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Global regularity for the Monge-Ampère equation with natural boundary condition

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Abstract

In this paper, we establish the global $C^{2,\alpha}$ and $W^{2,p}$ regularity for the Monge-Ampère equation $\det D^2u = f$ subject to boundary condition $Du(\Omega) = \Omega^*$, where Ω and Ω^* are bounded convex domains in the Euclidean space $\mathbb{R}^n$ with $C^{1,1}$ boundaries, and f is a Hölder continuous function. This boundary value problem arises naturally in optimal transportation and many other applications.

1. Introduction

In this paper we establish the global $C^{2,\alpha}$ and $W^{2,p}$ regularity for the Monge-Ampère equation

$$\det D^2 u(x) = f(x) \tag{1.1}$$

subject to the boundary condition

$$Du(\Omega) = \Omega^*, \tag{1.2}$$

where Ω, Ω^* are bounded convex domains in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and f is a positive function. We also assume that $f \in C^0(\overline{\Omega})$ for the global $W^{2,p}$ estimate ($p \geq 1$), and $f \in C^\alpha(\overline{\Omega})$ for the global $C^{2,\alpha}$ estimate ($\alpha \in (0,1)$).

The boundary value problem (1.1) and (1.2) arises naturally in optimal transportation with the quadratic cost function. It is a fundamental problem in the area and has received much attention due to its wide range of applications, such as in fluid mechanics, meteorology, image recognition, reflector design, and also in geometry and probability [14], [16], [26], [33], [34]. In particular, it was recently found that the problem (1.1) and (1.2) plays a fundamental role in Wasserstein generative adversarial networks, a fast growing technique

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in machine learning [22]. The existence and uniqueness of solutions to the problem (1.1) and (1.2) were obtained by Brenier in his pioneering work [3]. Since then the regularity of solutions has been a focus of attention in this area [33], [34] and has been studied in [5], [6], [7], [15], [32]. When Ω and Ω^* are bounded convex domains, and $f \geq 0$ satisfies the doubling condition, the global $C^{1,\alpha}$ regularity for the solution was obtained by Caffarelli [5]. In a landmark paper [7], Caffarelli established the global $C^{2,\alpha'}$ regularity for the problem (1.1) and (1.2), assuming that Ω and Ω^* are uniformly convex with C^2 boundary, $f \in C^\alpha(\overline{\Omega})$ and $f > 0$. When Ω and Ω^* are uniformly convex and $C^{3,1}$ smooth, and $f \in C^{1,1}(\overline{\Omega})$, the global smooth solution was first obtained by Delanoë [15] in 1991 for dimension two and later extended to high dimensions by Urbas [32]. The results of Caffarelli, Delanoë, and Urbas were used by Brendle and Warren [1], [2] to study the minimal Lagrangian graphs. These results may also be applied to the problem of convex hypersurfaces with prescribed spherical map [25].

The uniform convexity of domains is a natural condition for the regularity of solutions to boundary value problems of the Monge-Ampère equation. In fact, the uniform convexity is a necessary condition for the global regularity of solutions to the Dirichlet problem [9], [28], [31]. It was used extensively and played a critical role in the proof for both the Dirichlet problem and the second boundary value problem (1.1), (1.2) in the above mentioned papers [7], [9], [15], [28], [31], [32], and also in the paper on the Neumann problem [23].

Surprisingly, we found that for the boundary value problem (1.1) and (1.2), the uniform convexity of domains can be dropped. In this paper we obtain the global $C^{2,\alpha}$ regularity for the problem (1.1) and (1.2), assuming both Ω and Ω^* are convex only (instead of uniformly convex). From [6], [24, §7.3] it is known that for arbitrary positive and smooth functions f, the convexity of domains is necessary for the global C^1 regularity.

Not only the uniform convexity of domains can be dropped; we also prove that the boundary smoothness can be reduced to $C^{1,1}$. Note that if the boundaries are C^2 and uniformly convex, they will become quadratic polynomials after blowing-up, but if the boundaries are only $C^{1,1}$, we have to deal with the possibility that limit shape is not even C^1 smooth after the blowing-up. This is the situation that gives rise to substantial difficulties in our proof (see Remark 4.2). By blowing-up, we mean to normalize a sequence of sub-level sets of the solution.

Under the above assumptions on domains, in this paper we obtain the sharp boundary $C^{2,\alpha}$ regularity when $f \in C^\alpha(\overline{\Omega})$ and $f > 0$, and C^2 regularity when f is Dini continuous. For the Dirichlet problem, the sharp boundary $C^{2,\alpha}$ estimate was obtained in [31], [28], and the interior $C^{2,\alpha}$ estimate was obtained in [4].

THEOREM 1.1. *Assume that* Ω *and* Ω^* *are bounded convex domains in* $\mathbb{R}^n$ *with* $C^{1,1}$ *boundary, and assume that* $f \in C^\alpha(\overline{\Omega})$ *is positive for some* $\alpha \in (0,1)$. *Let* u *be a convex solution to* (1.1) *and* (1.2). *Then we have the estimate*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \tag{1.3}$$

where C *is a constant depending only* $n, \alpha, f, \Omega,$ *and* Ω^*.

Our argument also leads to the global $W^{2,p}$ estimate for the solution.

THEOREM 1.2. *Assume that* Ω *and* Ω^* *are bounded convex domains in* $\mathbb{R}^n$ *with* $C^{1,1}$ *boundary, and assume that* $f \in C^0(\overline{\Omega})$ *is positive. Let* u *be a convex solution to* (1.1) *and* (1.2). *Then we have the estimate*

$$\|u\|_{W^{2,p}(\overline{\Omega})} \leq C \tag{1.4}$$

for all $p \geq 1$, *where* C *is a constant depending only* $n, p, f, \Omega,$ *and* Ω^*.

The interior $W^{2,p}$ estimate for the Monge-Ampère equation was proved by Caffarelli [4]. The $W^{2,p}$ estimate at the boundary was obtained by Savin [27] for the Dirichlet problem, and by Figalli and the first author [10] for the boundary condition (1.2). The proof in [10] relies on the estimates in [7] and thus required the domains to be C^2 smooth and uniformly convex. In this paper, we assume that the domains are convex with $C^{1,1}$ boundary; see Remark 6.2 as well.

Relaxing the uniform convexity of domains to convexity does make sense in applications. For example, in Wasserstein generative adversarial networks, a typical case is when the domains Ω and Ω^* are squares or cubes [22], [30]. Theorems 1.1 and 1.2 imply the regularity of the solution on the faces of the cube. We will prove the $C^{3,\alpha}$ regularity at the corner in a separate paper. In dimension two, the $C^{3,\alpha}$ regularity was proved in [20] and it is optimal.

The proof of Theorems 1.1 and 1.2 is based on delicate analysis on sub-level sets of the solution near the boundary and uses various techniques on the Monge-Ampère equation [17], [19], in particular those from Caffarelli's papers. The uniform density in Section 2 was introduced by Caffarelli [7] but a different proof is needed for non-uniformly convex domains. The key estimate of the paper is the following uniform obliqueness.

LEMMA 1.1. *Assume that* Ω, Ω^* *are two convex domains with* $C^{1,1}$ *boundaries, and that* f *is positive and continuous. Let* $0 \in \partial\Omega$ *and the image* $Du(0) = 0 \in \partial\Omega^*$. *Then there exists a positive constant* μ *such that*

$$\langle \nu(0), \nu^*(Du(0))\rangle \geq \mu > 0, \tag{1.5}$$

where ν *and* ν^* *are the unit inner normals of* Ω *and* Ω^*, *respectively.*

Lemma 1.1 will be proved in Sections 4 and 5. To prove (1.5) for non-uniformly convex domains, we need to introduce a completely different and new idea. We also provide different proof for the boundary $C^{2,\alpha}$ estimate in Section 6 for convex domains with $C^{1,1}$ boundary. These new techniques may apply to other problems related to Monge-Ampère type equations. In particular, we have recently established the $C^{2,\alpha}$ regularity of free boundaries in optimal transportation [12], thus resolving an open problem raised by Caffarelli and McCann in [8].

This paper is organized as follows. In Section 2, we introduce some properties on the sub-level sets of solutions to the problem (1.1) and (1.2) and prove the uniform density property. In Section 3, we obtain the tangential $C^{1,\alpha}$ regularity for any given $\alpha \in (0,1)$. In Sections 4 and 5, we prove the uniform obliqueness in dimension two and high dimensions, respectively, which is the key ingredient for the proof of the global $C^{2,\alpha}$ and $W^{2,p}$ regularity. Finally in Section 6 we complete the proof of Theorems 1.1 and 1.2.

2. **Uniform density**

Consider the optimal transport with density f in Ω and density 1 in Ω^*. We assume that f satisfies $\lambda^{-1} < f < \lambda$ for a constant $\lambda > 0$ and $\int_\Omega f(x)dx = \int_{\Omega^*} dy$. Let u and v be the potential functions in Ω and Ω^*, respectively. Then u is a solution to (1.1) and (1.2). We extend u, v to convex functions in $\mathbb{R}^n$ as follows:

$$\tilde{u}(x) := \sup\{\ell(x) : \ \ell \text{ is affine}, \ \ell \le u \text{ in } \Omega, \ \nabla\ell \in \Omega^*\} \quad \text{for } x \in \mathbb{R}^n,$$
$$\tilde{v}(y) := \sup\{\ell(y) : \ \ell \text{ is affine}, \ \ \ell \le v \text{ in } \Omega^*, \ \nabla\ell \in \Omega\} \quad \text{for } y \in \mathbb{R}^n.$$

For simplicity of notation, we denote the extended functions $\tilde{u}, \tilde{v}$ as u, v. Let $0 \in \partial\Omega$ be a boundary point. By subtracting a linear function, we assume that $u(0) = 0$ and $u \ge 0$. Correspondingly, one has $0 \in \partial\Omega^*$, $v(0) = 0$ and $v \ge 0$ as well.

We introduce two different sub-level sets of u at $x_0 \in \overline{\Omega}$. One is

$$S_h[u](x_0) = \Big\{x \in \Omega : u(x) < \ell_{x_0}(x) + h\Big\},$$

which may be abbreviated as $S_h[u]$ or $S_h(x_0)$ when no confusion arises, where ℓ_{x_0} is a support function of u at x_0. The other one is the *centered* sub-level set

$$S^c_h[u](x_0) = \Big\{x \in \mathbb{R}^n : u(x) < \hat{\ell}(x) + h\Big\},$$

or simply denoted as $S^c_h[u]$ or $S^c_h(x_0)$, where the affine function $\hat{\ell}$ is chosen such that $\hat{\ell}(x_0) = u(x_0)$ and x_0 is the mass center for $S^c_h[u](x_0)$. The existence of such a linear function is proved in [5]. Note that $S_h(x_0)$ is contained in Ω, but $S^c_h(x_0)$ may contain both points in and out of Ω.

The extended function $u \in C^1(\mathbb{R}^n)$ satisfies $\det D^2 u = f\chi_\Omega$ in $\mathbb{R}^n$ if Ω, Ω^* are convex. The following lemma was established by Caffarelli [7, Cor. 2.2].

LEMMA 2.1. *Assume that Ω, Ω^* are convex and bounded. Given a centered sub-level set $S^c_h(x_0)$ with $x_0 \in \overline{\Omega}$, let T be a linear transform such that $B_1(0) \subset S^* =: T(S^c_h(x_0)) \subset B_n(0)$. Then $\hat{u}(x) =: h^{-1}[u-\ell](T^{-1}(x))$ satisfies*

$$B_r(0) \subset \nabla\hat{u}\left(\frac{1}{2}S^*\right) \subset \hat{u}(S^*) \subset B_{r^{-1}}(0), \tag{2.1}$$

where ℓ is the linear function such that $u = \ell$ on $\partial S^c_h(x_0)$. Scaling back, there is an ellipsoid E centered at $\nabla\ell$ such that

$$rE \subset \nabla u(S^c_h(x_0)) \subset r^{-1}E, \tag{2.2}$$

where αE denotes the α-dilation of E with respect to its center, and the constant $r > 0$ depends only on $n, \lambda, \Omega, \Omega^*$, but is independent of h and u.

Lemma 2.1 implies that $\nabla\ell$ is a true interior point of $\nabla u(S^c_h(x_0))$; namely, it has a positive distance from the boundary after normalization. The first inclusion in (2.1) also follows from the strict convexity of u [7, Cor. 2.3]; namely,

$$u(x) \geq u(x_0) + \nabla u(x_0)(x-x_0) + c_0 h \quad \forall\, x \in \partial S^c_h(x_0) \cap \Omega, \tag{2.3}$$

where c_0 is a constant depending on $n, \lambda, \Omega, \Omega^*$ but independent of h, u. The last inclusion in (2.1) is due to the doubling condition of $\mu_{\hat{u}}$, where $\mu_{\hat{u}}$ is the Monge-Ampère measure of $\hat{u}$.

Let $x_0 = 0 \in \partial\Omega$. We shall describe a geometric implication of (2.2). Denote $w = u - \hat{\ell}$, and assume that w attains its minimum at p_0. Let ϕ be a convex function whose graph is a convex cone with vertex at $(p_0, w(p_0))$ and satisfies $\phi = w$ on $\partial S^c_h(0)$. Then we have

$$\nabla\phi(x)\cdot(x-p_0) \leq \nabla w(x)\cdot(x-p_0) \leq c\nabla\phi(x)\cdot(x-p_0) \quad \forall\, x \in \partial S^c_h(0). \tag{2.4}$$

The first inequality is due to the convexity of w and the second one is due to (2.2).

Let $p \in \partial S^c_h(0)$ such that $p \cdot e_1 = \sup\{x \cdot e_1 :\ x \in S^c_h(0)\}$, where e_k denotes the unit vector on the x_k-axis, for $k = 1, 2, \ldots, n$. Then $\nabla w(p) = |\nabla w(p)|e_1$ and (2.4) implies that

$$|\nabla w(p)| \approx \frac{h}{(p-p_0)\cdot e_1}. \tag{2.5}$$

By the convexity, one sees that (2.5) holds if p_0 is replaced by any point in $\frac{1}{2}S^c_h(0)$. In particular, it holds when $p_0 = 0$. We will use (2.5) in the proof of Lemma 2.3 below.

In this paper we use the notation $a \gtrsim b$ (resp. $a \lesssim b$) if there exists a constant $C > 0$ depending only on n, f, Ω, Ω^* such that $a \geq Cb$ (resp. $a \leq Cb$), and $a \approx b$ means that $C^{-1}b \leq a \leq Cb$. For a convex set A, we also use the notation $A \sim E$, where E is an ellipsoid, if $C^{-1}E \subset A \subset CE$. For two convex sets A_1 and A_2, we denote $A_1 \sim A_2$ if there is an ellipsoid E such that $A_1 \sim E$ and $A_2 \sim E$. If $A \sim B$ for a ball B, we also say that A has a good shape.

The following lemma shows an equivalence relation between these two sub-level sets.

LEMMA 2.2. *Under the hypotheses of Lemma 2.1, for $h > 0$ small, we have*

$$S^c_{b^{-1}h}(0) \cap \Omega \subset S_h(0) \subset S^c_{bh}(0) \cap \Omega, \tag{2.6}$$

where the constant $b \geq 1$ depends only on $n, \lambda, \Omega, \Omega^*$, but is independent of h and u.

Proof. To prove the first inclusion, it suffices to prove that for any $x \in S^c_h(0)$, we have $u(x) \leq Ch$ for a constant $C > 0$ depending only on n. Indeed, assume that $\sup\{u(x) : x \in S^c_h(0)\}$ is attained at $p \in \partial S^c_h(0)$. Let $q = -\beta p$, where $\beta > 0$, be a point on $\partial S^c_h(0)$ such that $p, q, 0$ stay on a line segment. Since 0 is the center of $S^c_h(0)$, we have $c_n^{-1} \leq \beta \leq c_n$ for a constant c_n depending only on n. Noting that $u(0) = 0$ and $u = \ell$ on $\partial S^c_h(0)$ for a linear function ℓ, we have

$$\ell(q) + \beta\ell(p) = (1+\beta)\ell(0) = (1+\beta)h.$$

If $\ell(p) = u(p) > Ch$ for a large C, we have $u(q) = \ell(q) < 0$, which is a contradiction.

The second inclusion follows readily from the strict convexity, (2.3). □

The following uniform density was introduced and proved by Caffarelli in [7], assuming that Ω is polynomially convex. Here we relax the polynomial convexity to the convexity of domains with $C^{1,1}$ boundary.

LEMMA 2.3. *Assume that Ω, Ω^* are bounded convex domains with $C^{1,1}$ boundary, and that $0 \in \partial\Omega$. Then*

$$\frac{\text{Vol}\,(\Omega \cap S^c_h(0))}{\text{Vol}\left(S^c_h(0)\right)} \geq \delta_0 > 0 \tag{2.7}$$

for some positive constant δ_0 depending on $n, \lambda, \Omega, \Omega^*$, but independent of u and h.

Proof. Assume that $\{x_n = 0\}$ is the tangential plane of $\partial\Omega$ at 0 and $\Omega \subset \{x_n > 0\}$. Let S'_h and $S'_{\Omega,h}$ be respectively the projections of S^c_h and $S^c_h \cap \Omega$ on $\{x_n = 0\}$. To prove (2.7), it suffices to prove

$$|S'_{\Omega,h}| \geq C|S'_h|. \tag{2.8}$$

In fact, let $\tilde{p} = r_n e_n \in \partial S_h^c$; see Figure 1 below. Then we have $\mathrm{Vol}(S_h^c) \le C_1 r_n |S_h'|$ and $\mathrm{Vol}(\Omega \cap S_h^c) \ge C_2 r_n |S'_{\Omega,h}|$, where the constants C_1, C_2 only depend on the dimension n.

For any unit vector $e \in \{x_n = 0\}$, denote

$$\begin{aligned}\lambda_e &= \sup\{(x-y)\cdot e : \ x, y \in S'_{\Omega,h}\},\\ r_e &= \sup\{t : te \in S'_h\}.\end{aligned}$$

Note that λ_e is the width of projection of $S'_{\Omega,h}$ in the direction e, and r_e is the distance from 0 to the boundary $\partial S'_h$ in the direction e. We *claim* that if there is a positive constant C such that

$$\frac{\lambda_e}{r_e} \ge C \quad \forall\ e \in \partial B_1(0) \cap \{x_n = 0\}, \tag{2.9}$$

then (2.8) holds.

To prove this claim, we use induction on dimensions. Let E be the minimum ellipsoid of S'_h with principal radii $r_1 \le \cdots \le r_{n-1}$ and principal axes $e_1, \ldots, e_{n-1}$.

