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Representability theorem in derived analytic geometry

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Abstract. We prove the representability theorem in derived analytic geometry. The theorem asserts that an analytic moduli functor is a derived analytic stack if and only if it is compatible with Postnikov towers, has a global analytic cotangent complex, and its truncation is an analytic stack. Our result applies to both derived complex analytic geometry and derived nonarchimedean analytic geometry (rigid analytic geometry). The representability theorem is of both philosophical and practical importance in derived geometry. The conditions of representability are natural expectations for a moduli functor. So the theorem confirms that the notion of derived analytic space is natural and sufficiently general. On the other hand, the conditions are easy to verify in practice. So the theorem enables us to enhance various classical moduli spaces with derived structures, thus providing plenty of down-to-earth examples of derived analytic spaces. For the purpose of proof, we study analytification, square-zero extensions, analytic modules and cotangent complexes in the context of derived analytic geometry. We will explore applications of the representability theorem in our subsequent works. In particular, we will establish the existence of derived mapping stacks via the representability theorem.

Keywords. Representability, deformation theory, analytic cotangent complex, derived geometry, rigid analytic geometry, complex geometry, derived stacks

Contents

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1. Introduction

Derived algebraic geometry is a far reaching enhancement of algebraic geometry. We refer to Toën [23] for an overview, and to Lurie [5, 8] and Toën–Vezzosi [24, 25] for foundational works. A fundamental result in derived algebraic geometry is Lurie's representability theorem. It gives sufficient and necessary conditions for a moduli functor to possess the structure of a derived algebraic stack. The representability theorem enables us to enrich various classical moduli spaces with derived structures, thus bringing derived geometry into the study of important moduli problems. Examples include derived Picard schemes, derived Hilbert schemes, Weil restrictions, derived Betti moduli spaces, derived de Rham moduli spaces, and derived Dolbeault moduli spaces . [11, 5, 13, 20, 21, 3].

Algebraic geometry is intimately related to analytic geometry. In [7], Lurie proposed a framework for derived complex analytic geometry. In [19], we started to develop the foundation for derived nonarchimedean analytic geometry. Our motivation comes from enumerative problems in the study of mirror symmetry of Calabi–Yau manifolds. We refer to the introduction of [19] for a more detailed discussion on the motivations. Our results in [19] include the existence of fiber products, and a comparison theorem between discrete derived analytic spaces and nonarchimedean analytic Deligne–Mumford stacks.

As in the algebraic case, the theory of derived analytic geometry cannot be useful without a representability theorem. So we establish the representability theorem in derived analytic geometry in this paper. We cover both the complex analytic case and the nonarchimedean analytic case using a unified approach. Let us now state our main result:

Theorem 1.1 (Representability, cf. Theorem 7.1). *Let F be a stack over the étale site of derived analytic spaces. The following are equivalent:*

(1) *F is a derived analytic stack.*
(2) *F is compatible with Postnikov towers, has a global analytic cotangent complex, and its truncation $\mathrm{t}_0(F)$ is an (underived) analytic stack.*

As in derived algebraic geometry, the representability theorem is of both philosophical and practical importance. Since the conditions in Theorem 1.1(2) are natural expectations for a moduli functor F, the theorem confirms that our notion of derived analytic space is natural and sufficiently general. On the other hand, these conditions are easy to verify in practice. So Theorem 1.1 provides us at the same time with plenty of down-to-earth examples of derived analytic spaces.

The main ingredient in the proof of the representability theorem is *derived analytic deformation theory*, which we develop in this paper. Central to this theory is the notion of *analytic cotangent complex*. Although this concept is similar to its algebraic counterpart, new ideas are needed in the analytic setting, especially in the nonarchimedean case when the ground field has positive characteristic.

Let us give an informal account of the ideas involved. Intuitively, a derived analytic space is a topological space equipped with a sheaf of derived analytic rings. A derived analytic ring is a derived ring (e.g. a simplicial commutative ring) equipped with an extra analytic structure. The analytic structure consists of information about norms, convergence of power series, as well as composition rules for convergent power series. In [7, 19], this heuristic idea is made precise using the theory of pregeometry and structured topos introduced by Lurie [8] (we recall it in Section 2). All analytic information is encoded in a pregeometry $\mathcal{T}_{\mathrm{an}}(k)$, where k is either $\mathbb{C}$ or a nonarchimedean field. Then a derived analytic space X is a pair $(\mathcal{X}, \mathcal{O}_X)$ consisting of an ∞-topos $\mathcal{X}$ and a $\mathcal{T}_{\mathrm{an}}(k)$-structure $\mathcal{O}_X$ on $\mathcal{X}$ satisfying some local finiteness condition (cf. Definition 2.3). One should think of $\mathcal{X}$ as the underlying topological space, and $\mathcal{O}_X$ as the structure sheaf. A derived analytic ring is formally defined as a $\mathcal{T}_{\mathrm{an}}(k)$-structure on a point.

Intuitively, the analytic cotangent complex of a derived analytic space represents the derived cotangent spaces. We will construct it via the space of derivations. Recall that given a k-algebra A and an A-module M, a *derivation* from A into M is a k-linear map $d\colon A \to M$ satisfying the Leibniz rule

$$d(ab) = bd(a) + ad(b).$$

In the context of derived analytic geometry, we take A to be a derived analytic ring. Let A^{alg} denote the underlying derived ring of A, obtained by forgetting the analytic structure. We define A-modules to be simply A^{alg}-modules, (we will see later that this is a reasonable definition.) Let M be an A-module and we want to define analytic derivations from A into M. However, the Leibniz rule above is problematic in derived analytic geometry for two reasons.

The first problem concerns analytic geometry. It follows from the Leibniz rule that for any $a \in A$ and any polynomial f in one variable, we have

$$d(f(a)) = f'(a)d(a).$$

In analytic geometry, it is natural to demand the same formula not only for polynomials but also for every convergent power series f. This means that we have to add infinitely many new rules.

The second problem concerns derived geometry. For derived rings, we are obliged to demand the Leibniz rule up to homotopy. This results in an infinite chain of higher homotopies and becomes impossible to manipulate.

In order to solve the two problems, note that in the classical case, a derivation from A into M is equivalent to a section of the projection from the split square-zero extension $A \oplus M$ to A. So we can reduce the problem of formulating the Leibniz rule involving convergent power series as well as higher homotopies to the problem of constructing split square-zero extensions of derived analytic rings. In other words, given a derived analytic ring A and an A-module M, we would like to construct a derived analytic ring structure on the direct sum $A \oplus M$.

For this purpose, we need to interpret the notion of A-module in a different way, which is the content of the following theorem.

Theorem 1.2 (Reinterpretation of modules, cf. Theorem 4.5). *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space. We have an equivalence of stable ∞-categories*

$$\mathcal{O}_X\text{-Mod} \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X})),$$

where $\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}$ denotes the ∞-category of sheaves of derived k-analytic rings on $\mathcal{X}$ over $\mathcal{O}_X$, $\mathrm{Ab}(-)$ denotes the ∞-category of abelian group objects, and $\mathrm{Sp}(-)$ denotes the ∞-category of spectrum objects.

We have natural functors

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}))) \xrightarrow{\Omega^\infty} \mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}) \xrightarrow{U} \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}.$$

We will show that given $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$, the underlying sheaf of derived rings of $U(\Omega^\infty(\mathcal{F}))$ is equivalent to the algebraic split square-zero extension of $\mathcal{O}_X^{\mathrm{alg}}$ by $\mathcal{F}$ (cf. Corollary 5.16). So we define $U(\Omega^\infty(\mathcal{F}))$ to be the *analytic split square-zero extension* of $\mathcal{O}_X$ by $\mathcal{F}$, and we denote it by $\mathcal{O}_X \oplus \mathcal{F}$.

Theorem 1.2 also confirms that our definition of module over a derived analytic ring A as A^{alg}-module is reasonable because it can be reinterpreted in a purely analytic way without forgetting the analytic structure.

Let us now explain the necessity of taking abelian group objects in the statement of Theorem 1.2. Given an $\mathbb{E}_\infty$-ring A, the ∞-category of A-modules is equivalent to the ∞-category $\mathrm{Sp}(\mathbb{E}_\infty\text{-Ring}_{/A})$, where $\mathbb{E}_\infty\text{-Ring}_{/A}$ denotes the ∞-category of $\mathbb{E}_\infty$-rings over A (cf. [12, 7.3.4.14]). However, our approach to derived analytic geometry via structured topoi is simplicial in nature. For a simplicial commutative ring A, the ∞-category of A-modules is in general not equivalent to the ∞-category $\mathrm{Sp}(\mathrm{CRing}_{/A})$, where $\mathrm{CRing}_{/A}$ denotes the ∞-category of simplicial commutative rings over A. This problem can be solved by taking abelian group objects before taking spectrum objects. More precisely, in Section 8.1, for any simplicial commutative ring A, we prove the following equivalence of stable ∞-categories:

$$A\text{-Mod} \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_{/A})). \tag{1.3}$$

The proof of Theorem 1.2 is rather involved. Let us give a quick outline for the convenience of the reader: By (1.3), we are reduced to proving the equivalence

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X})) \xrightarrow{\sim} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}})). \tag{1.4}$$

The functor above is induced by the underlying algebra functor forgetting the analytic structure

$$(-)^{\mathrm{alg}} \colon \mathrm{AnRing}_k(\mathcal{X}) \to \mathrm{CRing}_k(\mathcal{X}).$$

Via a series of reduction steps in Section 4.2, we can deduce (1.4) from the equivalence

$$(-)^{\mathrm{alg}} \colon \mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X} \simeq \mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}.$$

In Section 4.3, we make a further reduction to the case of a point, i.e. when $\mathcal{X}$ is the ∞-category of spaces, $\mathcal{S}$. The proof is completed in Section 4.4 via flatness arguments.

With the preparations above, we are ready to introduce the notions of analytic derivation and analytic cotangent complex.

Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space, and let $\mathcal{A} \in \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}$ and $\mathcal{F} \in \mathcal{O}_X\text{-}\mathrm{Mod}^{\geq 0}$. The space of *$\mathcal{A}$-linear analytic derivations from $\mathcal{O}_X$ into $\mathcal{F}$* is by definition

$$\mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{F}) \coloneqq \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{O}_X \oplus \mathcal{F}).$$

In Section 5.2, we show that the functor $\mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}_X, -)$ is representable by an $\mathcal{O}_X$-module which we denote by $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{A}}$, and call the *analytic cotangent complex* of $\mathcal{O}_X/\mathcal{A}$. For a map $f \colon X \to Y$ of derived analytic spaces, we define its analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ to be $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

Important properties of the analytic cotangent complex are established in Section 5, and we summarize them in the following theorem:

Theorem 1.5 (Properties of the analytic cotangent complex). *The analytic cotangent complex satisfies the following properties:*

(1) *For any map $f \colon X \to Y$ of derived analytic spaces, the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is connective and coherent.*

(2) *For any sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of maps, we have a fiber sequence*

$$f^*\mathbb{L}^{\mathrm{an}}_{Y/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Y}.$$

(3) *For any pullback square of derived analytic spaces*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y \end{array}$$

we have a canonical equivalence

$$\mathbb{L}^{\mathrm{an}}_{X'/Y'} \simeq g^*\mathbb{L}^{\mathrm{an}}_{X/Y}.$$

(4) *For any derived algebraic Deligne–Mumford stack X locally almost of finite presentation over k, its analytification X^{an} is a derived analytic space* (*cf.* *Section* 3). *We have a canonical equivalence*

$$(\mathbb{L}_X)^{\mathrm{an}} \simeq \mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}}.$$

(5) *For any closed immersion $X \hookrightarrow Y$ of derived analytic spaces, we have a canonical equivalence*

$$\mathbb{L}^{\mathrm{an}}_{X/Y} \simeq \mathbb{L}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}.$$

(6) (Analytic Postnikov tower) *For any derived analytic space X, every $n \geq 0$, the canonical map $\mathrm{t}_{\leq n}(X) \hookrightarrow \mathrm{t}_{\leq n+1}(X)$ is an analytic square-zero extension. In other words, there exists an analytic derivation*

$$d\colon \mathbb{L}^{\mathrm{an}}_{\tau_{\leq n}\mathcal{O}_X} \to \pi_{n+1}(\mathcal{O}_X)[n+2]$$

such that the square

$$\begin{array}{ccc} \tau_{\leq n+1}\mathcal{O}_X & \longrightarrow & \tau_{\leq n}\mathcal{O}_X \\ \downarrow & & \downarrow{\scriptstyle \eta_d} \\ \tau_{\leq n}\mathcal{O}_X & \xrightarrow{\eta_0} & \tau_{\leq n}\mathcal{O}_X \oplus \pi_{n+1}(\mathcal{O}_X)[n+2] \end{array}$$

is a pullback, where η_d is the map associated to the derivation d and η_0 is the map associated to the zero derivation.

(7) *A morphism $f\colon X \to Y$ of derived analytic spaces is smooth if and only if its truncation $\mathrm{t}_0(f)$ is smooth and the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is perfect and in tor-amplitude* 0.

Properties (1)–(7) correspond respectively to Corollary 5.40, Proposition 5.10, Proposition 5.27, Theorem 5.21, Corollary 5.33, Corollary 5.44 and Proposition 5.50.

Using properties (2), (4) and (5), we can compute the analytic cotangent complex of any derived analytic space via local embeddings into affine spaces.

In Section 6, we use the analytic Postnikov tower decomposition to construct pushouts of derived analytic spaces along closed immersions:

Theorem 1.6 (Gluing along closed immersions, cf. Theorem 6.5). *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle j}\downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout square of $\mathcal{T}_{\mathrm{an}}(k)$-structured topoi. Suppose that i and j are closed immersions and X, X', Y are derived analytic spaces. Then Y' is also a derived analytic space.

In other words, the theorem asserts that derived analytic spaces can be glued together along closed immersions. In particular, it has the following important consequence:

Corollary 1.7 (Representability of analytic square-zero extensions). *Let X be a derived analytic space and let $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$. Let $X[\mathcal{F}]$ be the analytic split square-zero extension of X by $\mathcal{F}$. Let $i_d \colon X[\mathcal{F}] \to X$ be the map associated to an analytic derivation d of $\mathcal{O}_X$ into $\mathcal{F}$. Let $i_0 \colon X[\mathcal{F}] \to X$ be the map associated to the zero derivation. Then the pushout*

$$\begin{array}{ccc} X[\mathcal{F}] & \xrightarrow{i_d} & X \\ {\scriptstyle i_0}\downarrow & & \downarrow \\ X & \longrightarrow & X_d[\mathcal{F}] \end{array}$$

is a derived analytic space.

The corollary gives one more evidence that our notions of analytic derivation and analytic cotangent complex are correct. If we replace d by an algebraic derivation, the pushout will no longer be a derived analytic space in general.

Now let us give a sketch of the proof of Theorem 1.1, the main theorem of this paper.

The implication (1)⇒(2) is worked out in Section 7.1. We first prove that (2) holds for derived analytic spaces. We deduce it from the various properties of the analytic cotangent complex explained above as well as the gluing along closed immersions. Then we prove (2) for derived analytic stacks by induction on the geometric level.

The proof of (2)⇒(1) is more involved. By induction on the geometric level of the truncation $\mathrm{t}_0(F)$, it suffices to lift a smooth atlas $U_0 \to \mathrm{t}_0(F)$ of $\mathrm{t}_0(F)$ to a smooth atlas $\widetilde{U} \to F$ of F. To obtain such a lifting, we proceed by constructing successive approximations:

$$U_0 \xhookrightarrow{j_0} U_1 \xhookrightarrow{j_1} \cdots \xhookrightarrow{j_{m-1}} U_m \xhookrightarrow{j_m} \cdots, \qquad u_0\colon U_0 \to F,\ u_1\colon U_1 \to F,\ \dots,\ u_m\colon U_m \to F$$

where

- U_m is m-truncated;
- $U_m \to U_{m+1}$ induces an equivalence on m-truncations;
- $\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat to order $m+1$ (cf. Definition 8.4).

The construction goes by induction on m. The notion of flatness to order n is the key idea behind the induction step. Indeed, combining the fact that $\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat to order $m+1$ with the fact that U_m is m-truncated guarantees that the truncation $\tau_{\leq m}\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat as a sheaf on U_m. It is not hard to deduce from this that $\tau_{\leq m}\mathbb{L}^{\mathrm{an}}_{U_m/F}$ must be perfect. Hence, we can choose a splitting

$$\mathbb{L}^{\mathrm{an}}_{U_m/F} \simeq \tau_{\leq m}\mathbb{L}^{\mathrm{an}}_{U_m/F} \oplus \tau_{\geq m+1}\mathbb{L}^{\mathrm{an}}_{U_m/F}.$$

The choice of the splitting determines the passage to the next level of approximation. We remark that the splitting above is in general not unique, and thus the choice of the lifting $\widetilde{U} \to F$ of $U_0 \to \mathrm{t}_0(F)$ is not unique. When F is Deligne–Mumford, the lifting is unique, in other words, an atlas of $\mathrm{t}_0(F)$ canonically determines an atlas of F.

To complete the proof, we set $\widetilde{U} := \operatorname{colim}_m U_m$. The construction above guarantees that $U_m \simeq \mathrm{t}_{\leq m}(\widetilde{U})$. Since F is compatible with Postnikov towers, we obtain a canonical map $\widetilde{U} \to F$. The induction hypothesis on the geometric level of F guarantees that this map is representable by geometric stacks. In order to check that the map $\widetilde{U} \to F$ is also smooth, we use an infinitesimal lifting property that we establish in Proposition 7.11.

Finally, we stress that our approach to the representability theorem in derived analytic geometry is by no means a simple repetition of the proof of the representability theorem in derived algebraic geometry. As explained above, the presence of the extra analytic structure has obliged us to take different paths at various stages. This also leads to a more conceptual understanding of the proof of the representability theorem in derived algebraic geometry.

We will explore applications of the representability theorem in our subsequent works. In particular, we will establish the existence of derived mapping stacks via the representability theorem (cf. [18]).

Notations and terminology. We refer to [19] for derived nonarchimedean analytic geometry, and to [7] for derived complex analytic geometry. We give a unified review of the basic notions in Section 2.

The letter k denotes either the field $\mathbb{C}$ of complex numbers or a nonarchimedean field with a nontrivial valuation. By *k-analytic spaces* (or simply *analytic spaces*), we mean complex analytic spaces when $k = \mathbb{C}$, and rigid k-analytic spaces when k is nonarchimedean.

We denote by An_k the category of k-analytic spaces, and by dAn_k the ∞-category of derived k-analytic spaces. We denote by Afd_k the category of k-affinoid spaces when k is nonarchimedean, and the category of Stein spaces when $k = \mathbb{C}$. We denote by dAfd_k the ∞-category of derived affinoid spaces when k is nonarchimedean, and the ∞-category of derived Stein spaces when $k = \mathbb{C}$.

For $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{A}^n_k$ the algebraic n-dimensional affine space over k, by $\mathbf{A}^n_k$ the analytic n-dimensional affine space over k, and by $\mathbf{D}^n_k$ the n-dimensional closed unit polydisk over k.

For an ∞-topos $\mathcal{X}$, we denote by $\mathrm{AnRing}_k(\mathcal{X})$ the ∞-category of sheaves of derived k-analytic rings over $\mathcal{X}$, and by $\mathrm{CRing}_k(\mathcal{X})$ the ∞-category of sheaves of simplicial commutative k-algebras over $\mathcal{X}$.

We denote by $\mathcal{S}$ the ∞-category of spaces. An ∞-site $(\mathcal{C}, \tau)$ consists of a small ∞-category $\mathcal{C}$ equipped with a Grothendieck topology τ. A stack over an ∞-site $(\mathcal{C}, \tau)$ is by definition a hypercomplete sheaf with values in $\mathcal{S}$ over the ∞-site (cf. [17, §2]). We denote by $\mathrm{St}(\mathcal{C}, \tau)$ the ∞-category of stacks over $(\mathcal{C}, \tau)$.

Throughout this paper, we use homological indexing conventions, i.e., the differential in chain complexes lowers the degree by 1.

A commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & \mathcal{C}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathcal{D} & \xrightarrow{q} & \mathcal{D}' \end{array}$$

is called *left adjointable* if the functors g and g' have left adjoints $f: \mathcal{D} \to \mathcal{C}$, $f': \mathcal{D}' \to \mathcal{C}'$ and the push-pull transformation

$$\gamma: f' \circ q \to p \circ f$$

is an equivalence (cf. [6, 7.3.1.1]).

2. Basic notions of derived analytic geometry

In this section we review the basic notions of derived complex analytic geometry and derived nonarchimedean geometry in a unified framework.

First we recall the notions of pregeometry and structured topos introduced by Lurie [8].

Definition 2.1 ([8, 3.1.1]). A *pregeometry* is an ∞-category $\mathcal{T}$ equipped with a class of *admissible* morphisms and a Grothendieck topology generated by admissible morphisms, satisfying the following conditions:

(i) The ∞-category $\mathcal{T}$ admits finite products.
(ii) The pullback of an admissible morphism along any morphism exists.
(iii) The class of admissible morphisms is closed under composition, pullback and retract. Moreover, for morphisms f and g, if g and $g \circ f$ are admissible, then f is admissible.

Definition 2.2 ([8, 3.1.4]). Let $\mathcal{T}$ be a pregeometry, and let $\mathcal{X}$ be an ∞-topos. Then a $\mathcal{T}$-*structure* on $\mathcal{X}$ is a functor $\mathcal{O}: \mathcal{T} \to \mathcal{X}$ with the following properties:

(i) The functor $\mathcal{O}$ preserves finite products.
(ii) Suppose

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

is a pullback diagram in $\mathcal{T}$, where f is admissible. Then the induced diagram

$$\begin{array}{ccc} \mathcal{O}(U') & \longrightarrow & \mathcal{O}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X') & \longrightarrow & \mathcal{O}(X) \end{array}$$

is a pullback square in $\mathcal{X}$.
(iii) Let $\{U_\alpha \to X\}$ be a covering in $\mathcal{T}$ consisting of admissible morphisms. Then the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \to \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$.

A morphism of $\mathcal{T}$-structures $\mathcal{O} \to \mathcal{O}'$ on $\mathcal{X}$ is *local* if for every admissible morphism $U \to X$ in $\mathcal{T}$, the resulting diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}'(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \mathcal{O}'(X) \end{array}$$

is a pullback square in $\mathcal{X}$. We denote by $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ the ∞-category of $\mathcal{T}$-structures on $\mathcal{X}$ with local morphisms.

A $\mathcal{T}$-*structured* ∞-*topos* X is a pair $(\mathcal{X}, \mathcal{O}_X)$ consisting of an ∞-topos $\mathcal{X}$ and a $\mathcal{T}$-structure $\mathcal{O}_X$ on $\mathcal{X}$. We denote by ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ the ∞-category of $\mathcal{T}$-structured ∞-topoi (cf. [8, Definition 1.4.8]). Note that a 1-morphism $f\colon (\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ consists of a geometric morphism of ∞-topoi $f_*\colon \mathcal{X} \rightleftarrows \mathcal{Y} : f^{-1}$ and a local morphism of $\mathcal{T}$-structures $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Let k denote either the field $\mathbb{C}$ of complex numbers or a complete nonarchimedean field with nontrivial valuation. We introduce three pregeometries $\mathcal{T}_{\mathrm{an}}(k)$, $\mathcal{T}_{\mathrm{disc}}(k)$ and $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ that are relevant to derived analytic geometry.

The pregeometry $\mathcal{T}_{\mathrm{an}}(k)$ is defined as follows:

(i) The underlying category of $\mathcal{T}_{\mathrm{an}}(k)$ is the category of smooth k-analytic spaces.
(ii) A morphism in $\mathcal{T}_{\mathrm{an}}(k)$ is admissible if and only if it is étale.
(iii) The topology on $\mathcal{T}_{\mathrm{an}}(k)$ is the étale topology. (Note that in the complex analytic case, the étale topology is equivalent to the usual analytic topology.)

The pregeometry $\mathcal{T}_{\mathrm{disc}}(k)$ is defined as follows:

(i) The underlying category of $\mathcal{T}_{\mathrm{disc}}(k)$ is the full subcategory of the category of k-schemes spanned by affine spaces $\mathbb{A}^n_k$.
(ii) A morphism in $\mathcal{T}_{\mathrm{disc}}(k)$ is admissible if and only if it is an isomorphism.
(iii) The topology on $\mathcal{T}_{\mathrm{disc}}(k)$ is the trivial topology, i.e. a collection of admissible morphisms is a covering if and only if it is nonempty.

The pregeometry $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ is defined as follows:

(i) The underlying category of $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ is the category of smooth k-schemes.
(ii) A morphism in $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ is admissible if and only if it is étale.
(iii) The topology on $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ is the étale topology.

We have a natural functor $\mathcal{T}_{\mathrm{disc}}(k) \to \mathcal{T}_{\mathrm{an}}(k)$ induced by analytification. Composing with this functor, we obtain an "algebraization" functor

$$(-)^{\mathrm{alg}}\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

In virtue of [8, Example 3.1.6, Remark 4.1.2], we have an equivalence induced by evaluation on the affine line,

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{CRing}_k}(\mathcal{X}),$$

where $\mathrm{Sh}_{\mathrm{CRing}_k}(\mathcal{X})$ denotes the ∞-category of sheaves on $\mathcal{X}$ with values in the ∞-category of simplicial commutative k-algebras.

Definition 2.3. A *derived k-analytic space X* is a $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_X)$ such that $\mathcal{X}$ is hypercomplete and there exists an effective epimorphism from $\coprod_i U_i$ to a final object of $\mathcal{X}$ satisfying the following conditions, for every index i:

(i) The pair $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_X^{\mathrm{alg}}|U_i))$ is equivalent to the ringed ∞-topos associated to the étale site of a k-analytic space X_i.

(ii) For each $j \geq 0$, $\pi_j(\mathcal{O}_X^{\mathrm{alg}}|U_i)$ is a coherent sheaf of $\pi_0(\mathcal{O}_X^{\mathrm{alg}}|U_i)$-modules on X_i.

We denote by dAn_k the full subcategory of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ spanned by derived k-analytic spaces.

Definition 2.4. When k is nonarchimedean, a *derived k-affinoid space* is by definition a derived k-analytic space $(\mathcal{X}, \mathcal{O}_X)$ whose truncation $(\mathcal{X}, \pi_0(\mathcal{O}_X))$ is a k-affinoid space. A *derived Stein space* is by definition a derived $\mathbb{C}$-analytic space whose truncation is a Stein space. We denote the ∞-category of derived k-affinoid (resp. Stein) spaces by dAfd_k (resp. $\mathrm{dAfd}_{\mathbb{C}}$).

3. Derived analytification

In this section, we study the analytification of derived algebraic Deligne–Mumford stacks.

Let ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote the full subcategory of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ spanned by $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi whose underlying ∞-topos is hypercomplete. By [19, Lemma 2.8], the inclusion ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \hookrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ admits a right adjoint $\mathrm{Hyp}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$.

By analytification, we have a transformation of pregeometries

$$(-)^{\mathrm{an}}\colon \mathcal{T}_{\mathrm{\acute{e}t}}(k) \to \mathcal{T}_{\mathrm{an}}(k).$$

Precomposition with $(-)^{\mathrm{an}}$ induces a forgetful functor

$$(-)^{\mathrm{alg}}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k)),$$

which admits a right adjoint in virtue of [8, Theorem 2.1.1]. Composing with the right adjoint $\mathrm{Hyp}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, we obtain a functor

$${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k)) \to {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

We call this functor the *derived analytification functor*, and we denote it by $(-)^{\mathrm{an}}$ again. This notation is justified by the lemma below.

Lemma 3.1. (1) *The diagram*

$$\begin{array}{ccc} \mathcal{T}_{\mathrm{\acute{e}t}}(k) & \xrightarrow{(-)^{\mathrm{an}}} & \mathcal{T}_{\mathrm{an}}(k) \\ \downarrow & & \downarrow \\ {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k)) & \xrightarrow{(-)^{\mathrm{an}}} & {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \end{array}$$

commutes.

(2) *Let* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\text{ét}}(k))$ (*resp.* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\mathrm{an}}(k))$) *be the full subcategory of* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$ (*resp.* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$) *spanned by those* $(\mathcal{X}, \mathcal{O}_X)$ *such that* $\mathcal{O}_X$ *is* n*-truncated. The diagram*

$$\begin{array}{ccc} {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k)) & \xleftarrow{(-)^{\mathrm{alg}}} & {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \\ \uparrow & & \uparrow \\ {}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\text{ét}}(k)) & \xleftarrow{(-)^{\mathrm{alg}}} & {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\mathrm{an}}(k)) \end{array}$$

commutes, and the vertical arrows are left adjoint to the truncation functor $\mathrm{t}_{\leq n}$.

(3) *The functor* $(-)^{\mathrm{alg}} \colon {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\text{ét}}(k))$ *admits a right adjoint which we denote by* Ψ_n*, and moreover the diagram*

$$\begin{array}{ccc} \mathrm{Sch}(\mathcal{T}_{\text{ét}}(k)) & \xrightarrow{(-)^{\mathrm{an}}} & {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \\ \downarrow{\scriptstyle \mathrm{t}_{\leq n}} & & \downarrow{\scriptstyle \mathrm{t}_{\leq n}} \\ \mathrm{Sch}^{\leq n}(\mathcal{T}_{\text{ét}}(k)) & \xrightarrow{\Psi_n} & {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\mathrm{an}}(k)) \end{array}$$

is left adjointable, where $\mathrm{Sch}^{\leq n}(\mathcal{T}_{\text{ét}}(k)) := \mathrm{Sch}(\mathcal{T}_{\text{ét}}(k)) \cap {}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq n}(\mathcal{T}_{\text{ét}}(k))$*, and* $\mathrm{Sch}(\mathcal{T}_{\text{ét}}(k))$ *denotes the* ∞*-category of* $\mathcal{T}_{\text{ét}}(k)$*-schemes* (*cf.* [8, 3.4.8]).

Proof. Recall from [6, 6.5.2.9] that truncated objects in an ∞-topos are hypercomplete. Then statement (1) follows from [8, Proposition 2.3.8]. Statement (2) is a consequence of the compatibility of $\mathcal{T}_{\text{ét}}(k)$ and $\mathcal{T}_{\mathrm{an}}(k)$ with n-truncations for $n \geq 0$ (for $\mathcal{T}_{\text{ét}}(k)$, we refer to [8, 4.3.28]; for $\mathcal{T}_{\mathrm{an}}(k)$, we refer to [19, Theorem 3.23] in the nonarchimedean case and to [7, Proposition 11.4] in the complex case). Finally, statement (3) follows from [14, Proposition 6.2]. □

Corollary 3.2. *Let* $j \colon Y \hookrightarrow X$ *be a closed immersion in* $\mathcal{T}_{\text{ét}}(k)$*. The induced map* $j^{\mathrm{an}} \colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$ *is a closed immersion in* ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$.

Proof. Recall from [19, Lemma 5.2] that the hypercompletion functor Hyp preserves closed immersions of ∞-topoi. Hence, in the nonarchimedean case, the corollary is a consequence of Lemma 3.1(1) and of [19, Theorem 5.4]. In the complex case, the corollary is a consequence of Lemma 3.1(1) and of [6, 7.3.2.11]. □

Let us recall that a *derived algebraic Deligne–Mumford stack* over k is by definition a $\mathcal{T}_{\text{ét}}(k)$-scheme, which is in particular a $\mathcal{T}_{\text{ét}}(k)$-structured topos (cf. [8, 4.3.20]). We refer to [16] for a comparison with the definition of Deligne–Mumford stack via the functor of points.

Definition 3.3. A derived algebraic Deligne–Mumford stack $X = (\mathcal{X}, \mathcal{O}_X)$ is said to be *locally almost of finite presentation* if its truncation $\mathrm{t}_0(X) = (\mathcal{X}, \pi_0(\mathcal{O}_X))$ is an underived algebraic Deligne–Mumford stack of finite presentation, and $\pi_i(\mathcal{O}_X)$ is a coherent $\pi_0(\mathcal{O}_X)$-module for every i.

