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Martin boundary covers Floyd boundary

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Abstract For a random walk on a finitely generated group G we obtain a generalization of a classical inequality of Ancona. We deduce as a corollary that the identity map on G extends to a continuous equivariant surjection from the Martin boundary to the Floyd boundary, with preimages of conical points being singletons. This provides new results for Martin compactifications of relatively hyperbolic groups.

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1 Introduction

1.1 The main results

It is a common thread in geometric group theory to relate asymptotic properties of random walks on a group to the dynamics of its action on some geometric boundary.

Every probability measure μ on G determines a random walk on G. Assume that μ is such a measure whose support generates G as a semigroup. The Green (pseudo-)metric $d_{\mathcal{G}}$ (not necessarily symmetric) is defined to be minus the logarithm of the probability that a sample path starting at the first point ever reaches the second [2]. The Busemann (horospheric) compactification $\overline{G}_{\mathcal{M}}$ of G with respect to $d_{\mathcal{G}}$ is called the *Martin compactification* of G, and its remainder $\partial_{\mathcal{M}} G = \overline{G}_{\mathcal{M}} \backslash G$ is called the *Martin boundary* (see Sect. 2 for more details).

The geometric object that we consider is the *Floyd compactification* of G [12]. The Floyd metric δ_o^f at a basepoint $o \in G$ is obtained by rescaling the word metric d by a suitable scalar function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ (called a *Floyd function*). The Cauchy completion $\overline{G}_f$ of the Cayley graph of G equipped with the metric δ_o^f is called the *Floyd compactification* of G and $\partial_f G = \overline{G}_f \backslash G$ is its *Floyd boundary*. The restrictions imposed on the Floyd function f imply that $\overline{G}_f$ is compact and that left multiplication extends to a convergence action of G on $\overline{G}_f$ by homeomorphisms (see Sect. 3 for more details).

One of the main results of the paper is the following inequality which relates the probabilistic metric $d_{\mathcal{G}}$ to the geometric metric δ_o^f.

Theorem 1.1 *Let G be a finitely generated group and f a Floyd function on G. Let μ be a probability measure on G whose finite support generates G as a semigroup. Let $d_{\mathcal{G}}$ be the Green metric associated to μ.*

Then there exists a decreasing function $A : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\forall x, w, y \in G$ one has:

$$d_{\mathcal{G}}(x, w) + d_{\mathcal{G}}(w, y) \leq d_{\mathcal{G}}(x, y) + A(\delta_w^f(x, y)). \tag{1}$$

The finite support condition on the measure μ can be relaxed using the techniques developed by Gouëzel in [23]. To do it we need a few more assumptions.

Let the norm $||g||$ denote the length of the minimal word representing g in a fixed finite system of generators of G. A measure μ has *exponential moment* if

$$\sum_{g \in G} c^{||g||} \mu(g) < \infty$$

for some constant $c > 1$ and *superexponential moment* if the above series converges for all $c > 1$.

We say that the Floyd function f is *of order greater than* 2 if $r^{2+\iota} f(r) \to 0$ as $r \to \infty$ for some fixed $\iota > 0$.

Theorem 1.2 *Let μ be a probability measure with an infinite support, generating G as a semigroup. Assume that μ has a superexponential moment. Then the inequality* (1) *holds for a Floyd function f of order greater than* 2.

It is shown in [23] that Theorem 1.2 already fails in the case when G is a free group if μ is only assumed to have exponential moment.

We now provide a short history of the problem. An analog of the inequality (1) in the context of word hyperbolic groups is due to A. Ancona [1] and it states that there exists a constant C such that one has

$$d_{\mathcal{G}}(x, w) + d_{\mathcal{G}}(w, y) \leq d_{\mathcal{G}}(x, y) + C, \qquad \text{(Ancona)}$$

for all points x, w, y lying in this order on a geodesic in the word metric of the Cayley graph. Ancona also deduced from this inequality that the Martin compactification of G is equivariantly homeomorphic to the Gromov compactification of G.

The Ancona inequality reflects the hyperbolic nature of the metric $d_{\mathcal{G}}$ and it has sparked a fruitful line of research (see e.g. [3,22–24,28]). Ancona's original proof used the theory of elliptic operators. A different proof using elementary probability and hyperbolic geometry was given by Gouëzel and Lalley [24] for surface groups and Gouëzel [22] for general hyperbolic groups. The generalization to infinitely supported measures with superexponential moment (again in the hyperbolic group setting) was obtained by Gouëzel in [23].

There are several essential differences between the inequality (1) and the Ancona inequality. Unlike the function $A(\cdot)$, the constant C in the Ancona inequality is a uniform constant (depending on the hyperbolicity constant of the group). On the other hand, in the inequality (1) the points $\{x, y, z\}$ are arbitrary and do not necessarily belong to the same geodesic. Furthermore, in Theorem 1.1 we do not assume that the group is hyperbolic.

We also note that Ancona's theorem is valid for any hyperbolic graph and that the group action is not needed. To illustrate the interest of Theorem 1.1 we provide below a short proof that it implies the inequality (Ancona) for hyperbolic groups. Before we get to that let us mention some further issues motivating this work.

It turns out that Theorems 1.1 or 1.2 follow from the following statement (see the proof of Proposition 2.4) which is of independent interest.

Theorem 1.3 *Suppose that G, μ and f satisfy the assumptions of Theorem* 1.1 *or* 1.2. *For every $\varepsilon \in (0, 1)$ there exists a decreasing function*

$R_\varepsilon : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *such that the probability* $P_{x,y}$ *that a random path from* x *to* y *passes through a ball in the word metric centered at* w *of radius* $R_\varepsilon(\delta_w^f(x, y))$ *is at least* $1 - \varepsilon$.

The probability $P_{x,y}$ mentioned in the Theorem refers to the measures on the set of finite trajectories between the vertices x and y (see definition (4) in the next section).

Another inequality, related to our discussion, is due to Karlsson [29]. Let G be a finitely generated group and f a Floyd rescaling function. Then there exists a decreasing function $K : \mathbb{R} \to \mathbb{R}$ such that for every $x, y, v \in G$ and every geodesic segment $[x, y]$ joining x and y in the Cayley graph equipped with the word distance d one has

$$d(v, [x, y]) \leq K(\delta_v^f(x, y)). \tag{Karlsson}$$

One can also restate Karlsson's inequality in the following form affirming that with the probability equal to one Theorem 1.3 holds if one replaces the sample paths by geodesics:

Karlsson's lemma [29, Lemma 2.1]. *For every* $\varepsilon > 0$ *there exists* $R = R(\varepsilon)$ *such that the condition* $\delta_v^f(x, y) > \varepsilon$ *implies that* $d(v, [x, y]) \leq R$ *for every geodesic* $[x, y]$ *joining* x *and* y.

The Karlsson inequality admits many corollaries for relatively hyperbolic groups (see [17,18,20,21]). It was one of our initial motivations to relate the Martin and Floyd compactifications.

Finally let us recall the classical Gromov inequality for geodesic δ-hyperbolic spaces:

$$d(z, [x, y]) - \delta \leq (x._z y) \leq d(z, [x, y]), \tag{Gromov}$$

where $(x._z y)$ is the Gromov product $\frac{1}{2}(d(z, x)+d(z, y)-d(x, y))$ [26].

Note that the left-hand side of the Gromov inequality does not hold in general if the Cayley graph is not hyperbolic.

Theorem 1.1 has multiple consequences. One of them is a simple proof of the Ancona theorem for hyperbolic groups which we provide now.

Corollary 1.4 *If* G *is a hyperbolic group then the inequality* (1) *implies Ancona's inequality.*

Proof Assume that the group G is δ-hyperbolic and the inequality (1) holds. We apply it for the Floyd function $f(n) = e^{-an}$ $(n \in \mathbb{N})$ where a is a constant which is specified below. We need to show that for every three colinear points

x, w, y belonging to a word geodesic, the Floyd distance $\delta^f_w(x, y)$ is uniformly bounded below from zero. Since the Floyd distance is invariant under left multiplication in G (see Sect. 3) we can assume that $w = o$ is the basepoint in the Cayley graph.

The visual Gromov distance $\nu(x, y)$ defined on a hyperbolic graph is bilipschitz equivalent to $e^{-a\cdot(x.{}_o y)}$ [26]. By the (Gromov) inequality above it is also bilipschitz equivalent to $e^{-a\cdot d(o,[x,y])}$. Furthermore if $0 < a < a_0$ for some uniform constant a_0 (depending only on the hyperbolicity constant δ) the latter property extends to the Gromov boundary of the graph [25].

Using an equivalent definition of ν as a shortpath metric we obtain[1] that

$$\delta^f_o(x, y) \asymp_C \nu(x, y) \asymp_C e^{-a\cdot d(o,[x,y])}$$

where $A \asymp_C B$ means the double inequality $1/C \cdot B \leq A \leq C \cdot B$ between the quantities A and B for a uniform constant $C > 0$ depending only on δ and a. Since the points x, o, y are colinear, the above property implies that the distance $\delta^f_o(x, y)$ is bounded below by a uniform positive constant. The Corollary is proved. $\square$

Remarks. The above proof suggests that the inequality (1) applies when one can show that the Floyd distance $\delta^f_y(x, z)$ is bounded below by a strictly positive constant. This fact will be crucial for all further applications of our main results.

Since the functions $A(\cdot)$ and $R(\cdot)$ in Theorems 1.1 and 1.3 are both decreasing, then once the relevant statement is proved for a Floyd function f then it is also true for a Floyd function h if $h(r) \leq f(r)$ $(\forall r \in \mathbb{R}_{>0})$. This fact will be also used further on.

One of the main applications of the inequality (1) is our next result which relates two actions of a finitely generated group G: one on the Martin boundary $\partial_{\mathcal{M}} G$ associated to (G, μ) and the second one on the Floyd boundary $\partial_f G$. These actions are of a different nature, in particular the second one is a *convergence* action [29, Proposition 3], which is not always the case with the first one.

Recall that an action of $G \curvearrowright T$ on a compactum T is called convergence if the induced action of G on the space of distinct triples $\Theta^3(T)$ is discontinuous [6].[2] A point $x \in T$ is called *conical* if there exists a sequence $g_n \in G$ such that $g_n y \to b$ for all $y \in T\backslash\{x\}$ and $g_n x \to a$ such that $a \neq b$. The conical points are quite typical for convergence actions.

The title of the paper is explained by the following.

[1] The proof of the first part for hyperbolic groups is folklore and follows from different sources (e.g. [25,26,40]); a complete proof can be found in [35, Proposition A1, Appendix].

[2] if T contains at most two points then $\Theta^3(T) = \emptyset$ and the action is convergence by definition.

Theorem 1.5 ((Theorem 7.3 and Corollary 7.14) *Let G, μ and f be as in Theorem 1.1 or 1.2. Then the identity map* $\mathrm{id} : G \to G$ *extends to a continuous G-equivariant surjection*

$$\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f. \tag{2}$$

Moreover, the preimage by π of any conical point of $\partial_f G$ is a single point.

The subset $\partial_{\mathcal{M}}^{min} G$ of points of $\partial_{\mathcal{M}} G$ corresponding to minimal harmonic functions (see Sect. 7) is called the *minimal Martin boundary*.

Theorem 7.3 has several applications. One of them is the following Corollary describing the minimal G-invariant subsets of the Martin boundary.

Corollary 1.6 (Proposition 7.15 and Corollary 7.16). *Suppose that the hypotheses of Theorem 1.5 hold. Assume that the Floyd boundary $\partial_f G$ of G with respect to a rescaling function f contains at least three points. Then $\partial_{\mathcal{M}}^{min} G$ is contained in the closure of the G-orbit $\Xi = G\xi$ in $\partial_{\mathcal{M}} G$ for any $\xi \in \partial_{\mathcal{M}} G$.*

In particular, the set $\{\pi^{-1}(p) : p$ is a conical point in $\partial_f G\}$ is a dense subset of $\partial_{\mathcal{M}}^{min} G$.

Most of the other applications of our results deal with the class of relatively hyperbolic groups. Let us recall several main notions of this theory (see Sect. 3 for more details).

Let $G \curvearrowright T$ be a convergence action of a group G on a compactum T [14]. The action is called *minimal* if T is a minimal compactum (under inclusion) invariant under the group action. The action is *non-elementary* if it is minimal and T contains infinitely many points. The set ΛG of the accumulation points of the G-orbit is called the *limit set.* If the action is non-elementary then its limit set is the minimal closed subset invariant under the action and one has $T = \Lambda G$.

A point $p \in T$ is called *bounded parabolic* if the fixed-point set of the stabilizer $H_p = \{g \in G : gp = p\}$ is one point p and H_p acts cocompactly on $T \backslash \{p\}$. The subgroup H_p is then called *maximal parabolic*.

An action $G \curvearrowright T$ is called *geometrically finite* if either T contains at most two points or it is a minimal convergence action such that every point of T is either conical or bounded parabolic. A group G is said to be *hyperbolic relative to a system $\mathcal{P}$ of subgroups* (or simply *relatively hyperbolic*) if it admits a geometrically finite action on a compactum T such that the set of the maximal parabolic subgroups coincides with $\mathcal{P}$. Once the system $\mathcal{P}$ is fixed the compactum T is unique up to an equivariant homeomorphism [42] and is called the *Bowditch boundary* and denoted by $\partial_B G$. In particular if $G \curvearrowright T$ is geometrically finite and $\mathcal{P} = \emptyset$ then G is hyperbolic, the compactum T is equivariantly homeomorphic to the Gromov boundary ∂G and every point of T is conical [5].

The Floyd distance and related Floyd compactification of the Cayley graph of a finitely generated group was first introduced by Floyd [12]. He also proved that for a geometrically finite discrete (Kleinian) subgroup G of the isometry group of the hyperbolic space $\mathbb{H}^3$ of dimension 3 the identity map id : $G \to G$ extends to a surjective, equivariant and continuous map from the Floyd boundary to its limit set $T = \Lambda G \subset \mathbb{S}^2$. Since then this map has been called the *Floyd* map. Gerasimov proved [18] that if a finitely generated group G admits a geometrically finite action by homeomorphisms on a compactum T then there exists the Floyd map: $\varphi : \partial_f G \to T$. One of the corollaries of this result is that the Floyd boundary $\partial_f G$ of a relatively hyperbolic group G which admits a non-elementary geometrically finite action is an infinite set. The preimage $\varphi^{-1}(x)$ of every point $x \in T$ admits a complete description: for a conical point x it is a single point [18, Proposition 7.5.2]; and if $x = p \in T$ is a parabolic then $\varphi^{-1}(p)$ coincides with the Floyd boundary $\partial_f H_p$ of its stabilizer H_p [20, Corollary 7.8] for some Floyd function f. Note that T contains at most countably many non-conical points if the action $G \curvearrowright T$ is minimal and geometrically finite [17].

Composing the map π from Theorem 1.5 with the Floyd map φ above we obtain:

Corollary 1.7 *Assume that μ and f are as in Theorem 1.1 or 1.2. If the action $G \curvearrowright (T = \partial_B G)$ is geometrically finite then there exists a continuous G-equivariant surjective map*

$$\psi = \varphi \circ \pi : \partial_{\mathcal{M}} G \to \partial_B G.$$

Furthermore the preimage $\psi^{-1}(x)$ consists of one point for every conical $x \in \partial_B G$.

In the context of geometrically finite actions of groups on hyperbolic spaces our Theorem 1.1 can be formulated in a form very close to Ancona's original inequality. An action of a group G by isometries on a proper geodesic Gromov hyperbolic metric space (X, d_X) is geometrically finite if it is discontinuous on X and the action on the limit set $\Lambda G \subset \partial X$ is geometrically finite. We also say that the action is *non-elementary* if the limit set ΛG is an infinite set.

We fix a basepoint $o \in X$ and denote by $[x, y] \subset X$ a geodesic between two points x and y in X. We have the following.

Corollary 1.8 *(Proposition 9.3) Let $G \curvearrowright X$ be a geometrically finite, isometric and non-elementary action of a group G on a proper geodesic Gromov hyperbolic space X. Let μ and f as in Theorem 1.1 or 1.2.*

Then for every $D > 0$ there exists a constant $C = C(D) > 0$ such that for every triple g, h, w of elements of G with $d_X(ho, [go, wo]) \leq D$ the inequality:

$$d_{\mathcal{G}}(g, h) + d_{\mathcal{G}}(h, w) \leq d_{\mathcal{G}}(g, w) + C$$

holds on the Cayley graph of G.

Since the preimage of a conical point by the map ψ is a single point, the main problem in describing the Martin boundary $\partial_{\mathcal{M}} G$ is to describe the preimage of a parabolic fixed point. In general this problem remains largely open. However in [11] the authors use Theorem 1.1 as a crucial ingredient to give a precise description of the Martin boundary $\partial_{\mathcal{M}} G$ when G is relatively hyperbolic with respect to a system of virtually abelian subgroups. In particular, it is shown in [11] that in this case $\overline{G}_{\mathcal{M}}$ is obtained from $\overline{G}_f$ by replacing the parabolic fixed points by the spheres $\mathbb{S}^{d-1}$ where d is the rank of the stabilizer of the corresponding parabolic fixed point (*parabolic blow-up construction*).

