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Bi-Hölder Extensions of Quasi-isometries on Complex Domains

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Abstract
In this paper, we prove some results on bi-Hölder extensions not only for biholomorphisms but also for more general Kobayashi metric quasi-isometries between the domains. Furthermore, we establish the Gehring–Hayman type theorems on certain complex domains which play an important role through the paper. Then by applying the above results, we show the bi-Hölder equivalence between the Euclidean boundary and the Gromov boundary of bounded convex domains which are Gromov hyperbolic with respect to their Kobayashi metrics.

Keywords Quasi-isometries · Convex domains · Kobayashi metric · Gromov hyperbolicity

Mathematics Subject Classification 32F45 · 32F18 · 32T15

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1 Introduction

In this paper, we characterize the Kobayashi geodesic by generalizing the classical Gehring–Hayman Theorem. Then we apply these results to study the boundary extension problems for biholomorphisms isometries and for more generally rough quasi-isometries between domains in $\mathbb{C}^n (n \geq 2)$, under some (geo)metric assumptions. Moreover, we also investigate the boundary correspondence between the Gromov boundary and the Euclidean boundary of certain complex domains.

In the complex plane $\mathbb{C}$, suppose domains Ω_1 and Ω_2 are bounded by closed Jordan curves, then every biholomorphic map $f : \Omega_1 \to \Omega_2$ extends to a homeomorphism of $\overline{D}_1$ onto $\overline{D}_2$. In $\mathbb{C}^n,\ n > 1$, the boundary extension problem is more interesting. Suppose $\Omega_1,\ \Omega_2$ are C^2-smooth bounded pseudoconvex domains with

$$\frac{1}{C}\delta_{\Omega_1}^{\frac{1}{\nu_1}}(z) \leq \delta_{\Omega_2}(f(z)) \leq C\delta_{\Omega_1}^{\nu_1}(z), \quad \forall z \in \Omega_1,$$

and the Kobayashi metric

$$k_{\Omega_i}(\omega, v) \geq \frac{C|v|}{\delta_{\Omega_i}(\omega)^{\nu_i}}, \quad \forall \omega \in \Omega_i, \quad v \in \mathbb{C}^n,$$

for some $\nu_1,\ \nu_2,\ C > 0$, where $\delta_{\Omega_i}(z) := \inf\{|w - z|, w \in \partial\Omega_i\},\ i = 1, 2$, then the biholomorphic map $f : \Omega_1 \to \Omega_2$ extends to a bi-Hölder continuous map of $\overline{\Omega}_1$. This kind of result holds in particular if Ω_i are strictly pseudoconvex domains and more generally pseudoconvex domains with finite type. Moreover, Mercer in [20] introduced the class of *m-convex domains* which characterize the convex domain without extra regularities. He also proves boundary extensions of biholomorphisms between m-convex domains.

Theorem 1.1 (Propositions 2.6, [21]) *Let $\Omega_1,\ \Omega_2$ be bounded m-convex domains, and let $f : \Omega_1 \to \Omega_2$ be a biholomorphic map. Then f extends to a bi-Hölder continuous map on $\overline{\Omega}_1$.*

Usually, the Hopf Lemma and the estimates of $|\nabla f|$ play an important role in the proof of relevant results (see [17], for example). There are many other generalizations, and we refer the interested reader to the survey [11] by F. Forstnerič. In this paper, we prove a similar boundary extension result for isometries between m-convex domains instead of biholomorphisms, with some extra boundary regularities.

Theorem 1.2 *Let Ω_i, $i = 1, 2$, be bounded m-convex domains in $\mathbb{C}^n (n \geq 2)$ with Dini-smooth boundary, and let $\overline{\Omega}_i$ be their Euclidean compactifications. Let $f : \Omega_1 \to \Omega_2$ be an isometry with respect to the Kobayashi metrics K_{Ω_i}. Then f has a homeomorphic extension $\bar{f} : \overline{\Omega}_1 \to \overline{\Omega}_2$ such that the induced boundary map $\bar{f}\big|_{\partial\Omega_1} : \partial\Omega_1 \to \partial\Omega_2$ is bi-Hölder continuous with respect to the Euclidean metric.*

Remark 1.3 On strongly pseudoconvex domains with C^2 smooth boundary, the result was proved for more general rough quasi-isometries by Balogh and Bonk [1].

Note that every biholomorphic map is an isometry with respect to the Kobayashi metric. Thus, Theorem 1.2 clearly holds for biholomorphisms between the complex domains. However, we do not add any regularities on the isometries; thus, we cannot estimate $|\nabla f|$ here. The approach to Theorem 1.2 is also different from the discussions in [1,4,7,21,28]. We could not only show the continuous extension of the biholomorphisms and (quasi-)isometries in general but also enable one to determine the regularity of boundary extension map. Our strategy to prove Theorem 1.2 is to establish the *Gehring–Hayman Theorem* and the *Seperation property* on complex domains.

Recall the following classical Gehring–Hayman Theorem on planar domains.

Theorem 1.4 [13] *If Ω is a simply connected planar domain ($\Omega \neq \mathbb{C}$), then there exists $C > 0$ such that, for any $x, y \in \Omega$,*

$$l_d([x, y]) \le C l_d(\gamma),$$

where $[x, y]$ is the hyperbolic geodesic joining x and y, and $\gamma \subset \Omega$ is any curve with end points x and y, and l_d denotes the Euclidean length.

In this paper, we prove some results similar to Theorem 1.4 for *m-convex* domains (resp. *strongly pseudoconvex* domains) with respect to the Kobayashi geodesics. Our result in this direction is as follows, which shows that the Kobayashi geodesics (or quasi-geodesics) are essentially also short in the Euclidean sense.

Theorem 1.5 *Let Ω be a bounded m-convex domain in $\mathbb{C}^n$ ($n \ge 2$) with Dini-smooth boundary. Then for any $0 < c_2 < 1/(8m^2 - 4m)$, there exists a constant $c_1 > 0$ such that, for any $x, \; y \in \Omega$,*

$$l_d([x, y]) \le c_1 |x - y|^{c_2},$$

where $[x, y]$ is a Kobayashi geodesic joining x and y in Ω.

If in addition, $(\Omega, \; K_\Omega)$ is Gromov hyperbolic and γ is a Kobayashi λ-quasi-geodesic connecting x and y with $\lambda \ge 1$, then there exists a constant $c_1' > 0$ such that

$$l_d(\gamma) \le c_1' |x - y|^{c_2}.$$

In order to prove Theorem 1.5, we need the following result.

Lemma 1.6 *Let Ω be a bounded m-convex domain in $\mathbb{C}^n$ with $n \ge 2$, and let $[x, y] \subset \Omega$ be a Kobayashi geodesic joining x and y. Then for any $\alpha > 2m^2 - m$, there exists a constant $\tilde{C} > 0$ such that, for every $\omega \in [x, y]$,*

$$\delta_\Omega(\omega) \ge \tilde{C} \min\{l_d([x, \omega]), \; l_d([\omega, y])\}^\alpha, \tag{1}$$

where $\delta_\Omega(\omega)$ is the Euclidean distance from ω to $\partial\Omega$.

Note that we may compare Lemma 1.6 with the *Separation property*. It states that whenever $[x, y]$ is a geodesic in (Ω, K_Ω), $z \in [x, y]$ and γ is a curve in Ω connecting the subcurves $[x, z)$ and $(z, y]$ of $[x, y]$, then for some $a > 0$,

$$B(z, a\delta_\Omega(z)) \cap \gamma \neq \emptyset.$$

In fact, Lemma 1.6 gives that

$$B(z, \delta_\Omega^{\frac{1}{\alpha}}(z)/\tilde{C}) \cap \gamma \neq \emptyset.$$

We refer the reader to [2,6,16] for more background information in this topic.

We can also apply the above results to problems relevant to Gromov hyperbolicity. In [1], Balogh and Bonk investigated the *Gromov hyperbolicity* of the Kobayashi metric for bounded strictly pseudoconvex domains in $\mathbb{C}^n$. Recently, Zimmer [28] discussed the Gromov hyperbolicity and the Kobayashi metric on bounded convex domains of *finite type*.

In [1], Balogh and Bonk proved that a strongly pseudoconvex domain Ω endowed with its Kobayashi metric is Gromov hyperbolic, and its Gromov boundary coincides with its Euclidean boundary. Moreover, the Carnot-Carathéodory metric d_H on $\partial\Omega$ lies in (and, thus, determines) the canonical class of snowflake equivalent metrics on $\partial_G\Omega$. This actually means that the map between the Euclidean boundary and the Gromov boundary (equipped with a visual metric) is bi-Hölder.

Recently, Bracci, Gaussier, and Zimmer [7] demonstrated the following homeomorphic extension result on convex domains.

Theorem 1.7 (Theorem 1.4, [7]) *Let Ω be a $\mathbb{C}$-proper convex domain on $\mathbb{C}^n$. If (Ω, K_Ω) is Gromov hyperbolic, then the identity map id: $\Omega \to \Omega$ extends to a homeomorphism (still use the same name) id: $\overline{\Omega}^{\star} \to \overline{\Omega}^{G}$ where $\overline{\Omega}^{\star}$ denotes the Euclidean end compactification of Ω and $\overline{\Omega}^{G}$ is the Gromov compactification of the metric space (Ω, K_Ω).*

As an application of the Theorem 1.5 and Lemma 1.6, we obtain the following result:

Proposition 1.8 *Let Ω be a bounded domain in $\mathbb{C}^n$ $(n \geq 2)$ and suppose that Ω satisfies either*

(a) *Ω is convex with Dini-smooth boundary and (Ω, K_Ω) is Gromov hyperbolic; or*
(b) *Ω is strongly pseudoconvex with C^2-smooth boundary.*

Then the identity map $id : \Omega \to \Omega$ extends to a bi-Hölder homeomorphism

$$id : (\partial\Omega, |\cdot|) \to (\partial_G\Omega, \rho_G)$$

between the boundaries, where ρ_G is a visual metric on the Gromov boundary of (Ω, K_Ω).

