LOGARITHMIC RIEMANN–HILBERT CORRESPONDENCES FOR RIGID VARIETIES

HANSHENG DIAO, KAI-WEN LAN, RUOCHUAN LIU, AND XINWEN ZHU

ABSTRACT. On any smooth algebraic variety over a p-adic local field, we construct a tensor functor from the category of de Rham p-adic étale local systems to the category of filtered algebraic vector bundles with integrable connections satisfying the Griffiths transversality, which we view as a p-adic analogue of Deligne's classical Riemann–Hilbert correspondence. A crucial step is to construct canonical extensions of the desired connections to suitable compactifications of the algebraic variety with logarithmic poles along the boundary, in a precise sense characterized by the eigenvalues of residues; hence the title of the paper. As an application, we show that this p-adic Riemann–Hilbert functor is compatible with the classical one over all Shimura varieties, for local systems attached to representations of the associated reductive algebraic groups.

CONTENTS

1. Introduction 2
Outline of this paper 7
Acknowledgements 8
Notation and conventions 9
2. Log de Rham period sheaves 9
2.1. Basics of log adic spaces 9
2.2. Definitions of period sheaves 11
2.3. Local study of $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ 14
2.4. Consequences 20
3. Log Riemann–Hilbert Correspondences 21
3.1. Filtered log connections "relative to B_{dR}" 21
3.2. Statements of theorems 24
3.3. Coherence 26
3.4. Calculation of residues 31
3.5. Compatibility with pullbacks and pushforwards 37
3.6. Comparison of cohomology 38
3.7. Compatibility with nearby cycles 40

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4. Riemann–Hilbert functor for p-adic algebraic varieties 43
4.1. The functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ 43
4.2. Generalizations of Kodaira–Akizuki–Nakano vanishing 44
4.3. De Rham local systems at the boundary 46
5. Application to Shimura Varieties 48
5.1. The setup 48
5.2. Local systems on Shimura varieties 49
5.3. Statements of theorems 52
5.4. Proofs of theorems: preliminary reductions 54
5.5. Cases of real rank one, or of abelian type 57
5.6. Cases of real rank at least two 59
Appendix A. A formalism of decompletion 61
A.1. Results 61
A.2. Examples 66
References 69

1. Introduction

Let X be a connected smooth complex algebraic variety, X^{an} the associated analytic space and X^{top} the underlying topological space. The classical Riemann–Hilbert correspondence establishes (tensor) equivalences among the following:

- the category of finite-dimensional complex representations of $\pi_1(X^{\mathrm{top}}, x)$ (where x is a chosen based point), which by a well-known topological construction, is equivalent to the category of local systems (i.e., locally constant sheaves) of finite-dimensional $\mathbb{C}$-vector spaces on X^{top};
- the category of vector bundles with integrable connections on X^{an}; and
- the category of vector bundles with integrable connections on X, with regular singularities at infinity (which we shall simply call *regular integrable connections*, in what follows).

The equivalence of the first and second categories is a simple consequence of the Frobenius theorem: for a local system $\mathbb{L}$ on X^{top}, the associated vector bundle with an integrable connection is $(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d)$; and conversely, for a vector bundle with an integrable connection on X^{an}, its sheaf of horizontal sections is a local system on X^{top}. The equivalence of the second and third categories, however, is a deep theorem due to Deligne [Del70].

An analogous Riemann–Hilbert correspondence for varieties over a p-adic field is long desired but remains rather mysterious until recently. The situation is far more complicated. Let X be a smooth algebraic variety over a finite extension of $\mathbb{Q}_p$. In this setting, the second and third categories remain meaningful, and it is natural to replace the first with the category of p-adic étale local systems on X. However, after this replacement, one cannot expect an equivalence between the first and the second categories, as can already be seen when X is a point. Moreover, in general, the natural analytification functor from the third to the second categories is not an equivalence either. Nevertheless, one of the main goals of this paper is to prove the following result (as one step towards the p-adic Riemann–Hilbert correspondence):

Theorem 1.1. *Let X be a smooth algebraic variety over a finite extension k of $\mathbb{Q}_p$. Then there is a tensor functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ from the category of de Rham p-adic étale local systems $\mathbb{L}$ on X to the category of algebraic vector bundles on X with regular integrable connections and decreasing filtrations satisfying the Griffiths transversality. In addition, there is a canonical comparison isomorphism*

$$H^i_{\text{ét}}\big(X_{\overline{k}}, \mathbb{L}\big) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}\big(X, D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})\big) \otimes_k B_{\mathrm{dR}} \tag{1.2}$$

compatible with the canonical filtrations and the actions of $\mathrm{Gal}(\overline{k}/k)$ on both sides.