Let $p \in \partial S'_{\Omega,h}$ be a point satisfying $|p \cdot e_{n-1}| = \sup\{|x \cdot e_{n-1}| : x \in S'_{\Omega,h}\}$, and let $e_p := \frac{p}{|p|}$. By (2.9), $|p \cdot e_{n-1}| \ge C r_{n-1}$. Let $S''_{\Omega,h}$ be the projection of $S'_{\Omega,h}$ on $\{x : x \cdot e_p = 0\}$, and let $S''_h := S'_h \cap \{x : x \cdot e_p = 0\}$. Denoting

$$\begin{aligned}\lambda'_e &= \sup\{(x-y)\cdot e : \ x, y \in S''_{\Omega,h}\},\\ r'_e &= \sup\{t : te \in S''_h\}\end{aligned}$$

for any unit vector $e \in \mathrm{span}(e_1, \ldots, e_{n-1})$ and $e \perp e_p$, we still have $\frac{\lambda'_e}{r'_e} \ge C$. Observe that

$$|S'_{\Omega,h}| \approx |S''_{\Omega,h}||p| \ge C|S''_{\Omega,h}| r_{n-1} \quad \text{and} \quad |S'_h| \le C|S''_h| r_{n-1}.$$

Therefore, to prove (2.8) it suffices to prove

$$|S''_{\Omega,h}| \ge C|S''_h|.$$

By induction we can reduce it to one-dimensional case, in which the claim is trivial.

Let e_1 be the direction in which $\inf\left\{\frac{\lambda_e}{r_e} : \ e \in \{x_n = 0\}\right\}$ is attained. By the above claim, it suffices to prove that $\frac{\lambda_{e_1}}{r_{e_1}} \ge C$. Let ℓ be the linear function such that $u = \ell$ on $\partial S_h^c(0)$. By subtracting a linear function we assume that $\ell = 0$ (namely, we write $u - \ell$ as u). Assume u attains its minimum at p_0. Let p_l and p_r be the left and right ends of S_h^c, namely,

$$\begin{aligned}p_r \cdot e_1 &= \sup\{x \cdot e_1 : \ x \in S_h^c(0)\},\\ p_l \cdot e_1 &= \inf\{x \cdot e_1 : \ x \in S_h^c(0)\}.\end{aligned}$$

Denote $q_l = Du(p_l)$ and $q_r = Du(p_r)$. By definition, $r_{e_1}e_1 \in \partial S'_h$, and there exists $y \in \partial S^c_h(0)$ such that the projection of y on $\{x_n = 0\}$ is $r_{e_1}e_1$. Since $S^c_h(0)$ is balanced with respect to 0, we may assume $y_n = y \cdot e_n \geq 0$. Observe that $p_r \cdot e_1 \geq y \cdot e_1 = r_{e_1}$.

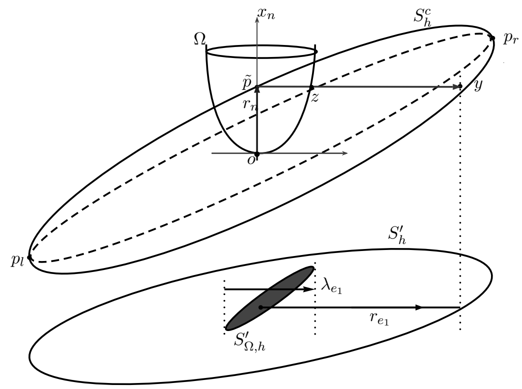

Figure 1.

If the ratio λ_{e_1}/r_{e_1} is sufficiently small, we have

(a) $\delta y \notin \Omega$ for some small $\delta > 0$.
(b) $\delta p_r, \delta p_l \notin \Omega$.
(c) $q_l, q_r \in \partial\Omega^*$.
(d) The segment $\overline{q_l q_r}$ is parallel to e_1.
(e) The point $q_0 := Du(p_0)$ lies on the segment $\overline{q_l q_r}$, and by (2.1), $|q_0 - q_l| \approx |q_0 - q_r|$.
(f) By the convexity of S^c_h, there is a unique number $r_n > 0$ such that $\widetilde{p} =: r_n e_n \in \partial S^c_h$. The line segment $\overline{\widetilde{p}y}$ intersects with $\partial\Omega$ at a point $z = (z_1, \dots, z_n)$. Since both points $\widetilde{p}, y \in \partial S^c_h$, we have $z \in S^c_h$. By definition we have $\lambda_{e_1} \geq |z'|$, where $z' = (z_1, \dots, z_{n-1})$. Hence by property (a) above and since $y_n \geq 0$, we infer that

$$z_n \geq \frac{r_{e_1} - |z'|}{r_{e_1}} r_n \geq \frac{1}{2} r_n.$$

Actually, in the triangle vertex at $(0, \tilde{p}, y)$, since $z \in \overline{\tilde{p}y}$, one has

$$\begin{aligned} z_n &\geq \frac{r_{e_1} - |z'|}{r_{e_1}} r_n \\ &\geq \frac{r_{e_1} - \lambda_{e_1}}{r_{e_1}} r_n \geq \frac{1}{2} r_n, \end{aligned}$$

as the ratio λ_{e_1}/r_{e_1} is sufficiently small.

(g) By the $C^{1,\delta}$ regularity of u, we have $r_n \geq Ch^{\frac{1}{1+\delta}}$. By the $C^{1,1}$ regularity of $\partial\Omega$ and property (f) above, we then have

$$r_{e_1} > |z'| \geq C z_n^{1/2} \geq C h^{\frac{1}{2(1+\delta)}}.$$

Let $q^* \in \partial\Omega^*$ be the point such that

$$|q_0 - q^*| = \inf\{|q - q_0| : \ q \in \partial\Omega^*\}.$$

Assume that $q^* = q_0 + \sigma e^*$ for a unit vector e^*. Note that $|p_l - p_r|$ is small if h is small. Hence by the $C^{1,1}$ smoothness of $\partial\Omega^*$ and property (e) above, we see that

$$\sigma = |q^* - q_0| \leq C|q_l - q_r|^2 \ \text{ as } h \to 0. \tag{2.10}$$

By (2.5) (note that (2.5) holds when p_0 is replaced by any point in $\frac{1}{2}S_h^c(0)$),

$$|q_r - q_l| = |Du(p_l) - Du(p_r)| \leq C\frac{h}{p_r \cdot e_1} \leq C\frac{h}{r_{e_1}} \leq C h^{\frac{1+2\delta}{2+2\delta}}.$$

Hence

$$\sigma = |q_0 - q^*| \leq C|q_r - q_l|^2 \leq C h^{\frac{1+2\delta}{1+\delta}} = C h^{1+\frac{\delta}{1+\delta}}.$$

But by (2.5) again, we also have

$$\sigma \approx \frac{h}{d_{e^*}}, \qquad \text{where} \quad d_{e^*} := \sup\{x \cdot e^* : x \in S_h^c(0)\}.$$

Hence

$$d_{e^*} \approx \frac{h}{\sigma} \geq h^{-\frac{\delta}{1+\delta}} \to \infty \ \text{ as } h \to 0.$$

This is apparently a contradiction, because $d_{e^*} \to 0$ as $h \to 0$, by the strict convexity of the solution. □

Remark 2.1. As mentioned before Lemma 2.3, the uniform density was proved by Caffarelli [7, Rem. 2, Th. 3.1], assuming that Ω is polynomially convex. In dimension two, a bounded convex domain is polynomially convex. Hence when $n = 2$, the uniform density holds for any bounded convex domains. No regularity on the boundaries $\partial\Omega$ and $\partial\Omega^*$ is needed.

From the uniform density property, we then have [7]

$$\mathrm{Vol}\,(S_h^c(0)) \approx \mathrm{Vol}\,(S_h^c(0) \cap \Omega) \approx h^{\frac{n}{2}} \tag{2.11}$$

for any $h > 0$ small. By Lemma 2.2, we also have

$$\mathrm{Vol}\,(S_h(0)) \approx h^{\frac{n}{2}}. \tag{2.12}$$

The following duality result can be found in [7, Cor. 3.2].

COROLLARY 2.1 (Duality). *Let T be a unimodular linear transform such that $B_{h^{1/2}} \subset T\{S_h^c[u](0)\} \subset B_{nh^{1/2}}$. Then we have*

$$B_{Ch^{1/2}} \subset T^*\{S_h^c[v](0)\} \subset B_{C^{-1}h^{1/2}}, \tag{2.13}$$

where $T^* = (T')^{-1}$ is the inverse of the transpose of T.

Proof. As the inner product $x \cdot y$ is invariant under the transforms T and T^*, to prove (2.13) one may assume directly that T is the identity mapping. Then (2.13) follows from (2.11) and (2.1). ☐

From Corollary 2.1, we also have the following corollaries, which will be used in Section 5.1.

COROLLARY 2.2. *For any $h > 0$ small, we have*

$$|x \cdot y| \le Ch \qquad \forall\ x \in S_h^c[u](0),\ y \in S_h^c[v](0). \tag{2.14}$$

Moreover, for any $x \in \partial S_h^c[u](0)$, there exists $y \in \partial S_h^c[v](0)$ such that

$$x \cdot y \ge C^{-1}h, \tag{2.15}$$

where C is a constant independent of u and h.

Remark 2.2. Similarly to Corollary 2.2, by Lemma 2.2 we also have the following relation between $S_h[u](0)$ and $S_h[v](0)$:

$$|x \cdot y| \le Ch \qquad \forall\ x \in S_h[u](0),\ y \in S_h[v](0). \tag{2.16}$$

Remark 2.3. Given any unit vector $e \in \mathbb{R}^n$, let

$$d_1 := \sup\{|x \cdot e| : x \in S_h^c[u](0)\}, \quad d_2 := \sup\{|x \cdot e| : x \in S_h[u](0)\}$$

be the width of $S_h^c[u](0)$ and $S_h[u](0)$, respectively, in the e direction. Note that $S_{bh}^c(0) \subset CbS_h^c(0)$ and $S_h^c(0) \subset CbS_{b^{-1}h}^c(0)$, where b is the constant in Lemma 2.2 and C is a constant independent of h; see [7, Observation (b) in Lemma 4.1]. Then by Lemma 2.2, Lemma 2.3, (2.11) and (2.12), we can obtain $d_1 \approx d_2$.

Remark 2.4. The estimates in this section are invariant under affine transforms. Let $S_{h_j}[u](x_j)$ be a sequence of sub-level sets, and let T_j be a linear transform such that $T_j\big(S_{h_j}[u](x_j)\big)$ has a good shape and $T_j(x_j) = 0$, where $x_j \in \overline{\Omega}$ and $h_j \to 0$ as $j \to \infty$. Denote

$$u_j(x) := \frac{1}{h_j} u(T_{h_j}^{-1}x) \quad \text{and} \quad \Omega_j := T_j(\Omega). \tag{2.17}$$

Then the estimates in Lemmas 2.1, 2.2 and 2.3 also hold for u_j, Ω_j with the same constants r, b, δ_0 independent of the sequence h_j. Assume that u_j, Ω_j sub-converge as $j \to \infty$ to limits u_0, Ω_0. One sees that these estimates hold for u_0 near 0 as well, again with the same constants r, b, δ_0, which depend only on $n, \lambda, \Omega, \Omega^*$, but are independent of Ω_0. Similarly, by taking limits, the estimates in Corollaries 2.1 and 2.2 for the centered sub-level sets $S^c_{h_j}[u], S^c_{h_j}[v]$ also hold for the limits u_0, v_0.

Furthermore, by Caffarelli's geometric decay estimate (see [5, Lemma 4], [7, Lemma 2.2]), one infers the strict convexity and $C^{1,\delta}$ regularity of solutions, namely,

$$C^{-1}|x|^{1+\delta^{-1}} \leq u(x) \leq C|x|^{1+\delta} \quad \forall\, x \in S_1[u](0), \tag{2.18}$$

if $u(0) = 0, Du(0) = 0$ and $S_1[u](0)$ is normalized, where $C, \delta > 0$ depend only on n, λ (assuming that $\partial\Omega \cap \partial S_1[u](0)$ and $\partial\Omega^*$ are convex). Formula (2.18) also holds for the sequence u_j and its limit u_0 near 0, with the same constants.

3. Tangential $C^{1,\alpha}$ regularity

The tangential $C^{1,\alpha}$ regularity of u, for any given $\alpha \in (0,1)$, was established in [7], where Ω is assumed to be a uniformly convex domain with C^2 boundary. But the same strategy applies to convex domains with $C^{1,1}$ boundary. To see this, let us outline the proof here.

Let $0 \in \partial\Omega$ be a boundary point. We assume that locally $\partial\Omega$ is given by $\{x_n = \rho(x')\}$ for some convex function $\rho \in C^{1,1}$ satisfying

$$\rho(0) = 0, \qquad D\rho(0) = 0,$$
$$\rho(x') \leq C|x'|^2,$$

where $x' = (x_1, \ldots, x_{n-1})$. In this section, we assume that $0 < f \in C^0(\overline{\Omega})$ and $f(0) = 1$. To prove the tangential $C^{1,\alpha}$, it suffices to prove

LEMMA 3.1. *For any given $\alpha \in (0,1)$, there exists a small constant $C = C_\alpha > 0$ depending only on n, the modulus of continuity of f and $\|\partial\Omega\|_{C^{1,1}}$, but independent of h, such that for the centered sub-level set $S^c_h(0)$, we have*

$$S^c_h(0) \cap \{x_n = 0\} \supset B_{C_\alpha h^{1/(1+\alpha)}}(0) \cap \{x_n = 0\}. \tag{3.1}$$

The idea of the proof is as follows. For each $h > 0$, there is an ellipsoid E_h such that

$$S^c_h(0) \sim E_h = \Big\{ \sum_{i=1}^{n-1} \Big(\frac{x_i - k_i x_n}{a_i}\Big)^2 + \Big(\frac{x_n}{a_n}\Big)^2 \leq 1 \Big\},$$

where $a_1 \leq \cdots \leq a_{n-1}$, namely, $\beta E_h \subset S^c_h(0) \subset \beta^{-1} E_h$ for some constant β depending only on n. Let be_n be the intersection of the positive x_n-axis and ∂E_h.

We first make a linear transform

$$\mathcal{T}_1 : \begin{cases} y_i = x_i - k_i x_n & i < n, \\ y_n = x_n. \end{cases} \tag{3.2}$$

This transformation $\mathcal{T}_1$ moves the center of $E_h \cap \{x_n = b\}$ to the point be_n. Hence, the "slope" k_i is bounded by

$$k_i \le \frac{a_i}{b} \quad \text{for } i = 1, \dots, n-1. \tag{3.3}$$

If the inclusion (3.1) does not hold, let $h_0 > 0$ be the largest constant such that (3.1) holds for $h > h_0$ and $\partial S^c_{h_0}(0) \cap \{x_n = 0\}$ touches $\partial B_{C_0 h_0^{1/(1+\alpha)}}$, where the constant C_0 is chosen small so that h_0 is also small. Then

$$a_1 \le C_0 h_0^{1/(1+\alpha)}. \tag{3.4}$$

By the $C^{1,\delta}$ regularity of u [5], we have

$$a_n \ge b \ge C h_0^{1/(1+\delta)}. \tag{3.5}$$

Next we make the linear transform

$$\mathcal{T}_2: \qquad z_i = y_i / a_i \qquad i = 1, \dots, n, \tag{3.6}$$

such that the sub-level set $S^c_{h_0}(0)$ is "normalized." Denote $\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$. The next lemma shows that near the origin, the $\mathcal{T}(\Omega)$ tends to be flat in the e_1 direction as $h_0 \to 0$.

Lemma 3.2. *For any given $R > 0$, the limit of $\mathcal{T}(\partial\Omega) \cap B_R(0)$ (as $h_0 \to 0$) is flat in the e_1 direction.*

Proof. Let $p' = (h_0^\gamma, 0, \dots, 0)$ be a point on the x_1-axis, where γ is chosen so that $\frac{1}{2(1+\delta)} < \gamma < \frac{1}{1+\alpha}$. Denote $p = (p', \rho(p'))$ and $q = \mathcal{T}(p)$. By direct computation we have

$$\begin{aligned} q_1 &= \frac{1}{a_1}(h_0^\gamma - k_1 \rho(p')), \\ q_i &= -\frac{1}{a_i} k_i \rho(p'), \qquad i = 2, \dots, n-1, \\ q_n &= \frac{1}{a_n} \rho(p'). \end{aligned}$$

Note that $\rho(p') \le C h_0^{2\gamma}$. By (3.3), (3.4) and (3.5) we have

$$k_1 \rho(p') \lesssim h_0^{\frac{1}{1+\alpha} - \frac{1}{1+\delta} + 2\gamma} \ll h_0^\gamma,$$

where the last inequality is due to the choice of γ. Hence $q_1 \to \infty$ as $h_0 \to 0$. It is also easy to verify that $|q_i| \le \frac{1}{b} h_0^{2\gamma} \to 0$ $(i = 2, \dots, n-1)$, and $q_n \to 0$, as $h_0 \to 0$. Note that the above computation still works if $p' = (-h_0^\gamma, 0, \dots, 0)$. Therefore, the limit of $\mathcal{T}(\partial\Omega)$ (as $h_0 \to 0$) contains the x_1 axis. By convexity, we see that the limit of $\mathcal{T}(\Omega)$ is independent of the e_1 direction. □

Since $\mathcal{T}\{S^c_h(0)\}$ is normalized, the domain $\mathcal{T}\big(\Omega \cap S^c_{Mh}(0)\big)$ has a good shape, where $M > 1$ is chosen such that $S^c_h(0) \subset \frac{1}{2} S^c_{Mh}(0)$. By the above discussion, the boundary part $\mathcal{T}\{\partial\Omega\cap S^c_{Mh}(0)\}$ becomes flat in the e_1 direction as $h\to 0$. As in [7], we denote

$$D_h = \{z \in \mathcal{T}(S^c_{Mh}(0)) :\ z = \hat{z} + te_1 \text{ for some } \hat{z} \in \mathcal{T}(S^c_{Mh}(0)\cap\Omega) \text{ and } t \in \mathbb{R}\}$$

by erasing the dependence on x_1. Then

$$\mathcal{T}(S^c_{Mh}(0)\cap\Omega) \subset D_h \subset \mathcal{T}(S^c_{Mh}(0)) \cap \{x_n > 0\},$$

and near the origin, ∂D_h is flat in the x_1-direction.

Let w be the solution to

$$\begin{cases} \det D^2 w = \chi_{D_{h_0}} \ \text{ in } \mathcal{T}(S^c_{Mh_0}(0)), \\ w = \tilde{u} \ \text{ on } \ \partial\{\mathcal{T}(S^c_{Mh_0}(0))\}, \end{cases} \tag{3.7}$$

where $\tilde{u}(z) = |\mathcal{T}|^{2/n} u(\mathcal{T}^{-1}(z))$. A key observation is that Pogorelov's interior second derivative estimate applies to w_{11}, even though the right-hand side of (3.7) is discontinuous in $(z_2,\dots,z_n)$, and no regularity of $\mathcal{T}(\partial\Omega)$ in $(z_2,\dots,z_n)$ is assumed. Therefore, w is $C^{1,1}$ in z_1. By the maximum principle one can give an estimate for $|w-\tilde{u}|$:

$$|w-\tilde{u}| \le C[\delta_0 + V_{h_0}]^{1/n}, \tag{3.8}$$

where $\delta_0 = \sup\{|f(x)-1| : x \in S^c_{Mh_0}(0)\cap\Omega\}$ and $V_{h_0} = \mathrm{Vol}\{D_{h_0} - \mathcal{T}(S^c_{Mh_0}(0)\cap\Omega)\}$ $= o(h_0)$.Changing back one obtains an estimate for u from the estimate $\partial_1^2 w$ $\le C$, from which one infers the tangential $C^{1,\alpha}$ for any given $\alpha\in(0,1)$. For details, see [7].