Remark 1.9 (1) The assertion for Case (b) in Theorem 1.8 follows from Balogh and Bonk's result in [1], and the assertion for Case (a) can be deduced from the recent

result of Zimmer [29, Propsition 12.2] even without the Dini-smooth assumptions. We still write down the proof since the approach is different.
(2) Gromov boundary equipped with any two visual metrics is power quasisymmetrically and so bi-Hölder equivalent to each other. Thus, the boundary extension of the identity map in Proposition 1.8 is bi-Hölder with respect to any visual metric on the Gromov boundary.

We can now apply this boundary correspondence to investigate boundary extension results for quasi-isometries with respect to the Kobayashi metrics between the domains. In [1], Balogh and Bonk generalized this kind of results for rough quasi-isometries in the Kobayashi metrics. Then, Baharali and Zimmer defined a class of complex domains named 'Goldilock' domains and they showed the following:

Theorem 1.10 (Theorem 1.7, [4]) *Let Ω_1 be a bounded domain in $\mathbb{C}^n$ and suppose that (Ω_1, K_{Ω_1}) is Gromov hyperbolic. Let $\Omega_2 \Subset \mathbb{C}^n$ be a Goldilock domain. If $f : (\Omega_1, K_{\Omega_1}) \to (\Omega_2, K_{\Omega_2})$ is a continuous quasi-isometric embedding, then f extends continuously to a continuous map $\bar{f} : \overline{\Omega}_1^G \to \overline{\Omega}_2$.*

Moreover, recently Bracci, Gaussier, and Zimmer [7] proved the following result:

Theorem 1.11 *Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$. Assume*

(1) *Ω_1 is either a bounded, C^2-smooth strongly pseudoconvex domain, or a convex $\mathbb{C}$-proper domain such that $\left(\Omega_1, K_{\Omega_1}\right)$ is Gromov hyperbolic,*
(2) *Ω_2 is convex.*

Then every roughly quasi-isometric homeomorphism $f : \left(\Omega_1, K_{\Omega_1}\right) \to \left(\Omega_2, K_{\Omega_2}\right)$ extends to homeomorphism $\bar{f} : \overline{\Omega}_1^\star \to \overline{\Omega}_2^\star$, where $\overline{\Omega}_i^\star$ is the Euclidean end compactification of Ω_i, $i = 1, 2$.

As a corollary of Proposition 1.8, we prove the following bi-Hölder homeomorphism extension result, which gives the regularity of $\bar{f}\big|_{\partial\Omega_1}$.

Corollary 1.12 *For $i = 1, 2$, suppose that $\Omega_i \subset \mathbb{C}^n$ $(n \geq 2)$ are bounded domains, and Ω_i satisfy either condition (1) or (2):*

(1) *Ω_i is a convex domain with Dini-smooth boundary and (Ω_i, K_{Ω_i}) is Gromov hyperbolic;*
(2) *Ω_i is a strongly pseudoconvex domain with C^2-smooth boundary.*

Let $f : \Omega_1 \to \Omega_2$ be a homeomorphism that is a rough quasi-isometry with respect to the Kobayashi metrics K_{Ω_i}. Then f has a homeomorphic extension $\bar{f} : \overline{\Omega_1} \to \overline{\Omega_2}$ such that the induced boundary map $\bar{f}\big|_{\partial\Omega_1} : \partial\Omega_1 \to \partial\Omega_2$ is bi-Hölder with respect to the Euclidean metric.

The rest of this paper is organized as follows. In Sect. 2, we recall some definitions and preliminary results. Section 3 focuses on the proofs of Theorem 1.5 and Lemma 1.6. In Sect. 4, we prove similar results for strongly pseudoconvex domains. Section 5 is devoted to the proof of Theorem 1.2 and Sect. 6 is devoted to proving Proposition 1.8 and Corollary 1.12.

2 Preliminaries

2.1 Notation

(1) For $z \in \mathbb{C}^n$, let $|\cdot|$ and d denote the standard Euclidean norm, and let $|z_1 - z_2|$ and $d(z_1, z_2)$ be the standard Euclidean distance of $z_1, z_2 \in \mathbb{C}^n$.

(2) Given an open set $\Omega \subsetneq \mathbb{C}^n$, $x \in \Omega$ and $v \in \mathbb{C}^n \backslash \{0\}$, denote

$$\delta_\Omega(x) = \inf \{d(x, \xi) : \xi \in \partial\Omega\}$$

as before, and denote

$$\delta_\Omega(x, v) = \inf\{d(x, \xi) : \xi \in \partial\Omega \cap (x + \mathbb{C}v)\}.$$

(3) For any curve σ, we denote its Euclidean length by $l_d(\sigma)$ and the Kobayashi length by $l_k(\sigma)$.

(4) For any $z_0 \in \mathbb{C}^n$ and $\epsilon > 0$, we denote by $B_\epsilon(z_0)$ or $B(z_0, \epsilon)$ the open ball $B_\epsilon(z_0) = \{z \in \mathbb{C}^n |\, |z - z_0| < \epsilon\}$.

(5) For all real numbers a, b, we denote $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

2.2 *m*-Convex Domains and Strongly Pseudoconvex Domains

In [20], Mercer introduced the class of *m-convex* domains. Now we give the definition of m-convex domains as follows.

Definition 2.1 A bounded convex domain $\Omega \subset \mathbb{C}^n$ with $n \geq 2$ is called m-convex for some $m \geq 1$ if there exists $C > 0$ such that, for any $z \in \Omega$ and non-zero $v \in \mathbb{C}^n$,

$$\delta_\Omega(z; v) \leq C\delta_\Omega^{\frac{1}{m}}(z). \tag{2}$$

Note that the m-convexity is related to the finite type by the following proposition.

Proposition 2.2 (Proposition 9.1, [28]) *Given a bounded convex domain $\Omega \subset \mathbb{C}^n (n \geq 2)$ with smooth boundary, then Ω is m-convex for some $m \in \mathbb{N}$ if and only if $\partial\Omega$ has finite line type in the sense of D'Angelo.*

Definition 2.3 A domain $\Omega = \{z|\rho(z) < 0\}$ in $\mathbb{C}^n (n \geq 2)$ with C^2-smooth boundary is called *strongly pseudoconvex* if the Levi form of the boundary

$$L_\rho(p; v) = \sum_{\nu,\mu=1}^{n} \frac{\partial^2 \rho}{\partial z_\nu \partial \bar{z}_\mu}(p) v_\nu \bar{v}_\mu, \quad \text{for} \quad v = (v_1, \ldots, v_n) \in \mathbb{C}^n$$

is positive definite for every $p \in \partial\Omega$.

2.3 The Kobayashi Metric

Given a domain $\Omega \subset \mathbb{C}^n (n \geq 2)$, the (infinitesimal) Kobayashi metric is the pseudo-Finsler metric defined by

$$k_\Omega(x; v) = \inf \{|\xi| : f \in \mathrm{Hol}(\mathbb{D}, \Omega), \ \ \text{with } f(0) = x, d(f)_0(\xi) = v\} .$$

Define the Kobayashi length of any curve $\sigma : [a, b] \to \Omega$ to be

$$l_k(\sigma) = \int_a^b k_\Omega \left(\sigma(t); \sigma'(t)\right) dt.$$

It is a consequence of a result due to Venturini [26], which is based on an observation by Royden [23] that the Kobayashi pseudo-distance can be given by

$$\begin{aligned} K_\Omega(x, y) = \inf_\sigma \big\{ l_k(\sigma) | \, \sigma : [a, b] \to \Omega \text{ is any absolutely continuous curve} \\ \text{with } \sigma(a) = x \text{ and } \sigma(b) = y \big\}. \end{aligned}$$

There are some estimates concerning the Kobayashi metric on convex domains.

