E, \theta)$ and the identity map on $(\bar{H}, \bar{\nabla}, \overline{Fil})$ give rise to an isomorphism

$$\phi : (E_1, \theta_1, \bar{H}, \bar{\nabla}, \overline{Fil}, \mathrm{Id}) \cong (E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\phi})$$

and therefore an isomorphism $\tilde{\phi} := \mathcal{T}_n(\phi)$ of twisted flat modules by Proposition 4.11:

$$(\tilde{H}, \tilde{\nabla}) := \mathcal{T}_n(E_1, \theta_1, \bar{H}, \bar{\nabla}, \overline{Fil}, \mathrm{Id}) \cong \mathcal{T}_n(E, \theta, \bar{H}, \bar{\nabla}, \overline{Fil}, \bar{\phi}) := (\tilde{H}_{-1}, \tilde{\nabla}_{-1}).$$

Note that the isomorphism $\mathcal{T}_1(\phi)$ for $n = 1$ is nothing but the original isomorphism ϕ between graded Higgs bundles. Then for an open affine covering of X_{n+1} and the set of Frobenius liftings as given in Lemma 5.6, following the method in the characteristic p case, we simply define an $\mathcal{O}_{U_i}$-isomorphism by

$$\Phi_i = F_{i,n}^*(\tilde{\phi}) : F_{i,n}^* \tilde{H} \cong F_{i,n}^*(\tilde{H}_{-1}),$$

where the latter module is just $H|_{U_{i,n}} = C_n^{-1}(E, \theta)|_{U_{i,n}}$. At this point, the triple $(H|_{U_{i,n}}, Fil|_{U_{i,n}}, \Phi_i)$ is a local p-torsion Fontaine module without connection (i.e. it is an object in the local category $MF(R_{i,n})$ with $R_{i,n} = \Gamma(U_{i,n}, \mathcal{O}_{U_{i,n}})$ as given in [4, p. 31]).

By [4, Theorem 2.1(ii)], one knows that H is a *locally free* $\mathcal{O}_{X_n}$-module. Moreover, by the construction of the functor $\mathcal{F}_n$, Φ_i is indeed an isomorphism of flat modules:

$$\Phi_i : (F_{i,n}^* \tilde{H}|_{U_{i,n}}, F_{i,n}^* \tilde{\nabla}|_{U_{i,n}}) \cong (H, \nabla)|_{U_{i,n}}.$$

This gives the horizontal property as required by an evaluation of the relative Frobenius (see Variant 1, §2). It remains to explain that the Φ_is are related via the Taylor formula (5.6.1). Let e be a local section of $\tilde{H}$ over $U_{i,n}$. Then over $U_{ij,n} := U_{i,n} \cap U_{j,n}$ one computes

$$\begin{aligned} G_{ij} \circ \Phi_i(e \otimes 1) &= \sum_J \frac{\tilde{\nabla}_{-1}(\partial)^J(\tilde{\phi}(e))}{J!} \otimes z^J = \sum_J \frac{\tilde{\phi}(\tilde{\nabla}(\partial)^J(e))}{J!} \otimes z^J \\ &= \Phi_j \left(\sum_J \frac{\tilde{\nabla}(\partial)^J(e)}{J!} \otimes z^J \right) = \Phi_j \circ \varepsilon_{ij}(e \otimes 1). \end{aligned}$$

The second equality follows from the property that $\tilde{\phi}$ respects the twisted connections, and the last equality follows from the transition over $U_{ij,n}$ of a local section of $\tilde{H}$ over $U_{i,n}$ to a local section of $\tilde{H}$ over $U_{j,n}$. So we obtain a relative Frobenius Φ from local Φ_is and thus a strict p^n-torsion Fontaine module over X_n:

$$\mathcal{IC}_n(E, \theta, Fil, \phi, \bar{E}, \bar{\theta}, \overline{Fil}, \bar{\phi}) := (H, \nabla, Fil, \Phi).$$

That the functor $\mathcal{IC}_n$ lifts $\mathcal{IC}_n$ follows from the lifting property of the inverse Cartier transform and the construction of the relative Frobenius.

Equivalence of categories: This is done by induction on n and the constructions of the natural transformations in the proof of Proposition 3.3. □

At this point, we have completed our theory of the Higgs correspondence in positive and mixed characteristic (§3–§5). In [13] (see also [12, Remark 5.5]), N. Katz established the following result.

Theorem 5.7 ([13, Proposition 4.1.1]). *Let X_n/W_n be an irreducible smooth affine scheme over W_n, equipped with an absolute Frobenius lifting F_{X_n}. Then there is an equivalence of categories between the category of $W_n(\mathbb{F}_{p^f})$-representations of $\pi_1(X_n)$ and the category of pairs (E, ϕ) consisting of a locally free sheaf E of finite rank over X_n together with an isomorphism $\phi : F_{X_n}^{*f}(E) \to E$.*

Katz's correspondence has been further developed in the work [3] of Emerton–Kisin, where the role of unit F-crystals is emphasized. A pair (E, ϕ) in Theorem 5.7 is called a *Frobenius-periodic* vector bundle over X_n. This is the prototype of the notion of periodic Higgs–de Rham flow. Indeed, to a Frobenius-periodic vector bundle (E, ϕ), one associates the periodic Higgs–de Rham flow $(E, 0, Fil_{\mathrm{tr}}, \ldots, Fil_{\mathrm{tr}}, \phi)$ over X_n of level zero, and this association is an equivalence of categories. Take an arbitrary W-lifting $X = \operatorname{Spec} R$ of X_n and Γ (resp. Γ^{ur}) the Galois group of the maximal extension of R étale over $R[1/p]$ (resp. over R) [4, Ch. II]. Then one may show that the corresponding $W_n(\mathbb{F}_{p^f})$-representation of Γ to $(E, 0, Fil_{\mathrm{tr}}, \ldots, Fil_{\mathrm{tr}}, \phi)$ over X_n according to Theorem 5.3 and [4, Theorem 2.6] factors through the natural quotient $\Gamma \to \Gamma^{\mathrm{ur}} = \pi_1(X_n)$, and the resulting representation of $\pi_1(X_n)$ coincides with the representation corresponding to (E, ϕ) in view of Theorem 5.7. Moreover, our theory extends to the case where only the existence of W_{n+1}-lifting of X_n is assumed. Namely, one may show that given a W_{n+1}-lifting X_{n+1} of X_n, there is an equivalence of categories between the category of f-periodic Higgs–de Rham flows over X_n of level zero and the category of $W_n(\mathbb{F}_{p^f})$-representations of $\pi_1(X_n)$.

To demonstrate the use of the theory in a higher Hodge–Tate weight situation, we provide an immediate construction of a p-divisible group over a geometric base over W whose Kodaira–Spencer map is an isomorphism.

Example 5.8. Let A_1 be any ordinary abelian variety defined over k of dimension g. By Serre–Tate theory, it has the canonical lifting A over $W(k)$ with the Frobenius lifting $F : A \to A$. Consider the following Higgs bundle (E, θ) over A/W:

$$E^{1,0} \oplus E^{0,1} = \Omega_A \oplus \mathcal{O}_A, \quad \theta^{1,0} = \mathrm{Id} : \Omega_A \to \mathcal{O}_A \otimes \Omega_A.$$

We will show that this Higgs bundle is one-periodic. We set $(E_n, \theta_n) := (E, \theta) \otimes \mathbb{Z}/p^n\mathbb{Z}$ and $F_n = F \otimes \mathbb{Z}/p^n\mathbb{Z}$. First we construct a one-periodic Higgs–de Rham flow on A_1: since A_2 has the global Frobenius lifting F_2, it follows that

$$C_1^{-1}(E_1, \theta_1) = (H_1, \nabla_1),$$

with

$$H_1 := F_1^* E_1 \quad \text{and} \quad \nabla_1 = \nabla_{\mathrm{can}} + \frac{dF_2}{p}(F_1^* \theta_1).$$