Corollary 3.1. *Assume that the function f, defined in Ω, is a positive constant near the origin, and both $\partial\Omega$ and $\partial\Omega^*$ are flat near the origin in a direction e. Then near the origin, u is $C^{1,1}$ and uniformly convex in the direction e.*

Proof. From the assumption, one can see that δ_0 and V_{h_0} vanish in estimate (3.8), thus $u \in C^{1,1}$ in the direction e. Since $\partial\Omega^*$ is also flat in the direction e, by Corollary 2.1 we have $B_{Ch^{1/2}}(0) \subset S^c_h[v](0)$ along the e direction. Then by the duality in Corollary 2.2, we have $S^c_h[u](0) \subset B_{Ch^{1/2}(0)}$ along the e direction. Hence u is uniformly convex in the direction e. □

4. Uniform obliqueness in dimension two

The uniform obliqueness (Lemma 1.1) is a key ingredient in proving the boundary $C^{2,\alpha}$ and $W^{2,p}$ estimates. The proof is technically rather complicated. For the reader's convenience, we divide the proof into two sections. In this section we prove Lemma 1.1 in dimension two. In dimension two, we assume that Ω,Ω^* are bounded convex domains with $C^{1,\gamma}$ boundaries for a small $\gamma>0$, and $f\in C^0(\overline{\Omega})$. In the next section we prove Lemma 1.1 in high dimensions. In dimension two, our proof consists of the following four steps:

(i) If the uniform obliqueness does not hold at the origin, we express the boundaries of Ω and Ω^* by (4.1) and prove a "balance property" of the sub-level set $S_h[u](0)$ in Lemma 4.1. It implies the decay estimates (4.12) and (4.13).
(ii) We introduce a blow-up argument so that the inhomogeneous term f becomes a positive constant in the limit.
(iii) The blow-up limit u_0 may not be smooth. We construct a smooth sequence $\{u_k\}$, which converges to u_0 locally uniformly.
(iv) We introduce the auxiliary function $w = \partial_1 u_0 + u_0 - x_1\partial_1 u_0$. By Steps (ii), (iii) and the maximum principle, the function $\underline{w}(t) = \inf w(t, \cdot)$ is concave near the origin. The concavity and the decay estimate (4.13) imply that $\underline{w} \equiv 0$ for $t > 0$ small, which contradicts to the strict convexity of u_0. Hence we infer the uniform obliqueness.

4.1. *Balance property and decay estimate.* Assume that $0 \in \partial\Omega$ and $\Omega \subset \{x_2 > 0\}$. To prove the uniform obliqueness, by the global $C^{1,\delta}$ regularity [5], we may assume to the contrary that $u(0) = 0$, $Du(0) = 0 \in \partial\Omega^*$ and $\Omega^* \subset \{y_1 > 0\}$. Then we have

(i) $u_1 =: u_{x_1} > 0$ in Ω and $v_2 =: v_{y_2} > 0$ in Ω^*; it implies that
(ii) if $x \in S_h(0)$, then $x - te_1 \in S_h(0)$ for all $t > 0$, provided $x - te_1 \in \Omega$,

where $S_h(0) = S_h[u](0)$ is the sub-level set of u, introduced in Section 2. Accordingly, the boundaries $\partial\Omega$ and $\partial\Omega^*$ near the origin can be expressed as

$$\begin{aligned} \partial\Omega &= \{x_2 = \rho(x_1)\}, \\ \partial\Omega^* &= \{y_1 = \rho^*(y_2)\}, \end{aligned} \tag{4.1}$$

with the following properties:

($\mathbf{H}_1$) $\rho, \rho^* \geq 0$ are convex functions defined in an interval $(-r_0, r_0)$ and satisfying $\rho(0) = 0$ and $\rho^*(0) = 0$, where $r_0 > 0$ is a constant.
($\mathbf{H}_2$) Denote $\sigma(t) = |t|^{1+\gamma}$. By the assumption $\partial\Omega \in C^{1,\gamma}$, we have

$$\rho(t) \leq C\sigma(t) \quad \text{for } t \leq 0. \tag{4.2}$$

Remark 4.1. (i) We will derive a contradiction from ($\mathbf{H}_1$) and ($\mathbf{H}_2$). Note that it suffices to assume $t \leq 0$ in (4.2). By Lemma 4.1 below, we can show that (4.2) holds for $t > 0$ as well. Hence $\partial\Omega$ is C^1 at 0.

(ii) As the reader will see, in our argument below we will not use any boundary regularity for $\partial\Omega^*$. For any point $p \in \partial\Omega \cap \{x_1 < 0\}$, since the inner product $\langle \nu(p), \nu^*(Du(p))\rangle \geq 0$, ($\mathbf{H}_2$) implies that $\rho^*(t) = o(t)$ for $t \geq 0$.

(iii) For clarity, in this section we will always assume that $\partial\Omega, \partial\Omega^* \in C^{1,\gamma}$ for a small $\gamma > 0$. By Remark 2.4, the constants in this section depend on $n, \lambda, \Omega, \Omega^*$ (inner and outer radii of Ω, Ω^* and γ). In the approximation $\{u_k\}$ in Section 4.3, we also allow that the constants depend on k. But all the constants are independent of h and u (for $h > 0$ small). The continuity of f is used only in the blow-up process, such that the right-hand side of (4.18) is a constant. In this section we do not use the tangential $C^{1,\alpha}$ regularity of Section 3.

Let $q = (q_1, q_2)$ and $\xi = (\xi_1, \xi_2)$ be two points on $\partial S_h(0) \cap \overline{\Omega}$ such that

$$\begin{aligned} \langle q, e_1\rangle &= \sup\{\langle x, e_1\rangle : x \in S_h(0)\}, \\ \langle \xi, e_1\rangle &= \inf\{\langle x, e_1\rangle : x \in S_h(0)\}. \end{aligned} \tag{4.3}$$

Apparently $q_1 > 0$ and $\xi_1 < 0$; see Figure 4.1. Note that $u_{x_2}(p) < 0$ for any boundary point $p \in \partial\Omega \cap \{x_1 > 0\}$. Hence q is an interior point of Ω. The following lemma shows that the area of $S_h[u](0) \cap \{x_1 > 0\}$ can balance that of $S_h[u](0) \cap \{x_1 < 0\}$.

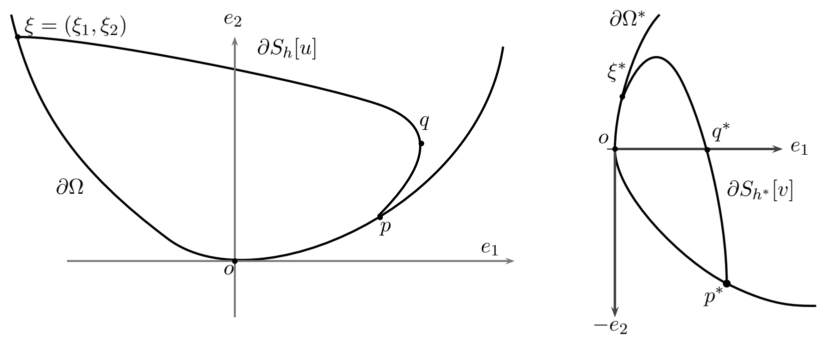

Figure 2.

LEMMA 4.1. *For all $h > 0$ small, we have the "balance" property*

$$q_1 \geq \delta_0 |\xi_1|, \tag{4.4}$$

where $\delta_0 > 0$ is a constant independent of h.

Proof. To prove (4.4), suppose to the contrary that

$$q_1 = o(|\xi_1|) \tag{4.5}$$

for a sequence $h \to 0$. Denote $t_0 = \frac{1}{2}(\xi_1 + q_1)$. There is a unique $s_0 > \rho(t_0)$ such that $u(t_0, s_0) = h$. Denote

$$x^c = (x_1^c, x_2^c) := \big(t_0, \frac{1}{2}(s_0 + \rho(t_0))\big).$$

The point x^c can be regarded as the center of $S_h(0)$. Denote

$$\begin{cases} \lambda_1 = q_1 - \xi_1, \\ \lambda_2 = s_0 - \rho(t_0). \end{cases}$$

Apparently

$$\text{Vol}(S_h(0)) \approx \lambda_1\lambda_2. \tag{4.6}$$

Moreover, by (4.2) and property (i), we have

$$\lambda_2 \le \xi_2 \le \sigma(\xi_1) \le \sigma(\lambda_1). \tag{4.7}$$

By (4.5), we have $\frac{1}{2}\xi_1 < x_1^c < \frac{1}{4}\xi_1$ for $h > 0$ small. Let us make the first change

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= x_2 - \frac{x_2^c}{x_1^c} x_1 \end{aligned} \tag{4.8}$$

such that $S_h[u] \subset \{x \in \mathbb{R}^2 : \xi_1 < x_1 < q_1, |x_2| < 4\lambda_2\}$. Note that such a change does not change the ratio $\frac{q_1}{|\xi_1|}$ in (4.4). We then make the change

$$\begin{aligned} z_1 &= y_1/\lambda_1, \\ z_2 &= y_2/\lambda_2, \end{aligned} \tag{4.9}$$

and accordingly,

$$u_h = u/h, \quad \text{where } h = u(\xi). \tag{4.10}$$

By the volume estimate (4.6), the sub-level set $S_h(0)$ has a good shape after changes (4.8) and (4.9). By Lemmas 2.2 and 2.3 (in dimension two, the uniform density holds for any bounded convex domains — see Remark 2.1), the centered sub-level set $S_h^c(0)$ also has a good shape after the change. Hence,

$$\|u_h\|_{L^\infty(B_1(0))} \le C \tag{4.11}$$

for a constant C independent of u and h.

Let q_h, ξ_h be the corresponding points of q, ξ after the above changes. Assume that $q_h \to q_0 = (q_{0,1}, q_{0,2})$, $u_h \to u_0$ as $h \to 0$. By (4.5) we have $q_{0,1} = 0$; namely, q_0 is on the x_2-axis. On the other hand, after the change (4.9), the line $\{x_1 = q_{h,1}\}$ is tangent to $S_1[u_h]$ at q_h, and thus $u_h(x) \ge 1$ for all $x \in \{x_1 \ge q_{h,1}\}$. Passing to the limit we have $u_0(x) \ge 1$ for $x \in \{x_1 \ge 0\}$ and $u_0(0) = \lim_{h\to 0} u_h(0) = 0$, which is a contradiction by (4.11). In fact, since u_0 is a limit of a sequence of locally uniformly bounded convex functions u_h, the function u_0 must be continuous at the origin. □

Corollary 4.1. *For $t > 0$, denote*

$$\underline{u}(t) = \inf\{u(t, x_2) : x_2 \ge \rho(t)\}.$$

We have the asymptotic estimate

$$\underline{u}(t) \le Ct\sigma(t) \quad \textit{for } t > 0 \textit{ small.} \tag{4.12}$$

Proof. By the strict convexity of u, the sub-level set $S_h(0)$ shrinks to the origin as $h \to 0$. Hence, for any $t > 0$, there exists a unique $h > 0$ such that $\{x_1 = t\}$ is tangential to $\partial S_h(0)$ at the point $q = (q_1, q_2)$. This implies that $q_1 = t$ and $\underline{u}(t) = h$.

From (4.4),

$$\lambda_1 = q_1 - \xi_1 \leq Ct$$

for some constant C independent of h. Then from (4.7) and (4.6) we have

$$\lambda_2 \leq C\sigma(t) \quad \text{and} \quad \mathrm{Vol}(S_h(0)) \leq Ct\sigma(t).$$

Hence by (2.12) we obtain that

$$\underline{u}(t) = h \leq Ct\sigma(t).$$

□

COROLLARY 4.2. *For $t > 0$, denote*

$$\underline{\partial_1 u}(t) = \inf\{\partial_1 u(t, x_2) : x_2 \geq \rho(t)\}.$$

Then we have the asymptotic behavior for $t > 0$ small,

$$\underline{\partial_1 u}(t) \leq C\sigma(t). \tag{4.13}$$

Proof. This is a direct consequence of (4.12). In fact, by the convexity of u, for $t > 0$ small,

$$\partial_1 u(t, x_2) \leq \frac{u(2t, x_2) - u(t, x_2)}{t} \leq \frac{u(2t, x_2)}{t}.$$

Then taking the infimum in x_2, from (4.12), we obtain that

$$\underline{\partial_1 u}(t) = \inf_{x_2} \partial_1 u(t, x_2) \leq \frac{\underline{u}(2t)}{t} \leq C\sigma(t).$$

□

4.2. *A blow-up sequence.* Assume that $f > 0$ is continuous. The purpose of blow-up is such that f becomes a positive constant in the limit.

From the proof of Lemma 4.1, the sub-level set $S_h[u](0)$ has a good shape under the following normalization $\mathcal{T}$:

$$\begin{aligned} y_1 &= x_1/\lambda_1, \quad &\text{with } \lambda_1 = q_1 - \xi_1, \\ y_2 &= x_2/\lambda_2, \quad &\text{with } \lambda_2 = \rho(\xi_1). \end{aligned} \tag{4.14}$$

In fact, as shown in the proof of Lemma 4.1, we have $\mathrm{Vol}(S_h[u]) \approx \lambda_1\lambda_2$. Hence $\mathcal{T}(S_h[u]) \approx 1$. Also by the proof of Lemma 4.1, $\mathcal{T}(S_h[u]) \subset [-1, 1] \times [0, 1]$. Hence $\mathcal{T}(S_h[u])$ has a good shape.

Accordingly we make the change $u \to u_h$, where

$$u_h(x) = u(\mathcal{T}^{-1}x)/h. \tag{4.15}$$

After the change, the domain Ω is changed to Ω_h, and the boundary $\{x_2 = \rho(x_1)\}$ is changed to $\left\{x_2 = \rho_h(x_1) = \frac{1}{\lambda_2}\rho(\lambda_1 x_1)\right\}$. By Lemmas 2.2 and 2.3 we also have

$$B_{\frac{1}{C}}(0) \subset \mathcal{T}(S_h^c[u](0)) \subset B_C(0) \tag{4.16}$$

for some constant C depending only on n, the constants b in Lemma 2.2 and δ_0 in Lemma 2.3, but independent of h. In (4.16) the centered sub-level set S_h^c can be replaced by the usual sub-level set S_h if the center of the concentric ball is properly chosen.

By (4.12), the limit $\lim_{t\to 0} \frac{\underline{u}(t)}{t\sigma(t)} < \infty$. Hence for any fixed small $\bar{\varepsilon} > 0$ (we may fix $\bar{\varepsilon} = 1$), there is a sequence $t_j \to 0$ ($t_j > 0$) such that

$$\frac{\underline{u}(t)}{t\sigma(t)} \le (1+\bar{\varepsilon})\frac{\underline{u}(t_j)}{t_j\sigma(t_j)} \quad \forall\, t \in (0, t_j). \tag{4.17}$$

Denote $h_j = \underline{u}(t_j)$. Since $\mathcal{T}(S_h[u])$ has a good shape, for any $R > 0$, $\Omega_{h_j} \cap B_R(0)$ converges in Hausdorff distance to a limit. Hence by passing to a subsequence, Ω_{h_j} converges to a limit Ω_0 as $h_j \to 0$, which is an unbounded convex domain in $\mathbb{R}^2$.

Next we show that u_{h_j} sub-converges to a limit u_0 as $h_j \to 0$. Indeed, by the geometric decay of sections ([7, Lemma 2.2]), for any $k > 0$, there exists a constant M_k such that

$$kS_h^c[u] \subset S_{M_k h}^c[u] \quad \text{ for } h > 0 \text{ small.}$$

Hence by the convexity of u_{h_j} and the estimate (4.16), u_{h_j} is locally uniformly bounded, which implies the sub-convergence.

By the weak convergence of the Monge-Ampère operator, u_0 satisfies the equation

$$\det D^2 u_0 = c_1 \chi_{\Omega_0} \quad \text{ in } \mathbb{R}^2 \tag{4.18}$$

for a positive constant c_1. There is no loss of generality in assuming that $c_1 = 1$.

By the change (4.14), we have $0 \in \partial\Omega_0$. Let $\mathcal{J}$ be the projection of Ω_0 on the x_1-axis. By Lemma 4.1, there is an interval $(0, r_0) \subset \mathcal{J}$. Hence the *lower boundary* of $\Omega_0 \cap \{0 < x_1 < r_0\}$ can be represented by a convex function

$$x_2 = \rho_0(x_1), \tag{4.19}$$

and ρ_0 is the limit of ρ_{h_j}, passing to a subsequence if necessary.

Remark 4.2. (i) In Lemma 4.1, we proved that $|\xi_1| \le Cq_1$, but the possibility $|\xi_1| = o(q_1)$ as $h \to 0$ has not been ruled out. Hence, even when $\partial\Omega \in C^{1,1}$, the limit Ω_0 may be contained in the first quadrant, i.e., $\Omega_0 \subset \{x_1 > 0, x_2 > 0\}$. In this case, ρ_0 is defined in $\{x_1 > 0\}$.

(ii) No matter whether Ω_0 is contained in the first quadrant, we point out that the whole positive x_2 axis is contained in $\overline{\Omega}_0$. To see this, notice that there exists a constant β_0 such that $\beta e_2 \in \Omega$ for all $\beta \in (0, \beta_0)$. By the transform $\mathcal{T}$

in (4.14), we have $\beta e_2 \in \Omega_h$ for all $\beta \in (0, \frac{\beta_0}{\lambda_2})$. By the strict convexity of u, $\lambda_2 \to 0$ and $\beta_0/\lambda_2 \to \infty$ as $h \to 0$. Hence for any $R > 0$, $Re_2 \in \Omega_h$ provided $h > 0$ is small enough. Passing to the limit, we have $Re_2 \in \overline{\Omega_0}$.

(iii) In Corollary 4.4 below, we will show that $\rho_0(t) \le C\sigma(t)$ for $t > 0$ small. But ρ_0 may not be smooth. In comparison, if $\partial\Omega \in C^2$ is uniformly convex, then ρ_0 is a quadratic polynomial [7]. The lack of smoothness of ρ_0 in our case makes the problem much more complicated.

By our choice of the sequences t_j and $h_j = \underline{u}(t_j)$ in (4.17), the asymptotic behavior (4.12) holds for the limits u_0 and ρ_0. Namely, we have the following estimates.

Corollary 4.3. *Denote*

$$\underline{u_0}(t) = \inf\{u_0(t, x_2) : x_2 \ge \rho_0(t)\} \quad t > 0.$$

We have

$$\underline{u_0}(t) \le Ct\sigma(t) \quad \textit{for } t > 0 \textit{ small.} \tag{4.20}$$

Proof. Let q, ξ be the points defined in (4.3) with $h = h_j$, and let $\lambda_1 = q_1 - \xi_1, \lambda_2 = \rho(\xi_1)$ as in (4.14). From (4.4), $t_j = q_1 \approx \lambda_1$.