Lemma 2.4 [15]. *If $\Omega \subset \mathbb{C}^n$ is a bounded convex domain, then for all $x \in \Omega$ and for every $v \in \mathbb{C}^n$,*

$$\frac{|v|}{2\delta_\Omega(x; v)} \leq k_\Omega(x; v) \leq \frac{|v|}{\delta_\Omega(x; v)}. \tag{3}$$

Lemma 2.5 (Proposition 2.4, [20]) *Suppose that $\Omega \subset \mathrm{C}^n$ is a bounded convex domain, for any $x, \ y \in \Omega$, we have*

$$K_\Omega\,(x, y) \geq \frac{1}{2} \left| \log \frac{\delta_\Omega(x)}{\delta_\Omega(y)} \right| . \tag{4}$$

Recall that a C^1-smooth boundary point p of a domain Ω in $\mathbb{C}^n$ is said to be *Dini-smooth* (or *Lyapunov-Dini-smooth*), if the inner unit normal vector $\mathbf{n}$ to $\partial\Omega$ near p is a Dini-continuous function. This means that there exists a neighborhood U of p such that

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty,$$

where

$$\omega(t) = \omega(\mathbf{n}, \partial\Omega \cap U, t) := \sup \left\{ |\mathbf{n}_x - \mathbf{n}_y| : |x - y| < t, \ \ x, y \in \partial\Omega \cap U \right\}$$

is the respective modulus of continuity. Note that Dini-smooth is a weaker condition than $C^{1,\epsilon}$-smooth. Here a *Dini-smooth domain* means that each boundary point of Ω is a Dini-smooth point. Then we have

Lemma 2.6 (Corollary 8, [22]) *Let Ω be a Dini-smooth bounded domain in $\mathbb{C}^n$ and $x,\ y \in \Omega$. Then there exists a constant $A > 1 + \sqrt{2}/2$ such that*

$$K_\Omega(x, y) \leq \log\left(1 + \frac{A|x-y|}{\sqrt{\delta_\Omega(x)\delta_\Omega(y)}}\right). \tag{5}$$

Proposition 2.7 [3] *If Ω is a $\mathbb{C}$-proper convex domain in $\mathbb{C}^n$ $(n \geq 2)$, then the Kobayashi metric K_Ω is complete.*

Here $\mathbb{C}$-*proper* means that Ω does not contain any entire complex affine lines. Since all bounded domains are $\mathbb{C}$-proper; thus, for a bounded convex domain Ω, the Kobayashi metric K_Ω of Ω is a complete length metric. Therefore, $(\Omega,\ K_\Omega)$ is a geodesic space.

2.4 Quasi-geodesic

Definition 2.8 Suppose that $(X,\ \rho)$ is a metric space and $I \subset \mathbb{R}$ is an interval. A map $\sigma : I \to X$ is called a *geodesic* if for all $s,\ t \in I$,

$$\rho(\sigma(s),\ \sigma(t)) = |t - s|.$$

For $\lambda \geq 1$ and $\kappa \geq 0$, a curve $\sigma : I \to \Omega$ is called a (λ, κ)-*quasi-geodesic*, if for all $s,\ t \in I$,

$$\frac{1}{\lambda}|t - s| - \kappa \leq \rho(\sigma(s),\ \sigma(t)) \leq \lambda|t - s| + \kappa.$$

In particular if $\kappa = 0$, it is called a $(\lambda,\ 0)$-quasi-geodesic or λ-quasi-geodesic.

2.5 Uniformly Squeezing Property

Following Liu et al. [18,19], a domain $\Omega \subset \mathbb{C}^n$ with $n \geq 2$ is said to be *holomorphic homogeneous regular* (HHR) or *uniformly squeezing* (USq), if there exists $s > 0$ with the following property: for every $z \in \Omega$, there exists a holomorphic embedding $\phi : \Omega \to \mathbb{C}^n$ with $\phi(z) = 0$ and

$$B_s(0) \subset \phi(\Omega) \subset B_1(0),$$

where $B_1(0) \subset \mathbb{C}^n$ is the unit ball.

Examples of USq domains include

(1) $T_{g,n}$, the Teichmüller space of hyperbolic surfaces with genus $g > 1$ and n punctures;
(2) bounded convex domains [12];
(3) strongly pseudoconvex domains [9,10].

It was shown in [18,19,27] that in a HHR/USq domain Ω, the Carathéodory metric, Kobayashi metric, Bergman metric and Kähler-Einstein metric of Ω are bilipschitzly equivalent to each other.

2.6 Gromov Product and Gromov Hyperbolicity

Definition 2.9 Let $(X,\ \rho)$ be a metric space. Given three points $x,\ y,\ o \in X$, the *Gromov product* of $x,\ y$ with respect to o is given by

$$(x|y)_o = \frac{1}{2}\Big(\rho(x,\ o) + \rho(o,\ y) - \rho(x,\ y)\Big).$$

A proper geodesic metric space $(X,\ \rho)$ is called *Gromov hyperbolic* (or δ-hyperbolic), if there exists $\delta \geq 0$ such that, for all $o,\ x,\ y,\ z \in X$,

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta.$$

By the triangle inequality, we know that

$$(x|y)_o \leq \rho(o, [x, y]),$$

where $[x,\ y]$ is a geodesic connecting x and y in $(X,\ \rho)$. Moreover, if X is Gromov hyperbolic, then

$$|(x|y)_o - \rho(o, [x, y])| \leq \delta' \tag{6}$$

for some $\delta' > 0$.

Note that the large-scale behavior of quasi-geodesics in Gromov hyperbolic spaces mimics that of geodesics rather closely.

Theorem 2.10 (Stability of quasi-geodesics, p. 401, [8]). *For all $\delta > 0,\ \lambda \geq 1$ and $\epsilon > 0$, there exists a constant $R = R(\delta, \lambda, \epsilon)$ with the following property:*

If X is a δ-hyperbolic geodesic space, γ is a $(\lambda,\ \epsilon)$-quasi-geodesic in X and $[x,\ y]$ is a geodesic segment joining the endpoints of γ, then the Hausdorff distance between $[x,\ y]$ and the image of γ is no more than R.

Now we introduce the definition of rough quasi-isometric maps as follows.

Definition 2.11 Let $f : X \to Y$ be a map between metric spaces X and Y, and let $L \geq 1$ and $M \geq 0$ be constants.

(1) If for all $x,\ y \in X$,

$$L^{-1}d_X(x, y) - M \leq d_Y(f(x), f(y)) \leq Ld_X(x, y) + M,$$

then f is called an (L, M)-*roughly quasi-isometric map* (cf. [5]). If $L = 1$, then f is called an M-*roughly isometric*.

(2) Moreover, if f is a homeomorphism and $M = 0$, then it is called an L-*bilipschitz* or L-*quasi-isometry*.

The following result states that Gromov hyperbolicity is preserved under rough quasi-isometries.

Theorem 2.12 [8, p. 402] *Let X and X' be geodesic metric spaces and $f : X \to X'$ be a rough quasi-isometry. If X is Gromov hyperbolic, then X' is also Gromov hyperbolic.*

Remark 2.13 In [1], Balogh and Bonk proved that every bounded strongly pseudoconvex domain in $\mathbb{C}^n (n \geq 2)$ equipped with the Kobayashi metric is Gromov hyperbolic. Recently, Zimmer [28] demonstrated that smooth bounded convex domains in $\mathbb{C}^n (n \geq 2)$ equipped with the Kobayashi metrics are Gromov hyperbolic if and only if they are of finite type.

3 *m*-Convex Domains

In this section, we will investigate the geometric properties of the Kobayashi geodesics on m-convex domains. We first prove Lemma 1.6 via the idea from [6,14], the differences between the Kobayashi metric and the quasihyperbolic metric necessitate some changes in the proof.

In order to prove Lemma 1.6, we need to verify the following result.

Lemma 3.1 *Suppose that Ω is a bounded m-convex domain in $\mathbb{C}^n (n \geq 2)$, and that $\gamma \subset \Omega$ is a λ-quasi-geodesic in the Kobayashi metric K_Ω connecting y_1 and y_2 with $\lambda \geq 1$. Then for any $\alpha > 2m^2 - m$, there exists a constant $\tilde{C} > 0$ such that, for every $\omega = \gamma(t) \in \gamma$,*

$$\delta_\Omega(\omega) \geq \tilde{C}\, (l_d(\gamma|[0,\ t]) \wedge l_d(\gamma|[t,\ 1]))^\alpha\ .$$

Remark 3.2 This result tells us that for any curve γ' connecting $\gamma|[0, t]$ and $\gamma|[t, 1]$, we always have

$$B(\omega, \delta_\Omega^{\frac{1}{\alpha}}(\omega)/\tilde{C}) \cap \gamma' \neq \emptyset.$$

Proof Put

$$D = \max_{z \in \gamma} \delta_\Omega(z).$$

For $i = 1,\ 2$, let N_i denote the unique integer such that

$$\frac{D}{2^{N_i+1}} \leq \delta_\Omega(y_i) < \frac{D}{2^{N_i}}.$$

For $k = 0,\ 1,\ \ldots,\ N_1$, let x_k^1 be the first point on γ with

$$\delta_\Omega(x_k^1) = \frac{D}{2^k}$$

when a point travels from y_1 towards y_2.