A Hodge filtration of level one on (H_1, ∇_1) is defined by $Fil_1^1 = F_1^* \Omega_{A_1} = \Omega_{A_1}$. Set

$$(E_1', \theta_1') := \mathrm{Gr}_{Fil_1}(H_1, \nabla_1).$$

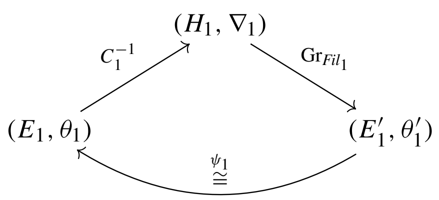
Then

$$E_1' = \Omega_{A_1} \oplus \mathcal{O}_{A_1}, \quad \theta_1'^{1,0} = \frac{dF_2}{p}(F_1^* \theta_1).$$

Because of the ordinariness of A_1, the Hasse–Witt map

$$\frac{dF_2}{p} : H^0(A_1, \Omega_{A_1}) \to H^0(A_1, \Omega_{A_1})$$

is bijective. So $\theta_1'^{1,0}$ has to be an isomorphism. Then we proceed to show (E_1', θ_1') is isomorphic to (E_1, θ_1). Indeed, there is one natural choice $\psi_1 : E_1' \to E_1$ of isomorphism described as follows: its (0, 1)-component mapping $\mathcal{O}_{A_1}$ to itself is the identity, and its (1, 0)-component mapping Ω_{A_1} to itself is the unique isomorphism commuting with the Higgs fields. Therefore, we have obtained a one-periodic flow over A_1 as claimed:



Next we proceed to the W_2-level. Using the fact that A_3 has the Frobenius lifting F_3, one computes that $\tilde{H}_{-1,2} = \Omega_{A_2} \oplus \mathcal{O}_{A_2}$ and $\tilde{\nabla}_{-1,2} = p\nabla_{\mathrm{can}} + \theta_2$. Using Remark 4.17, it follows that

$$C_2^{-1}(E_2, \theta_2) = (H_2, \nabla_2)$$

with $H_2 = F_2^* E_2 = \Omega_{A_2} \oplus \mathcal{O}_{A_2}$ and ∇_2 the connection defined by

$$\nabla_2(f \otimes e) = df \otimes e + f \cdot \left(\frac{dF_3}{p} \otimes 1\right)(1 \otimes \tilde{\nabla}_{-1,2}(e)),$$

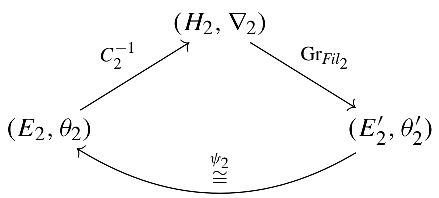
where f (resp. e) is a local section of $\mathcal{O}_{A_2}$ (resp. $\tilde{H}_{-1,2}$) (see Claim 4.16). Now we take the filtration $Fil_2^1 = \Omega_{A_2}$, which lifts Fil_1^1. Then the associated graded Higgs bundle is

$$E_2' = \Omega_{A_2} \oplus \mathcal{O}_{A_2}, \qquad \theta_2'^{1,0} = \frac{dF_3}{p}(F_2^* \theta_2),$$

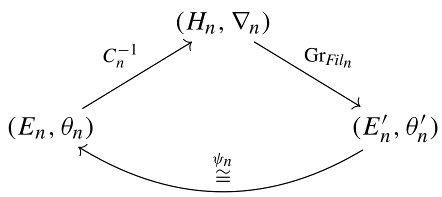
which lifts (E_1', θ_1'). Again there is an obvious isomorphism

$$\psi_2 : (E_2', \theta_2') \to (E_2, \theta_2)$$

which lifts ψ_1 and whose (0, 1)-component is the identity map. So we obtain a one-periodic flow over A_2:



Then one continues and constructs inductively a one-periodic flow over A_n, $n \geq 1$:



Passing to the limit, one therefore obtains a one-periodic Higgs–de Rham flow over A/W, and hence a rank $g + 1$ crystalline $\mathbb{Z}_p$-representation of Hodge–Tate weight one of the generic fiber A^0 of A/W, which by [4, Theorem 7.1] corresponds to a p-divisible group over A/W.

6. Strongly semistable Higgs modules

Let X/k be a smooth projective variety over k, equipped with an ample divisor Z. Semistability in this section means μ_Z-slope semistability. Recall that a vector bundle over X is said to be *strongly semistable* if the bundle as well as its pull-back under any power of Frobenius are semistable. The relation between strongly semistable bundles with trivial Chern classes and representations of the algebraic fundamental group was first revealed by Lange–Stuhler in the curve case [18, §1]. It asserts that a semistable vector bundle E of degree zero over a curve can be trivialized after a finite morphism if and only if it is strongly semistable [18, Satz 1.9]. Note that E being strongly semistable of degree zero is equivalent to there being a pair (e, f) of integers with e nonnegative and f positive such that

$$F_X^{*e+f}(E) \cong F_X^{*e}(E),$$

where $F_X : X \to X$ denotes the absolute Frobenius morphism as usual. It corresponds to a representation of $\pi_1(X)$ in $\mathrm{GL}(k)$ if and only if e in the above isomorphism can be taken to be zero [18, Proposition 1.2, Satz 1.4]. The result of Lange–Stuhler has been generalized to singular curves by Deninger–Werner [2, Theorem 18] and this generalization played a key role in their partial p-adic analogue of Narasimhan–Seshadri theory. Besides the intimate relation to representations of π_1, the notion of strong semistability is also useful in other situations, for example, in Langer's proof of boundedness of semistable sheaves and Bogomolov's inequality in positive characteristic [19]. Therefore, it is a natural question to generalize this notion to Higgs modules. Interestingly enough, it turns out that our generalization (especially Theorem 6.5 below) has played a key role in the very recent result, due to A. Langer [22], on the Bogomolov–Gieseker inequality for semistable Higgs bundles and the Miyaoka–Yau inequality for surfaces in positive characteristic.

The key of the generalization is to replace the Frobenius pull-back with the inverse Cartier transform of Ogus–Vologodsky [26], as seen in the following

Definition 6.1. Let X be a smooth projective variety over k together with a fixed W_2-lifting of X. A Higgs module (E, θ) is called *strongly semistable* if it appears in the initial term of a semistable Higgs–de Rham flow, that is, all Higgs terms (E_i, θ_i) in the flow are semistable and defined over a common finite subfield of k.[2]

A torsion Higgs module is by definition automatically semistable, which is however uninteresting in the current setting. Therefore, a semistable Higgs module is tacily assumed to be torsion-free in this paper. Clearly, a strongly semistable vector bundle E is strongly Higgs semistable: one simply takes the Higgs–de Rham flow

$$\begin{array}{ccccccc} & (F_X^*E, \nabla_{\mathrm{can}}) & & & (F_X^{*2}E, \nabla_{\mathrm{can}}) & & \\ \nearrow^{F_X^*} & & \searrow^{\mathrm{Gr}_{Fil_{\mathrm{tr}}}} & \nearrow^{F_X^*} & & \searrow^{\mathrm{Gr}_{Fil_{\mathrm{tr}}}} & \\ (E, 0) & & & (F_X^*E, 0) & & & \dots \end{array}$$

[2] This definition corrects an error in [17, Definition 2.1].

where ∇_{can} is the canonical connection in the Cartier descent theorem and Fil_{tr} stands for the trivial filtration as before.

The first result on the Higgs semistability of the graded Higgs module associated to a strict p-torsion Fontaine module is due to Ogus–Vologodsky [26, Proposition 4.19].

Proposition 6.2 (Ogus–Vologodsky). *Let X/k be a smooth projective curve of genus g. Let (H, ∇, Fil, Φ) be a strict p-torsion Fontaine module (with respect to some W_2-lifting of X). Assume that*

$$n(\mathrm{rank}\, H - 1)\max\{2g-2, 1\} < p - 1.$$

Then $\mathrm{Gr}_{Fil}(H, \nabla)$ is a semistable Higgs bundle.