Lemma 3.4. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived algebraic Deligne–Mumford stack locally almost of finite presentation over k. Let $X^{\mathrm{an}} = (\mathcal{X}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ be its analytification. Then $\mathrm{t}_0(X^{\mathrm{an}}) = (\mathcal{X}^{\mathrm{an}}, \pi_0(\mathcal{O}_{X^{\mathrm{an}}}))$ is an underived analytic Deligne–Mumford stack.*

Proof. By [8, Lemma 2.1.3], the question is local on X. So we can assume that X is affine. Furthermore, using Lemma 3.1(2), we see that there is a canonical equivalence

$$\mathrm{t}_0(X^{\mathrm{an}}) \simeq \Psi_0(\mathrm{t}_0(X)).$$

Since X is an affine scheme, we can find an *underived* pullback diagram

$$\begin{array}{ccc} \mathrm{t}_0(X) & \longrightarrow & \mathbb{A}^n_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{0} & \mathbb{A}^m_k \end{array}$$

Let Y denote the derived pullback of the above diagram. Then $\mathrm{t}_0(Y) \simeq \mathrm{t}_0(X)$. Unramifiedness of $\mathcal{T}_{\text{ét}}(k)$ implies that

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^n_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{0} & \mathbb{A}^m_k \end{array}$$

remains a pullback diagram when viewed in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$. Since $(-)^{\mathrm{an}}$ is a right adjoint, it follows that

$$\begin{array}{ccc} Y^{\mathrm{an}} & \longrightarrow & (\mathbb{A}^n_k)^{\mathrm{an}} \\ \downarrow & & \downarrow \\ (\mathrm{Spec}(k))^{\mathrm{an}} & \longrightarrow & (\mathbb{A}^m_k)^{\mathrm{an}} \end{array}$$

is a pullback diagram in ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. Using Lemma 3.1(1), we see that $(\mathrm{Spec}(k))^{\mathrm{an}} \simeq \mathrm{Sp}(k)$, $(\mathbb{A}^n_k)^{\mathrm{an}} \simeq \mathbf{A}^n_k$ and $(\mathbb{A}^m_k)^{\mathrm{an}} \simeq \mathbf{A}^m_k$. Moreover, Corollary 3.2 implies that the morphism $\mathrm{Sp}(k) \to \mathbf{A}^m_k$ is again a closed immersion. Since dAn_k is closed in ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ under pullback by closed immersions by [19, Proposition 6.2] and [7, Proposition 12.10], we conclude that Y^{an} is a derived analytic space. So it follows from [19, Corollary 3.24] that $\mathrm{t}_0(Y^{\mathrm{an}})$ is an analytic space. Finally, using the chain of equivalences provided by Lemma 3.1(3)

$$\mathrm{t}_0(Y^{\mathrm{an}}) \simeq \Psi_0(\mathrm{t}_0(Y)) \simeq \Psi_0(\mathrm{t}_0(X)) \simeq \mathrm{t}_0(X^{\mathrm{an}}),$$

we conclude that $\mathrm{t}_0(X^{\mathrm{an}})$ is an analytic space. □

Corollary 3.5. *Let X be an underived algebraic Deligne–Mumford stack locally of finite presentation over k. Then X^{an} is a derived analytic space and it is equivalent to the classical analytification of X.*

Proof. The question is local on X and therefore we can assume that X is affine. Using Lemma 3.1(3), we see that the structure sheaf of X^{an} is discrete. Thus, $X^{\mathrm{an}} \simeq \mathrm{t}_0(X^{\mathrm{an}})$ is an analytic space in virtue of Lemma 3.4. Moreover, Lemma 3.1(2) shows that $\mathrm{t}_0(X^{\mathrm{an}}) \simeq \Psi_0(\mathrm{t}_0(X)) \simeq \Psi_0(X)$. Using the universal property of Ψ_0 and the fact that X^{an} is an analytic space, we see that for every analytic space Y, there is an equivalence

$$\mathrm{Map}_{\mathrm{An}_k}(Y, X^{\mathrm{an}}) \simeq \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))}(Y^{\mathrm{alg}}, X).$$

This shows that X^{an} can be identified with the classical analytification of X. □

Corollary 3.6. *Let $j\colon X \to Y$ be a closed immersion of derived algebraic Deligne–Mumford stacks locally almost of finite presentation over k. Then $j^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a closed immersion in ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$.*

Proof. It is enough to prove that $\mathrm{t}_0(j^{\mathrm{an}})\colon \mathrm{t}_0(X^{\mathrm{an}}) \to \mathrm{t}_0(Y^{\mathrm{an}})$ is a closed immersion. Since $\mathrm{t}_0(j^{\mathrm{an}}) \simeq \Psi_0(\mathrm{t}_0(j))$, the statement is now a consequence of Corollary 3.5. □

We are now ready to state and prove the main result of this section:

Proposition 3.7. *Let $X = (\mathcal{X}, \mathcal{O}_X) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))$ be a derived algebraic Deligne–Mumford stack locally almost of finite presentation over k. Then X^{an} is a derived analytic space.*

Proof. Using [8, Lemma 2.1.3], we can reason étale locally on X and therefore assume that X is affine. Let $\mathrm{dAff}_k^{\mathrm{afp}}$ denote the ∞-category of derived affine k-schemes almost of finite presentation. Let $\mathcal{C}$ be the full subcategory of $\mathrm{dAff}_k^{\mathrm{afp}}$ spanned by those derived affines X such that $X^{\mathrm{an}} \in \mathrm{dAn}_k$. Let us remark that $\mathcal{C}$ has the following properties:

(1) $\mathcal{C}$ contains $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ in virtue of Lemma 3.1(1).
(2) $\mathcal{C}$ is closed under pullbacks along closed immersions. Indeed, if

$$\begin{array}{ccc} W & \hookrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \overset{j}{\hookrightarrow} & X \end{array}$$

is a pullback diagram in $\mathrm{dAff}_k^{\mathrm{afp}}$ and j is a closed immersion, then the unramifiedness of $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$ implies that the image of this diagram in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))$ is again a pullback square. Since $(-)^{\mathrm{an}}$ is a right adjoint, we see that

$$\begin{array}{ccc} W^{\mathrm{an}} & \longrightarrow & Z^{\mathrm{an}} \\ \downarrow & & \downarrow \\ Y^{\mathrm{an}} & \xrightarrow{j^{\mathrm{an}}} & X^{\mathrm{an}} \end{array}$$

is a pullback square in ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. Using Corollary 3.6, we see that j^{an} is a closed immersion. Since dAn_k is closed under pullback along closed immersions in ${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ (see [19, Proposition 6.2] for the nonarchimedean case and [7, Proposition 12.10] for the complex case), we conclude that if $X, Y, Z \in \mathcal{C}$, then $W \in \mathcal{C}$ as well.

(3) $\mathcal{C}$ is closed under finite limits. Indeed, it follows from [19, §6] that general pullbacks can be constructed in terms of products of affine spaces and pullbacks along closed immersions. Since $(-)^{\mathrm{an}}$ takes $\mathbb{A}^n_k$ to $\mathbf{A}^n_k$ by Lemma 3.1(1), we see that $(-)^{\mathrm{an}}\colon \mathcal{C} \to \mathrm{dAn}_k$ commutes with products of affine spaces. Since $\mathcal{C}$ is furthermore closed under pullbacks along closed immersions by the previous point, the conclusion follows.

(4) $\mathcal{C}$ is closed under retractions. Indeed, let $X \in \mathcal{C}$ and let $Y \xrightarrow{j} X \xrightarrow{p} Y$ be a retraction diagram in $\mathrm{dAff}^{\mathrm{afp}}_k$. By assumption, $X^{\mathrm{an}} \in \mathrm{dAn}_k$ and Lemma 3.4 shows that $\mathrm{t}_0(Y^{\mathrm{an}}) \in \mathrm{dAn}_k$. It is therefore sufficient to show that $\pi_i(\mathcal{O}_{Y^{\mathrm{an}}})$ is a coherent sheaf over $\pi_0(\mathcal{O}_{Y^{\mathrm{an}}})$. But, $\pi_i(\mathcal{O}_{Y^{\mathrm{an}}})$ is a retract of $j^{-1}\pi_i(\mathcal{O}_{X^{\mathrm{an}}})$, which is locally of finite presentation over $j^{-1}\pi_0(\mathcal{O}_{X^{\mathrm{an}}})$. It follows that $\pi_i(\mathcal{O}_{Y^{\mathrm{an}}})$ is locally of finite presentation over $j^{-1}\pi_0(\mathcal{O}_{X^{\mathrm{an}}})$. Since $\pi_0(\mathcal{O}_{Y^{\mathrm{an}}})$ is a retract of $\pi_0(\mathcal{O}_{X^{\mathrm{an}}})$ and $\pi_i(\mathcal{O}_{Y^{\mathrm{an}}})$ has a canonical $\pi_0(\mathcal{O}_{Y^{\mathrm{an}}})$-structure, we conclude that $\pi_i(\mathcal{O}_{Y^{\mathrm{an}}})$ is of finite presentation over $\pi_0(\mathcal{O}_{Y^{\mathrm{an}}})$ as well. The conclusion now follows from the fact that $\pi_0(\mathcal{O}_{Y^{\mathrm{an}}})$ is coherent.

Let now $X \in \mathrm{dAff}^{\mathrm{afp}}_k$ and write $X \simeq \mathrm{Spec}(A)$ for a simplicial commutative k-algebra A almost of finite presentation. We want to prove that $X \in \mathcal{C}$. Since Lemma 3.4 guarantees that $\mathrm{t}_0(X^{\mathrm{an}})$ is an analytic space, we only have to show that $\pi_i(\mathcal{O}_{X^{\mathrm{an}}})$ is a coherent sheaf of $\pi_0(\mathcal{O}_{X^{\mathrm{an}}})$-modules.

In particular, for every $n \geq 0$ the algebra $\tau_{\leq n}(A)$ is a compact object in the ∞-category $\mathrm{CRing}^{\leq n}_k$ of n-truncated simplicial commutative k-algebras. It follows that there exists a *finite* diagram of free simplicial commutative k-algebras

$$F\colon I \to \mathrm{CRing}_k$$

such that $\tau_{\leq n}A$ is a retraction of $\tau_{\leq n}(B)$, where

$$B := \operatorname*{colim}_I F \in \mathrm{CRing}_k.$$

Since $\mathcal{C}$ is closed under finite limits, we see that $\mathrm{Spec}(B) \in \mathcal{C}$. Now, using Lemma 3.1(2) we conclude that

$$(\mathrm{t}_{\leq n}(\mathrm{Spec}(A)))^{\mathrm{an}} \simeq \Psi_n(\mathrm{t}_{\leq n}(\mathrm{Spec}(A)))$$

is a retract of

$$\Psi_n(\mathrm{t}_{\leq n}(\mathrm{Spec}(B))) \simeq \mathrm{t}_{\leq n}(\mathrm{Spec}(B)^{\mathrm{an}}).$$

Property (3) implies that this is a derived analytic space. Therefore, it follows from (4) that $(\mathrm{t}_{\leq n}(\mathrm{Spec}(A)))^{\mathrm{an}}$ is a derived analytic space as well.

Since we further have

$$\Psi_n(\mathrm{t}_{\leq n}(\mathrm{Spec}(A))) \simeq \mathrm{t}_{\leq n}(\mathrm{Spec}(A)^{\mathrm{an}}),$$

we conclude that $\pi_i(\mathcal{O}_{\mathrm{Spec}(A)^{\mathrm{an}}})$ is a coherent sheaf of $\pi_0(\mathcal{O}_{\mathrm{Spec}(A)^{\mathrm{an}}})$-modules for all $0 \leq i \leq n$. Repeating the same reasoning for every n, we now conclude that $\mathrm{Spec}(A)^{\mathrm{an}}$ is a derived analytic space. □

4. Analytic modules

In this section we study modules over derived analytic rings. The main result is Theorem 4.5. We refer to Section 1 for motivations and for a sketch of the proof.

Let us introduce a few notations before stating the main theorem.

Let $\mathcal{X}$ be an ∞-topos. In virtue of [8, Example 3.1.6, Remark 4.1.2], we have an equivalence of ∞-categories induced by the evaluation on the affine line,

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{CRing}_k}(\mathcal{X}),$$

where $\mathrm{Sh}_{\mathrm{CRing}_k}(\mathcal{X})$ denotes the ∞-category of sheaves on $\mathcal{X}$ with values in the ∞-category of simplicial commutative k-algebras.

This motivates the following definition:

Definition 4.1. Let $\mathcal{X}$ be an ∞-topos. We denote $\mathrm{CRing}_k(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X})$, and call it the ∞-category of *sheaves of simplicial commutative k-algebras on $\mathcal{X}$*. We denote $\mathrm{AnRing}_k(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, and call it the ∞-category of *sheaves of derived k-analytic rings on $\mathcal{X}$*. We have an algebraization functor

$$(-)^{\mathrm{alg}} \colon \mathrm{AnRing}_k(\mathcal{X}) \to \mathrm{CRing}_k(\mathcal{X})$$

induced by the analytification functor $\mathcal{T}_{\mathrm{disc}}(k) \to \mathcal{T}_{\mathrm{an}}(k)$.

Definition 4.2. Let Ab be the 1-category of abelian groups. Let $\mathcal{T}_{\mathrm{Ab}}$ denote the opposite of the full subcategory of Ab spanned by free abelian groups of finite rank. Let $\mathcal{C}$ be an ∞-category with finite products. The *∞-category of abelian group objects in* $\mathcal{C}$ is by definition the ∞-category

$$\mathrm{Ab}(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}),$$

where the right hand side denotes the full subcategory of $\mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C})$ spanned by product preserving functors.

Definition 4.3. For a $\mathcal{T}_{\mathrm{disc}}(k)$-structured topos $X = (\mathcal{X}, \mathcal{O}_X)$, we define $\mathcal{O}_X$-Mod to be the ∞-category of left $\mathcal{O}_X$-module objects of $\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})}(\mathcal{X})$, where $\mathcal{D}(\mathrm{Ab})$ denotes the derived ∞-category of abelian groups.

Definition 4.4. For a $\mathcal{T}_{\mathrm{an}}(k)$-structured topos $X = (\mathcal{X}, \mathcal{O}_X)$, we define $\mathcal{O}_X\text{-Mod} := \mathcal{O}_X^{\mathrm{alg}}\text{-Mod}$. In particular, an $\mathcal{O}_X$-module is by definition an $\mathcal{O}_X^{\mathrm{alg}}$-module.

The goal of this section is to prove the following result:

Theorem 4.5. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space. We have an equivalence of stable ∞-categories*

$$\mathcal{O}_X\text{-Mod} \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X})),$$

where $\mathrm{Sp}(-)$ *denotes the ∞-category of spectrum objects in a given ∞-category.*

We split the proof into several steps.

4.1. *Construction of the functor*

Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space. The transformation of pregeometries

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{disc}}(k) \to \mathcal{T}_{\mathrm{an}}(k)$$

induces a functor

$$\Phi \colon \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X} \to \mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}}.$$

Note that the following diagram is commutative by construction:

$$\begin{array}{ccc} \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X} & \xrightarrow{\Phi} & \mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}} \\ \downarrow & & \downarrow \\ \mathrm{AnRing}_k(\mathcal{X}) & \xrightarrow{(-)^{\mathrm{alg}}} & \mathrm{CRing}_k(\mathcal{X}) \end{array} \tag{4.6}$$

Lemma 4.7. *The functor Φ has the following properties:*

(1) *It is conservative.*
(2) *It commutes with limits and with sifted colimits.*

Proof. The first property follows from [7, Proposition 11.9] in the complex analytic case and from [19, Lemma 3.13] in the nonarchimedean case. The second property is a consequence of [14, Proposition 1.17]. □

Lemma 4.8. *Let $f \colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with finite products. If f preserves finite products, then it induces a well-defined functor* $\mathrm{Ab}(f) \colon \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D})$. *Furthermore, suppose f has one of the following properties:*

(1) *f is conservative.*
(2) *f commutes with limits.*
(3) *f commutes with sifted colimits.*

Then $\mathrm{Ab}(f)$ *has the same property.*

Proof. Unraveling the definitions we see that composition with f restricts to a well-defined functor

$$\mathrm{Ab}(\mathcal{C}) = \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) \to \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{D}) = \mathrm{Ab}(\mathcal{D}).$$

This functor fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) & \xrightarrow{\mathrm{Ab}(f)} & \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) & \xrightarrow{f_*} & \mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{D}) \end{array}$$

The vertical morphisms are fully faithful and furthermore they commute with limits and with sifted colimits. Observe now that if f has one of the listed properties, then f_* shares the same property for formal reasons. The commutativity of the above diagram then allows one to deduce that also $\mathrm{Ab}(f)$ inherits these properties. □

Since Φ commutes with limits, Lemma 4.8 implies that Φ induces a well-defined functor

$$\mathrm{Ab}(\Phi)\colon \mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}) \to \mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}}).$$

Moreover, $\mathrm{Ab}(\Phi)$ is conservative and commutes with limits and sifted colimits.

Corollary 4.9. *The functor* $\mathrm{Ab}(\Phi)$ *induces a well-defined functor of stable ∞-categories*

$$\partial_{\mathrm{Ab}}(\Phi)\colon \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}})). \tag{4.10}$$

Proof. Recall from [12, 1.4.2.8] that given an ∞-category $\mathcal{C}$, the ∞-category of spectra in $\mathcal{C}$ is equivalent to

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}),$$

where $\mathcal{S}_*^{\mathrm{fin}}$ denotes the ∞-category of pointed finite spaces and Exc_* denotes the ∞-category of strongly excisive functors from $\mathcal{S}_*^{\mathrm{fin}}$ to $\mathcal{C}$, that is, functors $f\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ satisfying the following two conditions:

(i) f takes final objects to final objects.
(ii) f takes pushout diagrams to pullback diagrams.

Since $\mathrm{Ab}(\Phi)$ commutes with limits, it is clear that composition with $\mathrm{Ab}(\Phi)$ induces the functor (4.10). □

By Corollary 8.3, we have an equivalence of stable ∞-categories

$$\mathcal{O}_X\text{-Mod} \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{/A})).$$

Therefore, we can reduce Theorem 4.5 to the following theorem:

Theorem 4.11. *The functor*

$$\partial_{\mathrm{Ab}}(\Phi)\colon \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X}))_{/\mathcal{O}_X} \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X}))_{/\mathcal{O}_X^{\mathrm{alg}}}$$

is an equivalence of stable ∞-categories.

4.2. Reduction to connected objects

By the construction of $\partial_{\mathrm{Ab}}(\Phi)$, in order to prove Theorem 4.11, it would be enough to prove that $\mathrm{Ab}(\Phi)$ is an equivalence. In fact, it is sufficient to prove that $\mathrm{Ab}(\Phi)$ is an equivalence up to a finite number of suspensions. Let us explain this reduction step precisely.

Observe that the functor $\Phi\colon \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X} \to \mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}}$ induces a well-defined functor

$$\Phi_*\colon \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{O}_X//\mathcal{O}_X} \to \mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}.$$

Lemma 4.12. (1) *Let $\mathcal{C}$ be an ∞-category with finite limits and let $*_{\mathcal{C}}$ denote a final object for $\mathcal{C}$. Write $\mathcal{C}_* \coloneqq \mathcal{C}_{*_{\mathcal{C}}/}$. Then the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces equivalences*

$$\mathrm{Ab}(\mathcal{C}_*) \to \mathrm{Ab}(\mathcal{C}) \quad \textit{and} \quad \mathrm{Sp}(\mathcal{C}_*) \to \mathrm{Sp}(\mathcal{C}).$$

(2) *Let $f: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with finite limits. Suppose that f commutes with finite limits. Then f induces a well-defined functor $f_*: \mathcal{C}_* \to \mathcal{D}_*$. Moreover, the diagrams*

$$\begin{array}{ccc} \mathrm{Ab}(\mathcal{C}_*) & \xrightarrow{\mathrm{Ab}(f_*)} & \mathrm{Ab}(\mathcal{D}_*) \\ \downarrow & & \downarrow \\ \mathrm{Ab}(\mathcal{C}) & \xrightarrow{\mathrm{Ab}(f)} & \mathrm{Ab}(\mathcal{D}) \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathrm{Sp}(\mathcal{C}_*) & \xrightarrow{\partial(f_*)} & \mathrm{Sp}(\mathcal{D}_*) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(\mathcal{C}) & \xrightarrow{\partial(f)} & \mathrm{Sp}(\mathcal{D}) \end{array}$$

commute. In particular, $\mathrm{Ab}(f)$ (resp. $\partial(f)$) is an equivalence if and only if $\mathrm{Ab}(f_*)$ (resp. $\partial(f_*)$) is one.

Proof. The forgetful functor $\mathcal{C}_* \to \mathcal{C}$ commutes with limits. Therefore, the existence of f_* is a consequence of Lemma 4.8. Hence the second statement is a direct consequence of the first one.

We now prove the first statement. The case of spectra has been discussed in [12, 1.4.2.25]. As for abelian groups, let $F: \mathcal{T}_{\mathrm{Ab}} \to \mathcal{C}$ be an ∞-functor that preserves products. Since $\mathcal{T}_{\mathrm{Ab}}$ has a zero object, we see that F factors canonically as

$$\widetilde{F}: \mathcal{T}_{\mathrm{Ab}} \to \mathcal{C}_*.$$

This produces a functor $\mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{C}_*)$ that is readily checked to be the inverse to the canonical functor $\mathrm{Ab}(\mathcal{C}_*) \to \mathrm{Ab}(\mathcal{C})$. □

We need a digression on connected objects in ∞-categories. We refer to [6, 5.5.6.18] for the definition of truncation functors $\tau_{\leq n}$ in a presentable ∞-category.

Definition 4.13. Let $\mathcal{C}$ be a presentable ∞-category. For any $n \geq 1$, we say that an object $X \in \mathcal{C}$ is *n-connected* if the canonical map $X \to *_{\mathcal{C}}$ induces an equivalence

$$\tau_{\leq n-1} X \xrightarrow{\sim} *_{\mathcal{C}}.$$

We denote by $\mathcal{C}^{\geq n}$ the full subcategory of $\mathcal{C}$ spanned by n-connected objects.

Lemma 4.14. *Let $\mathcal{C}$ be a presentable ∞-category. Suppose that there exists an ∞-topos $\mathcal{X}$ and a functor $F: \mathcal{C} \to \mathcal{X}$ such that*

- *F is conservative;*
- *F commutes with finite limits;*
- *F commutes with the truncation functors.*

Then:

(1) *$\mathcal{C}^{\geq n}$ is closed under finite products in $\mathcal{C}$;*
(2) *there is a canonical equivalence of ∞-categories $\mathrm{Ab}(\mathcal{C}^{\geq n}) \simeq \mathrm{Ab}(\mathcal{C})^{\geq n}$.*

Proof. Recall from [6, 6.5.1.2] that the truncation functor $\tau_{\leq n}\colon \mathcal{X} \to \mathcal{X}$ commutes with finite products. The hypotheses on F guarantee that the same holds for the truncation functor $\tau_{\leq n}\colon \mathcal{C} \to \mathcal{C}$. Now, the first statement follows immediately.

Let us now prove the second statement. We start by recalling that there is an equivalence

$$\mathrm{Ab}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}),$$

where $\mathcal{T}_{\mathrm{Ab}}$ is the opposite category of free abelian groups of finite rank. We denote the free abelian group of rank n by A^n.

We claim that an object $F \in \mathrm{Ab}(\mathcal{C})$ belongs to $\mathrm{Ab}(\mathcal{C})^{\geq n}$ if and only if its image in $\mathcal{C}$ belongs to $\mathcal{C}^{\geq n}$. To see this, let $0 \in \mathrm{Ab}(\mathcal{C})$ denote the constant functor associated to $*_{\mathcal{C}}$. Let furthermore $F\colon \mathcal{T}_{\mathrm{Ab}} \to \mathcal{C}$ be a product preserving functor. Since $\tau_{\leq n}$ commutes with finite products, $\tau_{\leq n} \circ F$ is again a product preserving functor. It follows that the morphism $\tau_{\leq n} \circ F \to 0$ is an equivalence if and only if it is an equivalence when evaluated on $A^1 \in \mathcal{T}_{\mathrm{Ab}}$. Since the forgetful functor $\mathrm{Ab}(\mathcal{C}) \to \mathcal{C}$ coincides (by definition) with the evaluation at A^1, this completes the proof of the claim.

Now we remark that statement (1) implies that the inclusion

$$i\colon \mathcal{C}^{\geq n} \hookrightarrow \mathcal{C}$$

commutes with finite products. Using [4, Lemma 5.2], we see that the induced functor

$$\mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}^{\geq n}) \to \mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C})$$

is fully faithful. It follows that the induced functor

$$\mathrm{Ab}(i)\colon \mathrm{Ab}(\mathcal{C}^{\geq n}) \to \mathrm{Ab}(\mathcal{C})$$

is fully faithful as well. Moreover, the diagram

$$\begin{array}{ccc} \mathrm{Ab}(\mathcal{C}^{\geq n}) & \longrightarrow & \mathrm{Ab}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{\geq n} & \xrightarrow{\ i\ } & \mathcal{C} \end{array}$$

commutes. It follows that $\mathrm{Ab}(i)$ factors through

$$j\colon \mathrm{Ab}(\mathcal{C}^{\geq n}) \to \mathrm{Ab}(\mathcal{C})^{\geq n},$$

and that j is also fully faithful. It remains to prove that j is essentially surjective. Let $F \in \mathrm{Ab}(\mathcal{C})^{\geq n}$. Then by the above claim, the image of F in $\mathcal{C}$ belongs to $\mathcal{C}^{\geq n}$. We can therefore see F as an element in $\mathrm{Ab}(\mathcal{C}^{\geq n})$, completing the proof. □

Since the functor

$$\Phi_*\colon \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{O}_X // \mathcal{O}_X} \to \mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}} // \mathcal{O}_X^{\mathrm{alg}}}$$

commutes with limits and sifted colimits, it admits a left adjoint

$$\Psi_*\colon \mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}} // \mathcal{O}_X^{\mathrm{alg}}} \to \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{O}_X // \mathcal{O}_X}.$$

Lemma 4.15. *The functor* Ψ_* *takes* $\mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}$ *to* $\mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X}$, *where* $(-)^{\geq 1}$ *is in the sense of Definition 4.13.*

Proof. It is enough to remark that the functor

$$\pi_0 \circ \Psi_* \colon \mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} \to \mathrm{AnRing}_k^{\leq 0}(\mathcal{X})_{\pi_0(\mathcal{O}_X)//\pi_0(\mathcal{O}_X)}$$

is naturally equivalent to the functor

$$\pi_0 \circ \Psi_* \circ \pi_0 \colon \mathrm{CRing}_k^{\leq 0}(\mathcal{X})_{\pi_0(\mathcal{O}_X^{\mathrm{alg}})//\pi_0(\mathcal{O}_X^{\mathrm{alg}})} \to \mathrm{AnRing}_k^{\leq 0}(\mathcal{X})_{\pi_0(\mathcal{O}_X)//\pi_0(\mathcal{O}_X)},$$

where $(-)^{\leq 0}$ denotes the full subcategory spanned by 0-truncated objects (cf. [6, 5.5.6.1]). □

In particular, Ψ_* induces a functor

$$\Psi_*^{\geq 1} \colon \mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} \to \mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X},$$

and moreover $\Psi_*^{\geq 1}$ is a left adjoint to $\Phi_*^{\geq 1}$.

The main goal of this subsection is to reduce the proof of Theorem 4.11 to the following statement:

Theorem 4.16. *The adjoint functors*

$$\Phi_*^{\geq 1} \colon \mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X} \leftrightarrows \mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} \colon \Psi_*^{\geq 1}$$

form an equivalence.

The next two subsections will be devoted to the proof of Theorem 4.16. Now let us explain how to deduce Theorem 4.11 from Theorem 4.16:

Proof of Theorem 4.11 assuming Theorem 4.16. Since $\Phi_*^{\geq 1}$ is an equivalence, so is

$$\mathrm{Ab}(\Phi_*^{\geq 1}) \colon \mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X}) \to \mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}).$$

Notice that Theorem 4.16 guarantees, in particular, that $\Psi_*^{\geq 1}$ commutes with finite limits. In particular, composition with $\Psi_*^{\geq 1}$ induces a well-defined functor

$$\mathrm{Ab}(\Psi_*^{\geq 1}) \colon \mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}) \to \mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X})$$

which is left adjoint to $\mathrm{Ab}(\Phi_*^{\geq 1})$.

In order to prove that

$$\partial_{\mathrm{Ab}}(\Phi_*) \colon \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{O}_X//\mathcal{O}_X})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}))$$

is an equivalence, it is enough to prove that for any

$$M \in \mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}})$$

the canonical map

$$\Sigma(M) \to \mathrm{Ab}(\Phi_*)(\mathrm{Ab}(\Psi_*)(\Sigma(M)))$$

is an equivalence. Here Σ denotes the suspension functor (see the discussion around [12, 1.1.2.6]).

Notice that the natural inclusion

$$\mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} \hookrightarrow \mathrm{Fun}(\mathcal{T}_{\mathrm{disc}}(k), \mathcal{X})_{/\mathcal{O}_X}$$

is conservative, commutes with limits and with truncations. In particular, we can apply Lemma 4.14 to deduce the equivalence

$$\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}})^{\geq 1} \simeq \mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}).$$

Observe now that

$$\Sigma(M) \in \mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}})^{\geq 1}.$$

In particular

$$\mathrm{Ab}(\Psi_*)(\Sigma(M)) \simeq \mathrm{Ab}(\Psi_*^{\geq 1})(\Sigma(M)) \in \mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}}).$$

As a consequence,

$$\mathrm{Ab}(\Phi_*)(\mathrm{Ab}(\Psi_*)(\Sigma(M)) \simeq \mathrm{Ab}(\Phi_*^{\geq 1})(\mathrm{Ab}(\Psi_*^{\geq 1})(\Sigma(M)).$$

Since $\mathrm{Ab}(\Phi_*^{\geq 1})$ is an equivalence and $\mathrm{Ab}(\Psi_*^{\geq 1})$ is its left adjoint, the conclusion follows. □

4.3. Reduction to the case of spaces

Here we explain how to reduce the proof of Theorem 4.16 to the case where $\mathcal{X}$ is the ∞-category of spaces, $\mathcal{S}$.

In order to prove Theorem 4.16, it is enough to prove that the pair of functors $(\Psi_*^{\geq 1}, \Phi_*^{\geq 1})$ form an equivalence of categories. Fix a geometric point $x^{-1} \colon \mathcal{X} \leftrightarrows \mathcal{S} : x_*$. Invoking [15, Theorem 1.12] we conclude that the induced diagram

$$\begin{array}{ccc} \mathrm{AnRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X//\mathcal{O}_X} & \xrightarrow{x^{-1}} & \mathrm{AnRing}_k(\mathcal{S})^{\geq 1}_{x^{-1}\mathcal{O}_X//x^{-1}\mathcal{O}_X} \\ \Phi_*^{\geq 1}\downarrow & & \downarrow\Phi_*^{\geq 1} \\ \mathrm{CRing}_k(\mathcal{X})^{\geq 1}_{\mathcal{O}_X^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} & \xrightarrow{x^{-1}} & \mathrm{CRing}_k(\mathcal{S})^{\geq 1}_{x^{-1}\mathcal{O}_X^{\mathrm{alg}}//x^{-1}\mathcal{O}_X^{\mathrm{alg}}} \end{array}$$

commutes and it is left adjointable. Since $\mathcal{X}$ has enough points (see [19, Remark 3.3]), we see that it is enough to check that the adjunction

$$\Phi_*^{\geq 1} \colon \mathrm{AnRing}_k(\mathcal{S})^{\geq 1}_{x^{-1}\mathcal{O}_X//x^{-1}\mathcal{O}_X} \leftrightarrows \mathrm{CRing}_k(\mathcal{S})^{\geq 1}_{x^{-1}\mathcal{O}_X^{\mathrm{alg}}//x^{-1}\mathcal{O}_X^{\mathrm{alg}}} : \Psi_*^{\geq 1}$$

is an equivalence. We can therefore take $\mathcal{X} = \mathcal{S}$. To simplify notation, we set

$$A \coloneqq x^{-1}\mathcal{O}_X.$$

Furthermore, we write AnRing_k instead of $\mathrm{AnRing}_k(\mathcal{S})$, and similarly CRing_k for $\mathrm{CRing}_k(\mathcal{S})$.

4.4. *Flatness*

Here we will complete the proof of Theorem 4.16, that the functor

$$\Phi_*^{\geq 1} \colon \mathrm{AnRing}_{A//A}^{\geq 1} \to \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}^{\geq 1}$$

is an equivalence. We already observed that $\Phi_*^{\geq 1}$ has a left adjoint $\Psi_*^{\geq 1}$. Furthermore, we know that Φ_* is conservative, and hence so is $\Phi_*^{\geq 1}$. Therefore, it is enough to prove that for every $B \in \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}^{\geq 1}$, the unit transformation

$$\eta \colon B \to \Phi_*^{\geq 1}(\Psi_*^{\geq 1}(B))$$

is an equivalence. Notice that

$$\pi_0(B) \simeq \pi_0(A^{\mathrm{alg}}) \simeq \pi_0(\Phi_*^{\geq 1}(\Psi_*^{\geq 1}(B))).$$

In particular, $\pi_0(\eta)$ is an isomorphism. In order to complete the proof of Theorem 4.16, it is therefore sufficient to prove the following result:

Proposition 4.17. *For every $B \in \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$, the canonical map*

$$\eta \colon B \to \Phi_*(\Psi_*(B))$$

is a flat map of simplicial commutative rings.

Notation 4.18. To simplify notation, in virtue of the commutative diagram (4.6), let us denote from now on Φ_* by $(-)^{\mathrm{alg}}$. Moreover, we denote Ψ_* by $(-)^{\mathrm{an}}_A$ and call it the *functor of analytfication relative to A*.

Remark 4.19. In the complex case, a proof of the above result already appeared in [15, Appendix A]. In this section, we expand the proof given in loc. cit. and we introduce slight modifications in order to obtain a uniform proof that works both in the nonarchimedean case and in the complex case.

The proof of Proposition 4.17 occupies the remainder of this subsection. We start by introducing the full subcategory $\mathcal{C}_A$ of $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ spanned by those $B \in \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ such that the canonical map

$$B \to (B^{\mathrm{an}}_A)^{\mathrm{alg}}$$

is flat. We observe that $\mathcal{C}_A$ is closed under various operations:

Lemma 4.20. *The full subcategory $\mathcal{C}_A$ enjoys the following properties:*

(1) $A^{\mathrm{alg}} \in \mathcal{C}_A$.
(2) *$\mathcal{C}_A$ is closed under retracts.*
(3) *$\mathcal{C}_A$ is closed under filtered colimits.*

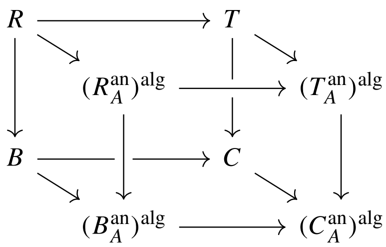
(4) *Let* $R \to T$ *is an effective epimorphism in* $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ *such that the square*

$$\begin{array}{ccc} R & \longrightarrow & T \\ \downarrow & & \downarrow \\ (R_A^{\mathrm{an}})^{\mathrm{alg}} & \longrightarrow & (T_A^{\mathrm{an}})^{\mathrm{alg}} \end{array}$$

is a pushout. Let $R \to B$ *be any map in* $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$. *If* B, R *and* T *belong to* $\mathcal{C}_A$, *then so does the pushout* $B \otimes_R T$.