Similarly, we can define x_k^2 for $k = 0, 1, \ldots, N_2$ with travel direction from y_2 to y_1. By using points x_k^1 and x_k^2 together with the end points y_1 and y_2, we can divide γ into N_1+N_2+3 non-overlapping (modulo end points) subcurves γ_ν, $\nu \in [-N_1-1, N_2+1]$. Note that a curve containing one end point of γ, as well as the middle subcurve between x_0^1 and x_0^2, may degenerate. All subcurves γ_ν are Kobayashi λ-quasi-geodesics between their respective end points, and

$$\begin{aligned} \delta_\Omega(z) &\leq \frac{D}{2^{|\nu|-1}}, \text{ if } z \in \gamma_\nu, \\ \delta_\Omega(z) &\geq \frac{D}{2^{|\nu|}}, \text{ if } z \text{ is one end point of } \gamma_\nu. \end{aligned} \tag{7}$$

It, thus, follows from (5),(7) and the definition of the quasi-geodesic that there exists a constant $A > 2$ such that

$$l_k(\gamma_\nu) \leq \lambda \log\left(1 + A\frac{2^{|\nu|}}{D} l_d(\gamma_\nu)\right). \tag{8}$$

And by (2) and (3), we have

$$l_k(\gamma_\nu) \geq \frac{l_d(\gamma_\nu)}{2C\left(\frac{D}{2^{|\nu|-1}}\right)^{\frac{1}{m}}} = \frac{2^{\frac{|\nu|-1}{m}}}{2CD^{\frac{1}{m}}} l_d(\gamma_\nu),$$

where C is the constant in Definition 2.1. It is easy to see that, for any $N \in \mathbb{N}$, there exists $C(N) > 0$ such that

$$\log(1+x) \leq C(N)x^{1/N},$$

for $x \geq 0$. Then for all $N > m$, clearly

$$\frac{2^{\frac{|\nu|-1}{m}}}{2CD^{\frac{1}{m}}} l_d(\gamma_\nu) \leq l_k(\gamma_\nu) \leq \lambda C(N) A^{\frac{1}{N}} \frac{2^{\frac{|\nu|}{N}}}{D^{\frac{1}{N}}} l_d^{\frac{1}{N}}(\gamma_\nu), \tag{9}$$

which implies that

$$l_d(\gamma_\nu) \leq C' \left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{m}-\frac{1}{N}}{1-\frac{1}{N}}}, \tag{10}$$

where $C' = \left(2^{1+\frac{1}{m}}\lambda C(N) A^{\frac{1}{N}} C\right)^{\frac{1}{1-1/N}}$. Hence, if $\gamma(t) \in \gamma_\nu$,

$$l_d(\gamma|[0,t]) \wedge l_d(\gamma|[t,1]) \leq C' \sum_{j \geq |\nu|} \left(\frac{D}{2^j}\right)^{\frac{\frac{1}{m}-\frac{1}{N}}{1-\frac{1}{N}}} \leq 2C' \left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{m}-\frac{1}{N}}{1-\frac{1}{N}}}. \tag{11}$$

Moreover, by formulas (8) and (10), we obtain

$$l_k(\gamma_\nu) \leq \log\left(1 + 2AC'\left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{m}-1}{1-\frac{1}{N}}}\right).$$

Therefore, we only need to estimate $\delta_\Omega(\omega)$ for $\omega = \gamma(t) \in \gamma_\nu$.

Let x be one end point of γ_ν. By the estimate (4), we conclude that

$$K_\Omega(x, \omega) \geq \frac{1}{2}\left|\log\frac{\delta_\Omega(x)}{\delta_\Omega(\omega)}\right|.$$

Therefore, it follows from Definition 2.1 that

$$\frac{1}{2}\log\frac{\delta_\Omega(x)}{\delta_\Omega(\omega)} \leq K_\Omega(x, \omega) \leq l_k(\gamma_\nu) \leq \log\left(1 + 2AC'\left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{m}-1}{1-\frac{1}{N}}}\right).$$

This guarantees that

$$\frac{\delta_\Omega(x)}{\delta_\Omega(\omega)} \leq \left(1 + 2AC'\left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{m}-1}{1-\frac{1}{N}}}\right)^2,$$

and

$$\delta_\Omega(\omega) \geq \frac{\delta_\Omega(x)}{C''\left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{m}-1}{1-\frac{1}{N}}\cdot 2}} \geq \frac{1}{C''}\left(\frac{D}{2^{|\nu|}}\right)^{1-\frac{2-2m}{1-1/N}},$$

where $C'' = (4AC')^2$. Hence by (11), it follows that

$$\delta_\Omega(\omega) \geq \tilde{C}\left(l_d(\gamma|[0, t]) \wedge l_d(\gamma|[t, 1])\right)^\alpha, \tag{12}$$

where

$$\alpha = \frac{2m - 1 - \frac{1}{N}}{\frac{1}{m} - \frac{1}{N}} \quad \text{and} \quad \tilde{C} = \frac{C''}{2C'}.$$

By taking $N \to \infty$, we have

$$\alpha \to 2m^2 - m.$$

Therefore, for any $\alpha > 2m^2 - m$, there exists $\tilde{C}$ such that (12) holds.

This completes the proof. □

The following result is a direct consequence of the estimate (4).

Lemma 3.3 *Let Ω be a bounded convex domain in $\mathbb{C}^n$ with $n \geq 2$ and $\omega_0 \in \Omega$. Then there exists $K > 0$ such that the Kobayashi metric*

$$K_\Omega(z, \omega_0) \geq \frac{1}{2} \log \frac{1}{\delta_\Omega(z)} - K. \tag{13}$$

By using Lemma 3.1, we are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality, we may assume that $\mathrm{diam}(\Omega) \leq 1$ by scaling Ω. Fix $\omega \in \Omega$. By (5) and (13), it follows that the Gromov product $(x|y)_\omega$ satisfies

$$\begin{aligned} 2(x|y)_\omega &= K_\Omega(x, \omega) + K_\Omega(y, \omega) - K_\Omega(x, y) \\ &\geq \frac{1}{2} \log \frac{1}{\delta_\Omega(x)} + \frac{1}{2} \log \frac{1}{\delta_\Omega(y)} - \frac{1}{2} \log \frac{(\sqrt{\delta_\Omega(x)\delta_\Omega(y)} + A|x-y|)^2}{\delta_\Omega(x)\delta_\Omega(y)} - 2K \\ &= \frac{1}{2} \log \frac{1}{\delta_\Omega(x)\delta_\Omega(y) + |x-y|(2A\sqrt{\delta_\Omega(x)\delta_\Omega(y)} + A^2|x-y|)} - 2K \\ &\geq \frac{1}{2} \log \frac{1}{\delta_\Omega(x)\delta_\Omega(y) + (A^2+2A)|x-y|} - 2K, \end{aligned}$$

where A is the constant in Lemma 2.6 such that (5) holds.

In order to estimate the Euclidean length of the geodesic $[x, y]$, there are two cases to consider

Case a $|x-y| \geq (\delta_\Omega(x)\delta_\Omega(y))^2$. Hence,

$$\begin{aligned} (x|y)_\omega &\geq \frac{1}{4} \log \frac{1}{(A+1)^2|x-y|^{\frac{1}{2}}} - K \\ &\geq \frac{1}{8} \log \frac{1}{|x-y|} - K', \end{aligned}$$

where $K' = K - \frac{1}{4} \log \frac{1}{(A+1)^2}$. By the definition of the Gromov product, it follows that

$$K_\Omega(\omega, [x, y]) \geq (x|y)_\omega \geq \frac{1}{8} \log \frac{1}{|x-y|} - K'. \tag{14}$$

Thus, by Lemma 2.6, for any $z \in [x, y]$, we see that there exists $K'' > 0$ such that

$$\frac{1}{2} \log \frac{1}{\delta_\Omega(z)} + K'' \geq K_\Omega(\omega, z) \geq \frac{1}{8} \log \frac{1}{|x-y|} - K'.$$

Choosing a point $z \in [x, y]$ with

$$l_d([x, z]) = l_d([z, y]) = \frac{1}{2} l_d([x, y]),$$

then by Lemma 3.1, we now have

$$l_d([x,y]) \le \left(\frac{2e^{K'+K''}}{\tilde{C}}\right)^{\frac{1}{\alpha}} |x-y|^{\frac{1}{4\alpha}}.$$

Therefore, **Case a** is proved.

Case b $|x-y| \le (\delta_\Omega(x)\delta_\Omega(y))^2$. By $\operatorname{diam}(\Omega) < 1$, it follows from Lemma 2.4 that

$$k_\Omega(z; v) \ge \frac{|v|}{2\delta_\Omega(z)} \ge \frac{|v|}{2}.$$

Thus,

$$\begin{aligned}\frac{1}{2} l_d([x,y]) \le l_k([x,y]) = K_\Omega(x,y) &\le \log\left(1 + A\frac{|x-y|}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right)\\ &\le \log(1 + A|x-y|^{\frac{1}{2}})\\ &\le A|x-y|^{\frac{1}{2}},\end{aligned} \tag{15}$$

which implies that

$$l_d([x,y]) \le c_1|x-y|^{c_2},$$

where

$$c_1 = \left(\frac{2e^{K'+K''}}{\tilde{C}}\right)^{\frac{1}{\alpha}} \vee 2A \quad \text{and} \quad c_2 = \frac{1}{4\alpha} < \frac{1}{8m^2 - 4m}.$$

This completes the first part of the proof.

For the second part, we need only a minor modification of (14) and (15).

Assume that $(\Omega,\ K_\Omega)$ is Gromov hyperbolic (δ-hyperbolic). Then it follows from Theorem 2.10 that the Kobayashi Hausdorff distance between $[x,\ y]$ and the image of γ is no more than $R = R(\delta,\ \lambda)$. Thus, we take

$$K_\Omega(\omega, \gamma) + R \ge K_\Omega(\omega, [x,y]) \ge (x|y)_\omega \ge \frac{1}{8}\log\frac{1}{|x-y|} - K'$$

instead of (14) and take

$$\frac{1}{2\lambda} l_d(\gamma) \le \frac{1}{\lambda} l_k(\gamma) \le l_k([x,y]) = K_\Omega(x,y) \le \log\left(1 + A\frac{|x-y|}{\delta_\Omega(x)\wedge\delta_\Omega(y)}\right)$$

instead of (15). Furthermore, noting that these changes make no difference on c_2, we complete the proof of the second part. $\square$

Remark 3.4 Suppose that Ω is a bounded m-convex complex domain. For any two points $x,\ y \in \Omega$, there exists a complex geodesic which contains $x,\ y$ in its image. Due to a well-known result of Hardy and Littlewood, any complex geodesic in Ω extends continuously to its boundary (see [20]).