Their result can be generalized as follows (see [27, Proposition 0.2] for a generalization in the geometric case as given in Example 2.2, and also [29, Proposition 3.7]):

Proposition 6.3. *Let X/k be a smooth projective variety. Let (H, ∇, Fil, Φ) be a strict p-torsion Fontaine module (with respect to some W_2-lifting of X). Then the graded Higgs bundle $(E, \theta) := \mathrm{Gr}_{Fil}(H, \nabla)$ is Higgs semistable. Moreover, any Higgs subsheaf $(G, \theta) \subset (E, \theta)$ of slope zero is strongly semistable.*

Proof. Let us first recall that we have proven (see [16, Proposition 1.4]) that there is a natural isomorphism

$$\tilde{\Phi} : C_1^{-1}(E, \theta) \cong (H, \nabla).$$

Therefore, one obtains a natural isomorphism $\mathrm{Gr}_{Fil} \circ C_1^{-1}(E, \theta) \cong (E, \theta)$. It implies that $p\mu(E) = \mu(E)$, and thus $\mu(E) = 0$. For the first statement, we argue by contradiction. Let (F, θ) be the maximal destabilizing Higgs subsheaf of (E, θ) with positive slope $\mu(F)$. Then $\mathrm{Gr}_{Fil} \circ C_1^{-1}(F, \theta)$ is naturally a Higgs subsheaf of slope $p\mu(F) > \mu(F)$, a contradiction. For the second statement, first notice that a Higgs subsheaf (G, θ) of slope zero is automatically semistable, as first observed by C. Seshadri. As $\mathrm{Gr}_{Fil} \circ C_1^{-1}(G, \theta)$ is naturally a Higgs subsheaf of (E, θ) which is again of slope zero and hence semistable, (G, θ) is strongly semistable. □

It is surprising to have the following

Proposition 6.4. *Any rank two nilpotent semistable Higgs module is strongly semistable.*

Proof. Let (E, θ) be a rank two nilpotent semistable Higgs module over X. For the reason of rank, θ is nilpotent of exponent ≤ 1. Write (H, ∇) for $C_1^{-1}(E, \theta)$, and HN for the Harder–Narasimhan filtration on H. We need to show that the graded Higgs module $\mathrm{Gr}_{HN}(H, \nabla)$ is again semistable.

If H is semistable, there is nothing to prove: in this case, HN is trivial and hence the induced Higgs field is zero, and $\mathrm{Gr}_{HN}(H, \nabla) = (H, 0)$ is Higgs semistable. Otherwise denote by $L_1 \subset H$ the invertible subsheaf of maximal slope and $L_2 = H/L_1$ the quotient sheaf. Then L_1 cannot be ∇-invariant. Indeed, (H, ∇) is ∇-semistable as a general property: Let $M \subset H$ be any ∇-invariant subsheaf. Then $(F, \theta|_F) := C_1(M, \nabla|_M) \subset C_1(H, \nabla) = (E, \theta)$ is a Higgs subsheaf of slope $\mu(F) = \mu(M)/p$, where C_1 is the

Cartier transform of Ogus–Vologodsky (see also [16]). As (E, θ) is Higgs semistable, it follows that

$$\mu(M) = p\mu(F) \leq p\mu(E) = \mu(H).$$

So the natural map

$$\theta' = \mathrm{Gr}_{HN}\nabla : L_1 \to L_2 \otimes \Omega_{X_1}$$

is nonzero. A nontrivial proper Higgs subsheaf $L \subset \mathrm{Gr}_{HN}(H, \nabla)$ is simply an invertible subsheaf with $\theta'(L) = 0$. So $L \subset L_2$ and

$$\mu(L) \leq \mu(L_2) < \mu(H) = \mu(\mathrm{Gr}_{HN} H).$$

In this case, $\mathrm{Gr}_{HN}(H, \nabla)$ is actually Higgs stable. □

Motivated by Proposition 6.4, we proposed a conjecture on strong semistability of a nilpotent semistable Higgs module of higher rank in the first version of the paper (see [17, Conjecture 2.8]). The conjecture was proven by A. Langer [21] in the case of small rank, which is crucial to his algebraic proof of the Bogomolov–Gieseker inequality and the Miyaoka–Yau inequality. In the Appendix, we shall provide an independent proof of the conjecture in the case of small rank.

Theorem 6.5 (Theorem A.1, [21, Theorem 5.1]). *Any nilpotent semistable Higgs module of rank $\leq p$ over X is strongly semistable.*

After establishing the notion of strong semistability and exhibiting ample examples, we shall concentrate on those strongly semistable Higgs modules with trivial Chern classes, as guided by the theorem of Lange–Stuhler. It turns out they are quite close to being periodic.

Theorem 6.6. *Let X/k be a smooth projective variety with a fixed W_2-lifting of X. A preperiodic Higgs module is strongly semistable with trivial Chern classes. Conversely, a strongly semistable Higgs module with trivial Chern classes is preperiodic.*

Proof. For a Higgs module $(E, \theta) \in HIG_{p-1}(X)$, let $(H, \nabla) = C_1^{-1}(E, \theta)$ be the corresponding flat module. It follows from [26, proof of Theorem 4.17] that

$$c_l(H) = p^l c_l(E), \quad l \geq 0.$$

Since for any Griffiths transverse filtration Fil on (H, ∇) the associated graded Higgs module $(E', \theta') = \mathrm{Gr}_{Fil} \circ C_1^{-1}(E, \theta)$ has the same Chern classes as H, it follows that

$$c_l(E') = p^l c_l(E), \quad l \geq 0.$$

Therefore, in a Higgs–de Rham flow one has, for $i \geq 0$,

$$c_l(E_{i+1}) = p^l c_l(E_i), \quad l \geq 0.$$

This forces the Chern classes of a preperiodic Higgs module to be trivial. Also, a slope λ Higgs subsheaf in (E_i, θ_i) gives rise to a slope $p\lambda$ Higgs subsheaf in (E_{i+1}, θ_{i+1}). This implies that, in a preperiodic Higgs–de Rham flow, each Higgs term (E_i, θ_i) contains no

Higgs subsheaf of positive degree. So each (E_i, θ_i) is semistable. This shows the first statement.

Conversely, let (E, θ) be a strongly semistable Higgs module with trivial Chern classes, and let $(E_i, \theta_i)_{i\geq 0}$ be the Higgs terms appearing in a semistable Higgs–de Rham flow with initial term (E, θ). As discussed above, each (E_i, θ_i) has trivial Chern classes (and the same rank as E). By [22, Lemma 5] and [21, Theorem 4.4], the moduli space of semistable Higgs modules with trivial Chern classes over X/k is bounded and defined over k. Let $k' \subset k$ be the common finite subfield for the infinite sequence $\{(E_i, \theta_i)\}_{i\geq 0}$. Since any scheme of finite type over k has only finitely many k'-rational points, there must exist a pair of integers $(e, f) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ such that there is an isomorphism (over k) of Higgs modules $(E_{e+f}, \theta_{e+f}) \cong (E_e, \theta_e)$. Certainly, one can make the above isomorphism into an isomorphism of graded Higgs modules by adjusting one of their gradings. Therefore, (E, θ) is preperiodic. □

Using the notion of strongly semistable Higgs module as bridge, we can produce crystalline k-representations (up to isomorphism) from semistable nilpotent Higgs bundles (of small rank) with trivial Chern classes.

Theorem 6.7. *Let X/W be a smooth projective scheme. Then for any rank $r \leq p-1$ semistable nilpotent Higgs bundle (E, θ) over X_k with trivial Chern classes, there is a unique r-dimensional crystalline k-representation of $\pi_1(X_K)$ up to isomorphism.*

The proof relies on the next result which follows directly from Theorems 6.5 and 6.6.

Corollary 6.8. *With notation as Theorem 6.7, any rank $r \leq p$ semistable nilpotent Higgs bundle over X_k with trivial Chern classes is preperiodic.*

It is a nontrivial problem to decide when a small rank semistable graded Higgs bundle with trivial Chern classes is periodic (and to find its period when it is indeed the case), albeit it is always preperiodic by the corollary. We shall discuss this problem as well as the basic properties of representations constructed in Theorem 6.7 on a later occasion.

More preparations are needed before we can prove Theorem 6.7. A (pre)periodic Higgs bundle may lead to more than one (pre)periodic Higgs–de Rham flow. However, in the setting of Corollary 6.8, there is a natural choice of filtrations in a preperiodic Higgs–de Rham flow.