Denote by $\partial^{\mathcal{M}} H$ the set of accumulation points of a subgroup H of G in $\overline{G}_{\mathcal{M}}$. In the following result we describe the subset of minimal points of the preimage of a bounded parabolic point.

Proposition 1.9 *(Proposition 9.6) Let $G \curvearrowright T$ be a minimal geometrically finite action on a compactum T, and $\pi : \partial_{\mathcal{M}} G \to \partial_f G$ be the map from Theorem 1.5. Let $H < G$ be the stabilizer of a bounded parabolic fixed point $p \in T$. Then*

$$\pi^{-1}(p) \cap \partial_{\mathcal{M}}^{min} G \subseteq \partial^{\mathcal{M}} H. \tag{3}$$

Theorems 1.1 and 1.5 have also several applications to the theory of harmonic measures on boundaries of hyperbolic spaces which we will now briefly mention. These results were originally included in the previous version of our preprint [15], however keeping in mind that they are valid in a different context we decided to put them in a separate paper. There are two natural classes of measures on the Gromov boundary ∂X associated with the action. One consists of quasiconformal, or Patterson-Sullivan measures and the other consists of stationary or harmonic measures, which are limits of convolution powers of measures on G. As a consequence of Theorems 1.1 and 1.5 we obtain that if G contains at least one parabolic subgroup then the harmonic and Patersson-Sullivan measures are singular [15, Theorem 11.3]. We note that the result is already new in the case of rank 1 symmetric spaces.

Another application concerns geometrically finite actions on Riemannian manifolds of negative curvature bounded away from 0 or more general

CAT(-1) spaces. If X is such a space and G is a group acting geometrically finitely on X (or equivalently on ∂X) then our result affirms that every stationary probability measure on ∂X can be extended to a product measure on the unit tangent bundle of X and which projects to a finite measure on its G-quotient [15, Theorem 10.4].

1.2 Organization of the paper

We will now briefly describe the sections of the paper and their dependence.

In Sect. 2 we recall several standard notions concerning random walks on groups. Using these notions we prove there two technical statements needed further on. First we prove Proposition 2.3 which estimates the probability that a sample path between given two points is sufficiently long. Then we show (Proposition 2.4) that Theorem 1.3 implies the inequality (1), and thus Theorems 1.1 and 1.2 depending on whether the measure μ is of finite support or of infinite support with superexponential moment. The further strategy consists of proving Theorem 1.3.

In Sect. 3 we provide some background information about the Floyd compactification and convergence actions.

Sections 4 and 5 are devoted to the proof of Theorem 1.3 in the case of a non-amenable group. Section 4 contains the proof of geometric Proposition 4.3 which is the main tool to prove Theorem 1.3. In Sect. 5 we show how to deduce Theorem 1.3 from Proposition 4.3 in the case when the support of a measure is finite. This will finish the proof of Theorem 1.1 in the case when the group is non-amenable.

In Sect. 6 we treat the case of infinitely supported measures. In Sect. 6.1 we obtain the proof of Theorem 1.2 for a non-amenable group using the superexponential moment condition. Section 6.2 deals with the case of an amenable group.

Starting with Sect. 7 we obtain different applications of the inequality (1). In Sect. 7.1 we construct the continuous and equivariant map $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$. Then in Sect. 7.2 we prove that the π-preimage of every conical point in $\partial_f G$ contains points from the minimal Martin boundary $\partial^{min}_{\mathcal{M}} G$ (Corollary 7.10); we then deduce that this preimage consists in fact of a single point (Corollary 7.14). This concludes the proof of Theorem 1.5. We finish this section with few further applications. The first is given in Sect. 7.3 where we prove Corollary 1.6 describing minimal G-invariant closed subsets of the Martin boundary. In Sect. 7.4 we apply the obtained results to describe a link with the end compactification of a finitely generated group.

In Sect. 8 we prove Proposition 8.1 giving a partial description of the Martin boundary of quasiconvex subgroups of G. This Proposition will be applied in the next section to prove Proposition 1.9 describing the preimages of parabolic fixed points when G is relatively hyperbolic.

In Sect. 9 composing the map π from Theorem 1.5 with the Floyd map φ we obtain Corollary 1.8 (Proposition 9.3) which gives an analog of the original Ancona inequality for geometrically finite actions on Gromov hyperbolic spaces.

2 Random walks on groups

Let G be a finitely generated infinite group. We fix a finite symmetric generating system S of G and let $\Gamma = Cay(G, S)$ be its Cayley graph. We equip the graph Γ with the word distance d and fix a basepoint $o \in \Gamma$. Whenever we need to consider the norm $||g||$ of an element $g \in G$ we will identify the basepoint o of Γ with the neutral element of G and put $||g|| = d(o, g)$.

Let μ be a probability measure on G whose support generates G as a semigroup. This defines a G-invariant Markov chain on G with n step transition probabilities $p_n(x, y) = \mu^{*n}(x^{-1}y)$.

Recall that μ has finite support if

$$\mathsf{supp}(\mu) = \{g \in G : \mu(g) > 0\}$$

is a finite set. We say μ has exponential (resp. superexponential) moment if

$$\sum_{g \in G} c^{||g||}\mu(g) < \infty$$

for some (resp. for all) $c > 1$. We define the reflected measure by $\hat{\mu}(g) = \mu(g^{-1})$.

The measure μ is said to be symmetric if $\hat{\mu} = \mu$. A trajectory τ of length n, denoted by length(τ), is a sequence $g_0, ..., g_{n-1}$ of elements of G. Such a trajectory is said to have jump size bounded by K if $d(g_i, g_{i+1}) \leq K$ for all i, if the jump size is equal to 1 the trajectory is called a *path*.

A trajectory

$$\tau = g_0, g_1, ..., g_n$$

in G is called admissible if $\mu(g_i^{-1} g_{i+1}) > 0$ for each i. Note, if μ has finite support, an admissible trajectory has jump size bounded by $K = \max_{g \in \mathsf{supp}(\mu)} ||g||$.

Given an admissible trajectory τ, its weight is defined to be

$$w(\tau) = \mu(g_0^{-1} g_1)\mu(g_1^{-1} g_2)...\mu(g_{n-1}^{-1} g_n).$$

Let $\mathsf{Traj}(x, y)$ denote the set of all admissible trajectories in G which begin at x and end at y. Let $\mathsf{Traj}_r(x, y) \subset \mathsf{Traj}(x, y)$ consist of trajectories of length r. The Green function associated to μ is defined as

$$\mathcal{G}(x, y) = \sum_{\tau \in \mathsf{Traj}(x,y)} w(\tau).$$

The random walk is said to be *transient* if the probability of ever returning to the start point is less than 1. In this case, $\mathcal{G}(x, y) < \infty$ for all $x, y \in G$; in the opposite case the random walk is called *recurrent* [40].

By work of Varopoulos [39, Theorem 4.6], if there is a measure μ on G whose support generates G as a semigroup and the random walk is recurrent, then G is either finite or contains $\mathbb{Z}$ or $\mathbb{Z}^2$ as a finite index subgroup. We will from now on assume that the random walk is transient.

For a subset $V \subset \mathsf{Traj}(x, y)$ let $\mathcal{G}(V) = \sum_{\tau \in V} w(\tau)$ be the total weight of trajectories in V. For each $x, y \in G$ one can define a probability measure $P_{x,y}$ on the set $\mathsf{Traj}(x, y)$ of trajectories from x to y as follows

$$P_{x,y}(V) = \frac{\mathcal{G}(V)}{\mathcal{G}(x, y)}, \quad V \subset \mathsf{Traj}(x, y). \tag{4}$$

For $U \subset G$ let $\mathcal{G}(x, y, U)$ be the total weight of trajectories from x to y whose interior is contained in U. For a real number t define

$$\mathcal{G}(x, y|t) = \sum_{n=0}^{\infty} t^n p^n(x, y) \tag{5}$$

It is easy to see that $\mathcal{G}(., .|t)$ is G equivariant, i.e.

$$\mathcal{G}(gx, gy|t) = \mathcal{G}(x, y|t)$$

for all $x, y, g \in G$, $t > 0$.

When the support of μ generates G as a semigroup, the convergence of the series in (5) does not depend on x, y (see e.g. [40, Lemma 1.7]). Consequently, the radius $r(\mu)$ of convergence of $\mathcal{G}(x, y|.)$ is independent of $x, y \in G$. Note,

$$r(\mu) = \liminf_{n\to\infty} p^n(x, y)^{-1/n}.$$

The number $\rho(\mu) = 1/r(\mu)$ is called the spectral radius of μ. Kesten [31], [32] and Day [7] proved that $\rho(\mu) < 1$ whenever G is non-amenable and the support of μ generates G as a semigroup.

The following is a (local) Harnack inequality:

Lemma 2.1 *Assume that G is a finitely generated group equipped with a probability measure μ whose support generates G a semigroup.*

Then for each $t \in (0, r(\mu))$ there is a $\lambda = \lambda_t \in (0, 1)$ such that for all $x, y, z \in G$ one has $\mathcal{G}(x, y|t) \geq \mathcal{G}(x, z|t)\lambda^{d(y,z)}$ and similarly $\mathcal{G}(x, y|t) \geq \mathcal{G}(z, y|t)\lambda^{d(x,z)}$.

This easily implies:

Corollary 2.2 *For each $t \in (0, r(\mu))$ there is an $L_t > 1$ such that*

$$L_t^{-d(x,y)} \leq \mathcal{G}(x, y|t) \leq L_t^{d(x,y)}$$

for all $x, y, z \in G$.

We will need the following.

Proposition 2.3 *If G is non-amenable and the support of μ generates G as a semigroup, there exists $0 < \phi < 1$ and $D > 0$ such that for any $x, y \in G$ and $M \in \mathbb{N}$ one has*

$$P_{x,y}(\gamma \in \mathit{Traj}(x, y) : length(\gamma) \geq M) \leq \phi^{M-Dd(x,y)}. \tag{6}$$

Proof Since Γ is non-amenable, $r(\mu) > 1$. Let $t \in (1, r(\mu))$. Then

$$\mathcal{G}(x, y|t) = \sum_{n=0}^{\infty} t^n p^n(x, y)$$

converges for all $x, y \in G$. Let $\phi = 1/t$ and $L = \max(L_1, L_t, t)$.
We have

$$\sum_{n \geq M} p^n(x, y) \leq t^{-M} \sum_{n \geq M} t^n p^n(x, y) \leq t^{-M}\mathcal{G}(x, y|t) \leq \phi^M L^{d(x,y)}.$$

On the other hand,

$$\mathcal{G}(x, y) \geq L^{-d(x,y)}.$$

Thus we obtain

$$\sum_{n \geq M} p^n(x, y) \leq \phi^M L^{2d(x,y)}\mathcal{G}(x, y) = \phi^{M-Dd(x,y)}\mathcal{G}(x, y)$$

where $D = 2 \log_t L > 0$. ☐

The following Proposition shows that Theorem 1.3 implies the inequality (1). This fact determines our further strategy to prove Theorem 1.3 which will be done in Sects. 4–6.

Proposition 2.4 *The conclusion of Theorem* 1.3 *implies the inequality* (1).

Proof By applying the exponential function to the inequality (1) we can restate it in the following multiplicative form:

$$\mathcal{G}(x, y) \leq S(\delta_w^f(x, y))\mathcal{G}(x, w)\mathcal{G}(w, y), \tag{7}$$

where $S(\delta_w^f(x, y)) = e^{A(\delta_w^f(x,y))}/\mathcal{G}(o, o)$ is a positive non-increasing function. So it is enough to show that the inequality (7) follows from Theorem 1.3.

Let $\varepsilon = 1/2$ and $R = R_{\frac{1}{2}}(\delta_w^f(x, y))$ given by Theorem 1.3. It implies that

$$P_{x,y}(\gamma \in \mathsf{Traj}(x, y) : \gamma \cap B(w, R) \neq \emptyset) \geq 1/2.$$

So

$$\mathcal{G}(x, y) \leq 2 \sum_{z \in B(w,R)} \mathcal{G}(x, z)\mathcal{G}(z, y).$$

Since the support $\mathsf{supp}(\mu)$ generates G as a semigroup by Lemma 2.1 there is a finite number L (depending only on (G, μ)) such that for all $x, y \in G$ and $z \in B(w, R)$ one has

$$L^{-d(z,w)} \leq \mathcal{G}(x, z)/\mathcal{G}(x, w) \leq L^{d(z,w)} \text{ and}$$
$$L^{-d(z,w)} \leq \mathcal{G}(z, y)/\mathcal{G}(w, y) \leq L^{d(z,w)}.$$

Thus,

$$\mathcal{G}(x, y) \leq 2L^{2R}|B(w, R)|\mathcal{G}(x, w)\mathcal{G}(w, y),$$

where $|\cdot|$ denotes the cardinality of a set. So the inequality (7) follows. □

3 Background on the Floyd compactifications

By a *graph* we mean a pair (Γ^0, Γ^1) where Γ^0 is a set and Γ^1 is a set of subsets of Γ^0 of cardinality 2.

A *path* in Γ is a map $\gamma : J \to \Gamma^0$ where J is a finite non-empty interval in $\mathbb{Z}$, such that $\{\gamma(i), \gamma(i+1)\} \in \Gamma^1$ for all $i \in J \setminus \{\max J\}$. The *length* of such a path γ is the number $\max J - \min J$.

For $x, y \in \Gamma^0$ let $\mathsf{Path}(x, y) = \{\gamma : J \to \Gamma^0 : \gamma(\mathsf{min}J) = x, \gamma(\mathsf{max}J) = y\}$.

Suppose that Γ is connected. Then the distance function d on Γ^0 is given by

$$d_\Gamma(x, y) = d(x, y) = \mathsf{min}\{\mathsf{length}(\gamma) : \gamma \in \mathsf{Path}(x, y)\}.$$

Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ be a non-increasing continuous function. We use f to rescale the distance d as follows. For a fixed vertex $v \in \Gamma^0$ and every edge $e \in \Gamma^1$ we declare that the (f, v)-*length* of the edge e is equal to $\mathsf{length}_v^f(e) = f(d(e, v))$.[3] The (f, v)-*length* of a path $J \xrightarrow{\gamma} \Gamma^0$ is the number

$$\mathsf{length}_v^f(\gamma) = \sum_{j \in J \setminus \{\mathsf{max}J\}} \mathsf{length}_v^f\{\gamma(j), \gamma(j+1)\}.$$

Then the quantity

$$\delta_v^f(x, y) = \mathsf{min}\{\mathsf{length}_v^f \gamma : \gamma \in \mathsf{Path}(x, y)\}$$

defines a distance called the *Floyd distance*. If the rescaling function f is fixed we will use the notation $\delta_v(x, y)$.

We suppose that the graph Γ is *locally finite*, i.e, the set of edges containing any given vertex $v \in \Gamma^0$ is finite.

If the rescaling function f satisfies the condition

$$\sum_{k=0}^{\infty} f(k) < \infty \tag{8}$$

then the Cauchy completion of the metric space (Γ^0, δ_v^f) is compact.

Now we impose on f one more condition:

$$\exists \kappa \geq 1 \, \forall n \in \mathbb{N} : 1 \leq \frac{f(n)}{f(n+1)} \leq \kappa. \tag{9}$$

Any nonincreasing function satisfying (8) and (9) is called a *Floyd rescaling function*. For such a function, the Cauchy completion $\overline{\Gamma}_f$ (called the *Floyd compactification* of Γ with respect to f) does not depend on the choice of the base point v and every isometry of the metric space (Γ, d) is uniformly

[3] In most cases the function f needs to only be defined on $\mathbb{N} \cup \{0\}$, to cover all cases we consider it on $\mathbb{R}_{\geqslant 0}$.

continuous with respect to the Floyd distance δ_v^f and hence extends to a homeomorphism $\overline{\Gamma}_f \to \overline{\Gamma}_f$.

The complement $\partial_f \Gamma = \overline{\Gamma}_f \setminus \Gamma^0$ is the *Floyd boundary* of Γ corresponding to f.

Suppose that $\Gamma = \mathrm{Cay}(G, S)$ is a Cayley graph of a group G with respect to a finite generating set S. For a fixed system S and a vertex $v \in G$ rescaling the word distance $d(v, \text{edge})$ by a function f we obtain in the same way the Floyd compactification of G and its boundary denoted respectively by $\overline{G}_f$ and $\partial_f G$.

As mentioned in the introduction, if we have two Floyd functions f and h such that $h(r) \le f(r)$ $(r \in \mathbb{R}_{\geqslant 0})$, then once our main theorems are true for f they are also true for h. We will further need the following.

Lemma 3.1 *For every Floyd function f there exists a Floyd function g such that $f(n) \le g(n)$ and $g(n)/g(2n)$ is uniformly bounded above for all $n \in \mathbb{N}$.*

The proof of the Lemma follows from a more general proposition below. Let us introduce the following sets:

$\mathcal{A} = \{$the continuous non-increasing functions $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}\}$.
For a fixed $\sigma \in \mathbb{R}_{>1}$, let

$$\mathcal{A}_\sigma = \{f \in \mathcal{A} : \forall x, y \in \mathbb{R}_{\geqslant 0}\ x \le y \Rightarrow f(x)x^\sigma \le f(y)y^\sigma\}$$

and finally

$$\mathcal{B} = \{f \in \mathcal{A} : \int_0^\infty f(t)\mathrm{d}t < \infty\}.$$

Proposition 3.2 *For all $f \in \mathcal{B}$ there exists $g \in \mathcal{B} \cap \mathcal{A}_\sigma$ such that $\forall r \ge 0$: $f(r) \le g(r)$ (denoted below by $f \le g$).*

The Proposition implies the Lemma. Indeed, if f is a Floyd function, then $f \in \mathcal{B} \cap \mathcal{A}$. Then by the Proposition there exists $g \in \mathcal{B} \cap \mathcal{A}_\sigma$: $f \le g$. Since $g \in \mathcal{A}_\sigma$ we have $g(n)n^\sigma \le g(2n)(2n)^\sigma$ and so $g(n)/g(2n) \le (1/2)^\sigma$ $(n \in \mathbb{N})$ implying the Lemma.