Conversely, Mercer [20] proved that for any two points $x, y \in \overline{\Omega}$, there is a complex geodesic whose continuous extension contains $\{x, y\}$ in its image. Thus, the first part of Theorem 1.5 also holds for $x,\ y \in \overline{\Omega}$.

4 Strongly Pseudoconvex Domains

In this part, we will establish a similar result for strongly pseudoconvex domains.

Theorem 4.1 *Let Ω is a bounded strongly pseudoconvex domain with C^2 smooth boundary. Then for any $c_2 < \frac{1}{16}$ and $\lambda > 1$, there exists $c_1 > 0$ such that $\forall x,\ y \in \Omega$,*

$$l_d(\gamma) \leq c_1|x-y|^{c_2}$$

where γ is a λ-quasi-geodesic joining x and y.

At first, we need some auxiliary results.

Lemma 4.2 (Lemma 4.1, [1]) *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n (n \geq 2)$ with C^2-smooth boundary. There exists $C > 0$ such that for any $x,\ y \in \Omega$,*

$$K_\Omega(x, y) \geq \frac{1}{2}\left|\log\frac{\delta_\Omega(x)}{\delta_\Omega(y)}\right| - C. \tag{16}$$

The estimate also shows that $(\Omega,\ K_\Omega)$ is complete and, thus, it is a geodesic space.

Lemma 4.3 (Proposition 1.2, [1]) *Let Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n (n \geq 2)$ with C^2-smooth boundary, Then, for every $\epsilon > 0$, there exists $\epsilon_0 > 0$ and $C \geq 0$ such that, for all $z \in N_{\epsilon_0}(\partial\Omega) \cap \Omega$ and all $v \in \mathbb{C}^n$,*

$$\begin{aligned}\left(1 - C\delta_\Omega^{1/2}(z)\right)\left(\frac{|v_N|^2}{4\delta_\Omega^2(z)} + (1-\epsilon)\frac{L_\rho(\pi(z); v_H)}{\delta_\Omega(z)}\right)^{1/2} &\leq K(z; v)\\ \leq \left(1 + C\delta_\Omega^{1/2}(z)\right)\left(\frac{|v_N|^2}{4\delta_\Omega^2(z)} + (1+\epsilon)\frac{L_\rho(\pi(z); v_H)}{\delta_\Omega(z)}\right)^{1/2}&,\end{aligned} \tag{17}$$

where $\pi : \Omega \to \partial\Omega$ is a map satisfying $|x - \pi(x)| = \delta_\Omega(x)$, v_H is in the complex tangential plane $H_{\pi(z)}\partial\Omega$, v_N in the complex one-dimensional subspace orthogonal to $H_{\pi(z)}\partial\Omega$ and $v = z_H + v_N$.

By using Lemma 4.3, we immediately obtain that, for some $C_1 > 0$,

$$k_\Omega(z; v) \geq C_1\frac{|v|}{\delta_\Omega^{\frac{1}{2}}(z)}. \tag{18}$$

If $z \in N_{\epsilon_0}(\partial\Omega) \cap \Omega$, then (17) obviously implies (18). For those $z \in \Omega \backslash N_{\epsilon_0}(\partial\Omega)$, we obtain

$$k_\Omega\left(z, \frac{v}{|v|}\right) \geq \delta_0 > 0$$

for some $\delta_0 > 0$ and $\delta_\Omega(z) > \epsilon_0$. Thus, we obtain the inequality (18) for some $C_1 > 0$.

The following result is similar to Lemma 3.1, which can be viewed as the (separation property) geometric characteristic of the Kobayashi (quasi-) geodesic.

Lemma 4.4 *Suppose that Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 2)$ with C^2-smooth boundary. And suppose that γ is a Kobayashi λ-quasi-geodesic joining y_1 and y_2 with $\lambda \geq 1$. Then for any $\alpha > 4$, there exists a constant $\tilde{C} > 0$ such that, for every $\omega = \gamma(t) \in \gamma$,*

$$\delta_\Omega(\omega) \geq \tilde{C}(l_d(\gamma|[0,\ t]) \wedge l_d(\gamma|[t,\ 1]))^\alpha. \tag{19}$$

Proof Put

$$D = \max_{z \in [y_1, y_2]} \delta_\Omega(z).$$

For $i = 1, 2$, let N_i be the unique integer such that

$$\frac{D}{2^{N_i+1}} \leq \delta_\Omega(y_i) < \frac{D}{2^{N_i}}.$$

Define x_k^1, x_k^2 and γ_ν as in the proof of Lemma 3.1 and

$$\begin{aligned} &\delta_\Omega(z) \leq \frac{D}{2^{|\nu|-1}},\ \text{if } z \in \gamma_\nu, \\ &\delta_\Omega(z) \geq \frac{D}{2^{|\nu|}},\ \text{if } z \text{ is one end point of } \gamma_\nu. \end{aligned} \tag{20}$$

It, thus, follows from (5) and (20) and the definition of λ-quasi-geodesic that there exists a constant $A > 0$ such that

$$l_k(\gamma_\nu) \leq \lambda \log\left(1 + A\frac{2^{|\nu|}}{D} l_d(\gamma_\nu)\right), \tag{21}$$

where A is the constant from Lemma 2.6 such that (5) holds.

By the estimate (18), we have

$$l_k(\gamma_\nu) \geq \frac{C_1 l_d(\gamma_\nu)}{\left(\frac{D}{2^{|\nu|-1}}\right)^{\frac{1}{2}}} = \frac{C_1 2^{\frac{|\nu|-1}{2}}}{D^{\frac{1}{2}}} l_d(\gamma_\nu).$$

It is easy to see that for any $N \in \mathbb{N}$, there exists $C(N) > 0$ such that

$$\log(1+x) \leq C(N)x^{\frac{1}{N}},$$

for $x \geq 0$. Thus, if we take $N > 2$, we have

$$\frac{2^{\frac{|\nu|-1}{2}}}{C_1 D^{\frac{1}{2}}} l_d(\gamma_\nu) \leq l_k(\gamma_\nu) \leq \lambda C(N) A^{\frac{1}{N}} \frac{2^{\frac{|\nu|}{N}}}{D^{\frac{1}{N}}} l_d^{\frac{1}{N}}(\gamma_\nu),$$

which implies that

$$l_d(\gamma_\nu) \leq C' \left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{2}-\frac{1}{N}}{1-\frac{1}{N}}}, \tag{22}$$

where $C' = (2^{\frac{1}{2}} \lambda C(N) A^{\frac{1}{N}} C_1)^{\frac{N}{N-1}}$. Therefore, if $\gamma(t) \in \gamma_\nu$,

$$l_d(\gamma|[0,\ t]) \wedge l_d(\gamma|[t,\ 1]) \leq C' \sum_{j \geq |\nu|} \left(\frac{D}{2^j}\right)^{\frac{\frac{1}{2}-\frac{1}{N}}{1-\frac{1}{N}}} \leq 2C' \left(\frac{D}{2^{|\nu|}}\right)^{\frac{\frac{1}{2}-\frac{1}{N}}{1-\frac{1}{N}}}. \tag{23}$$

Moreover, by formulas (21) and (22), we get

$$l_k(\gamma_\nu) \leq \log\left(1 + 2AC' \left(\frac{D}{2^{|\nu|}}\right)^{\frac{-\frac{1}{2}}{1-\frac{1}{N}}}\right).$$

Suppose x is one end point of γ_ν. Denoting $\omega = \gamma(t) \in \gamma_\nu$, the estimation $\delta_\Omega(\omega)$ consists of two cases. With the constant C in Lemma 4.2, we have

Case I $\frac{1}{2}|\log \frac{\delta_\Omega(x)}{\delta_\Omega(\omega)}| \leq NC$. Hence,

$$\delta_\Omega(\omega) \geq e^{-2NC} \delta_\Omega(x) = e^{-2NC} \frac{D}{2^{|\nu|}}.$$

Case II $\frac{1}{2}|\log \frac{\delta_\Omega(x)}{\delta_\Omega(\omega)}| > NC$. By Lemma 4.2, we obtain

$$K_\Omega(x,\omega) \geq \frac{1}{2}\left|\log \frac{\delta_\Omega(x)}{\delta_\Omega(\omega)}\right| - C \geq \frac{N-1}{2N}\left|\log \frac{\delta_\Omega(x)}{\delta_\Omega(\omega)}\right|,$$

which implies that

$$\frac{N-1}{2N} \log \frac{\delta_\Omega(x)}{\delta_\Omega(\omega)} \leq K(x,\omega) \leq l_k(\gamma_\nu) \leq \log\left(1 + 2AC' \left(\frac{D}{2^{|\nu|}}\right)^{\frac{-\frac{1}{2}}{1-\frac{1}{N}}}\right).$$

Thus, it follows that

$$\frac{\delta_\Omega(x)}{\delta_\Omega(\omega)} \le \left(1 + 2AC'\left(\frac{D}{2^{|\nu|}}\right)^{\frac{-\frac{1}{2}}{1-\frac{1}{N}}}\right)^{\frac{2N}{N-1}}$$

and

$$\delta_\Omega(\omega) \ge \frac{1}{C''}\left(\frac{D}{2^{|\nu|}}\right)^{1+\frac{N^2}{(N-1)^2}},$$

where $C'' = (4AC')^{\frac{2N}{N-1}}$. Now (23) implies that

$$\delta_\Omega(\omega) \ge \tilde{C}\left(l_d(\gamma|[0,t]) \wedge l_d(\gamma|[t,1])\right)^\alpha, \tag{24}$$

where

$$\tilde{C} = \frac{C'' \wedge e^{-2NC}}{C'} \quad \text{and} \quad \alpha = \left(1 + \frac{N^2}{(N-1)^2}\right) \cdot \frac{1-\frac{1}{N}}{\frac{1}{2}-\frac{1}{N}}.$$

Noting that $\lim_{N\to\infty} \alpha = 4$, thus for any $\alpha > 4$, there exists $\tilde{C} > 0$ such that (24) holds, which completes the proof. □

We remark that the proof of Theorem 4.1 follows almost the same line as the proof of Theorem 1.5. We need only a minor modification for the estimate of the Kobayashi metric. For the sake of completeness, we present its simple proof here.