Proposition 6.9 (Theorem A.4, Remark A.9; [21, Theorem 5.5]). *Let (H, ∇) be a ∇-semistable flat bundle over X_k. Then there exists a unique reduced gr-semistable filtration on (H, ∇) which is preserved by any automorphism of (H, ∇).*

The reader is referred to the Appendix for the definition of a gr-semistable (resp. reduced) filtration. We shall call the filtration in the above result the *Simpson filtration* and denote it by Fil_S. Note that the Simpson filtration in the rank two case is nothing but the Harder–Narasimhan filtration on H, but it may well differ from the Harder–Narasimhan filtration for higher ranks. A periodic Higgs–de Rham flow whose de Rham terms are all equipped with the Simpson filtration is unique up to lengthening, as proven below. The unicity is shared more generally by a periodic Higgs–de Rham flow satisfying

Assumption 6.10. Let $(E, \theta, Fil_0, \ldots, Fil_{f-1}, \varphi)$ be a periodic Higgs–de Rham flow over X_k. Assume that, for each $0 \leq i \leq f-1$, the filtration Fil_i on H_i is preserved by any automorphism of (H_i, ∇_i).

Recall that in the definition of lengthening, one has the following isomorphisms of graded Higgs modules induced by $\varphi : (E_f, \theta_f) \cong (E_0, \theta_0)$:

$$(\mathrm{Gr}\, C_1^{-1})^{nf}(\varphi) : (E_{(n+1)f}, \theta_{(n+1)f}) \cong (E_{nf}, \theta_{nf}), \qquad n \in \mathbb{N}.$$

Set, for $-1 \leq i < j$,

$$\begin{aligned}\varphi_{j,i} = (\mathrm{Gr}\, C_1^{-1})^{(i+1)f}(\varphi) \circ \cdots \circ (\mathrm{Gr}\, C_1^{-1})^{jf}(\varphi) : (E_{(j+1)f}, \theta_{(j+1)f}) \\ \cong (E_{(i+1)f}, \theta_{(i+1)f}).\end{aligned}$$

Note that $\varphi_{j,-1}$ is just the isomorphism φ_j in the j-fold lengthening of the starting periodic flow.

Lemma 6.11. *Let $\phi : (E_f, \theta_f) \cong (E_0, \theta_0)$ be another isomorphism of graded Higgs modules. Then there exists a pair (i, j) with $0 \leq i < j$ such that $\phi_{j,i} \circ \varphi_{j,i}^{-1} = \mathrm{Id}$.*

Proof. If we denote $\tau_s = \phi_s \circ \varphi_s^{-1}$, then τ_s is an automorphism of (E_0, θ_0). Moreover, each element in the set $\{\tau_s\}_{s \in \mathbb{N}}$ is defined over the same finite field in k. As this is a finite set, there are $j > i \geq 0$ such that $\tau_j = \tau_i$. So the lemma follows. □

Proposition 6.12. *Assume that Assumption 6.10 holds. Let (i, j) be a pair given by Lemma 6.11 for two given isomorphisms $\varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0)$. Then there is an isomorphism in $HDF_{n,(j-i)f}(X_2/W_2)$:*

$$(E, \theta, Fil_0, \ldots, Fil_{(j-i)f-1}, \varphi_{j-i-1}) \cong (E, \theta, Fil_0, \ldots, Fil_{(j-i)f-1}, \phi_{j-i-1}),$$

where both sides are obtained by $j-i-1$-fold lengthening.

Proof. Put $\beta = \phi_i \circ \varphi_i^{-1} : (E_0, \theta_0) \cong (E_0, \theta_0)$. We shall check that it induces an isomorphism in $HDF_{n,(j-i)f}(X_2/W_2)$. By Assumption 6.10, $C_1^{-1}(\mathrm{Gr}\, C_1^{-1})^m(\beta)$ for $m \geq 0$ always respects the filtrations. We need only check that β is compatible with ϕ_{j-i-1} and φ_{j-i-1}. So it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} E_{(j-i)f} & \xrightarrow{\varphi_{j-i-1}} & E_0 \\ {\scriptstyle \varphi_{j,j-i-1}^{-1}}\downarrow & & \downarrow{\scriptstyle \varphi_i^{-1}} \\ E_{(j+1)f} & & E_{(i+1)f} \\ {\scriptstyle \phi_{j,j-i-1}}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ E_{(j-i)f} & \xrightarrow{\phi_{j-i-1}} & E_0 \end{array}$$

Clearly, it is equivalent to show the commutativity of

$$\begin{array}{ccc}
E_{(j-i)f} & \xrightarrow{\varphi_{j-i-1}^{-1}} & E_0 \\
{\scriptstyle \varphi_{j,j-i-1}^{-1}}\downarrow & & \downarrow{\scriptstyle \varphi_i^{-1}} \\
E_{(j+1)f} & & E_{(i+1)f} \\
{\scriptstyle \phi_{j,j-i-1}}\downarrow & & \downarrow{\scriptstyle \phi_i} \\
E_{(j-i)f} & \xrightarrow{\phi_{j-i-1}} & E_0
\end{array}$$

In the above diagram, the anti-clockwise direction is

$$\phi_{j-i-1} \circ \phi_{j,j-i-1} \circ \varphi_{j,j-i-1}^{-1} \circ \varphi_{j-i-1}^{-1} = \phi_j \circ \varphi_j^{-1} = \phi_i \circ (\phi_{j,i} \circ \varphi_{j,i}^{-1}) \circ \varphi_i.$$

By the requirement for (i, j), we have $\phi_{j,i} \circ \varphi_{j,i}^{-1} = \mathrm{Id}$, so the anti-clockwise direction is $\phi_i \circ \varphi_i$, which is exactly the clockwise direction. So β is indeed compatible with ϕ_{j-i-1} and φ_{j-i-1}. □

Proof of Theorem 6.7. We assemble the previous results. First, by Theorem 6.8, there exists a preperiodic flow with initial term (E, θ). But there are several choices. In order to make a unique choice, we shall apply Proposition 6.9 at each step, that is, we use the Simpson filtration Fil_S on each flat bundle and then we obtain the uniqueness on the filtrations in the flow. Now let $e \in \mathbb{N}_0$ be the minimal number such that $(\mathrm{Gr}_{Fil_S} \circ C_1^{-1})^e(E, \theta)$ is periodic and let $f \in \mathbb{N}$ be its period. Thus from (E, θ) we have obtained a periodic Higgs–de Rham flow

$$((\mathrm{Gr}_{Fil_S} \circ C_1^{-1})^e(E, \theta), Fil_0 = Fil_S, \ldots, Fil_{f-1} = Fil_S, \phi),$$

unique up to the choice of ϕ. Fix one choice of ϕ and let ρ be the corresponding representation according to Corollary 3.10. As Fil_Ss satisfy Assumption 6.10, it follows from Proposition 6.12 and Corollary 3.11(ii) that the isomorphism class of $\rho \otimes k$ is independent of the choice of ϕ. Finally, the rank condition $\leq p-1$ allows us to apply Theorem 2.3 to conclude the proof. □

7. Rigidity theorem for Fontaine modules

Let X/W be a smooth and projective scheme, equipped with a W-ample divisor Z. To a (p-torsion) Fontaine module (H, ∇, Fil, Φ), one naturally associates the graded Higgs bundle $(E, \theta) := \mathrm{Gr}_{Fil}(H, \nabla)$ by taking the grading of (H, ∇) with respect to the filtration Fil. In this section, we show that the gr-functor is faithful over those mod p stable objects (with respect to the μ_{Z_1}-slope).

For a Griffiths transverse filtration Fil of level w, we extend $Fil^i = Fil^0$ for $-i \in \mathbb{N}$ and $Fil^{w+j} = 0$ for $j \in \mathbb{N}$.

Lemma 7.1. *Let Y be a smooth projective variety over an algebraically closed field k and let (V,∇) be a flat bundle over Y. If there exists a Griffiths transverse filtration Fil on (V,∇) such that the associated graded Higgs module (E,θ) is stable, then it is unique up to a shift of index.*

Proof. Suppose on the contrary that there is another gr-semistable filtration $\overline{Fil}$ on V which differs from Fil after arbitrary index shifting. Set $(\bar{E},\bar{\theta}) := \mathrm{Gr}_{\overline{Fil}}(V,\nabla)$.