Here B_{dR} is Fontaine's p-adic period ring, and H_{dR} is the algebraic de Rham cohomology. The notion of *de Rham p-adic étale local systems* was first introduced by Scholze in [Sch13, Def. 8.3] (generalizing earlier work of Brinon [Bri08]) using some relative de Rham period sheaf. However, it turns out that this notion satisfies a rather surprising rigidity property: by [LZ17, Thm. 3.9], a p-adic étale local system $\mathbb{L}$ on X^{an} is de Rham if and only if, on each connected component of X, there exists *some* classical point x such that, for some (and hence every) geometric point $\overline{x}$ over x, the corresponding p-adic representation $\mathbb{L}_{\overline{x}}$ of the absolute Galois group of the residue field of x is de Rham in the classical sense. In this situation, it follows that the same is also true at *every* classical point x of X. Note that the functor denoted by D_{dR} in [LZ17, Thm. 3.9] is the composition of the functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ in Theorem 1.1 with the analytification functor. Compared with [Sch13] and [LZ17], the algebraicity of the integrable connection is an important new contribution of this paper. In particular, this allows us to go further to compare p-adic theory with Deligne's complex theory mentioned above, as we shall see shortly.

Theorem 1.1 also includes a new *de Rham comparison isomorphism* for smooth algebraic varieties over k with nontrivial coefficients, which implies that $H^i_{\text{ét}}\big(X_{\overline{k}}, \mathbb{L}\big)$ is a *de Rham* representation of $\mathrm{Gal}(\overline{k}/k)$. The de Rham comparison for smooth varieties has a long history, which we shall not attempt to review—see, for example, [Fal89, Fal02, Tsu99, Kis02, Niz08, Niz09, Yam11, Bei12, AI13, Sch13, Sch16, CN17, LP18]. All these earlier works either imposed some strong assumption on the coefficient $\mathbb{L}$ (and in fact most works assumed that $\mathbb{L}$ is trivial) or assumed that the variety X is proper. But we require *neither*. In this generality, without first constructing the corresponding $D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})$ as in Theorem 1.1, it was not even clear how to formulate the comparison isomorphism! Once $D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})$ is constructed, we can adapt Scholze's approach in [Sch13] and obtain the desired comparison for arbitrary nontrivial coefficients on arbitrary smooth varieties.

Besides taking cohomology, the functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ is compatible with many other operations of sheaves. For example, it commutes with taking nearby cycles in the simplest situation where our formulation is available.

Theorem 1.3. *Let $f : X \to \mathbb{A}^1$ be a smooth morphism and let $D := f^{-1}(0)$. Let $\mathbb{L}$ be a de Rham p-adic étale local systems on $X - D$. Then there is a canonical isomorphism of integrable connections*

$$D_{\mathrm{dR}}^{\mathrm{alg}}\big(R\Psi_f(\mathbb{L})\big) \cong R\Psi_f\big(D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})\big)$$

on D, compatible with the filtrations on both sides. (Here $R\Psi_f$ at the two sides of the isomorphism denotes the nearby cycle functors in the étale and D-module settings, respectively.) In particular, $R\Psi_f(\mathbb{L})$ is a de Rham local system on $D_{\text{ét}}$.

See Theorem 4.3.2 for a slightly more general statement, and see Corollary 4.3.4 for a concrete interpretation when X is a smooth curve over k. These results suggest a strong relation between our work and classical Hodge theory, such as Schmid's theorem on limit Hodge structures [Sch73]. Another manifestation of this relation is that we prove some cohomology vanishing results for p-adic algebraic varieties (see Theorem 4.2.1), similar to those that can be obtained from complex Hodge theory. In particular, we obtain a new proof of the Kodaira–Akizuki–Nakano vanishing theorem (together with some generalizations) by p-adic Hodge-theoretic method.