Proof. Statement (1) follows directly from the fact that $(A^{\mathrm{alg}})_A^{\mathrm{an}} \simeq A$. Statement (2) follows because flat maps are stable under retracts. Statement (3) is a consequence of the following two facts: on the one hand, flat maps are stable under filtered colimits and, on the other hand, the functors $(-)^{\mathrm{alg}}$ and $(-)_A^{\mathrm{an}}$ commute with filtered colimits.

We now prove statement (4). Set $C := B \otimes_R T$ and consider the commutative cube



Since $(-)_A^{\mathrm{an}}$ is a left adjoint, we see that

$$\begin{array}{ccc} R_A^{\mathrm{an}} & \longrightarrow & T_A^{\mathrm{an}} \\ \downarrow & & \downarrow \\ B_A^{\mathrm{an}} & \longrightarrow & C_A^{\mathrm{an}} \end{array}$$

is a pushout diagram in $\mathrm{AnRing}_{A//A}$. Moreover, since the top square in the above cube is a pushout by assumption, the map $R_A^{\mathrm{an}} \to T_A^{\mathrm{an}}$ is an effective epimorphism. Therefore, the unramifiedness of $\mathcal{T}_{\mathrm{an}}(k)$ implies that the front square in the above cube is a pushout as well (cf. [19, Corollary 3.11 and Proposition 3.17]). It follows that the outer square in the diagram

$$\begin{array}{ccccc} R & \longrightarrow & B & \longrightarrow & (B_A^{\mathrm{an}})^{\mathrm{alg}} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & C & \longrightarrow & (C_A^{\mathrm{an}})^{\mathrm{alg}} \end{array}$$

is a pushout. Since the left square is a pushout by construction, we conclude that the right square is a pushout as well. Since flat maps are stable under base change and $B \to (B_A^{\mathrm{an}})^{\mathrm{alg}}$ is flat, so is $C \to (C_A^{\mathrm{an}})^{\mathrm{alg}}$. In other words, $C \in \mathcal{C}_A$. □

Motivated by statement (4) in the above lemma, we introduce the following temporary definition:

Definition 4.21. Let $p\colon R \to T$ be an effective epimorphism in $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$. We say that p has the *property* (P_A) if the diagram

$$\begin{array}{ccc} R & \longrightarrow & T \\ \downarrow & & \downarrow \\ (R_A^{\mathrm{an}})^{\mathrm{alg}} & \longrightarrow & (R_A^{\mathrm{an}})^{\mathrm{alg}} \end{array}$$

is a pushout.

With this terminology, Lemma 4.20 immediately implies the following:

Corollary 4.22. *Suppose that there exists a collection* $S = \{B_\alpha\}_{\alpha\in I}$ *of objects in* $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ *such that:*

(1) *every object in* $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ *is a retract of an* S*-cell complex;*
(2) *the structural morphisms* $B_\alpha \to A^{\mathrm{alg}}$ *have the property* (P_A)*;*
(3) *each* B_α *belongs to* $\mathcal{C}_A$.

Then $\mathcal{C}_A = \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$.

We are therefore reduced to finding a set S of objects in $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ with the above properties. In order to achieve this goal, we need a further reduction step: we want to replace A^{alg} with the ring of germs of holomorphic functions at any geometric point of $\mathbf{D}_k^n$ in the nonarchimedean case, and of $\mathbf{A}_{\mathbb{C}}^n$ in the complex case.

We start by observing that the collection of A^{alg}-linear morphisms

$$A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}$$

for various m is a set S_A of elements in $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ with the property that every other object is a retract of an S-cell complex. The following is the key reduction step:

Lemma 4.23. *Let* $f\colon R \to A$ *be an effective epimorphism in* AnRing_k.

(1) *If* $B \in \mathrm{CRing}_{R^{\mathrm{alg}}//R^{\mathrm{alg}}}$ *belongs to* $\mathcal{C}_R$*, then* $B \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}} \in \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ *belongs to* $\mathcal{C}_A$.
(2) *If* $B \to C$ *is an effective epimorphism in* $\mathrm{CRing}_{R^{\mathrm{alg}}//R^{\mathrm{alg}}}$ *that satisfies the property* (P_R)*, then the induced morphism*

$$B \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}} \to C \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}} \in \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$$

satisfies the property (P_A).
(3) *The base change functor*

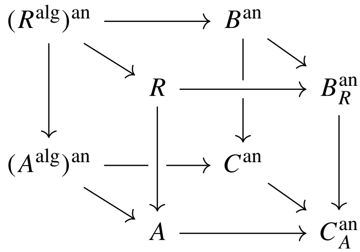
$$- \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}} \colon \mathrm{CRing}_{R^{\mathrm{alg}}//R^{\mathrm{alg}}} \to \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$$

takes S_R *to* S_A*. Furthermore every object in* S_A *lies in the essential image of* S_R *via this functor.*

Proof. We start by proving (1). Denote by $(-)^{\mathrm{an}}$ the left adjoint to the underlying algebra functor

$$(-)^{\mathrm{alg}}\colon \mathrm{AnRing}_{/A} \to \mathrm{CRing}_{/A}.$$

We therefore obtain the following commutative cube:



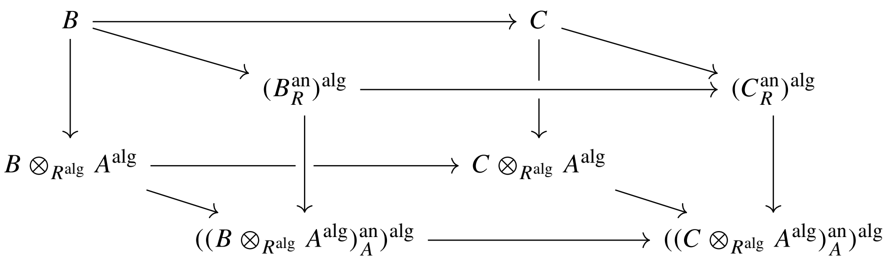
The universal property of the relative analytifications $(-)^{\mathrm{an}}_R$ and $(-)^{\mathrm{an}}_A$ shows that the top and the bottom squares are pushout squares. Furthermore, since $(-)^{\mathrm{an}}$ is a left adjoint, the back square is a pushout as well. The transitivity property for pushouts implies that the front square is a pushout.

Since the morphism $f\colon R \to A$ is an epimorphism, unramifiedness of $\mathcal{T}_{\mathrm{an}}(k)$ implies that the functor $(-)^{\mathrm{alg}}$ preserves the pushout at the front. Consider now the commutative diagram

$$\begin{array}{ccccc} R^{\mathrm{alg}} & \longrightarrow & B & \longrightarrow & (B^{\mathrm{an}}_R)^{\mathrm{alg}} \\ \downarrow & & \downarrow & & \downarrow \\ A^{\mathrm{alg}} & \longrightarrow & C & \longrightarrow & (C^{\mathrm{an}}_A)^{\mathrm{alg}} \end{array}$$

The left square is a pushout by definition, and we proved above that the outer square is also a pushout. It follows that the right square is a pushout as well. Since $B \to (B^{\mathrm{an}}_R)^{\mathrm{alg}}$ is flat, so is $C \to (C^{\mathrm{an}}_A)^{\mathrm{alg}}$, completing the proof.

We now prove statement (2). Consider the commutative cube



The hypotheses guarantee that the top and the back squares are pushouts. As a consequence, $B^{\mathrm{an}}_R \to C^{\mathrm{an}}_R$ is an effective epimorphism. We claim that the front square is a pushout as well. Indeed, we have the commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & B_R^{\mathrm{an}} & \longrightarrow & C_R^{\mathrm{an}} \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & (B \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}})_A^{\mathrm{an}} & \longrightarrow & (C \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}})_A^{\mathrm{an}} \end{array}$$

The argument in the proof of (1) implies that the outer and the left squares are pushouts. Therefore, so is the right square. Since $B_R^{\mathrm{an}} \to C_R^{\mathrm{an}}$ is an effective epimorphism, the unramifiedness of $\mathcal{T}_{\mathrm{an}}(k)$ guarantees that $(-)^{\mathrm{alg}}$ commutes with this pushout. Therefore, the front square in the previous commutative cube is a pushout as well. The transitivity property of pushout squares then implies that the bottom square is also a pushout. In other words, the map

$$B \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}} \to C \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}}$$

has the property (P_A).

Finally, we prove statement (3). Let

$$p\colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}$$

be an A^{alg}-linear morphism. This morphism chooses m elements $a_1, \dots, a_m \in \pi_0(A^{\mathrm{alg}})$. Since the map $\pi_0(p)\colon \pi_0(R^{\mathrm{alg}}) \to \pi_0(A^{\mathrm{alg}})$ is surjective, we can find $r_1, \dots, r_m \in \pi_0(R^{\mathrm{alg}})$ such that

$$\pi_0(p)(r_i) = a_i$$

for $1 \leq i \leq m$. We can now choose a morphism

$$q\colon R^{\mathrm{alg}}[X_1, \dots, X_m] \to R^{\mathrm{alg}}$$

that selects the elements $r_1, \dots, r_m$. Applying the base change functor $- \otimes_{R^{\mathrm{alg}}} A^{\mathrm{alg}}$ we obtain a new map

$$p'\colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}.$$

Observe that both p and p' define elements in

$$\pi_0 \operatorname{Map}_{\mathrm{CRing}_{A^{\mathrm{alg}}}}(A^{\mathrm{alg}}[X_1, \dots, A_m], A^{\mathrm{alg}}) \simeq \pi_0(A^{\mathrm{alg}})^m.$$

The construction reveals that p and p' coincide as elements in the above set. In other words, we can find a homotopy $p \simeq p'$ in $\mathrm{CRing}_{A^{\mathrm{alg}}}$. This completes the proof. □

Combining Lemma 4.23 and Corollary 4.22, we deduce that whenever $R \to A$ is an effective epimorphism in AnRing_k, if $\mathcal{C}_R = \mathrm{CRing}_{R^{\mathrm{alg}}//R^{\mathrm{alg}}}$, then $\mathcal{C}_A = \mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ as well.

We now use the hypothesis that A is the stalk of a derived analytic space $X = (\mathcal{X}, \mathcal{O}_X)$ at a geometric point $x_*\colon \mathcal{S} \leftrightarrows \mathcal{X} : x^{-1}$. In particular, using [19, Lemma 6.3] in the nonarchimedean case and [7, Proposition 12.13] in the complex case, we can suppose that X admits a closed embedding into a smooth analytic space

$$j\colon X \hookrightarrow U.$$

In the nonarchimedean case, we can take U to be a polydisk $\mathbf{D}^n_k$, while in the complex case we can take U to be an affine space $\mathbf{A}^n_{\mathbb{C}}$. In either case, let

$$y_* \colon \mathcal{S} \leftrightarrows \mathcal{X} : y^{-1}$$

be the geometric point defined as the composition $y_* := j_* \circ x_*$. Set

$$R := y^{-1}\mathcal{O}_U$$

and observe that the induced map $f \colon R \to A$ is an effective epimorphism. The above argument allows us to replace A by R. In other words, we can assume from the very beginning that A is of the form $x^{-1}\mathcal{O}_U$ for some geometric point of U, where U is a polydisk $\mathbf{D}^n_k$ in the nonarchimedean case and it is $\mathbf{A}^n_k$ in the complex case. Using Corollary 4.22, we are therefore reduced to proving that for every A^{alg}-linear morphism

$$f \colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}$$

the following properties are satisfied:

(1) $A^{\mathrm{alg}}[X_1, \dots, X_m]$ belongs to $\mathcal{C}_A$.
(2) The morphism f has the property (P_A).

In order to prove these statements, we need a geometric characterization of the relative analytification

$$A^{\mathrm{alg}}[X_1, \dots, X_m]^{\mathrm{an}}_A \in \mathrm{AnRing}_{A//A}.$$

The map $f \colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}$ selects m elements $a_1, \dots, a_m \in A^{\mathrm{alg}}$. Since $A = x^{-1}\mathcal{O}_U$ is the ring of germs of holomorphic functions around the point x, we can find an étale neighborhood V of x such that the elements $a_1, \dots, a_m$ extend to holomorphic functions $\widetilde{a_1}, \dots, \widetilde{a_m}$ on V. In both cases, we can interpret these holomorphic functions as a section of the relative algebraic space

$$\pi \colon V^{\mathrm{alg}} \times \mathbb{A}^m_k \to V^{\mathrm{alg}}.$$

We denote the section determined by the functions $\widetilde{a_1}, \dots, \widetilde{a_m}$ by $s \colon V^{\mathrm{alg}} \to V^{\mathrm{alg}} \times \mathbb{A}^m_k$. The analytification relative to V takes s to a section

$$s^{\mathrm{an}}_V \colon V \to V \times \mathbf{A}^m_k.$$

Denote by y the point of $V \times \mathbf{A}^m_k$ which is the image of the point $x \in V$ via s^{an}_V. Since $V \times \mathbf{A}^m_k$ is the analytification of $V^{\mathrm{alg}} \times \mathbb{A}^m_k$ relative to V, there is a canonical map

$$q \colon (V \times \mathbf{A}^m_k)^{\mathrm{alg}} \to V^{\mathrm{alg}} \times \mathbb{A}^m_k$$

making the following diagram commutative:

$$\begin{array}{ccccc} & & V^{\mathrm{alg}} & & \\ & \swarrow^{(s^{\mathrm{an}}_V)^{\mathrm{alg}}} & & \searrow^{s} & \\ (V \times \mathbf{A}^m_k)^{\mathrm{alg}} & & \xrightarrow{\quad q \quad} & & V^{\mathrm{alg}} \times \mathbb{A}^m_k \\ & \searrow_{(\pi^{\mathrm{an}}_V)^{\mathrm{alg}}} & & \swarrow_{\pi} & \\ & & V^{\mathrm{alg}} & & \end{array}$$

By passing to the stalk at x the map q induces a well-defined map

$$\alpha\colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to x^{-1}\mathcal{O}^{\mathrm{alg}}_{V\times \mathbf{A}^m_k} \to y^{-1}\mathcal{O}^{\mathrm{alg}}_{V\times \mathbf{A}^m_k}.$$

We can now prove the following result:

Proposition 4.24. *The map* $\alpha\colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to y^{-1}\mathcal{O}^{\mathrm{alg}}_{V\times \mathbf{A}^n_k}$ *exhibits* $y^{-1}\mathcal{O}_{V\times \mathbf{A}^n_k}$ *as analytification of* $A^{\mathrm{alg}}[X_1, \dots, X_m]$ *relative to* A*. In particular, it induces an equivalence*

$$A^{\mathrm{alg}}[X_1, \dots, X_m]^{\mathrm{an}}_A \simeq y^{-1}\mathcal{O}_{V\times \mathbf{A}^m_k}$$

in $\mathrm{AnRing}_{A//A}$.

Proof. To simplify notation, we set $R := A^{\mathrm{alg}}[X_1, \dots, X_m]$ and $B := y^{-1}\mathcal{O}_{V\times \mathbf{A}^m_k}$. Furthermore, we denote by $\mathrm{Map}_{A//A}$ and $\mathrm{Map}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ the mapping spaces in the ∞-categories $\mathrm{AnRing}_{A//A}$ and $\mathrm{CRing}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$, respectively.

We have to check that for any $C \in \mathrm{AnRing}_{A//A}$, the map

$$\mathrm{Map}_{A//A}(B, C) \to \mathrm{Map}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}(R, C^{\mathrm{alg}})$$

induced by $\alpha\colon R \to B^{\mathrm{alg}}$ is an equivalence. Let us introduce the following temporary notation: given an object C in either AnRing_k or CRing_k, we denote by $\mathcal{S}_C$ the structured ∞-topos $(\mathcal{S}, C)$. When $C \in \mathrm{AnRing}_k$, we set, as usual, $\mathcal{S}^{\mathrm{alg}}_C := (\mathcal{S}, C^{\mathrm{alg}})$. Moreover, we denote by $\mathrm{Map}_{\mathcal{S}_A//\mathcal{S}_A}$ and $\mathrm{Map}_{\mathcal{S}^{\mathrm{alg}}_A//\mathcal{S}^{\mathrm{alg}}_A}$ the mapping spaces in the ∞-categories ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))_{\mathcal{S}_A//\mathcal{S}_A}$ and ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k))_{\mathcal{S}^{\mathrm{alg}}_A//\mathcal{S}^{\mathrm{alg}}_A}$, respectively. The very definition of the mapping spaces in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ and in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k))$ yields the natural equivalences

$$\mathrm{Map}_{A//A}(B, C) \simeq \mathrm{Map}_{\mathcal{S}_A//\mathcal{S}_A}(\mathcal{S}_C, \mathcal{S}_B)$$

and

$$\mathrm{Map}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}(R, C^{\mathrm{alg}}) \simeq \mathrm{Map}_{\mathcal{S}^{\mathrm{alg}}_A//\mathcal{S}^{\mathrm{alg}}_A}(\mathcal{S}^{\mathrm{alg}}_C, \mathcal{S}_R).$$

Finally, we represent the $\mathcal{T}_{\mathrm{an}}(k)$-structured topoi V and $V \times \mathbf{A}^m_k$ as the pairs $(\mathcal{V}, \mathcal{O}_V)$ and $(\mathcal{Y}, \mathcal{O}_{V\times \mathbf{A}^m_k})$, respectively. We represent the $\mathcal{T}_{\mathrm{disc}}(k)$-structured topos $V^{\mathrm{alg}} \times \mathbb{A}^n_k$ as the pair $(\mathcal{Z}, \mathcal{O}_{V^{\mathrm{alg}}\times \mathbb{A}^m_k})$. Form the pullbacks of topoi

$$\begin{array}{ccccc} \mathcal{W}_2 & \longrightarrow & \mathcal{W}_1 & \longrightarrow & \mathcal{S} \\ {\scriptstyle g_{2*}}\downarrow & & \downarrow{\scriptstyle g_{1*}} & & \downarrow{\scriptstyle x_*} \\ \mathcal{Y} & \xrightarrow{q_*} & \mathcal{Z} & \xrightarrow{\pi_*} & \mathcal{V} \end{array}$$

Using [8, Lemma 2.1.3], we see that $W_2 := (\mathcal{W}_2, g_2^{-1}\mathcal{O}_{V\times \mathbf{A}^m_k})$ is the analytification of $W_1 := (\mathcal{W}_1, g_1^{-1}\mathcal{O}_{V^{\mathrm{alg}}\times \mathbb{A}^m_k})$ relative to $\mathcal{S}_A$. In particular, for every $C \in \mathrm{AnRing}_{A//A}$, we obtain an equivalence

$$\mathrm{Map}_{\mathcal{S}_A//\mathcal{S}_A}(\mathcal{S}_C, W_2) \simeq \mathrm{Map}_{\mathcal{S}^{\mathrm{alg}}_A//\mathcal{S}^{\mathrm{alg}}_A}(\mathcal{S}^{\mathrm{alg}}_C, W_1).$$

In order to complete the proof, it is now sufficient to show that there are equivalences

$$\mathrm{Map}_{\mathcal{S}_A^{\mathrm{alg}}//\mathcal{S}_A^{\mathrm{alg}}}(\mathcal{S}_C^{\mathrm{alg}}, W_1) \simeq \mathrm{Map}_{\mathcal{S}_A^{\mathrm{alg}}//\mathcal{S}_A^{\mathrm{alg}}}(\mathcal{S}_C^{\mathrm{alg}}, \mathcal{S}_R)$$

and

$$\mathrm{Map}_{\mathcal{S}_A//\mathcal{S}_A}(\mathcal{S}_C, W_2) \simeq \mathrm{Map}_{\mathcal{S}_A//\mathcal{S}_A}(\mathcal{S}_C, \mathcal{S}_B).$$

We argue for the first one. The map $s\colon V^{\mathrm{alg}} \to V^{\mathrm{alg}} \times \mathbb{A}_k^m$ induces a map $s_1\colon \mathcal{S}_A \to W_1$, and there is a canonical equivalence

$$R \simeq s_1^{-1} g_1^{-1} \mathcal{O}_{V^{\mathrm{alg}} \times \mathbb{A}_k^m}.$$

Consider the natural fiber sequence

$$\mathrm{Map}_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}(s_1^{-1} g_1^{-1} \mathcal{O}_{V^{\mathrm{alg}} \times \mathbb{A}_k^n}, C^{\mathrm{alg}}) \to \mathrm{Map}_{\mathcal{S}_A^{\mathrm{alg}}//\mathcal{S}_A^{\mathrm{alg}}}(\mathcal{S}_C, W_1) \to \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}_{\mathcal{S}//\mathcal{S}}}(\mathcal{S}, \mathcal{W}_1).$$

Since $\mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}_{\mathcal{S}//\mathcal{S}}}(\mathcal{S}, \mathcal{W}_1) \simeq *$, we conclude that the first map is an equivalence. The second equivalence is proved in a similar way. □

Now we move to the next step of the proof of Proposition 4.17:

Corollary 4.25. *For every A^{alg}-linear map*

$$f\colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}},$$

the canonical map $\eta\colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}[X_1, \dots, X_m]_A^{\mathrm{an}}$ is flat.

Proof. Using Proposition 4.24, we can describe $A^{\mathrm{alg}}[X_1, \dots, X_m]_A^{\mathrm{an}}$ as the ring of germs of holomorphic functions $y^{-1}\mathcal{O}_{V \times \mathbf{A}_k^n}$.

Let us deal with the nonarchimedean case first. In this case, we have

$$A^{\mathrm{alg}} \simeq k\langle T_1, \dots, T_n \rangle_x$$

and

$$y^{-1}\mathcal{O}_{V \times \mathbf{A}_k^n} \simeq k\langle T_1, \dots, T_n, X_1, \dots, X_m \rangle_y.$$

We have to prove that the canonical map

$$k\langle T_1, \dots, T_n \rangle_x [X_1, \dots, X_m] \to k\langle T_1, \dots, T_n, X_1, \dots, X_m \rangle_y$$

is flat. Since the passage to germs preserves flatness, it is enough to prove that the map of commutative rings

$$i\colon k\langle T_1, \dots, T_n \rangle [X_1, \dots, X_m] \to k\langle T_1, \dots, T_n, X_1, \dots, X_m \rangle$$

is flat. Since both rings are noetherian, it is enough to check flatness after passing to the formal completion at every maximal ideal of $k\langle T_1, \dots, T_n, X_1, \dots, X_m \rangle$. If $\mathfrak{m}$ is such a maximal ideal, then we have equivalences

$$(k\langle T_1, \dots, T_n \rangle [X_1, \dots, X_m])^{\wedge}_{i^{-1}(\mathfrak{m})} \simeq \kappa(\mathfrak{m})[\![T_1, \dots, T_n, X_1, \dots, X_m]\!]$$

and

$$(k\langle T_1, \dots, T_n, X_1, \dots, X_m\rangle)^{\wedge}_{\mathfrak{m}} \simeq \kappa(\mathfrak{m})[\![T_1, \dots, T_n, X_1, \dots, X_m]\!],$$

where $\kappa(\mathfrak{m})$ denotes the residue field. It follows that i induces an isomorphism on the formal completions, and therefore i is flat.

Let us now deal with the complex case. In this case, we have

$$A^{\mathrm{alg}} \simeq \mathbb{C}\{T_1, \dots, T_n\}_x$$

and

$$y^{-1}\mathcal{O}_{V \times \mathbf{A}^n_k} \simeq \mathbb{C}\{T_1, \dots, T_n, X_1, \dots, X_m\}_y$$

where the right hand sides denote the rings of germs of holomorphic functions on V at x and on $V \times \mathbf{A}^n_k$ at y, respectively. Thus, we have to prove that the natural map

$$\mathbb{C}\{T_1, \dots, T_n\}_x[X_1, \dots, X_m] \to \mathbb{C}\{T_1, \dots, T_n, X_1, \dots, X_m\}_y \tag{4.26}$$

is flat. Consider the map

$$f \colon \mathbb{C}\{T_1, \dots, T_n\}_x[X_1, \dots, X_m] \to \mathbb{C}\{T_1, \dots, T_n\}_x,$$

and let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{C}\{T_1, \dots, T_n\}_x$. Since f is $\mathbb{C}\{T_1, \dots, T_n\}_x$-linear, we see that $f^{-1}(\mathfrak{m})$ is again a maximal ideal of $\mathbb{C}\{T_1, \dots, T_n\}_x[X_1, \dots, X_m]$ and the map (4.26) induces a canonical map

$$(\mathbb{C}\{T_1, \dots, T_n\}_x[X_1, \dots, X_m])_{f^{-1}(\mathfrak{m})} \to \mathbb{C}\{T_1, \dots, T_n, X_1, \dots, X_m\}_y. \tag{4.27}$$

Since the localization map

$$\mathbb{C}\{T_1, \dots, T_n\}_x[X_1, \dots, X_m] \to (\mathbb{C}\{T_1, \dots, T_n\}_x[X_1, \dots, X_m])_{f^{-1}(\mathfrak{m})}$$

is flat, it is enough to prove that (4.27) is flat. Observe that both the source and the target of that map are noetherian local rings. In particular, it is enough to check that (4.27) becomes flat after passing to the formal completions at the maximal ideals. Since we can identify both formal completions with the ring $\mathbb{C}[\![T_1, \dots, T_n, X_1, \dots, X_m]\!]$ of formal power series, the conclusion follows. □

The last step of the proof of Proposition 4.17 is provided by the following:

Corollary 4.28. *Every A^{alg}-linear map*

$$f \colon A^{\mathrm{alg}}[X_1, \dots, X_m] \to A^{\mathrm{alg}}$$

has the property (P_A).

Proof. Unraveling the definitions, we see that we have to prove that the square

$$\begin{array}{ccc} A^{\mathrm{alg}}[X_1, \dots, X_m] & \xrightarrow{f} & A^{\mathrm{alg}} \\ \eta \downarrow & & \downarrow \mathrm{id} \\ A^{\mathrm{alg}}[X_1, \dots, X_m]^{\mathrm{an}}_A & \longrightarrow & A^{\mathrm{alg}} \end{array}$$

is a pushout in CRing_k. Using Proposition 4.24, $A^{\mathrm{alg}}[X_1, \dots, X_m]^{\mathrm{an}}_A$ can be described as $y^{-1}\mathcal{O}_{V \times \mathbf{A}^m_k}$, where the notations are introduced right before Proposition 4.24. Therefore, the square above is a pushout in the category of (underived) rings. By Corollary 4.25, the map η is flat, so the square is a pushout in CRing_k. □

5. Analytic cotangent complex

In this section we introduce the analytic cotangent complex and we establish its basic properties. In the first subsection, we work in the general framework of structured topoi for a given pregeometry. The main tool we employ is Lurie's formalism of tangent category. However, an adaptation is needed due to our framework of analytic modules in Section 4. In Subsection 5.2, we specialize the general formalism to the setting of derived analytic geometry. The remaining subsections concern various properties of the analytic cotangent complex.

5.1. *The cotangent complex formalism*

Let Cat_∞ denote the ∞-category of ∞-categories. Let $\mathrm{Cat}_\infty^{\mathrm{lex}}$ denote the subcategory of Cat_∞ spanned by those ∞-categories having finite limits and by those functors that preserve them. Let $\mathcal{T}_{\mathrm{Ab}}$ be the Lawvere theory of abelian groups (cf. Definition 4.2). For $n \geq 0$, we denote by A^n the free abelian group on n elements seen as an element in $\mathcal{T}_{\mathrm{Ab}}$.

By Lemma 4.8, the assignment $\mathcal{C} \mapsto \mathrm{Ab}(\mathcal{C})$ can be promoted to an ∞-functor

$$\mathrm{Ab}(-) := \mathrm{Fun}^\times(\mathcal{T}_{\mathrm{Ab}}, -) \colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{lex}},$$

which we call the *abelianization functor*.

Let $\mathcal{C}$ be an ∞-category with finite limits and consider the Cartesian fibration

$$p \colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}.$$

Observe that the associated ∞-functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through $\mathrm{Cat}_\infty^{\mathrm{lex}}$. Let $\mathcal{C}_{\mathrm{Ab}}$ be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}, \mathcal{C})$ spanned by those functors

$$F \colon \Delta^1 \times \mathcal{T}_{\mathrm{Ab}} \to \mathcal{C}$$

satisfying the following conditions:

(1) The restriction $F|_{\{0\}\times\mathcal{T}_{\mathrm{Ab}}}$ commutes with fiber products.
(2) The canonical map $F(0, A^0) \to F(1, A^0)$ is an equivalence.
(3) For every $A^n \in \mathcal{T}_{\mathrm{Ab}}$, the canonical map $F(1, A^n) \to F(1, A^0)$ is an equivalence.

Let $e \colon \Delta^1 \to \Delta^1 \times \mathcal{T}_{\mathrm{Ab}}$ be the functor selecting the morphism

$$(0, A^1) \to (0, A^0).$$

Finally, we consider the composition

$$q \colon \mathcal{C}_{\mathrm{Ab}} \hookrightarrow \mathrm{Fun}(\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) \xrightarrow{e_*} \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{p} \mathcal{C},$$

where e_* is given by precomposition with e.

Lemma 5.1. *The functor $q \colon \mathcal{C}_{\mathrm{Ab}} \to \mathcal{C}$ is a Cartesian fibration. Furthermore:*

(1) *A morphism f in $\mathcal{C}_{\mathrm{Ab}}$ is q-Cartesian if and only if $e_*(f)$ is p-Cartesian in* $\mathrm{Fun}(\Delta^1, \mathcal{C})$.
(2) *For any $x \in \mathcal{C}$, the fiber $(\mathcal{C}_{\mathrm{Ab}})_x$ is equivalent to* $\mathrm{Ab}(\mathcal{C}_{/x})$.
(3) *A diagram $g \colon K^{\triangleleft} \to \mathcal{C}_{\mathrm{Ab}}$ is a (co)limit diagram if and only if g is a q-(co)limit diagram and $q \circ g$ is a (co)limit diagram in $\mathcal{C}$.*

Proof. We first remark that if $\mathcal{D}$ is an ∞-category with final object $*_{\mathcal{D}}$ then evaluation at $*_{\mathcal{D}}$ induces a Cartesian fibration

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathcal{C},$$

and moreover a natural transformation $f \colon F \to G$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is a Cartesian edge if and only if for every object $x \in \mathcal{D}$, the square

$$\begin{array}{ccc} F(x) & \xrightarrow{f} & G(x) \\ \downarrow & & \downarrow \\ F(*_{\mathcal{D}}) & \xrightarrow{f} & G(*_{\mathcal{D}}) \end{array}$$

is a pullback square in $\mathcal{C}$. It follows that evaluation at $(1, A^0) \in \Delta^1 \times \mathcal{T}_{\mathrm{Ab}}$ induces a Cartesian fibration

$$\mathrm{Fun}(\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) \to \mathcal{C},$$

and moreover

$$e_* \colon \mathrm{Fun}(\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$$

preserves Cartesian edges.

Let now $G \in \mathcal{C}_{\mathrm{Ab}}$ and suppose $f \colon F \to G$ is a Cartesian edge in $\mathrm{Fun}(\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}, \mathcal{C})$. We claim that $F \in \mathcal{C}_{\mathrm{Ab}}$ as well. Indeed, the morphism $(1, A^n) \to (1, A^0)$ induces a pullback square

$$\begin{array}{ccc} F(1, A^n) & \xrightarrow{f} & G(1, A^n) \\ \downarrow & & \downarrow \\ F(1, A^0) & \xrightarrow{f} & G(1, A^0) \end{array}$$

Since the right vertical morphism is an equivalence, so is the left one. The same reasoning applied to the morphism $(0, A^0) \to (1, A^0)$ shows that

$$F(0, A^0) \to F(1, A^0)$$

is an equivalence. It remains to prove that $F(0, A^{n+m}) \simeq F(0, A^n) \times F(0, A^m)$. Consider the diagram

$$\begin{array}{ccccc} F(0, A^{n+m}) & \longrightarrow & F(0, A^n) & \longrightarrow & F(0, A^0) \\ \downarrow & & \downarrow & & \downarrow \\ G(0, A^{n+m}) & \longrightarrow & G(0, A^n) & \longrightarrow & G(0, A^0) \end{array}$$

Since f is a Cartesian edge, we see that the outer square and the right one are pullbacks. It follows that so is the left one. Since $G(0, A^{n+m}) \simeq G(0, A^n) \times G(0, A^m)$, we conclude that $F(0, A^{n+m}) \simeq F(0, A^n) \times F(0, A^m)$ as well.

Recall now that for objects $F \in \mathcal{C}_{\mathrm{Ab}}$ the canonical morphism $F(0, A^0) \to F(1, A^0)$ is an equivalence. Therefore the functor $q \colon \mathcal{C}_{\mathrm{Ab}} \to \mathcal{C}$ is a Cartesian fibration and the composition

$$\mathcal{C}_{\mathrm{Ab}} \hookrightarrow \mathrm{Fun}(\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}, \mathcal{C}) \xrightarrow{e_*} \mathrm{Fun}(\Delta^1, \mathcal{C})$$

preserves Cartesian edges. Let now $f \colon F \to G$ be a morphism in $\mathcal{C}_{\mathrm{Ab}}$ and suppose that $e_*(f)$ is p-Cartesian. Since both F and G belong to $\mathcal{C}_{\mathrm{Ab}}$, it is enough to check that squares of the form

$$\begin{array}{ccc} F(0, A^n) & \xrightarrow{f} & G(0, A^n) \\ \downarrow & & \downarrow \\ F(1, A^0) & \xrightarrow{f} & G(1, A^0) \end{array}$$

are pullback diagrams. When $n = 0$, this is true because both vertical maps are equivalences, and when $n = 1$ it follows from the hypothesis that $e_*(f)$ is p-Cartesian. The general case follows by induction, using the fact that $F(0, A^{n+1}) \simeq F(0, A^n) \times F(0, A^1)$ and $G(0, A^{n+1}) \simeq G(0, A^n) \times G(0, A^1)$. This completes the proof of (1).