The proof of Theorem 4.1. Scaling domain as necessary, we assume without loss of generality that $\mathrm{diam}(\Omega) \le 1$. Fix a point $\omega \in \Omega$. By using (16), we obtain

$$K_\Omega(z,\omega) \ge \frac{1}{2}\log\frac{1}{\delta_\Omega(z)} - K,$$

for some $K > 0$. Then by (5), we deduce that the Gromov product $(x|y)_\omega$ satisfies

$$\begin{aligned} 2(x|y)_\omega &= K_\Omega(x,\ \omega) + K_\Omega(y,\ \omega) - K_\Omega(x,\ y) \\ &\ge \frac{1}{2}\log\frac{1}{\delta_\Omega(x)\delta_\Omega(y) + (A^2+2A)|x-y|} - 2K, \end{aligned}$$

where A is the constant from Lemma 2.6 such that (5) holds.

Next, to estimate the Euclidean length of γ, we consider two cases:

Case A $|x-y| \geq (\delta_\Omega(x)\delta_\Omega(y))^2$. By the assumption, it follows that

$$\begin{aligned}(x|y)_\omega &\geq \frac{1}{4}\log\frac{1}{(A+1)^2|x-y|^{\frac{1}{2}}} - K\\ &\geq \frac{1}{8}\log\frac{1}{|x-y|} - K',\end{aligned}$$

where $K' = K - \frac{1}{4}\log\frac{1}{(A+1)^2}$. By the definition of the Gromov product, we obtain

$$K_\Omega(\omega, [x, y]) \geq (x|y)_\omega \geq \frac{1}{8}\log\frac{1}{|x-y|} - K',$$

where $[x,\ y]$ denotes a Kobayashi geodesic connecting x and y in Ω. Now it follows from Theorem 2.10 and Remark 2.13 that, there exists $R > 0$ such that

$$K_\Omega(\omega, [x, y]) \leq K_\Omega(\omega, \gamma) + R.$$

Thus by (5) for any $z \in \gamma$, there exists $K'' > 0$ such that

$$\frac{1}{2}\log\frac{1}{\delta_\Omega(z)} + K'' \geq K_\Omega(\omega, z) \geq \frac{1}{8}\log\frac{1}{|x-y|} - K'.$$

Take a point $z = \gamma(t)$ with

$$l_d(\gamma|[0, t]) = l_d(\gamma|[t, 1]) = \frac{1}{2}l_d(\gamma).$$

Therefore, Lemma 4.4 gives

$$l_d(\gamma) \leq \left(\frac{2e^{K'+K''}}{\tilde{C}}\right)^{\frac{1}{\alpha}} |x-y|^{\frac{1}{4\alpha}}.$$

Case B $|x-y| \leq (\delta_\Omega(x)\delta_\Omega(y))^2$. By the estimate (18) and the fact that $\operatorname{diam}(\Omega) \leq 1$, we obtain that, for any $z \in \Omega$ and $0 \neq v \in \mathbb{C}^n$,

$$k_\Omega(z; v) \geq C_1|v|,$$

which implies that

$$\begin{aligned}\frac{C_1}{\lambda}l_d(\gamma) \leq \frac{1}{\lambda}l_k(\gamma) \leq K_\Omega(x, y) &\leq \log\left(1 + A\frac{|x-y|}{\delta_\Omega(x)\wedge\delta_\Omega(y)}\right)\\ &\leq \log(1 + A|x-y|^{\frac{1}{2}})\\ &\leq A|x-y|^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$l_d([x, y]) \le c_1|x - y|^{c_2},$$

where

$$c_1 = \left(\frac{2e^{K'+K''}}{\tilde{C}}\right)^{\frac{1}{\alpha}} \vee \frac{A\lambda}{C_1} \quad \text{and} \quad c_2 = \frac{1}{4\alpha}.$$

This completes the proof. □

Recall that bounded convex domains and strongly pseudoconvex domains are both uniformly squeezing (see Sect. 2.5 for the precise definition). Therefore, the Kobayashi metric, Bergman metric, Carathéodory metric, and Kähler-Einstein metric are bilipschitzly equivalent to each other. Thus, we have

Corollary 4.5 *Suppose that Ω is a bounded domain in $\mathbb{C}^n (n \ge 2)$ and that Ω satisfies either*

(a) *Ω is m-convex domain with Dini-smooth boundary; or*
(b) *Ω is strongly pseudoconvex with C^2-smooth boundary.*

Then there exists $c_1, c_2 > 0$ such that, $\forall x, y \in \Omega$,

$$l_d(\gamma) \le c_1|x - y|^{c_2},$$

where γ is a λ-quasi-geodesic from x to y with respect to the metric ϱ_Ω, and ϱ_Ω is one of the Kobayashi metric, Bergman metric, Carathéodory metric and Kähler-Einstein metric of Ω.

Proof Let γ be a λ-quasi-geodesic joining x and y with respect to one of the Bergman metric, Carathéodory metric, and Kähler-Einstein metric of Ω. Then by using the fact recorded in Sect. 2.5, we know that γ is also a $(C\lambda)$-quasi-geodesic for the Kobayashi metric for some $C > 0$. Thus, we can complete the proof by using Theorems 1.5 and 4.1. □

5 Proof of Theorem 1.2

By Remark 3.4, for any $x, y \in \overline{\Omega}$, there exists a geodesic joining x and y; thus, we can write $[x, y]$ for $x, y \in \overline{\Omega}$. Now we prove the Theorem by the results proved in Sect. 3.

First, we need the following result which characterizes the distance from a fixed point to the geodesic $[x, y]$.

Lemma 5.1 *Let Ω be a bounded m-convex domain in $\mathbb{C}^n (n \ge 2)$ with Dini-smooth boundary. Suppose $diam(\Omega) < 1$, then for any $x \in \overline{\Omega}$, $y \in \partial\Omega$ and x, y close to each* *other,*

$$K_\Omega(\omega,\ [x,\ y]) \asymp \log \frac{1}{|x-y|}, \tag{25}$$

Here, we write $f \asymp g$ for two functions if there exists a constant $C \geq 1$ such that $(1/C)f \leq g \leq Cf$.

Proof Fix a point $\omega \in \Omega$. On one hand, choose a point $z \in [x,\ y]$ such that

$$l_d([x,\ z]) = l_d([z,\ y]).$$

Then Lemma 2.6 implies that there exists $K_1 > 0$ such that

$$K_\Omega(\omega, z) \leq \frac{1}{2}\log\frac{1}{\delta_\Omega(z)} + K_1.$$

Lemma 3.1 implies

$$\begin{aligned} K_\Omega(\omega, z) &\leq \frac{\alpha}{2}\log\frac{1}{l_d([x, y])} + \frac{1}{2}\log\frac{2^\alpha}{\tilde{C}} + K_1 \\ &\leq \frac{\alpha}{2}\log\frac{1}{|x-y|} + \frac{1}{2}\log\frac{2^\alpha}{\tilde{C}} + K_1. \end{aligned}$$

Since $\mathrm{diam}(\Omega) < 1$, there exists $C > 0$ such that

$$K_\Omega(\omega, [x, y]) = K_\Omega(\omega, z) \leq C\log\frac{1}{|x-y|}.$$

On the other hand, choose a point $z \in [x,\ y]$ such that $K_\Omega(\omega,\ [x,\ y]) = K_\Omega(\omega,\ z)$. By Theorem 1.5,

$$\begin{aligned} K_\Omega(\omega, z) \geq \frac{1}{2}\log\frac{\delta_\Omega(\omega)}{\delta_\Omega(z)} &\geq \frac{1}{2}\log\frac{\delta_\Omega(\omega)}{l_d([x, y]) + \delta_\Omega(x)\wedge\delta_\Omega(y)} \\ &\gtrsim \log\frac{1}{c_1|x-y|^{c_2}}. \end{aligned} \tag{26}$$

Thus, we have

$$K_\Omega(\omega, [x, y]) \asymp \log\frac{1}{|x-y|}.$$

$\square$

Remark 5.2 By the Triangle inequality,

$$(x|y)_\omega \leq K_\Omega(\omega, [x, y]) \leq \log\frac{1}{|x-y|}.$$

on the other hand, the Gromov hyperbolicity implies

$$|(x|y)_\omega - K_\Omega(\omega, [x, y])| \le \delta,$$

which means that: if $(\Omega,\ K_\Omega)$ is also Gromov hyperbolic, then

$$(x|y)_\omega \asymp \log \frac{1}{|x-y|}.$$

***Proof of Theorem* 1.2** Fix $\omega \in \Omega_1$. Since f is a isometry, and the continuity of Kobayashi metric, we know that f is a homeomorphism. Also, for any $x,\ y \in \Omega_1$, we have

$$K_{\Omega_2}(f(\omega),\ f([x,\ y])) = K_{\Omega_1}(\omega,\ [x,\ y])$$

and $f([x,\ y])$ is a geodesic between $f(x)$ and $f(y)$. Take two sequences $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$ in Ω_1 such that $x_k \to \xi \in \partial\Omega_1$ and $y_k \to \eta \in \partial\Omega$. Just repeat the proof of Lemma 5.1 for x_k and y_k, only note that $\delta_{\Omega_1}(x_k) \wedge \delta_{\Omega_1}(y_k) \to 0$ and $\delta_{\Omega_2}(f(x_k)) \wedge \delta_{\Omega_2}(f(y_k)) \to 0$ in (26), it is easy to see that if $\xi = \eta$, then $|f(x_k) - f(y_k)| \to 0$ which means f extends continuously to the boundary.