Case 1: Suppose that there exists an integer N such that $Fil^i \subset \overline{Fil}^{i+N}$ for every i. Also, we can assume N is so chosen that $Fil^{i_0} \subsetneq \overline{Fil}^{i_0+N+1}$ for some i_0. Then the inclusion induces a natural morphism of Higgs bundles

$$f : (E,\theta) \to (\bar{E},\bar{\theta}).$$

Clearly, the morphism f cannot be injective at each closed point: otherwise, $Fil^i = \overline{Fil}^{i+N}$ for all i, which contradicts the assumption that the two filtrations are different. So f is not injective. Neither is f zero, since it would imply $Fil^i \subset \overline{Fil}^{i+N+1}$ for all i, which contradicts the assumption on N. Therefore, on the one hand, as $\mathrm{Im}(f)$ is a quotient of (E,θ), we have $\mu(\mathrm{Im}(f)) > \mu(E) = \mu(V)$; on the other hand, $\mathrm{Im}(f)$ is also a subobject of $(\bar{E},\bar{\theta})$, so $\mu(\mathrm{Im}(f)) \leq \mu(\bar{E}) = \mu(V)$, a contradiction.

Case 2: Otherwise, let a be the largest integer such that Fil^a is not contained in $\overline{Fil}^a$, and b the largest integer such that Fil^{a-i} is contained in $\overline{Fil}^{b-i}$ for all $i \geq 0$. Certainly $a > b$. Then we define a morphism of Higgs bundles $f : (E,\theta) \to (\bar{E},\bar{\theta})$ as follows: for $i > 0$, $f|_{E^{a+i}} = 0$, and for $j \geq 0$, $f|_{E^{a-j}}$ is given by

$$E^{a-j} = Fil^{a-j}/Fil^{a-j+1} \to \overline{Fil}^{b-j}/\overline{Fil}^{b-j+1}.$$

This is well defined as $Fil^{a+1} \subset \overline{Fil}^{a+1} \subset \overline{Fil}^{b+1}$ and $Fil^{a-j} \subset \overline{Fil}^{b-j}$ for $j \geq 0$. Clearly f cannot be injective. Also, f cannot be zero: otherwise, one would get the relation $Fil^{a-j} \subset \overline{Fil}^{b+1-j}$ for all $j \geq 0$, which contradicts the maximality of b. The remaining argument goes exactly as in Case 1. □

The above statement, true for any characteristic, has the following nice consequence in the current setting.

Proposition 7.2. *Let (E,θ) be a Higgs bundle over X_1. Suppose (E,θ) is stable and one-periodic. Then there is a unique one-periodic Higgs–de Rham flow with initial term (E,θ), up to isomorphism.*

Proof. Let (E,θ,Fil,ϕ) be a one-periodic Higgs–de Rham flow. First, because of the stability assumption on (E,θ), it follows from Lemma 7.1 that the Hodge filtration Fil on $C_1^{-1}(E,\theta)$ is unique up to a possible shift of index. However, ϕ is an isomorphism of *graded* Higgs modules, so Fil has to be unique. Second, for any other choice φ making

$(E, \theta, Fil, \varphi)$ periodic, the composite $\varphi \circ \phi^{-1}$ is an automorphism of (E, θ). Since (E, θ) is stable, one must have $\varphi = \lambda\phi$ for a nonzero constant $\lambda \in k$. As there is an obvious isomorphism

$$(E, \theta, Fil, \phi) \cong (E, \theta, Fil, \lambda\phi)$$

in the category *HDF*, the proposition follows. □

An $\mathbb{F}_p$-representation ρ of $\pi_1(X_K)$ is said to be *absolutely irreducible* if the k-representation $\rho \otimes k$ is irreducible. A direct consequence of the previous proposition is the following

Corollary 7.3. *With notation as above, there is a natural equivalence of categories between the category of absolutely irreducible crystalline $\mathbb{F}_p$-representations of $\pi_1(X_K)$ with Hodge–Tate weights $\leq p-2$ and the category of one-periodic stable Higgs bundles in $HIG_{p-2}(X_1)$.*

Proof. Proposition 7.2 embeds the category of one-periodic stable Higgs bundles over X_1 into the category of one-periodic Higgs–de Rham flows over X_1 as a full subcategory. Under the Higgs correspondence and Fontaine–Laffaille–Faltings correspondence (Proposition 3.3 and Theorem 2.3), the category of one-periodic stable Higgs bundles in $HIG_{p-2}(X_1)$ corresponds to a full subcategory of crystalline $\mathbb{F}_p$-representations of $\pi_1(X_K)$. The image is characterized by the absolute irreducibility of the associated $\mathbb{F}_p$-representations, as shown by [29, Theorem 1.3]. □

Remark 7.4. Over $\mathbb{C}$, the gr-functor is an equivalence of categories between the category of irreducible complex polarized variations of Hodge structure and the category of stable graded Higgs bundles. The above corollary is a characteristic p analogue of this equivalence. However, compared with the transcendental nature of the quasi-inverse functor in the complex case, the one in the characteristic p case is much more constructive: indeed, it has already been noticed by Ogus–Vologodsky [26, Definition 4.16 and the paragraph thereafter] that the flat bundle (H, ∇) of a strict p-torsion Fontaine module is reconstructed by $C_1^{-1}(E, \theta)$, whether (E, θ) stable or not (but it is at least semistable as explained in Proposition 6.3). This point has also been explicitly emphasized in [16, §4]. Furthermore, in Theorem A.4 of the Appendix, we show how to construct a gr-semistable filtration on (H, ∇). By the proof of Lemma 7.1, it has to coincide with the Hodge filtration *Fil* up to a possible shift of index, which is also uniquely determined by the gradings in E. The construction of the relative Frobenius from an isomorphism $\mathrm{Gr}_{Fil}(H, \nabla) \cong (E, \theta)$ is the major content of the functor $\mathcal{C}^{-1}$ in the proof of Proposition 3.3.

The above results may be lifted to W_n for n arbitrary.

Proposition 7.5. *Let (E, θ) be the initial term of a one-periodic Higgs–de Rham flow over X_n. If the mod p reduction of (E, θ) is stable over X_1, then there is a unique one-periodic Higgs–de Rham flow for (E, θ) up to isomorphism.*

Proof. We use induction on n. The $n = 1$ case is Proposition 7.2. The induction hypothesis is as follows: Let $(\bar{E}, \bar{\theta})$ be the mod p^{n-1} reduction of (E, θ), and $(\bar{E}, \bar{\theta}, \overline{Fil}, \bar{\psi})$ be a one-periodic Higgs–de Rham flow over X_{n-1}. Then $\overline{Fil}$ is the unique lifting of the Hodge filtration in characteristic p, and $\bar{\psi}$ is the unique lifting of the isomorphism in characteristic p up to a scalar in W_{n-1}. Now let $(E, \theta, Fil_i, \psi_i)$, $i = 1, 2$, be two one-periodic Higgs–de Rham flows over X_n lifting $(\bar{E}, \bar{\theta}, \overline{Fil}, \bar{\psi})$ over X_{n-1}. It suffices to show the following claims:

(i) $Fil_1 = Fil_2$;
(ii) $\psi_1 = \lambda\psi_2$ for some $\lambda \in W_n$.

Without loss of generality, we may assume that $\overline{Fil}$ is reduced. Assume the contrary of (i). Let a be the largest integer such that Fil_1^a differs from Fil_2^a, and then let b be the largest integer such that $Fil_1^{a-i} \subseteq Fil_2^{b-i}$ for each $i \geq 0$.

Case 1: $b > a$. Then as

$$Fil_1^a \subset Fil_2^b \subset Fil_2^{a+1} = Fil_1^{a+1} \subset Fil_1^a,$$

we get $Fil_1^{a+1} = Fil_1^a$, and in particular $\overline{Fil}^{a+1} = \overline{Fil}^a$, a contradiction.

Case 2: $b = a$. As $Fil_1^a \subset Fil_2^a$ and $Fil_1^a = Fil_2^a \bmod p^{n-1}$, it follows that $Fil_1^a = Fil_2^a$, a contradiction.