Proof of the Proposition. The proof is rather elementary but not obvious, so we provide it here. For $f \in \mathcal{A}$ define the function $f^{(\sigma)}$ such that

$$f^{(\sigma)}(0) = f(0) \text{ and } f^\sigma(t) = \sup\{f(r)(r/t)^\sigma : r \le t\},\ t > 0.$$

Denote $g = f^\sigma$. We have $g \ge f$ and $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ is a continuous function. Let us show that g is non-increasing. Indeed for $y \le x$ and $y > 0$ there exists $x_0 \in [0, x]$: $g(x) = f(x_0)(x_0/x)^\sigma$. Then if $x_0 \in [y, x]$ we have $f(x_0) \le f(y)$ so $g(x) \le f(y) \le g(y)$; and if $x_0 < y$ then $g(x) < f(x_0)(x_0/y)^\sigma \le g(y)$.

It is a direct verification that $g \in \mathcal{A}_\sigma$.

It remains to prove that $f \in \mathcal{B} \Rightarrow f^{(\sigma)} \in \mathcal{B}$.

Denote $O = \{t : g(t) \ne f(t)\}$. We will prove that $\int_O g(t)\mathrm{d}t < \frac{1}{\rho}\int_0^\infty f(t)\mathrm{d}t$ where $\rho = \sigma - 1 > 0$.

Let $\mathcal{C}$ be the set of the connected components of O. For $I = (a, b) \in \mathcal{C}$ we have $f(a) = g(a)$, $f(b) = g(b)$ and $\forall t \in I : f(t) < g(t)$. Then

$$\exists\, t_0 \in [0, t) \ : \ g(t) = f(t_0)(t_0/t)^\sigma. \tag{10}$$

It follows that $g(t_0) = f(t_0)$, as otherwise there is $x_o < t_0$ such that $f(x_0)(x_0/t_0)^\sigma > f(t_0)$. Then $f(t_0)(t_0/t)^\sigma < f(x_0)(x_0/t)^\sigma$ which is impossible by (10).

Furthermore, $t_0 = a$ as if $c < a$ and for $t \in I$ we have $f(c)(c/t)^\sigma < f(c)(c/a)^\sigma \le g(a) = f(a)$. Hence, $g(t) = f(a)(a/t)^\sigma$ for all $t \in I$. By continuity of g we also have $g(b) = f(a)(a/b)^\sigma = f(b)$.

Denote $\zeta = 1 - f(b)/f(a) \in (0, 1)$ then $a/b = (f(b)/f(a))^{1/\sigma} = (1 - \zeta)^{1/\sigma}$. We have

$$\int_a^b g(t)\mathrm{d}t = f(a)a^\sigma \int_a^b (\mathrm{d}t/t^\sigma) = (f(a)a/\rho)(1 - (1-\zeta)^{\rho/\sigma}) < \\ < \frac{1}{\rho} a f(a)\zeta = \frac{1}{\rho} a(f(a) - f(b)).$$

Thus

$$\int_O g(t)\mathrm{d}t < \frac{1}{\rho} \sum_{(a,b)\in\mathcal{C}} a(f(a) - f(b)) < \frac{1}{\rho}\int_0^\infty f(t)\mathrm{d}t.$$

The latter inequality takes place as the expression $\sum_{(a,b)\in\mathcal{C}} a(f(a) - f(b))$ is the area of the union of pairwise disjoint rectangles $[0, a]\times(f(b), f(a))$ situated below the graph of f. The Proposition is proved. □

In the introduction the dynamical definition of the notion of relative hyperbolicity was given. This definition, due to B. Bowditch, states that a finitely generated group is relatively hyperbolic if it admits a minimal geometrically finite action on a compactum T. Recall that a convergence action $G \curvearrowright T$ is geometrically finite and minimal if every point of T is either conical or bounded parabolic [4]. By minimality of the action, the compactum T coincides with the limit set ΛG of the action. Furthermore this action extends to a convergence action on the compactum $\overline{G}_\mathcal{B} = G \sqcup \Lambda G$, called the *Bowditch* *compactification* and ΛG is called the *Bowditch boundary*. The equivalence of this definition to several other definitions of relative hyperbolicity has been discussed in series of papers e.g. [17,18,20,27,42].

The Floyd compactification has been instrumental in studying *relatively hyperbolic groups*. Indeed, the following result will be often used in this context (see Sect. 9):

Theorem 3.3 [18, Proposition 3.4.6] *Let $G \curvearrowright T$ be a non-elementary geometrically finite minimal action on a compactum $T = \Lambda G$. Then there exists a positive* $\lambda \in (0, 1)$ *such that for every function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ satisfying the conditions* (8)*,* (9) *and $f(r) \leq \lambda^r$ $(r \in \mathbb{R})$ there is a continuous equivariant surjection $\varphi : \partial_f G \to T$.*

We note that this theorem was first proved by W. Floyd [12] in the context of Kleinian groups acting on 3-dimensional hyperbolic space $\mathbb{H}^3$ and for the rescaling function $f(n) = \frac{1}{1+n^2}$. In [18] it is proved for the exponential function $f_0(n) = \lambda^n$ (for a fixed $\lambda \in (0, 1)$ and $n \in \mathbb{N}$). Theorem 3.3 remains valid for every Floyd function greater than f_0 (in particular for f).

4 Proof of Theorem 1.1: geometric part

The goal of this and the next sections is to prove Theorem 1.1 for a finitely generated non-amenable group G equipped with a probability measure of finite support which generates G as a semigroup. The amenable case will be treated in Sect. 6.2.

By Proposition 2.4 to prove Theorem 1.1 it is enough to prove Theorem 1.3. In this section we will prove Proposition 4.3 below and in the next section we will use it to obtain Theorem 1.3. To state this Proposition we need few more preliminaries.

Let d denote the word distance on G. Given a symmetric probability measure μ on G, Blachere and Brofferio [2] introduced a metric $d_{\mathcal{G}}$ on G, called the Green metric, given by

$$d_{\mathcal{G}}(x, y) = -\ln \frac{\mathcal{G}(x, y)}{\mathcal{G}(o, o)},$$

where o is a basepoint in G. If μ is symmetric this expression defines a metric if the Markov chain defined by μ is transient. When μ is not symmetric, $d_{\mathcal{G}}$ defines a pseudo-metric on G (which we still call *metric*). This metric is *proper* (i.e every ball of finite radius is a finite set) unless G contains a finite index copy of $\mathbb{Z}$ [8, Theorem 25]. If G is non-amenable and μ has exponential moment $d_{\mathcal{G}}$ is quasi-isometric to the word metric on G (see [3, Lemma 3.6] for the symmetric case and [16, Proposition 7.8] in general).

Recall that δ_z^f denotes the Floyd distance based at a point z with respect to the Floyd rescaling function f. Since δ_z^f is invariant under left multiplication we can assume that $z = o$ is a fixed basepoint. Let $\theta > 1$ be a fixed constant (it will suffice throughout to consider $\theta = 2$).

We need the following elementary Lemma.

Lemma 4.1 *There exists a strictly decreasing function $e : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that as $r \to \infty$ we have:*

$$e(r) \to 0, \tag{11}$$

and

$$\frac{e(r) - e(\theta r)}{r^2 f(r)} \to \infty. \tag{12}$$

Proof Consider auxiliary functions

$$\eta(s) = \int_s^\infty f(t)dt$$

and

$$g(t) = \frac{f(t/\theta)}{\eta(t/\theta)^{1/2}}.$$

The condition (8) of the last section implies that $\eta(\cdot)$ is bounded and $\lim_{s\to\infty} \eta(s) = 0$. Set

$$e(r) = \int_r^\infty g(t)dt. \tag{13}$$

Since $\frac{d\eta}{ds} = -f(s)$ for every $M > 0$ we obtain

$$\int_0^M \frac{f(t/\theta)}{\eta(t/\theta)^{1/2}} dt = \theta \int_{\eta(M/\theta)}^{\eta(0)} \frac{d\eta}{\sqrt{\eta}} \leq 2\theta\sqrt{\eta(0)}.$$

So the function $e(\cdot)$ is well-defined, strictly decreasing and satisfies (11). By the mean value theorem there is an $s \in [r, \theta r]$ with $e(r) - e(\theta r) = (\theta r - r)g(s)$. Since f is decreasing we have

$$\frac{e(r) - e(\theta r)}{rf(r)} = \frac{(\theta - 1)g(s)}{f(r)} = (\theta - 1)\frac{f(s/\theta)}{f(r)\eta(s/\theta)^{1/2}} \geq \frac{\theta - 1}{\eta(s/\theta)^{1/2}} \to \infty$$

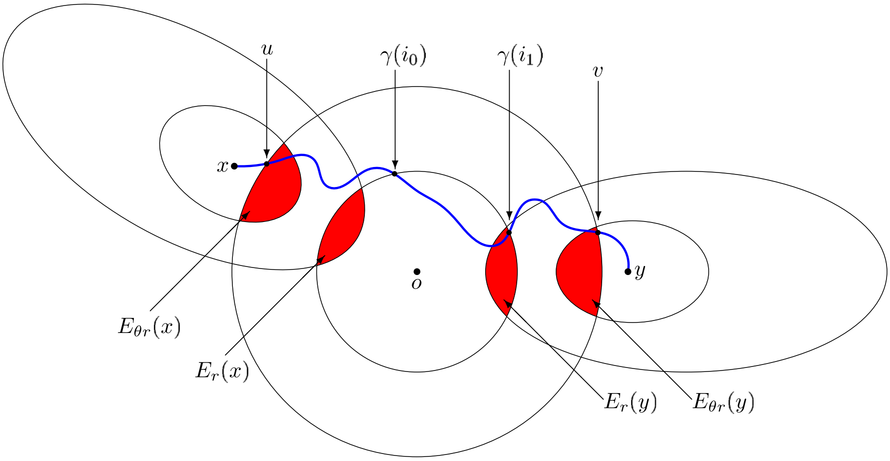
as $r \to \infty$. □

Remark 4.2 If f satisfies $f(r) \leq r^{-1-\varepsilon}$ for some $\varepsilon > 0$ we can use in the argument above the simpler expression $e(r) = 1/\ln r$.

Let $e : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function satisfying Lemma 4.1. For a basepoint $o \in \Gamma$ and for every $x \in \Gamma^0$ denote by $B^f(x, e(r))$ the open ball around x in the Floyd metric δ_o^f of radius $e(r)$; and by $B(o, r)$ the open ball of radius r around o in the word metric. Set

$$E_r(x) = B(o, r) \cap B^f(x, e(r)).$$

The sets are indicated on the figure below.



The following geometric estimate is crucial for the proof of Theorem 1.3.

Proposition 4.3 *There are functions $R_0 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\lim_{r\to\infty} h(r)/r = \infty$ such that for all $x, y \in G$ and all $r > R_0(\delta_o^f(x, y))$, for each $u \in E_{\theta r}(x)$ and $v \in E_{\theta r}(y)$, any path from u to v disjoint from one of $E_r(x)$ or $E_r(y)$ has length at least $h(r)$.*

Proof For fixed $x, y \in G$ and the basepoint o denote by δ the Floyd distance $\delta_o^f(x, y)$. Let $\gamma = \gamma_0, \ldots, \gamma_N$ be a path of word length N from $u = \gamma_0 \in E_{\theta r}(x)$ to $v = \gamma_N \in E_{\theta r}(y)$ disjoint from $E_r(x)$.

Suppose first that γ does not pass through $B(o, r)$ then $\delta_o^f(\gamma_{i-1}, \gamma_i) \leq f(r)$ $(i \in \{0, \ldots, N\})$, implying:

$$\delta_o^f(u, v) \leq l_o^f(\gamma) = \sum_{i=1}^{N} \delta_o^f(\gamma_{i-1}, \gamma_i) \leq N f(r).$$

So in this case we have

$$\delta = \delta_o^f(x, y) \leq \delta_o^f(u, v) + 2e(\theta r) \leq N f(r) + 2e(\theta r),$$

and so

$$N \geq \frac{\delta - 2e(\theta r)}{f(r)}. \tag{14}$$

Suppose now that γ passes through $B(o,r)$. We have $\gamma_0 = u \in E_{\theta r}(x) \backslash E_r(x)$, so $\delta_o^f(x,u) < e(\theta r) < e(r)$ as the function $e(\cdot)$ in (13) is strictly decreasing. Since $\gamma \cap E_r(x) = \emptyset$ then $u = \gamma_0 \notin B(o,r)$. Let γ_{i_0} be the first intersection point of γ with $B(o,r)$, $0 < i_0 \leq N$. Then for all $i \in \{1, ..., i_0\}$ we still have $\delta_o^f(\gamma_{i-1}, \gamma_i) \leq f(r)$.

Since $\gamma_{i_0} \in B(o,r) \backslash E_r(x)$ it follows $\delta_o^f(\gamma_{i_0}, x) \geq e(r)$. Thus, $\delta_o^f(u, \gamma_{i_0}) \geq e(r) - e(\theta r)$. Summarizing all this we obtain

$Nf(r) \geq i_0 f(r) \geq \sum_{1 \leq i \leq i_0} \delta_o^f(\gamma_{i-1}, \gamma_i) \geq \delta_o^f(u, \gamma_{i_0}) \geq e(r) - e(\theta r)$.

By Lemma 4.1 $\lim\limits_{r \to 0} e(r) = 0$ so there exists $R_0 = R_0(\delta)$ such that for all $r \geq R_0$ we have

$$\delta \geq e(r) + e(\theta r). \tag{15}$$

Therefore in both cases we obtain

$$N \geq \frac{e(r) - e(\theta r)}{f(r)}$$

Set

$$h(r) = \frac{e(r) - e(\theta r)}{f(r)}.$$

It follows from (12) that $h(r)/r \to \infty$ as $r \to \infty$. The same argument works if the path γ is disjoint from $E_r(y)$. The proposition is proved. $\square$

5 Conclusion of the proof of Theorem 1.1.

In this section we still assume that G equipped with a measure μ of finite support generating G as a semigroup. We also assume that G is a non-amenable group, the case when G is amenable will be treated in Sect. 6.2.

Using Lemma 3.1 and the remark preceding it, we may assume without lost of generality that the Floyd function f satisfies the following condition:

$$\frac{f(n)}{f(2n)} \leq D, \; n \in \mathbb{N}, \tag{16}$$

for a uniform constant $D \in (1, +\infty)$.

Let S be a finite symmetric system of generators of G. The following lemma shows that we can also assume that $S \supset \mathsf{supp}(\mu)$.

Lemma 5.1 *Let S' denote a symmetric generating system of G containing S and $\mathsf{supp}(\mu)$. If the inequality (1) is satisfied on the Cayley graph $\Gamma' = \mathrm{Cay}(G, S')$ then it is also true on $\Gamma = \mathrm{Cay}(G, S)$.*

Proof Since both generating systems S and S' are finite, the identity map $id : \Gamma' \to \Gamma$ is quasi-isometric. Then by [19, Lemma 2.5] it induces a Lipschitz map $\varphi : (\Gamma', \delta') \to (\Gamma, \delta)$ where δ' and δ are the Floyd distances on the graphs Γ' and Γ respectively, based at the same vertex of both graphs, and with respect to the same Floyd rescaling function f. Then there exists a uniform constant $\varepsilon > 0$ such that for every triple of distinct points $x, y, z \in G$ and for every Floyd function f satisfying (16) one has (see formula (7) in [19]):

$$\delta'_y(x, z) \geq \varepsilon \cdot \delta_y(x, z).$$

The formula (1) is valid for Γ' with the decreasing function $A'(\cdot)$ for the remainder term. Since $A'(\delta'_y(x, z)) \leq A'(\varepsilon \cdot \delta_y(x, z))$ for all $x, y, z \in G$, the formula (1) holds on the graph Γ with the remainder term $A(t) = A'(\varepsilon \cdot t)$. □

Let $\tau = \{g_0 = x, , g_1, ..., g_n = y\} \in \mathsf{Traj}(x, y)$ such that $g_{i-1}^{-1} g_i \in \mathsf{supp}\mu$. By Lemma 5.1 we may assume that $\mathsf{supp}(\mu) \subset S$ so $d(g_{i-1}, g_i) = 1, i \in \{1, ..., n\}$. Then $\tau \in \mathsf{Path}(x, y)$ and $\mathsf{Traj}(x, y) \subset \mathsf{Path}(x, y)$ for all $x, y \in G$.