Thus by Lemma 5.1, when ξ, η close enough, we have

$$K_{\Omega_1}(\omega, [\xi, \eta]) \asymp \log \frac{1}{|\xi - \eta|},$$

and

$$K_{\Omega_2}(f(\omega), f([\xi, \eta])) \asymp \log \frac{1}{|f(\xi) - f(\eta)|}.$$

Thus, by combining these two inequalities, we know that there exists $C_1 > 1,\ 1 > C_2 > 0$ such that

$$\frac{1}{C_1}|\xi - \eta|^{\frac{1}{C_2}} \le |f(\xi) - f(\eta)| \le C_1|\xi - \eta|^{C_2},$$

which means that f not only extends to a homeomorphism on $\overline{\Omega}_1$ but also bi-Hölder continuous on $\partial\Omega_1$. □

6 Boundary Correspondence and Extensions of Maps

The aim of this section is to show Proposition 1.8 and Corollary 1.12. First, we prove the bi-Hölder equivalence between the Euclidean boundary and the Gromov boundary on certain complex domains. Second, we use this boundary correspondence to obtain some extension results not only for biholomorphisms but also for more general rough quasi-isometries with respect to the Kobayashi metrics between the domains.

We begin with some necessary definitions and auxiliary results concerning Gromov hyperbolic geometry and morphisms between their boundaries at infinity.

Let (X, ρ) be a δ-hyperbolic space. Fix a base point o in X. Recall that the Gromov product of x, y with respect to o is

$$(x|y)_o = \frac{1}{2}\left(\rho(x, o) + \rho(y, o) - \rho(x, y)\right).$$

(1) A sequence $\{x_i\}$ in X is called a *Gromov sequence* if $(x_i|x_j)_o \to \infty$ as $i, j \to \infty$.
(2) Two such sequences $\{x_i\}$ and $\{y_j\}$ are said to be *equivalent* if $(x_i|y_i)_o \to \infty$ as $i \to \infty$.
(3) The *Gromov boundary* $\partial_G X$ of X is defined to be the set of all equivalence classes of Gromov sequences, and $\overline{X}^G = X \cup \partial_G X$ is called the *Gromov closure* of X.
(4) For $a \in X$ and $b \in \partial_G X$, the Gromov product $(a|b)_o$ of a and b is defined by

$$(a|b)_o = \inf\left\{\liminf_{i\to\infty}(a|b_i)_o : \{b_i\} \in b\right\}.$$

(5) For $a, b \in \partial_G X$, the Gromov product $(a|b)_o$ of a and b is defined by

$$(a|b)_o = \inf\left\{\liminf_{i\to\infty}(a_i|b_i)_o : \{a_i\} \in a \text{ and } \{b_i\} \in b\right\}.$$

We recall the following basic results about the Gromov product on the Gromov closure $\overline{X}^G$.

Proposition 6.1 (Lemma 5.11, [25]) *Let X be a δ-hyperbolic space, o, $z \in X$, and $\xi, \xi' \in \partial_G X$. Then for any sequences $\{y_i\} \in \xi$, $\{y_i'\} \in \xi'$, we have*

(1) $(z|\xi)_o \le \liminf_{i\to\infty}(z|y_i)_o \le \limsup_{i\to\infty}(z|y_i)_o \le (z|\xi)_o + \delta$;
(2) $(\xi|\xi')_o \le \liminf_{i\to\infty}(y_i|y_i')_o \le \limsup_{i\to\infty}(y_i|y_i')_o \le (\xi|\xi')_o + 2\delta$.

The next result is known as the standard estimate on Gromov hyperbolic spaces (see for instance (3.2) in [6]). If X is a proper geodesic δ-hyperbolic space, then for all $x, y, w \in X$, we have

$$\left|(x|y)_w - \rho(w, [x, y])\right| \le 8\delta, \tag{27}$$

where $[x, y]$ is a geodesic with end points x and y.

Note that if (X, ρ) is proper and geodesic, then its Gromov boundary is also equivalent to the geodesic boundary(cf. [8]). Here, the geodesic boundary is defined as the set of all equivalence classes of geodesic rays, where two geodesic rays are said to be *equivalent* if they have finite Hausdorff distance (cf. [8]).

For a Gromov hyperbolic space X, one can define a class of *visual metrics* on $\partial_G X$ via the extended Gromov products, see [5,8]. For any metric ρ_G in this class, there exist a parameter $\epsilon > 0$ and a base point $w \in X$ such that

$$\rho_G(a, b) \asymp \exp\left(-\epsilon(a|b)_w\right), \quad \text{for } a, b \in \partial_G X. \tag{28}$$

Recall that $f \asymp g$ for two functions if there exists a constant $C \geq 1$ such that $(1/C)f \leq g \leq Cf$. Any two metrics d_1 and d_2 in the canonical class are called *snowflake equivalent*, i.e., the identity map $\mathrm{id} : (\partial_G X,\ d_1) \to (\partial_G X,\ d_2)$ is a snowflake map. Note that a homeomorphism $\phi : (X_1, d_1) \to (X_2, d_2)$ between two metric spaces is said to be *snowflake* if there exist $\lambda,\ \kappa > 0$ such that, for any $x,\ y \in X_1$,

$$(1/\lambda)d_1(x, y)^\kappa \leq d_2(\phi(x), \phi(y)) \leq \lambda d_1(x, y)^\kappa.$$

Now we recall the definitions of Hölder and power quasisymmetric mappings as follows. A homeomorphism $\phi : (X_1,\ d_1) \to (X_2,\ d_2)$ between two metric spaces is said to be *Hölder* if there exist $\lambda,\ \kappa > 0$ such that, for any $x,\ y \in X_1$,

$$d_2(\phi(x),\ \phi(y)) \leq \lambda d_1(x,\ y)^\kappa.$$

Moreover, ϕ is called *bi-Hölder* if there exist $\lambda \geq 1, 0 < \alpha \leq 1$ such that, for any $x, y \in X_1$,

$$(1/\lambda)d_1(x,\ y)^{1/\alpha} \leq d_2(\phi(x),\ \phi(y)) \leq \lambda d_1(x, y)^\alpha.$$

For a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary, we know that the mapping between the Carnot-Carathéodory metric and the Euclidean metric on $\partial\Omega$ are bi-Hölder equivalent (See [1] and the references given there for more information). That is,

$$C_1|p - q| \leq d_H(p,\ q) \leq C_2|p - q|^{1/2}, \quad \text{for } p,\ q \in \partial\Omega.$$

Thus, the visual metric of $\partial_G\Omega$ and the Euclidean metric of $\partial\Omega$ are bi-Hölder equivalent to each other.

Definition 6.2 Let $\phi : (X_1,\ d_1) \to (X_2,\ d_2)$ be a homeomorphism between metric spaces, and $\lambda \geq 1, \kappa > 0$ be constants.

If for all distinct points $x,\ y,\ z \in X_1$,

$$\frac{d_2(\phi(x), \phi(z))}{d_2(\phi(x), \phi(y))} \leq \eta_{\kappa,\lambda}\left(\frac{d_1(x, z)}{d_1(x, y)}\right),$$

then ϕ is called a (κ, λ)-*power quasisymmetry*. Here, we have used the notation

$$\eta_{\kappa,\lambda}(t) = \begin{cases} \lambda t^{1/\kappa} & \text{for } 0 < t < 1, \\ \lambda t^\kappa & \text{for } t \geq 1. \end{cases}$$

It is easy to see that every snowflake mapping is bi-Hölder. By carefully checking the proof of Theorem 6.15 in [24], we obtain the following result:

Proposition 6.3 *A power quasisymmetry between two bounded metric spaces is bi-Hölder.*

Also, we need an auxiliary result for our later use.

Proposition 6.4 (Section 6, [5]) *Suppose that $f : X \to Y$ is a rough quasi-isometry between two geodesic Gromov hyperbolic spaces X and Y. Then f sends every Gromov sequence in X to a Gromov sequence in Y, and f induces a power quasisymmetric boundary mapping $\tilde{f}_G : \partial_G X \to \partial_G Y$, where $\partial_G X$ and $\partial_G Y$ are equipped with certain visual metrics.*

Recently Zimmer proved that

Theorem 6.5 (Corollary 7.2, [29]) *If Ω is a bounded convex domain in $\mathbb{C}^n$ $(n \geq 2)$ and $(\Omega,\ K_\Omega)$ is Gromov hyperbolic, then Ω is m-convex for some $m \geq 1$.*

Now we are ready to show Proposition 1.8. For the convenience of the reader, we restate the propositions as follows:

Proposition 6.6 *Suppose that Ω is a bounded domain in $\mathbb{C}^n$ $(n \geq 2)$ and satisfies either*

(a) *Ω is convex with Dini-smooth boundary and $(\Omega,\ K_\Omega)$ is Gromov hyperbolic; or*
(b) *Ω is strongly pseudoconvex with C^2-smooth boundary.*

Then the identity map $id : \Omega \to \Omega$ extends to a bi-Hölder homeomorphism of the boundaries

$$id : (\partial\Omega,\ |\cdot|) \to (\partial_G\Omega,\ \rho_G)$$

(for simplicity of notation, here use the same notation), where ρ_G is a certain visual metric on the Gromov boundary of the domain $(\Omega,\ K_\Omega)$ (see (28)).