Case 3: $b < a$. Let $(H, \nabla) = C_n^{-1}(E, \theta)$. We define a morphism

$$f : \mathrm{Gr}_{Fil_1}(H, \nabla) \to \mathrm{Gr}_{Fil_2}(H, \nabla)$$

as follows: for $i > 0$, f on the factor $Fil_1^{a+i}/Fil_1^{a+i+1}$ is simply zero; for $i \leq 0$, $f : Fil_1^{a+i}/Fil_1^{a+i+1} \to Fil_2^{b+i}/Fil_2^{b+i+1}$ is the natural morphism. It is easy to check that this gives a morphism of Higgs bundles. Because of the choice of b, f is nonzero. As its mod p^{n-1} reduction is clearly the zero map, we obtain a nonzero morphism $f/[p^{n-1}]$ on the mod p reductions of both sides of the morphism f, which are isomorphic to the stable Higgs bundle $(E, \theta)_1$ in characteristic p. Clearly, $f/[p^{n-1}]$ is not an isomorphism. A contradiction. Therefore, $Fil_1 = Fil_2$, and $\mathrm{Gr}_{Fil_1}(H, \nabla) = \mathrm{Gr}_{Fil_2}(H, \nabla)$.

We now show (ii). For this, we consider the composite

$$\phi := \psi_1 \circ \psi_2^{-1} : (E, \theta) \cong (E, \theta).$$

By the induction hypothesis, $\phi \bmod p^{n-1} = \bar{\lambda} \in W_{n-1}$. Take any lifting $\lambda \in W_n$ of $\bar{\lambda}$ and consider the endomorphism $\phi' := \phi - \lambda$ of (E, η). As $\phi' \bmod p^{n-1} = 0$, we get an endomorphism of the stable Higgs bundle in characteristic p:

$$\phi'/[p^{n-1}] : (E, \theta)_1 \to (E, \theta)_1,$$

which has to be a scalar $\mu \in k$. So we get $\phi = \lambda + p^{n-1}\mu \in W_n$, and (ii) follows. □

The following rigidity theorem for Fontaine modules follows immediately from the previous proposition and the Higgs correspondence.

Corollary 7.6. *Let $(H_i, \nabla_i, Fil_i, \Phi_i)$, $i = 1, 2$, be two Fontaine modules (torsion-free or not) over X/W, and (E_i, θ_i) the associated graded Higgs bundles. If (E_1, θ_1) is isomorphic to (E_2, θ_2) and mod p Higgs stable, then $(H_i, \nabla_i, Fil_i, \Phi_i)$, $i = 1, 2$, are isomorphic.*

Combining Proposition 7.5 with Theorem 5.3 and Corollary 7.3, we obtain

Corollary 7.7. *There is a natural equivalence of categories between the category of crystalline $\mathbb{Z}_p$- (resp. $W_n(\mathbb{F}_p)$-) representations of $\pi_1(X_K)$ with Hodge–Tate weight $\leq p-2$ whose mod p reduction is absolutely irreducible and the category of one-periodic Higgs bundles over X/W (resp. X_n/W_n) whose exponent is $\leq p-2$ and whose mod p reduction is stable.*

Appendix (by Guitang Lan, Mao Sheng, Yanhong Yang and Kang Zuo): Semistable Higgs bundles of small rank are strongly Higgs semistable

In this Appendix, we shall prove the following result.

Theorem A.1. *Let (E, θ) be a nilpotent semistable Higgs module over a smooth projective variety X defined over $\bar{\mathbb{F}}_p$. If* rank $E \leq p$, *then (E, θ) is strongly semistable.*

The same result has been obtained independently by A. Langer [21, Theorem 5.12]. In the early version of [17], we have proposed the following conjecture which inspired the above theorem.

Conjecture A.2 ([17, Conjecture 2.8]). *A nilpotent semistable Higgs module of exponent $\leq p$ is strongly Higgs semistable.*

We had shown the rank two case in [17, Theorem 2.6], which is Proposition 6.4 in the current version. Shortly after the appearance of [17], Lingguang Li [23] verified the conjecture in the rank three case. The conjecture requires modification in order to be true in the higher rank case.

The key step in the proof is Theorem A.4 below, a positive characteristic generalization of Simpson's result [34, Theorem 2.5], which states that over a complex smooth projective curve, every vector bundle with an integrable holomorphic connection admits a Griffiths transverse filtration such that the associated graded Higgs bundle is semistable. This generalization is proved similarly to [34, Theorem 2.5].

Throughout the Appendix, we assume that Y is a smooth projective variety over an algebraically closed field k, H is an ample divisor of Y, Higgs modules as well as flat modules are torsion-free, and semistability means $\mu = \mu_H$-semistability.

Definition A.3. A flat module (V, ∇) is called ∇-*semistable* if $\mu(V_1) \leq \mu(V)$ for every submodule $V_1 \subset V$ with $\nabla(V_1) \subset V_1 \otimes_{\mathcal{O}_Y} \Omega_Y$.

The goal of this section is to prove the following theorem.

Theorem A.4. *Let (V, ∇) be a ∇-semistable flat module over a smooth projective variety Y over an algebraically closed field k. Then there exists a Griffiths transverse filtration Fil such that the graded Higgs module associated to (V, ∇, Fil) is semistable.*

A Griffiths transverse filtration on (H, ∇) with semistable graded Higgs module is said to be *gr-semistable*.

Remark A.5. By [35], every holomorphic vector bundle that admits a connection is of degree 0, thus in this case every flat bundle (V, ∇) is automatically ∇-semistable. So the above result generalizes [34, Theorem 2.5]. On the other hand, the ∇-semistability condition in the statement is indeed necessary for its truth over a general field. Let k be a field of characteristic $p \geq 3$ and $V = \mathcal{O} \oplus \mathcal{O}(p)$ be the rank two vector bundle over $\mathbb{P}^1_k$, equipped with the canonical connection ∇_{can} of the Cartier descent theorem. Then $(V, \nabla_{\mathrm{can}})$ admits *no* gr-semistable filtration.

To prove the above theorem, we need some lemmas.

Lemma A.6. *Let (E, θ) be a graded Higgs module on Y. If (E, θ) is unstable as a Higgs module, then its maximal destabilizing subsheaf $I \subset E$ is saturated, and it is a graded Higgs submodule, that is, $I = \bigoplus_{i=0}^n I^i$ with $I^i := I \cap E^i$.*

Proof. The saturation property of the Higgs subsheaf $I \subset E$ follows from maximality. To show the second property, one chooses $t \in k$ such that $t^i \neq 1$ for $0 < i \leq n$. Note that there is an isomorphism $f : (E, \theta) \to \left(E, \frac{1}{t}\theta\right)$ given by $f|_{E^i} = t^i \mathrm{Id}$. Because of the uniqueness of the maximal destabilizing subobject, we see that $f(I) = I$. Let s be any local section of I. Write s as $\sum_{i=0}^n s^i$, where s^i is a local section of E^i. Then for $j \geq 0$,

$$f^j(s) = \sum_{i=0}^n t^{ji} s^i \in I.$$

Consider

$$\begin{pmatrix} s \\ f(s) \\ \vdots \\ f^n(s) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & t & t^2 & \cdots & t^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t^n & t^{2n} & \cdots & t^{n^2} \end{pmatrix} \cdot \begin{pmatrix} s^0 \\ s^1 \\ \vdots \\ s^n \end{pmatrix}.$$

By assumption on t, the coefficient matrix is invertible; thus all s^i's are local sections of I and $I = \bigoplus_{i=0}^n I \cap E^i$. ◻

Let (V, ∇) be a flat module over Y. Start with an arbitrary Griffiths transverse filtration *Fil* of level n and consider the associated Higgs module $(\mathrm{Gr}_{Fil}(V), \theta)$. If $(\mathrm{Gr}_{Fil}(V), \theta)$ is unstable, let I_{Fil} be its maximal destabilizing subobject. By Lemma A.6,

$$I_{Fil} = \bigoplus_{i=0}^n I^i_{Fil}, \qquad I^i_{Fil} \subset Fil^i / Fil^{i+1} \subset V / Fil^{i+1}. \tag{A.6.1}$$