By (4.15), $u_{h_j}(x_1, x_2) = \frac{u(\lambda_1 x_1, \lambda_2 x_2)}{\underline{u}(t_j)}$. Hence, by (4.17),

$$\begin{aligned} \inf\{u_{h_j}(t, x_2) : x_2 \ge \rho_{h_j}(t)\} &= \frac{\inf\{u(\lambda_1 t, \lambda_2 x_2) : x_2 \ge \rho(t)\}}{\underline{u}(t_j)} \\ &= \frac{\underline{u}(\lambda_1 t)}{\underline{u}(t_j)} \\ &\le (1 + \bar{\varepsilon}) \frac{(\lambda_1 t)\sigma(\lambda_1 t)}{t_j \sigma(t_j)} \\ &\le Ct\sigma(t), \end{aligned}$$

where the constant $C > 0$ is independent of j. The above inequality implies that $\underline{u_{h_j}}(t) \le Ct\sigma(t)$. Passing to the limit, we obtain (4.20). □

Corollary 4.4. *We have*

$$\rho_0(t) \le C\sigma(t) \quad \textit{for } t > 0 \textit{ small.} \tag{4.21}$$

Proof. For any given $h > 0$ small, as in (4.3) we introduce two points ξ and q for the sub-level set $S_h[u_0]$. Let $z = \beta e_2$ be the point on the x_2-axis such that $u_0(z) = h$. By Corollary 4.3,

$$h = u(q) \le Cq_1\sigma(q_1).$$

By (2.12) and Remark 2.4, we have

$$\frac{1}{2}\beta q_1 \le \mathrm{Vol}(S_h[u_0]) \le Ch.$$

Hence $\beta \le C\sigma(q_1)$. Noting that $\partial_{x_1} u_0 \ge 0$, we infer that $q_2 \le \beta \le C\sigma(q_1)$, and so (4.21) follows. □

Denote $\mathcal{T}^* = \frac{1}{h}(\mathcal{T}')^{-1}$, the dual affine transform for v, where $\mathcal{T}'$ is the transpose of $\mathcal{T}$ in (4.14). As in (4.14), we denote $v_h(y) = \frac{1}{h} v((\mathcal{T}^*)^{-1}y)$ and $\Omega_h^* = \mathcal{T}^*(\Omega^*)$. The boundary $\partial\Omega_h^*$ near the origin is given by $\{y_1 = \rho_h^*(y_2)\}$. Similarly, $\Omega_{h_j}^*$ converges to an unbounded convex domain Ω_0^* and v_{h_j} converges to a convex function v_0, locally uniformly. Moreover,

$$\det D^2 v_0 = c_2 \chi_{\Omega_0^*} \quad \text{in } \mathbb{R}^2 \tag{4.22}$$

for a positive constant c_2.

Remark 4.3. As pointed out in Remark 4.2, we need to deal with the case when Ω_0^* is contained in the first quadrant. (Note that by Lemma 4.1, it is again the first quadrant, not the fourth quadrant.) By Remark 4.2, the whole positive y_1-axis is contained in $\overline{\Omega_0^*}$.

Although Ω_0 is unbounded, by Remark 2.4, u_0 is locally strictly convex and $u_0 \in C_{loc}^{1,\delta}(\overline{\Omega}_0)$. By the convexity of Ω_0 and Ω_0^*, we have $u_0 \in C^1(\mathbb{R}^2)$. By the interior regularity for the Monge-Ampère equation (4.18), u_0 is C^∞ smooth inside Ω_0.

4.3. *Smooth approximation.* In this subsection we shall construct a smooth approximation sequence $\{u_k\}$ converging to u_0 in a small neighborhood of the origin. The $C^{2,\alpha}$-smoothness of u_k is needed for deriving the contradiction in Section 4.4. Note that we just need the $C^{2,\alpha}$-smoothness of u_k but not a uniform upper bound for the $C^{2,\alpha}$-norm of u_k.

Let $V = B_r \cap \Omega_0^*$ for a small constant $r > 0$, and let $U = Dv_0(V)$. By the local strict convexity of v_0, there exists $r_0 > 0$ such that $B_{r_0} \cap \Omega_0 \subset U$. Let U_k, U_k^* be C^∞ smooth, bounded domains such that

(a) $0 \in \partial U_k$, $0 \in \partial U_k^*$ for all $k \ge 1$;
(b) $U_k \subset \{x_2 > 0\}$, $U_k^* \subset \{y_1 > 0\}$ for all $k \ge 1$;
(c) $U_k \to U$, $U_k^* \to V$, as $k \to \infty$, in Hausdorff's sense.

Moreover, we assume the lower part of the boundary $\partial U_k \cap B_{r_0}$ is the graph of a smooth, uniformly convex function ρ_k in the direction e_2, that is,

$$\Gamma_k =: \{x_2 = \rho_k(x_1)\} \cap B_{r_0}, \tag{4.23}$$

where ρ_k is defined on $\mathcal{J}_k$ that is the projection of $\partial U_k \cap B_{r_0}$ on the x_1-axis. In our construction, $0 \in \mathcal{J}_k$ is an interior point of $\mathcal{J}_k$ for all $k \ge 1$. By Corollary 4.4, $(0, \frac{1}{2}r_0) \subset \mathcal{J}_k$. From the above conditions (a)–(c) and uniform convexity, the function ρ_k satisfies

$$\begin{aligned} &\rho_k(0) = 0, \quad (\rho_k)'(0) = 0, \\ &(\rho_k)'(x_1) > 0 \ \text{ for } x_1 > 0, \quad (\rho_k)'(x_1) < 0 \ \text{ for } x_1 < 0. \end{aligned} \tag{4.24}$$

Meanwhile, the left part of the boundary $\partial U_k^* \cap B_{r_0}$ is the graph of a smooth, uniformly convex function ρ_k^* in the direction e_1, that is,

$$\Gamma_k^* =: \{y_1 = \rho_k^*(y_2)\} \cap B_{r_0}, \tag{4.25}$$

where ρ_k^* is defined on $\mathcal{J}_k^*$ that is the projection of $\partial U_k^* \cap B_{r_0}$ on the y_2-axis. In our construction, $0 \in \mathcal{J}_k^*$ is an interior point of $\mathcal{J}_k^*$ for all $k \geq 1$. The function ρ_k^* satisfies

$$\begin{aligned} &\rho_k^*(0) = 0; \quad (\rho_k^*)'(0) = 0; \\ &(\rho_k^*)'(y_2) > 0 \ \text{ for } y_2 > 0; \quad (\rho_k^*)'(y_2) < 0 \ \text{ for } y_2 < 0. \end{aligned} \tag{4.26}$$

Let u_k be the potential function for the optimal transport from $(U_k, 1)$ to (U_k^*, g_k), where the density $g_k = \frac{|U_k|}{|U_k^*|}$ is a constant. Subtracting a constant we have $u_k(0) = 0$. Since U_k^* is convex, we can extend u_k to $\mathbb{R}^2$ by

$$u_k(x) := \sup\{\ell(x) : \ \ell \text{ is affine, } \ell \leq u_k \text{ in } U_k, \ \nabla \ell \in U_k^*\} \quad \text{for } x \in \mathbb{R}^2. \tag{4.27}$$

Since $u_0(0) = 0$, by the uniqueness of potential functions we have $u_k \to u_0$ uniformly in $B_{r_0}(0)$ for a different $r_0 > 0$ small. (In this subsection, the constant r_0 may change from line to line but they have a uniform positive lower bound independent of k.) By the uniqueness of optimal mapping Du_0, we also have $Du_k \to Du_0$ in $B_{r_0}(0)$.

By (2.18) in Remark 2.4, u_k are uniformly $C^{1,\delta}$ smooth in $\overline{U_k} \cap B_{r_0}(0)$, and the extended $u_k \in C^1(B_{r_0}(0))$, by the convexity of U_k^*. Let v_k be the dual of u_k, namely, the potential function for the optimal transport from (U_k^*, g_k) to $(U_k, 1)$. Similarly to (4.27), let

$$\hat{v}_k(y) := \sup\{\ell(y) : \ell \text{ is affine, } \ell \leq v_k \text{ in } Du_k(B_{r_0}(0)), \ \nabla \ell \in B_{r_0}(0)\} \text{ for } y \in \mathbb{R}^2. \tag{4.28}$$

Then $\hat{v}_k = v_k$ in $\overline{U_k^*} \cap B_{r_0}(0)$, $\hat{v}_k$ is uniformly $C^{1,\delta}$ smooth in $\overline{U_k^*} \cap B_{r_0}(0)$, and $\hat{v}_k \in C^1(B_{r_0}(0))$ for a different r_0.

We have constructed the sequences of functions ρ_k, ρ_k^* defined on $\mathcal{J}_k, \mathcal{J}_k^*$, respectively, with the properties (4.24) and (4.26). Moreover, $[0, r_0/2] \subset \mathcal{J}_k, \mathcal{J}_k^*$. The next lemma shows that u_k satisfies the obliqueness condition and is smooth on Γ_k for all k.

LEMMA 4.2.

(i) *For each $k \geq 1$, we have*

$$\nu_k(x) \cdot \nu_k^*(Du_k(x)) > 0 \quad \forall\, x \in \Gamma_k, \tag{4.29}$$

where ν_k and ν_k^* are the unit inner normals of the domains $\{x_2 > \rho_k(x_1)\} \cap B_{r_0}$ and $\{y_1 > \rho_k^*(y_2)\} \cap B_{r_0}$, respectively.

(ii) *For each $k \geq 1$, u_k is smooth, locally uniformly convex, and $\det D^2 u_k$ is a positive constant in $B_{r_0}(0) \cap \{x_2 \geq \rho_k(x_1)\}$ (up to the boundary Γ_k).*

Proof. The proof is as follows (for a fixed k):

(a) (ii) actually follows from (i). In fact, if the obliqueness (4.29) holds, we can apply the argument in Section 6 to obtain the smoothness in (ii). So, it suffices to prove (i).

(b) Suppose to the contrary that the obliqueness fails for u_k at a point $p_0 = (t_0, \rho_k(t_0)) \in \Gamma_k$ with $t_0 < r_0$. We *claim* that there is an interval $(\bar t, \bar t + \varepsilon)$, where $\bar t \geq t_0$, such that the obliqueness fails for u_k at $\bar p =: (\bar t, \rho_k(\bar t))$, but holds at all points on $\Gamma_k \cap \{\bar t < x_1 < \bar t + \varepsilon\}$ (Lemma 4.3). Then by the regularity in Section 6, u_k is smooth up to the boundary $\Gamma_k \cap \{\bar t < x_1 < \bar t + \varepsilon\}$.

(c) Regard $\bar p$ as the origin. Applying the argument in Section 4.4 to u_k on $\Gamma_k \cap \{\bar t < x_1 < \bar t + \varepsilon\}$, we reach a contradiction at the point $\bar p$. It implies that the obliqueness must hold for u_k at all points on Γ_k; namely, (i) is proved.

To apply the argument in Section 4.4 to u_k, we do not need to carry out the blow-up for u_k in Section 4.2, as $\det D^2 u_k$ is already a positive constant. We also point out that the $C^{2,\alpha}$ regularity proved in Section 6 is localized. That is, let $x_0 \in \partial\Omega$ and $y_0 = Du(x_0) \in \partial\Omega^*$. If $\partial\Omega$ and $\partial\Omega^*$ are $C^{1,1}$ and convex near x_0 and y_0, respectively, then u is $C^{2,\alpha}$ smooth near x_0.

Therefore, it remains to verify the claim in (b), which will be proved in Lemma 4.3 below. □

Remark 4.4. Instead of using Lemma 4.3 below and the regularity in Section 6, we can also use Caffarelli's localized boundary $C^{2,\alpha}$ regularity [7] to conclude that u_k is smooth up to Γ_k. However, the proof in [7] is rather involved. Here we use Lemma 4.3 to make our proof self-contained. This lemma will also be used in Section 5 for high dimensions.

LEMMA 4.3. *Assume that the obliqueness fails for u_k at a point $p_0 = (t_0, \rho_k(t_0)) \in \Gamma_k$ (for any fixed $k \geq 1$). Then there is a boundary point $\bar p = (\bar t, \rho_k(\bar t)) \in \Gamma_k$ with $\bar t \geq t_0$, such that the obliqueness fails at $\bar p$, but holds at all points $p \in \Gamma_k \cap \{\bar t < x_1 < \bar t + \varepsilon\}$ for a constant $\varepsilon > 0$.*

Proof. Since the obliqueness fails at p_0, by a change of coordinates and subtracting a linear function to u_k, we can assume that $p_0 = 0$, $Du_k(0) = 0$, and also

$$0 \in \partial U_k, \quad U_k \subset \{x_2 > 0\}; \quad 0 \in \partial U_k^*, \quad U_k^* \subset \{y_1 > 0\}.$$

We still use ρ_k, ρ_k^* to denote the boundary $\partial U_k, \partial U_k^*$ near 0, which are smooth, uniformly convex near 0. Correspondingly, we have $\rho_k(0) = 0, \rho_k \geq 0$, and $\rho_k^*(0) = 0, \rho_k^* \geq 0$.

For any boundary point $p = (t, \rho_k(t)) \in \Gamma_k$, let $\zeta = \zeta(p)$ denote the unit tangential vector of Γ_k at p. Denote by τ_1 and η the tangential and normal vectors of $\partial S_h[u_k]$ at p, respectively, where $h = u_k(p)$. Let α be the angle between η and ν, and let β be the angle between ν^* and τ_1 (see Figure 3). Then,

$$\nu(p) \cdot \nu^*(Du_k(p)) = \cos\left(\frac{\pi}{2} - \alpha - \beta\right) \geq \frac{1}{2}(\alpha + \beta). \tag{4.30}$$

By definition, the obliqueness holds at p if and only if $\nu(p) \cdot \nu^*(Du_k(p)) > 0$.

We claim that $\beta > 0$. Indeed, since ρ_k^* is smooth, $(\rho_k^*)'(0) = 0$. Noticing that $p^* = Du_k(p)$, we see that $\overline{op^*}$ is parallel to η. Hence by the uniform convexity of ∂U_k^*,

$$\beta = \arctan\left(\left|(\rho_k^*)'(p_2^*)\right|\right) - \arctan\left(\frac{p_1^*}{|p_2^*|}\right) > 0.$$

Hence to prove Lemma 4.3, we just need to prove that there is an interval of Γ_k in which $\alpha \geq 0$. From Figure 3, it is easy to see that $\alpha \geq 0$ if and only if $\partial_\zeta u_k(p) \geq 0$. Since $u_k(0) = 0$ and $u_k(x) > 0$ for all $x \in \Gamma_k \setminus \{0\}$, there exists a point $\hat{p} = (\hat{t}, \rho_k(\hat{t})) \in \Gamma_k$ with $\hat{t} > 0$ such that $\partial_\zeta u_k(\hat{p}) > 0$. Hence there exists $\varepsilon > 0$ such that

$$\partial_\zeta u_k(p) > 0 \quad \text{for } p = (t, \rho_k(t)) \in \Gamma_k,\ t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon). \tag{4.31}$$

That means $\alpha \geq 0$, and so the obliqueness holds for $p = (t, \rho_k(t)) \in \Gamma_k$ with $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$.

Let

$$\bar{t} = \inf\{t : \text{ the obliqueness holds at } p = (s, \rho_k(s)) \in \Gamma_k \text{ for all } s \in (t, t+\varepsilon)\}. \tag{4.32}$$

By the $C^{1,\delta}$ regularity [5], $\bar{t} \geq 0$ is well defined and the obliqueness holds for $t \in (\bar{t}, \bar{t} + \varepsilon)$ but not at $\bar{p} = (\bar{t}, \rho_k(\bar{t}))$. This finishes the proof. □

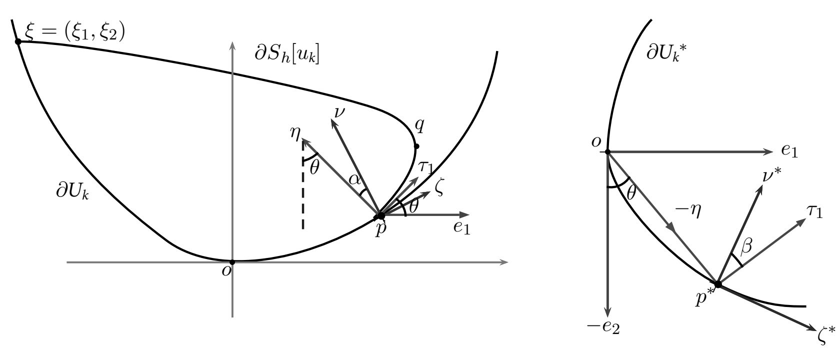

Figure 3. ζ, ζ^* are tangential vectors, and ν, ν^* are inner unit normals of ∂U_k, ∂U_k^* at p, $p^* = Du_k(p)$, respectively. η and τ_1 are the normal and tangential vectors of $S_h[u_k]$ at p. Hence we have $\cos\alpha = \langle \eta, \nu \rangle = \langle \tau_1, \zeta \rangle$.

4.4. *Contradiction.* Now we derive a contradiction with the help of the approximation sequence u_k. By our construction, $0 \in \Gamma_k$ and $0 \in \Gamma_k^*$, where Γ_k and Γ_k^* are the curves given in (4.23) and (4.25). First we point out that

$$\partial_{x_2} u_k(p) < 0 \quad \text{for} \quad p \in \Gamma_k \cap \{x_1 > 0\}. \tag{4.33}$$

Indeed, for any given point $p \in \Gamma_k \cap \{x_1 > 0\}$, the inner normal ν of U_k at p lies in the second quadrant. Let ν^* be the inner normal of U_k^* at $p^* = Du_k(p) \in \Gamma_k^*$. By (4.29), we have $\nu \cdot \nu^* > 0$. By (4.24), it implies $p^* \in \Gamma_k^* \cap \{y_2 < 0\}$. Hence we have (4.33).

Lemma 4.4. *We have*

$$\partial_{x_1x_2} u_k(t, \rho_k(t)) < 0 \qquad \textit{for} \quad t \in (0, r_0). \tag{4.34}$$

Proof. By the boundary condition $Du_k(\partial U_k) = \partial U_k^*$, we have

$$\partial_{x_1} u_k(t, \rho_k(t)) = \rho_k^*(\partial_{x_2} u_k(t, \rho_k(t))).$$

Differentiating the above equation, we have

$$\partial_{x_1x_1} u_k + \partial_{x_1x_2} u_k \, \rho_k' = (\rho_k^*)'(\partial_{x_1x_2} u_k + \partial_{x_2x_2} u_k \, \rho_k').$$

Namely,

$$\partial_{x_1x_1} u_k - \partial_{x_2x_2} u_k \, \rho_k' \, (\rho_k^*)' = ((\rho_k^*)' - \rho_k')\partial_{x_1x_2} u_k. \tag{4.35}$$

By the above approximation, $D^2 u_k$ is positive definite and continuous on the boundary. For $t > 0$ small, by (4.33) we have $\rho_k' > 0$ and $(\rho_k^*)' < 0$. Hence, the left-hand side of (4.35) is always positive, and the coefficient on the right-hand side is $(\rho_k^*)' - \rho_k' < 0$. Therefore, we obtain (4.34). □

We introduce the function

$$w_k(x) := \partial_{x_1} u_k + u_k - x_1 \partial_{x_1} u_k. \tag{4.36}$$

By Lemma 4.2, $\det D^2 u_k$ is a positive constant. Hence w_k satisfies the equation

$$M^{ij} D_{ij} w_k = 0 \tag{4.37}$$

in $B_{r_0} \cap U_k$, where $\{M^{ij}\}$ is the cofactor matrix of $D^2 u_k$.