Recall that by Proposition 2.4 Theorem 1.3 implies Theorem 1.1. So the rest of this section is devoted to proving Theorem 1.3 in case when the support of μ is contained in the finite generating set S. We restate Theorem 1.3 here in a more precise form:

Theorem 5.2 *Let μ be a probability measure on a non-amenable group G with support generating G as a semigroup and contained in a finite generating set S of G. For every $\varepsilon \in (0, 1)$ there exists a decreasing function $R_\varepsilon : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all $x, y, w \in G$ one has*

$$P_{x,y}(\tau \in \mathit{Path}(x, y) : \tau \cap B(w, R) \neq \emptyset) > 1 - \varepsilon, \tag{17}$$

where $B(w, R)$ is the ball in the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ centered at w of radius $R = R_\varepsilon(\delta^f_w(x, y))$ in the word metric.

By equivariance we can assume that w is the basepoint $o \in G$. A sample path from x to y will be called r-regular if it intersects both $E_r(x)$ and $E_r(y)$ (defined in the previous section). Denote by $Reg_r(x, y)$ the set of r-regular paths from x to y and by $Q_r(x, y)$ the set of paths which are (θr)-regular but not r-regular (an element of the set Q_r is indicated on the figure from the previous section).

Recall that $\theta > 1$ is a fixed number. We have the following.

Lemma 5.3 *For all $x, y \in G$ and $r > R_0 = R_0(\delta_o^f(x, y))$ we have*

$$P_{x,y}(Q_r(x, y)) \leq \phi^{h(r)-2D\theta r}, \tag{18}$$

where the functions h and R_0 come from Proposition 4.3, and the constants D and ϕ from Proposition 2.3.

Proof We first claim that for $V_r = E_r(x)$ or $V_r = E_r(y)$ we have

$$\sup\left\{\frac{\mathcal{G}(u, v, V_r^c)}{\mathcal{G}(u, v)} : (u\in E_{\theta r}(x) \wedge v\in E_{\theta r}(y)) \vee (u\in E_{\theta r}(y) \wedge v\in E_{\theta r}(x))\right\} \\ \leq \phi^{h(r)-2D\theta r}, \tag{19}$$

where $V_r^c = \mathsf{Path}(x, y)\backslash V_r$.

Indeed, by Proposition 4.3, if $r > R_0(\delta^f(x, y))$, any path from $u \in E_{\theta r}(x)$ to $v \in E_{\theta r}(y)$ (or from $u \in E_{\theta r}(y)$ to $v \in E_{\theta r}(x)$) disjoint from $E_r(x)$ (respectively $E_r(y)$) has length at least $h(r)$ while $d(u, v) \leq 2\theta r$. Thus, Proposition 2.3 implies:

$$\frac{\mathcal{G}(u, v, V^c)}{\mathcal{G}(u, v)} \leq \phi^{h(r)-2\theta r D},$$

proving the claim.

We now proceed to prove (18).

For a θr-regular path $\gamma = (\gamma_0, ..., \gamma_{j_0}, ..., \gamma_{j_1}, ..., \gamma_N)$ between $x = \gamma_0$ and $y = \gamma_N$ let $u = \gamma(j_0)$ be the first intersection point of γ with $E_{\theta r}(x)$ or with $E_{\theta r}(y)$; and $v = \gamma(j_1)$ be the last intersection point of γ respectively with $E_{\theta r}(y)$ or with $E_{\theta r}(x)$.

Consider the following product:

$$U_r = (E_r(x) \times E_r(y)) \cup (E_r(y) \times E_r(x)).$$

We have

$$\mathcal{G}(\gamma \in Reg_{\theta r}(x, y)) \\ = \sum_{(u,v)\in U_{\theta r}} \mathcal{G}(x, u, E_{\theta r}^c(x) \cap E_{\theta r}^c(y))\mathcal{G}(u, v)\mathcal{G}(v, y, V_{\theta r}^c), \tag{20}$$

where in the last term $v \in V_{\theta r} = E_{\theta r}(y)$ if $u \in E_{\theta r}(x)$; and $v \in V_{\theta r} = E_{\theta r}(x)$ if $u \in E_{\theta r}(y)$.

In the special case when the initial point x belongs to one of the sets $E_{\theta r}(x)$ or $E_{\theta r}(y)$ by definition of u we have $u = \gamma(j_0) = x$. Then $\mathcal{G}(x, x, E_{\theta r}^c(x)) = 1$

and $\mathcal{G}(u, v) = \mathcal{G}(x, v)$ in (20). Similarly if $y \in E_{\theta r}(x) \cup E_{\theta r}(y)$ then $v = y$ and the third factor in (20) is 1.

For the paths which avoid $V_r \in \{E_r(x), E_r(y)\}$ by the same rules we have

$$\mathcal{G}(\gamma \in Q_r(x, y)) \\ = \sum_{\substack{(u,v)\in U_{\theta r} \\ V_r \in \{E_r(x), E_r(y)\}}} \mathcal{G}(x, u, E^c_{\theta r}(x) \cap E^c_{\theta r}(y))\mathcal{G}(u, v, V_r^c)\mathcal{G}(v, y, V^c_{\theta r}). \tag{21}$$

Note that the only difference between the formulas (20) and (21) is in their middle factors. Applying (19) estimating the ratio of these factors we obtain:

$$P_{x,y}(\gamma \in Q_r(x, y)) \leq \frac{\mathcal{G}(\gamma \in Q_r(x, y))}{\mathcal{G}(\gamma \in Reg_{\theta r}(x, y))} \leq \phi^{h(r)-2\theta r D}. \tag{22}$$

The Lemma is proved. □

Proof of Theorem 5.2. For a given $\varepsilon > 0$ we need to find R such that the inequality (17) holds. Let $R_0 = R_0(\delta^f_o(x, y))$ be the number given by Lemma 5.3. We first choose R such that

$$R > R_0 = R_0(\delta^f_o(x, y)). \tag{*}$$

By Proposition 4.3 $h(R)/R \to \infty$ $(R \to \infty)$, so choosing R sufficiently large we can also assume that

$$h(Rt) \geq (2D + 2)Rt, \tag{**}$$

for all $t \geq 1$. Putting $t = \theta^i$ $(i \in \mathbb{N})$ we obtain $h(\theta^i R) - 2\theta^i RD \geq 2\theta^i R \geq (i + 1)R$ for each $i \geq 0$.

Thus we obtain

$$\sum_{i=0}^{\infty} \phi^{h(\theta^i R)-2\theta^i RD} \leq \sum_{i=0}^{\infty} \phi^{(i+1)R} = \phi^R/(1 - \phi) < \varepsilon \tag{23}$$

when R is large enough.

Let us fix R such that the inequalities $(*)$, $(**)$ and (23) are true. Then any path in $\mathsf{Path}(x, y)$ either passes through $B(o, R)$ or is an element of the set $\bigcup_{i=0}^{\infty} Q_{\theta^i R}(x, y)$. Indeed, if $\gamma \cap B(o, R) = \emptyset$ then there exists $m \in \mathbb{N}$ such that $x \in E_{\theta^m R}(x)$ and $y \in E_{\theta^m y}$, so γ is $\theta^m R$-regular. Assume now that m is the minimal number with this property. We have $m > 0$ as otherwise $\gamma \cap B(o, R) \neq \emptyset$. So γ is not $(\theta^{m-1} R)$-regular and belongs to $Q_{\theta^{m-1} R}$.

By (18) and (23) we have

$$P_{x,y}\left(\bigcup_{i=0}^{\infty} Q_{\theta^i R}(x, y)\right) \sum_{i=0}^{\infty} P_{x,y}(Q_{\theta^i R}(x, y)) \\ \leq \sum_{i=0}^{\infty} \phi^{h(\theta^i R) - 2\theta^i RD} < \varepsilon. \tag{24}$$

Therefore:

$$P_{x,y}(\gamma \in \mathsf{Path}(x, y) : \gamma \cap B(o, R(\delta_o^f(x, y)) \neq \emptyset) \geq 1 - \varepsilon. \tag{25}$$

This completes the proof of Theorem 5.2 and so that of Theorem 1.1 if G is non-amenable. □

6 Proof of Theorem 1.2: extension to infinite support

6.1 The case of infinite support: non-amenable case

The goal of this subsection is to prove Theorem 1.2 for measures with infinite support but superexponential moment when G is a non-amenable group. The amenable case is treated in the next subsection.

Recall that μ has superexponential moment if it satisfies

$$\sum_{g \in G} c^{||g||} \mu(g) < \infty$$

for all $c > 1$ and $||g|| = d(o, g)$ where o is a fixed basepoint in the Cayley graph which we identify with the neutral element of G.

Recall that a function $W : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to decay superexponentially if $\lim_{t \to \infty} c^t W(t) = 0$ for each $c > 1$ or equivalently $\lim_{t \to \infty} t^{-1} \ln W(t) = -\infty$. It is an elementary observation that if μ has superexponential moment then the remainder

$$\sum_{g \in G : ||g|| > t} \mu(g)$$

decays superexponentially in t.

The goal is to prove Theorem 1.3 for measures of superexponential moment which will imply the inequality (1).

We assume in this subsection that the Floyd function f decays at least as fast as $r \to r^{-2-\eta}$ for a fixed $\eta > 0$. It suffices to only consider functions of the form $f(r) = r^{-2-\eta}$ ($\eta > 0$). Indeed the function $R(\cdot)$ from Theorem 1.3 is decreasing so once we prove Theorem 1.3 for a fixed Floyd function, the analogue for faster decaying Floyd functions follows automatically. Let $\theta > 1$. We will use the following modification of Lemma 4.1.

Lemma 6.1 *There exists a function $e : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that as $r \to \infty$ we have $e(r) \to 0$ and*

$$\frac{e(r) - e(\theta r)}{r^2 f(r)} \to \infty$$

Proof An easy computation shows that $e(r) = \frac{1}{\ln r}$ satisfies the claim. □

Since the measure is infinitely supported we will consider general trajectories having different jumps and not paths with every jump of length one as it was in the previous section. We have the following adaptation of Proposition 4.3 (this is where we use the assumption on the Floyd function).

Proposition 6.2 *Let $f(r) = 1/r^{2+\eta}$ ($\eta > 0$) be a Floyd function. Then there are functions $R_0 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $h(r)/r \to \infty$ as $r \to \infty$ such that for all $x, y \in G$ and all $r > R_0(\delta_o^f(x, y))$, for each $u \in E_{\theta r}(x)$ and $v \in E_{\theta r}(y)$, any path from u to v disjoint from $E_r(x)$ with jump size bounded by r has length at least $h(r)$.*

Proof Denote by δ the Floyd distance $\delta_o^f(x, y)$. Let $\gamma = \gamma_0, ..., \gamma_N$ be a trajectory from $u = \gamma(0) \in E_{\theta r}(x)$ to $v = \gamma(N) \in E_{\theta r}(y)$ not intersecting $E_r(x)$ and with jump size bounded by r (which may be arbitrarily large).

First consider the case when γ does not pass through $B(o, r)$. Since γ has jump size bounded by r, by the triangle inequality we have

$$d(o, \gamma|_{[n,n+1]}) > r/2 \text{ for all } n.$$

Hence,

$$Nrf(r/2) \geq \delta_o^f(u, v) \geq \delta - \delta_o^f(x, u) - \delta_o^f(v, y) \geq \delta - 2e(\theta r). \quad (26)$$

Suppose now that γ does pass through $B(o, r)$ and let $\gamma(i_0)$ be the first intersection of γ with $B(o, r)$. Then by the argument used in the proof of Proposition 4.3 and applied to the interval $[0, i_0)$, we have

$$Nrf(r/2) \geq ri_0 f(r/2) \geq \delta_o^f(\gamma(i_0), u)). \quad (27)$$

As γ does not intersect $E_r(x)$ we must have $\delta_o^f(x, \gamma(i_0)) \geq e(r)$. Since $u \in E_{\theta r}(x)$ we have $\delta_o^f(x, u) \leq e(\theta r)$, and so

$$\delta_o^f(u, \gamma(i_0)) \geq e(r) - e(\theta r). \tag{28}$$

Then (27) and (28) imply

$$Nrf(r/2) \geq e(r) - e(\theta r). \tag{29}$$

Since the function $e(r)$ decays to zero there exists $R_0 = R_0(\delta)$ such that for all $r \geq R_0$ we have

$$\delta \geq e_r + e_{\theta r.} \tag{30}$$

Therefore from (26) and (29) we obtain

$$N \geq \frac{e(r) - e(\theta r)}{rf(r/2)} \geq \mathrm{const}\frac{e(r) - e(\theta r)}{rf(r)} = h(r).$$

Indeed $f(r) = r^{-2-\eta}$ so $f(r)/f(r/2) = (1/2)^{2+\eta}$. By Lemma 6.1 $h(r)/r \to \infty$ and the Proposition follows. □

For each n we can write $\mu = \mu_n + \sigma_n$ where μ_n is the restriction of μ to the ball $B(0, n) = \{g \in G : ||g|| \leq n\}$ and $\sigma_n = \mu - \mu_n$.

The contribution to $\mathcal{G}(x, y)$ of trajectories of length M, with exactly m jumps of size greater than n is bounded by

$$\binom{M}{m}|\sigma_n|^m|\mu_n|^{M-m}$$

where for a measure λ we use the notation $|\lambda|$ to denote $\lambda(G)$.

We will need the following lemma whose proof is a straightforward exercise.

Lemma 6.3 *The function* $W : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *given by*

$$W(t) = \sqrt{t \ln |\sigma_t|^{-1}}.$$

has the following properties:

(1) $W(t)/t \to \infty$ *as* $t \to \infty$;
(2) *for each* $c > 1$ *the quantity* $c^{W(t)}|\sigma_t|$ *decays superexponentially.*

Proof of Theorem 1.3 for measures of superexponential moment. We assume that $z = o$ is the basepoint. The only step where we need to deviate from the finite support case is in estimating (using our superexponential moment condition) the quantities $\mathcal{G}(u, v, E_r^c(x))$ and $\mathcal{G}(u, v, E_r^c(x))$ for each $u \in E_{\theta r}(x)$ and $v \in E_{\theta r}(x)$.

We need to prove that $\mathcal{G}(u, v, E_r^c(x)) \leq \Psi(r)\mathcal{G}(u, v)$ and $\mathcal{G}(u, v, E_r^c(y)) \leq \Psi(r)\mathcal{G}(u, v)$ where $\Psi(r)$ decays superexponentially in r. Indeed, if this is true then proceeding as in the proof of Theorem 1.3 in the finite support case, we obtain the inequality $P_{x,y}(Q_r(x, y)) \leq 2\Psi(r)$ (see Lemma 5.3). Since $\Psi(r)$ decays super-exponentially, we have

$$\sum_{i=0}^{\infty} \Psi(\theta^i R) < \varepsilon$$

for large enough $R = R_\varepsilon(\delta_o^f(x, y))$. Thus exhausting the complementary $B^c(o, R)$ of the ball $B(o, R)$ by the sets $Q_{\theta r}$ as in (24) we will obtain the requested result:

$$\mathcal{G}(x, y, B^c(o, R)) < \sum_{i=0}^{\infty} \mathcal{G}(x, y)\Psi(\theta^i r) \leq \varepsilon\mathcal{G}(x, y).$$

So let us estimate $\mathcal{G}(u, v, E_r^c(x))$ (the estimate for $\mathcal{G}(u, v, E_r^c(y))$ is similar).

We have $d(u, v) \leq 2\theta r$. By Proposition 6.2 any trajectory with no jumps of length greater than r has length at least $h(r)$. Hence, by Proposition 2.3 the contribution to $\mathcal{G}(u, v, E_r^c(x))$ of trajectories with no jumps greater than r is at most $\phi^{h(r)-2D\theta r}\mathcal{G}(u, v)$.

Also by Proposition 2.3, the contribution of trajectories of length at least $W(r)$ is at most $\phi^{W(r)-2D\theta r}\mathcal{G}(u, v)$.

It remains to control the contribution to $\mathcal{G}(u, v, E_r^c(x))$ of trajectories of length at most $W(r)$ with at least one jump of size at least r. Then it is bounded above by

$$\sum_{m=1}^{W(r)} \sum_{k=0}^{m-1} \binom{m}{k} |\sigma_r|^{m-k} |\mu_r|^k \tag{31}$$

Since $m-k \geq 1$, in the above expression we have $|\sigma_r|^{m-k} \leq |\sigma_r|$ so (31) is bounded above by

$$|\sigma_r| \sum_{m=1}^{W(r)} \sum_{k=0}^{m-1} \binom{m}{k} |\mu_r|^k \leq$$
$$\leq |\sigma_r| \sum_{m=1}^{W(r)} (1+|\mu_r|)^m \leq |\sigma_r| \sum_{m=0}^{W(r)} 2^m \leq 2^{W(r)+1}|\sigma_r|.$$

By the Harnack inequality there is a universal $1 > \lambda > 0$ such that

$$\mathcal{G}(u,v) \geq \mathcal{G}(o,o)\lambda^{d(u,v)} \geq \mathcal{G}(o,o)\lambda^{2\theta r}.$$

Thus the contribution to $\mathcal{G}(u, v, E_r^c(x))$ of trajectories of length at most $W(r)$ with at least one jump of size at least r is bounded above by

$$2^{W(r)+1}|\sigma_r|\lambda^{-2\theta r}\mathcal{G}(o,o)^{-1}\mathcal{G}(u,v).$$

Since $W(r)/r \to \infty$ $(r \to \infty)$ the above quantity is bounded above by $C^{W(r)}|\sigma_r|$ for a constant C.