We shall call the functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ in Theorem 1.1 the (algebraic) *p-adic Riemann–Hilbert functor*. It is natural to ask whether this functor is compatible with Deligne's classical Riemann–Hilbert correspondence in a suitable sense. We shall formulate our expectation in the Conjecture 1.4 below. Let us begin with some preparations.

Let X be a smooth algebraic variety over a number field E. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and a field homomorphism $\sigma : E \to \mathbb{C}$, and write $\sigma X = X \otimes_{E,\sigma} \mathbb{C}$. There is a tensor functor from the category of p-adic étale local systems on X to the category of regular integrable connections on σX as follows. Note that $\mathbb{L}|_{\sigma X}$ is an étale local system on σX, corresponding to a p-adic representation of the étale fundamental group of each connected component of σX, which is the profinite completion of the fundamental group of the corresponding connected component of $(\sigma X)^{\mathrm{top}}$. It follows that $\mathbb{L}|_{\sigma X} \otimes_{\mathbb{Q}_p, \iota} \mathbb{C}$ can be regarded as a classical local system on $(\sigma X)^{\mathrm{top}}$, denoted by $\iota \mathbb{L}_\sigma$. Then Deligne's Riemann–Hilbert correspondence produces a regular integrable connection on σX. On the other hand, the composition $\iota^{-1} \circ \sigma : E \to \overline{\mathbb{Q}}_p$ determines a p-adic place v. Let E_v be the completion of E with respect to v, and assume that $\mathbb{L}|_{X_{E_v}}$ is de Rham. Then $D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L}|_{X_{E_v}}) \otimes_{E_v, \iota} \mathbb{C}$ is another regular integrable connection, with an additional decreasing filtration $\mathrm{Fil}^\bullet$ satisfying the Griffiths transversality. We would like to compare the above two constructions. In order to do so, we need to impose a further restriction on $\mathbb{L}$.

We say that $\mathbb{L}$ is *geometric* if, at each geometric point $\overline{x}$ above a closed point x of X, the p-adic representation $\mathbb{L}_{\overline{x}}$ of $\mathrm{Gal}\big(\overline{k(x)}/k(x)\big)$ is *geometric in the sense of Fontaine–Mazur* (see [FM97, Part I, §1]). Note that geometric p-adic étale local systems on X form a full tensor subcategory of the category of all étale local systems. If $\mathbb{L}$ is geometric, then $\mathbb{L}|_{X_{E_v}}$ is de Rham (by [LZ17, Thm. 3.9]).

Conjecture 1.4. *The above two tensor functors from the category of geometric p-adic étale local systems on X to the category of regular integrable connections on σX are canonically isomorphic. In addition, $(D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L}|_{X_{E_v}}) \otimes_{E_v,\iota} \mathbb{C}, \mathrm{Fil}^\bullet)$ is a complex variation of Hodge structures.*

This is closely related to a relative version of the Fontaine–Mazur conjecture proposed in [LZ17], but it might be more approachable because it is stated purely in terms of sheaves. Even so, it seems to be currently out of reach. Nevertheless, in the case of Shimura varieties, we can partially verify this conjecture. Let (G, X) be a Shimura datum, $K \subset \mathrm{G}(\mathbb{A}_f)$ a neat open compact subgroup, and $\mathrm{Sh}_K = \mathrm{Sh}_K(\mathrm{G}, \mathsf{X})$ the associated Shimura variety, defined over the reflex field $E = E(\mathrm{G}, \mathsf{X})$. Let G^c be the quotient of G by the maximal subtorus of the center of G that is $\mathbb{Q}$-anisotropic but $\mathbb{R}$-split. Recall that there is a tensor functor from the category $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G}^c)$ of algebraic representations of G^c over $\mathbb{Q}_p$ to the category of p-adic étale local systems on Sh_K (see, for example, [LS18b, Sec. 3] or [LZ17, Sec. 4.2]), whose essential image consists of only *geometric* p-adic étale local systems (see [LZ17, Thm. 1.2]).