We now turn to statement (2). Recall that

$$\mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}}, \mathcal{C}_{/x}) \simeq \mathrm{Fun}_x(\mathcal{T}_{\mathrm{Ab}}^{\triangleright}, \mathcal{C}).$$

We can identify $\mathcal{T}_{\mathrm{Ab}}^{\triangleright}$ with the full subcategory of $\Delta^1 \times \mathcal{T}_{\mathrm{Ab}}$ spanned by $\{0\} \times \mathcal{T}_{\mathrm{Ab}}$ and the object $(1, A^0)$. Using [6, 4.3.2.15] twice, we see that restriction along $\mathcal{T}_{\mathrm{Ab}}^{\triangleright} \hookrightarrow \Delta^1 \times \mathcal{T}_{\mathrm{Ab}}$ induces an equivalence

$$\mathcal{C}_{\mathrm{Ab}} \simeq \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}^{\triangleright}, \mathcal{C}).$$

Passing to the fiber at $x \in \mathcal{C}$, we obtain the equivalence $(\mathcal{C}_{\mathrm{Ab}})_x \simeq \mathrm{Ab}(\mathcal{C}_{/x})$ we were looking for.

For (3), the proof of [12, 7.3.1.12] applies. □

Definition 5.2. Let $\mathcal{C}$ be a presentable ∞-category. The *abelianized tangent bundle* of $\mathcal{C}$ is by definition the stabilization of the Cartesian fibration $q \colon \mathcal{C}_{\mathrm{Ab}} \to \mathcal{C}$ constructed above. It is denoted by $\mathrm{T}_{\mathrm{Ab}}(\mathcal{C})$.

Using Lemma 5.1, we see that the abelianized tangent bundle to $\mathcal{C}$ is a Cartesian fibration

$$\pi \colon \mathrm{T}_{\mathrm{Ab}}(\mathcal{C}) \to \mathcal{C},$$

whose fiber at $x \in \mathcal{C}$ is equivalent to $\mathrm{Sp}(\mathrm{Ab}(\mathcal{C}_{/x}))$.

Now let us explain how to use the language of the abelianized tangent bundle to introduce the analytic cotangent complex. We have:

Lemma 5.3. *Let $\mathcal{C}$ be a presentable ∞-category. Then:*

(1) $\mathrm{T}_{\mathrm{Ab}}(\mathcal{C})$ *is a presentable ∞-category.*
(2) *The canonical map $q \colon \mathrm{T}_{\mathrm{Ab}}(\mathcal{C}) \to \mathcal{C}$ commutes with limits and filtered colimits.*

Proof. It follows from the proof of Lemma 5.1 that $\mathcal{C}_{\mathrm{Ab}}$ can be realized as an accessible localization of $\mathrm{Fun}(\mathcal{T}_{\mathrm{Ab}} \times \Delta^1, \mathcal{C})$. In particular, $\mathcal{C}_{\mathrm{Ab}}$ is presentable. Moreover, Lemma 5.1(3) implies that the map $q \colon \mathcal{C}_{\mathrm{Ab}} \to \mathcal{C}$ preserves both limits and colimits. We are therefore reduced to proving the following statements. Let $p \colon \mathcal{X} \to S$ be a presentable fibration which preserves limits and filtered colimits and where $\mathcal{X}$ is presentable.[1] Then:

- The ∞-category $\mathrm{Stab}(p)$ is presentable.
- The functor $\pi : \mathrm{Stab}(p)$ commutes with limits and filtered colimits.
- The functor $\pi : \mathrm{Stab}(p) \to S$ is a presentable fibration.

The last condition follows from the definition of $\mathrm{Stab}(p)$ [12, 7.3.1.1, 7.3.1.7]. The first two follow from the fact that $\mathrm{Stab}(p)$ can be realized as an accessible localization of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X})$. Indeed, let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X})$ spanned by those functors $g \colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{X}$ such that:

- g is excisive;
- if $s = p(g(*)) \in S$, then $g(*)$ is a final object for $\mathcal{X}_s$;
- $p \circ g \colon \mathcal{S}_*^{\mathrm{fin}}$ factors through $S^{\simeq}$, the maximal ∞-groupoid contained in S.

Observe that the inclusion

$$\mathcal{E} \hookrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X})$$

commutes with limits and filtered colimits. It follows that $\mathcal{E}$ is an accessible localization of the presentable ∞-category $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X})$ and that the projection

$$\mathcal{E} \hookrightarrow \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}) \to \mathcal{X} \xrightarrow{p} S$$

induced by evaluation at $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$ commutes with limits and filtered colimits.

It remains to identify $\mathcal{E}$ with $\mathrm{Stab}(p)$. Reasoning as in the proof of Lemma 5.1, we see that the map $\mathcal{E} \to \mathcal{X}$ takes Cartesian edges to Cartesian edges. Furthermore, the fiber at $s \in S$ can be canonically identified with the full subcategory of

$$\mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}_s)$$

spanned by those functors that take final objects to final objects. In other words, $\mathcal{E}_s \simeq \mathrm{Sp}(\mathcal{X}_s)$. This completes the proof. □

Now let $\mathcal{T}$ be any pregeometry and let $X := (\mathcal{X}, \mathcal{O}_X)$ be a $\mathcal{T}$-structured topos. Recall from [14, Proposition 1.15] that the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$ is presentable. Let $\mathcal{A} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$ be any $\mathcal{T}$-structure equipped with a local morphism to $\mathcal{O}_X$. Then the ∞-category

$$\mathcal{T}_{X,\mathcal{A}} := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X}$$

is again presentable. As a consequence, we can apply the above results to see that

$$\pi : \mathrm{T}_{\mathrm{Ab}}(\mathcal{T}_{X,\mathcal{A}}) \to \mathcal{T}_{X,\mathcal{A}}$$

[1] This last condition is redundant: see [4, Theorem 10.3].

is a functor between presentable categories that preserves limits and colimits. It fits in a commutative triangle

$$\begin{array}{ccc} \mathrm{T}_{\mathrm{Ab}}(\mathcal{T}_{X,\mathcal{A}}) & \xrightarrow{\quad G \quad} & \mathrm{Fun}(\Delta^1, \mathcal{T}_{X,\mathcal{A}}) \\ & {\scriptstyle \pi}\searrow \quad \swarrow & \\ & \mathcal{T}_{X,\mathcal{A}} & \end{array}$$

where G is the natural functor. Observe that the fiber of G at an object $\mathcal{B} \in \mathcal{T}_{X,\mathcal{A}}$ can be identified with the composition

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{\mathcal{A}//\mathcal{B}})) \xrightarrow{\Omega^\infty} \mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{\mathcal{A}//\mathcal{B}}) \xrightarrow{U} \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{\mathcal{A}//\mathcal{B}},$$

where U denotes the forgetful functor. Let us denote by $\Omega^\infty_{\mathrm{Ab}}$ the composition $U \circ \Omega^\infty$. In particular, it admits a left adjoint, which we denote $\Sigma^\infty_{\mathrm{Ab}}$. We can therefore combine Lemma 5.1 and [12, 7.2.3.11] to conclude that G admits a left adjoint relative to $\mathcal{T}_{X,\mathcal{A}}$ (in the sense of [12, 7.3.2.2]). We denote this left adjoint by F. Finally, we let

$$s\colon \mathcal{T}_{X,\mathcal{A}} \to \mathrm{Fun}(\Delta^1, \mathcal{T}_{X,\mathcal{A}})$$

be the functor defined informally by sending $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{X}$ to the diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mathrm{id}_{\mathcal{A}}} & \mathcal{A} & \longrightarrow & \mathcal{O}_X \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{O}_X}} \\ \mathcal{A} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{O}_X \end{array}$$

Notice that the existence of the functor s is a direct application of [6, 4.3.2.15].

Definition 5.4. Let $X := (\mathcal{X}, \mathcal{O}_X)$ be a $\mathcal{T}$-structured topos and let $\mathcal{A} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$. The *$\mathcal{T}$-theoretic cotangent complex functor relative to X and $\mathcal{A}$* is the composition

$$\mathbb{L}^{\mathcal{T}}_{X,\mathcal{A}}\colon \mathcal{T}_{X,\mathcal{A}} \xrightarrow{s} \mathrm{Fun}(\Delta^1, \mathcal{T}_{X,\mathcal{A}}) \xrightarrow{F} \mathrm{T}_{\mathrm{Ab}}(\mathcal{T}_{X,\mathcal{A}}).$$

Let $\mathcal{B} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$ and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a morphism. The *relative $\mathcal{T}$-theoretic cotangent complex of φ*, denoted by $\mathbb{L}^{\mathcal{T}}_{\varphi}$, or by $\mathbb{L}^{\mathcal{T}}_{\mathcal{B}/\mathcal{A}}$ when the morphism is clear from the context, is the object

$$\mathbb{L}^{\mathcal{T}}_{X,\mathcal{A}}(\mathcal{B}) \in \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{\mathcal{A}//\mathcal{B}})).$$

When $\mathcal{A}$ is an initial object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$ we refer to $\mathbb{L}^{\mathcal{T}}_{X,\mathcal{A}}$ as the *absolute cotangent complex* and we omit $\mathcal{A}$ from the above notations.

Let $\mathcal{T}$ be a pregeometry, $\mathcal{X}$ an ∞-topos and $\mathcal{O}$ a $\mathcal{T}$-structure on $\mathcal{X}$. Since $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}}$ is presentable, it admits pushouts. We denote by $\mathcal{B}_1 \otimes^{\mathcal{T}}_{\mathcal{A}} \mathcal{B}_2$ the pushout of the diagram

$$\mathcal{B}_1 \leftarrow \mathcal{A} \rightarrow \mathcal{B}_2$$

in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}}$. Furthermore, we can rewrite the $\mathcal{T}$-theoretic cotangent complex of $\varphi\colon \mathcal{A} \to \mathcal{B}$ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ as

$$\mathbb{L}^{\mathcal{T}}_{\mathcal{B}/\mathcal{A}} \simeq \Sigma^\infty_{\mathrm{Ab}}(\mathcal{B} \otimes^{\mathcal{T}}_{\mathcal{A}} \mathcal{B}).$$

Definition 5.5. Let $\mathcal{T}$ be a pregeometry and let $X = (\mathcal{X}, \mathcal{O}_X)$ and $Y = (\mathcal{Y}, \mathcal{O}_Y)$ be $\mathcal{T}$-structured topoi. Let $f = (f_*, f^\sharp)\colon (\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)$ be a morphism in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$. The *relative* $\mathcal{T}$*-theoretic cotangent complex of* f, denoted by $\mathbb{L}^{\mathcal{T}}_f$, is defined to be the relative $\mathcal{T}$-theoretic cotangent complex of $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ in the sense of Definition 5.4. We will denote $\mathbb{L}^{\mathcal{T}}_f$ by $\mathbb{L}^{\mathcal{T}}_{X/Y}$ when the morphism f is clear from the context.

We now deduce some basic properties of the cotangent complex using the formal properties in [12, §7.3.3]. We start by fixing some notations.

Let $X := (\mathcal{X}, \mathcal{O}_X)$, $Y := (\mathcal{Y}, \mathcal{O}_Y)$ be $\mathcal{T}$-structured topoi and let $f\colon X \to Y$ be a morphism between them. We denote the underlying geometric morphism of ∞-topoi by

$$f_*\colon \mathcal{X} \leftrightarrows \mathcal{Y} : f^{-1},$$

and the underlying local morphism of $\mathcal{T}$-structures by

$$f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

Since the functor f^{-1} commutes with finite limits, composition with it induces a well-defined functor

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y}. \tag{5.6}$$

Observe that this functor commutes again with limits and sifted colimits. In particular, it induces a functor

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y})),$$

which we still denote by f^{-1}.

On the other hand, composition with $f^\sharp$ induces a functor

$$f^\sharp_!\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}. \tag{5.7}$$

Although this functor does not commute with finite limits, pullback along $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ provides a right adjoint to $f^\sharp_!$, which we denote by $f^\sharp_*$. Notice that $f^\sharp_*$ commutes with filtered colimits. Composition with $f^\sharp_*$ induces a functor

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y}))$$

that commutes with limits and filtered colimits. The adjoint functor theorem then guarantees the existence of a left adjoint, which we denote by

$$f^{\sharp *}\colon\ \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X})).$$

Finally, composing $f^{\sharp *}$ and f^{-1} provides a functor

$$f^*\colon\ \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y})) \xrightarrow{f^{-1}} \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y})) \xrightarrow{f^{\sharp *}} \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X})).$$

Lemma 5.8. *Let $f\colon X \to Y$ be a morphism of $\mathcal{T}$-structured topoi. Then the diagram*

$$\begin{array}{ccc}
\mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y})) & \xrightarrow{f^{-1}} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y})) \\
\Sigma^\infty_{\mathrm{Ab}}\big\uparrow & & \Sigma^\infty_{\mathrm{Ab}}\big\uparrow \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y} & \xrightarrow{f^{-1}} & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y}
\end{array}$$

commutes. In particular, $f^{-1}(\mathbb{L}^{\mathrm{an}}_Y) \simeq \mathbb{L}^{\mathrm{an}}_{f^{-1}\mathcal{O}_Y}$.

Proof. Introduce the ∞-category $\mathrm{Str}'_{\mathcal{T}}(\mathcal{X})$ whose objects are functors $\mathcal{O}\colon \mathcal{T} \to \mathcal{X}$ that commute with products and admissible pullbacks, and whose morphisms are natural transformations $\varphi\colon \mathcal{O} \to \mathcal{O}'$ such that for every admissible morphism $j\colon U \to V$ in $\mathcal{T}$ the square

$$\begin{array}{ccc}
\mathcal{O}(U) & \longrightarrow & \mathcal{O}(U) \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}(V) & \longrightarrow & \mathcal{O}'(V)
\end{array}$$

is a pullback square. Then the natural functor $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X} \to \mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$ is fully faithful. Let $\mathcal{O} \in \mathrm{Str}'_{\mathcal{T}}(\mathcal{X})$ and let $\varphi\colon \mathcal{O} \to \mathcal{O}_X$ be a morphism. Let $\{U_i \to U\}$ be an admissible cover in $\mathcal{T}$. Then the diagram

$$\begin{array}{ccc}
\coprod \mathcal{O}(U_i) & \longrightarrow & \mathcal{O}(U) \\
\big\downarrow & & \big\downarrow \\
\coprod \mathcal{O}_X(U_i) & \longrightarrow & \mathcal{O}_X(U)
\end{array}$$

is a pullback. Since the bottom horizontal morphism is an effective epimorphism, so is the top one. In other words, $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$. This shows that there is a canonical equivalence

$$\mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X} \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}. \tag{5.9}$$

We can now argue as follows. Composition with $f_*\colon \mathcal{X} \to \mathcal{Y}$ induces a well-defined functor

$$f_*\colon \mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y} \to \mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/f_*f^{-1}\mathcal{O}_Y}.$$

Moreover, pullback along the natural transformation $\mathcal{O}_Y \to f_*f^{-1}\mathcal{O}_Y$ yields a functor

$$\mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/f_*f^{-1}\mathcal{O}_Y} \to \mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y}.$$

Composing these two functors and using the equivalence (5.9) we obtain a functor

$$f_*\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y} \simeq \mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y} \to \mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y} \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y}.$$

This is the right adjoint for the functor

$$f^{-1}\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y}.$$

It follows that composition with f_* induces a functor

$$f_* \colon \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y}))$$

that is right adjoint to the functor f^{-1} constructed above. It is now enough to check that the diagram of right adjoints

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y})) & \xleftarrow{f_*} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y})) \\ {\scriptstyle \Omega^\infty_{\mathrm{Ab}}}\downarrow & & \downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} \\ \mathrm{Str}'_{\mathcal{T}}(\mathcal{Y})_{/\mathcal{O}_Y} & \xleftarrow{f_*} & \mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y} \end{array}$$

commutes. This follows because, given $F \in \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}'_{\mathcal{T}}(\mathcal{X})_{/f^{-1}\mathcal{O}_Y}))$, we have natural identifications

$$f_*(\Omega^\infty_{\mathrm{Ab}}(F)) \simeq f_* \circ F(S^0, A^1) \simeq (f_* \circ F)(S^0, A^1) \simeq \Omega^\infty_{\mathrm{Ab}}(f_* F). \qquad \square$$

Proposition 5.10. *Let $\mathcal{T}$ be a pregeometry and let $f \colon X \to Y$ and $g \colon Y \to Z$ be morphisms of $\mathcal{T}$-structured topoi, where $X = (\mathcal{X}, \mathcal{O}_X)$. Then there is a fiber sequence*

$$f^* \mathbb{L}^{\mathcal{T}}_{Y/Z} \to \mathbb{L}^{\mathcal{T}}_{X/Z} \to \mathbb{L}^{\mathcal{T}}_{X/Y}$$

in $\mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}))$.

Proof. By using Lemmas 5.3 and 5.8, the proof of [12, 7.3.3.6] applies. □

Corollary 5.11. *Let $\mathcal{T}$ be a pregeometry. If $f \colon X \to Y$ is an étale morphism of $\mathcal{T}$-structured topoi (cf. [8, Definition 2.3.1]), then $\mathbb{L}^{\mathcal{T}}_{Y/X} \simeq 0$.*

Proof. It follows from the transitivity sequence of Proposition 5.10 by taking Z to be a point and localizing on X. □

Proposition 5.12. *Suppose*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y \end{array}$$

is a pullback diagram in the category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$. Then the natural morphism

$$g^*(\mathbb{L}^{\mathcal{T}}_{X/Y}) \to \mathbb{L}^{\mathcal{T}}_{X'/Y'}$$

is an equivalence.

Proof. By using Lemmas 5.3 and 5.8, the proof of [12, 7.3.3.7] applies. □

Remark 5.13. The above proposition works for any pregeometry $\mathcal{T}$. Nevertheless, we are seldom interested in working with the full ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$. For example, when $\mathcal{T} = \mathcal{T}_{\text{ét}}(k)$ is the étale pregeometry, we are only interested in working with the full subcategory of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$ spanned by derived algebraic Deligne–Mumford stacks. Similarly, when

$\mathcal{T} = \mathcal{T}_{\mathrm{an}}(k)$, we are interested in working with the full subcategory of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ spanned by derived analytic spaces. In general, the inclusion of these full subcategories does not commute with pullbacks. In other words, Proposition 5.12 has to be proven again in the cases of interest.

The complex analytic case is an exception. Indeed, [7, Proposition 12.12] guarantees that the inclusion $\mathrm{dAn}_{\mathbb{C}} \hookrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(\mathbb{C}))$ commutes with pullbacks. The case of derived algebraic Deligne–Mumford stacks can also be dealt with easily: the question being local, one can reduce to the affine case, where the result follows directly from [12, 7.3.3.7]. However, the nonarchimedean analytic case is trickier and requires techniques that will be introduced in the next subsection. We refer to Proposition 5.27 for the proof.

5.2. The analytic cotangent complex

From this point on, we will specialize to the pregeometry $\mathcal{T}_{\mathrm{an}}(k)$. If $f\colon X \to Y$ is a morphism in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, we write $\mathbb{L}^{\mathrm{an}}_{X/Y}$ instead of $\mathbb{L}^{\mathcal{T}_{\mathrm{an}}(k)}_{X/Y}$. It is an element in $\mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}))$. Nonetheless, using the equivalence

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X})) \simeq \mathcal{O}_X\text{-Mod}$$

provided by Theorem 4.5, we consider $\mathbb{L}^{\mathrm{an}}_{X/Y}$ as an element in $\mathcal{O}_X$-Mod. Since this stable ∞-category has a canonical t-structure (cf. [9, 1.7]), we have the cohomology sheaves $\pi_i(\mathbb{L}^{\mathrm{an}}_{X/Y})$.

As in the algebraic setting, the analytic cotangent complex is closely related to analytic derivations.

Definition 5.14. Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space and let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}^{\geq 0}$. The *analytic split square-zero extension of* $\mathcal{O}_X$ *by* $\mathcal{F}$ is the derived analytic ring

$$\mathcal{O}_X \oplus \mathcal{F} := \Omega^\infty_{\mathrm{Ab}}(\mathcal{F}) \in \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}.$$

This definition is motivated by [12, 7.3.4.15]. Let us show that the notion of analytic split square-zero extension is compatible with the underlying algebra:

Lemma 5.15. *Let $\varphi\colon \mathcal{T}' \to \mathcal{T}$ be a transformation of pregeometries and let $X = (\mathcal{X}, \mathcal{O}_X)$ be a $\mathcal{T}$-structured topos. Then the functor*

$$\varphi_*\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{X})_{/\mathcal{O}_X \circ \varphi}$$

given by precomposition with φ induces a commutative square

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{X})_{/\mathcal{O}_X \circ \varphi})) & \xleftarrow{\varphi_*} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X})) \\ \downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} & & \downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} \\ \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{X})_{/\mathcal{O}_X \circ \varphi} & \xleftarrow{\varphi_*} & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X} \end{array}$$

Proof. Since $\varphi_*\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{X})_{/\mathcal{O}_X}$ commutes with limits, composition with φ_* induces a well-defined functor

$$\varphi_*\colon \ \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{X})_{/\mathcal{O}_X})).$$

Let $\mathcal{F}\colon \mathcal{S}_*^{\mathrm{fin}} \times \mathcal{T}_{\mathrm{Ab}} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}$ be an element in $\mathrm{Sp}(\mathrm{Ab}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\mathcal{O}_X}))$. Then

$$\Omega^\infty_{\mathrm{Ab}}(\varphi_*(F)) \simeq (\varphi_*(F))(S^0, A^1) \simeq F(S^0, A^1) \circ \varphi \simeq \varphi_*(\Omega^\infty_{\mathrm{Ab}}(F)).$$

The proof is thus complete. □

Corollary 5.16. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space and let $\mathcal{F} \in \mathcal{O}_X\text{-}\mathrm{Mod}^{\geq 0}$. Then $(\mathcal{O}_X \oplus \mathcal{F})^{\mathrm{alg}}$ is the split square-zero extension of $\mathcal{O}_X^{\mathrm{alg}}$ by $\mathcal{F}$.*

Proof. Applying Lemma 5.15 to the transformation of pregeometries

$$(-)^{\mathrm{an}}\colon \mathcal{T}_{\text{ét}}(k) \to \mathcal{T}_{\mathrm{an}}(k),$$

we get the conclusion directly. □

Definition 5.17. Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space and let $\mathcal{A} \in \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}$. Let $\mathcal{F} \in \mathcal{O}_X\text{-}\mathrm{Mod}^{\geq 0}$. The space of *$\mathcal{A}$-linear analytic derivations from $\mathcal{O}_X$ into $\mathcal{F}$* is the space

$$\mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{F}) := \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{O}_X \oplus \mathcal{F}).$$

With this definition, we have the following characterization of the analytic cotangent complex:

Proposition 5.18. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space and let $\mathcal{A} \in \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}$. Then for any $\mathcal{F} \in \mathcal{O}_X\text{-}\mathrm{Mod}^{\geq 0}$ there is a canonical equivalence*

$$\mathrm{Map}_{\mathcal{O}_X\text{-}\mathrm{Mod}}(\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{A}}, \mathcal{F}) \simeq \mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{F}).$$

Proof. We have

$$\begin{aligned}\mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{F}) &\simeq \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{O}_X \oplus \mathcal{F})\\ &= \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X}}(\mathcal{O}_X, \Omega^\infty_{\mathrm{Ab}}(\mathcal{F}))\\ &\simeq \mathrm{Map}_{\mathcal{O}_X\text{-}\mathrm{Mod}}(\Sigma^\infty_{\mathrm{Ab}}(\mathcal{O}_X \widehat{\otimes}_{\mathcal{A}} \mathcal{O}_X), \mathcal{F}) \simeq \mathrm{Map}_{\mathcal{O}_X\text{-}\mathrm{Mod}}(\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{A}}, \mathcal{F}).\end{aligned}$$ □

To conclude this section, we discuss the behavior of the equivalence

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X})) \simeq \mathcal{O}_X^{\mathrm{alg}}\text{-}\mathrm{Mod}$$

under pullback along a morphism of derived analytic spaces.

Proposition 5.19. *Let $f\colon X \to Y$ be a morphism of derived analytic spaces. Let $\mathcal{A} \to f^{-1}\mathcal{O}_Y$ be a morphism in $\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}_X}$. Denote by*

$$(-)^{\mathrm{an}}\colon \mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} \to \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X}$$

the left adjoint to the underlying algebra functor. Then:

(1) *The diagram*

$$\begin{array}{ccccc} \mathrm{AnRing}_k(\mathcal{Y})_{\mathcal{A}//\mathcal{O}_Y} & \xrightarrow{f^{-1}} & \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//f^{-1}\mathcal{O}_Y} & \xrightarrow{f_!^\sharp} & \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}_X} \\ \uparrow{\scriptstyle (-)^{\mathrm{an}}} & & \uparrow{\scriptstyle (-)^{\mathrm{an}}} & & \uparrow{\scriptstyle (-)^{\mathrm{an}}} \\ \mathrm{CRing}_k(\mathcal{Y})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{O}_Y^{\mathrm{alg}}} & \xrightarrow{f^{-1}} & \mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//f^{-1}\mathcal{O}_Y^{\mathrm{alg}}} & \xrightarrow{f_!^\sharp} & \mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{O}_X^{\mathrm{alg}}} \end{array}$$

commutes.

(2) *The diagram*

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{/f^{-1}\mathcal{O}_Y^{\mathrm{alg}}})) & \xrightarrow{f^{\sharp *}} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{/\mathcal{O}_X^{\mathrm{alg}}})) \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ f^{-1}\mathcal{O}_Y^{\mathrm{alg}}\text{-Mod} & \xrightarrow{-\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X} & \mathcal{O}_X^{\mathrm{alg}}\text{-Mod} \end{array}$$

commutes.

(3) *There is a natural equivalence* $f^*\mathbb{L}_Y^{\mathrm{an}} \simeq \mathbb{L}^{\mathrm{an}}_{f^{-1}\mathcal{O}_Y} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

Proof. The first two statements follow from the commutativity of the corresponding diagrams of right adjoints. The last statement is a formal consequence of the previous ones and Lemma 5.8. □

5.3. *Cotangent complex and analytification*

The goal of this subsection is to show that the cotangent complex is compatible with analytification. This result allows us to compute the first examples of analytic cotangent complexes (Corollary 5.26). Finally, we will use these computations in order to prove the base change property of the analytic cotangent complex in the nonarchimedean setting (Proposition 5.27).

Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived algebraic Deligne–Mumford stack locally almost of finite presentation over k. Recall from Section 3 that the analytification functor

$$(-)^{\mathrm{an}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k)) \to {}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$$

is right adjoint to the algebraization functor

$${}^{\mathrm{R}}\mathcal{H}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k)).$$

The counit of the adjunction produces a canonical map

$$p \colon (\mathcal{X}^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}) \to (\mathcal{X}, \mathcal{O}_X).$$

Definition 5.20. We refer to the induced functor

$$p^* \colon \mathcal{O}_X\text{-Mod} \to \mathcal{O}_{X^{\mathrm{an}}}\text{-Mod}$$

as the *analytification functor*, and we denote it by $(-)^{\mathrm{an}}$.

Theorem 5.21. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived algebraic Deligne–Mumford stack locally almost of finite presentation over k. There is a canonical morphism*

$$\varphi \colon \mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}} \to (\mathbb{L}_X)^{\mathrm{an}}$$

*in $\mathcal{O}_{X^{\mathrm{an}}}$-*Mod. *Moreover, φ is an equivalence.*

Proof. Applying Lemma 5.8 with $\mathcal{T} = \mathcal{T}_{\text{ét}}(k)$ to the morphism $p \colon (\mathcal{X}^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}) \to (\mathcal{X}, \mathcal{O}_X)$, we see that $p^{-1}\mathbb{L}_X \simeq \mathbb{L}_{p^{-1}\mathcal{O}_X}$, where we write $\mathbb{L}_{p^{-1}\mathcal{O}_X}$ instead of $\mathbb{L}^{\mathcal{T}_{\text{ét}}(k)}_{p^{-1}\mathcal{O}_X}$. On the other hand, pulling back along the morphism $p^\sharp \colon p^{-1}\mathcal{O}_X \to \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}$ induces the commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/p^{-1}\mathcal{O}_X})) & \xleftarrow{p^\sharp_*} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}})) \\ \Big\downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} & & \Big\downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} \\ \mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/p^{-1}\mathcal{O}_X} & \xleftarrow{p^\sharp_*} & \mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}}. \end{array} \tag{5.22}$$

Passing to the left adjoints and applying Proposition 5.19(2), we obtain

$$(\mathbb{L}_X)^{\mathrm{an}} \simeq \mathbb{L}_{p^{-1}\mathcal{O}_X} \otimes_{p^{-1}\mathcal{O}_X} \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}} \simeq p^{\sharp *}(\mathbb{L}_{p^{-1}\mathcal{O}_X}).$$

Now we apply Lemma 5.15 to the canonical transformation of pregeometries

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\text{ét}}(k) \to \mathcal{T}_{\mathrm{an}}(k)$$

to deduce that the square

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}})) & \xleftarrow{\sim} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}_X})) \\ \Big\downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} & & \Big\downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} \\ \mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}} & \xleftarrow{(-)^{\mathrm{alg}}} & \mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}^{\mathrm{an}}_X} \end{array} \tag{5.23}$$

commutes, where the top morphism is an equivalence in virtue of Theorem 4.11.

Combining diagrams (5.22) and (5.23), we obtain the commutativity of

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/p^{-1}\mathcal{O}_X})) & \longleftarrow & \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}_{X^{\mathrm{an}}}})) \\ \Big\downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} & & \Big\downarrow{\scriptstyle \Omega^\infty_{\mathrm{Ab}}} \\ \mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/p^{-1}\mathcal{O}_X} & \xleftarrow{\Phi} & \mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}_{X^{\mathrm{an}}}} \end{array}$$

where Φ is the composition $p^\sharp_* \circ (-)^{\mathrm{alg}}$. Since both $(-)^{\mathrm{alg}}$ and $p^\sharp_*$ are right adjoint, Φ has a left adjoint

$$\Psi \colon \mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/p^{-1}\mathcal{O}_X} \to \mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}_{X^{\mathrm{an}}}}.$$

To complete the proof, it is enough to prove that $\Psi(p^{-1}\mathcal{O}_X) \simeq \mathcal{O}_{X^{\mathrm{an}}}$. Let

$$(-)^{\mathrm{an}} \colon \mathrm{CRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}} \to \mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}_{X^{\mathrm{an}}}}$$

denote the left adjoint to $(-)^{\mathrm{alg}}$. Unraveling the definitions, we see that it is enough to prove that

$$(p^{-1}\mathcal{O}_X)^{\mathrm{an}} \simeq \mathcal{O}_{X^{\mathrm{an}}}.$$

This amounts to proving that for every $\mathcal{O} \in \mathrm{AnRing}_k(\mathcal{X}^{\mathrm{an}})_{/\mathcal{O}_{X^{\mathrm{an}}}}$, $p^\sharp \colon p^{-1}\mathcal{O}_X \to \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}$ induces an equivalence

$$\mathrm{Map}_{/\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \mathcal{O}) \simeq \mathrm{Map}_{/\mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}}}(p^{-1}\mathcal{O}_X, \mathcal{O}^{\mathrm{alg}}). \tag{5.24}$$

Consider the commutative square

$$\begin{array}{ccc}
\mathrm{Map}_{X^{\mathrm{an}}/}((\mathcal{X}^{\mathrm{an}}, \mathcal{O}), (\mathcal{X}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})) & \longrightarrow & \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}}(\mathcal{X}^{\mathrm{an}}, \mathcal{X}^{\mathrm{an}}) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{Map}_{(\mathcal{X}^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}})/}((\mathcal{X}^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}), (\mathcal{X}^{\mathrm{an}}, p^{-1}\mathcal{O}_X)) & \longrightarrow & \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}}(\mathcal{X}^{\mathrm{an}}, \mathcal{X}^{\mathrm{an}}) \\
\downarrow{\scriptstyle \beta} & & \downarrow \\
\mathrm{Map}_{(\mathcal{X}^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}_{X^{\mathrm{an}}})/}((\mathcal{X}^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}), (\mathcal{X}, \mathcal{O}_X)) & \longrightarrow & \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}}(\mathcal{X}^{\mathrm{an}}, \mathcal{X})
\end{array}$$

The fiber of the top (resp. middle) horizontal morphism at the identity of $\mathcal{X}^{\mathrm{an}}$ is canonically equivalent to the left (resp. right) hand side of (5.24). It is therefore enough to prove that the map α becomes an equivalence after passing to the fiber at $p_* \colon \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$. This follows from two observations: on the one hand, $\beta \circ \alpha$ is an equivalence in virtue of the universal property of analytification; on the other hand, β becomes an equivalence after passing to the fiber at p_*. □

Corollary 5.25. *Let $f \colon X \to Y$ be a morphism of derived algebraic Deligne–Mumford stacks locally almost of finite presentation over k. Then there is a canonical morphism $\varphi \colon \mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}/Y^{\mathrm{an}}} \to (\mathbb{L}_{X/Y})^{\mathrm{an}}$ and moreover φ is an equivalence.*

Proof. Both statements follow at once by combining Theorem 5.21, Proposition 5.10 and Proposition 5.19. □

Corollary 5.26. *The analytic cotangent complex of $\mathbf{A}^n_k$ is free of rank n. In particular, it is perfect and in tor-amplitude* 0.