Following the construction of Simpson [34, §3], we define an operation ξ on the set of Griffiths transverse filtrations. The new filtration $\xi(Fil)$ of V is given by

$$\xi(Fil)^{i+1} := \mathrm{Ker}\left(V \to \frac{V/Fil^{i+1}}{I^i_{Fil}}\right) \quad \text{for } 0 \leq i \leq n, \qquad \xi(Fil)^0 = V. \tag{A.6.2}$$

Note that $Fil^i \supset \xi(Fil)^{i+1} \supset Fil^{i+1}$, and there is a short exact sequence

$$0 \to \mathrm{Gr}^i_{Fil}(V)/I^i_{Fil} \to \mathrm{Gr}^i_{\xi(Fil)}(V) \to I^{i-1}_{Fil} \to 0 \quad \text{for } 0 \leq i \leq n+1. \tag{A.6.3}$$

Altogether, we obtain a short exact sequence of graded Higgs modules

$$0 \to \mathrm{Gr}_{Fil}(V)/I_{Fil} \to \mathrm{Gr}_{\xi(Fil)}(V) \xrightarrow{h} I_{Fil}^{[1]} \to 0, \tag{A.6.4}$$

where $E^{[k]}$ denotes the graded Higgs module E with index shifted so that $(E^{[k]})^i = E^{i-k}$. If (E,θ) is unstable, let $\mu_{\max}(E)$ and $r_{\max}(E)$ denote respectively the slope and rank of the maximal destabilizing subobject of E; otherwise, let $\mu_{\max}(E) = \mu(E)$ and $r_{\max}(E) = \mathrm{rk}(E)$. By (A.6.4), we have

Lemma A.7. *The following statements hold:*

(1) $\mu_{\max}(\mathrm{Gr}_{\xi(Fil)}(V)) \leq \mu_{\max}(\mathrm{Gr}_{Fil}(V))$.
(2) *If* $\mu_{\max}(\mathrm{Gr}_{\xi(Fil)}(V)) = \mu_{\max}(\mathrm{Gr}_{Fil}(V))$*, then*

$$r_{\max}(\mathrm{Gr}_{\xi(Fil)}(V)) \leq r_{\max}(\mathrm{Gr}_{Fil}(V)).$$

(3) *If* $r_{\max}(\mathrm{Gr}_{\xi(Fil)}(V)) = r_{\max}(\mathrm{Gr}_{Fil}(V))$*, then the composite map* $I_{\xi(Fil)} \to \mathrm{Gr}_{\xi(Fil)}(V) \to I_{Fil}^{[1]}$ *is injective and is an isomorphism outside a codimension two closed subset of* Y*, where* $I_{\xi(Fil)}$ *is the maximal destabilizing subobject of* $\mathrm{Gr}_{\xi(Fil)}(V)$.

Proof. Use the exact sequence

$$0 \to \mathrm{Ker}(h) \cap I_{\xi(Fil)} \to I_{\xi(Fil)} \to h(I_{\xi(Fil)}) \to 0, \tag{A.7.1}$$

which is induced from (A.6.4), and where $h(I_{\xi(Fil)})$ is a subsheaf of $I_{Fil}^{[1]}$. □

The following lemma is the key to our theorem.

Lemma A.8. *Let Fil be a Griffiths transverse filtration of level n on a flat module* (V,∇)*. Assume* (V,∇) *is* ∇*-semistable. If the associated graded Higgs module* $(\mathrm{Gr}_{Fil}(V),\theta)$ *is unstable, then at least one of the following two strict inequalities holds:*

$$\mu_{\max}(\mathrm{Gr}_{\xi^{n+1}(Fil)}(V)) < \mu_{\max}(\mathrm{Gr}_{Fil}(V)), \qquad r_{\max}(\mathrm{Gr}_{\xi^{n+1}(Fil)}(V)) < r_{\max}(\mathrm{Gr}_{Fil}(V)).$$

Proof. Put $\mu_k = \mu_{\max}(\mathrm{Gr}_{\xi^k(Fil)}(V))$ and $r_k = r_{\max}(\mathrm{Gr}_{\xi^k(Fil)}(V))$ for $k \geq 0$. By Lemma A.7(1)–(2), (μ_k, r_k) decreases in the lexicographic ordering when k grows. Suppose on the contrary that $\mu_{n+1} = \mu_0$ and $r_{n+1} = r_0$. Then, for $0 \leq k \leq n$, $\mu_{k+1} = \mu_k$ and $r_{k+1} = r_k$, and, by Lemma A.7(3), $I_{\xi^{k+1}(Fil)} \subseteq I_{\xi^k(Fil)}^{[1]} \subseteq I_{Fil}^{[k+1]}$, which are locally free and coincide with each other away from a codimension two closed subset $Z \subset Y$. Hence they have the same slope.

A direct calculation on the short exact sequences (A.6.4) for $\mathrm{Gr}_{\xi^{k+1}(Fil)}(V)$, for k running from 0 to n, reveals the following fact: the short exact sequence (A.6.4) of *graded* Higgs modules for $\mathrm{Gr}_{\xi^{n+1}(Fil)}(V)$ takes a special form:

$$0 \to \bigoplus_{i=0}^{n} E^i \to \bigoplus_{i=0}^{2n+1} E^i \to \bigoplus_{i=n+1}^{2n+1} E^i \to 0, \tag{A.8.1}$$

where

$$\mathrm{Gr}_{\xi^{n+1}(Fil)}(V) = \mathrm{Gr}_{\xi^{n+1}(Fil)}(V) = \bigoplus_{i=0}^{2n+1} E^i,$$

and $\mathrm{Gr}_{\xi^n(Fil)}(V)/I_{\xi^n(Fil)} = \bigoplus_{i=0}^{n} E^i$ and $I^{[1]}_{\xi^n(Fil)} = \bigoplus_{i=n+1}^{2n+1} E^i$. Since over the open subset $U = Y - Z$,

$$I^{[1]}_{\xi^n(Fil)}|_U = I_{\xi^{n+1}(Fil)}|_U \subset \mathrm{Gr}_{\xi^{n+1}(Fil)}(V)|_U,$$

it follows that (A.6.4) splits over U as a sequence of graded Higgs modules. Thus,

$$\theta|_U(E^{n+1}|_U) = 0.$$

Hence, θ is indeed zero on E^{n+1}, which just means that $V' := (\xi^{n+1}(Fil))^{n+1}$ is ∇-invariant. On the other hand,

$$\mu(V') = \mu(\mathrm{Gr}_{\xi^{n+1}(Fil)}(V')) = \mu(I^{[1]}_{\xi^n(Fil)}) = \mu(I^{[n+1]}_{Fil}) = \mu(I_{Fil}) > \mu(V).$$

The strict inequality contradicts the ∇-semistability of (V, ∇). ◻

Proof of Theorem A.4. One takes an arbitrary Griffiths transverse filtration Fil of (V, ∇) (e.g. the trivial filtration), and then applies consecutively the operator ξ on Fil. The meaning of μ_k and r_k for $k \geq 0$ is the same as in the previous lemma. Lemma A.7 says that the pairs (μ_k, r_k), $k \geq 0$, decrease in the lexicographic ordering as k grows. So for a certain $k_0 \geq 0$, the sequence $\{(\mu_k, r_k)\}_{k \geq k_0}$ becomes constant. Then Lemma A.8 asserts that $\xi^{k_0}(Fil)$ has to be a gr-semistable filtration of (V, ∇). ◻

Remark A.9. In the proof of Theorem A.4, if we start with the trivial filtration $V = Fil^0 \supset Fil^1 = 0$, the resulting gr-semistable filtration has the extra property of being invariant under any automorphism and hence has the same definition field as (V, ∇). In a Griffiths transverse filtration $Fil : V = Fil^0 \supseteq Fil^1 \supseteq \cdots$ on (V, ∇), we call a term Fil^i, $i \geq 1$, *redundant* if $Fil^{i-1} = Fil^i$. One can remove all redundant terms from the filtration and shift the indices correspondingly so that the resulting filtration is a strictly decreasing filtration of the form $Fil_{\mathrm{red}} : V = Fil^0 \supsetneq Fil^1 \supsetneq Fil^2 \supsetneq \cdots$. We call this operation the *reduction* of a filtration, and a filtration is *reduced* if it is equal to its reduction. It is not hard to observe that $\mathrm{Gr}_{Fil}(V, \nabla)$ and $\mathrm{Gr}_{Fil_{\mathrm{red}}}(V, \nabla)$ are isomorphic as Higgs modules. Thus, the reduction of a gr-semistable filtration on (V, ∇) is again gr-semistable.