Theorem 1.5. *The conjecture holds for the* (p*-adic*) *étale local systems on* Sh_K *coming from* $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G}^c)$ *as above.*

Note that this theorem applies to *all* Shimura varieties, on which étale local systems are not (yet) known to be related to motives in general. Crucial ingredients in our proof include Margulis's *superrigidity theorem* [Mar91], and a construction credited to Piatetski-Shapiro by Borovoi [Bor84] and Milne [Mil83]. This theorem itself has applications to the arithmetic of Shimura varieties, such as the following:

Corollary 1.6. *The Grothendieck–Messing period map for* Sh_K *is étale.*

This implies that the local geometry of Shimura varieties is controlled by the moduli spaces of p-adic shtukas, which were recently constructed by Scholze (see [SW17, Sec. 23.3]). (Note that, from the definitions, it was not clear how these moduli spaces are related to Shimura varieties.) Some other applications of Theorem 1.5 will appear in [LLZ].

Now let us explain our strategy for proving Theorem 1.1. As mentioned above, in [LZ17, Thm. 3.9], a tensor functor $D_{\mathrm{dR}}^{\mathrm{an}}$ was constructed from the category of de Rham p-adic étale local systems on X^{an} to the category of filtered vector bundles on X^{an} with integrable connections and decreasing filtrations satisfying the Griffiths transversality. In order to prove Theorem 1.1, a natural idea is to fix a smooth compactification $\overline{X}$ of X with a normal crossings boundary divisor, and extend the filtered vector bundles with integrable connections in *loc. cit.* to filtered vector bundles on $\overline{X}^{\mathrm{an}}$ with integrable log connections (i.e., connections with log poles along the boundary divisor). However, rather unlike the complex analytic situation in [Del70], *not* every integrable connection on X^{an} is extendable and hence algebraizable (see [AB01, Ch. 4, Rem. 6.8.3] for some counter-example).

Instead, we shall directly construct a functor from the category of de Rham p-adic étale local systems on X^{an} to the category of filtered vector bundles with integrable log connections on $\overline{X}^{\mathrm{an}}$. We shall work in the realm of log analytic geometry as in [DLLZ], and construct a *log Riemann–Hilbert correspondence*, which is a crucial step in this paper. Compared with [LZ17], many new ingredients and essential new ideas are needed for this construction, and many new difficulties have to be overcome. Let us begin with a rough summary. Based on [DLLZ], the starting point is the construction of the log de Rham period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$, generalizing the de Rham period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ as in [Sch13] and [LZ17]. Then we define the log Riemann–Hilbert functors in a way similar to [LZ17]. However, for our purpose, we also need to develop a very general formalism of decompleting pairs and decompletion systems, generalizing the one introduced in [KL16, Sec. 5] and many other classical works. After these, the major divergence of the methods from [LZ17] occurs. One of the key facts used in *loc. cit.*, that a coherent module with an integrable connection is automatically locally free, completely breaks down for log connections in general. Our new idea is to study a collection of important invariants attached to a log connection—i.e., the residues along the irreducible components of the boundary, using the above-mentioned decompletion formalism. This allows us to prove a lot of favorable properties of the log connections constructed from the de Rham local systems. In particular, we can canonically extend the filtration on the integrable connection (as constructed in [Sch13] and [LZ17]) to the boundary. Moreover, the residues play an essential role in our study of nearby cycles, as in Theorem 1.3.

Let us also mention that, in the classical situation over $\mathbb{C}$, Illusie–Kato–Nakayama [IKN05, IKN07] developed a theory of quasi-unipotent log Riemann–Hilbert correspondence, which obtained as a byproduct Deligne's Riemann–Hilbert correspondence for local systems with quasi-unipotent monodromy at infinity.