Proof. Since $\mathbf{A}^n_k \simeq (\mathbb{A}^n_k)^{\mathrm{an}}$, this is an immediate consequence of Theorem 5.21. □

Proposition 5.27. *For any pullback square*

$$\begin{array}{ccc}
X' & \longrightarrow & Y' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{u} & Y
\end{array} \tag{5.28}$$

in dAn_k, *we have a canonical equivalence*

$$g^*\mathbb{L}^{\mathrm{an}}_{X'/Y'} \xrightarrow{\sim} \mathbb{L}^{\mathrm{an}}_{X/Y}.$$

Proof. In the complex case, this is an immediate consequence of Proposition 5.12 and of Remark 5.13. Let us now turn to the nonarchimedean case. Using the transitivity fiber sequence, we see that there is a canonical map

$$g^* \mathbb{L}^{\mathrm{an}}_{X'/Y'} \to \mathbb{L}^{\mathrm{an}}_{X/Y},$$

and we claim that this map is an equivalence. This question is local on X and on Y, so we can suppose that $u\colon X \to Y$ factors as

$$X \overset{j}{\hookrightarrow} Y \times \mathbf{D}^n_k \overset{p}{\to} Y,$$

where j is a closed immersion and p is the projection. We therefore get the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i} & Y' \times \mathbf{D}^n_k & \xrightarrow{q} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{j} & Y \times \mathbf{D}^n_k & \xrightarrow{p} & Y \end{array}$$

It induces a morphism of fiber sequences

$$\begin{array}{ccccc} g^* j^* \mathbb{L}^{\mathrm{an}}_{Y \times \mathbf{D}^n_k / Y} & \longrightarrow & g^* \mathbb{L}^{\mathrm{an}}_{X/Y} & \longrightarrow & g^* \mathbb{L}^{\mathrm{an}}_{X/Y \times \mathbf{D}^n_k} \\ \downarrow & & \downarrow & & \downarrow \\ i^* \mathbb{L}^{\mathrm{an}}_{Y' \times \mathbf{D}^n_k / Y'} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X'/Y'} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X'/Y' \times \mathbf{D}^n_k} \end{array}$$

Since $g^* j^* \simeq i^* h^*$, we are reduced to proving the following statements:

(1) The morphism $h^* \mathbb{L}^{\mathrm{an}}_{Y \times \mathbf{D}^n_k / Y} \to \mathbb{L}^{\mathrm{an}}_{Y' \times \mathbf{D}^n_k / X'}$ is an equivalence.
(2) The morphism $g^* \mathbb{L}^{\mathrm{an}}_{X/Y \times \mathbf{D}^n_k} \to \mathbb{L}^{\mathrm{an}}_{X'/Y' \times \mathbf{D}^n_k}$ is an equivalence.

In other words, we have to prove the proposition in the special case where u is either a closed immersion or a projection of the form $Y \times \mathbf{D}^n_k \to Y$.

We first deal with the case of a closed immersion. Using [19, Proposition 6.2], we see that the above pullback square remains a pullback when considered in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. We can therefore apply Proposition 5.12.

Let us now deal with the case of the projection $p\colon Y \times \mathbf{D}^n_k \to Y$. Consider the ladder of pullback squares

$$\begin{array}{ccc} Y' \times \mathbf{D}^n_k & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Y \times \mathbf{D}^n_k & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbf{D}^n_k & \longrightarrow & \mathrm{Sp}(k) \end{array}$$

Reasoning as before, it is enough to prove that the proposition holds true for the outer square and the bottom one. By symmetry, it is sufficient to handle the bottom square. Since the question is local on Y, we can choose a closed immersion $j\colon Y \hookrightarrow \mathbf{D}^m_k$. We can then further decompose the bottom square as

$$\begin{array}{ccccc} Y \times \mathbf{D}^n_k & \hookrightarrow & \mathbf{D}^{n+m}_k & \longrightarrow & \mathbf{D}^n_k \\ \downarrow & & \downarrow & & \downarrow \\ Y & \overset{j}{\hookrightarrow} & \mathbf{D}^m_k & \longrightarrow & \mathrm{Sp}(k) \end{array}$$

Once again, it is sufficient to prove the proposition for the left and the right squares. Since j is a closed immersion, we already know that the proposition holds true for the left square. For the right square, since the maps $\mathbf{D}^{n+m}_k \to \mathbf{D}^m_k$ and $\mathbf{D}^{n+m}_k \to \mathbf{D}^n_k$ are projections, they are restriction of maps

$$\mathbf{A}^{n+m}_k \to \mathbf{A}^m_k, \quad \mathbf{A}^{n+m}_k \to \mathbf{A}^n_k.$$

Furthermore, the inclusions $\mathbf{D}^l_k \to \mathbf{A}^l_k$ are étale. As a consequence, we can replace the polydisks by affine spaces. In this case, the proposition is a direct consequence of Corollary 5.25. □

5.4. The analytic cotangent complex of a closed immersion

The main result of this subsection asserts that the analytic cotangent complex of a closed immersion can be computed as the algebraic cotangent complex after forgetting the analytic structures. We will deduce from this result the connectivity estimates on the analytic cotangent complex.

Here is the precise statement:

Theorem 5.29. *Let $\mathcal{X}$ be an ∞-topos and let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism in* $\mathrm{AnRing}_k(\mathcal{X})$*. If f is an effective epimorphism, then there is a canonical equivalence*

$$(\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}})^{\mathrm{alg}} \simeq \mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}$$

in $\mathcal{B}^{\mathrm{alg}}$-Mod, *where $(\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}})^{\mathrm{alg}}$ denotes the image of $\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}}$ under the functor*

$$(-)^{\mathrm{alg}}\colon \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}})) \to \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}})).$$

The proof relies on the following lemma:

Lemma 5.30. *Let $\mathcal{X}$ be an ∞-topos and let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism in* $\mathrm{AnRing}_k(\mathcal{X})$*. Suppose that f is an effective epimorphism. Then the commutative diagram*

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}})) & \longleftarrow & \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}})) \\ \Omega^\infty_{\mathrm{Ab}} \downarrow & & \downarrow \Omega^\infty_{\mathrm{Ab}} \\ \mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}} & \xleftarrow{(-)^{\mathrm{alg}}} & \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}} \end{array} \tag{5.31}$$

is left adjointable.

Proof. Using the canonical equivalences

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}})) \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{B}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}})),$$
$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}})) \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{B}//\mathcal{B}})),$$

we can decompose the square (5.31) as

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}})) & \longleftarrow & \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}})) \\ \downarrow {\scriptstyle \Omega^\infty_{\mathrm{Ab}}} & & \downarrow {\scriptstyle \Omega^\infty_{\mathrm{Ab}}} \\ \mathrm{CRing}_k(\mathcal{X})_{\mathcal{B}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}} & \xleftarrow{(-)^{\mathrm{alg}}} & \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{B}//\mathcal{B}} \\ \downarrow {\scriptstyle f_!^{\mathrm{alg}}} & & \downarrow {\scriptstyle f_!} \\ \mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}} & \xleftarrow{(-)^{\mathrm{alg}}} & \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}} \end{array} \tag{5.32}$$

It is then enough to prove that both the upper and the lower squares are left adjointable.

For the lower one, this is a consequence of the unramifiedness of the pregeometry $\mathcal{T}_{\mathrm{an}}(k)$: see [7, Proposition 11.12] for the complex case and [19, Proposition 3.17(iii)] for the nonarchimedean case. Indeed, the left adjoints of $f_!$ and of $f_!^{\mathrm{alg}}$ can respectively be described as the functors

$$\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}} \ni \mathcal{O} \mapsto \mathcal{O}\widehat{\otimes}_{\mathcal{A}}\mathcal{B} \in \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{B}//\mathcal{B}}$$

and

$$\mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}} \ni \mathcal{O} \mapsto \mathcal{O} \otimes_{\mathcal{A}^{\mathrm{alg}}} \mathcal{B}^{\mathrm{alg}} \in \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{B}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}}.$$

Since $f : \mathcal{A} \to \mathcal{B}$ is an effective epimorphism, unramifiedness of $\mathcal{T}_{\mathrm{an}}(k)$ implies that

$$(\mathcal{O}\widehat{\otimes}_{\mathcal{A}}\mathcal{B})^{\mathrm{alg}} \simeq \mathcal{O}^{\mathrm{alg}} \otimes_{\mathcal{A}^{\mathrm{alg}}} \mathcal{B}^{\mathrm{alg}}.$$

For the upper square, it is enough to observe that given $\mathcal{O} \in \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{B}//\mathcal{B}}$, the canonical map $\mathcal{O} \to \mathcal{B}$ has a section and it is therefore an effective epimorphism. In particular, using the unramifiedness of $\mathcal{T}_{\mathrm{an}}(k)$ once again, we obtain

$$(\Sigma(\mathcal{O}))^{\mathrm{alg}} \simeq (\mathcal{B} \mathbin{\widehat{\otimes}_{\mathcal{O}}} \mathcal{B})^{\mathrm{alg}} \simeq \mathcal{B}^{\mathrm{alg}} \otimes_{\mathcal{O}^{\mathrm{alg}}} \mathcal{B}^{\mathrm{alg}} \simeq \Sigma(\mathcal{O}^{\mathrm{alg}}).$$

It follows that the upper square of (5.32) is left adjointable as well. □

Proof of Theorem 5.29. Applying Lemma 5.30 to the morphism f, we see that the square

$$\begin{array}{ccc} \mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}})) & \xleftarrow{(-)^{\mathrm{alg}}} & \mathrm{Sp}(\mathrm{Ab}(\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}})) \\ \uparrow {\scriptstyle \Sigma^\infty_{\mathrm{Ab}}} & & \uparrow {\scriptstyle \Sigma^\infty_{\mathrm{Ab}}} \\ \mathrm{CRing}_k(\mathcal{X})_{\mathcal{A}^{\mathrm{alg}}//\mathcal{B}^{\mathrm{alg}}} & \xleftarrow{(-)^{\mathrm{alg}}} & \mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{B}} \end{array}$$

is commutative. Since $\mathcal{B}$ is sent to $\mathcal{B}^{\mathrm{alg}}$ by the lower horizontal morphism, we conclude that

$$\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}} \simeq \Sigma^\infty_{\mathrm{Ab}}(\mathcal{B}^{\mathrm{alg}}) \simeq (\Sigma^\infty_{\mathrm{Ab}}(\mathcal{B}))^{\mathrm{alg}} \simeq (\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}})^{\mathrm{alg}}.$$ □

Corollary 5.33. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ and $Y = (\mathcal{Y}, \mathcal{O}_Y)$ be derived analytic spaces and let $f: X \to Y$ be a closed immersion. There is a canonical equivalence $\mathbb{L}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \simeq \mathbb{L}^{\mathrm{an}}_{X/Y}$, where X^{alg} and Y^{alg} denote the $\mathcal{T}_{\text{ét}}(k)$-structured topoi $(\mathcal{X}, \mathcal{O}_X^{\mathrm{alg}})$ and $(\mathcal{Y}, \mathcal{O}_Y^{\mathrm{alg}})$ respectively.*

Proof. $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is by definition the analytic cotangent complex of the morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ in $\mathrm{AnRing}_k(\mathcal{X})$. Since f is a closed immersion, this morphism is an effective epimorphism. The statement now follows from Theorem 5.29. □

An important consequence of this fact is the connectivity estimates on the analytic cotangent complex.

Proposition 5.34. *Let $\mathcal{X}$ be an ∞-topos and let $f: \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{AnRing}_k(\mathcal{X})$. Let $\mathrm{cofib}(f)$ denote the cofiber of the underlying map of $\mathcal{D}(\mathrm{Ab})$-valued sheaves. If $\mathrm{cofib}(f)$ is n-connective for $n \geq 1$, then there is a canonical $(2n)$-connective map*

$$\varepsilon_f : \mathrm{cofib}(f) \otimes_{\mathcal{A}^{\mathrm{alg}}} \mathcal{B}^{\mathrm{alg}} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}}.$$

Proof. Since $\pi_0(\mathrm{cofib}(f)) = 0$, we see that f is an effective epimorphism. Therefore, Theorem 5.29 implies that $\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}} \simeq \mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}$. Now, the statement follows immediately from [12, 7.4.3.1]. □

Corollary 5.35. *Let $\mathcal{X}$ be an ∞-topos and let $f: \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{AnRing}_k(\mathcal{X})$. Assume that $\mathrm{cofib}(f)$ is n-connective for some $n \geq 1$. Then $\mathbb{L}^{\mathrm{an}}_{B/A}$ is n-connective. The converse holds provided that f induces an isomorphism $\pi_0(A) \to \pi_0(B)$.*

Proof. This follows from Theorem 5.29 and [12, 7.4.3.2]. □

Lemma 5.36. *Let $f: A \to B$ be a morphism in $\mathrm{AnRing}_k(\mathcal{S})$. Then $\mathbb{L}^{\mathrm{an}}_{B/A}$ is connective.*

Proof. Let $M \in B\text{-Mod}$. Then $\Omega^\infty_B(M) \simeq \Omega^\infty_B(\tau_{\geq 0} M)$. In particular, we obtain

$$\begin{aligned} \mathrm{Map}_{B\text{-Mod}}(\mathbb{L}^{\mathrm{an}}_{B/A}, M) &\simeq \mathrm{Map}_{A//B}(B, \Omega^\infty_B(M)) \simeq \mathrm{Map}_{A//B}(B, \Omega^\infty_B(\tau_{\geq 0} M)) \\ &\simeq \mathrm{Map}_{B\text{-Mod}}(\mathbb{L}^{\mathrm{an}}_{B/A}, \tau_{\geq 0} M) \end{aligned}$$

We conclude that for all $M \in B\text{-Mod}$, we have

$$\mathrm{Map}_{B\text{-Mod}}(\mathbb{L}^{\mathrm{an}}_{B/A}, \tau_{\leq -1} M) \simeq 0.$$

So $\mathbb{L}^{\mathrm{an}}_{B/A}$ is connective. □

Corollary 5.35 has the following important consequence:

Corollary 5.37. *Let $f: X \to Y$ be a morphism of derived analytic spaces. Then f is étale if and only if $\mathrm{t}_0(f)$ is étale and $\mathbb{L}^{\mathrm{an}}_{X/Y} \simeq 0$.*

Proof. If f is étale then Corollary 5.11 shows that $\mathbb{L}^{\mathrm{an}}_{X/Y} \simeq 0$. In this case, we also have $\mathrm{t}_0(X) \simeq \mathrm{t}_0(Y) \times_Y X$ and therefore $\mathrm{t}_0(f)$ is étale. Conversely, if $\mathrm{t}_0(f)$ is étale, we see that the underlying morphism of ∞-topoi is étale. Moreover, $f^\sharp \colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ induces an equivalence on π_0 by hypothesis, and its cotangent complex vanishes. It follows from Corollary 5.35 that it is an equivalence, completing the proof. □

Using the results obtained so far, we can also prove the following important property of the analytic cotangent complex:

Proposition 5.38. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback square in dAn_k. *Then the canonical diagram*

$$\begin{array}{ccc} q^* f^* \mathbb{L}^{\mathrm{an}}_Y & \longrightarrow & q^* \mathbb{L}^{\mathrm{an}}_X \\ \downarrow & & \downarrow \\ p^* \mathbb{L}^{\mathrm{an}}_{Y'} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X'} \end{array}$$

is a pushout square in $\mathcal{O}_{X'}$-Mod.

Proof. Notice that if both f and g are closed immersions, the statement is a direct consequence of [19, Proposition 3.17], of Corollary 5.33 and of [12, 7.3.2.18]. Furthermore, the question is local on X, Y and Y'. We can therefore suppose that f and g factor respectively as

$$X \xhookrightarrow{i} Y \times \mathbf{D}^n_k \xrightarrow{\pi} Y, \qquad X' \xhookrightarrow{j} Y \times \mathbf{D}^n_k \xrightarrow{\pi'} Y,$$

where i and j are closed immersions and π, π' are the canonical projections. Since we already dealt with the case where both morphisms are closed immersions, we are reduced to proving the result for the following pullback square:

$$\begin{array}{ccc} Y \times \mathbf{D}^{n+m}_k & \longrightarrow & Y \times \mathbf{D}^n_k \\ \downarrow & & \downarrow \\ Y \times \mathbf{D}^m_k & \longrightarrow & Y \end{array}$$

Since the canonical inclusions $\mathbf{D}^l_k \hookrightarrow \mathbf{A}^l_k$ are étale, Corollary 5.37 implies that we can replace the disks by the analytic affine spaces. The result is now a direct consequence of Theorem 5.21 and Proposition 5.27. □

We conclude this subsection by proving a finiteness result for the analytic cotangent complex.

Definition 5.39. Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space. The stable ∞-category $\mathcal{O}_X$-Mod is naturally equipped with a t-structure (cf. [10, 2.1.3]). We define the stable ∞-category $\mathrm{Coh}(X)$ of *coherent sheaves* on X to be the full subcategory of $\mathcal{O}_X$-Mod spanned by $\mathcal{F} \in \mathcal{O}_X$-Mod such that $\pi_i(\mathcal{F})$ is a coherent sheaf of $\pi_0(\mathcal{O}_X^{\mathrm{alg}})$-modules for every i. Furthermore, for every $n \in \mathbb{Z}$, we set

$$\mathrm{Coh}^{\geq n}(X) := \mathrm{Coh}(X) \cap \mathcal{O}_X\text{-Mod}^{\geq n}, \quad \mathrm{Coh}^{\leq n}(X) := \mathrm{Coh}(X) \cap \mathcal{O}_X\text{-Mod}^{\leq n},$$
$$\mathrm{Coh}^{+}(X) := \mathrm{Coh}(X) \cap \mathcal{O}_X\text{-Mod}^{+}, \quad \mathrm{Coh}^{-}(X) := \mathrm{Coh}(X) \cap \mathcal{O}_X\text{-Mod}^{-}.$$

Corollary 5.40. *Let $f\colon X \to Y$ be a morphism of derived analytic spaces. Then $\mathbb{L}^{\mathrm{an}}_{X/Y}$ belongs to* $\mathrm{Coh}^{\geq 0}(X)$.

Proof. Using Proposition 5.10, we see that it is enough to prove the statement in the absolute case. Moreover, the question is local on X.

We first deal with the nonarchimedean case. Since we are working locally on X, we can use [19, Lemma 6.3] to show the existence of a closed immersion $j\colon X \hookrightarrow \mathbf{D}^n_k$. Corollary 5.33 guarantees that $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{D}^n_k}$ belongs to $\mathrm{Coh}^{\geq 0}(X)$. Using the transitivity fiber sequence

$$j^*\mathbb{L}^{\mathrm{an}}_{\mathbf{D}^n_k} \to \mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/\mathbf{D}^n_k},$$

we are therefore reduced to proving that the same holds for $j^*\mathbb{L}^{\mathrm{an}}_{\mathbf{D}^n_k}$, and hence for $\mathbb{L}^{\mathrm{an}}_{\mathbf{D}^n_k}$. For the latter statement, we observe that there is a canonical morphism $\mathbf{D}^n_k \hookrightarrow \mathbf{A}^n_k$ which is an affinoid domain and in particular it is étale. As a consequence, it is enough to prove that $\mathbb{L}^{\mathrm{an}}_{\mathbf{A}^n_k} \in \mathrm{Coh}^{\geq 0}(\mathbf{A}^n_k)$. This is a consequence of Corollary 5.26.

In the complex analytic situation, the same proof works. We simply notice that we can always find, locally on X, a closed embedding $X \hookrightarrow \mathbf{A}^n_{\mathbb{C}}$ (cf. [7, Lemma 12.13]). □

5.5. Postnikov towers

An invaluable tool in derived algebraic geometry is the Postnikov tower associated to a derived scheme. More precisely, the fact that the transition maps in this tower are square-zero extensions allows one to translate many problems in derived geometry into deformation-theoretic questions. This technique is extremely useful also in derived analytic geometry and we will use it repeatedly in the rest of this paper.

Definition 5.41. Let $X := (\mathcal{X}, \mathcal{O}_X)$ be a $\mathcal{T}_{\mathrm{an}}(k)$-structured topos and let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}^{\geq 1}$ be an $\mathcal{O}_X$-module. An *analytic square-zero extension of X by $\mathcal{F}$* is a structured topos $X' := (\mathcal{X}, \mathcal{O})$ equipped with a morphism $f\colon X \to X'$ satisfying the following conditions:

(1) The underlying geometric morphism of ∞-topoi is equivalent to the identity of $\mathcal{X}$.
(2) There exists an analytic derivation $d\colon \mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}[1]$ such that

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O}_X \\ {\scriptstyle f^\sharp}\downarrow & & \downarrow{\scriptstyle \eta_d} \\ \mathcal{O}_X & \xrightarrow{\eta_0} & \mathcal{O}_X \oplus \mathcal{F}[1] \end{array}$$

is a pullback square in $\mathrm{AnRing}_k(\mathcal{X})$.

Notation 5.42. Let $X := (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space, $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$ be a coherent sheaf and $d \colon \mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}$ an analytic derivation. We denote by $\mathcal{O}_X \oplus_d \mathcal{F}$ the pullback

$$\begin{array}{ccc} \mathcal{O}_X \oplus_d \mathcal{F} & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \eta_d \\ \mathcal{O}_X & \xrightarrow{\eta_0} & \mathcal{O}_X \oplus \mathcal{F} \end{array}$$

We denote by $X_d[\mathcal{F}]$ the $\mathcal{T}_{\mathrm{an}}(k)$-structured topos $(\mathcal{X}, \mathcal{O}_X \oplus_d \mathcal{F})$. Notice that when d is the zero derivation, $\mathcal{O}_X \oplus_d \mathcal{F}$ coincides with the split square-zero extension $\mathcal{O}_X \oplus \mathcal{F}[-1]$. We denote $X[\mathcal{F}] := X_0[\mathcal{F}[1]]$, and call it the *split square-zero extension* of X by $\mathcal{F}$.

Recall that if $X := (\mathcal{X}, \mathcal{O}_X)$ is a derived analytic space, then

$$\tau_{\leq n}\mathcal{O}_X \colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$$

is again a $\mathcal{T}_{\mathrm{an}}(k)$-structure (see [19, Theorem 3.23] for the nonarchimedean case and [7, Proposition 11.4] for the complex case). In particular, the n-th truncation

$$\mathrm{t}_{\leq n}(X) := (\mathcal{X}, \tau_{\leq n}\mathcal{O}_X)$$

is again a derived analytic space. The main goal of this subsection is to prove that the canonical morphisms $\mathrm{t}_{\leq n}(X) \hookrightarrow \mathrm{t}_{\leq n+1}(X)$ are analytic square-zero extensions. We will deduce it from the following more general result:

Theorem 5.43. *Let $\mathcal{X}$ be an ∞-topos and let $f \colon \mathcal{B} \to \mathcal{A}$ be an effective epimorphism in* $\mathrm{AnRing}_k(\mathcal{X})$. *Let n be a nonnegative integer and suppose that $f^{\mathrm{alg}} \colon \mathcal{B}^{\mathrm{alg}} \to \mathcal{A}^{\mathrm{alg}}$ is an n-small extension in the sense of* [12, 7.4.1.18]. *Then f is an analytic square-zero extension.*

Proof. Consider the analytic derivation

$$d \colon \mathbb{L}^{\mathrm{an}}_{\mathcal{A}} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}} \to \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}}$$

and introduce the associated analytic square-zero extension

$$\begin{array}{ccc} \mathcal{B}' & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \eta_d \\ \mathcal{A} & \xrightarrow{\eta_0} & \mathcal{A} \oplus \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}} \end{array}$$

We claim that the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{f} & \mathcal{A} \\ \downarrow f & & \downarrow \eta_d \\ \mathcal{A} & \xrightarrow{\eta_0} & \mathcal{A} \oplus \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}} \end{array}$$

is commutative. Indeed, the space of morphisms in $\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{A}}$ from $\mathcal{B}$ to $\mathcal{A} \oplus \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}}$ is equivalent to the space

$$\mathrm{Map}_{\mathcal{B}\text{-Mod}}(\mathbb{L}^{\mathrm{an}}_{\mathcal{B}}, \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}}).$$

The composition $\eta_d \circ f$ corresponds to the composition

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{B}} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{B}} \otimes_{\mathcal{B}^{\mathrm{alg}}} \mathcal{A}^{\mathrm{alg}} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{A}} \xrightarrow{d} \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}},$$

and it is therefore homotopic to zero. This produces a canonical map $g\colon \mathcal{B} \to \mathcal{B}'$. We claim that g is an equivalence.

Recall that the functor $(-)^{\mathrm{alg}}$ is conservative (see [19, Lemma 3.13] for the nonarchimedean case and [7, Proposition 11.9] for the complex case). In particular, it is enough to check that g^{alg} is an equivalence. Using Corollary 5.16, we can identify $(\mathcal{A} \oplus \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}})^{\mathrm{alg}}$ with the split square-zero extension

$$\mathcal{A}^{\mathrm{alg}} \oplus \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{B}/\mathcal{A}}.$$

As a consequence, η_d^{alg} corresponds to the algebraic derivation

$$\mathbb{L}_{\mathcal{A}} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{A}} \to \tau_{\leq 2n}\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}}.$$

Since f is an effective epimorphism, we can apply Theorem 5.29 to deduce that

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{A}/\mathcal{B}} \simeq \mathbb{L}_{\mathcal{A}^{\mathrm{alg}}/\mathcal{B}^{\mathrm{alg}}}.$$

Using [12, 7.4.1.26], we conclude that the canonical morphism $g^{\mathrm{alg}}\colon \mathcal{B}^{\mathrm{alg}} \to (\mathcal{B}')^{\mathrm{alg}}$ is an equivalence. □

Corollary 5.44. *For any derived analytic space X, every $n \geq 0$, the canonical map $\mathrm{t}_{\leq n}(X) \hookrightarrow \mathrm{t}_{\leq n+1}(X)$ is an analytic square-zero extension.*

Proof. Using [19, Theorem 3.23] in the nonarchimedean case and [7, Proposition 11.4] in the complex case, we deduce that there are natural equivalences

$$(\tau_{\leq n}\mathcal{O}_X)^{\mathrm{alg}} \simeq \tau_{\leq n}(\mathcal{O}_X^{\mathrm{alg}}).$$

The result is then a direct consequence of Theorem 5.43. □

5.6. The cotangent complex of a smooth morphism

As an application of the results obtained so far, we prove in this subsection that the cotangent complex of a smooth morphism of derived analytic spaces is perfect and in tor-amplitude 0.

Definition 5.45. Let $\mathcal{X}$ be an ∞-topos and let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{AnRing}_k(\mathcal{X})$. We say that f is *strong* if the morphism $f^{\mathrm{alg}}\colon \mathcal{A}^{\mathrm{alg}} \to \mathcal{B}^{\mathrm{alg}}$ is strong, i.e. for every $i \geq 0$, it induces an equivalence

$$\pi_i(\mathcal{A}^{\mathrm{alg}}) \otimes_{\pi_0(\mathcal{B}^{\mathrm{alg}})} \pi_0(\mathcal{A}^{\mathrm{alg}}) \xrightarrow{\sim} \pi_i(\mathcal{B}^{\mathrm{alg}}).$$

Definition 5.46. Let $f\colon X \to Y$ be a morphism of derived analytic spaces. We say that f is *smooth* if it satisfies the following two conditions:

(1) Locally on both X and Y, $\mathrm{t}_0(f)$ is a smooth morphism of ordinary analytic spaces.
(2) The morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is strong.

Lemma 5.47. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of derived analytic spaces. If g and $g \circ f$ are strong, then so is f.*

Proof. Since f^{-1} commutes with homotopy groups and it is monoidal, we see that it preserves strong morphisms. Therefore we are reduced to proving the following statement: if A, B, C are sheaves of connective $\mathbb{E}_\infty$-rings on $\mathcal{X}$ and $\alpha\colon A \to B$ and $\beta\colon B \to C$ are such that α and $\beta \circ \alpha$ are strong, then so is β. Since $\mathcal{X}$ has enough points, we are immediately reduced to the analogous statement for connective $\mathbb{E}_\infty$-rings. In this case, we only need to remark that

$$\pi_i(B) \otimes_{\pi_0(B)} \pi_0(C) \simeq \pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \otimes_{\pi_0(B)} \pi_0(C) \simeq \pi_i(B),$$

so that the statement follows. □

The following lemma is a generalization of [19, Lemma 6.3] and of [7, Lemma 12.13]:

Lemma 5.48. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived affinoid (resp. Stein) space. Suppose that $Y = (\mathcal{Y}, \mathcal{O}_Y)$ is discrete and that $\mathbb{L}^{\mathrm{an}}_Y$ is perfect and in tor-amplitude 0. Then any map $f\colon \mathrm{t}_0(X) \to Y$ admits an extension $\widetilde{f}\colon X \to Y$.*

Proof. We proceed by induction on the Postnikov tower of X. In other words, we will construct a sequence of maps making the following diagram commutative:

$$\begin{array}{ccccccccc}
\mathrm{t}_0(X) & \xrightarrow{j_0} & \mathrm{t}_{\le 1}(X) & \xrightarrow{j_1} & \cdots & \longrightarrow & \mathrm{t}_{\le n}(X) & \xrightarrow{j_n} & \cdots \\
 & \searrow^{f_0} & & \searrow^{f_1} & & & & \searrow^{f_n} & \\
 & & & & & & & & Y
\end{array}$$

Note that the morphisms j_n induce the identity on the underlying ∞-topos. In particular, all the maps f_n are forced to have the same underlying geometric morphism of ∞-topoi, which we simply denote by

$$f^{-1}\colon \mathcal{Y} \leftrightarrows \mathcal{X} : f_*.$$

For the base step, we simply set $f_0 = f$. Suppose now that f_n has been constructed. Recall that there is a fiber sequence

$$\mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})}(f^{-1}\mathcal{O}_Y, \tau_{\le n+1}\mathcal{O}_X) \to \mathrm{Map}_{\mathrm{dAn}_k}(\mathrm{t}_{\le n+1}(X), Y) \to \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}),$$

the fiber being taken at the geometric morphism (f^{-1}, f_*). Let $\varphi_n\colon f^{-1}\mathcal{O}_Y \to \tau_{\le n}\mathcal{O}_X$ be the morphism induced by f_n. We are thus reduced to solving the lifting problem

$$\begin{array}{ccc}
 & & \tau_{\le n+1}\mathcal{O}_X \\
 & \overset{\varphi_{n+1}}{\nearrow} & \downarrow \\
f^{-1}\mathcal{O}_Y & \xrightarrow{\varphi_n} & \tau_{\le n}(\mathcal{O}_X)
\end{array} \tag{5.49}$$

Set $\mathcal{F} := \pi_{n+1}(\mathcal{O}_X)[n+2]$. Using Corollary 5.44, we see that there exists an analytic derivation $d\colon \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leq n} X} \to \mathcal{F}$ such that

$$\begin{array}{ccc} \tau_{\leq n+1}\mathcal{O}_X & \longrightarrow & \tau_{\leq n}\mathcal{O}_X \\ \downarrow & & \downarrow \eta_d \\ \tau_{\leq n}\mathcal{O}_X & \xrightarrow{\eta_0} & \tau_{\leq n}\mathcal{O}_X \oplus \mathcal{F} \end{array}$$

is a pullback square in $\mathrm{AnRing}_k(\mathcal{X})$, where η_0 and η_d correspond to the zero derivation and to d, respectively. This shows that the obstruction to solving (5.49) lives in

$$\pi_0 \operatorname{Map}_{f^{-1}\mathcal{O}_Y\text{-Mod}}(f^{-1}\mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}) \simeq \pi_0 \operatorname{Map}_{\mathrm{Coh}^+(\mathrm{t}_{\leq n} X)}(f_n^* \mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}).$$

It is then enough to prove that the above mapping space vanishes. Since X is a derived affinoid (resp. Stein), it is enough to check that

$$\mathcal{H}\mathrm{om}_{\mathrm{Coh}^+(X)}(f_n^* \mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}) \in \mathrm{Coh}^{\geq 1}(X).$$

We can therefore reason locally on $\mathrm{t}_{\leq n}(X)$. As a consequence, we can assume that $f_n^* \mathbb{L}^{\mathrm{an}}_Y$ is a retract of a free sheaf of $\mathcal{O}_X$-modules. In this case, the statement follows because $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$. Therefore, the obstruction to the lifting vanishes and we obtain the map $f_{n+1}\colon \mathrm{t}_{\leq n+1}(X) \to Y$ we were looking for. □

Proposition 5.50. *Let $f\colon X \to Y$ be a morphism of derived analytic spaces. The following conditions are equivalent:*

(1) *f is smooth.*
(2) *$\mathrm{t}_0(f)$ is smooth and $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is perfect and in tor-amplitude 0.*
(3) *Locally on both X and Y, f can be factored as*

$$X \xrightarrow{g} Y \times \mathbf{A}^n_k \xrightarrow{p} Y,$$

where g is étale and p is the canonical projection.