By Lemma 6.3 the latter quantity tends to 0 superexponentially fast as $r \to \infty$.

Putting everything together, we obtain

$$\mathcal{G}(u, v, E_r^c(x)) \leq \Psi(r)\mathcal{G}(u,v)$$

where

$$\Psi(r) = \max\left(C^{W(r)}|\sigma_r|\,,\ \phi^{W(r)-2D\theta r},\ \phi^{h(r)-2D\theta r}\right)$$

tends to zero superexponentially fast as $r \to \infty$. □

6.2 Amenable case

Assume now that G is amenable. If G is not virtually cyclic, Theorem 1.3 holds with no assumption on the moment of μ. Indeed, A. Karlsson showed that $|\partial_f G| \leq 2$ [29, Corollary 2] for any amenable group G. If $\partial_f G$ is empty then G is finite so every random walk on a finite group is recurrent and the result holds.

If $|\partial_f G| = 1$, for a fixed $\delta > 0$, consider a triple $x, y, z \in G$ such that $\delta_z(x,y) = \delta$. Since the boundary is a single point, the points x and y cannot both tend to the boundary. Hence, there exists $R = R(\delta)$ such that $\{x, y\} \cap B(z, R) \neq \emptyset$ and the theorem trivially holds in this case too.

If $|\partial_f G| = 2$ then G is virtually cyclic. Indeed, the action $G \curvearrowright \overline{G}_f$ is convergence [29, Proposition 3], and up to passing to a subgroup of index 2 we may assume that both points of $\partial_f G$ are fixed by the group. Then G contains a loxodromic element which generates a cyclic subgroup of finite index of G [38, Theorem 2I].

Suppose G contains a cyclic subgroup H such that $|G : H| < \infty$. Identifying H with $\mathbb{Z}$ we denote by $[w]$ the projection of $w \in G$ to H and $|[w]|$ its length. Then $||w|| \asymp_C |[w]|$ for a uniform constant $C > 0$ (where the symbol $\asymp_C$ means C-bilipschitz equivalent).

Let $E_r^+ = \{g \in G \ : \ 0 \leq [g] \leq r\}$ and $E_r^- = \{g \in G \ : \ -r \leq [g] \leq 0\}$. Let $E_r = E_r^- \cup E_r^+,\ r \in \mathbb{R}_{>0}$.

To prove Theorem 1.3 in this case we fix a basepoint $o \in G$ and consider two points $x, y \in G$ such that $\delta_o^f(x, y) \geq \delta$ for some constant $\delta > 0$. As in the case above x and y cannot approach the same boundary point of $\partial_f G$ so without loss of generality we may assume that $[x] < -r$ and $[y] > r$ $(r \in \mathbb{R}_{>0})$.

Similarly to Sect. 5, a trajectory from x to y is called *r-regular* if it intersects both E_r^+ and E_r^- and denote by Q_r the set of trajectories which are $2r$-regular but not r-regular. Then a simplified version of the proof in the non-amenable case (see Propositions 4.3 and 6.2) will give an estimate on the measure of Q_r.

If the support of μ is finite then every trajectory between x and y has uniformly bounded jumps so for sufficiently big r we have $Q_r = \emptyset$, and Theorem 1.3 follows in this case.

In the case when μ is infinitely supported but has a superexponential moment it is enough to find a superexponentially decaying function Ψ such that $P_{x,y}(Q_r) < \Psi(r)$. As in the proof of Lemma 5.3, it suffices to show that for $V = E_r^{\pm}$ we have:

$$\sup_{u \in E_{2r}^-, v \in E_{2r}^+} \left\{ \max \left(\frac{\mathcal{G}(u, v, V^c)}{\mathcal{G}(u, v)}, \frac{\mathcal{G}(v, u, V^c)}{\mathcal{G}(v, u)} \right) \right\} \leq \Psi(r), \tag{32}$$

For $u \in E_{2r}^-$, $v \in E_{2r}^+$ we consider a trajectory $\gamma \in \mathsf{Traj}(u, v)$ disjoint from E_r^+. Every such trajectory contains a jump from some w_1 with $k = [w_1] < 0$ to some w_2 with $l = [w_2] > r$.

$$\mathcal{G}(u, v, (E_r^+)^c) \leq \sum_{k \leqslant 0, l > r} \mathcal{G}(u, w_1) p(w_1, w_2) \mathcal{G}(w_2, v).$$

Since the random walk is transient the function $\mathcal{G}$ is bounded. The superexponential moment condition implies that for any $d \in (0, 1)$ we have $p(w_1, w_2) \leq \text{const} \cdot d^{|k-l|}$ where the constant depends on the constant C

above. Therefore,

$$\mathcal{G}(u, v, (E_r^+)^c) \leq \text{const} \cdot \sum_{k \leq 0, l > r} d^{|k-l|} \leq \text{const} \cdot d^r.$$

By the Harnack inequality (Corollary 2.2) there is some constant $\lambda \in (0, 1)$ such that $\mathcal{G}(u, v) \geq \lambda^{d(u,v)} \geq \lambda^{C|[u]-[v]|} \geq \lambda^{4Cr}$ for all $u \in E_{2r}^-$, $v \in E_{2r}^+$. So we obtain

$$\sup_{u \in E_{2r}^-, v \in E_{2r}^+} \frac{\mathcal{G}(u, v, (E_r^+)^c)}{\mathcal{G}(u, v)} \leq \text{const} \cdot d^{r/2}$$

for all $d < \lambda^{8C}$. By an identical argument the same is true with $(E_r^-)^c$ in place of $(E_r^+)^c$, and also when the order of u and v is reversed. Letting $\Psi(r)$ be the supremum of these four superexponentially decaying quantities we obtain the estimate (32). The proof of Theorem 1.3 is finished.

7 A map from the Martin boundary to the Floyd boundary

7.1 Construction of the map

As before, we consider a finitely generated group G with a probability measure μ on G whose support generates G as a semigroup. Denote by $\mathcal{G}$ the associated Green function. Recall that the Green metric on G is given by $d_{\mathcal{G}}(x, y) = -\ln \dfrac{\mathcal{G}(x, y)}{\mathcal{G}(o, o)}$, where o ia a basepoint in G. The horofunction compactification of $(G, d_{\mathcal{G}})$ is called the Martin compactification and denoted by $\overline{G}_{\mathcal{M}}$. The boundary

$$\partial_{\mathcal{M}} G = \overline{G}_{\mathcal{M}} \backslash G$$

is called the Martin boundary of (G, μ) [36]. This means that $\partial_{\mathcal{M}} G$ consists of all functions $\alpha : G \to \mathbb{R}$ such that there exists an unbounded sequence $x_n \in G$ with

$$\alpha(x) = \lim_{n \to \infty} d_{\mathcal{G}}(x, x_n) - d_{\mathcal{G}}(o, x_n)$$

for all $x \in G$. The Martin boundary is intimately related to the set of μ−harmonic functions on (G, μ).

Recall that a function $h : G \to \mathbb{R}$ is called μ-harmonic (or simply harmonic when there is no ambiguity) if for all $x \in G$,

$$\sum_{g \in G} h(xg)\mu(g) = h(x).$$

For $p, q, x \in G$ we set $\Delta(p, q, x) = d_{\mathcal{G}}(p, x) - d_{\mathcal{G}}(q, x)$ and extend it by continuity: for $\alpha \in \partial G_{\mathcal{M}}$ we let $\Delta(p, q, \alpha) = \lim_{\substack{x_n \to \alpha \\ x_n \in G}} \Delta(p, q, x_n)$.

The Martin kernel is $K_y(\cdot) = K(\cdot, y) = e^{\Delta(\cdot, o, y)} = \dfrac{\mathcal{G}(\cdot, y)}{\mathcal{G}(o, y)}$ and its limit gives a harmonic function on the Martin boundary described as follows.

Lemma 7.1 *If μ has superexponential moment, then the function defined by*

$$K_\alpha(\cdot) = K_\alpha(\cdot) = e^{-\Delta(\cdot,\quad o,\quad \alpha)} = \lim_{x_n \to \alpha} \frac{\mathcal{G}(\cdot, x_n)}{\mathcal{G}(o, x_n)}$$

is harmonic for all $\alpha \in \partial G_{\mathcal{M}}$.

Proof When μ has finite support this is noted by Woess in [40, Lemma 24.16].

As in Sect. 6, for each R we can write $\mu = \mu_R + \sigma_R$ where μ_R is the restriction of μ to the ball $B(o, R)$ centered at o of radius R. Define the linear operator $P = P_\mu$ defined on the space $C(G, \mathbb{R})$ of functions $\omega : G \to \mathbb{R}$ by

$$P\omega(x) = \sum_{y \in G} p(x, y)\omega(y).$$

Consider a sequence $y_n \in G$ converging to $\alpha \in \partial_{\mathcal{M}} G$. We want to prove that K_α is μ-harmonic, i.e. $PK_\alpha = K_\alpha$. For this, it suffices to show that for every $x \in G$ one has:

$$PK_{y_n}(x) \to PK_\alpha(x) \;\text{ and }\; PK_{y_n}(x) \to K_\alpha(x) \tag{33}$$

The second property is obvious as $PK_{y_n}(x) = K_{y_n}$ for $y_n \neq x$.

To prove the first one in (33), denote $\Upsilon_n(x) = |K_{y_n}(x) - K_\alpha(x)|$. Let us show that $P\Upsilon_n(x) \to 0$ for all $x \in G$. For a fixed $x \in G$ and $R > ||x||$ we have $P_\mu = P_{\mu_R} + P_{\sigma_R}$ and $P_\mu \Upsilon_n = P_{\mu_R}\Upsilon_n + P_{\sigma_R}\Upsilon_n$.

By the Harnack inequality there is a uniform $C > 1$ with $K_z(x) \leq C^{||x||}$ for all $z \in G$ (see Lemma 2.1 in the case when the second variable of the Green function is fixed). We have $\Upsilon_n(x) \leq 2C^{||x||}$, and hence

$$P_{\sigma_R}\Upsilon_n(x) \leq \sum_{y \notin B(x,R)} \mu(x^{-1}y)2C^{||y||} \leq \sum_{z \notin B(o,R)} 2\mu(z)C^{R\cdot||z||} \to 0$$

uniformly in n as $R \to \infty$. Indeed, since μ has superexponential moment we have $\mu(z) \leq (1/C^2)^{||z||} \cdot \varepsilon(z)$ and $\varepsilon(z) \to 0$ when $||z|| \to \infty$.

On the other hand, for each fixed R the set $B(x, R)$ is finite so

$$P_{\mu_R}\Upsilon_n(x) = \sum_{y \in B(x,R)} p(x, y)|K_{y_n}(y) - K_\alpha(y)| \to 0 \text{ as } n \to \infty.$$

It follows that $\limsup_{n\to\infty} P_\mu \Upsilon_n(x) \to 0$ proving the lemma. □

The following assumptions on the measure μ will be used further on.

Assumption 1 The inequality (1) is satisfied.

Assumption 2 The support of μ generates G as a semigroup, and for every $\alpha \in \partial_{\mathcal{M}} G$, K_α is harmonic.

By Theorems 1.1, 1.2 and Lemma 7.1 these axioms are satisfied when μ has finite support, or μ has superexponential moment and $f(r) \leq r^{-2-\iota}$ for some $\iota > 0$.

The following Lemma is well-known, we indicate its short proof for completeness.

Lemma 7.2 *If G is not a virtually abelian group of rank less or equal 2 then $\mathcal{G}(o, z) \to 0$ once $d(o, z) \to \infty$.*

Proof If the claim is not true then there exist a constant $c > 0$ and infinitely many $g \in G$ with $\mathcal{G}(o, g) > c$, or in other words $d_{\mathcal{G}}(o, g) < D = -\log(c/\mathcal{G}(o, o))$. However if $\mathrm{rk}(G) > 2$ by [3, Proposition 3.1] we obtain that the Green metric $d_{\mathcal{G}}$ is proper: i.e. every $d_{\mathcal{G}}$-ball of finite radius contains finitely many elements. The obtained contradiction proves the Lemma. □

We are now ready to prove:

Theorem 7.3 *The identity map on G extends to a continuous equivariant surjection $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$.*

Proof Step 1. *The identity map* $\mathrm{id} : G \to G$ *extends to a map* $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_{\mathcal{M}}$.

Without loss of generality we may assume that the group G is not virtually abelian. Indeed if G is virtually abelian of rank $k > 1$ then by [12, Lemma 7] the Floyd boundary is a point, and so the map is constant. If $\mathsf{rk}(G) = 1$ then the Floyd boundary contains 2 points, so G is virtually cyclic; by [33] and [9] the Martin boundary is then the zero-dimensional sphere $\mathbb{S}^0$ which is homeomorphic to the Floyd boundary.

We first prove that every sequence $(x_n) \subset G$ converging to a point $\alpha \in \partial_{\mathcal{M}} G$ converges to a point $\mathfrak{p} \in \partial_f G$, and furthermore this limit does not depend on the choice of the sequence converging to the point α.

Suppose by contradiction that it is not true. Since the Floyd completion $\overline{G}_f$ is compact up to passing to subsequences we obtain two sequences (x_n) and (x'_n) in G converging to $\alpha \in \partial_{\mathcal{M}} G$ which converge to distinct points $\mathfrak{p}, \mathfrak{p}' \in \partial_f G$. Let $\delta_0 = \delta_o^f(\mathfrak{p}, \mathfrak{p}') > 0$. Fix an arbitrary $g \in G$. By the triangle inequality we have either $\delta_o^f(g, \mathfrak{p}) > \delta_0/2$ or $\delta_o^f(g, \mathfrak{p}') > \delta_0/2$. Up to passing to a new subsequence we may assume that for one of them, for example (x_n) we have $\forall n \in \mathbb{N} : \ \delta_o^f(g, x_n) > \delta_0/2$.

By the inequality (7) we have $\mathcal{G}(g, x_n) \le A \cdot \mathcal{G}(g, o) \cdot \mathcal{G}(o, x_n)$ where $A = A(\delta_0) < +\infty$. Since $x_n \to \alpha$ in $\overline{G}_{\mathcal{M}}$, taking limits we get $K_\alpha(g) \le A \cdot \mathcal{G}(g, o)$ for all $g \in G$. Then by Lemma 7.2 $K_\alpha(g) \to 0$ once $d(o, g) \to \infty$, and by Lemma 7.1 the function K_α is harmonic. Since $K_\alpha(o) = 1$ we obtain that K_α attains its maximum inside G which is impossible [40, 1.15]. We have proved that if $x_n \to \alpha \in \partial G_{\mathcal{M}}$ and $x_n \in G$ then there exists $\mathfrak{p} \in \overline{G}_f$ such that $x_n \to \mathfrak{p}$. Furthermore this limit does not depend on the choice of the sequence $x_n \in G$. Denote $\mathfrak{p} = \pi(\alpha)$.

Step 2. *π is a well-defined, surjective, continuous and equivariant map from $\overline{G}_{\mathcal{M}}$ to $\overline{G}_f$.*

After Step 1 we need only to show that if $(x_n) \subset \partial_{\mathcal{M}} G$ is a sequence which converges to $\alpha \in \partial_{\mathcal{M}} G$ then $\pi(x_n) \to \pi(\alpha)$ $(n \to \infty)$.

It follows from the classical diagonal procedure. Indeed the space $\overline{G}_{\mathcal{M}}$ is a metrisable Cauchy completion with respect to a metric θ generating the topology of $\overline{G}_{\mathcal{M}}$ [40, Section 24.5]. For every $n \in \mathbb{N}$ choose a sequence $x_{n,m}$ in G tending to x_n in $\overline{G}_{\mathcal{M}}$ when $m \to \infty$. By Step 1 for every $n \in \mathbb{N}$ there is a point $\pi(x_n) = \lim_{m \to \infty} x_{n,m} \in \partial_f G$. Using the diagonal procedure we can choose a subsequence $z_n = x_{n,m_n}$ of $x_{n,m}$ such that

$$\theta(z_n, x_n) < 1/n \text{ and } \delta_o^f(z_n, \pi(x_n)) < 1/n, \ \text{ for all } n > 0.$$

The first inequality implies that $z_n \to \alpha$ in $\overline{G}_{\mathcal{M}}$, hence $z_n \to \pi(\alpha)$ $(n \to \infty)$ in $\overline{G}_f$ by Step 1. Together with the second inequality it yields:

$$\forall \varepsilon > 0 \, \exists n_0 \, \forall n > n_0 \ : \delta_o^f(\pi(x_n), \mathfrak{p}) \le \delta_o^f(\pi(x_n), z_n) + \delta_o^f(z_n, \pi(\alpha)) < \varepsilon.$$

So $\pi(x_n) \to \pi(\alpha)$ and the map $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ is well defined and continuous.

Since π is obtained by the continuous extension of the identity map of G it is necessarily equivariant and surjective. The Theorem is proved. □

7.2 Minimal points: preimages of conical points

Let $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ be the map obtained in Theorem 7.3. Our next goal is to study the fibers of π over the points of the Floyd boundary $\partial_f G$. The rest of this section is devoted to proving that the π-preimage of every conical point in $\partial_f G$ is a single point. In the next section we study the fibers of π over the parabolic points of $\partial_f G$.