Corollary 4.5. *There exists a constant $\epsilon_0 > 0$ independent of k such that for $t \in (0, \epsilon_0)$, the function $w_k(t, \cdot)$ has an interior local minimum.*

Proof. From Lemma 4.4 we have $\partial_{x_1x_2} u_k(t, \rho_k(t)) < 0$ for $t \in (0, r_0)$. Hence, by (4.33),

$$\partial_{x_2} w_k(t, \rho_k(t)) = (1 - t)\partial_{x_1x_2} u_k(t, \rho_k(t)) + \partial_{x_2} u_k(t, \rho_k(t)) < 0 \tag{4.38}$$

for all $t \in (0, r_0)$.

On the other hand, by our assumption $U_k^* \subset \{y_1 > 0\}$, $\partial_{x_1} u_k \geq 0$. Hence by the strict convexity of u_0 and $u_k \to u_0$ uniformly in B_{r_0}, there exists a constant $\delta_0 > 0$ such that

$$w_k(x) = u_k(x) + (1 - x_1)\partial_{x_1} u_k > u_k(x) \geq \delta_0$$

when $x \in \partial B_{r_0} \cap U_k$. Hence there is a small $\epsilon_0 > 0$ such that for any $t \in (0, \epsilon_0)$, $w_k(t, \cdot)$ has a local minimum that is smaller than the boundary value $w_k(t, \rho_k(t))$. □

Hence we can define the following function:

$$\underline{w_k}(t) = \inf\{w_k(t, x_2) : \ x_2 > \rho_k(t), \ (t, x_2) \in U_k\}, \quad t \in (0, \epsilon_0). \tag{4.39}$$

By Corollary 4.5, the infimum cannot be attained on $\partial U_k \cap \{0 < x_1 < \epsilon_0\}$, and $\underline{w_k}$ is well defined for all $t \in (0, \epsilon_0)$.

LEMMA 4.5. *For $t \in (0, \epsilon_0)$, $\underline{w_k}$ is concave.*

Proof. If $\underline{w_k}$ is not concave, then there exist constants $0 < r_1 < r_2 < \epsilon_0$ and an affine function $L(t)$ such that $\underline{w_k}(r_i) = L(r_i)$ for $i = 1, 2$, and the set $\{t \in (r_1, r_2) : w_k(t) < L(t)\} \neq \emptyset$. Extend L to an affine function $\hat{L}$ defined in $\mathbb{R}^2$ such that $\hat{L}(t, s) = L(t)$. Denote

$$D_\varepsilon = \{x \in U_k : x_1 \in (r_1, r_2) \text{ and } w_k(x) < \hat{L}(x) - \varepsilon\}.$$

By our definition of $\underline{w_k}$ and Corollary 4.5, we can choose $\varepsilon > 0$ such that

$$\emptyset \neq D_\varepsilon \Subset U_k. \tag{4.40}$$

Indeed, by our choice of $\hat{L}$, $D_{\varepsilon\,|\,\varepsilon=0} \neq \emptyset$. Let $\varepsilon_1 = \sup\{\varepsilon : D_\varepsilon \neq \emptyset\}$. Then if $\varepsilon < \varepsilon_1$ and sufficiently close to ε_1, we have $D_\varepsilon \neq \emptyset$. By Corollary 4.5, the infimum in (4.39) is attained at an interior point. Hence we also have $D_\varepsilon \Subset U_k$.

By equation (4.37) and the boundary condition $w_k = \hat{L} - \varepsilon$ on ∂D_ε, we apply the maximum principle to w_k in D_ε and conclude that $w_k = \hat{L} - \varepsilon$ in D_ε. However, by our definition of D_ε, we have $w_k < \hat{L} - \varepsilon$ in D_ε. We reach a contradiction. □

Note that for any fixed $t \in (0, \epsilon_0)$, the minimum point in Corollary 4.5 may not be unique. In this case, the domain D_ε in the above proof may contain more than one connected component. But each component is compactly contained in U_k. Hence we can still use the maximum principle.

We have now established Lemmas 4.4 and 4.5 for the approximation sequence u_k. Denote

$$w_0 = \partial_{x_1} u_0 + u_0 - x_1 \partial_{x_1} u_0 = \lim_{k \to \infty} w_k.$$

Let

$$\underline{w_0}(t) = \inf\{w_0(t, x_2) : \ x_2 > \rho_0(t), \ (t, x_2) \in \Omega_0\}, \quad t \in (0, \epsilon_0).$$

From (4.20), we have $\underline{w_0}(t) \to 0$ as $t \to 0$. More precisely,

$$\underline{w_0}(t) \le C\sigma(2t) \quad \text{for } t > 0 \text{ small}.$$

To see this, let $q = (2t, q_2)$ such that $\underline{u_0}(2t) = u_0(q)$, where $\underline{u_0}$ is defined in Corollary 4.3. By (4.20), we have $u_0(q) \le Ct\sigma(2t)$. Let $\hat{q} = (t, q_2)$. Since $\partial_{x_1} u_0 \ge 0$ (i.e., $\Omega_0^* \subset \{y_1 > 0\}$), we have

$$u_0(\hat{q}) \le u_0(q) \le Ct\sigma(2t).$$

By the convexity of u_0, one also has

$$\partial_{x_1} u_0(\hat{q}) \le \frac{u_0(q) - u_0(\hat{q})}{t} \le C\sigma(2t).$$

Hence,

$$\begin{aligned} \underline{w_0}(t) &= \inf w_0(t, \cdot) \le w_0(\hat{q}) \\ &= (1-t)\partial_{x_1} u_0(\hat{q}) + u_0(\hat{q}) \le C\sigma(2t). \end{aligned} \tag{4.41}$$

Recall that $\sigma(t) = |t|^{1+\gamma}$. Hence by the concavity of $\underline{w_0}$ (Lemma 4.5, taking the limit $k \to \infty$), we conclude that $\underline{w_0}(t) \le 0$ for all $t \in (0, \epsilon_0)$. On the other hand, since $\partial_{x_1} u_0 \ge 0$, by the strictly convexity of u_0, we have

$$w_0(x) = u_0(x) + (1 - x_1)\partial_{x_1} u_0(x) \ge u_0(x) > 0 \ \text{ if } x \in \Omega_0,\ x_1 \in (0, \epsilon_0).$$

It implies that $\underline{w_0}(t) > 0$ when $t > 0$. We reach a contradiction. Therefore, the uniform obliqueness in dimension two is proved. □

5. Uniform obliqueness in high dimensions

In this section we prove the uniform obliqueness in high dimensions. Suppose the domains Ω and Ω^* are bounded, convex, with $C^{1,1}$ boundaries, and $f \in C^0(\overline{\Omega})$. The proof of obliqueness uses the ideas in Section 4, and also the following:

(i) Suppose the obliqueness fails at the point $0 \in \partial\Omega$ and $0 = Du(0) \in \partial\Omega^*$. To understand the geometry of the sub-level set $S_h[u](0)$, we prove that $S_h[u](0)$ is contained in a cuboid Q with volume $|Q| \le C|S_h[u](0)|$. Then by a rescaling of the coordinates such that Q becomes a cube, $S_h[u](0)$ changes accordingly to a convex set with good shape. In particular, we show that the boundary $\partial\Omega$ becomes flat in directions orthogonal to the inner normals ν and ν^* as $h \to 0$. This property enables us to employ the techniques in Section 4.

(ii) As in dimension two, we need to construct a smooth approximation sequence to derive the contradiction. The construction in high dimensions is more complicated.

Similarly as in Remark 4.1(iii), the constants in this section depend on $n, \lambda, \Omega, \Omega^*$ (diameters and $C^{1,1}$ norm of Ω, Ω^*). They also depend on k in the argument on the approximation sequence $\{u_k\}$, but are independent of h and u for small $h > 0$. The continuity of f is used only in the blow-up process. The tangential $C^{1,\alpha}$ regularity of Section 3 is used only in Section 5.1 for u and in the approximation sequence $\{u_k\}$, where the domains are $C^{1,1}$ smooth. We do not need the tangential $C^{1,\alpha}$ regularity for the limit u_0.

5.1. *The limit profile.* To prove the uniform obliqueness, by the $C^{1,\delta}$ regularity [5], we may suppose to the contrary that $0 \in \partial\Omega$, $u(0) = 0$, $Du(0) = 0 \in \partial\Omega^*$, and locally

$$\begin{aligned} &\{x_n > C|x'|^2\} \subset \Omega \subset \{x_n > 0\}, \quad \text{where } x' = (x_1, \dots, x_{n-1}), \\ &\{y_1 > C|\tilde{y}|^2\} \subset \Omega^* \subset \{y_1 > 0\}, \quad \text{where } \tilde{y} = (y_2, \dots, y_n). \end{aligned} \tag{5.1}$$

Corresponding to properties (i) and (ii) in Section 4.1, similarly we have

(i) $u_1 > 0$ in Ω and $v_n > 0$ in Ω^*;

(ii) if $x \in S_h[u]$, then $x - te_1 \in S_h[u]$ for all $t > 0$, provided $x - te_1 \in \Omega$.

First we prove a lemma that strengthens Lemma 4.1.

LEMMA 5.1. *For any given point $p \in \partial S_h[u] \cap \Omega$, let $\mathcal{H}$ be the tangential plane of $S_h[u]$ at p. Assume $\mathcal{H} = \{x \in \mathbb{R}^n : \ x \cdot \gamma = a\}$ for a unit vector γ, where a is a positive constant. Then*

$$x \cdot \gamma \geq -Ca \quad \forall\ x \in S_h[u], \tag{5.2}$$

where $C > 0$ is a constant independent of h $(h > 0$ small$)$ and u.

Proof. Denote $b = \inf\{x \cdot \gamma : \ x \in S_h[u]\}$, and denote $\mathcal{H}_1 = \{x \in \mathbb{R}^n : x \cdot \gamma = b\}$. Suppose to the contrary that the ratio $\frac{a}{|b|} \to 0$ as $h \to 0$. Let $\mathcal{A}_h$ be an affine transformation such that $\mathcal{A}_h(S_h[u]) \sim B_1(z)$ for some point z. Note that the transform does not change the ratio $\frac{a}{|b|}$.

Accordingly, let $u_h(x) = u(\mathcal{A}_h^{-1}x)/h$. By Caffarelli's geometric decay estimate, similarly as in (4.11), u_h is locally uniformly bounded and sub-converges to a limit u_0 as $h \to 0$, and $u_0(0) = \lim_{h\to 0} u_h(0) = 0$.

On the other hand, by passing to a subsequence, we have

$$\mathcal{A}_h(\mathcal{H}) \to \mathcal{H}^*, \qquad \mathcal{A}_h(\mathcal{H}_1) \to \mathcal{H}_1^*$$

as $h \to 0$. Observe that

$$\frac{\mathrm{dist}(0, \mathcal{H}^*)}{\mathrm{dist}(0, \mathcal{H}_1^*)} = \lim_{h\to 0} \frac{a}{|b|} = 0.$$

Hence $\mathcal{H}^*$ passes through the origin. It implies that $u_0 \geq 1$ on one side of $\mathcal{H}^*$, which is a contradiction since u_0 is continuous and $u_0(0) = 0$. □

Similarly as in (4.3), let $q, \xi \in \partial S_h(0)$ such that

$$q_1 = \langle q, e_1\rangle = \sup\{\langle x, e_1\rangle : x \in S_h(0)\}, \tag{5.3}$$

$$\xi_1 = \langle \xi, e_1\rangle = \inf\{\langle x, e_1\rangle : x \in S_h(0)\}. \tag{5.4}$$

Obviously, $q_1 > 0$ and $\xi_1 < 0$. We point out that q is an interior point of Ω. To see this, let $\ell^* =: \{te_1 : \ t \in (0, t_0)\}$ be a line segment in Ω^*. Let $\ell = (Du)^{-1}(\ell^*) \subset \Omega$ be the pre-image of ℓ^*. Then q is the intersection of ℓ with $\partial S_h[u](0)$. From Lemma 5.1,

Corollary 5.1. *We have*

$$q_1 \geq \delta_0|\xi_1| \tag{5.5}$$

for some constant $\delta_0 > 0$ independent of h and u.

Having obtained the balance property (5.5), one would expect a decay estimate like Corollary 4.1 as in dimension two. But to obtain the decay estimate in high dimensions, one needs a bound of $S_h[u]$ along the $x_2, \ldots, x_{n-1}$ directions.

Denote $S^c_{h;1}[u] = S^c_h[u] \cap \{x_1 = 0\}$, and denote $B'_r(0)$ as the ball of radius r in $\mathbb{R}^{n-2} = \text{span}(e_2, \ldots, e_{n-1})$. We have the following estimates for $S^c_h[u]$ in the $x_2, \ldots, x_{n-1}$ directions.

Lemma 5.2. *For any given $\varepsilon > 0$ small, we have*

$$B'_{C^{-1}h^{1/2+\varepsilon}}(0) \subset S^c_{h;1}[u] \cap \{x_n = 0\} \subset B'_{Ch^{1/2-\varepsilon}}(0). \tag{5.6}$$

In particular, for any $i = 2, \ldots, n-1$ and any $x \in S^c_h[u]$,

$$|x_i| = |x \cdot e_i| \leq Ch^{\frac{1}{2}-\varepsilon}, \tag{5.7}$$

provided $h > 0$ is sufficiently small, where the constant $C = C(\varepsilon)$ is independent of h and u.

Proof. By the tangential $C^{1,1-\varepsilon}$ regularity for u, we have $S^c_{h;1}[u] \cap \{x_n = 0\} \supset B'_{C^{-1}h^{1/2+\varepsilon}}(0)$, which is the first inclusion of (5.6).

For any points $x \in S^c_h[u]$ and $y \in S^c_h[v]$, by (2.14) we have $|x \cdot y| \leq Ch$. By the tangential $C^{1,1-\varepsilon}$ regularity for v, we have $C^{-1}h^{\frac{1}{2}+\varepsilon}e_i \in S^c_h[v]$, $(i = 2, \ldots, n-1)$. Hence $(h^{\frac{1}{2}+\varepsilon}e_i) \cdot x \leq Ch$ for all $x \in S^c_h[u]$; namely,

$$|x_i| = |x \cdot e_i| \leq Ch^{\frac{1}{2}-\varepsilon} \quad \forall\, x \in S^c_h[u]. \tag{5.8}$$

We obtain (5.7) and the second inclusion of (5.6). □

From (5.7) and Remark 2.3, we also obtain

$$|x \cdot e_i| \leq Ch^{\frac{1}{2}-\varepsilon} \quad \forall\, x \in S_h[u] \text{ and } i = 2, \ldots, n-1. \tag{5.9}$$

The same estimate is true for $S_h[v], S^c_h[v]$ as well. From (5.9) we will derive two consequences: one is a decomposition (5.12), the other one is the decay estimate in Corollary 5.3.

We shall first derive the decomposition (5.12). Note that in high dimensions, there may be a small portion of $S_h[u] \cap \{x_1 > 0\}$, whose projection on the plane $\{x_1 = 0\}$ is not contained in $S_{h;1} := S_h[u] \cap \{x_1 = 0\}$. Nevertheless, we have the following inclusion.

COROLLARY 5.2. *Let $S'_{h;1}[u]$ be the projection of $S_h[u] \cap \{x_1 > 0\}$ on the plane $\{x_1 = 0\}$. Then*

$$S'_{h;1}[u] \subset (1 + o(1))S_{h;1}[u] \tag{5.10}$$

as $h \to 0$, where the dilation is with respect to z, the center of $S_{h;1}[u]$.

Proof. Let $\tilde{x} = (0, x'', x_n) \in S'_{h;1}$, where $x'' = (x_2, \ldots, x_{n-1})$. By definition of $S'_{h;1}[u]$, there is $t > 0$ such that $x = (t, x'', x_n) \in S_h[u]$ and $u(x) < h$. If $\tilde{x} \in \Omega$, since $u_1 > 0$, one must have $u(\tilde{x}) < h$, and thus $\tilde{x} \in S_{h;1}[u]$.

If $\tilde{x} \notin \Omega$, let z be the center of $S_{h;1}[u]$. By the $C^{1,\delta}$ regularity of u, we have $z_n = z \cdot e_n \geq Ch^{\frac{1}{1+\delta}}$. From (5.9), $|x''| \leq Ch^{\frac{1}{2}-\epsilon}$. Since $\tilde{x} \notin \Omega$ and $\partial\Omega \in C^{1,1}$,

$$x_n \leq C|x''|^2 \leq Ch^{1-2\epsilon} = o(h^{\frac{1}{1+\delta}}) = o(z_n). \tag{5.11}$$

Let ℓ be the segment connecting z and $\tilde{x}$, and let y be the intersection of ℓ and $\partial\Omega$. Since $u(z) < h$, $u(\tilde{x}) < h$, we have $u(y) < h$ and thus $y \in S_{h;1}[u]$. Write $y = (0, y'', y_n)$. By (5.9) again, we have $|y''| \leq Ch^{\frac{1}{2}-\epsilon}$. Then since $y \in \partial\Omega$, one has $y_n \leq Ch^{1-2\epsilon} \ll z_n$. Therefore,

$$\lim_{h\to 0} \frac{|z\tilde{x}|}{|zy|} = \lim_{h\to 0} \frac{|z_n - x_n|}{|z_n - y_n|} = 1,$$

from which one easily obtains (5.10). □

Next we estimate the size of $S_h[u] \cap \{x_1 < 0\}$. We introduce a cone with vertex q (see (5.3)) and passing through $S_{h;1}[u]$, namely,

$$\mathcal{V} = \{q + t(x - q) : x \in S_{h;1}[u], t \geq 0\}.$$

By the convexity of $S_h[u]$, we have $S_h[u] \cap \{x_1 < 0\} \subset \mathcal{V}$. Hence by Corollaries 5.1 and 5.2,

$$S_h[u] \subset [\xi_1, q_1] \times \beta S_{h;1}[u] \tag{5.12}$$

for a constant $\beta > 0$ independent of h, where $\beta S_{h;1}[u]$ denotes the β-dilation with respect to the center of $S_{h;1}[u]$. Indeed, by performing an affine transform in the $e_2, \ldots, e_n$ directions, we may assume $S_{h;1}[u]$ is normalized. Then by Corollary 5.2, we have that $q' = (0, q_2, \ldots, q_n) \in (1 + o(1))S_{h;1}[u]$. Hence, by Corollary 5.1 and using the fact that $S_h[u] \cap \{x_1 < 0\} \subset \mathcal{V}$, we have (5.12).

Remark 5.1. Replacing the e_1-direction by the e_n-direction, the same argument for (5.12) also applies to $S_h[v]$ and yields

$$S_h[v] \subset [\xi_n^*, q_n^*] \times \beta^* S_{h;n}[v] \tag{5.13}$$

for a constant $\beta^* > 0$ independent of h, where $\xi^*, q^* \in \partial S_h[v]$ is defined analogously to (5.3) (where e_1 is replaced by e_n) and $S_{h;n}[v] := S_h[v] \cap \{y_n = 0\}$.

As another consequence of (5.9), we next derive a decay estimate analogous to Corollary 4.1.