Proof. Let us start by proving (1)⇔(3). The projection $p\colon Y \times \mathbf{A}^n_k \to Y$ is a smooth morphism, and every étale morphism is smooth. Therefore, if locally on X and Y we can exhibit such a factorization, we can deduce that f is smooth. Let us prove the converse. By definition of smooth morphism and up to localizing on X and Y, we can suppose that we are already given a factorization of $\mathrm{t}_0(f)$ as

$$\mathrm{t}_0(X) \xrightarrow{g_0} Y \times \mathbf{A}^n_k \xrightarrow{p} Y.$$

Let $q\colon Y \times \mathbf{A}^n_k \to \mathbf{A}^n_k$ be the second projection. It follows from Corollary 5.26 and Lemma 5.48 that we can extend $q \circ g_0$ to a morphism $h\colon X \to \mathbf{A}^n_k$. This determines a map $g := f \times h\colon X \to Y \times \mathbf{A}^n_k$, which clearly extends g_0. By construction, $p \circ g \simeq f$. In particular, Lemma 5.47 implies that g is strong. This means that the canonical morphism

$$g^\sharp\colon g^{-1}\mathcal{O}_{Y \times \mathbf{A}^n_k} \to \mathcal{O}_X$$

is strong. It is moreover an equivalence on π_0. It follows that $g^\sharp$ is an equivalence. In particular, g is an étale morphism.

We now prove the equivalence of (1) and (2). Assume first that (1) holds. Then $\mathrm{t}_0(f)$ is smooth, and thus all we have to prove is that $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is perfect and in tor-amplitude 0. This statement is local on both X and Y. We can therefore use (3) to factor f as $p \circ g$, where $g\colon X \to Y \times \mathbf{A}^n_k$ is étale and $p\colon Y \times \mathbf{A}^n_k \to Y$ is the canonical projection. It follows from Corollary 5.37 that $\mathbb{L}^{\mathrm{an}}_{X/Y \times \mathbf{A}^n_k}$ vanishes. In particular, $\mathbb{L}^{\mathrm{an}}_{X/Y} \simeq f^* \mathbb{L}^{\mathrm{an}}_{Y \times \mathbf{A}^n_k / Y}$. Since f is flat, it is therefore sufficient to prove the same statement for p. Applying Proposition 5.27 to the pullback square

$$\begin{array}{ccc} Y \times \mathbf{A}^n_k & \xrightarrow{p} & Y \\ \big\downarrow{\scriptstyle q} & & \big\downarrow \\ \mathbf{A}^n_k & \longrightarrow & \mathrm{Sp}(k) \end{array}$$

we get a canonical equivalence

$$\mathbb{L}^{\mathrm{an}}_{Y \times \mathbf{A}^n_k / Y} \simeq q^* \mathbb{L}^{\mathrm{an}}_{\mathbf{A}^n_k}.$$

The statement is therefore a consequence of Corollary 5.26.

Now assume that $\mathrm{t}_0(f)$ is smooth and that $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is perfect and in tor-amplitude 0. We prove that f is strong. The question is local on both X and Y, and therefore we can localize at a point in X, thus reducing to the analogous statement in $\mathrm{AnRing}_k := \mathrm{AnRing}_k(\mathcal{S})$. In other words, we are given a morphism $\varphi\colon A \to B$ in AnRing_k whose analytic cotangent complex is perfect in tor-amplitude 0, and we want to prove that φ is strong. Form the pushout

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ \pi_0(A) & \longrightarrow & C \end{array}$$

Observe that since $A \to \pi_0(A)$ is an effective epimorphism, $C^{\mathrm{alg}} \simeq B^{\mathrm{alg}} \otimes_{A^{\mathrm{alg}}} \pi_0(A^{\mathrm{alg}})$. We have a canonical map $C \to \pi_0(B)$, and we claim that it is an equivalence. Indeed, suppose it is not. Let $i > 0$ be the smallest integer such that $\pi_i(C) \neq 0$. Let $C_i := \tau_{\leq i}(C)$. We have a fiber sequence

$$\mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(A)} \otimes_{C_i} \pi_0(C) \to \mathbb{L}^{\mathrm{an}}_{\pi_0(C)/\pi_0(A)} \to \mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(C)}.$$

Since $\pi_0(C) \simeq \pi_0(B)$ and since by hypothesis $\mathrm{t}_0(f)$ is smooth, we conclude that $\mathbb{L}^{\mathrm{an}}_{\pi_0(C)/\pi_0(A)}$ is perfect and concentrated in degree 0. In particular, we obtain a canonical identification

$$\pi_i(\mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(A)}) \simeq \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(C)}).$$

Note that Corollary 5.33 and [25, 2.2.2.8] imply together that

$$\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(C)}) \simeq \pi_i(C).$$

Using the connectivity estimates for the analytic cotangent complex provided by Corollary 5.35, we deduce that

$$\pi_i(\mathbb{L}^{\mathrm{an}}_{C/\pi_0(A)} \otimes_C \pi_0(C)) \simeq \pi_i(\mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(A)} \otimes_{C_i} \pi_0(C)) \simeq \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{C_i/\pi_0(C)}) \simeq \pi_i(C) \neq 0.$$

On the other hand, $\mathbb{L}^{\mathrm{an}}_{C/\pi_0(A)} \simeq \mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B C$. In particular, it is perfect and in tor-amplitude 0. Therefore, so is $\mathbb{L}^{\mathrm{an}}_{C/\pi_0(A)} \otimes_C \pi_0(C)$. This is a contradiction, and so $C \simeq \pi_0(C)$. Since $\pi_0(A) \to \pi_0(B)$ is a flat map of ordinary rings, we can now apply [12, 7.2.2.13] to conclude that $\varphi\colon A \to B$ is strong, which completes the proof. □

We conclude the subsection with the following useful lemma.

Lemma 5.51. *Let X and Y be underived analytic spaces, and assume that Y is smooth. Let $f\colon X \to Y$ be a closed immersion. Let $\mathcal{J}$ be the ideal sheaf on Y defining X. Then $\tau_{\leq 1}\mathbb{L}^{\mathrm{an}}_X$ is noncanonically quasi-isomorphic to the complex*

$$\cdots \to 0 \to \mathcal{J}/\mathcal{J}^2 \xrightarrow{\delta} f^*\Omega^{\mathrm{an}}_Y \to 0 \to \cdots,$$

where the map δ is induced by

$$\mathcal{J} \to \mathcal{O}_Y \xrightarrow{d} \Omega^{\mathrm{an}}_Y.$$

Proof. We start with some general discussion. Let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Let

$$M \to N \to P$$

be a fiber sequence. Assume that $M \in \mathcal{C}^{\heartsuit}$ and $N \in \mathcal{C}_{\geq 0}$ and $P \in \mathcal{C}_{\geq 1}$. Let

$$\delta\colon \pi_1(P) \to \pi_0(M)$$

be the natural map. Write $P_1 := \pi_1(P)$ (seen as an object in $\mathcal{C}^{\heartsuit}$). As $M \in \mathcal{C}^{\heartsuit}$, we have a canonical equivalence $M \simeq \pi_0(M)$. We can therefore view δ as a map $\delta\colon P_1 \to M$. Observe that the composition

$$P_1 \to M \to N$$

induces the zero map on homotopy groups. Since the t-structure is complete, we deduce that the above composition is nullhomotopic. For any nullhomotopy α, we thus obtain a canonical map

$$g_\alpha\colon \operatorname{cofib}(P_1 \xrightarrow{\delta} M) \to N.$$

Write $Q := \operatorname{cofib}(P_1 \xrightarrow{\delta} M)$. The five-lemma implies that g_α induces an isomorphism on π_1 and on π_0. We therefore obtain an equivalence (depending on α)

$$h_\alpha\colon Q \simeq \tau_{\leq 1}N.$$

Let us apply this reasoning with $\mathcal{C} = \mathrm{Coh}^+(X)$ and to the fiber sequence

$$f^*\mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/Y}.$$

Notice that $f^*\mathbb{L}^{\mathrm{an}}_Y \in \mathrm{Coh}^{\heartsuit}(X)$ because X is underived and Y is smooth. On the other hand, since $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is surjective, Corollary 5.35 implies that $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Coh}^{\geq 1}(X)$. We therefore obtain a (noncanonical) quasi-isomorphism

$$\tau_{\leq 1}\mathbb{L}^{\mathrm{an}}_X \simeq \mathrm{cofib}(\pi_1(\mathbb{L}^{\mathrm{an}}_{X/Y}) \xrightarrow{\delta} j^*\Omega^{\mathrm{an}}_Y).$$

To complete the proof, we observe that there is a commutative square

$$\begin{array}{ccc} \mathbb{L}_{X/Y} & \xrightarrow{\delta^{\mathrm{alg}}} & j^*\Omega_Y \\ \downarrow & & \downarrow \\ \mathbb{L}^{\mathrm{an}}_{X/Y} & \xrightarrow{\delta} & j^*\Omega^{\mathrm{an}}_Y \end{array}$$

Since $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is surjective, Theorem 5.29 implies that the left vertical map is an equivalence. Furthermore, the morphism δ^{alg} is obtained via the transitivity sequence for algebraic cotangent complexes for the morphism of locally ringed topoi

$$(\mathcal{X}, \mathcal{O}^{\mathrm{alg}}_X) \to (\mathcal{Y}, \mathcal{O}^{\mathrm{alg}}_Y).$$

We can therefore canonically identify δ^{alg} with the inclusion of the conormal sheaf of $f^{-1}\mathcal{O}^{\mathrm{alg}}_Y \to \mathcal{O}^{\mathrm{alg}}_X$ into $j^*\Omega_Y$. Recall now that the conormal sheaf is canonically identified with $\mathcal{J}/\mathcal{J}^2$ and the map to $j^*\Omega_Y$ is the one induced by

$$\mathcal{J} \to \mathcal{O}_Y \xrightarrow{d^{\mathrm{alg}}} \Omega_Y.$$

Recall also that the diagram

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{d^{\mathrm{alg}}} & \Omega_Y \\ & \searrow & \downarrow \\ & & \Omega^{\mathrm{an}}_Y \end{array}$$

commutes. Thus δ coincides with the map induced by $\mathcal{J} \to \mathcal{O}_Y \xrightarrow{d} \Omega^{\mathrm{an}}_Y$. □

6. Gluing along closed immersions

In this section we prove that the ∞-category dAn_k of derived analytic spaces is closed under pushout along closed immersions. Using the Postnikov tower machinery provided by Corollary 5.44, we can decompose the problem into two smaller tasks. First, we need to know that the category An_k of underived analytic spaces is closed under pushout along closed immersions; second, we need to know that any analytic square-zero extension of a derived analytic space is again a derived analytic space. This second problem is also a good testing ground for our notion of analytic derivation, hence our construction of the analytic cotangent complex. The reason is that the square-zero extension of a derived analytic space by an arbitrary algebraic derivation is in general no longer a derived analytic space.

Proposition 6.1. *Let $X := (\mathcal{X}, \mathcal{O}_X)$ be an underived analytic space. Let $\mathcal{F} \in \mathrm{Coh}^{\heartsuit}(X)$ and let $X' := (\mathcal{X}, \mathcal{O}')$ be an analytic square-zero extension of X by $\mathcal{F}$. Then X' is an underived analytic space.*

Proof. By definition, there exists an analytic derivation $\mathbb{L}_X^{\mathrm{an}} \to \mathcal{F}[1]$ such that

$$\begin{array}{ccc} \mathcal{O}' & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \eta_d \\ \mathcal{O}_X & \xrightarrow{\eta_0} & \mathcal{O}_X \oplus \mathcal{F}[1] \end{array}$$

is a pullback square in $\mathrm{AnRing}_k(\mathcal{X})$. Here η_0 corresponds to the zero derivation and η_d corresponds to d.

It follows that there is a fiber sequence

$$\mathcal{F} \to \mathcal{O}' \to \mathcal{O}_X.$$

Since both $\mathcal{O}_X$ and $\mathcal{F}$ are discrete, so is $\mathcal{O}'$. It remains to check that X' is an analytic space. This question is local on X and we can therefore suppose that X is an affinoid (resp. Stein) space and admits a closed embedding $j\colon X \hookrightarrow Y$, where Y is either $\mathbf{D}_k^n$ or $\mathbf{A}_{\mathbb{C}}^n$.

Let $\mathcal{J}$ denote the sheaf of ideals defining X as a closed subspace of Y. It follows from Lemma 5.51 that $\mathbb{L}_X^{\mathrm{an}}$ satisfies the relation

$$\tau_{\leq 1}\mathbb{L}_X^{\mathrm{an}} \simeq (\cdots \to 0 \to \mathcal{J}/\mathcal{J}^2 \to j^*\Omega_Y^{\mathrm{an}} \to 0 \to \cdots).$$

In particular, we can describe $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{L}_X^{\mathrm{an}}, \mathcal{F})$ as the cokernel of the map

$$\mathrm{Hom}_{\mathcal{O}_X}(j^*\Omega_Y^{\mathrm{an}}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2, \mathcal{F}).$$

Fix $\alpha\colon \mathcal{J}/\mathcal{J}^2 \to \mathcal{F}$. We can describe the associated extension as follows. Let Z denote the closed analytic subspace of Y defined by the sheaf of ideals $\mathcal{J}^2$. Then we can see $\mathcal{F}$ as a coherent sheaf on Z and we introduce the split square-zero extension $Z[\mathcal{F}]$. Let $\gamma\colon \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_Y/\mathcal{J}^2 \simeq \mathcal{O}_Z$ be the natural map and consider the morphism of $\mathcal{O}_{Z[\mathcal{F}]}$-modules $\beta\colon \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_Z \oplus \mathcal{F}$ defined by $x \mapsto (\gamma(x), \alpha(x))$. The image of β is an ideal $\mathcal{I}$, and we have $\mathcal{O}' = \mathcal{O}_{Z[\mathcal{F}]}/\mathcal{I}$. Since $Z[\mathcal{F}]$ was an analytic space, so is X'. □

Lemma 6.2 (Artin–Tate). *Let S be a k-affinoid algebra. If $T \subset S$ is a k-subalgebra such that S is a finite T-module, then T is k-affinoid.*

Proof. We follow the proof of [22, Tag 00IS]. Choose a system of affinoid generators $x_1, \ldots, x_n$ of S. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the k-affinoid subalgebra generated by a_{ij} and b_{ijk}. Consider the k-affinoid algebra

$$S' = T'\langle Y_1, \ldots, Y_m\rangle / \Big(Y_i Y_j - \sum b_{ijk} Y_k\Big).$$

Note that S' is a finite T'-module. The T'-algebra homomorphism $S' \to S$ sending Y_i to y_i is surjective by construction. So S is finite over T'. Since T' is noetherian, we conclude that $T \subset S$ is also finite over T', and so by [1, §6.1, Proposition 6], T is k-affinoid. □

Lemma 6.3. *Let $A' \to A$ and $B \to A$ be maps of k-affinoid algebras. Let $B' := A' \times_A B$ as rings. Assume that A is a finite A'-module and $B \to A$ is surjective. Then B' is a k-affinoid algebra.*

Proof. We follow the proof of [22, Tag 00IT]. Choose $y_1, \dots, y_n \in A$ which generate A as an A'-module. Choose $x_i \in B$ mapping to y_i. Then $1, x_1, \dots, x_n$ generate B as a B'-module. Since A' is also a finite B'-module, the product (i.e. direct sum) $A' \times B$ is a finite B'-module. As a corollary of [1, §6.1, Proposition 6], $A' \times B$ is k-affinoid. Note that $B' \subset A' \times B$ and $A' \times B$ is finite as a B'-module. Now use Lemma 6.2. □

Proposition 6.4. *Let $i\colon X \to X'$ and $j\colon X \to Y$ be closed immersions of underived analytic spaces. Then the pushout*

$$\begin{array}{ccc} X & \xhookrightarrow{i} & X' \\ {\scriptstyle j}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & Y' \end{array}$$

exists in An_k. *Furthermore, the forgetful functor* $\mathrm{An}_k \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ *preserves this pushout.*

Proof. In the complex case, this is a special case of [2, Théorème 3]. Let us now prove the nonarchimedean case. Since [7, Theorem 5.1] guarantees the existence of the pushout in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, we are reduced to checking that Y' is an underived analytic space. This question is local on Y', so it is also local on X' and on Y. In other words, we can assume from the beginning that X, X' and Y are affinoid spaces. Let us write

$$X = \mathrm{Sp}(A), \qquad X' = \mathrm{Sp}(A'), \qquad Y = \mathrm{Sp}(B).$$

Furthermore, denote by $f\colon A' \to A$ and $g\colon B \to A$ the maps corresponding to i and j respectively. Since i and j are closed immersions, f and g are surjective. Let B' be the (discrete) commutative ring defined via the following pullback diagram in CRing_k:

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ A' & \xrightarrow{f} & A \end{array}$$

By Lemma 6.3, B' is a k-affinoid algebra. Let $\mathcal{X}_A$ denote the étale ∞-topos of A and define $\mathcal{X}_{A'}$, $\mathcal{X}_B$ and $\mathcal{X}_{B'}$ similarly. In order to complete the proof, it suffices to show that the natural diagram

$$\begin{array}{ccc} \mathcal{X}_A & \xrightarrow{i_*} & \mathcal{X}_{A'} \\ {\scriptstyle j_*}\big\downarrow & & \big\downarrow \\ \mathcal{X}_B & \longrightarrow & \mathcal{X}_{B'} \end{array}$$

is a pushout diagram in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$. Now the proof of [7, Corollary 6.5] applies, with the only caveat that one should use [19, Proposition 3.10] instead of [10, 1.2.7]. □

We are now ready for the main theorem of this section:

Theorem 6.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & Y' \end{array}$$

be a pushout square in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$*. Suppose that* i *and* j *are closed immersions and* X, X', Y *are derived analytic spaces. Then* Y' *is also a derived analytic space.*

Before starting the proof, we need the following technical lemma:

Lemma 6.6. *Let* $j_* \colon \mathcal{X} \leftrightarrows \mathcal{Y} : j^{-1}$ *be a closed immersion of* ∞*-topoi. Then* j_* *commutes with truncations. In other words, there are natural equivalences*

$$j_* \circ \tau_{\leq n}^{\mathcal{X}} \simeq \tau_{\leq n}^{\mathcal{Y}} \circ j_* \quad \text{for every } n \geq 0.$$

Proof. By definition of closed immersion, we can find a (-1)-truncated object $U \in \mathcal{Y}$ and an equivalence $\mathcal{X} \simeq \mathcal{Y}/U$. The functor $j_* \colon \mathcal{Y}/U \to \mathcal{Y}$ is fully faithful and [6, 7.3.2.5] guarantees that an object $V \in \mathcal{Y}$ belongs to $\mathcal{Y}/U$ if and only if $V \times U \simeq U$. Now let $V \in \mathcal{Y}/U$. Since U is (-1)-truncated, we see that $\tau_{\leq n}^{\mathcal{Y}}(U) \simeq U$ and therefore

$$\tau_{\leq n}^{\mathcal{Y}}(V) \times U \simeq \tau_{\leq n}^{\mathcal{Y}}(V) \times \tau_{\leq n}^{\mathcal{Y}}(U) \simeq \tau_{\leq n}^{\mathcal{Y}}(V \times U) \simeq \tau_{\leq n}^{\mathcal{Y}}(U) \simeq U.$$

In other words, $\tau_{\leq n}^{\mathcal{Y}}(V)$ belongs to $\mathcal{Y}/U$. Since furthermore j_* is fully faithful and commutes with n-truncated objects, we conclude that $\tau_{\leq n}^{\mathcal{Y}}(V) \simeq \tau_{\leq n}^{\mathcal{X}}(V)$. □

Proof of Theorem 6.5. The question is local on Y', so it is also local on Y and on X'. We can therefore assume that X, X' and Y are derived affinoid (resp. Stein) spaces.

Write

$$X = (\mathcal{X}, \mathcal{O}_X), \qquad X' = (\mathcal{X}', \mathcal{O}_{X'}), \qquad Y = (\mathcal{Y}, \mathcal{O}_Y), \qquad Y' = (\mathcal{Y}', \mathcal{O}_{Y'}).$$

The morphisms i and j induce closed immersions of the underlying ∞-topoi

$$i_* \colon \mathcal{X} \leftrightarrows \mathcal{X}' : i^{-1}, \qquad j_* \colon \mathcal{X} \leftrightarrows \mathcal{Y} : j^{-1}.$$

Using [7, Theorem 5.1], we can identify $\mathcal{Y}'$ with the pushout

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i_*} & \mathcal{X}' \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathcal{Y} & \xrightarrow{q_*} & \mathcal{Y}' \end{array}$$

computed in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$. Let $h \colon X \to Y'$ denote the compositions $p \circ i \simeq q \circ j$. We can use [7, Theorem 5.1] once more to identify $\mathcal{O}_{Y'}$ with the pullback

$$\begin{array}{ccc} \mathcal{O}_{Y'} & \longrightarrow & p_*\mathcal{O}_{X'} \\ \downarrow & & \downarrow \\ q_*\mathcal{O}_Y & \longrightarrow & h_*\mathcal{O}_X \end{array} \tag{6.7}$$

In particular, we obtain a long exact sequence of homotopy groups

$$\pi_1(p_*\mathcal{O}_{X'}^{\mathrm{alg}}) \oplus \pi_1(q_*\mathcal{O}_Y^{\mathrm{alg}}) \to \pi_1(h_*\mathcal{O}_X^{\mathrm{alg}}) \to \pi_0\mathcal{O}_{Y'}^{\mathrm{alg}} \to \pi_0(p_*\mathcal{O}_{X'}^{\mathrm{alg}}) \oplus \pi_0(q_*\mathcal{O}_Y^{\mathrm{alg}}) \to \pi_0(h_*\mathcal{O}_X^{\mathrm{alg}}) \to 0 \tag{6.8}$$

Now consider the truncations $\mathrm{t}_0(X)$, $\mathrm{t}_0(X')$, $\mathrm{t}_0(Y)$ and let Y'' be the pushout

$$\begin{array}{ccc} \mathrm{t}_0(X) & \xrightarrow{\mathrm{t}_0(i)} & \mathrm{t}_0(X') \\ {\scriptstyle \mathrm{t}_0(j)}\downarrow & & \downarrow \\ \mathrm{t}_0(Y) & \longrightarrow & Y'' \end{array}$$

in An_k, whose existence is guaranteed by Proposition 6.4. Furthermore, Proposition 6.4 ensures that the ∞-topos underlying Y'' coincides with $\mathcal{Y}'$ and that the structure sheaf $\mathcal{O}_{Y''}$ fits in the pullback diagram

$$\begin{array}{ccc} \mathcal{O}_{Y''} & \longrightarrow & p_*\pi_0(\mathcal{O}_{X'}) \\ \downarrow & & \downarrow \\ q_*\pi_0(\mathcal{O}_Y) & \longrightarrow & h_*\pi_0(\mathcal{O}_X) \end{array}$$

Using Lemma 6.6, we deduce that there are canonical equivalences

$$p_*\pi_0(\mathcal{O}_{X'}) \simeq \pi_0(p_*\mathcal{O}_{X'}), \qquad q_*\pi_0(\mathcal{O}_X) \simeq \pi_0(q_*\mathcal{O}_X), \qquad h_*\pi_0(\mathcal{O}_X) \simeq \pi_0(h_*\mathcal{O}_X).$$

We can therefore split the long exact sequence (6.8) into

$$0 \to \mathcal{J} \to \pi_0(\mathcal{O}_{Y'}^{\mathrm{alg}}) \to \mathcal{O}_{Y''}^{\mathrm{alg}} \to 0,$$

where

$$\mathcal{J} := \mathrm{coker}\big(\pi_1(p_*\mathcal{O}_{X'}^{\mathrm{alg}}) \oplus \pi_1(q_*\mathcal{O}_Y^{\mathrm{alg}}) \to \pi_1(h_*\mathcal{O}_X^{\mathrm{alg}})\big).$$

Using Lemma 6.6 once more, we deduce that there are natural equivalences

$$\pi_1(p_*\mathcal{O}_{X'}) \simeq p_*(\pi_1\mathcal{O}_{X'}), \qquad \pi_1(q_*\mathcal{O}_Y^{\mathrm{alg}}) \simeq q_*(\pi_1\mathcal{O}_Y^{\mathrm{alg}}), \qquad \pi_1(h_*\mathcal{O}_X^{\mathrm{alg}}) \simeq h_*\pi_1(\mathcal{O}_X^{\mathrm{alg}}).$$

This implies that the above sheaves are coherent sheaves of $\mathcal{O}_{Y''}^{\mathrm{alg}}$-modules. As a consequence, $\mathcal{J}$ is also a coherent sheaf of $\mathcal{O}_{Y''}^{\mathrm{alg}}$-modules. Finally, $\pi_0(\mathcal{O}_{Y'}^{\mathrm{alg}})$ and $\mathcal{O}_{Y''}^{\mathrm{alg}}$ have the same support. This implies that $\mathcal{J}$ is (locally) a nilpotent sheaf of ideals of $\pi_0(\mathcal{O}_{Y'}^{\mathrm{alg}})$. Proceeding by induction, we can therefore suppose that $\mathcal{J}^2 = 0$.

We are therefore reduced to the case where $\pi_0(\mathcal{O}_{Y'})$ is a square-zero extension of $\mathcal{O}_{Y''}$. In this case, we can invoke Theorem 5.43 to conclude that $\pi_0(\mathcal{O}_{Y'})$ is an *analytic* square-zero extension of $\mathcal{O}_{Y''}$. Using Proposition 6.1, we conclude that the $\mathcal{T}_{\mathrm{an}}(k)$-structured topos $(\mathcal{Y}', \pi_0(\mathcal{O}_{Y'}))$ is an analytic space. To complete the proof, we only have to prove that each $\pi_i(\mathcal{O}_{Y'})$ is coherent over $\pi_0(\mathcal{O}_{Y'})$. Observe that

$$\pi_0(\mathcal{O}_{Y'}) \to \pi_0(p_*\mathcal{O}_{X'}), \qquad \pi_0(\mathcal{O}_{Y'}) \to \pi_0(h_*\mathcal{O}_X), \qquad \pi_0(\mathcal{O}_{Y'}) \to \pi_0(q_*\mathcal{O}_Y)$$

are epimorphisms. The conclusion now follows from the long exact sequence associated to the pullback diagram (6.7). □

7. The representability theorem

The goal of this section is to prove the main theorem of this paper, i.e. the representability theorem in derived analytic geometry.

Let k be either the field $\mathbb{C}$ of complex numbers, or a complete nonarchimedean field with nontrivial valuation.

Let Afd_k denote the category of k-affinoid spaces when k is nonarchimedean, and the category of Stein spaces when $k = \mathbb{C}$. Let dAfd_k denote the ∞-category of derived k-affinoid spaces when k is nonarchimedean, and the ∞-category of derived Stein spaces when $k = \mathbb{C}$.

Let us state the theorem before giving the precise definitions of the notions involved.

Theorem 7.1. *Let F be a stack over the ∞-site* $(\mathrm{dAfd}_k, \tau_{\text{ét}})$. *The following are equivalent:*

(1) *F is an n-geometric stack with respect to the geometric context* $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$.
(2) *F is compatible with Postnikov towers, has a global analytic cotangent complex, and its truncation $\mathrm{t}_0(F)$ is an n-geometric stack with respect to the geometric context* $(\mathrm{Afd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$.

We refer to [17, §2] for the notions of geometric context and geometric stack with respect to a given geometric context. Recall that a geometric context $(\mathcal{C}, \tau, \mathbf{P})$ consists of a small ∞-category $\mathcal{C}$ equipped with a Grothendieck topology τ and a class $\mathbf{P}$ of morphisms in $\mathcal{C}$, satisfying a short list of axioms. In the statement of Theorem 7.1, $\tau_{\text{ét}}$ is the étale topology and $\mathbf{P}_{\mathrm{sm}}$ the class of smooth morphisms.

A *stack* over an ∞-site $(\mathcal{C}, \tau)$ is by definition a hypercomplete sheaf with values in spaces over the ∞-site. We denote by $\mathrm{St}(\mathcal{C}, \tau)$ the ∞-category of stacks over $(\mathcal{C}, \tau)$.

Given a geometric context $(\mathcal{C}, \tau, \mathbf{P})$ and an integer $n \geq -1$, the notion of n-geometric stack is defined by induction on the geometric level n. We refer to [17, §2.3] for the details. Let us simply recall that a (-1)-geometric stack is by definition a representable stack.

Definition 7.2. A *derived analytic stack* is an n-geometric stack with respect to the geometric context $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$ for some n.

The following definitions are analytic analogues of the algebraic notions introduced in [11, 25].

Definition 7.3. Let $f\colon F \to G$ be a morphism in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. We say that f is *infinitesimally Cartesian* if for every derived affinoid (resp. Stein) space $X \in \mathrm{dAfd}_k$, every coherent sheaf $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$ and every analytic derivation $d\colon \mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}$, the square

$$\begin{array}{ccc} F(X_d[\mathcal{F}]) & \longrightarrow & G(X_d[\mathcal{F}]) \\ \downarrow & & \downarrow \\ F(X) \times_{F(X[\mathcal{F}])} F(X) & \longrightarrow & G(X) \times_{G(X[\mathcal{F}])} G(X) \end{array}$$

is a pullback square. We say that a stack $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ is *infinitesimally Cartesian* if the canonical map $F \to *$ is infinitesimally Cartesian, where $*$ denotes a final object of $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$.

Definition 7.4. Let $f\colon F \to G$ be a morphism in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. We say that f is *convergent* (or *nil-complete*) if for every derived affinoid (resp. Stein) space $X = (\mathcal{X}, \mathcal{O}_X) \in \mathrm{dAfd}_k$, the square

$$\begin{array}{ccc} F(X) & \longrightarrow & \lim_n F(\mathrm{t}_{\leq n} X) \\ \downarrow & & \downarrow \\ G(X) & \longrightarrow & \lim_n G(\mathrm{t}_{\leq n} X) \end{array}$$

is a pullback square. We say that a stack $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ is *convergent* if the canonical map $F \to *$ is convergent, where $*$ denotes a final object of $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$.

Definition 7.5. A morphism $f\colon F \to G$ is said to be *compatible with Postnikov towers* if it is infinitesimally Cartesian and convergent.

Let $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. Let $X \in \mathrm{dAfd}_k$ and let $x\colon X \to F$ be a morphism of sheaves. For every coherent sheaf $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(X)$, we denote by $\mathrm{Der}^{\mathrm{an}}_F(X, \mathcal{F})$ the fiber at x of the canonical map

$$F(X[\mathcal{F}]) \to F(X).$$

This assignment is functorial in $\mathcal{F}$ and therefore provides a functor

$$\mathrm{Der}^{\mathrm{an}}_F(X, -)\colon \mathrm{Coh}^{\geq 0}(X) \to \mathcal{S}.$$

If $f\colon F \to G$ is a morphism of sheaves, we obtain a natural transformation

$$\eta\colon \mathrm{Der}^{\mathrm{an}}_F(X, -) \to \mathrm{Der}^{\mathrm{an}}_G(X, -)$$

for every fixed $X \in \mathrm{dAfd}_k$ and every fixed morphism $x\colon X \to F$. For every $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(X)$, the space $\mathrm{Der}^{\mathrm{an}}_G(X, \mathcal{F})$ has a distinguished element: the zero derivation. Let us denote the fiber of $\eta_{\mathcal{F}}$ at the zero derivation by $\mathrm{Der}^{\mathrm{an}}_{F/G}(X, \mathcal{F})$. It is naturally functorial in $\mathcal{F}$. We denote the corresponding functor by

$$\mathrm{Der}^{\mathrm{an}}_{F/G}(X, -)\colon \mathrm{Coh}^{\geq 0}(X) \to \mathcal{S}.$$

Definition 7.6. Let $f\colon F \to G$ be a morphism in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$.

(1) Let $X \in \mathrm{dAfd}_k$ and let $x\colon X \to F$ be a morphism. We say that f *has an analytic cotangent complex at* x if the functor

$$\mathrm{Der}^{\mathrm{an}}_{F/G}(X, -)\colon \mathrm{Coh}^{\geq 0}(X) \to \mathcal{S}$$

is corepresentable by an object in $\mathrm{Coh}^{+}(X)$. In this case, we denote this object by $\mathbb{L}^{\mathrm{an}}_{F/G,x}$.

(2) We say that f *has a global analytic cotangent complex* if the following conditions are satisfied:

(a) f has an analytic cotangent complex at every morphism $x\colon X \to F$ for every $X \in \mathrm{dAfd}_k$.

(b) For any morphism $g\colon X \to Y$ in dAfd_k, any morphism $y\colon Y \to F$, denote $x := y \circ g$. Then the canonical morphism

$$g^* \mathbb{L}^{\mathrm{an}}_{F/G,y} \to \mathbb{L}^{\mathrm{an}}_{F/G,x}$$

is an equivalence in $\mathrm{Coh}^+(X)$.

To prove Theorem 7.1, we will address the implication (1)⇒(2) in Section 7.1, and the implication (2)⇒(1) in Section 7.2.

7.1. *Properties of derived analytic stacks*

In this subsection, we prove the implication (1)⇒(2) of Theorem 7.1. We will first prove that (2) holds for derived analytic spaces, and then we will prove it for derived analytic stacks by induction on the geometric level.

Lemma 7.7. *Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space and let $F_X \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ be the associated stack via the Yoneda embedding. Then F_X is infinitesimally Cartesian, convergent and it admits a global analytic cotangent complex.*

Proof. Let $Y \in \mathrm{dAfd}_k$ be a derived affinoid (resp. Stein) space. Let $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(Y)$ and let $d\colon \mathbb{L}^{\mathrm{an}}_Y \to \mathcal{F}$ be an analytic derivation. It follows from Theorem 6.5 that

$$\begin{array}{ccc} Y[\mathcal{F}] & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_d[\mathcal{F}] \end{array}$$

is a pushout square in dAn_k. As a consequence, $F_X = \mathrm{Map}_{\mathrm{dAn}_k}(-, X)$ takes this diagram into a pullback square. In other words, F_X is infinitesimally Cartesian.