We now explain our construction in more details. We will work over a smooth rigid analytic variety Y over k (viewed as an adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$), together with a normal crossings divisor $D \subset Y$, and view Y as a log adic space by equipping it with the natural log structure defined by D. (For applications to our previous setup, we take $Y = \overline{X}^{\mathrm{an}}$, and take D to be the analytification of the boundary divisor $\overline{X} - X$ with its reduced subscheme structure.) Any Kummer étale $\mathbb{Z}_p$-local system $\mathbb{L}$ on Y induces a $\widehat{\mathbb{Z}}_p$-local system $\widehat{\mathbb{L}}$ on $Y_{\text{prokét}}$. Let $\mu : Y_{\text{prokét}} \to Y_{\mathrm{an}}$ denote the natural projection from the pro-Kummer étale site to the analytic site. (Note that as a subscript "an" means the analytic site, while as a superscript it means the analytification of an algebraic object.) Let $\Omega^1_Y(\log D)$ denote the sheaf of differentials with log poles along D, as usual. The following theorem is an abbreviated version of Theorem 3.2.7, from which Theorem 1.1 will be deduced.

Theorem 1.7. *Let Y and μ be as above. Consider the functor $D_{\mathrm{dR},\log}$, which sends a Kummer étale $\mathbb{Z}_p$-local system $\mathbb{L}$ on Y to*

$$D_{\mathrm{dR},\log}(\mathbb{L}) := \mu_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}).$$

Then $D_{\mathrm{dR},\log}(\mathbb{L})$ is a vector bundle on Y_{an} equipped with an integrable log connection

$$\nabla_{\mathbb{L}} : D_{\mathrm{dR},\log}(\mathbb{L}) \to D_{\mathrm{dR},\log}(\mathbb{L}) \otimes_{\mathcal{O}_Y} \Omega^1_Y(\log D)$$

and a decreasing filtration (by coherent subsheaves) satisfying the Griffiths transversality, which extends the vector bundle $D_{\mathrm{dR}}(\mathbb{L})$ with its integrable connection in [LZ17, Thm. 3.9]. *Moreover, all eigenvalues of the residues of $D_{\mathrm{dR},\log}(\mathbb{L})$ along the irreducible components of D are rational numbers in $[0, 1)$. In particular, $(D_{\mathrm{dR},\log}(\mathbb{L}), \nabla_{\mathbb{L}})$ is the* **canonical extension** *of the $(D_{\mathrm{dR}}(\mathbb{L}), \nabla_{\mathbb{L}})$; i.e., the unique (if existent) extension of $(D_{\mathrm{dR}}(\mathbb{L}), \nabla_{\mathbb{L}})$ with such eigenvalues of residues.*

If $\mathbb{L}|_{Y-D}$ is a **de Rham** *étale $\mathbb{Z}_p$-local system, then $\mathrm{gr}\, D_{\mathrm{dR},\log}(\mathbb{L})$ is a vector bundle on Y_{an} of rank $\mathrm{rk}_{\mathbb{Z}_p}(\mathbb{L})$, and we have the de Rham (resp. Hodge–Tate) comparison isomorphism between the Kummer étale cohomology of $\mathbb{L}$ and the log de Rham (resp. log Hodge) cohomology of $D_{\mathrm{dR},\log}(\mathbb{L})$.*

Note that, unlike the functors $D^{\mathrm{alg}}_{\mathrm{dR}}$ and D_{dR} in Theorem 1.1 and [LZ17, Thm. 3.9], the functor $D_{\mathrm{dR},\log}$ fails to be a tensor functor in general, as the eigenvalues of the residues of $D_{\mathrm{dR},\log}(\mathbb{L}_1) \otimes_{\mathcal{O}_Y} D_{\mathrm{dR},\log}(\mathbb{L}_2)$ might be outside $[0, 1)$, and therefore $D_{\mathrm{dR},\log}(\mathbb{L}_1) \otimes_{\mathcal{O}_Y} D_{\mathrm{dR},\log}(\mathbb{L}_2)$ might not be isomorphic to $D_{\mathrm{dR},\log}(\mathbb{L}_1 \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{L}_2)$. This failure is caused by the failure of the surjectivity of the canonical morphism

$$\mu^*\big(D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_{\mathcal{O}_{Y_{\text{prokét}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \tag{1.8}$$

in general, even when $\mathrm{rk}_{\mathcal{O}_Y}(D_{\mathrm{dR},\log}(\mathbb{L})) = \mathrm{rk}_{\mathbb{Z}_p}(\mathbb{L})$. This phenomenon is not present in the usual comparison theorems in p-adic Hodge theory, but is consistent with the complex Riemann–Hilbert correspondence. Nevertheless, we will see in Theorem 3.2.12 that $D_{\mathrm{dR},\log}$ restricts to a natural *tensor functor* from the subcategory of de Rham local systems with *unipotent monodromy* along the boundary to the subcategory of integrable log connections with *nilpotent residues* along the boundary.