LEMMA 5.3. *For any given $\varepsilon > 0$ small, we have $q_1 \geq h^{\frac{1}{3}+\varepsilon}$, provided h is sufficiently small.*

Proof. For any $x \in S_h[u]$, by (5.9) we have $|x_i| \leq Ch^{\frac{1}{2}-\epsilon}$ for $i = 2, \dots, n-1$. By Corollary 5.1 we also have $q_1 \geq C|x_1|$. Since $u_1 > 0$, we see that $\sup\{e_n \cdot x : x \in S_h[u]\}$ must be attained on the boundary $\partial\Omega$. Since $\partial\Omega \in C^{1,1}$, we have

$$x_n \leq C \sum_{i=1}^{n-1} x_i^2 \leq C(q_1^2 + h^{1-2\varepsilon}) \qquad \forall\, x \in S_h[u] \cap \partial\Omega.$$

From (2.12), the volume $|S_h[u]| \approx h^{\frac{n}{2}}$. Hence

$$h^{\frac{n}{2}} \approx |S_h[u]| \leq Cq_1(q_1^2 + h^{1-2\varepsilon})h^{\frac{n-2}{2}-(n-2)\varepsilon}.$$

Therefore, $q_1 \geq h^{\frac{1}{3}+\varepsilon}$ for any given $\varepsilon > 0$ small. □

From Lemma 5.3, similarly to (4.12), we have the following corollary.

COROLLARY 5.3. *For $t > 0$ small, denote*

$$\underline{u}(t) = \inf\{u(t, x_2, \dots, x_n) : (t, x_2, \dots, x_n) \in \Omega\},$$
$$\underline{\partial_1 u}(t) = \inf\{\partial_1 u(t, x_2, \dots, x_n) : (t, x_2, \dots, x_n) \in \Omega\}.$$

We have the asymptotic behavior

$$\begin{aligned} \underline{u}(t) &\leq Ct^{3-\varepsilon}, \\ \underline{\partial_1 u}(t) &\leq Ct^{2-\varepsilon} \end{aligned} \tag{5.14}$$

for $t > 0$ small, where $\varepsilon > 0$ is any given small constant.

Remark 5.2. By Lemmas 2.2 and 5.3, we have

$$s \gtrsim q_1 \gtrsim h^{\frac{1}{3}+\varepsilon}, \qquad \text{where } s := \sup\{x \cdot e_1 : x \in S_{bh}^c[u]\}.$$

Let $d = \sup\{x \cdot e_n : x \in S_{bh}^c[u],\ x_1 = 0\}$. Then by Lemma 5.2 and (2.11), we have

$$h^{\frac{n}{2}} \approx |S_{bh}^c[u]| \gtrsim h^{\frac{n-2}{2}(1+\epsilon)} sd,$$

which implies $d \lesssim h^{\frac{2}{3}-\epsilon}$. By Lemma 2.2 again, we obtain that

$$\sup\{x \cdot e_n : x \in S_{h;1}[u]\} \lesssim h^{\frac{2}{3}-\epsilon}. \tag{5.15}$$

In order to bound the sub-level set $S_h[u]$ by a cuboid, we need to further decompose $S_{h;1}[u]$ in (5.12) along the e_n direction. Denote

$$S^c_{h;1,0} = S^c_{h;1}[u] \cap \{x_n = 0\},$$

where $S^c_{h;1}[u] = S^c_h[u] \cap \{x_1 = 0\}$ was introduced above.

LEMMA 5.4. *Let P_h be the projection of $S_{h;1}[u]$ on $\{x_n = 0\}$. Then we have*

$$P_h \subset \beta S^c_{h;1,0} \tag{5.16}$$

for a constant β independent of h and u.

Proof. Let $e \in \mathrm{span}\{e_2, \dots, e_{n-1}\}$ be a unit vector, and denote $r_e := \sup\{t : te \in S^c_{h;1,0}[u]\}$. To prove (5.16), it suffices to show that

$$|x \cdot e| \le \beta r_e \quad \forall\, x \in S_h[u] \cap \mathrm{span}\{e_n, e\} \tag{5.17}$$

for all unit vectors $e \in \mathrm{span}\{e_2, \dots, e_{n-1}\}$. By Lemma 5.2, we have $r_e \ge C^{-1} h^{\frac{1}{2}+\epsilon}$.

Given a unit vector $e \in \mathrm{span}\{e_2, \dots, e_{n-1}\}$ and a point $p \in S_h[u] \cap \mathrm{span}\{e_n, e\}$, up to a rotation of coordinates, we assume $e = e_2$ and $p = (0, p_2, 0, \dots, 0, p_n)$ with $p_2 > 0$. By Remark 5.2, we have

$$p_n \le C h^{\frac{2}{3}-\epsilon} \tag{5.18}$$

for ϵ as small as we want. In order to prove (5.17), it suffices to show that $p_2 \le \beta r_{e_2}$. If $p_2 \ll h^{\frac{1}{2}+\epsilon}$, we readily have $p_2 = p \cdot e_2 \le r_{e_2}$. Hence it suffices to consider the case

$$p_2 \ge C h^{\frac{1}{2}+\epsilon} \gg p_n \quad \text{(and thus } \; p_2 \approx |p| \; \text{ for } h \text{ small).} \tag{5.19}$$

By Remark 2.2, we have

$$|y \cdot p| \le Ch \quad \forall\, y \in S_h[v]. \tag{5.20}$$

In particular, when $y \in S_{h;n}[v] := S_h[v] \cap \{y_n = 0\}$, $y \cdot p = y_2 p_2$. Thus we obtain

$$\sup\{|y_2| : y \in S_{h;n}[v]\} \le C \frac{h}{p_2}.$$

By Remark 5.1, $\sup\{|y_2| : y \in S_h[v]\} \le \beta^* \sup\{|y_2| : y \in S_{h;n}[v]\}$. Therefore, we obtain

$$\sup\{|y_2| : y \in S_h[v]\} \le C\beta^* \frac{h}{p_2}. \tag{5.21}$$

By the definition of r_{e_2}, we have $r_{e_2} e_2 \in \partial S^c_h[u]$. Hence by (2.15), there exists $z^* \in \partial S^c_h[v]$ such that

$$z^* \cdot (r_{e_2} e_2) \ge C^{-1} h. \tag{5.22}$$

By (5.21) and Remark 2.3, we have

$$z^* \cdot e_2 \le \sup\{y \cdot e_2 : y \in S^c_h[v]\} \le C\beta^* \frac{h}{p_2}. \tag{5.23}$$

Hence from (5.22) and (5.23), we obtain the desired inequality

$$r_{e_2} \ge \frac{C^{-1}h}{z^* \cdot e_2} \ge \frac{1}{C\beta^*} p_2 \tag{5.24}$$

for a different constant $C > 0$. This finishes the proof with $\beta = C\beta^*$. □

Thanks to (5.12) and Lemma 5.4, we can now show that $S_h[u]$ is contained in a cuboid as follows. Denote

$$d_n = \sup\{e_n \cdot x : \ x \in S^c_{h;1}[u]\} \tag{5.25}$$

to be the height of $S^c_h[u]$ on the section $\{x_1 = 0\}$. We have

$$\begin{aligned} d_n &\gtrsim \sup\{e_n \cdot x : \ x \in S_{b^{-1}h;1}[u]\} \\ &\gtrsim \sup\{e_n \cdot x : \ x \in S_{b^{-1}h}[u]\} \\ &\gtrsim \sup\{e_n \cdot x : \ x \in S_h[u]\}, \end{aligned} \tag{5.26}$$

where the first inequality is due to Lemma 2.2, the second inequalities follows from (5.12), and the last inequality is due to the convexity of u, which implies that $S_h[u] \subset bS_{b^{-1}h}[u]$, $(b > 1)$.

Let $\tilde{q} \in \partial S^c_h[u]$ be the point such that

$$\tilde{q}_1 = \tilde{q} \cdot e_1 = \sup\{x \cdot e_1 : x \in S^c_h[u]\}. \tag{5.27}$$

By Remark 2.3 we have that

$$\sup\{|x \cdot e_1| : x \in S_h[u]\} \lesssim \tilde{q}_1. \tag{5.28}$$

Let

$$\mathcal{R}_h = [-\tilde{q}_1, \tilde{q}_1] \times E'_h \times [-d_n, d_n] \tag{5.29}$$

be a cuboid, where $E'_h \subset \mathbb{R}^{n-2}$ is an ellipsoid centered at 0 such that $E'_h \sim S^c_{h;1,0} = S^c_{h;1}[u] \cap \{x_n = 0\}$. By Lemma 5.4, (5.12) and (5.28), we have

$$S_h[u] \subset C\mathcal{R}_h \tag{5.30}$$

for some constant C independent of h. Moreover, by (5.25) and (5.27), we have the volume

$$|S^c_h[u]| \gtrsim |E'_h| d_n \tilde{q}_1 \gtrsim |\mathcal{R}_h|.$$

Hence,

$$C^{-1}|\mathcal{R}_h| \le |S^c_h[u]| \approx |S_h[u]| \le C|\mathcal{R}_h|. \tag{5.31}$$

Now we make a linear transform $\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$ such that the sub-level set $S_h[u]$ has a good shape, where $\mathcal{T}_1$ is a linear transform normalising E'_h to the unit ball in $\mathbb{R}^{n-2} = \text{span}(e_2, \dots, e_{n-1})$ while leaving x_1 and x_n unchanged, and $\mathcal{T}_2$ is given by

$$\mathcal{T}_2 : \quad \begin{cases} \tilde{x}_1 = x_1/\tilde{q}_1, \quad \tilde{x}_n = x_n/d_n, \\ \tilde{x}_i = x_i \quad \text{ for } 2 \le i \le n-1. \end{cases} \tag{5.32}$$

It is easy to see that $\mathcal{T}_2 \circ \mathcal{T}_1 = \mathcal{T}_1 \circ \mathcal{T}_2$.

After the transform $\mathcal{T}$, the set $\mathcal{T}(S_h[u])$ is contained in the cube $\mathcal{D} = [-C, C]^n$, and the volume $|\mathcal{T}(S_h[u])| \ge \delta_0$ for a positive constant δ_0 independent of h and u. Hence $\mathcal{T}(S_h[u])$ has a good shape. By Lemmas 2.2 and 2.3, we see that $\mathcal{T}(S^c_h[u])$ also has a good shape. By rescaling back and using Lemma 2.2 again, we have

$$C^{-1}\mathcal{R}_n \cap \Omega \subset S_h[u] \subset C\mathcal{R}_n \tag{5.33}$$

for a constant C independent of h.

Having made the transform $\mathcal{T}$ (note that $\mathcal{T} = \mathcal{T}_h$ depends on h), accordingly we also make the change $u_h(x) = u(\mathcal{T}^{-1}x)/h$.

Let $\underline{u}(t)$ be the function introduced in Corollary 5.3. Similarly to (4.17), we choose a sequence $\{t_j\} \to 0$ such that

$$\frac{\underline{u}(t)}{\underline{u}(t_j)} \le 2\Big(\frac{t}{t_j}\Big)^{3-\varepsilon} \qquad \forall\, t \in (0, t_j), \tag{5.34}$$

where $\varepsilon > 0$ is the small constant in (5.14). Denote $\mathcal{T}_j = \mathcal{T}_{h_j}$, $u_j = u_{h_j}$, where $h_j = \underline{u}(t_j)$.

Similarly as in Section 4, by passing to a subsequence, $\Omega_{h_j} := \mathcal{T}_j(\Omega)$ converges to a limit Ω_0 as $j \to \infty$, and Ω_0 is an unbounded convex domain in $\mathbb{R}^n$. Also, u_j converges to a limit u_0 as $j \to \infty$, which satisfies the Monge-Ampère equation (4.18) in $\mathbb{R}^n$.

By the proof of Corollary 4.3, u_0 satisfies the asymptotic behaviors (5.14). Moreover, u_0 is strictly convex and $C^{1,\alpha}$ regular in $B_k \cap \overline{\Omega_0}$ for any $k > 0$, and $u_0 \in C^1(\mathbb{R}^n)$ (see Remark 4.3).

Thanks to the above cuboid decomposition (5.29), we can prove that the boundary $\partial\Omega_0$ is flat in the $x_2, \dots, x_{n-1}$ directions.

LEMMA 5.5. *Assume Ω_h sub-converges as $h \to 0$ to a convex domain Ω_0, locally in the sense of Hausdorff. Then $\Omega_0 = \mathbb{R}^{n-2} \times \omega_0$, where ω_0 is a convex set in the* 2*-dim space span*$\{e_1, e_n\}$.

Proof. By the global $C^{1,\delta}$ regularity [5], we have $d_n \ge Ch^{\frac{1}{1+\delta}}$ for some $\delta > 0$, where $d_n = d_n(h)$ is given in (5.25) and (5.26). By (5.6), we have

$$S^c_{h;1,0}[u] = S^c_{h;1}[u] \cap \{x_n = 0\} \subset B'_{Ch^{1/2-\varepsilon}}(0).$$

Hence by the $C^{1,1}$ regularity of the boundary $\partial\Omega$, the height of $S^c_{h;1}[u] \cap \partial\Omega$ satisfies

$$d_{n,b}(h) =: \sup\{e_n \cdot x : \ x \in S^c_{h;1}[u] \cap \partial\Omega\} \leq Ch^{1-2\varepsilon},$$

where $\varepsilon > 0$ is fixed but can be as small as we want. Hence,

$$\frac{d_{n,b}(h^{1-\delta/2})}{d_n(h)} \to 0 \quad \text{as } h \to 0.$$

Note that by (5.6),

$$\operatorname{diam}(S^c_{h;1,0}[u]) = o(1)\operatorname{diam}(S^c_{h^{1-\delta/2};1,0}[u]) \qquad \text{when } h \to 0.$$

The above formula implies that $\mathcal{T}_h(\partial\Omega \cap S^c_{h^{1-\delta/2};1}[u])$ is becoming flat and so its limit is the plane $\operatorname{span}(e_2, \dots, e_{n-1})$. Namely, $\mathcal{T}_h(\partial\Omega \cap B_R(0) \cap \{x_1 = 0\})$ becomes flat as $h \to 0$ for all $R > 0$.

It is well known that if a convex set G contains a straight line ℓ, then G can be expressed as a product $G = G' \times \ell$. The lemma is proved. □

Denote $\mathcal{T}^* = \frac{1}{h}(\mathcal{T}')^{-1}$, the dual affine transformation for v, where $\mathcal{T}'$ is the transpose of $\mathcal{T}$. Similarly, we denote $v_h(y) = \frac{1}{h}v((\mathcal{T}^*)^{-1}y)$ and $\Omega^*_h = \mathcal{T}^*(\Omega^*)$. Applying Lemma 5.5 to v, we see that Ω^*_h converges to an unbounded convex domain $\Omega^*_0 = \mathbb{R}^{n-2} \times \omega^*_0$, where $\omega^*_0 \subset \operatorname{span}\{e_1, e_n\}$ is a convex set. Also, v_{h_j} converges to a convex function v_0 locally uniformly, which satisfies the equation (4.22) correspondingly.

5.2. *Smooth approximation.* First we construct a smooth approximation sequence $\{u_k\}$ converging to u_0 in a small neighborhood of the origin similarly as in Section 4.3.

Let $V = B_r \cap \Omega^*_0$ for a small constant r, and let $U = Dv_0(V)$. Then $B_{r_0} \cap \Omega_0 \subset U$ for a small constant $r_0 > 0$. By Lemma 5.5 we approximate U, V by a sequence of bounded smooth sets U_k, U^*_k respectively such that

(a) $0 \in \partial U_k$, $0 \in \partial U^*_k$ for all $k \geq 1$;
(b) $U_k \subset \{x_n > 0\}$, $U^*_k \subset \{y_1 > 0\}$ for all $k \geq 1$;
(c) $U_k \to U$, $U^*_k \to V$, as $k \to \infty$, in Hausdorff's sense;
(d) there exist smooth, uniformly convex sets $\hat\omega_k, \hat\omega^*_k \subset \operatorname{span}\{e_1, e_n\}$ such that

$$U_k \cap B_{r_0} = (\hat\omega_k \times \mathbb{R}^{n-2}) \cap B_{r_0},$$
$$U^*_k \cap B_{r_0} = (\hat\omega^*_k \times \mathbb{R}^{n-2}) \cap B_{r_0}$$

for a different, smaller constant $r_0 > 0$.

In this subsection, the constant r_0 may change from line to line but they have a uniform positive lower bound independent of k.

Moreover, we assume the lower part of the boundary $\partial U_k \cap B_{r_0}$ is the graph of a smooth, uniformly convex function ρ_k in the direction e_n, that is,

$$\Gamma_k =: \{x \in \mathbb{R}^n : x_n = \rho_k(x_1)\} \cap B_{r_0}, \tag{5.35}$$

where ρ_k is defined on $\mathcal{J}_k$, which is the projection of $\partial U_k \cap B_{r_0}$ on the x_1-axis with $[0, \frac{1}{2}r_0) \subset \mathcal{J}_k$. The function ρ_k satisfies (4.24) as in dimension two.

Similarly, the left part of the boundary $\partial U_k^* \cap B_{r_0}$ is the graph of a smooth, uniformly convex function ρ_k^* in the direction e_1, that is,

$$\Gamma_k^* =: \{y \in \mathbb{R}^n : y_1 = \rho_k^*(y_n)\} \cap B_{r_0}, \tag{5.36}$$

where ρ_k^* is defined on $\mathcal{J}_k^*$, which is the projection of $\partial U_k^* \cap B_{r_0}$ on the y_n-axis containing the origin inside. The function ρ_k^* satisfies (4.26) as in dimension two.

Let u_k be the potential function for the optimal transport from $(U_k, 1)$ to (U_k^*, g_k), where the density $g_k = \frac{|U_k|}{|U_k^*|}$ is a constant. Subtracting a constant we have $u_k(0) = 0$. Since U_k^* is convex, as before we can extend u_k to $\mathbb{R}^n$ by

$$u_k(x) := \sup\{\ell(x) : \ \ell \text{ is affine}, \ \ell \le u_k \text{ in } U_k, \ \nabla \ell \in U_k^*\} \quad \text{for } x \in \mathbb{R}^n.$$

Since $u_0 \in C^1(\mathbb{R}^n)$ and $u_0(0) = 0$, by the uniqueness of potential functions, $u_k \to u_0$ uniformly in $B_{r_0}(0)$ for a different $r_0 > 0$ small. In addition we have $\|u_k - u_0\|_{C^1(B_{r_0/2})} \to 0$ as $k \to \infty$.

LEMMA 5.6.

(i) *For each $k \ge 1$, we have*

$$\nu_k(x) \cdot \nu_k^*(Du_k(x)) > 0 \quad \forall\, x \in \Gamma_k, \tag{5.37}$$

where ν_k and ν_k^* are the unit inner normals of the domains $\{x \in \mathbb{R}^n : x_n > \rho_k(x_1)\} \cap B_{r_0}$ and $\{y \in \mathbb{R}^n : y_1 > \rho_k^*(y_n)\} \cap B_{r_0}$, respectively.