Let $Y = (\mathcal{Y}, \mathcal{O}_Y) \in \mathrm{dAfd}_k$. Since $\mathcal{Y}$ is hypercomplete, the canonical map

$$\operatorname*{colim}_n \mathrm{t}_{\leq n}(Y) \to Y$$

is an equivalence in dAn_k. In particular,

$$\begin{aligned} F_X(Y) = \mathrm{Map}_{\mathrm{dAn}_k}(Y, X) &\simeq \mathrm{Map}_{\mathrm{dAn}_k}\Big(\operatorname*{colim}_n \mathrm{t}_{\leq n}(Y), X\Big) \\ &\simeq \lim_n \mathrm{Map}_{\mathrm{dAn}_k}(\mathrm{t}_{\leq n}(Y), X) \simeq \lim_n F_X(\mathrm{t}_{\leq n}(Y)). \end{aligned}$$

It follows that F_X is convergent.

Let us now show that F_X admits a global cotangent complex. Let $\mathbb{L}^{\mathrm{an}}_X$ be the analytic cotangent complex of X introduced in Section 5.2. It follows from Corollary 5.40

that $\mathbb{L}_X^{\mathrm{an}} \in \mathrm{Coh}^{\geq 0}(X)$. It will therefore be sufficient to prove that for every derived affinoid (resp. Stein) space $Y = (\mathcal{Y}, \mathcal{O}_Y)$ and every map $y\colon Y \to F_X$, the object $y^*\mathbb{L}_X^{\mathrm{an}} \in \mathrm{Coh}^{\geq 0}(Y)$ satisfies the universal property of the analytic cotangent complex. Recall now that derived analytic spaces embed fully faithfully in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$: in the nonarchimedean case, this follows from [19, Theorem 7.9], while in the complex case it is a consequence of [14, Theorem 3.7]. Therefore the map y corresponds to a unique (up to a contractible space of choices) map $f_y\colon Y \to X$ in dAn_k. Using again the fully faithfulness of the embedding of derived analytic spaces in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$, we conclude that

$$\mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}) = \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{Y})_{/\mathcal{O}_Y}}(f_y^{-1}\mathcal{O}_X, \mathcal{O}_Y \oplus \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(Y)}(y^*\mathbb{L}_X^{\mathrm{an}}, \mathcal{F}).$$

This completes the proof. □

We will now show that the above conditions are also satisfied by derived analytic stacks. Our arguments are similar to [25, §1.4.3].

Lemma 7.8. *Let $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. If F is infinitesimally Cartesian, then for every $X \in \mathrm{dAfd}_k$, every point $x\colon X \to F$ and every connective coherent sheaf $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(X)$, the canonical morphism*

$$\mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}) \to \Omega\mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}[1])$$

is an equivalence.

Proof. Let $X \in \mathrm{dAfd}_k$ be a derived affinoid (resp. Stein) space, and let $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(X)$. Since F is infinitesimally Cartesian, we have a pullback square

$$\begin{array}{ccc} F(X[\mathcal{F}]) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & F(X[\mathcal{F}[1]]) \end{array}$$

We have a canonical map $F(X[\mathcal{F}[1]]) \to F(X)$ induced by the closed immersion $X \to X[\mathcal{F}[1]]$. Taking fibers at $x \in \pi_0(F(X))$, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}) & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}[1]) \end{array}$$

Hence, we conclude that $\mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}) \simeq \Omega\mathrm{Der}_F^{\mathrm{an}}(X, \mathcal{F}[1])$. □

Proposition 7.9. *Let $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ be an n-geometric stack with respect to the geometric context $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$. If F is infinitesimally Cartesian, then it has a global cotangent complex, which is $(-n)$-connective.*

Proof. We follow closely the proof of [25, 1.4.1.11]. We proceed by induction on n. If $n = -1$, then the statement follows from Lemma 7.7. Let therefore $n \geq 0$ and let F be an n-geometric stack and $x \colon X \to F$ be a point, with $X \in \mathrm{dAfd}_k$. Consider the natural morphisms

$$\delta \colon X \to X \times X, \qquad \delta_F \colon X \to X \times_F X.$$

By induction, both $X \times X$ and $X \times_F X$ have analytic cotangent complexes at δ and at δ_F; denote them by $\mathbb{L}$, $\mathbb{L}'$, respectively. Since δ factors through δ_F, there is a canonical map $f \colon \mathbb{L} \to \mathbb{L}'$ in $\mathrm{Coh}^+(X)$. Let $\mathbb{L}'' := \mathrm{cofib}(f)$. By definition, for any $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(X)$, the space $\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}'', \mathcal{F})$ is the fiber of

$$\mathrm{Der}^{\mathrm{an}}_X(X \times_F X, \mathcal{F}) \to \mathrm{Der}^{\mathrm{an}}_X(X \times X, \mathcal{F}).$$

Now, $\mathrm{Der}^{\mathrm{an}}_X(X \times X, \mathcal{F}) \simeq \{*\}$, while

$$\mathrm{Der}^{\mathrm{an}}_X(X \times_F X, \mathcal{F}) \simeq \mathrm{Der}^{\mathrm{an}}_X(X, \mathcal{F}) \times_{\mathrm{Der}^{\mathrm{an}}_F(X, \mathcal{F})} \mathrm{Der}^{\mathrm{an}}_X(X, \mathcal{F}) \simeq \Omega \mathrm{Der}^{\mathrm{an}}_F(X, \mathcal{F}).$$

As a consequence,

$$\mathrm{Map}_{\mathrm{Coh}^+(X)}(\Omega(\mathbb{L}''), \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}'', \mathcal{F}[1]) \simeq \Omega \mathrm{Der}^{\mathrm{an}}_F(X, \mathcal{F}[1]) \simeq \mathrm{Der}^{\mathrm{an}}_F(X, \mathcal{F}).$$

It follows that F has a cotangent complex at x. Moreover, the inductive hypothesis shows that both $\mathbb{L}$ and $\mathbb{L}'$ are $(-n+1)$-connective. It follows that $\mathbb{L}''$ is $(-n+1)$-connective as well, and therefore $\Omega(\mathbb{L}'') = \mathbb{L}''[-1]$ is $(-n)$-connective. The argument of [25, 1.4.1.12] shows that F has a global cotangent complex. □

Our next task is to show that any n-geometric stack with respect to the geometric context $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$ is infinitesimally Cartesian.

Let us recall that the notion of smooth morphism between derived analytic spaces is local on both source and target. Therefore, we can extend it to representable morphisms in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ (cf. [17, Remark 2.10]). More explicitly, an n-representable morphism $f \colon F \to G$ in $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ is smooth if and only if for every $U \in \mathrm{dAfd}_k$ and every map $U \to G$, there exists an atlas $\{V_i\}$ of $U \times_G F$ such that the compositions $V_i \to U$ are smooth morphisms of derived analytic spaces.

Proposition 7.10. (1) *Any n-representable morphism of stacks is infinitesimally Cartesian.*

(2) *Let $f \colon F \to G$ be an n-representable morphism. If f is smooth, then for any $X \in \mathrm{dAfd}_k$ and any $x \colon X \to F$ there exists an étale covering $x' \colon X' \to X$ such that for any $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X')$ the canonical map*

$$\pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X')}(\mathbb{L}^{\mathrm{an}}_{X/G, x'}, \mathcal{F}) \to \pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X')}(\mathbb{L}^{\mathrm{an}}_{F/G, x \circ x'}, \mathcal{F})$$

is zero.

Proof. We proceed by induction on n. If $n = -1$, then (1) follows from Lemma 7.7 and (2) follows from Proposition 5.50.

Let now $n \geq 0$. To prove (1), it is enough to show that if F is n-geometric then it is infinitesimally Cartesian. Let $X \in \mathrm{dAfd}_k$, $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$ and $d\colon X[\mathcal{F}] \to X$ be an analytic derivation. Let x be a point in $\pi_0(F(X) \times_{F(X[\mathcal{F}])} F(X))$ with projection $x_1 \in \pi_0(F(X))$ on the first factor. We will prove that the fiber at x of

$$F(X_d[\mathcal{F}]) \to F(X) \times_{F(X[\mathcal{F}])} F(X)$$

is contractible. This implies that the above morphism is an equivalence and therefore F is infinitesimally Cartesian.

We claim that this statement is local for the étale topology on $X_d[\mathcal{F}]$. Indeed, if $j'\colon U' \to X_d[\mathcal{F}]$ is an étale map in dAfd_k, let

$$j\colon U := U' \times_{X_d[\mathcal{F}]} X \to X$$

be the étale map obtained by base change. Since the formation of analytic square-zero extension is local on any structured topos, we obtain

$$U' \times_{X_d[\mathcal{F}]} X[\mathcal{F}] \simeq U[j^*\mathcal{F}], \qquad U' \simeq U_{d'}[j^*\mathcal{F}].$$

As a consequence, we are free to replace X by any étale cover.

Choose an $(n-1)$-atlas $\{U_i \to F\}_{i\in I}$ of F. Thanks to the above claim, we can assume that the point $x_1 \in \pi_0(F(X))$ lifts to a point $y_1 \in \pi_0(U_i(X))$ for some index $i \in I$. Write simply $U := U_i$. Consider the diagram

$$\begin{array}{ccc} U(X_d[\mathcal{F}]) & \longrightarrow & U(X) \times_{U(X[\mathcal{F}])} U(X) \\ \downarrow & & \downarrow{\scriptstyle f} \\ F(X_d[\mathcal{F}]) & \longrightarrow & F(X) \times_{F(X[\mathcal{F}])} F(X) \end{array}$$

The induction hypothesis applied to the $(n-1)$-representable morphism $\pi\colon U \to F$ shows that the above square is a pullback. Moreover, the top horizontal morphism is an equivalence. It follows that the fibers of the bottom horizontal morphism is either empty or contractible. In order to complete the proof of (1), it is thus sufficient to prove that the fiber of f at x is non-empty. Consider the diagram

$$\begin{array}{ccccc} \mathrm{fib}(g) & \longrightarrow & \mathrm{fib}(p) & \xrightarrow{g} & \mathrm{fib}(q) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{fib}(f) & \longrightarrow & U(X) \times_{U(X[\mathcal{F}])} U(X) & \xrightarrow{f} & F(X) \times_{F(X[\mathcal{F}])} F(X) \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ & & U(X) & \longrightarrow & F(X) \end{array}$$

where the fiber of q (resp. p) is taken at x_1 (resp. y_1), while the horizontal fibers are taken at x. The commutativity of the diagram shows that it is enough to prove that $\mathrm{fib}(g)$ is non-empty. Now, g is equivalent to the canonical map

$$\Omega_{d,0}\mathrm{Der}^{\mathrm{an}}_U(X,\mathcal{F}) \to \Omega_{d,0}\mathrm{Der}^{\mathrm{an}}_F(X,\mathcal{F}),$$

and therefore $\mathrm{fib}(g) \simeq \Omega_{d,0}\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{U/F,y_1},\mathcal{F})$. The composition $X \to U \to F$ gives rise to the exact sequence

$$\begin{aligned}\pi_0\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/F,x_1},\mathcal{F}) \to \pi_0\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{U/F,y_1},\mathcal{F})&\\ \to \pi_{-1}\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/U,y_1},\mathcal{F}).&\end{aligned}$$

Using (2) at rank $n-1$ for the map $\pi\colon U \to F$ and up to covering X with an étale atlas, we can therefore suppose that the first map vanishes. On the other hand, the image of d via the second map is zero. Therefore, d lies in the image of $\pi_0\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/F,x_1})$, i.e. d is in the connected component of 0. In particular, we can find a path from d to 0 in $\mathrm{Map}(\mathbb{L}^{\mathrm{an}}_{F/U,y_1},\mathcal{F})$. This shows that $\Omega_{d,0}\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{U/F,y_1},\mathcal{F})$ is nonempty and concludes the proof of (1).

We now turn to the proof of (2) for rank n. We can assume that G is a final object. Let $U \to F$ be an n-atlas and let $x\colon X \to F$ be a point, with $X \in \mathrm{dAfd}_k$. Up to choosing an étale cover of X, we can suppose that x factors through some $u\colon X \to U$. Therefore, the map $\mathbb{L}^{\mathrm{an}}_{F,x} \to \mathbb{L}^{\mathrm{an}}_X$ factors as

$$\mathbb{L}^{\mathrm{an}}_{F,x} \to \mathbb{L}^{\mathrm{an}}_{U,u} \to \mathbb{L}^{\mathrm{an}}_X.$$

Since U is smooth, Proposition 5.50 shows that $\mathbb{L}^{\mathrm{an}}_{U,u}$ is perfect and concentrated in degree 0. Therefore, for every $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$, we have

$$\pi_0\,\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{U,u},\mathcal{F}) = 0,$$

thus completing the proof. □

In order to prove the convergence property of n-representable maps, we need a characterization of smooth morphisms in terms of infinitesimal lifting properties.

Proposition 7.11. *Let $f\colon F \to G$ be an n-representable morphism in $\mathrm{St}(\mathrm{dAfd}_k,\tau_{\text{ét}})$ with respect to the geometric context $(\mathrm{dAfd}_k,\tau_{\text{ét}},\mathbf{P}_{\mathrm{sm}})$. Then f is smooth if and only if it satisfies the following conditions:*

(1) $\mathrm{t}_0(f)$ *is smooth.*
(2) *For any derived affinoid (resp. Stein) space $X \in \mathrm{dAfd}_k$, any $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$ and any $d \in \mathrm{Der}^{\mathrm{an}}(X,\mathcal{F})$, every lifting problem*

$$\begin{array}{ccc} X & \xrightarrow{x} & F \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ X_d[\mathcal{F}] & \longrightarrow & G \end{array} \tag{7.12}$$

has a solution.

Proof. First suppose that f is smooth. Then there exists an affinoid atlas $\{U_i\}$ of G and affinoid atlases $\{V_{ij}\}$ of $F \times_G U_i$ such that the maps $V_{ij} \to U_i$ are smooth. In particular, the truncations $\mathrm{t}_0(V_{ij}) \to \mathrm{t}_0(U_i)$ are smooth. Since $\{\mathrm{t}_0(U_i)\}$ is an atlas of $\mathrm{t}_0(G)$ and $\{\mathrm{t}_0(V_{ij})\}$ is an atlas of $\mathrm{t}_0(F)$, we deduce that the truncation $\mathrm{t}_0(f)$ is smooth. Let us now prove that the second condition is satisfied as well. We proceed by induction on n. Suppose first $n = -1$ and consider the lifting problem (7.12). Set

$$F' \coloneqq X_d[\mathcal{F}] \times_G F.$$

Let $x' \colon X \to F'$ be the morphism induced by the universal property of the pullback. Then the lifting problem (7.12) is equivalent to the following one:

$$\begin{array}{ccc} X & \xrightarrow{x'} & F' \\ \downarrow & \nearrow & \downarrow \\ X_d[\mathcal{F}] & \xrightarrow{\mathrm{id}} & X_d[\mathcal{F}] \end{array}$$

In other words, we can assume that G, and hence F, is (-1)-representable. Recall that, by definition, $X_d[\mathcal{F}]$ is the pushout

$$\begin{array}{ccc} X[\mathcal{F}] & \xrightarrow{d_0} & X \\ \downarrow{\scriptstyle d} & & \downarrow \\ X & \longrightarrow & X_d[\mathcal{F}] \end{array}$$

in the category dAn. Since F is (-1)-representable, to produce a solution $X_d[\mathcal{F}] \to F$ of the lifting problem is equivalent to producing a path between the two morphisms

$$X[\mathcal{F}] \underset{d_0}{\overset{d}{\rightrightarrows}} X \xrightarrow{x} F$$

in the category $\mathrm{dAn}_{X//G}$. Observe that these two morphisms in $\mathrm{dAn}_{X//G}$ define two elements $\alpha, \beta \in \pi_0 \mathrm{Der}^{\mathrm{an}}_{F/G}(X; \mathcal{F})$. In order to solve the original lifting problem, it is enough to find a path between α and β in the space

$$\mathrm{Der}^{\mathrm{an}}_{F/G}(X; \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(x^* \mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F}).$$

It is enough to prove that

$$\pi_0 \, \mathrm{Map}_{\mathrm{Coh}^+(X)}(x^* \mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F}) \simeq 0. \tag{7.13}$$

Let us first prove (7.13) in the nonarchimedean analytic case. By Proposition 5.50, $x^* \mathbb{L}^{\mathrm{an}}_{F/G}$ is perfect and in tor-amplitude 0. This implies that it is a retract of a free module of finite rank. In particular, $\pi_0 \, \mathrm{Map}_{\mathrm{Coh}^+(X)}(x^* \mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F})$ is a retract of $\pi_0(\mathcal{F}^n) \simeq 0$ for some nonnegative integer n. This completes the proof of (7.13) in the nonarchimedean case.

Now let us prove (7.13) in the complex analytic case. Consider the internal Hom $\mathcal{H}\mathrm{om}(x^*\mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F})$ in $\mathrm{Coh}^+(X)$, and remark that

$$\mathrm{Map}_{\mathrm{Coh}^+(X)}(x^*\mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F}) \simeq \tau_{\geq 0}\Gamma(X, \mathcal{H}\mathrm{om}(x^*\mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F})).$$

Since X is Stein, Cartan's Theorem B shows that it is enough to check that

$$\mathcal{H}\mathrm{om}(x^*\mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F}) \in \mathrm{Coh}^{\geq 1}(X).$$

This condition is local and it can therefore be checked after shrinking X. Since f is smooth, it follows from Proposition 5.50 that $x^*\mathbb{L}^{\mathrm{an}}_{F/G}$ is perfect and in tor-amplitude 0. Therefore, locally on X, we can express $x^*\mathbb{L}^{\mathrm{an}}_{F/G}$ as a retract of a free module of finite rank. It follows that, locally on X, the sheaf $\mathcal{H}\mathrm{om}(x^*\mathbb{L}^{\mathrm{an}}_{F/G}, \mathcal{F})$ is a retract of $\mathcal{F}^n$ for some nonnegative integer n. Since $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$, this completes the proof of (7.13).

We now assume that $n \geq 0$ and that the statement has already been proven for $m < n$. Base-changing to $X_d[\mathcal{F}]$ we can assume once again that G is representable and therefore that F is n-geometric. In particular, F is infinitesimally Cartesian in virtue of Proposition 7.10. It will therefore be sufficient to prove that $\mathbb{L}^{\mathrm{an}}_{F/G}$ is perfect and in tor-amplitude $[0, n]$. This follows by induction on n, and the proof of [25, 2.2.5.2] applies.

We now prove the converse. Assume that $\mathrm{t}_0(f)$ is smooth and the lifting problem (2) always has a solution. By base change, we can assume that G is itself representable and therefore F is n-geometric. Let $U \to F$ be a smooth atlas for F. Since $U \to F$ is smooth, the lifting problem (7.12) for this map has a solution. It follows that the composition $U \to F \to G$ has the same property. We are thus reduced to the case where both $F = X$ and $G = Y$ are representable. In virtue of Proposition 5.50(2), it will be enough to show that $\mathbb{L}^{\mathrm{an}}_{F/G}$ is perfect and concentrated in tor-amplitude 0. Notice that these conditions can be checked locally on X.

The lifting condition implies that for any $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(X)$ we have

$$\pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) = 0.$$

Using Corollary 5.40, up to shrinking X in the complex analytic case, we can choose a map $\phi\colon \mathcal{O}^n_X \to \mathbb{L}^{\mathrm{an}}_{X/Y}$ which is surjective on π_0. Let $K := \mathrm{fib}(\phi)$. We therefore obtain an exact sequence

$$\pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{O}^n_X) \to \pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathbb{L}^{\mathrm{an}}_{X/Y}) \\ \to \pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Y}, K[1]).$$

Since $\pi_0 \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Y}, K[1]) = 0$, we conclude that $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is a retraction of $\mathcal{O}^n_X$, and as a consequence it is perfect and in tor-amplitude 0. □

We complete the proof of the implication (1)⇒(2) in Theorem 7.1 by the following lemma, analogous to [25, C.0.10].

Lemma 7.14. *Let* $f\colon F \to G$ *be an* n*-representable morphism in* $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. *Then for any* $X \in \mathrm{dAfd}_k$*, the square*

$$\begin{array}{ccc} F(X) & \longrightarrow & \lim_m F(\mathrm{t}_{\leq m} X) \\ \downarrow & & \downarrow \\ G(X) & \longrightarrow & \lim_m G(\mathrm{t}_{\leq m} X) \end{array}$$

is a pullback.

Proof. We start by remarking that in the special case where $G = *$ and f is (-1)-representable, the statement follows directly from the fact that

$$X \simeq \operatorname*{colim}_m \mathrm{t}_{\leq m} X \quad \text{in } \mathrm{dAn}_k.$$

Let us now turn to the general case. We want to prove that the canonical map

$$F(X) \to G(X) \times_{\lim G(\mathrm{t}_{\leq m} X)} \lim F(\mathrm{t}_{\leq m} X)$$

is an equivalence. For this, it is enough to prove that its fibers are contractible. Fix a point $x \in G(X) \times_{\lim G(\mathrm{t}_{\leq m} X)} \lim F(\mathrm{t}_{\leq m} X)$. The projection of x in $G(X)$ determines a map $f\colon X \to G$. We can then replace G by X and F by the fiber product $X \times_G F$. Hence G is (-1)-representable and therefore the map

$$G(X) \to \lim_m G(\mathrm{t}_{\leq m} X)$$

is an equivalence. We are therefore reduced to proving that the map

$$F(X) \to \lim_m F(\mathrm{t}_{\leq m} X)$$

is an equivalence. In other words, we can assume that $G = *$ and F is n-geometric.

We proceed by induction on the geometric level n. When $n = -1$, we have already proved that the statement is true. Suppose $n \geq 0$ and let $u\colon U \to F$ be an n-atlas. We will prove that the fibers of the morphism

$$F(X) \to \lim_m F(\mathrm{t}_{\leq m} X)$$

are contractible. Let $x \in \lim_m F(\mathrm{t}_{\leq m} X)$ be a point and let $x_m\colon \mathrm{t}_{\leq m} X \to F$ be the morphism classified by the projection of x in $F(\mathrm{t}_{\leq m} X)$. Since F is a sheaf and limits commute with limits, we see that this statement is local on X. We can therefore suppose that x_0 factors as

$$\begin{array}{ccc} & & U \\ & \overset{y_0}{\nearrow} & \downarrow{\scriptstyle u} \\ \mathrm{t}_0(X) & \xrightarrow{x_0} & F \end{array}$$

We claim that there exists a point $y \in \lim_m U(\mathrm{t}_{\leq m} X)$ whose image in $\lim_m F(\mathrm{t}_{\leq m} X)$ is x. In order to see this, we construct a compatible sequence of maps $y_m\colon \mathrm{t}_{\leq m} X \to U$

by induction on m. We have already handled $m = 0$. Now, observe that since u is smooth and since the morphisms $\mathrm{t}_{\leq n} X \hookrightarrow \mathrm{t}_{\leq n+1} X$ are analytic square-zero extensions by Corollary 5.44, Proposition 7.11 implies that the lifting problem

$$\begin{array}{ccc} \mathrm{t}_{\leq m} X & \xrightarrow{y_m} & U \\ \big\downarrow & \overset{y_{m+1}}{\nearrow} & \big\downarrow u \\ \mathrm{t}_{\leq m+1} X & \xrightarrow{x_{m+1}} & X \end{array}$$

has a solution. This proves the claim. We now consider the diagram

$$\begin{array}{ccc} U(X) & \longrightarrow & \lim_m U(\mathrm{t}_{\leq m} X) \\ \big\downarrow & & \big\downarrow \\ F(X) & \longrightarrow & \lim_m F(\mathrm{t}_{\leq m} X) \end{array}$$

Since $u \colon U \to F$ is $(n-1)$-representable, the induction hypothesis implies that the above diagram is a pullback square. We can therefore identify the fiber at $y \in \lim_m U(\mathrm{t}_{\leq m} X)$ of the top morphism with the fiber at $x \in \lim_m F(\mathrm{t}_{\leq m} X)$ of the bottom morphism. On the other hand, since U is representable, we see that the top morphism is an equivalence, which completes the proof. □

7.2. Lifting atlases

In this subsection, we prove the implication (2)⇒(1) of Theorem 7.1.

Lemma 7.15. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Let $f \colon M \to N$ be a morphism between eventually connective objects. Let m be an integer. If for every $P \in \mathcal{C}^{\heartsuit}$ the canonical map*

$$\mathrm{Map}_{\mathcal{C}}(N, P[m]) \to \mathrm{Map}_{\mathcal{C}}(M, P[m])$$

is an equivalence, then $\tau_{\leq m} M \to \tau_{\leq m} N$ is an equivalence as well.

Proof. Up to replacing M and N by $M[-m]$ and $N[-m]$, we can suppose $m = 0$. Moreover, since $\mathrm{Map}_{\mathcal{C}}(\tau_{\geq 1} M, P) \simeq \mathrm{Map}_{\mathcal{C}}(\tau_{\geq 1} N, P) \simeq \{*\}$ for every $P \in \mathcal{C}^{\heartsuit}$, we can further replace M and N by $\tau_{\leq 0} M$ and $\tau_{\leq 0} N$, respectively. In other words, we can suppose that $\pi_i(M) = \pi_i(N) = 0$ for every $i > 0$.

Let n be the largest integer such that at least one of $\pi_{-n}(M)$ and $\pi_{-n}(N)$ is not zero. We proceed by induction on n. If $n = 0$, then $M, N \in \mathcal{C}^{\heartsuit}$ and the statement follows from the Yoneda lemma. Let now $n > 0$. Choosing $P = \pi_n(M)$, we obtain an element $\gamma \in \pi_n \mathrm{Map}_{\mathcal{C}}(M, P)$. The corresponding element in $\pi_n \mathrm{Map}_{\mathcal{C}}(N, P)$ can be represented by a morphism $g \colon N \to P[-n]$. Inspection reveals that $\pi_n(g) \colon \pi_n(N) \to \pi_n(M)$ is an inverse for $\pi_n(f)$. We now consider the morphism of fiber sequences

$$\begin{array}{ccccc} \tau_{\geq -n+1} M & \longrightarrow & M & \longrightarrow & \pi_{-n}(M) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \tau_{\geq -n+1} N & \longrightarrow & N & \longrightarrow & \pi_{-n}(N) \end{array}$$

Fix $P \in \mathcal{C}^{\heartsuit}$. Applying the functor $\mathrm{Map}_{\mathcal{C}}(-, P)$ and then taking the long exact sequence of homotopy groups, we conclude that

$$\mathrm{Map}_{\mathcal{C}}(\tau_{\geq -n+1} N, P) \to \mathrm{Map}_{\mathcal{C}}(\tau_{\geq -n+1} M, P)$$

is an equivalence for every choice of P. We can therefore invoke the induction hypothesis to deduce that $\tau_{\geq -n+1}(f)$ is an equivalence. As we have already argued that $\pi_n(f)$ is an equivalence, so is f itself, completing the proof. □

Lemma 7.16. *Let $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ be a stack satisfying the conditions in Theorem 7.1(2). Let $j \colon \mathrm{t}_0(F) \to F$ be the canonical morphism. Then $\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F}$ belongs to* $\mathrm{Coh}^{\geq 2}(\mathrm{t}_0(F))$.

Proof. We follow closely the proof of [11, Theorem 3.1.2]. Let $\eta \colon U \to \mathrm{t}_0(F)$ be a smooth morphism from an affinoid (resp. Stein) space U. For every discrete coherent sheaf $\mathcal{F}$ on U, the canonical map

$$\mathrm{Map}_{\mathrm{Coh}^+(U)}(\eta^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Coh}^+(U)}(\eta^* j^* \mathbb{L}^{\mathrm{an}}_F, \mathcal{F}) \tag{7.17}$$

is obtained by passing to vertical fibers in the commutative diagram

$$\begin{array}{ccc} \mathrm{t}_0(F)(U[\mathcal{F}]) & \longrightarrow & F(U[\mathcal{F}]) \\ \downarrow & & \downarrow \\ \mathrm{t}_0(F)(U) & \longrightarrow & F(U) \end{array}$$

Since $\mathcal{F}$ is discrete, $U[\mathcal{F}]$ is an underived affinoid (resp. Stein) space. As a consequence, the horizontal morphisms are equivalences. It follows that the same holds for the map (7.17). Therefore, Lemma 7.15 shows that $\tau_{\leq 0} \eta^* j^* \mathbb{L}^{\mathrm{an}}_F \to \tau_{\leq 0} \eta^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)}$ is an equivalence. We conclude that $\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F}$ is 1-connective.

We now prove that it is also 2-connective. We have an exact sequence

$$\pi_1(j^* \mathbb{L}^{\mathrm{an}}_F) \to \pi_1(\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)}) \to \pi_1(\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F, j}) \to 0.$$

Let $\mathcal{F} := \pi_1(\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F, j})$. If $\mathcal{F} \neq 0$, then we obtain a nonzero map

$$\gamma \colon \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)} \to \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F} \to \mathcal{F}[1]$$

whose restriction to $j^* \mathbb{L}^{\mathrm{an}}_F$ vanishes. Choose a smooth morphism $\eta \colon U \to \mathrm{t}_0(F)$ such that $\eta^* \mathcal{F} \neq 0$. Then γ determines a nonzero morphism $\eta^* \mathbb{L}_{\mathrm{t}_0(F)} \to \eta^* \mathcal{F}[1]$. Since there is a fiber sequence

$$\mathbb{L}^{\mathrm{an}}_{U/\mathrm{t}_0(F)}[-1] \to \eta^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)} \to \mathbb{L}^{\mathrm{an}}_U$$

and since $\mathbb{L}^{\mathrm{an}}_{U/\mathrm{t}_0(F)}$ is perfect and in tor-amplitude 0, we conclude that the composition

$$\mathbb{L}^{\mathrm{an}}_{U/\mathrm{t}_0(F)}[-1] \to \eta^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)} \to \eta^* \mathcal{F}[1]$$

vanishes. In other words, we obtain a nonzero analytic derivation $d\colon \mathbb{L}_U^{\mathrm{an}} \to \eta^*\mathcal{F}[1]$. Let $U_d[\eta^*\mathcal{F}]$ be the associated square-zero extension. We now consider the diagram

$$\begin{array}{ccccc} U[\eta^*\mathcal{F}[1]] & \xrightarrow{d} & U & \xrightarrow{\eta} & \mathrm{t}_0(F) \\ \downarrow & & \downarrow i & \overset{\alpha}{\nearrow} & \downarrow j \\ U & \longrightarrow & U_d[\eta^*\mathcal{F}] & \overset{\beta}{\dashrightarrow} & F \end{array}$$

The left square is a pushout, so to produce the lifting α (resp. β) in the category $\mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})_{U/}$ is equivalent to producing a path in

$$\mathrm{Map}_{\mathrm{Coh}^+(U)}(\eta^*\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)}, \eta^*\mathcal{F}[1]) \quad (\text{resp. } \mathrm{Map}_{\mathrm{Coh}^+(U)}(\eta^* j^*\mathbb{L}^{\mathrm{an}}_F, \eta^*\mathcal{F}[1]))$$

between $\eta \circ d$ and $\eta \circ d_0$ (resp. $j \circ \eta \circ d$ and $j \circ \eta \circ d_0$). It follows from Proposition 6.1 that $U_d[\eta^*\mathcal{F}]$ is an underived affinoid (resp. Stein) space. In particular, the canonical map

$$\mathrm{t}_0(F)(U_d[\eta^*\mathcal{F}]) \to F(U_d[\eta^*\mathcal{F}])$$

is a homotopy equivalence. As a consequence, the existence of α is equivalent to the existence of β. Nevertheless:

(1) The map α cannot exist because $\eta \circ d_0$ is equivalent to the zero map $\eta^*\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)} \to \eta^*\mathcal{F}[1]$, while $\eta \circ d$ is nonzero by construction.
(2) The map β exists because both $j \circ \eta \circ d_0$ and $j \circ \eta \circ d_0$ correspond to the zero map $\eta^* j^*\mathbb{L}^{\mathrm{an}}_F \to \eta^*\mathcal{F}[1]$. This is because the composition $\eta^* j^*\mathbb{L}^{\mathrm{an}}_F \to \eta^*\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)} \to \eta^*\mathcal{F}[1]$ is zero.

This is a contradiction, and the lemma is proved. □

Lemma 7.18. *Let $F \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ be a stack satisfying the conditions in Theorem 7.1(2). Then for any $U_0 \in \mathrm{Afd}_k$ and any étale morphism $u_0\colon U_0 \to \mathrm{t}_0(F)$, there is $U \in \mathrm{dAfd}_k$, a morphism $u\colon U \to F$ satisfying $\mathbb{L}^{\mathrm{an}}_{U/F} \simeq 0$ and a pullback square*

$$\begin{array}{ccc} U_0 & \longrightarrow & \mathrm{t}_0(F) \\ \downarrow & & \downarrow \\ U & \longrightarrow & F \end{array}$$

Proof. We follow closely the proof of [25, Lemma C.0.11]. We will construct by induction a sequence of derived affinoid (resp. Stein) spaces

$$U_0 \to U_1 \to \cdots \to U_n \to \cdots \to F$$

satisfying the following properties:

(1) U_n is n-truncated.
(2) The morphism $U_n \to U_{n+1}$ induces an equivalence on n-th truncations.
(3) The morphisms $u_n\colon U_n \to F$ are such that $\pi_i(\mathbb{L}^{\mathrm{an}}_{U_n/F}) \simeq 0$ for every $i \leq n+1$.