We will deduce Theorem 1.7 from a geometric log Riemann–Hilbert correspondence. Let Y be as above, let K be a perfectoid field over k containing all roots of unity, and let $\mathrm{Gal}(K/k)$ abusively denote the group of continuous field automorphisms of K over k. Let $B_{\mathrm{dR}}^+ = \mathbb{B}_{\mathrm{dR}}^+(K, \mathcal{O}_K)$ and $B_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}(K, \mathcal{O}_K)$ (as in [LZ17, Sec. 3.1]), and consider the ringed spaces $\mathcal{Y}^+ = (Y_{\mathrm{an}}, \mathcal{O}_Y \widehat{\otimes} B_{\mathrm{dR}}^+)$ and $\mathcal{Y} = (Y_{\mathrm{an}}, \mathcal{O}_Y \widehat{\otimes} B_{\mathrm{dR}})$, where $\mathcal{O}_Y \widehat{\otimes} B_{\mathrm{dR}}^+$ and $\mathcal{O}_Y \widehat{\otimes} B_{\mathrm{dR}}$ are sheaves on Y_{an} which we interpret as the rings of functions on the not-yet-defined base changes "$Y \widehat{\otimes}_k B_{\mathrm{dR}}^+$" and "$Y \widehat{\otimes}_k B_{\mathrm{dR}}$", respectively. Let $\mu' : Y_{\text{prokét}/Y_K} \to Y_{\mathrm{an}}$ denote the natural morphism of sites. The following theorem is an abbreviated version of Theorem 3.2.3.

Theorem 1.9. *The functor*

$$\mathcal{RH}_{\log}(\mathbb{L}) := R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})$$

is an exact functor from the category of Kummer étale $\mathbb{Z}_p$-local systems on Y to the category of $\mathrm{Gal}(K/k)$-equivariant vector bundles on $\mathcal{Y}$, equipped with an integrable log connection $\nabla : \mathcal{RH}_{\log}(\mathbb{L}) \to \mathcal{RH}_{\log}(\mathbb{L}) \otimes_{\mathcal{O}_Y} \Omega^1_Y(\log D)$, and a decreasing filtration (by locally free $\mathcal{O}_Y \widehat{\otimes} B_{\mathrm{dR}}^+$-submodules) satisfying the Griffith transversality. Moreover, we have $H^i_{\text{két}}(Y_{\overline{k}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i_{\log \mathrm{dR}}(\mathcal{Y}, \mathcal{RH}_{\log}(\mathbb{L}))$ when $K = \widehat{\overline{k}}$.

Let $\mathcal{RH}^+_{\log}(\mathbb{L}) := R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})$, which is an $\mathcal{O}_Y \widehat{\otimes} B_{\mathrm{dR}}^+$-lattice in $\mathcal{RH}_{\log}(\mathbb{L})$ equipped (as in [LZ17, Rem. 3.2]) with the t-connection$\nabla^+ := t\nabla : \mathcal{RH}^+_{\log}(\mathbb{L}) \to \mathcal{RH}^+_{\log}(\mathbb{L}) \otimes_{\mathcal{O}_Y} \Omega^1_Y(\log D)(1)$, where $t \in B_{\mathrm{dR}}$ is an element on which $\mathrm{Gal}(K/k)$ acts via the cyclotomic character. By reduction modulo t, we obtain the log p-adic Simpson functor $\mathcal{H}_{\log}$, constructed in much greater generality by Faltings [Fal05] and Abbes–Gros–Tsuji [AGT16]. See Theorem 3.2.4 for more details.

Compared with the situation in [LZ17], the proof of Theorem 1.9 requires some decompletion statement beyond the scope of [KL16]. We have therefore developed a general decompletion formalism in Appendix A, which might be of some independent interest.