(ii) *For each $k \ge 1$, u_k is smooth, locally uniformly convex, and* $\det D^2 u_k$ *is a positive constant in $B_{r_0}(0) \cap \{x_n > \rho_k(x_1)\}$ (up to the boundary Γ_k).*

Proof. Similarly as in Lemma 4.2, (ii) follows from (i). That is, if the obliqueness (5.37) holds, by Section 6 we have the smoothness of u_k in (ii). The proof of (i) will be given in the following two lemmas. □

LEMMA 5.7. *For any fixed $k \ge 1$, assume that $u_k(0) = 0$ and $Du_k(0) = 0$. Then for any $x = (t, x'', \rho_k(t)) \in \Gamma_k$ with $t \le |x''|^{2/3}$, we have*

$$u_k(x) \approx |x''|^2. \tag{5.38}$$

Proof. Since the boundaries Γ_k, Γ_k^* are flat in $x'' = (x_2, \dots, x_{n-1})$, and ρ_k, ρ_k^* are smooth and uniformly convex, we can choose $\varepsilon = 0$ in Lemmas 5.2 and 5.3 (similarly as in Corollary 3.1). From (5.33), we have

$$C^{-1}Q \cap U_k \subset S_h[u_k] \subset CQ \quad \text{with } \ Q := [-\tilde{q}_1, \tilde{q}_1] \times B_{h^{1/2}}(0) \times [-d_n, d_n], \tag{5.39}$$

where $\tilde{q}, d_n$ are defined in (5.25), (5.27) respectively. Similarly to (5.3), let $q, \xi \in \partial S_h[u_k]$ be the points on $\partial S_h[u_k]$ such that

$$q_1 = \langle q, e_1 \rangle = \sup\{\langle x, e_1 \rangle : x \in S_h[u_k]\},$$
$$\xi_n = \langle \xi, e_n \rangle = \sup\{\langle x, e_n \rangle : x \in S_h[u_k]\}.$$

Since $Du_k(U_k) \subset U_k^* \subset \{y_1 \geq 0\}$, u_k is increasing in the e_1 direction. Hence ξ can be chosen on Γ_k. Then by (5.39), $C^{-1}\tilde{q}_1 \leq q_1 \leq C\tilde{q}_1$ (see also Remark 2.3) and $C^{-1}d_n < \xi_n \leq Cd_n$. Since $\rho_k \in C^2$, we have $\xi_n \leq C_1\xi_1^2$. By the uniformly convexity of ρ_k, we have $q_n \geq C_2 q_1^2$. By Corollary 5.1, we then obtain

$$\tilde{C}_1\tilde{q}_1^2 \geq \xi_n \geq q_n \geq \tilde{C}_2\tilde{q}_1^2, \quad \text{thus } d_n \approx \tilde{q}_1^2.$$

By the fact that $|S_h[u_k]| \approx h^{n/2}$, we then have $\tilde{q}_1 \approx h^{1/3}$. Hence, when $x = (t, x'', \rho_k(t)) \in \Gamma_k$ with $t \leq |x''|^{2/3}$, we obtain (5.38). □

Remark 5.3. In dimension two, we can use Caffarelli's regularity to conclude that u_k is $C^{2,\alpha}$ smooth up to Γ_k (Remark 4.4). In high dimensions, for the proof of Lemma 1.1 in Section 5.3, we have to choose the domains U_k, U_k^* that are flat in the $e_2, \ldots, e_{n-1}$ directions. Hence we cannot use Caffarelli's boundary $C^{2,\alpha}$ regularity [7] directly. But with the help of Lemma 5.7, one can modify Caffarelli's argument to prove that u_k is smooth up to Γ_k. In fact, Lemma 5.7 implies that the solution u_k (for any fixed k) behaves nicely in x'', and so the directions x'' would not cause us new troubles. Here we will not use the argument in [7] but provide an independent proof of (5.37), based on Lemma 4.3.

LEMMA 5.8. *For any fixed $k \geq 1$,* (5.37) *holds.*

Proof. Suppose to the contrary that (5.37) fails at a point $\hat{x} \in \Gamma_k$, that is,

$$\nu_k(\hat{x}) \cdot \nu_k^*(Du_k(\hat{x})) = 0.$$

By a change of coordinates and subtracting a linear function, we can assume $\hat{x} = 0$, $u_k(0) = 0$ and $Du_k(0) = 0$ such that the hypotheses of Lemma 5.7 are satisfied.

Consider the restriction of ∂U_k in span$\{e_1, e_n\}$. For a boundary point $p = (t, 0, \ldots, 0, \rho_k(t)) \in \Gamma_k$, let $h = u_k(p)$. Denote by $\tilde{\eta}$ the unit inner normal of $S_h[u_k]$ at p, and η the projection of $\tilde{\eta}$ on span$\{e_1, e_n\}$. Denote by ν the unit inner normal of ∂U_k at p, and α the angle between η and ν (see Figure 3). Note that by (d) in our domain constructions, $\partial U_k, \partial U_k^*$ are flat along the $e_2, \ldots, e_{n-1}$ directions near the origin. Hence the normal vectors $\nu(p), \nu^*(p^*)$ and the tangential vectors $\zeta(p), \zeta^*(p^*)$ are all in the 2-dim plane span$\{e_1, e_n\}$, where $p^* = Du_k(p)$. By the strict convexity of u_k and the proof of Lemma 4.3, there exists a small $t_0 > 0$ such that $\alpha \geq 0$ at $p_0 = (t_0, 0, \rho_k(t_0))$, which implies the obliqueness holds at p_0. Hence by the $C^{1,\delta}$ regularity, there is a small constant $\epsilon_0 > 0$ such that

$$\nu_k(p) \cdot \nu_k^*(Du_k(p)) > 0, \quad \forall\, p = (t, p'', \rho_k(t)) \ \text{ with } t \in (t_0 - \epsilon_0, t_0] \text{ and } |p''| \le \epsilon_0.$$

For any $t \in (0, t_0)$, denote

$$\mathcal{C}_t = \left\{(x_1, x'', 0) : t < x_1 < t_0,\ |x''| < \epsilon_0(x_1 - t)\right\},$$

which is an $(n-1)$-dimensional round open cone in the hyperplane $\{x_n = 0\}$ with vertex at $(t, 0, 0)$ and base on the disk $\{(t_0, x'', 0) : |x''| \le \epsilon_0(t_0 - t)\}$.

Let

$$\tilde{t} = \inf \{t :\ \text{the obliqueness holds } \forall\, p \in \Gamma_k,\ \text{provided } (p - p_n e_n) \in \mathcal{C}_t\},$$

where $p - p_n e_n$ is the projection of p on the plane $\{x_n = 0\}$. Obviously $\tilde{t} \ge 0$, and there is a point $(\tilde{x}_1, \tilde{x}'', 0) \in \partial \mathcal{C}_{\tilde{t}}$, with $\tilde{x}_1 < t_0$, such that the obliqueness fails at $\tilde{x} = (\tilde{x}_1, \tilde{x}'', \rho_k(\tilde{x}_1))$ but it holds in $\{(x_1, x'', \rho_k(x_1)) :\ (x_1, x'', 0) \in \mathcal{C}_{\tilde{t}}\}$.

Therefore, by a change of coordinates, we can assume that the obliqueness fails at the origin but it holds for all $x \in \Gamma_k$ whose projection $(x - x_n e_n)$ lies in $\mathcal{C}_0$, where $\mathcal{C}_0 = \mathcal{C}_{t\,|\,t=0}$ was the cone defined above. By subtracting a linear function, we again have $u_k(0) = 0$, $Du_k(0) = 0$.

Now, we introduce the auxiliary function

$$w = \partial_1 u_k + K\left(u_k - \frac{n}{2} x_1 \partial_1 u_k\right), \tag{5.40}$$

where K is a large constant to be determined. Let $\underline{w}$ be the function given by

$$\underline{w}(t) = \inf\{w(t, x_2, \ldots, x_n) : (t, x_2, \ldots, x_n) \in U_k \cap B_{r_0}\} \tag{5.41}$$

for $t > 0$ small.

We *claim* that the infimum in (5.41) cannot be attained on the boundary $\partial(U_k \cap B_{r_0})$ for $t > 0$ small. Indeed, as in the proof of Corollary 4.5, there exists a small constant $\tau_0 > 0$ such that for $t \in (0, \tau_0)$, the infimum in (5.41) cannot be attained on $U_k \cap \partial B_{r_0}$. Hence, it suffices to prove the claim over the part $\Gamma_k = \partial U_k \cap B_{r_0}$. For any given $0 < t < \min\{\tau_0, t_0\}$, denote

$$\Gamma_k \cap \{x_1 = t\} = \partial_{\text{in}}(t) \cup \partial_{\text{out}}(t),$$

where $\partial_{\text{in}}(t)$ denotes the boundary points $x \in \Gamma_k$ whose projection $(x - x_n e_n)$ lies in $\mathcal{C}_0$, while $\partial_{\text{out}}(t) = \Gamma_k \cap \{x_1 = t\} - \partial_{\text{in}}(t)$.

By our choice of the cone $\mathcal{C}_0$, the obliqueness holds at all points $x \in \partial_{\text{in}}(t)$. Hence u_k is smooth up to the boundary $\partial_{\text{in}}(t)$. Similarly to Lemma 4.4, we then infer that $\partial_{1n} u_k < 0$ and $\partial_n w < 0$ on $\partial_{\text{in}}(t)$. Hence the infimum in (5.41) cannot be attained on $\partial_{\text{in}}(t)$.

Next we show that the infimum cannot be attained on $\partial_{\text{out}}(t)$ either. On the one hand, since U_k, U_k^* satisfy condition (5.1) of Section 5.1, by Corollary 5.3 and (5.38), when $t > 0$ sufficiently small, we have

$$\underline{u_k}(t) = \inf\{u_k(t, x_2, \dots, x_n) : (t, x_2, \dots, x_n) \in U_k \cap B_{r_0}\} \leq Ct^3.$$

Note that due to the flatness of $\partial U_k, \partial U_k^*$ in the $e_2, \dots, e_{n-1}$ directions, we can choose $\varepsilon = 0$ in (5.14). Hence similarly to (4.41) we obtain

$$\underline{w}(t) \leq Ct^2 + CKt^3. \tag{5.42}$$

On the other hand, for any point $x = (t, x'', \rho_k(t)) \in \partial_{\text{out}}(t)$, we have $|x''| > \varepsilon_0 t$. Hence by (5.38), we have $u_k(x) \geq c_0|x''|^2 > c_0\varepsilon_0^2 t^2$. Since $\partial_1 u_k \geq 0$, we then obtain that for $t < 2(nK)^{-1}$ small,

$$w(x) > Ku_k(x) > Kc_0\varepsilon_0^2 t^2 \quad \forall\, x \in \partial_{\text{out}}(t). \tag{5.43}$$

Therefore, by choosing K sufficiently large, from (5.42) and (5.43) one can see that $w(x) > \underline{w}(t)$ for all $x \in \partial_{\text{out}}(t)$; namely, the infimum in (5.41) cannot be attained on $\partial_{\text{out}}(t)$.

Once the claim is proved, we can show that $\underline{w}$ is concave and reach a contradiction by a similar argument as in Section 4.4. The proof of Lemma 5.8 is finished. □

With the preparations in Sections 5.1 and 5.2, we are now in position to prove Lemma 1.1.

5.3. *Proof of Lemma* 1.1. By our construction, $0 \in \Gamma_k$, $0 \in \Gamma_k^*$, and (4.24), (4.26) hold for ρ_k, ρ_k^*, respectively. Similarly to (4.33), we have

$$\partial_{x_n} u_k(t, x'', \rho_k(t)) < 0 \quad \text{for} \quad (t, x'', \rho_k(t)) \in \Gamma_k \cap \{x_1 > 0\} \quad \text{near the origin.} \tag{5.44}$$

Now, we can prove Lemma 1.1 in a similar way as in Section 4, which is outlined as follows:

(i) By the computation as in Lemma 4.4, we have

$$\partial_{x_1 x_n} u_k(x) < 0 \quad \forall\, x \in \partial U_k \cap \{x \in B_{r_0} : x_1 > 0\}. \tag{5.45}$$

(ii) Define the auxiliary function

$$w_k(x) := \partial_{x_1} u_k + u_k - \frac{n}{2} x_1 \partial_{x_1} u_k$$

that satisfies

$$M^{ij} D_{ij} w_k = 0$$

in $B_{r_0} \cap U_k$, where $\{M^{ij}\}$ is the cofactor matrix of $D^2 u_k$.

(iii) By (5.45), similarly to Corollary 4.5, we see that there exists a constant ϵ_0 independent of k, such that for any given $t \in (0, \epsilon_0)$, the function $w_k(t, \cdot)$ has an interior local minimum. Hence we can define

$$\underline{w_k}(t) = \inf\{w_k(t, x_2, \ldots, x_n) : (t, x_2, \ldots, x_n) \in U_k\} \tag{5.46}$$

for $t \in (0, \epsilon_0)$. Note that by (5.45), the infimum cannot be attained on $\partial U_k \cap B_{\epsilon_0}$.

(iv) Similarly to Lemma 4.5, we can prove that $\underline{w_k}$ is concave in $(0, \epsilon_0)$.

(v) By letting $k \to \infty$, we have now obtained the function $\underline{w_0}$ that satisfies
   (a) $\underline{w_0} \geq 0$, and $\underline{w_0}(t) \to 0$ as $t \to 0$;
   (b) $\underline{w_0}$ is concave;
   (c) $\underline{w_0}(t) \leq Ct^{2-\varepsilon}$ for $t > 0$ small (by (5.14) that also holds for u_0 with the same constants).

   Therefore, $\underline{w_0} \equiv 0$, and we reach a contradiction analogous to that of dimension two. This completes the proof of Lemma 1.1. □

6. Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2, namely, the global $C^{2,\alpha}$ and $W^{2,p}$ estimates for the problem (1.1), (1.2). In [7], Caffarelli established the global $C^{2,\alpha'}$ estimate for some $\alpha' \in (0, \alpha)$. The exponent α' can be improved to α, using the global $C^{2,\alpha}$ estimate for the Dirichlet problem in [31], [28]. Here we give a direct proof. We also obtain the continuity of D^2u for Dini continuous and positive f.

Assume that $0 \in \partial\Omega$, $u(0) = 0$ and $Du(0) = 0 \in \partial\Omega^*$. By the uniform obliqueness (1.5) and a linear transform of the coordinates, we may assume that locally

$$\partial\Omega = \{x_n = \rho(x')\},$$
$$\partial\Omega^* = \{y_n = \rho^*(y')\},$$

where $\rho, \rho^* \in C^{1,1}$ satisfying $\rho, \rho^* \geq 0$ and $\rho(0) = \rho^*(0) = 0$. Note that this expression implies that $u_{x_n} > 0$ in Ω.

Extend u to $\mathbb{R}^n$ as at the beginning of Section 2. Denote

$$D^+_{h,a} = \{x \in \mathbb{R}^n : \ u(x) < h\} \cap \{x_n > a\}, \tag{6.1}$$

where $a \geq 0$ is a small constant. Let a_h be the smallest number such that $D^+_{h,a_h} \subset \Omega$, but $D^+_{h,a_h-\varepsilon} \not\subset \Omega$ for any $\varepsilon > 0$. One can see that $a_h \to 0$ as $h \to 0$. For simplicity, we denote D^+_{h,a_h} by D^+_h.

Let D^-_h be the reflection of D^+_h with respect to the plane $\{x_n = a_h\}$, and let $D_h := D^+_h \cup D^-_h$. Since $D_n u \geq 0$, the domain D_h is convex. Moreover, D_h shrinks to the origin as $h \to 0$.

LEMMA 6.1. *The shape of D_h is close to a ball of radius $h^{1/2}$, in the sense that*

$$B_{C^{-1}h^{\frac{1}{2}+\epsilon}}(x_h) \subset D_h \subset B_{Ch^{\frac{1}{2}-\epsilon}}(x_h) \tag{6.2}$$

for any given small $\epsilon > 0$, where the center $x_h = a_h e_n$.

Proof. First we show the centered sub-level set $S^c_h[u]$ is close to a ball of radius $h^{1/2}$; namely,

$$B_{C^{-1}h^{\frac{1}{2}+\epsilon}}(0) \subset S^c_h[u] \subset B_{Ch^{\frac{1}{2}-\epsilon}}(0) \tag{6.3}$$

for any small $\epsilon > 0$. Indeed, from Lemma 3.1,

$$B_{C^{-1}h^{\frac{1}{2}+\epsilon}}(0) \cap \{x_n = 0\} \subset S^c_h[u] \cap \{x_n = 0\} \tag{6.4}$$

for any small $\epsilon > 0$. Similarly, this also holds for centered sub-level sets $S^c_h[v]$, for the dual potential v.

Let e' be a unit tangential vector with $e' \perp e_n$, and let $t > 0$ such that $te' \in \partial S^c_h[v]$. Applying (6.4) to v, we have $t \geq C^{-1}h^{\frac{1}{2}+\epsilon}$. For any $x \in S^c_h[u]$, from (2.14),

$$|x \cdot e'| \leq C\frac{h}{t} \leq Ch^{\frac{1}{2}-\epsilon}, \tag{6.5}$$

which implies that $S^c_h[u]$ is contained in a vertical cylinder centered at the origin with radius $r' \leq Ch^{\frac{1}{2}-\epsilon}$, for any given small $\epsilon > 0$. Hence we have proved that

$$B_{C^{-1}h^{\frac{1}{2}+\epsilon}}(0) \cap \{x_n = 0\} \subset S^c_h[u] \cap \{x_n = 0\} \subset B_{Ch^{\frac{1}{2}-\epsilon}}(0).$$

Let $r_n e_n \in \partial S^c_h[u]$, and let $S'_h[u]$ be the projection of $S^c_h[u]$ on $\{x_n = 0\}$. By the convexity of u and noticing that $u_n \geq 0$,

$$r_n|S'_h[u]| = C\mathrm{Vol}(S^c_h[u]) = Ch^{\frac{n}{2}},$$

where for the last equality we use (2.11). By (6.5), $|S'_h[u]| \leq Ch^{(n-1)(\frac{1}{2}-\epsilon)}$, and thus we obtain

$$r_n \geq Ch^{\frac{1}{2}+\epsilon} \tag{6.6}$$

for a different small $\epsilon > 0$. Formula (6.6) is also true for the dual centered sub-level set $S^c_h[v]$. By (6.4), we have, similarly to (6.5),

$$|x \cdot e_n| \leq Ch^{\frac{1}{2}-\epsilon} \quad \forall\ x \in S^c_h[u]. \tag{6.7}$$

Combining (6.4)–(6.7) we obtain (6.3).

Next we show that there exist two constants b_1, b_2, independent of u and h, such that

$$S^{c,+}_{b_1h} \subset D^+_{h,0} \subset S^{c,+}_{b_2h}, \tag{6.8}$$

where $S^{c,+}_h = S^c_h[u] \cap \{x_n > 0\}$, and $D^+_{h,0}$ is given in (6.1). The first inclusion can be proved similarly to that of (2.6). Indeed, for any $x \in \partial S^c_h \cap \{x_n = 0\}$, by (6.3) and since $\partial\Omega \in C^{1,1}$, we have $\text{dist}(x, \partial S_h^c \cap \Omega) \le Ch^{1-\epsilon}$. By (6.3) we also have $|Du| \le Ch^{\frac{1}{2}-\epsilon}$ in S_h^c. Hence

$$u(x) \ge Ch \qquad \forall\, x \in \partial S_h^c \cap \{x_n = 0\},$$

and (6.8) follows. The second inclusion of (6.8) also follows from (2.6).