Assume that the sequence has already been constructed. Then all the derived affinoid (resp. Stein) spaces U_n share the same underlying ∞-topos $\mathcal{U}$. Moreover, the canonical morphisms $\mathcal{O}_{U_n} \to \pi_0(\mathcal{O}_{U_n}) \simeq \mathcal{O}_{U_0}$ are local. It follows that

$$\mathcal{O}_U := \lim_n \mathcal{O}_{U_n} \in \mathrm{AnRing}_k(\mathcal{U})_{/\mathcal{O}_{U_0}}$$

is a $\mathcal{T}_{\mathrm{an}}(k)$-structure satisfying $\tau_{\leq n}(\mathcal{O}_U) \simeq \mathcal{O}_{U_n}$. In particular, $U := (\mathcal{U}, \mathcal{O}_U)$ is a derived affinoid (resp. Stein) space. Since F is convergent, we obtain a canonical morphism $u\colon U \to F$. Let us check that $\mathbb{L}^{\mathrm{an}}_{U/F,u} \simeq 0$. Fix $\mathcal{F} \in \mathrm{Coh}^{\geq 0}(U)$. We have

$$\mathrm{Der}^{\mathrm{an}}_F(U, \mathcal{F}) \simeq \lim \mathrm{Der}^{\mathrm{an}}_F(U_n, \tau_{\leq n}(\mathcal{F})) \simeq \lim \mathrm{Map}_{\mathrm{Coh}^+(U_n)}(\mathbb{L}^{\mathrm{an}}_{U_n/F}, \tau_{\leq n}\mathcal{F}) \simeq 0.$$

Finally, the map $U \times_F \mathrm{t}_0(F) \to U$ enjoys the following universal property: for every underived X the map

$$\mathrm{Map}_{\mathrm{St}(\mathrm{dAfd}, \tau_{\text{ét}})}(X, U \times_F \mathrm{t}_0(F)) \to \mathrm{Map}_{\mathrm{St}(\mathrm{dAfd}, \tau_{\text{ét}})}(X, U)$$

is an equivalence. This allows us to identify $U \times_F \mathrm{t}_0(F)$ with $\mathrm{t}_0(U) \simeq U_0$.

It remains to construct the sequence U_n. We proceed by induction. If $n = 0$, we only have to prove that $\mathbb{L}^{\mathrm{an}}_{U_0/F}$ is 2-connective. Let $j\colon \mathrm{t}_0(F) \to F$ be the canonical map. Then we have a fiber sequence

$$u_0^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F} \to \mathbb{L}^{\mathrm{an}}_{U_0/F} \to \mathbb{L}^{\mathrm{an}}_{U_0/\mathrm{t}_0(F)}.$$

Since u_0 is étale, $\mathbb{L}^{\mathrm{an}}_{U_0/\mathrm{t}_0(F), u_0} \simeq 0$. Therefore, the statement follows from the fact that $\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F, j}$ is 2-connective, which is the content of Lemma 7.16.

Assume now that U_n has been constructed. Let $u_n\colon U_n \to F$ be the given morphism. Consider the composite map

$$d\colon \mathbb{L}^{\mathrm{an}}_{U_n} \to \mathbb{L}^{\mathrm{an}}_{U_n/F} \to \tau_{\leq n+2} \mathbb{L}^{\mathrm{an}}_{U_n/F} \simeq \pi_{n+2}(\mathbb{L}^{\mathrm{an}}_{U_n/F})[n+2].$$

This is an analytic derivation and thus it defines an analytic square-zero extension of U_n by $\pi_{n+2}(\mathbb{L}^{\mathrm{an}}_{U_n/F})[n+2]$. Let us denote it by U_{n+1}. It follows from Proposition 6.1 that U_{n+1} is a derived affinoid (resp. Stein) space. Moreover, since F is infinitesimally Cartesian, we see that there is a canonical map $u_{n+1}\colon U_{n+1} \to F$.

Then conditions (1) and (2) are met by construction. To prove (3), let $j_n\colon U_n \to U_{n+1}$ denote the canonical morphism. Since $\mathrm{t}_{\leq n}(j_n)$ is an equivalence, it will be sufficient to show that $j_n^* \mathbb{L}^{\mathrm{an}}_{U_{n+1}/F}$ is $(n+2)$-connective. This fits into a fiber sequence

$$j_n^* \mathbb{L}^{\mathrm{an}}_{U_{n+1}/F} \to \mathbb{L}^{\mathrm{an}}_{U_n/F} \xrightarrow{\phi} \mathbb{L}^{\mathrm{an}}_{U_n/U_{n+1}}.$$

Since j_n is n-connective and U_n is n-truncated, $\mathrm{cofib}(j_n)$ is $(n+1)$-connective. It follows from Corollary 5.35 that $\mathbb{L}^{\mathrm{an}}_{U_n/U_{n+1}}$ is $(n+1)$-connective. Moreover, since $n \geq 1$, we can combine Corollary 5.33 with [25, 2.2.2.8] to conclude that

$$\pi_{n+2}(\mathbb{L}^{\mathrm{an}}_{U_n/U_{n+1}}) \simeq \pi_{n+2}(\mathbb{L}^{\mathrm{an}}_{U_n/F}),$$

which completes the proof. □

We are finally ready to complete the proof of Theorem 7.1.

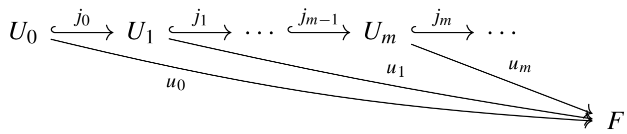
Proof of Theorem 7.1. The implication (1)⇒(2) follows from Proposition 7.10 and Lemma 7.14. Let now $F \in \mathrm{St}(\mathrm{dAfd}, \tau_{\text{ét}})$ be a stack satisfying (2). We will prove by induction on n that F is n-geometric.

If $n = -1$, then Lemma 7.18 allows us to lift the identity of $\mathrm{t}_0(F)$ to a morphism $U \to F$, where $U \in \mathrm{dAfd}_k$ and $\mathbb{L}^{\mathrm{an}}_{U/F} \simeq 0$. Let $X \in \mathrm{dAfd}_k$. By Corollary 5.44 and by induction on m, the canonical map

$$U(\mathrm{t}_{\le m} X) \to F(\mathrm{t}_{\le m} X)$$

is an equivalence for every m. Since F and U are convergent, we deduce that $U \simeq F$, so F is representable.

Let now $n \ge 0$. It follows from the induction hypothesis that the diagonal of F is $(n-1)$-representable. It remains to prove that F admits an atlas. Let $u \colon U_0 \to \mathrm{t}_0(F)$ be a smooth atlas and let $j \colon \mathrm{t}_0(F) \hookrightarrow F$ be the natural inclusion. We will construct a sequence of morphisms



satisfying the following properties:

(1) U_m is m-truncated.
(2) $U_m \to U_{m+1}$ induces an equivalence on m-truncations.
(3) $\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat to order $m+1$ (cf. Definition 8.4).

The construction is carried out by induction on m. When $m = 0$, we set $u_0 := j \circ u$. It suffices to check that $\mathbb{L}^{\mathrm{an}}_{U_0/F}$ is flat to order 1. Consider the fiber sequence

$$u^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F} \to \mathbb{L}^{\mathrm{an}}_{U_0/F} \to \mathbb{L}^{\mathrm{an}}_{U_0/\mathrm{t}_0(F)}.$$

Lemma 7.16 guarantees that $\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F}$ is 2-connective and therefore

$$u^* \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_0(F)/F} \in \mathrm{Coh}^{\ge 2}(U_0).$$

In particular, the natural morphism

$$\tau_{\le 1} \mathbb{L}^{\mathrm{an}}_{U_0/F} \to \tau_{\le 1} \mathbb{L}^{\mathrm{an}}_{U_0/\mathrm{t}_0(F)}$$

is an equivalence. Since $u \colon U_0 \to \mathrm{t}_0(F)$ is smooth and U_0 is discrete, we conclude that $\tau_{\le 1} \mathbb{L}^{\mathrm{an}}_{U_0/\mathrm{t}_0(F)} \simeq \mathbb{L}^{\mathrm{an}}_{U_0/\mathrm{t}_0(F)}$. In particular, $\tau_{\le 1} \mathbb{L}^{\mathrm{an}}_{U_0/F}$ is perfect and in tor-amplitude 0. Proposition 8.5(2) implies that $\mathbb{L}^{\mathrm{an}}_{U_0/F}$ is flat to order 1.

Assume now that $u_m \colon U_m \to F$ has been constructed. Since U_m is m-truncated and $\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat to order $m+1$, Proposition 8.6 implies that $\tau_{\le m+1} \mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat. Up to shrinking U_m in the complex case, we can assume that $\tau_{\leq m+1}\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is a retract of a free module. In particular,

$$\pi_0 \operatorname{Map}_{\mathrm{Coh}^+(U_m)}(\tau_{\leq m+1}\mathbb{L}^{\mathrm{an}}_{U_m/F}, \mathcal{F}) = 0$$

for every $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(U_m)$. Taking $\mathcal{F} = \tau_{\leq m+2}\mathbb{L}^{\mathrm{an}}_{U_m/F}[1]$, we conclude that the natural map

$$\tau_{\leq m+1}\mathbb{L}^{\mathrm{an}}_{U_m/F}[-1] \to \mathbb{L}^{\mathrm{an}}_{U_m/F}$$

is homotopic to zero. Consider now the diagram

$$\begin{array}{ccc}
\tau_{\leq m+1}\mathbb{L}^{\mathrm{an}}_{U_m}[-1] & \longrightarrow & \tau_{\leq m+1}\mathbb{L}^{\mathrm{an}}_{U_m/F}[-1] \\
\downarrow & & \downarrow{\scriptstyle \simeq 0} \\
\tau_{\geq m+2}\mathbb{L}^{\mathrm{an}}_{U_m} & \longrightarrow & \tau_{\geq m+2}\mathbb{L}^{\mathrm{an}}_{U_m/F} \\
\downarrow & \overset{\varphi}{\nearrow} & \downarrow \\
\mathbb{L}^{\mathrm{an}}_{U_m} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{U_m/F}
\end{array}$$

The universal property of the cofiber implies the existence of the dotted arrow.

Consider the composition

$$d\colon \mathbb{L}^{\mathrm{an}}_{U_m} \xrightarrow{\varphi} \tau_{\geq m+2}\mathbb{L}^{\mathrm{an}}_{U_m/F} \to \pi_{m+2}(\mathbb{L}^{\mathrm{an}}_{U_m/F}).$$

This map corresponds to an analytic derivation. We let U_{m+1} denote the associated analytic square-zero extension. By construction, U_{m+1} is $(m+1)$-truncated and the canonical map

$$j_m\colon U_m \to U_{m+1}$$

induces an equivalence on the m-th truncation. Furthermore, since F is infinitesimally Cartesian and the composition

$$u_m^*\mathbb{L}^{\mathrm{an}}_F \to \mathbb{L}^{\mathrm{an}}_{U_m} \to \mathbb{L}^{\mathrm{an}}_{U_m/F}$$

is homotopic to zero, there is a map $u_{m+1}\colon U_{m+1} \to F$ fitting in the commutative triangle

$$\begin{array}{ccc}
U_m & \xrightarrow{u_m} & F \\
{\scriptstyle j_m}\downarrow & \overset{u_{m+1}}{\nearrow} & \\
U_{m+1} & &
\end{array}$$

Conditions (1) and (2) are satisfied by construction. It remains to check that $\mathbb{L}^{\mathrm{an}}_{U_{m+1}/F}$ is flat to order $m+2$. Using Proposition 8.5(4), it is enough to check that $j_m^*\mathbb{L}^{\mathrm{an}}_{U_{m+1}/F}$ is flat to order $m+2$. Consider the transitivity fiber sequence

$$j_m^*\mathbb{L}^{\mathrm{an}}_{U_{m+1}/F} \to \mathbb{L}^{\mathrm{an}}_{U_m/F} \xrightarrow{\phi} \mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}}.$$

By the induction hypothesis, $\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat to order $m+1$. Moreover, $\mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}}$ is $(m+2)$-connective. It follows that $j_m^*\mathbb{L}^{\mathrm{an}}_{U_{m+1}/F}$ is flat to order $m+1$. Since U_m is m-truncated, Corollary 8.7 shows that $j_m^*\mathbb{L}^{\mathrm{an}}_{U_{m+1}/F}$ is flat to order $m+2$ if and only if

$$\pi_{m+2}(\mathbb{L}^{\mathrm{an}}_{U_{m+1}/F}) = 0.$$

To prove the latter, it is enough to show that the map ϕ induces an isomorphism on π_{m+2} and a surjection on π_{m+3}. Set

$$\mathcal{F} \coloneqq \pi_{m+2}(\mathbb{L}^{\mathrm{an}}_{U_m/F})[m+2].$$

Combining Corollary 5.33 and [25, Lemma 1.4.3.7], we see that $\mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}}$ can be computed as the pushout

$$\begin{array}{ccc} \mathcal{F} \otimes_{\mathcal{O}_{U_{m+1}}} \mathcal{F} & \xrightarrow{\ \mu\ } & \mathcal{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}} \end{array}$$

where μ is the multiplication map induced by $\mathcal{O}_{U_{m+1}}$. Using [12, 7.4.1.14], we see that μ is nullhomotopic. As a consequence,

$$\mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}} \simeq \mathcal{F} \oplus (\mathcal{F} \otimes_{\mathcal{O}_{U_{m+1}}} \mathcal{F}[1]).$$

Since $m > 0$, we have

$$\pi_{m+2}(\mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}}) \simeq \mathcal{F} = \pi_{m+2}(\mathbb{L}^{\mathrm{an}}_{U_m/F}), \qquad \pi_{m+3}(\mathbb{L}^{\mathrm{an}}_{U_m/U_{m+1}}) \simeq 0.$$

Hence ϕ has the required properties, which completes the construction of the sequence of maps $u_m \colon U_m \to F$.

The argument given in Lemma 7.18 shows that the colimit of the diagram

$$U_0 \overset{j_0}{\hookrightarrow} U_1 \overset{j_1}{\hookrightarrow} \cdots \hookrightarrow U_m \overset{j_m}{\hookrightarrow} \cdots$$

exists in dAfd_k. We denote it by $\widetilde{U}$. Since F is convergent, we can assemble the maps $u_m \colon U_m \to F$ into a canonical map $\widetilde{u} \colon \widetilde{U} \to F$. Let $i_m \colon U_m \to \widetilde{U}$ be induced map. Consider the fiber sequence

$$i_m^* \mathbb{L}^{\mathrm{an}}_{\widetilde{U}/F} \to \mathbb{L}^{\mathrm{an}}_{U_m/F} \to \mathbb{L}^{\mathrm{an}}_{U_m/\widetilde{U}}.$$

Since $\mathbb{L}^{\mathrm{an}}_{U_m/F}$ is flat to order $m+1$ by construction and $\mathbb{L}^{\mathrm{an}}_{U_m/U}$ is $(m+2)$-connective, it follows that $i_m^*\mathbb{L}^{\mathrm{an}}_{\widetilde{U}/F}$ is flat to order $m+1$. Using Proposition 8.5(2), we conclude that $\mathbb{L}^{\mathrm{an}}_{\widetilde{U}/F}$ is flat to order $m+1$. Since this holds for every m, we see that $\mathbb{L}^{\mathrm{an}}_{\widetilde{U}/F}$ has tor-amplitude 0. Since it is almost perfect, it is perfect and in degree 0. Using the lifting criterion of Proposition 7.11, we conclude that $\widetilde{u}$ is smooth. The proof of Theorem 7.1 is thus complete. □

8. Appendices

8.1. Modules over a simplicial commutative ring

Let CRing denote the ∞-category of simplicial commutative rings. Let $A \in \mathrm{CRing}$ and let $X := \mathrm{Spec}(A)$ be the associated derived scheme. We denote by $\mathrm{dSch}_{/X}$ the ∞-category of derived schemes over X. Let $\mathcal{T}_A$ be the discrete pregeometry whose underlying ∞-category is the full subcategory of $\mathrm{dSch}_{/X}$ spanned by the derived schemes $\mathbb{A}^n_X := \mathrm{Spec}(\mathrm{Sym}_A(A^n))$ for all $n \geq 0$. Moreover, let us define the discrete pregeometry $\mathcal{T}_A^{\mathrm{aug}} := (\mathcal{T}_A)_{X/}$, whose underlying ∞-category is the full subcategory of $(\mathrm{dSch}_k)_{X//X}$ spanned by objects $X \to Y \to X$ with $Y \in \mathcal{T}_A$.

Proposition 8.1. *We have the following equivalences of ∞-categories:*

(1) $\mathrm{CRing}_A \simeq \mathrm{Fun}^\times(\mathcal{T}_A, \mathcal{S})$.
(2) $\mathrm{CRing}_{A//A} \simeq \mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S})$.

Proof. The first equivalence is the content of [8, Definition 4.1.1 and Remark 4.1.2].

Let us now prove the second one. Observe that there is a forgetful functor $\varphi\colon \mathcal{T}_A^{\mathrm{aug}} \to \mathcal{T}_A$ that commutes with products. In particular, composition with φ induces a well-defined functor

$$\Phi\colon \mathrm{Fun}^\times(\mathcal{T}_A, \mathcal{S}) \to \mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}).$$

This functor commutes with limits and with sifted colimits. In particular, it has a left adjoint

$$\Psi\colon \mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \to \mathrm{Fun}^\times(\mathcal{T}_A, \mathcal{S}) \simeq \mathrm{CRing}_A.$$

Let $\mathcal{O}_A := \mathrm{Map}_{\mathcal{T}_A^{\mathrm{aug}}}(\mathrm{Spec}(A), -) \in \mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}})$. Since $X = \mathrm{Spec}(A)$ is a final object in $\mathcal{T}_A^{\mathrm{aug}}$, it follows that $\mathcal{O}_A$ is an initial object in $\mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}})$. In particular, $\Psi(\mathcal{O}_A)$ is an initial object in $\mathrm{Fun}^\times(\mathcal{T}_A, \mathcal{S}) \simeq \mathrm{CRing}_A$. In other words, $\Psi(\mathcal{O}_A) \simeq A$. On the other hand, X is also an initial object in $\mathcal{T}_A^{\mathrm{aug}}$. Thus, $\mathcal{O}_A$ is also a final object in $\mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S})$. It follows that Ψ factors through

$$F\colon \mathrm{Fun}(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \to \mathrm{CRing}_{A//A},$$

in such a way that the diagram

$$\begin{array}{ccc} \mathrm{CRing}_A & \xleftarrow{\sim} & \mathrm{Fun}^\times(\mathcal{T}_A, \mathcal{S}) \\ \uparrow & & \uparrow{\scriptstyle \Psi} \\ \mathrm{CRing}_{A//A} & \xleftarrow{F} & \mathrm{Fun}^\times(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \end{array}$$

commutes.

The functor F admits a right adjoint G that can be constructed as follows. Let $(B, f) \in \mathrm{CRing}_{A//A}$, where B is an A-algebra and $f\colon B \to A$ is the augmentation. We can review B as an object in $\mathrm{Fun}^{\times}(\mathcal{T}_A, \mathcal{S})$. Applying Φ we obtain a product preserving functor $\Phi(B)$ equipped with a map to $\Phi(A) \simeq \Phi(\Psi(\mathcal{O}_A))$. We can thus form the pullback

$$\begin{array}{ccc} G(B) & \longrightarrow & \Phi(B) \\ \downarrow & & \downarrow \\ \mathcal{O}_A & \longrightarrow & \Phi(\Psi(\mathcal{O}_A)) \end{array} \tag{8.2}$$

This construction shows immediately that G is a right adjoint to F. Let us now remark that for $B \in \mathcal{T}_A^{\mathrm{aug}} \subset \mathrm{CRing}_{A//A}$, we can canonically identify $G(B)$ with the functor $\mathcal{O}_B \colon \mathcal{T}_A^{\mathrm{aug}} \to \mathcal{S}$ defined by

$$\mathcal{O}_B(\mathbb{A}^n_X) = \mathrm{Map}_{X//X}(\mathrm{Spec}(B), \mathbb{A}^n_X).$$

Indeed, we remark that evaluating the diagram (8.2) of natural transformations on $f \colon X \to \mathbb{A}^n_X$, we get the pullback diagram

$$\begin{array}{ccc} G(B)(X \xrightarrow{f} \mathbb{A}^n_X) & \longrightarrow & \mathrm{Map}_{/X}(\mathrm{Spec}(B), \mathbb{A}^n_X) \\ \downarrow & & \downarrow \\ \{*\} & \xrightarrow{\quad f \quad} & \mathrm{Map}_{/X}(X, \mathbb{A}^n_X) \end{array}$$

In particular, we obtain a canonical identification

$$G(B)(X \xrightarrow{f} \mathbb{A}^n_X) \simeq \mathrm{Map}_{X//X}(\mathrm{Spec}(B), \mathbb{A}^n_X).$$

We now remark that both F and G commute with sifted colimits. By (1) and [6, 5.5.8.10], it is enough to check that for every $f \colon X \to \mathbb{A}^n_X$, the canonical maps

$$F(G(\mathbb{A}^n, f)) \to (\mathbb{A}^n_X, f), \qquad \mathcal{O}_f \to G(F(\mathcal{O}_f))$$

are equivalences. Observe that the functor

$$\Psi \colon \mathrm{Fun}^{\times}(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \to \mathrm{Fun}^{\times}(\mathcal{T}_A, \mathcal{S})$$

can be factored as

$$\mathrm{Fun}^{\times}(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \xrightarrow{\mathrm{Lan}_{\varphi}} \mathrm{Fun}(\mathcal{T}_A, \mathcal{S}) \xrightarrow{\pi} \mathrm{Fun}^{\times}(\mathcal{T}_A, \mathcal{S}).$$

Now, observe that $\mathrm{Lan}_{\varphi}(\mathcal{O}_f) = \mathrm{Map}_{/X}(\mathbb{A}^n_X, -)$. In particular, $\mathrm{Lan}_{\varphi}(\mathcal{O}_f)$ is still a product preserving functor. As a consequence,

$$\Psi(\mathcal{O}_f) = \pi(\mathrm{Lan}_{\varphi}(\mathcal{O}_f)) \simeq \mathrm{Lan}_{\varphi}(\mathcal{O}_f).$$

In particular, we obtain

$$F(\mathcal{O}_f) \simeq (\mathbb{A}^n_X, f).$$

The above discussion of the construction of G implies that $\mathcal{O}_f \simeq G(F(\mathcal{O}_f))$. Conversely, $G(\mathbb{A}^n_X, f) \simeq \mathcal{O}_f$, so that the above argument yields $(\mathbb{A}^n_X, f) \simeq F(G(\mathbb{A}^n_X, f))$. This completes the proof. ◻

Let $\mathcal{X}$ be an ∞-topos and let $\mathrm{CRing}(\mathcal{X}) := \mathrm{Sh}_{\mathrm{CRing}}(\mathcal{X})$ denote the ∞-category of sheaves of simplicial commutative rings on $\mathcal{X}$. Let $\mathcal{A} \in \mathrm{CRing}(\mathcal{X})$ and let $\mathcal{A}$-Mod denote the ∞-category of left $\mathcal{A}$-modules in $\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})}(\mathcal{X})$. The Dold–Kan correspondence induces a forgetful functor

$$\mathrm{CRing}(\mathcal{X}) \to \mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(\mathcal{X}).$$

Let $\mathcal{A}\text{-Mod}(\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(X))$ denote the ∞-category of left $\mathcal{A}$-modules in $\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(\mathcal{X})$. When $\mathcal{X} \simeq \mathcal{S}$, we have $\mathcal{A}\text{-Mod}(\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(X)) \simeq \mathcal{A}\text{-Mod}^{\geq 0}$, where $\mathcal{A}\text{-Mod}^{\geq 0}$ denotes the connective part of the canonical t-structure on $\mathcal{A}$-Mod. Note that the equivalence does not hold for general ∞-topos $\mathcal{X}$.

Corollary 8.3. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{A} \in \mathrm{CRing}(\mathcal{X})$ be a sheaf of simplicial commutative rings on $\mathcal{X}$. We have a canonical equivalence of ∞-categories*

$$\mathrm{Ab}(\mathrm{CRing}(\mathcal{X})_{/\mathcal{A}}) \simeq \mathcal{A}\text{-Mod}(\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(X)).$$

As a consequence, we have a canonical equivalence of stable ∞-categories.

$$\mathrm{Sp}(\mathrm{Ab}(\mathrm{CRing}(\mathcal{X})_{/\mathcal{A}})) \simeq \mathcal{A}\text{-Mod}.$$

Proof. The second statement follows from the first one. Indeed, it is enough to remark that

$$\mathrm{Sp}(\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(\mathcal{X})) \simeq \mathrm{Sh}_{\mathrm{Sp}(\mathcal{D}(\mathrm{Ab})^{\geq 0})}(\mathcal{X}) \simeq \mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})}(\mathcal{X}).$$

We are therefore reduced to proving the first statement.

Since $\mathcal{X}$ is an ∞-topos, we can choose a small ∞-category $\mathcal{C}$ such that $\mathcal{X}$ is a left exact and accessible localization of $\mathrm{PSh}(\mathcal{C})$. It follows that $\mathrm{Sh}_{\mathrm{CRing}}(\mathcal{X})$ and $\mathrm{Sh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(\mathcal{X})$ are localizations of $\mathrm{PSh}_{\mathrm{CRing}}(\mathcal{C})$ and of $\mathrm{PSh}_{\mathcal{D}(\mathrm{Ab})^{\geq 0}}(\mathcal{X})$, respectively. We can therefore replace $\mathcal{X}$ by $\mathrm{PSh}(\mathcal{C})$. For every $C \in \mathcal{C}$, let $\mathrm{ev}_C \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{S}$ be the functor given by evaluation at C. The collection $\{\mathrm{ev}_C\}_{C \in \mathcal{C}}$ of functors is jointly conservative. Furthermore, each ev_C is part of a geometric morphism of topoi. We are therefore reduced to proving the statement in the ∞-category of spaces $\mathcal{S}$, and we will write A instead of $\mathcal{A}$.

Recall from Definition 4.2 the Lawvere theory of abelian groups, $\mathcal{T}_{\mathrm{Ab}}$. Using Lemma 4.12, we have

$$\begin{aligned}\mathrm{Ab}(\mathrm{CRing}_{/A}) \simeq \mathrm{Ab}(\mathrm{CRing}_{A//A}) &\simeq \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathrm{CRing}_{A//A}) \\ &\simeq \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathrm{Fun}^{\times}(\mathcal{T}_A^{\mathrm{aug}}, \mathcal{S})) \simeq \mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}} \times \mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}).\end{aligned}$$

We can now invoke [6, 5.5.9.2] to obtain an equivalence

$$\mathrm{Fun}^{\times}(\mathcal{T}_{\mathrm{Ab}} \times \mathcal{T}_A^{\mathrm{aug}}, \mathcal{S}) \simeq \infty(\mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{Ab}} \times \mathcal{T}_A^{\mathrm{aug}}, \mathrm{sSet})),$$

where $\mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{Ab}} \times \mathcal{T}_A^{\mathrm{aug}}, \mathrm{sSet})$ is the category of strictly product preserving functors to sSet equipped with the projective model structure (whose existence is guaranteed by

[6, 5.5.9.1]), and where $\infty(-)$ denotes the underlying ∞-category of a simplicial model category (cf. [6, A.3.7]). We now remark that

$$\begin{aligned}\mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{Ab}} \times \mathcal{T}_A^{\mathrm{aug}}, \mathrm{sSet}) &\simeq \mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathrm{Funct}^{\times}(\mathcal{T}_A^{\mathrm{aug}}, \mathrm{sSet})) \\ &\simeq \mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{Ab}}, \mathrm{sCRing}_{A//A}) \\ &\simeq \mathrm{Ab}(\mathrm{sCRing}_{A//A}) \simeq A\text{-sMod},\end{aligned}$$

where $\mathrm{sCRing}_{A//A}$ denotes the simplicial model category of simplicial commutative A-algebras with an augmentation to A. Moreover, under this chain of equivalences, the model structure on $\mathrm{Funct}^{\times}(\mathcal{T}_{\mathrm{Ab}} \times \mathcal{T}_A^{\mathrm{aug}}, \mathrm{sSet})$ corresponds to the standard model structure on A-sMod. Finally, we can use the Dold–Kan equivalence in order to obtain the equivalence

$$\mathrm{Ab}(\mathrm{CRing}_{/A}) \simeq \infty(A\text{-sMod}) \overset{\mathrm{D-K}}{\simeq} A\text{-}\mathrm{Mod}^{\geq 0}. \qquad \square$$

8.2. *Flatness to order n*

In this section we introduce the notion of flatness to order n, which plays a key role in our proof of the representability theorem.

Definition 8.4. Let A be a simplicial commutative algebra and let $M \in A\text{-}\mathrm{Mod}^{\geq 0}$ be a connective A-module. We say that M is *flat to order n* if for every discrete A-module $N \in A\text{-}\mathrm{Mod}^{\heartsuit}$, we have

$$\pi_i(M \otimes_A N) = 0 \quad \text{for every } 0 < i < n+1.$$

Proposition 8.5. *Let A be a simplicial commutative algebra and let $M \in A\text{-}\mathrm{Mod}^{\geq 0}$ be a connective A-module.*

(1) *If M is flat to order n, then it is flat to order m for every $m \leq n$.*
(2) *M is flat to order n if and only if $\tau_{\leq n} M$ is flat to order n.*
(3) *If $f\colon A \to B$ is a morphism of simplicial commutative algebras and M is flat to order n, then $f^*(M) = M \otimes_A B$ is flat to order n.*
(4) *Let $m, n \geq 0$ be integers. Then M is flat to order n if and only if $M \otimes_A \tau_{\leq m} A$ is flat to order n.*

Proof. Statement (1) follows directly from the definitions. We prove (2). Consider the fiber sequence

$$\tau_{\geq n+1} M \to M \to \tau_{\leq n} M.$$

Let $N \in A\text{-}\mathrm{Mod}^{\heartsuit}$ and consider the induced fiber sequence

$$(\tau_{\geq n+1} M) \otimes_A N \to M \otimes_A N \to (\tau_{\leq n} M) \otimes_A N.$$

Since $\tau_{\geq n+1} M \otimes_A N \in A\text{-}\mathrm{Mod}^{\geq n+1}$, the conclusion follows from the long exact sequence of cohomology groups.

To prove (3), let $N \in B\text{-Mod}^{\heartsuit}$. Recall that the functor $f_*\colon B\text{-Mod} \to A\text{-Mod}$ is t-exact and conservative. In particular, it is enough to prove that $\pi_i(f_*(f^*(M) \otimes_B N)) = 0$ for $0 < i < n+1$. We have

$$f_*(f^*(M) \otimes_B N) \simeq M \otimes_A f_*(N).$$

The conclusion now follows from the fact that M is flat to order n.

Finally, we prove (4). Since $\pi_0(A) \simeq \pi_0(\tau_{\leq m}(A))$, it is enough to deal with the case $m = 0$. As the "only if" follows from (2), we handle the "if" direction. So suppose that $M \otimes_A \pi_0(A)$ is flat to order n. Let $N \in A\text{-Mod}^{\heartsuit}$. Since $A\text{-Mod}^{\heartsuit} \simeq \pi_0(A)\text{-Mod}^{\heartsuit}$, we see that N is naturally a $\pi_0(A)$-module. Therefore,

$$M \otimes_A N \simeq (M \otimes_A \pi_0(A)) \otimes_{\pi_0(A)} N.$$

Since $M \otimes_A \pi_0(A)$ is flat to order n, it follows that $\pi_i(M \otimes_A N) = 0$ for every $0 < i < n+1$. In other words, M is flat to order n. □

Proposition 8.6. *Let $A \in \mathrm{CRing}$ and let $M \in A\text{-Mod}^{\geq 0}$. Assume that M is flat to order n and A is m-truncated with $m \leq n$. Then $\tau_{\leq n} M$ is flat as an A-module.*

Proof. It follows from the proof of [12, 7.2.2.15, (3)⇒(1)] that

$$\pi_i(M) \simeq \pi_i(A) \otimes_{\pi_0(A)} \pi_0(M)$$

for $0 \leq i \leq n$. Moreover, since A is m-truncated and $m \leq n$, we see that

$$\pi_i(\tau_{\leq n} M) \simeq 0 \simeq \pi_i(A) \otimes_{\pi_0(A)} \pi_0(M)$$

for $i > n$. Therefore, $\tau_{\leq n} M$ is flat. □

Corollary 8.7. *Let $A \in \mathrm{CRing}$ and $M \in A\text{-Mod}^{\geq 0}$. Assume that M is flat to order n and A is m-truncated with $m \leq n$. Then M is flat to order $n+1$ if and only if $\pi_{n+1}(M) = 0$.*

Proof. Using Proposition 8.5(2), we deduce that $\tau_{\leq n}$ is flat to order n. For any $N \in A\text{-Mod}^{\heartsuit}$, consider the fiber sequence

$$\tau_{\geq n+1} M \otimes_A N \to M \otimes_A N \to \tau_{\leq n} M \otimes_A N.$$

Since A is m-truncated, Proposition 8.6 implies $\tau_{\leq n} M$ is flat. In particular, $\tau_{\leq n} M \otimes_A N$ is discrete. Therefore, passing to the long exact sequence of cohomology groups, we obtain

$$0 \to \pi_{n+1}(\tau_{\geq n+1} M \otimes_A N) \to \pi_{n+1}(M \otimes_A N) \to 0.$$

It follows from [12, 7.2.1.23] that

$$\pi_{n+1}(\tau_{\geq n+1} M \otimes_A N) \simeq \pi_{n+1}(M) \otimes_{\pi_0(A)} N.$$

Therefore, if $\pi_{n+1}(M) = 0$, then M is flat to order $n+1$. Conversely, if M is flat to order $n+1$, then choosing $N = \pi_0(A)$, we conclude that $\pi_{n+1}(M) = 0$. The proof is thus complete. □

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