Now we explain how to deduce Theorem 1.7 from Theorem 1.9, where some essential new ideas of this paper appear. By using the above-mentioned decompletion statement and an argument similar to the one in [LZ17], it is not difficult to show that $D_{\mathrm{dR},\log}(\mathbb{L}) \cong (\mathcal{RH}_{\log}(\mathbb{L}))^{\mathrm{Gal}(K/k)}$ is a coherent sheaf on Y_{an}. However, unlike the situation in [LZ17], the existence of a log connection does not guarantee the local freeness of $D_{\mathrm{dR},\log}(\mathbb{L})$. A priori, only its reflexiveness is clear. Nevertheless, there is a collection of important invariants attached to a log connection; i.e., the *residues* along the irreducible components of D. Somewhat surprisingly, by using the decompletion formalism again, we find that all the eigenvalues of the residues are *rational* numbers in $[0, 1)$. Together with the reflexiveness and a general fact about log connections (see [AB01, Ch. 1, Prop. 4.5 and Lem. 4.6.1]), this allows us to conclude that $D_{\mathrm{dR},\log}(\mathbb{L})$ is indeed locally free. We remark that residues also plays a vital role in deducing the comparison of cohomology in Theorem 1.7 from the comparison of cohomology in Theorem 1.9.

Finally, our results on residues also allow us to define V-filtrations and study nearby cycles in the p-adic setting, which in turn allows us to deduce Theorem 1.3.

Outline of this paper. Let us briefly describe the organization of this paper, and highlight the main themes in each section.

In Section 2, we study the *log de Rham period sheaves*, generalizing the usual ones studied in [Bri08, Sec. 5], [Sch13, Sec. 6], and [Sch16], with a subtle difference—see Remark 2.2.11. In Section 2.1, we recall some notation and basic results for log adic spaces developed in [DLLZ]. In Section 2.2, we present the general definitions of these log de Rham period sheaves. In Section 2.3, we describe their structures in detail, when there are good local coordinates. In Section 2.4, we record some consequences, including the *Poincaré lemma*. We note that results of this section hold for a class of log adic spaces larger than those considered in Theorem 1.7 and 1.9. This extra generality is useful for many applications (see, e.g., [LLZ]).

In Section 3, we establish the geometric and arithmetic versions of *log p-adic Riemann–Hilbert correspondences*, as well as the *log p-adic Simpson correspondence*, as explained above. We introduce some general terminologies for filtered log connections "relative to B_{dR}" in Section 3.1 and state the main results in Section 3.2, which are more detailed versions of Theorems 1.7 and 1.9. The proofs are given in the following subsections. In Section 3.3, we show that we obtain *coherent sheaves* in the various constructions. In Section 3.4, we calculate the *eigenvalues of residues* of the connections along the boundary. This is the technical heart of this paper. In Section 3.5, we show that our correspondences are compatible with certain pullbacks and pushforwards, by using our results on residues and the known compatibilities in [Sch13] and [LZ17]. In Section 3.6, we establish the comparisons of cohomology in our main theorems. In Section 3.7, we show that the formation of quasi-unipotent nearby cycles (in the rigid analytic setting, as introduced in [DLLZ, Sec. 6.4.1]) is compatible with the log Riemann–Hilbert functors.

In Section 4, we present our main results for algebraic varieties. In Section 4.1, we construct the p-adic Riemann–Hilbert functor and prove Theorem 1.1. We also establish the corresponding log Hodge–Tate comparison and the degeneration of (log) Hodge–de Rham spectral sequences, and record these latter results in Theorem 4.1.4. In Section 4.2, we present some vanishing theorem for p-adic algebraic varieties, by adapting complex Hodge-theoretic arguments in [Suh18] using our p-adic results. In Section 4.3, we show that the formation of algebraic (quasi-unipotent) nearby cycles is compatible with our p-adic Riemann–Hilbert functor.