We are ready to prove (6.2). Combining (6.3) and (6.8), there exists a constant C independent of u and h such that

$$B_{C^{-1}h^{\frac{1}{2}+\epsilon}}(0) \cap \{x_n > 0\} \subset D^+_{h,0} \subset B_{Ch^{\frac{1}{2}-\epsilon}}(0) \cap \{x_n > 0\} \tag{6.9}$$

for any given small $\epsilon > 0$. Since $\partial\Omega \in C^{1,1}$, by the definition of a_h (after (6.1)), one has $a_h < Ch^{1-\epsilon}$ for some $\epsilon > 0$ as small as we want. Recall that $D_h = D^+_{h,a_h} \cup D^-_{h,a_h}$ and D^-_{h,a_h} is an even extension of D^+_{h,a_h} with respect to $\{x_n = a_h\}$. We obtain (6.2) from (6.9). □

By (6.2), we infer that

Corollary 6.1. *For any given small $\varepsilon > 0$, $u \in C^{1,1-\varepsilon}(\overline{\Omega})$.*

From the above $C^{1,\alpha}$ regularity for all $\alpha < 1$, we can prove the global $W^{2,p}$ regularity (Theorem 1.2). As mentioned in the introduction, the global $W^{2,p}$ estimate for the problem (1.1)–(1.2) was obtained in [10], using the estimates of Caffarelli in [7]. Hence the domains are the uniform convexity with C^2 boundaries in [10]. By our estimates above, we can remove the uniform convexity condition and reduce the smoothness assumption on domains.

Proof of Theorem 1.2. The proof is based on the estimate (6.2) and uses the argument of Savin [27]; see also [10]. For completeness, let us outline the main steps. Given $x \in \Omega$, let $\bar{h}_x$ be the maximal value of h such that $S_h[u](x) \subset \Omega$, i.e.,

$$\bar{h}_x := \max\{h \ge 0 : S_h(x) \subset \Omega\}.$$

Let T be a unimodular linear transform such that $T(S_{\bar{h}_x}[u](x)) \sim B_{\bar{h}_x^{1/2}}$. By (6.2), one has $\|T\|, \|T^{-1}\| \lesssim \bar{h}_x^{-\varepsilon}$, for any small $\varepsilon > 0$. Hence, when $\bar{h}_x$ is small,

$$S_{\bar{h}_x}[u](x) \subset D_{C\bar{h}_x^{1/2}} := \{z \in \overline{\Omega} : \text{dist}(z, \partial\Omega) \le C\bar{h}_x^{1/2-\varepsilon}\} \tag{6.10}$$

for any small $\varepsilon > 0$.

By subtracting a linear function we may assume that $x = 0$, $u(0) = 0$ and $Du(0) = 0$. Let

$$\tilde{u}(x) = \frac{1}{\bar{h}} u(\bar{h}^{1/2} T^{-1} x),$$

where $x \in \tilde{S}_1(0) = \bar{h}^{-1/2} T(S_{\bar{h}}(0))$. The interior $W^{2,p}$ estimate for $\tilde{u}$ in $\tilde{S}_1(0)$ gives $\int_{\tilde{S}_{1/2}(0)} \|D^2\tilde{u}\|^p dx \le C$; hence by rescaling,

$$\int_{S_{\bar{h}/2}(0)} \|D^2 u\|^p \, dx = \int_{\tilde{S}_{1/2}(0)} \|T' D^2 \tilde{u} T\|^p \bar{h}^{n/2} \, dx \le C\bar{h}^{\frac{n}{2}-2\varepsilon p}. \tag{6.11}$$

From Vitali covering lemma, there exists a sequence of disjoint sub-level sets $\{S_{\delta\bar{h}_i}(x_i)\}$, $\bar{h}_i = \bar{h}_{x_i}$ such that $\Omega \subset \bigcup_{i=1}^{\infty} S_{\bar{h}_i/2}(x_i)$, where $\delta > 0$ is a small constant (see [27, Lemma 2.3]). Then

$$\int_{\Omega} \|D^2 u\|^p \, dx \le \sum_i \int_{S_{\bar{h}_i/2}(x_i)} \|D^2 u\|^p \, dx.$$

Note that it suffices to consider those $\bar{h}_i \le c_1$ for a small constant $c_1 > 0$. We can adopt the argument of Savin [27]: Consider the family $\mathcal{F}_d$ of those $S_{\bar{h}_i/2}(x_i)$ satisfying

$$d/2 < \bar{h}_i \le d$$

for a constant $d \le c_1$. By (6.11) and (2.12),

$$\int_{S_{\bar{h}_i/2}(x_i)} \|D^2 u\|^p \, dx \le C d^{-2\varepsilon p} |S_{\delta\bar{h}_i}(x_i)|.$$

By (6.10) and Vitali covering lemma, the sets $S_{\delta\bar{h}_i}(x_i) \subset D_{Cd^{1/2-\varepsilon}}$ are disjoint. Hence, we have

$$\sum_{i\in\mathcal{F}_d} \int_{S_{\bar{h}_i/2}(x_i)} \|D^2 u\|^p \, dx \le C d^{\frac{1}{2}-\varepsilon-2\varepsilon p} \le C d^{\frac{1}{2}-3\varepsilon p}.$$

Let $d = c_1 2^{-k}$, $k = 0, 1, 2, \ldots$. By adding the sequence of inequalities, we obtain

$$\int_{\Omega} \|D^2 u\|^p \, dx \le C + C_1 \sum_{k=0}^{\infty} 2^{-k(\frac{1}{2}-3\varepsilon p)}.$$

For any $p \ge 1$, as ε is arbitrarily small so that $3\varepsilon p < \frac{1}{4}$, therefore, the series is convergent. □

Now we continue with the proof of global $C^{2,\alpha}$ estimate. Let w be the solution of

$$\det D^2 w = 1 \text{ in } D_h,$$
$$w = h \text{ on } \partial D_h.$$

Denote by $\tilde{u}$ the even extension of u with respect to $\{x_n = 0\}$, namely,

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n \ge 0, \\ u(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

For simplicity, we still denote $\tilde{u}$ by u. The following lemma gives an estimate on the difference between the "*original*" solution u and the "*good*" solution w.

LEMMA 6.2. *Assume $|f - 1| \le h^{\delta}$ in $D_h \cap \Omega$ for some $\delta \in (0, 1/2)$. We have*

$$|u - w| \le C h^{1+\delta} \qquad \text{in } D_h \cap \Omega, \tag{6.12}$$

where the constant C is independent of h, δ.

Proof. Divide $\partial D_h^+ = \mathcal{C}_1 \cup \mathcal{C}_2$ into two parts, where $\mathcal{C}_1 \subset \{x_n > a_h\}$ and $\mathcal{C}_2 \subset \{x_n = a_h\}$. On $\mathcal{C}_1$ we have $u = w$. On $\mathcal{C}_2$, by symmetry we have $D_n w = 0$. As $a_h < Ch^{1-\epsilon}$, by Corollary 6.1 we have $0 \le D_n u \le C_1 h^{1-\epsilon}$ on $\mathcal{C}_2$ for any given small $\epsilon > 0$.

Let

$$\hat{w} = (1 - h^\delta)^{1/n} w - (1 - h^\delta)^{1/n} h + h.$$

Then

$$\begin{aligned} \det D^2 \hat{w} \le \det D^2 u \quad & \text{in } D_h^+, \\ \hat{w} = u = h \quad & \text{on } \mathcal{C}_1, \\ D_n \hat{w} = 0 < D_n u \quad & \text{on } \mathcal{C}_2. \end{aligned}$$

By the comparison principle we have $\hat{w} \ge u$ in D_h^+.

On the other hand, let

$$\check{w} = (1 + h^\delta)^{1/n} w - (1 + h^\delta)^{1/n} h + h + C_1(x_n - Ch^{1/2-\epsilon}) h^{1-\epsilon}.$$

Then

$$\begin{aligned} \det D^2 \check{w} \ge \det D^2 u \quad & \text{in } D_h^+, \\ \check{w} \le u = h \quad & \text{on } \mathcal{C}_1, \\ D_n \check{w} = C_1 h^{1-\epsilon} > D_n u \quad & \text{on } \mathcal{C}_2. \end{aligned}$$

Hence by the comparison principle, $\check{w} \le u$ in D_h^+.

Since $h > 0$ is small, $\delta < 1/2$, and $\epsilon > 0$ is small, we obtain

$$|u - w| \le Ch^{1+\delta} \qquad \text{in } D_h^+. \tag{6.13}$$

Next, we estimate $|u - w|$ in $D_h^- \cap \Omega$. For $x = (x', x_n) \in D_h^- \cap \Omega$, let $z = (x', 2a_h - x_n) \in D_h^+$. Then $|x - z| \le Ch^{1-\epsilon}$. From (6.13), $|u(z) - w(z)| \le Ch^{1+\delta}$. Since w is symmetric with respect to $\{x_n = a_h\}$, we have $w(x) = w(z)$. Since $u \in C^{1,1-\epsilon}(\overline{\Omega})$, we obtain

$$|u(x) - u(z)| \le \|Du\|_{L^\infty(D_h)} |x - z| \le Ch^{3/2-\epsilon}.$$

Therefore, as $\delta < 1/2$ is a given constant,

$$|u(x) - w(x)| \le |u(x) - u(z)| + |u(z) - w(z)| \le Ch^{1+\delta}.$$

Combining with (6.13) we thus obtain the desired L^∞ estimate

$$|u - w| \le Ch^{1+\delta} \qquad \text{in } D_h \cap \Omega. \tag{6.14}$$

□

We are now in position to prove the global $C^{2,\alpha}$ estimate. We will adopt the argument in [21]. Note that when f is Hölder continuous with exponent $\alpha \in (0, 1)$ and $f(0) = 1$, from Lemma 6.1 we have the oscillation

$$\omega_f(h) := \sup_{D_h \cap \Omega} |f - 1| \le Ch^\delta \tag{6.15}$$

for some

$$\delta \geq \alpha/2 - \varepsilon, \tag{6.16}$$

where $\varepsilon > 0$ is a small constant arising in (6.2). We point out that if $\varepsilon = 0$ in (6.2), then $\delta = \alpha/2$. We first quote two lemmas from [21].

LEMMA 6.3. *Let $u \in C^2$ be a convex solution of $\det D^2 u = 1$ in $\mathcal{D}$, vanishing on $\partial\mathcal{D}$. Suppose u attains its minimum at the origin, and $|D^2u(0)| \leq C_0$ for some constant $C_0 > 0$. Then the domain $\mathcal{D}$ is of good shape.*

LEMMA 6.4. *Let u_i, $i = 1, 2$, be two convex solutions of $\det D^2 u = 1$ in $B_1(0)$. Suppose $\|u_i\|_{C^4} \leq C_0$. Then if $|u_1 - u_2| \leq \delta$ in $B_1(0)$ for some constant $\delta > 0$, we have, for $1 \leq k \leq 3$,*

$$|D^k(u_1 - u_2)| \leq C\delta \quad \textit{in } B_{1/2}.$$

Proof of Theorem 1.1. We sketch the proof here as it is similar to that in [21]. Choose a sufficiently small initial height $h_0 > 0$, and normalize D_{h_0} by a transformation T such that $T(D_{h_0}) \sim B_1(0)$.

After the change, D_1 has a good shape. Denote

$$\omega(h) = \omega_f(h), \qquad \omega_k := \omega(4^{-k}) \tag{6.17}$$

for $k = 0, 1, 2, \ldots$, where $\omega_f(h)$ is given in (6.15). Define

$$\mathcal{D}_k := D_{4^{-k}}, \qquad f_k := \inf_{\mathcal{D}_k \cap \Omega} f > 0. \tag{6.18}$$

We *claim* that $\mathcal{D}_k \sim \mathcal{D}_{k+1}$; namely, there is a constant C depending only on n such that

$$C^{-1}\mathcal{D}_k \subset \mathcal{D}_{k+1} \subset C\mathcal{D}_k. \tag{6.19}$$

To see this, note that (before the change T) by (6.2) and $a_{h_0} < Ch_0^{1-\epsilon}$, one has $|D_{h_0}| \approx |S_{h_0}[u]| \approx h_0^{n/2}$ and $|D_{h_0/4} \cap \{x_n > a_{h_0}\}| \approx h_0^{n/2}$. Since $|\det T| \approx h_0^{-n/2}$, we have $|T(D_{h_0/4} \cap \{x_n > a_{h_0}\})| \approx 1$. By definition, $D_{h_0/4} \cap \{x_n > a_{h_0}\} \subset D_{h_0}$, thus $T(D_{h_0/4} \cap \{x_n > a_{h_0}\})$ is bounded. Therefore, by symmetry and the fact $a_{h_0} < Ch_0^{1-\epsilon} \ll h^{\frac{1}{2}+\epsilon}$ that is the width of $D_{h_0/4}$ in the e_n direction, we obtain $\mathcal{D}_1 \sim \mathcal{D}_0$. Similarly we can obtain (6.19) for all $k = 0, 1, 2, \ldots$.

Let u_k, $k = 0, 1, 2, \cdots$, be the convex solution of

$$\begin{aligned} \det D^2 u_k &= f_k && \text{in } \mathcal{D}_k, \\ u_k &= 4^{-k} && \text{on } \partial\mathcal{D}_k. \end{aligned} \tag{6.20}$$

When $k = 0$, since initially $\mathcal{D}_0$ has a good shape, by interior regularity [18],

$$\|u_0\|_{C^4(D_{3/4})} \leq C.$$

From Lemma 6.2,

$$\sup_{\mathcal{D}_0\cap\Omega} |u - u_0| \le C\omega_0,$$

which implies that $\mathcal{D}_1$ has a good shape (also shown in (6.19)), and thus

$$\|u_1\|_{C^4(D_{3/16})} \le C.$$

Hence, from Lemma 6.4 and (6.19),

$$|D^2u_0(x) - D^2u_1(x)| \le C\omega_0 \tag{6.21}$$

for $x \in C^{-1}\mathcal{D}_2$, where $1 \le k \le 3$. By Lemma 6.3, this estimate then implies that $\mathcal{D}_2$ has a good shape.

By induction and (6.12), (6.19), we obtain

$$|D^2u_k(x) - D^2u_{k+1}(x)| \le C\omega_k \tag{6.22}$$

for $x \in C^{-1}\mathcal{D}_{k+2}$.

Therefore, for any given point $z \in \overline{\Omega}$ near the origin such that $4^{-k-4} \le u(z) \le 4^{-k-3}$,

$$\begin{aligned} &|D^2u(z) - D^2u(0)| \le I_1 + I_2 + I_3 \\ &:= |D^2u_k(z) - D^2u_k(0)| + |D^2u_k(0) - D^2u(0)| + |D^2u(z) - D^2u_k(z)|. \end{aligned} \tag{6.23}$$

By (6.22),

$$I_2 \le C\sum_{j=k}^{\infty}\omega_j \le C\int_0^{|z|}\frac{\omega(r)}{r}. \tag{6.24}$$

Similarly to (6.24), as in [21] one can derive that

$$I_3 \le C\int_0^{|z|}\frac{\omega(r)}{r}. \tag{6.25}$$

To estimate I_1, denote $h_j = u_j - u_{j-1}$. By Lemma 6.4,

$$\left|D^2h_j(z) - D^2h_j(0)\right| \le C2^j\omega_j|z|.$$

Hence

$$\begin{aligned} I_1 &\le |D^2u_0(z) - D^2u_0(0)| + \sum_{j=1}^{k}|D^2h_j(z) - D^2h_j(0)| \\ &\le C|z|\Big(1 + \int_{|z|}^1\frac{\omega(r)}{r^2}\Big). \end{aligned} \tag{6.26}$$

Since f is Hölder continuous with exponent α, inserting (6.24)–(6.26) into (6.23) we obtain the Hölder continuity at the origin; namely, for any point z near the origin,

$$|D^2u(z) - D^2u(0)| \le C|z|^\alpha. \tag{6.27}$$

In (6.27), we obtained the Hölder continuity of D^2u at the boundary. If two points $x, y \in \Omega$ are both interior points, let $\hat{x}, \hat{y} \in \partial\Omega$ be the closest points to x, y, respectively. In the case $|x-y| \geq \delta_0\big(\text{dist}(x, \partial\Omega)+\text{dist}(y, \partial\Omega)\big)$ for some constant $\delta_0 > 0$, by (6.27) we have

$$|D^2u(x) - D^2u(y)| \leq |D^2u(x) - D^2u(\hat{x})| + |D^2u(\hat{x}) - D^2u(\hat{y})| + |D^2u(\hat{y}) - D^2u(y)| \leq C|x-y|^\alpha.$$

Otherwise, the estimate for $|D^2u(x) - D^2u(y)|$ has been established in [4], [21] for the interior $C^{2,\alpha}$ regularity.

We have proved the global $C^{2,\alpha-\varepsilon}$ regularity for problem (1.1), (1.2). To remove the small constant ε, observe that once the second derivatives are uniformly bounded, the inclusions (6.2) holds for $\varepsilon = 0$. Therefore, (6.16) can be improved to $\delta = \alpha/2$. Repeating the above argument, we then obtain the global $C^{2,\alpha}$ regularity for problem (1.1), (1.2). □

Remark 6.1. The above argument also implies that if f is Dini continuous, that is if

$$\int_0^1 \frac{\omega_f(t)}{t}\, dt < \infty,$$

where $\omega(t) = \sup\{|f(x) - f(y)| : |x-y| < t\}$, then the integrals in (6.24) and (6.26) are convergent. Hence D^2u is positive definite and continuous up to the boundary. Therefore, we have proved the following result.

THEOREM 6.1. *Assume that Ω and Ω^* are bounded convex domains in $\mathbb{R}^n$ with $C^{1,1}$ boundaries, and assume that f is positive and Dini continuous. Then the second derivatives of the solution u to the problem* (1.1) *and* (1.2) *are continuous in $\overline{\Omega}$.*

Remark 6.2. Checking the proof of the uniform density (Lemma 2.3), the tangential $C^{1,\alpha}$ regularity (Lemma 3.1), and the uniform obliqueness (Lemma 1.1), we find that the $C^{1,1}$ regularity of the boundaries $\partial\Omega$ and $\partial\Omega^*$ can be weakened to $C^{1,1-\theta}$ for some $\theta > 0$ depending on the constant δ, provided u is globally $C^{1,\delta}$ smooth [5]. Therefore, our main result, Theorem 1.1, holds for $C^{1,1-\theta}$ convex domains Ω, Ω^*. In particular, we prove that it suffices to assume $\partial\Omega, \partial\Omega^* \in C^{1,\alpha}$ in dimension two [11]. When $f \equiv 1$, very recently Savin and Yu [29] obtained the global $W^{2,p}$ estimate for arbitrary bounded convex domains $\Omega, \Omega^* \subset \mathbb{R}^2$. In general dimensions, it may be possible to relax the $C^{1,1}$ regularity of the boundaries to $C^{1,\alpha}$. Indeed, if one can manage this relaxation for the uniform density estimate and the tangential $C^{1,1-\varepsilon}$ estimate (for all $\varepsilon > 0$), then our method for the uniform obliqueness can be applied.

Remark 6.3. From [24, §7.3] it is known that for arbitrary positive and smooth functions f, the convexity of domains is necessary for the global C^1

regularity. However, for a fixed function $f > 0$, by Theorem 1.1 and a perturbation argument, we can prove that [13] the solution is smooth up to the boundary, if the domains Ω and Ω^* are smooth perturbations of convex ones, even though they are not convex themselves.

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