The following is a simple consequence of Theorem 1.1:

Lemma 7.4 *There is a non-increasing positive function $S_1 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all $y \in G$ and $\alpha \in \partial_{\mathcal{M}} G$ we have*

$$K_\alpha(y) \leq S_1(\delta_o^f(y, \pi(\alpha))\mathcal{G}(y, o).$$

Proof Let a sequence $(x_n) \subset G$ converge to α in $\overline{G}_{\mathcal{M}}$. Then by Theorem 7.3 $x_n \to \pi(\alpha)$ in $\overline{G}_f$, so for all $y \in G$ and large enough n we have

$$\delta_o^f(y, x_n) \geq \delta_o^f(y, \pi(\alpha)) - \delta_o^f(x_n, \pi(\alpha)) \geq \delta_o^f(y, \pi(\alpha))/2.$$

By the inequality (7) it follows that

$$\mathcal{G}(y, x_n)/\mathcal{G}(o, x_n) \leq S(\delta_o^f(y, x_n))\mathcal{G}(y, o) \leq S(\delta_o^f(y, \pi(\alpha))/2)\mathcal{G}(y, o),$$

for a non-increasing positive function $S(\cdot)$.

Passing to the limit we obtain

$$K_\alpha(y) \leq S_1(\delta_o^f(y, \pi(\alpha))\mathcal{G}(y, o)$$

for $S_1(t) = S(t/2)$. □

For a function $h : G \to \mathbb{R}_{\geqslant 0}$ define its Martin support to be

$$\mathsf{supp}_{\mathcal{M}}(h) = \{\zeta \in \partial_{\mathcal{M}} G : \limsup_{x \to \zeta} h(x) > 0\}$$

and its Floyd support

$$\mathsf{supp}_f(h) = \{\mathfrak{q} \in \partial_f G : \limsup_{x \to \mathfrak{q}} h(x) > 0\}.$$

By Theorem 7.3 $\mathsf{supp}_f(h) = \pi(\mathsf{supp}_{\mathcal{M}}(h))$. Clearly if $0 \leq u \leq h$ then $\mathsf{supp}_f(u) \subset \mathsf{supp}_f(h)$ and $\mathsf{supp}_{\mathcal{M}}(u) \subset \mathsf{supp}_{\mathcal{M}}(h)$. Lemma 7.4 implies the following.

Corollary 7.5 *For every $\alpha \in \partial_{\mathcal{M}} G$ one has $\mathsf{supp}_f(K_\alpha) = \pi(\alpha)$.*

Proof Indeed if $y \to \mathfrak{q} \in \partial_f G \setminus \pi(\alpha)$ then $\delta_0^f(\mathfrak{q}, \pi(\alpha)) > 0$. By Lemma 7.2 $\mathcal{G}(o, y) \to 0$ ($||y|| \to \infty$). It follows from Lemma 7.4 that $K_\alpha(\mathfrak{q}) = 0$.

If now $y \to \pi(\alpha)$ then $K_y(y) = \mathcal{G}(y, y)/\mathcal{G}(o, y)$ is not bounded so $\pi(\alpha) \in \mathsf{supp}_f(K_\alpha)$. □

Lemma 7.6 *Let $A_1, A_2 \subset \partial_{\mathcal{M}} G$ be closed subsets of the Martin boundary such that $\pi(A_i)$ are disjoint subsets of the Floyd boundary. Then for any sequence $x_n \to \alpha$ with $\alpha \in A_1$ the functions $\beta \to K_\beta(x_n)$ converge to* 0 *uniformly over $\beta \in A_2$.*

Proof Let U_i be closed neighborhoods of A_i in $\overline{G}_{\mathcal{M}}$ such that $\pi(U_i)$ are disjoint. Then there is $\delta > 0$ such that $\delta_o^f(u_1, u_2) > \delta$ for all $u_i \in U_i$.

By the inequality (7) there is a constant $C = C(\delta) > 0$ such that $K_{u_2}(u_1) < C \cdot \mathcal{G}(u_1, o)$ for all $u_i \in U_i \cap G$.

Then using a sequence $u_{2,n} \in U_2 \cap G$ such that $u_{2,n} \to \beta \in A_2$, we obtain $K_\beta(u_1) < C \cdot \mathcal{G}(u_1, o)$ for any $u_1 \in U_1$.

Assuming now that $u_{1,n} \to \alpha \in A_1$ we have $u_{1,n} \in U_1$ for large enough n. Thus, for each $\beta \in A_2$ we obtain $K_\beta(u_{1,n}) < C \cdot \mathcal{G}(u_{1,n}, o) \to 0$ by Lemma 7.2. □

Recall that a positive μ-harmonic function $h : G \to \mathbb{R}_{\geqslant 0}$ is called minimal harmonic if for every μ-harmonic function $q : G \to \mathbb{R}_{\geqslant 0}$ with $q \leq h$ we have $q = c \cdot h$ for some constant $c \in \mathbb{R}$. A point $\alpha \in \partial_{\mathcal{M}} G$ is called *minimal* if the corresponding function $K_\alpha(\cdot)$ is minimal. The minimal Martin boundary $\partial_{\mathcal{M}}^{min} G \subset \partial_{\mathcal{M}} G$ consist of minimal points. The following is the Martin representation theorem, see e.g. [36], [40, (24.7),(24.8)].

Theorem 7.7 *(Martin Representation Theorem) Any minimal harmonic function $h : G \to \mathbb{R}_{\geqslant 0}$ with $h(o) = 1$ is of the form $h(x) = K_\alpha(x)$ for some $\alpha \in \partial_{\mathcal{M}} G$. For any positive μ-harmonic function $h : G \to \mathbb{R}_{\geqslant 0}$ there is a finite measure ν^h on $\partial_{\mathcal{M}}^{min} G$ such that*

$$h(x) = \int_{\alpha \in \partial_{\mathcal{M}}^{min} G} K_\alpha(x) d\nu^h(\alpha) \tag{$\star$}$$

for every $x \in G$.

Proposition 7.8 *Let h be any positive harmonic function. Then the representing measure ν^h is supported on $\pi^{-1}\left(\overline{\mathsf{supp}_f h}\right)$ where $\overline{\mathsf{supp}_f h}$ is the closure of $\mathsf{supp}_f h$ in $\partial_f G$.*

Proof Suppose not. Then there is a closed subset $A \subset \partial_{\mathcal{M}} G \backslash \pi^{-1}\left(\overline{\mathsf{supp}_f h}\right)$ with $\nu^h(A) > 0$. Consider the positive harmonic function

$$h'(x) = \int_{\beta \in A} K_\beta(x) d\nu^h(\beta).$$

By the Martin representation theorem the set $\partial_{\mathcal{M}}^{min} G$ is a subset of $\partial_{\mathcal{M}} G$ of full ν^h-measure. The function h satisfies ($\star$), so we have $h' \leq h$ everywhere.

Since the non-constant harmonic function h' cannot attain a maximal value on G there is a sequence $x_n \in G$ converging to some $\eta \in \partial_{\mathcal{M}} G$ with $h'(x_n) \to c > 0$. This implies $\liminf_{n\to\infty} h(x_n) \geq c > 0$ so $\eta \in \pi^{-1}\left(\overline{\mathsf{supp}_f h}\right)$. Since A is a closed set disjoint from the closure of $\pi^{-1}(\mathsf{supp}_f h)$ we get by Lemma 7.6 that $K_\beta(x_n) \to 0$ uniformly for $\beta \in A$. This implies $h'(x_n) = \int_{\beta\in A} K_\beta(x_n) d\nu^h(\beta) \to 0$ $(x_n \to \beta)$ contradicting $h'(x_n) \to c > 0$. □

Corollary 7.9 *For every $\alpha \in \partial_{\mathcal{M}} G$, if $h = K_\alpha$ then ν^h is supported on $\pi^{-1}(\pi(\alpha))$.*

Proof By Corollary 7.5 $\pi(\alpha) = \mathsf{supp}_f(h)$, so the Corollary follows from Proposition 7.8. □

Corollary 7.10 *For every $\mathfrak{p} \in \partial_f G$, $\pi^{-1}(\mathfrak{p})$ contains a point of $\partial^{min}_{\mathcal{M}} G$.*

Proof Since the map π is surjective there exists some $\alpha \in \pi^{-1}(\mathfrak{p})$. For $h = K_\alpha$ it follows from Corollary 7.9 that ν^h gives full (hence nonzero) measure to $\pi^{-1}(\mathfrak{p}) \cap \partial^{min}_{\mathcal{M}} G$, so this set must be nonempty. □

Corollary 7.11 *If $\mathfrak{p} \in \partial_f G$ is a point such that there is a constant $C > 0$ with $K_\beta(x)/K_\alpha(x) \leq C$ for all $x \in G$ and $\alpha, \beta \in \pi^{-1}(\mathfrak{p})$ then $\pi^{-1}(\mathfrak{p})$ consists of a single point.*

Proof By Corollary 7.10 there exists $\alpha \in \pi^{-1}(\mathfrak{p})$ such that the function $h = K_\alpha(\cdot)$ is minimal. Let $\beta \in \pi^{-1}(\mathfrak{p})$ be any other point. By assumption $K_\beta(x)/K_\alpha(x) \leq C$ for all $x \in G$ and thus by minimality of $K_\alpha(\cdot)$ there exists a constant $C' > 0$ such that $K_\beta(x) = C' \cdot K_\alpha(x)$ for all $x \in G$. By the normalization assumption $K_\beta(o) = K_\alpha(o) = 1$ so $C' = 1$ and $\alpha = \beta$. □

We will use Corollary 7.11 to prove that if $\mathfrak{p} \in \partial_f G$ is conical, then $\pi^{-1}(\mathfrak{p})$ consists of a single point.

Proposition 7.12 *Assume $\mathfrak{p} \in \partial_f G$ is conical. Then there is a constant $D = D_{\mathfrak{p}}$ such that for each $x \in G$ there exists a closed neighborhood W of $\mathfrak{p}$ in $\overline{G}_f$ for which one has*

$$D^{-1} \leq K_y(x)/K_z(x) \leq D$$

for all $y, z \in W \cap G$.

Proof Since $\mathfrak{p} \in \partial_f G$ is conical there exists a sequence $g_n \in G$ and distinct points $\mathfrak{q}, \mathfrak{q}' \in \partial_f G$ such that $g_n\mathfrak{p} \to \mathfrak{q}'$ and $g_n\mathfrak{r} \to \mathfrak{q}$ for all $\mathfrak{r} \in \overline{G}_f \setminus \{\mathfrak{p}\}$.

Let $U, V \subset \overline{G}_f$ be disjoint closed neighborhoods of $\mathfrak{q}'$ and $\mathfrak{q}$ respectively and $0 < \varepsilon < \delta^f_o(U, V)$.

Fix $x \in G$. For n large enough we choose an element $g = g_n$ such that $gx, go \in V$ and $g\mathfrak{p} \in U$. Then $W = g^{-1}U$ is a closed neighborhood of $\mathfrak{p}$ in $\overline{G}_f$. For every $y \in W$ $\delta_o^f(gy, gx) > \varepsilon$ and $\delta_o^f(gy, go) > \varepsilon$. Thus,

$$\delta^f_{g^{-1}o}(y, x) > \varepsilon \text{ and } \delta^f_{g^{-1}o}(y, o) > \varepsilon.$$

Hence by (7) the constant $C = S(\varepsilon)$ satisfies

$$\mathcal{G}(y, g^{-1}o)\mathcal{G}(g^{-1}o, o) \leq \mathcal{G}(y, o) \leq C\mathcal{G}(y, g^{-1}o)\mathcal{G}(g^{-1}o, o)$$

and

$$\mathcal{G}(y, g^{-1}o)\mathcal{G}(g^{-1}o, x) \leq \mathcal{G}(y, x) \leq C\mathcal{G}(y, g^{-1}o)\mathcal{G}(g^{-1}o, x)$$

for all $y \in W$.

Hence,

$$C^{-1} \cdot \frac{\mathcal{G}(g^{-1}o, x)}{\mathcal{G}(g^{-1}o, o)} \leq K_y(x) \leq C \cdot \frac{\mathcal{G}(g^{-1}o, x)}{\mathcal{G}(g^{-1}o, o)}.$$

This is true for every $y \in W$ hence for distinct $y, z \in W$ we have

$$D^{-1} \leq K_y(x)/K_z(x) \leq D$$

for all $y, z \in W$ where $D = C^2$ is a constant. □

Corollary 7.13 *For each conical* $\mathfrak{p} \in \partial_f G$ *there is a constant* $D = D(\mathfrak{p})$ *such that for all* $\alpha, \beta \in \pi^{-1}\mathfrak{p}$ *and* $x \in G$ *we have* $K_\alpha(x)/K_\beta(x) \leq D$.

Proof Let $y_n, z_n \in G$ be two sequences with $y_n, z_n \to \mathfrak{p}$ in the Floyd compactification and $y_n \to \alpha$, $z_n \to \beta$ in the Martin compactification. Then by Proposition 7.12 there exists a uniform constant D such that for each $x \in G$ there is a neighborhood $W \subset \overline{G}_f$ such that for large enough n we have $y_n, z_n \in W$ and $D^{-1} \leq K_{y_n}(x)/K_{z_n}(x) \leq D$. Passing to the limits we obtain the result. □

Corollaries 7.11 and 7.13 imply:

Corollary 7.14 *If* $\mathfrak{q} \in \partial_f G$ *is conical,* $\pi^{-1}(\mathfrak{q})$ *consists of a single point.*

7.3 Minimal invariant subsets of the Martin boundary

The equivariant map $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ constructed in Theorem 7.3 relates the actions of G on these spaces even though these actions have different properties. The action $G \curvearrowright \overline{G}_f$ is convergence [29, Proposition 3] for which $\partial_f G$ is the limit set. Thus if $|\partial_f G| \neq 2$ then $\partial_f G$ is the minimal closed G-invariant set which coincides with the closure of the G-orbit of any $x \in \overline{G}_f$.

The action $G \curvearrowright \partial_{\mathcal{M}} G$ is in general not convergence, in particular $\partial_{\mathcal{M}} G$ can be non-trivial for the Cartesian products of groups [40, Section 28]. However the following Proposition shows that the minimal Martin boundary enjoys a weaker but similar property to that of the limit set of a convergence action.

Proposition 7.15 *Let G, μ and f be as in Theorem 1.1 or 1.2. Assume also that the Floyd boundary $\partial_f G$ contains at least three points. Then the minimal Martin boundary $\partial_{\mathcal{M}}^{min} G$ is contained in the closure of the G-orbit $\Xi = G\xi$ in $\partial_{\mathcal{M}} G$ for any $\xi \in \partial_{\mathcal{M}} G$.*

Proof Let us fix $\alpha \in \partial_{\mathcal{M}}^{min} G$ and the orbit $\Xi = G\xi,\ \xi \in \partial_{\mathcal{M}} G$. Our goal is to show that $\alpha \in \overline{\Xi}$. Let V_n be the ball $B(o, n) \subset G$ of radius $n \in \mathbb{N}$ in the word distance centered at the basepoint o. Denote by δ the diameter $\mathrm{diam}_{\delta_x}(\partial_f G)$ of $\partial_f G$ with respect to the Floyd distance δ_x based at the point $x \in G$. Since the left multiplication by xo^{-1} is an isometry $(G, \delta_o) \to (G, \delta_x)$, the quantity δ does not depend on $x \in G$.

For a given finite set $F \subset G$, one has $\mathrm{diam}_{\delta_x}(F) \to 0$ as $x \to \infty$. So there exists a sequence $\{x_n : n \in \mathbb{N}\}$ in G converging to α, such that

$$\mathrm{diam}_{\delta_{x_n}}(V_n) < \delta/4,\ \ n \in \mathbb{N}. \tag{34}$$

Since $\mathrm{diam}_{\delta_{x_n}}(\partial_f G) = \delta$, it follows from (34) that for each $n \in \mathbb{N}$, there exists an open $O_n \subset \overline{G}_f$ such that $\partial_f G \cap O_n \neq \varnothing$ and

$$\delta_{x_n}(V_n, O_n) \geqslant \delta/4. \tag{35}$$

The map $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ given by Theorem 7.3 is G-equivariant so the set $\pi(\Xi) \subset \partial_f G$ is G-invariant. As $|\partial_f G| > 2$ the convergence action $G \curvearrowright \partial_f G$ is non-elementary. So $\pi(\Xi)$ is dense in $\partial_f G$. Thus, there exists $\xi_n \in \Xi \cap \pi^{-1} O_n$.

The space $\overline{G}_{\mathcal{M}}$ is metrizable so there exists a metric ϱ defining its topology [36], [40]. Choose $y_n \in G \cap O_n$ such that

$$\varrho(y_n, \xi_n) \leqslant 2^{-n},\ \ n \in \mathbb{N}. \tag{36}$$

By (34) and Theorem 1.1, for every $z \in V_n$ we have

$$d_{\mathcal{G}}(z, x_n) + d_{\mathcal{G}}(x_n, y_n) \leq d_{\mathcal{G}}(z, y_n) + A(\delta/4).$$

Hence

$$d_{\mathcal{G}}(z, y_n) - d_{\mathcal{G}}(o, y_n) + A(\delta/4) \geq d_{\mathcal{G}}(z, x_n) - d_{\mathcal{G}}(o, x_n).$$

Since $o \in V_n$ the same inequality is also true if we permute the points o and z. Thus we obtain

$$|\Delta(z, o, x_n) - \Delta(z, o, y_n)| \leq A(\delta/4). \tag{37}$$

By compactness of $\overline{G}_{\mathcal{M}}$ there exists a subsequence $\{y_{n_k} : k \in \mathbb{N}\}$ converging to a point $\beta \in \partial_{\mathcal{M}} G$. So (37) implies that $K_\beta(z)/K_\alpha(z)$ is bounded above uniformly on z. By minimality of α it follows that $K_\beta/K_\alpha \equiv$ const and since $K_\beta(o) = K_\alpha(o) = 1$ we have $K_\alpha \equiv K_\beta$ and so $\beta = \alpha$.