Proof of Theorem A.1. Note first that the inverse Cartier transform (V, ∇) of a Higgs module (E, θ) is ∇-semistable if and only if E is θ-semistable. This is a direct consequence of the equivalence theorem for the (inverse) Cartier transform of Ogus–Vologodsky [26]. So for a nilpotent semistable Higgs module of rank $\leq p$, its inverse Cartier transform is a ∇-semistable flat module of rank $\leq p$. By Theorem A.4 and Remark A.9, there exists a Griffiths transverse filtration Fil on (V, ∇) (of level $\leq p - 1$ for the reason of rank) such that $\mathrm{Gr}_{Fil}(V, \nabla)$ is nilpotent semistable of the same rank $\leq p$ and defined over the same ground field of (E, θ). Therefore, we can obtain a semistable Higgs–de Rham flow with leading term (E, θ) by applying the previous construction inductively. This completes the proof. ◻

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References

[1] Deligne, P.: Un théorème de finitude pour la monodromie. In: Discrete Groups in Geometry and Analysis (New Haven, CN, 1984), Progr. Math. 67, Birkhäuser Boston, Boston, MA, 1–19 (1987) Zbl 0656.14010 MR 0900821

[2] Deninger, C., Werner, A.: Vector bundles on p-adic curves and parallel transport. Ann. Sci. École Norm. Sup. **38**, 553–597 (2005) Zbl 1087.14026 MR 2172951

[3] Emerton, M., Kisin, M.: The Riemann–Hilbert Correspondence for Unit F-crystals. Astérisque **293**, vi+257 pp. (2004) Zbl 1056.14025 MR 2071510

[4] Faltings, G.: Crystalline cohomology and p-adic Galois-representations. In: Algebraic Analysis, Geometry, and Number Theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 25–80 (1989) Zbl 0805.14008 MR 1463696

[5] Faltings, G.: Integral crystalline cohomology over very ramified valuation rings. J. Amer. Math. Soc. **12**, 117–144 (1999) Zbl 0914.14009 MR 1618483

[6] Faltings, G.: A p-adic Simpson correspondence. Adv. Math. **198**, 847–862 (2005) Zbl 1102.14022 MR 2183394

[7] Fontaine, J.-M., Laffaille, G.: Construction de représentations p-adiques. Ann. Sci. École Norm. Sup. **15**, 547–608 (1982) Zbl 0579.14037 MR 0707328

[8] Fontaine, J.-M., Messing, W.: p-adic periods and p-adic étale cohomology. In: Current Trends in Arithmetical Algebraic Geometry (Arcata, CA, 1985), Contemp. Math. 67, Amer. Math. Soc., Providence, RI, 179–207 (1987) Zbl 0632.14016 MR 0902593

[9] Griffiths, P.: Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. Inst. Hautes Études Sci. Publ. Math. **38**, 125–180 (1970) Zbl 0212.53503 MR 0282990

[10] Hitchin, N.: The self-duality equations on a Riemann surface. Proc. London Math. Soc. (3) **55**, 59–126 (1987) Zbl 0634.53045 MR 0887284

[11] Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves. Aspects of Math. E31, Vieweg (1997) Zbl 0872.14002 MR 1450870

[12] Katz, N.: Travaux de Dwork. In: Séminaire Bourbaki, 24ème année (1971/1972), exp. 409, Lecture Notes in Math. 317, Springer, Berlin, 167–200 (1973) Zbl 0259.14007 MR 0498577

[13] Katz, N.: p-adic properties of modular schemes and modular forms. In: Modular Functions of One Variable, III (Antwerp, 1972), Lecture Notes in Math. 350, Springer, Berlin, 69–190 (1973) Zbl 0271.10033 MR 0447119

[14] Lan, G.-T., Sheng, M., Yang, Y.-H., Zuo, K.: Semistable Higgs bundles of small ranks are strongly Higgs semistable. arXiv:1311.2405 (2013)

[15] Lan, G.-T., Sheng, M., Yang, Y.-H., Zuo, K.: Uniformization of p-adic curves via Higgs–de Rham flows. J. Reine Angew. Math. **747**, 63–108 (2019) Zbl 07012668 MR 3905130

[16] Lan, G.-T., Sheng, M., Zuo, K.: Nonabelian Hodge theory in positive characteristic via exponential twisting. Math. Res. Lett. **22**, 859–879 (2015) Zbl 1326.14016 MR 3350108

[17] Lan, G.-T., Sheng, M., Zuo, K.: Semistable Higgs bundles and representations of algebraic fundamental groups: Positive characteristic case. arXiv:1210.8280v1 (2012)

[18] Lange, H., Stuhler, U.: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe. Math. Z. **156**, 73–83 (1977) Zbl 0349.14018 MR 0472827

[19] Langer, A.: Semistable sheaves in positive characteristic. Ann. of Math. **159**, 251–276 (2004) Zbl 1080.14014 MR 2051393

[20] Langer, A.: Moduli spaces of sheaves in mixed characteristic. Duke Math. J. **124**, 571–586 (2004) Zbl 1086.14036 MR 2085175

[21] Langer, A.: Semistable modules over Lie algebroids in positive characteristic. Documenta Math. **19**, 509–540 (2014) Zbl 1330.14017 MR 3218782

[22] Langer, A.: Bogomolov's inequality for Higgs sheaves in positive characteristic. Invent. Math. **199**, 889–920 (2015) Zbl 1348.14048 MR 3314517

[23] Li, L.-G.: On a conjecture of Lan–Sheng–Zuo on semistable Higgs bundles: rank 3 case. Int. J. Math. **25**, no. 2, art. 1450013, 11 pp. (2014) Zbl 1286.14049 MR 3189770

[24] Mochizuki, S.: A theory of ordinary p-adic curves. Publ. RIMS Kyoto Univ. **32**, 957–1152 (1996) Zbl 0879.14009 MR 1437328

[25] Ogus, A.: F-crystals, Griffiths Transversality, and the Hodge Decomposition. Astérisque **221** (1994) Zbl 0801.14004 MR 1280543

[26] Ogus, A., Vologodsky, V.: Nonabelian Hodge theory in characteristic p. Inst. Hautes Études Sci. Publ. Math. **106**, 1–138 (2007) Zbl 1140.14007 MR 2373230

[27] Sheng, M., Xin, H., Zuo, K.: A note on the characteristic p nonabelian Hodge theory in the geometric case. Int. J. Math. **26**, no. 1, art. 1550011, 18 pp. (2015) Zbl 1331.14026 MR 3313656

[28] Sheng, M., Zhang, J.-J., Zuo, K.: Higgs bundles over the good reduction of a quaternionic Shimura curve. J. Reine Angew. Math. **671**, 223–248 (2012) Zbl 1272.14025

[29] Sheng, M., Zuo, K.: Periodic Higgs subbundles in positive and mixed characteristic. arXiv:1206.4865v1 (2012)

[30] Shiho, A.: Notes on generalizations of local Ogus–Vologodsky correspondence. J. Math. Sci. Univ. Tokyo **22**, 793–875 (2015) Zbl 1332.14032 MR 3408075

[31] Simpson, C.: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization. J. Amer. Math. Soc. **1**, 867–918 (1988) Zbl 0669.58008 MR 0944577

[32] Simpson, C.: Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math. **75**, 5–95 (1992) Zbl 0814.32003 MR 1179076

[33] Simpson, C.: Moduli of representations of the fundamental group of a smooth projective variety. I, II. Inst. Hautes Études Sci. Publ. Math. **79**, 47–129 (1994), **80**, 5–79 (1995) Zbl 0891.14006 MR 1320603

[34] Simpson, C.: Iterated destabilizing modifications for vector bundles with connections. In: Vector Bundles and Complex Geometry, Contemp. Math. 522, Amer. Math. Soc., Providence, RI, 183–206 (2010) Zbl 1215.14037 MR 2681730

[35] Weil, A.: Généralisation des fonctions abéliennes. J. Math. Pures Appl. **17**, 47–87 (1938) Zbl 0018.06302