Proof We first record some auxiliary results for later use. On the one hand, it follows from Theorems 1.5, 6.5, and 4.1 that in both cases (a) and (b), there are constants $c_1,\ c_2 > 0$ such that, for any $x,\ y \in \Omega$,

$$l_d([x, y]) \leq c_1|x - y|^{c_2}, \tag{29}$$

where $[x, y]$ is a Kobayashi geodesic joining x and y in Ω.

On the other hand, by using Lemmas 1.6 and 4.4 , we see that there exist constants $\tilde{C},\ \alpha > 0$ such that, for every $u \in [x,\ y]$,

$$\delta_\Omega(u) \geq \tilde{C}\left(l_d([x,\ u]) \wedge l_d([u,\ y])\right)^\alpha . \tag{30}$$

Next we want to show that the identity map extends to a bijection between the Euclidean boundary of Ω and the Gromov boundary of the space (Ω, K_Ω). That is to say, a sequence in Ω is a Gromov sequence if and only if it converges to certain boundary point in the Euclidean metric.

To this end, fix a point $\omega \in \Omega$. For any Gromov sequences $\{x_k\}, \{y_k\} \subset \Omega$ with $(x_k|y_k)_\omega \to \infty$ as $n \to \infty$. For each k, connect x_k and y_k by a Kobayashi geodesic $[x_k, y_k]$ in Ω. Then choose a point $z_k \in [x_k, y_k]$ such that

$$l_d([x_k, z_k]) = l_d([z_k, y_k]).$$

Lemma 2.6 implies that there exists $K_1 > 0$ such that

$$K_\Omega(\omega, z_k) \leq \frac{1}{2} \log \frac{1}{\delta_\Omega(z_k)} + K_1, \tag{31}$$

which implies that

$$(x_k|y_k)_\omega \leq K_\Omega(\omega, z_k) \leq \frac{1}{2} \log \frac{1}{\delta_\Omega(z_k)} + K_1.$$

Hence,

$$\delta_\Omega(z_k) \to 0 \quad \text{as } k \to \infty.$$

Now, by applying (30) to $[x_k, \ y_k]$, we obtain that

$$|x_k - y_k| \leq l_d([x_k, y_k]) \leq \frac{2}{\tilde{C}} \delta_\Omega^{\frac{1}{\alpha}}(z_k) \to 0, \tag{32}$$

as desired.

On the other hand, for every $x \in \partial\Omega$, choose a sequences $\{x_k\}, \{y_k\} \subset \Omega$ with $x_k \to x$ and $y_k \to x$ as $k \to \infty$. For each k, again we may join x_k and y_k by a Kobayashi geodesic $[x_k, \ y_k]$ in Ω. Choose a point $u_k \in [x_k, \ y_k]$ such that

$$K_\Omega(\omega, z_k) = K_\Omega(\omega, [x_k, y_k]).$$

Now, applying (29) to $[x_k, \ y_k]$, we conclude that

$$\begin{aligned} \delta_\Omega(z_k) &\leq \frac{1}{2} l_d([x_k, y_k]) + \delta_\Omega(x_k) \vee \delta_\Omega(y_k) \\ &\leq \frac{c_1}{2} |x_k - y_k|^{c_2} + \delta_\Omega(x_k) \vee \delta_\Omega(y_k). \end{aligned} \tag{33}$$

By using the estimates (13) and (16) in both cases (a) and (b), it follows that there exists $K_2 > 0$ such that

$$K_\Omega(\omega, z_k) \geq \frac{1}{2} \log \frac{1}{\delta_\Omega(z_k)} - K_2. \tag{34}$$

Since $|x_k - y_k| \to 0$ and $\delta_\Omega(x_k), \ \delta_\Omega(y_k) \to 0$, we have $K_\Omega(\omega, [x_k, y_k]) \to \infty$ as $k \to \infty$. Then we deduce from the standard estimate (27) that

$$(x_k|y_k)_\omega \to \infty \quad \text{as} \quad k \to \infty.$$

Therefore, we have proved that the identity map extends to a bijection between the Euclidean boundary of Ω and the Gromov boundary of the space $(\Omega, \ K_\Omega)$.

Now we are ready to show this boundary mapping $id : (\partial\Omega,\ |\cdot|) \to (\partial_G\Omega,\ \rho_G)$ is bi-Hölder continuous, where ρ_G is a visual metric on the Gromov boundary of $(\Omega,\ K_\Omega)$ with parameter $\varepsilon > 0$ and base point ω (refer to (28)).

Only note that the Gromov hyperbolicity of $(\Omega,\ K_\Omega)$ implies that

$$e^{-(x|y)_\omega} \asymp e^{-K_\Omega(\omega,[x,y])}.$$

Thus, we complete the proof by repeating the arguments in the proof of Lemma 5.1. □

Note the bilipschitz equivalence of the canonical metrics on Ω. Denoting by ϱ_Ω one of the Kobayashi metric, Bergman metric, Carathéodory metric, and Kähler-Einstein metric on Ω, we obtain

Corollary 6.7 *Suppose that Ω is a bounded domain in $\mathbb{C}^n (n \geq 2)$ and satisfies either*

(a) *Ω is convex with Dini-smooth boundary and $(\Omega,\ \varrho_\Omega)$ is Gromov hyperbolic; or*
(b) *Ω is strongly pseudoconvex with C^2-smooth boundary.*

Then the identity map $id : \Omega \to \Omega$ extends to a bi-Hölder homeomorphism

$$id : (\partial\Omega,\ |\cdot|) \to (\partial_G\Omega,\ \rho),$$

where ρ is certain visual metric on the Gromov boundary of $(\Omega,\ \varrho_\Omega)$.

Proof Note that, by using the fact recorded in Sect. 2.5, the Kobayashi metric K_Ω is bilipschitzly equivalent to ϱ_Ω under the identity map. Then by Theorem 2.12, if Ω is Gromov hyperbolic with respect to the metric ϱ_Ω, then it is also Gromov hyperbolic with respect to the Kobayashi metric K_Ω.

From Propositions 6.4 and 6.3 , it follows that the identity map

$$id : (\Omega,\ K_\Omega) \to (\Omega,\ \varrho_\Omega)$$

extends to a bi-Hölder homeomorphism between the Gromov boundary of $(\Omega,\ K_\Omega)$ and the Gromov boundary of $(\Omega,\ \varrho_\Omega)$ with respect to the visual metrics.

Moreover, it follows from Proposition 1.8 that the identity map extends to a bi-Hölder homeomorphism between the Euclidean boundary of Ω and the Gromov boundary of $(\Omega,\ K_\Omega)$.

Therefore, the conclusion follows easily from these results and the fact that the composition of bi-Hölder mappings is bi-Hölder as well. □

Finally, we conclude this section by showing Theorem 1.12.

Proof of Corollary 1.12. At first, one observes from Propositions 6.4 and 6.3 that

$$f : \left(\Omega_1, K_{\Omega_1}\right) \to \left(\Omega_2, K_{\Omega_2}\right)$$

extends to a homeomorphism such that every sequence $\{x_n\}$ in $(\Omega_1,\ K_{\Omega_1})$ is Gromov if and only if the image sequence $\{f(x_n)\}$ in $(\Omega_2,\ K_{\Omega_2})$ is Gromov. Moreover, the induced mapping

$$\widetilde{f} : \partial_G \Omega_1 \to \partial_G \Omega_2$$

is bi-Hölder when the Gromov boundaries $\partial_G \Omega_i$ of (Ω_i, K_{Ω_i}) are endowed with their visual metrics for $i = 1, 2$.

For each $i = 1, 2$, then by Proposition 1.8, the identity map $id_i : \Omega_i \to \Omega_i$ extends as a homeomorphism such that a sequence in Ω_i is a Gromov sequence if and only if it converges to a point in $\partial \Omega_i$. And the induced mapping

$$id_i : (\partial \Omega_i, |\cdot|) \to (\partial_G \Omega_i, \rho_i)$$

is bi-Hölder, where ρ_i is a visual metric on the Gromov boundary of (Ω_i, K_{Ω_i}).

Thus, we get a well-defined boundary mapping

$$\overline{f} = id_2^{-1} \circ \tilde{f} \circ id_1 : \partial \Omega_1 \to \partial \Omega_2$$

such that $\{x_n\}$ in Ω_1 converges to a point in $\partial \Omega_1$ if and only if $\{f(x_n)\}$ in Ω_2 converges to a point in $\partial \Omega_2$. This shows that $\overline{f}$ is the corresponding continuous extension mapping by f, which is a homeomorphism. Clearly, $\overline{f}$ is bi-Hölder because the composition of bi-Hölder mappings is also bi-Hölder, which completes the proof. □

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