By (36) $\xi_{n_k} \to \beta = \alpha$ $(k \to \infty)$ implying the Proposition. □

Corollary 7.16 *The orbit* $\{g\pi^{-1}(\mathfrak{q}) \ : \ g \in G\}$ *of the* π*-preimage of any conical point* $\mathfrak{q} \in \partial_f G$ *is a dense subset of* $\partial^{min}_{\mathcal{M}} G$.

Proof By Corollary 7.14 the preimage $\xi = \pi^{-1}(\mathfrak{q})$ is a single point and by Corollary 7.10 we have $\xi \in \partial^{min}_{\mathcal{M}} G$. Since every point $g(\mathfrak{q})$ is also conical by the same reason we have for any $g \in G$, $\pi^{-1}(g(\mathfrak{q})) = g(\xi) \in \partial^{min}_{\mathcal{M}} G$ as π is equivariant. So $G\xi \subset \partial^{min}_{\mathcal{M}} G$. By Proposition 7.15 $G\xi$ is a dense subset of $\partial^{min}_{\mathcal{M}} G$. □

We finish the subsection with the following substantial example[4]:

Remark 7.17 There exist groups having a non-trivial Floyd boundary which admit symmetric finitely supported measures whose minimal Martin boundary is a proper subset of the Martin boundary. Indeed, suppose G_1 is non-amenable, G_2 any finitely generated infinite group, and μ_i finitely supported generating measures on G_i. Let $G = G_1 \times G_2$ be the Cartesian product and $\mu = \mu_1 \times \mu_2$ be the product measure. Picardello and Woess show [34, Corollary 4.4] that the Martin boundary of $(G, t\mu)$ contains non-minimal points for any t up to and including the inverse of the spectral radius of μ. Then Theorems 26.18 and 26.21 of [40] imply that whenever (Γ, m) is any finitely generated group and m a finitely supported measure on Γ, the Martin boundary of the free product $(G * \Gamma, \mu + m)$ contains non-minimal points. Furthermore, if now one chooses Γ to be hyperbolic then $G * \Gamma$ is relatively hyperbolic with respect to G and so its Floyd boundary is non-trivial by Theorem 3.3.

[4] We thank Wolfgang Woess for indicating to us this example.

7.4 Connection with the Freudenthal compactification

Let G be a finitely generated group and Γ its Cayley graph. We denote by $\partial_{\mathcal{F}} G$ the *Freudenthal boundary* (*end space*) of G (or Γ) [13]. We refer to [37] for all standard definitions of the theory of ends.

Let $\xi \in \partial_{\mathcal{F}} G$ be an end. For a finite subset of edges $U \subset \Gamma^1$ denote by $C_U(\xi)$ the unique component of $\Gamma \backslash U$ containing ξ. Let $C_{U_i}(\xi)$ be a sequence of strictly sthrinking components: $C_{U_{i+1}}(\xi) \subset C_{U_i}(\xi)$. Then their closures $\bar{C}_{U_i}(\xi)$ give a neighborhood basis of ξ. Following Woess [41], we say that an end ξ is *thin* if there exists a sequence of finite subsets $U_i \subset \Gamma^1$ of bounded cardinality such that $U_{i+1} \subset C_{U_i}(\xi)$ for each i. We start with the following.

Proposition 7.18 *The identity map* id : $G \to G$ *extends to a continuous* G*-equivariant surjection* $\Phi : \partial_f G \to \partial_{\mathcal{F}} G$ *for any Floyd function* f*. Moreover, the set of thin ends of* $\partial_{\mathcal{F}} G$ *coincides with the set of its conical points.*

Proof It is enough to show that any two sequences $x_n, y_n \in G$ converging to the same point in $\partial_f G$ (i.e. $\delta_o^f(x_n, y_n) \to 0$ as $n \to \infty$ for a basepoint $o \in G$), also converge to the same point in $\partial_{\mathcal{F}} G$. Suppose not. By the description of the neighborhood basis of an end, there exists a finite set E of edges of Γ such that any path connecting x_n and y_n has to pass through E ($n \geq n_0$). Since the quantity $d(o, E)$ is bounded, by definition of the Floyd metric we have $\delta_o^f(x_n, y_n) \geq c$ for some constant $c = c(E) > 0$. This is a contradiction, so we obtain an equivariant continuous surjection Φ from $\overline{G}_f$ to the Freudenthal completion $\overline{G}_{\mathcal{F}} = G \cup \partial_{\mathcal{F}} G$.

As the action $G \curvearrowright \partial_f G$ is convergence, the surjective equivariant continuous map $\Phi : \partial_f G \to \partial_{\mathcal{F}} G$ implies that the action of G on $\partial_{\mathcal{F}} G$ is also convergence (this result is originally due to Stallings [37]).

If a point ξ in $\partial_{\mathcal{F}} G$ is conical then there exist $g_n \in G$ and distinct ends $\{a, b\} \subset \partial_{\mathcal{F}} G$ such that $g_n(\zeta, \xi) \to (a, b)$ for any $\zeta \in \partial_{\mathcal{F}} G \backslash \{\xi\}$. Thus, a and b are separated by a finite set U and hence ζ and ξ are separated by the set $U_n = g_n^{-1} U$ for all $n \geq n_0$. Since the action on $\partial_{\mathcal{F}} G$ is convergence, the sets U_n converge to ξ, so we can extract a subsequence, still denoted by U_n, such that $U_{n+1} \subset C(U_n, \xi)$. It follows that ξ is thin. If ξ is thin the previous argument is reversible, so it implies that ξ is conical. □

Putting $\Psi = \pi \circ \Phi$ where π is the map from Theorem 1.5, we obtain as a corollary the following result which is originally due to W. Woess.

Corollary 7.19 ([41]) *For a finitely generated group* G *there exists a continuous* G*-equivariant surjection* $\Psi : \partial_{\mathcal{M}} G \to \partial_{\mathcal{F}} G$*. The preimage* $\Psi^{-1}(\xi)$ *of every thin end* $\xi \in \partial_{\mathcal{F}} G$ *is a single point in the minimal Martin boundary* $\partial_{\mathcal{M}}^{min} G$*. Furthermore, if* G *has infinitely many ends then the* G*-orbit of the* Ψ*-preimage of any thin end is dense in* $\partial_{\mathcal{M}}^{min} G$*.*

Proof Let $\xi \in \partial_{\mathcal{F}} G$ be a thin end. Then, by the above Proposition, it is a conical point for the convergence action $G \curvearrowright \partial_{\mathcal{F}} G$. Then, by [18, Proposition 7.5.2], the set $\Phi^{-1}(\xi)$ consists of one point $\mathfrak{q} \in \partial_f G$ which is conical too.

If G is an infinitely ended group then the set $\partial_f G = \Phi^{-1}(\partial_{\mathcal{F}} G)$ is infinite too. By Corollary 7.16 the orbit $\Xi = G(\Psi^{-1}(\xi))$ is a dense subset of $\partial_{\mathcal{M}}^{min} G$.□

Remark. The dense subset indicated in the Corollary is much "thinner" than the preimage of all thin ends which is also a dense subset of $\partial_{\mathcal{M}}^{min} G$.

8 Preimages of parabolic points in the Martin boundaries

If G is a finitely generated group then by Theorem 7.3 there exists a surjective, equivariant, and continuous map $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ from the Martin to the Floyd completion of G. Furthermore by Corollary 7.14 the map π is injective on the subset π^{-1}(conical points) of $\overline{G}_{\mathcal{M}}$. To have a complete description of the Martin boundary we need to study the π-preimages of non-conical points.

The main result of this section is Proposition 8.1 below which gives a partial description of π-preimages of non-conical points. We note that this partial description was already essentially used in the paper [11] where a complete description of the Martin boundary of the class of groups hyperbolic relative to virtually abelian subgroups was deduced.

We denote by X the Floyd completion $\overline{G}_f$ of G with respect to a Floyd function f and the Floyd metric δ_v^f based at a point $v \in G$ (see Sect. 3).

By [29] the action $G \curvearrowright X$ is a convergence action. For a subgroup $H \leq G$ we denote by ΛH its limit set for the action on X. If $|\Lambda H| > 2$ it coincides with the subset of accumulation points of the H-orbit in X.

We will now introduce a few notions which are used in this and the next sections.

Consider a geodesic (infinite or not) in the Cayley graph equipped with the word metric d. A bi-infinite geodesic $\gamma : \mathbb{Z} \to G$ is called a *horocycle* at $p \in \partial X$ if $\lim\limits_{n\to\pm\infty} \gamma(n)=p$. By [19, Lemma 3.6] the unique limit point p of γ is not conical,[5] and is called the *base* of the horocycle. A *horosphere* P at the parabolic point p is the set of all horocycles based at p. So as in the classical case of discrete groups acting on hyperbolic spaces, a horosphere is the geodesic convex hull of a parabolic point.

We define the geodesic convex hull $\mathcal{H}$ in X of the limit set ΛH of an arbitrary subgroup $H < G$ in a similar way:

$$\mathcal{H} = \{\gamma : \mathbb{Z} \to G \text{ is a geodesic } : \lim_{n\to\pm\infty} \gamma(n) \in \Lambda H\}.$$

[5] In [19] this statement is formally stated for the Bowditch boundary but the proof equally works on the Floyd boundary as the only tool which is used is the Karlsson lemma (see Introduction).

Let $\partial_{\mathcal{M}} G$ be the Martin boundary of G with respect to a probability measure μ on G satisfying Assumptions 1 and 2 (see Sect. 7), and $\overline{G}_{\mathcal{M}} = G \sqcup \partial_{\mathcal{M}} G$ its Martin compactification. Let $\partial^{\mathcal{M}} H$ be the topological boundary of H in $\overline{G}_{\mathcal{M}}$, i.e. the set of accumulation points of H in $\overline{G}_{\mathcal{M}}$.

A subgroup H of G is called quasiconvex if any geodesic between two elements of H belongs to a uniform neighborhood of H. It is called *fully quasiconvex* if it is quasiconvex and for every parabolic subgroup P of G either $H \cap P$ is a subgroup of finite index of P or is finite. Note that if the group G is relatively hyperbolic the cocompactness of the action H on $X \backslash \Lambda H$ is equivalent to the full quasiconvexity of H [21, Theorem B].

By Corollary 7.10 for every point $\mathfrak{q} \in \partial_f G$ its preimage $\pi^{-1}(\mathfrak{q})$ contains points from the minimal Martin boundary $\partial_{\mathcal{M}}^{min} G$. A natural question is whether the π-preimage of the limit set $\Lambda H \subset \partial_f G$ of a subgroup H is a subset of the Martin boundary of H. The following proposition provides a partial answer to this question.

Proposition 8.1 *Let $\pi : \partial_{\mathcal{M}} G \to \partial_f G$ be a continuous equivariant map from the Martin boundary to the Floyd boundary of G. Let $H < G$ be a subgroup acting cocompactly on $X \backslash \Lambda H$. Then*

$$\pi^{-1}(\Lambda H) \cap \partial_{\mathcal{M}}^{min} G \subseteq \partial^{\mathcal{M}} H. \tag{38}$$

Proof of Proposition 8.1 In all arguments below the subgroup H acting cocompactly on $X \backslash \Lambda H$ is fixed. For a vertex $x \in G$ we denote by $\mathsf{Pr}_{\mathcal{H}}(x)$ the projection set $\{y \in \mathcal{H} : d(y, x) = d(x, \mathcal{H})\}$ of x to the convex hull $\mathcal{H}$ of H. Denote by o a basepoint in G. □

Lemma 8.2 *There exist two constants $D = D(H) < +\infty$ and $\delta = \delta(H) > 0$ such that for every sequence x_n converging to a point $\mathfrak{q} \in \Lambda H$ and for the sequence of projections $v_n \in \mathsf{Pr}_{\mathcal{H}}(x_n)$ and $n \geq n_0$ we have $\delta_{v_n}^f(o, x_n) \geq \delta$ and $d(v_n, \gamma_n) \leq D$ where $\gamma_n = [o, x_n]$ is a geodesic between o and x_n.*

Proof The set $\mathcal{H} \cup \Lambda H$ is a closed subset of X. Since the action of H on $X \backslash \Lambda H$ is cocompact, the quotient $\mathcal{H}/H$ is finite (we call the set $\mathcal{H}$ *weakly homogeneous* in this case) [21, Proposition 4.5]. Let $\mathcal{F}$ denote a compact fundamental set for the action of H on $X \backslash \Lambda H$. Since $\mathcal{F} \cap \Lambda H = \emptyset$ there exists a constant $\nu = \nu(H) > 0$ such that $\delta_o^f(\mathcal{F}, \Lambda H) \geq \nu$.

Let $F = \mathsf{Pr}_{\mathcal{H}}(\mathcal{F} \cap G)$. Since $\mathcal{H}$ is H-invariant and weakly homogeneous by [21, Proposition 3.5] the diameter $d = \operatorname{diam}(F)$ with respect to the word metric is finite and depends only on the constant ν above.

Let $\gamma_n : \mathbb{N} \to G$ be a geodesic between o and x_n such that $\lim\limits_{n\to\infty} x_n = \mathfrak{q} \in \Lambda H$. Then there exists a sequence $h_n \in H$ such that $y_n = h_n(x_n) \in \mathcal{F} \cap G$. Since the action of H on the Cayley graph of G is isometric, the images

$u_n = h_n(v_n)$ of the projections v_n of x_n to $\mathcal{H}$, are projections of y_n to $\mathcal{H}$. So the set $\{u_n\}_n \subset F$ is a finite subset of the graph of diameter at most $2d$.

Set $z_n = h_n(o)$ and fix a sufficiently small $\varepsilon \in (0, \nu/2)$. Denote by $\mathsf{N}^f_\varepsilon(\Lambda H)$ the ε-neighbourhood of ΛH in X with respect to the Floyd distance δ^f_o. We have $z_n \in \mathsf{N}^f_\varepsilon(\Lambda H)$ for $n > n_0$.

Using the inequality (9) for the finite set $\{u_n\}_n \subset F$ of diameter at most $2d$, we obtain

$$\delta^f_{u_n}(\mathcal{F}, \mathsf{N}_\varepsilon(\Lambda H)) \geq \delta, \text{ where } \delta = \frac{\nu - \varepsilon}{\kappa^{d(o,F)+2d}} \geq \frac{\nu/2}{\kappa^{d(o,F)+2d}} > 0. \quad (39)$$

Note that the lower bound (39) depends only on H and fixed $\varepsilon \in (0, \nu/2)$. From (39) it follows that $\delta^f_{u_n}(y_n, z_n) \geq \delta$, and applying h_n^{-1} we obtain $\delta^f_{v_n}(x_n, o) \geq \delta > 0$.

By Karlsson's lemma [29, Lemma 1] there exists a constant $D = D(H, \varepsilon)$ such that $d(v_n, \gamma_n) \leq D$. □

End of the proof of Proposition 8.1. By Corollary 7.10 for a point $\mathfrak{q} \in \Lambda H$ there exists a point $\alpha \in \pi^{-1}(\mathfrak{q}) \cap \partial^{min}_{\mathcal{M}} G$ such that the harmonic function K_α is minimal. Consider a sequence of points $x_n \to \alpha$ $(n \to \infty)$ and their projections $v_n \in \mathsf{Pr}_{\mathcal{H}}(x_n)$ to $\mathcal{H}$.

For a geodesic $\gamma_n = [o, x_n]$ by Lemma 8.2 we obtain points $w_n \in \gamma_n$ such that $d(v_n, w_n) = d(v_n, \gamma_n) \leq D$. Then applying the Harnack inequality (Lemma 2.1) we obtain a constant $\lambda \in (0, 1)$ such that for any $x \in G$ we have

$$\frac{K_{v_n}(x)}{K_{w_n}(x)} = \frac{\mathcal{G}(x, v_n) \cdot \mathcal{G}(o, w_n)}{\mathcal{G}(o, v_n) \cdot \mathcal{G}(x, w_n)} \leq \lambda^{-2d(w_n, v_n)} \leq \lambda^{-2D}. \quad (40)$$

We also have

$$\frac{K_{w_n}(x)}{K_{x_n}(x)} = \frac{\mathcal{G}(x, w_n) \cdot \mathcal{G}(o, x_n)}{\mathcal{G}(o, w_n) \cdot \mathcal{G}(x, x_n)} \leq S(\delta^f_{w_n}(o, x_n)). \quad (41)$$

Indeed, in the numerator of (41) by the inequality (7) we have:

$$\mathcal{G}(o, x_n) \leq S(\delta_{w_n}(o, x_n)) \cdot \mathcal{G}(o, w_n) \cdot \mathcal{G}(w_n, x_n);$$

and in the denominator we used the (triangle) inequality $\mathcal{G}(x, x_n) \geq \mathcal{G}(x, w_n) \cdot \mathcal{G}(w_n, x_n)$.

By Lemma 8.2 $\delta^f_{v_n}(o, x_n) \geq \delta$ and $d(v_n, w_n) \leq D$ so by (9) we have $\delta^f_{w_n}(o, x_n) \geq \kappa^{-D} \cdot \delta$ which is a uniform constant too. The function $S(\cdot)$ is decreasing so (40) and (41) imply

$$\frac{K_{v_n}(x)}{K_{x_n}(x)} \leq C, \text{ where } C = S(\kappa^{-D}\delta) \cdot \lambda^{-2D}. \tag{42}$$

Replacing the geodesic $[o, x_n]$ in the previous argument by a geodesic $[x, x_n]$ we similarly obtain the points $\tilde{w}_n \in [x, x_n]$ such that for the projections $v_n \in \mathsf{Pr}_{\mathcal{H}}(x_n)$ we have $d(\tilde{w}_n, v_n) \leq D$ and $\delta^f_{v_n}(x, x_n) \geq \delta$ for the same constants D and δ from Lemma 8.2. Then the previous argument implies the double inequality:

$$\frac{1}{C} \leq \frac{K_{v_n}(x)}{K_{x_n}(x)} \leq C, \tag{43}$$

where C is as in (42).

Up to passing to a subsequence we can assume that the sequence $v_n \in \mathcal{H}$ converges to some point $\beta \in \partial_{\mathcal{M}} G$. From (43) we obtain

$$\frac{1}{C} \leq \frac{K_{\beta}(x)}{K_{\alpha}(x)} \leq C. \tag{44}$$

Then $K_\beta \leq C \cdot K_\alpha$ and so $K_\beta = C' \cdot K_\alpha$ for some $C' > 0$ by minimality of α.

We have that $\alpha = \lim\limits_{n\to\infty} v_n = \beta$ and $v_n \in \mathcal{H}$. Since H is quasiconvex there exists a constant C_1 such that for every $v_n \in \mathcal{H}$ there exists $\tilde{v}_n \in H$ such that $d(v_n, \tilde{v}_n) \leq C_1$ [21, Proposition 4.5]. Applying again the Harnack inequality we obtain $\dfrac{K_{\tilde{v}_n}(x)}{K_{v_n}(x)} \leq C_1'$ for some $C_1' > 0$ depending on C_1. Since $\alpha \in \partial^{\min}_{\mathcal{M}} G$ is minimal the above argument yields $\lim\limits_{n\to\infty} \tilde{v}_n = \alpha$. We have proved that every minimal point in $\pi^{-1}(\Lambda H)$ is an accumulation point of the H-orbit. The Proposition is proved. □

Remarks 8.3 1. Note that the choice of the approximation sequence $(v_n) \subset \mathcal{H}$ as the projection of the approximating sequence $(x_n) \subset G$ is constructive. One can prove that $\lim\limits_{n\to\infty} \pi(v_n) = \lim\limits_{n\to\infty} \pi(x_n) = \mathfrak{q} \in \Lambda H$ without assuming that the limit point α on the Martin boundary is minimal. Indeed, if the sequence $\pi(x_n)$ converges to $\mathfrak{q}$ and $\pi(v_n)$ does not, then the word distance $d(v_n, x_n)$ is unbounded. By Lemma 8.2 there exists a point $b_n \in [x_n, \mathfrak{q}[$ such that $d(v_n, b_n) \leq D$. Since $\pi(x_n) \to \mathfrak{q}$ we obtain that the infinite geodesic rays $[x_n, \mathfrak{q}[$ converge to a geodesic horocycle l based at $\mathfrak{q} \in \Lambda H$ (in particular $\mathfrak{q}$ is not a conical point [19, Lemma 3.6]). But $l \subset \mathcal{H}$ so $d(x_n, v_n) > d(x_n, \mathcal{H})$ $(n > n_0)$ which is impossible by definition of v_n. However the same argument does not work for the sequences x_n and

v_n in $\overline{G}_{\mathcal{M}}$ (instead of $\overline{G}_f$), as there is a question whether the boundness of the distance $d_{\mathcal{G}}(x_n, v_n)$ implies that the convergence of the first yields the convergence of the second to the same limit. We call this property the *perspectivity property*, it is satisfied on $\overline{G}_f$ [29] and remains unknown on $\overline{G}_{\mathcal{M}}$.

2. Several corollaries of Proposition 8.1 for relatively hyperbolic groups as well as several open questions are stated in the next section.

9 Applications to relatively hyperbolic groups

The aim of this section is to provide several useful geometric consequences of our previous results for relatively hyperbolic groups.

9.1 Geometrically finite actions

Suppose G is relatively hyperbolic with respect to a collection $\mathcal{P}$ of subgroups. A point v on a (quasi-)geodesic γ is called an (ε, R)-*transition point* if for any horosphere P based at a parabolic fixed point (see Sect. 8 for the definition) one has $\gamma \cap B(v, R) \not\subset N_\varepsilon(P)$ where $B(v, R)$ denotes the ball centered at v of radius R and $N_\varepsilon(P)$ is an ε-neighborhood of P in the word distance.

The following Proposition provides a characterization of transition points in terms of the Floyd function f.

Proposition 9.1 ([21], Corollary 5.10) *For each $\varepsilon > 0$ and $R > 0$ there is a number $\delta > 0$ such that if y is an (ε, R)-transition point of a word geodesic from x to z then $\delta^f_y(x, z) > \delta$.*

As a result, the inequality (1) admits the following immediate corollary:

Corollary 9.2 *Let G be hyperbolic relative to a collection of subgroups. If $x, y, z \in G$ is an ordered triple of distinct points belonging to a word geodesic γ, and y is an (ε, R)-transition point then*

$$d_{\mathcal{G}}(x, y) + d_{\mathcal{G}}(y, z) \le d_{\mathcal{G}}(x, z) + A$$

where A depends only on (ε, R), and μ. □

We will now prove another corollary of Theorem 1.1 valid for geometrically finite actions on the hyperbolic spaces.

Let (X, d_X) denote Gromov hyperbolic space. For every two distinct points $x, y \in X$ we denote by $[x, y]$ a geodesic segment between them with respect to the hyperbolic metric d_X. Let us fix a basepoint $o \in X$.

Proposition 9.3 *(Corollary 1.8) Let $G \curvearrowright X$ be a geometrically finite, isometric, properly discontinuous and non-elementary action of a group G on a proper geodesic Gromov hyperbolic space X. Let μ and f satisfy the assumptions of Theorem 1.1 or 1.2.*

Then, for every $D > 0$ there exists a constant $C = C(D) > 0$ such that for every triple g, h, w of elements of G with $d_X(ho, [go, wo]) \leq D$ the inequality

$$d_{\mathcal{G}}(g, h) + d_{\mathcal{G}}(h, w) \leq d_{\mathcal{G}}(g, w) + C \tag{45}$$

holds on the Cayley graph of G.

Proof We first prove Proposition 9.3 under the additional assumption that $go \neq wo$, in which case it is an immediate consequence of the inequality (1) and the following lemma.

Lemma 9.4 *Suppose that the assumptions of Proposition 9.3 are satisfied.*

Then for each $D > 0$ there exists $\delta > 0$ such that for any $g, h, w \in G$ satisfying $go \neq wo$ and $d_X(ho, [go, wo]) \leq D$ one has $\delta_h^f(g, w) > \delta$.

Proof of the Lemma Up to multiplying by h^{-1} we may assume that $h = 1$. Suppose by contradiction that the statement is not true. Then we have sequences of elements $g_n, w_n \in G$ and distinct triples $g_n o, w_n o, o$ such that $d_X(o, [g_n o, w_n o]) < D$ and $\delta_o^f(g_n, w_n) \to 0$ $(n \to 0)$. After passing to subsequences and keeping the same notations we have that g_n and w_n converge to the same point $\mathfrak{q} \in \partial_f G$. At the same time $g_n o$ and $w_n o$ converge to two distinct points $\eta, \zeta \in \Lambda G$. Indeed it follows from the fact that the visual Gromov metric $\nu(g_n o, w_n o)$ is equivalent to the quantity $\exp(-a \cdot d(o, [g_n o, w_n o]))$ on $X \cup \Lambda G$ for some constant $a > 0$ (see the proof of Corollary 1.4 in the Introduction). So $\nu(g_n o, w_n o)$ is bounded below by a positive constant.

The action $G \curvearrowright X$ is properly discontinuous, and induces a convergence action on $X \cup \partial X$. So up to passing to new subsequences (and keeping the same notations) we have that $g_n y \to \eta$ and $w_n y \to \zeta$ for all points $y \in X \cup \partial X$ besides at most two exceptional points $y_i \in \partial X$ $(i = 1, 2)$.

By [29, Proposition 3] the action of G on $\overline{G}_f = G \cup \partial_f G$ is convergence too. Then again up to passing to further subsequences we may assume that for all $z \in \overline{G}_f \backslash \{z_1, z_2\}$ one has $\lim_{n\to\infty} g_n(z) = \lim_{n\to\infty} w_n(z) = \mathfrak{q}$ where $z_1, z_2 \in \partial_f G$ are two possible exceptional points for the sequences (g_n) and (w_n) respectively.

The group G admits a geometrically finite non-elementary action on X, so the limit set $\Lambda G \subset \partial X$ is an infinite set. Then by Theorem 3.3 there exists an equivariant continuous and surjective map $\varphi : \partial_f G \to \Lambda G$. Hence, we have $\varphi(g_n(z)) = g_n(\varphi(z)) \to \varphi(\mathfrak{q})$ $(n \to \infty)$ and also $w_n(\varphi(z)) \to \varphi(\mathfrak{q})$ $(z \in \partial_f G \backslash \{z_1, z_2\},\ n \to \infty)$. This contradicts to the fact that for $\varphi(z) \notin \{y_1, y_2\}$

these two sequences must converge to two distinct points η and ζ. The Lemma is proved.

Lemma 9.4 implies Proposition 9.3 if $go \neq wo$. If now $go = wo$ then $[go, wo] = go = wo$. The condition $d_X(ho, [go, wo]) < D$ implies that $h^{-1}g$ and $h^{-1}w$ both belong to the set $G(o, D)$ consisting of elements of G translating o in a distance d_X at most D. Since the action $G \curvearrowright X$ is properly discontinuous, $G(o, D)$ is finite. The stabilizer G_o of the point o is finite too. By assumption we have $d_X(ho, wo) = d_X(ho, go) = d_X(o, h^{-1}wo) = d_X(o, h^{-1}go) \leq D$. We add to the constant C obtained previously the supremum of the following expression:

$$\begin{aligned} &d_{\mathcal{G}}(g, h) + d_{\mathcal{G}}(h, w) - d_{\mathcal{G}}(g, w) = \\ &d_{\mathcal{G}}(h^{-1}g, o) + d_{\mathcal{G}}(o, h^{-1}w) - d_{\mathcal{G}}(o, g^{-1}w) \end{aligned} \tag{46}$$

taken over all such elements $\{h^{-1}w, h^{-1}g\} \subset G(o, D)$ and $g^{-1}w \in G_o$. The quantity (46) is bounded above since $G(o, D)$ is finite. Keeping the same notation C for the new constant, we obtain Proposition 9.3. □

Note that if a convergence group is not relatively hyperbolic, the above argument does not in general give a uniform constant C independent of the choice of basepoint o. In particular it was recently shown by M. Kapovich that the orders of the point stabilizers can be unbounded even in case when X is an Hadamard space of pinched negative curvature [30]. The aim of the next Corollary is to describe a subset $X_0 \subset X$ for which any choice of the basepoint $o \in X_0$ does not change the constant C given by Proposition 9.3.

Recall that by Gromov's original (equivalent) definition of a geometrically finite action, there exists a G-invariant collection of disjoint horoballs $\mathcal{B}$ based at parabolic fixed points such that the G-action on the *truncated* space $X_0 = X\backslash\mathcal{B}$ is cocompact [26], [4], [27, Definition 3.3].

Corollary 9.5 *Suppose that all the assumptions of Proposition 9.3 are satisfied. Then for every $D > 0$ there exists a constant $C = C(D) > 0$ such that for every basepoint $o \in X_0$ the condition $d_X(ho, [go, wo]) \leq D$ implies the inequality* (45).

Proof We only need to show that if the basepoint belongs to X_0 then Lemma 9.4 provides a uniform lower bound for the Floyd distance. Suppose this is not true. Then there is a sequence of points $o_n \in X_0$ and elements h_n, g_n, w_n such that on the space X we have $d_X(h_n o_n, [g_n o_n, w_n o_n]) \leq D$ and on the Cayley graph $\lim\limits_{n\to\infty} \delta^f_{h_n}(g_n, w_n) = 0$.

Since $o_n \in X_0$ there exists $b_n \in G$ such that $b_n o_n \in R \subset X_0$ where R is a compact fundamental domain for the action $G \curvearrowright X_0$. Precomposing the elements h_n, g_n, w_n with b_n and keeping the same notations, we may assume that $o_n \in R$. So up to passing to a subsequence we may also assume that the endpoints $h_n o_n$ and $g_n o_n$ of the geodesics $l_n = [g_n o_n, w_n o_n]$ tend to two distinct limit points on ∂X as $l_n \cap R \neq \emptyset$. Thus by the argument of Lemma 9.4 we obtain that the sequence $\delta^f_{h_n}(g_n, w_n)$ is bounded below by a positive constant. A contradiction. □

We note that Corollary 9.2 has been used in [11] to precisely determine the Martin boundary of relatively hyperbolic groups with virtually abelian parabolic subgroups. Furthermore, Proposition 9.3 has been used in [10] and [16] to study the connection between entropy, drift, and growth rate in relatively hyperbolic groups and geometrically finite manifolds.

9.2 Partial description of the preimages of parabolic points and open questions

In this subsection we assume that G admits a minimal geometrically finite action on a compactum T.

By Theorem 7.3 there exists an equivariant continuous map $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ from the Martin to the Floyd compactification. By Theorem 3.3 there also exists an equivariant continuous map φ from the Floyd compactification $\overline{G}_f$ to the Bowditch compactification $\overline{G}_B = G \sqcup T$. So we have an equivariant continuous map $\psi = \varphi \circ \pi : \overline{G}_{\mathcal{M}} \to \overline{G}_B$. The following Proposition partly completes the situation described in Sect. 8 for the preimages of non-conical points.

Proposition 9.6 *Let $p \in T$ be a bounded parabolic point and H the stabilizer of p for the action $G \curvearrowright T$. Then the following inclusion is satisfied for the map φ (compare with (38)) :*

$$\psi^{-1}(T) \cap \partial^{min}_{\mathcal{M}} G \subseteq \partial^{\mathcal{M}} H. \tag{47}$$

Furthermore, there exists a uniform constant $C > 0$ such that for every bounded parabolic point $p \in T$ and every $\alpha \in \varphi^{-1}(p)$ there is some $\beta \in \partial^{\mathcal{M}} H$ such that for every $x \in G$,

$$C^{-1} \leq K_\alpha(x)/K_\beta(x) \leq C. \tag{48}$$

Proof We need to show that the constant C can be chosen uniformly independently of the parabolic point. Indeed for every parabolic point $p \in T$ the action of its stabilizer H on T is cocompact on $T \backslash \{p\}$. Then $\varphi^{-1}(p)$ is the limit set ΛH for the action $H \curvearrowright \overline{G}_f$ [19, Theorem A]. Consequently $(\varphi^{-1}(p))^c = \partial_f G \backslash \partial_f H$. Since φ is equivariant and continuous and $\partial_f G$ is compact, H acts cocompactly on $(\varphi^{-1}(p))^c$. So by Proposition 8.1 we obtain the inequality (48) where $\alpha = \lim_{n\to\infty} v_n \in \partial_{\mathcal{M}} H$.

The constant C found in Proposition 8.1 depends only on the subgroup H. Furthermore the system of all horospheres $\{\mathcal{H}_H : H$ is maximal parabolic subgroup for the action $G \curvearrowright T\}$ is G-invariant and contains at most finitely many G-non-equivalent horospheres [17, Main Theorem.a]. Since $\delta_v^f(x, y) = \delta_{gv}^f(gx, gy)$ and $d(gx, gy) = d(x, y)$ $(g \in G)$, the constant C is the same for the conjugacy class of each maximal parabolic subgroup H. So the constant can be chosen uniformly for all maximal parabolic subgroups of G of the geometrically finite action $G \curvearrowright T$. □

We finish the discussion with some intriguing open questions motivated by the above discussion:

Questions. Let $H < G$ be a fully quasiconvex subgroup of a relatively hyperbolic group G.

(a) Is $\partial^{\mathcal{M}} H = \psi^{-1}(\Lambda H)$?
(b) Does the inequality (48) imply that the points α and β give rise to the same point at the Martin boundary of G (without assuming the minimality of one them)?

Note that $b) \Rightarrow a)$ by the proof of Proposition 8.1.

We also note that by the existence of the continuous extension $\pi : \overline{G}_{\mathcal{M}} \to \overline{G}_f$ of the identity map id : $G \to G$ (Theorem 7.3), we also have $\partial^{\mathcal{M}} H \subseteq \pi^{-1}(\Lambda H)$. The opposite inclusion remains unknown.

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