In Section 5, we compare two constructions of filtered regular connections on Shimura varieties, and deduce Theorem 1.5 and Corollary 1.6. In Section 5.1, we begin with the overall setup. In Section 5.2, we explain the two constructions, one complex analytic and the other p-adic. In Section 5.3, we state our main comparison theorem on these two constructions, and record some consequences. In Section 5.4, we reduce the theorem to a technical statement concerning representations of fundamental groups, which are then verified in the remaining two subsections. This section can be read largely independent of the rest of the paper.

In Appendix A, we generalize the *formalism of decompletion* developed in [KL16, Sec. 5], in order to treat the general Kummer towers. In Section A.1, we introduce the notions of *decompletion systems* and *decompleting pairs* in general, and show that every stably decompleting pair is a decompletion system. In Section A.2, we present several important examples of decompletion systems, which play crucial roles in Section 3 (in the proof of coherence and in the calculation of residues). The appendix can also be read largely independent of the rest of the paper.

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Notation and conventions. Unless otherwise specified, we always denote by k a nonarchimedean local field (i.e., a field complete with respect to a nontrivial nonarchimedean multiplicative norm $|\cdot| : k \to \mathbb{R}_{\geq 0}$) with residue field κ of characteristic $p > 0$, and $\mathcal{O}_k$ denotes the ring of integers in k. We also denote by $k^+ \subset \mathcal{O}_k$ an open valuation ring, whose choice depends on the context. Sometimes, we choose a *pseudo-uniformizer* (i.e., a topological nilpotent unit) ϖ of k contained in k^+.

By a *locally noetherian adic space* over k, we mean an adic space X over $\mathrm{Spa}(k, k^+)$ that admits an open covering by affinoids $U_i = \mathrm{Spa}(A_i, A_i^+)$ where each A_i is strongly noetherian. A *noetherian adic space* over k is a qcqs locally noetherian adic space over k. If X is locally noetherian, we denote by X_{an} its analytic site, by $X_{\text{ét}}$ its associated étale site, and by $\lambda : X_{\text{ét}} \to X_{\mathrm{an}}$ the natural projection of sites. We will regard rigid analytic varieties as adic spaces topologically of finite type over $\mathrm{Spa}(k, \mathcal{O}_k)$ (as in [Hub96]), in which case we will work with $k^+ = \mathcal{O}_k$.

By default, monoids are assumed to be commutative, and the monoid operations are written additively (rather than multiplicatively), unless otherwise specified. For a monoid P, let P^{gp} denote its group completion. If R is a commutative ring with unit and P is a monoid, we denote by $R[P]$ the monoid algebra over R associated with P. The image of $a \in P$ in $R[P]$ will be denoted by e^a.

Group cohomology will always mean continuous group cohomology.

2. Log de Rham period sheaves

In this section, we define and study the log de Rham period sheaves, generalizing the usual ones studied in [Bri08, Sec. 5], [Sch13, Sec. 6], and [Sch16]. We shall assume that k is of characteristic zero and residue characteristic $p > 0$.

2.1. Basics of log adic spaces.

We begin with a summary of some notation and basic results for log adic spaces developed in the companion paper [DLLZ], in slightly less generality than the one in *loc. cit.*, for the sake of simplicity.

Let X be any étale sheafy adic space. This means, in particular, that X admits a well-defined étale site $X_{\text{ét}}$ and that the étale structure presheaf $\mathcal{O}_{X_{\text{ét}}} : U \mapsto \mathcal{O}_U(U)$ is a sheaf. A *pre-log structure* on X is a pair $(\mathcal{M}_X, \alpha)$ consisting of a sheaf $\mathcal{M}_X$ of monoids on $X_{\text{ét}}$ and a morphism $\alpha : \mathcal{M}_X \to \mathcal{O}_{X_{\text{ét}}}$ of sheaves of monoids. (Here $\mathcal{O}_{X_{\text{ét}}}$ is equipped with the natural multiplicative monoid structure.) Such a pair is a *log structure* if $\alpha^{-1}(\mathcal{O}^\times_{X_{\text{ét}}}) \to \mathcal{O}^\times_{X_{\text{ét}}}$ is an isomorphism, in which case the triple $(X, \mathcal{M}_X, \alpha)$ is called a *log adic space*. The log structure is *trivial* when $\alpha^{-1}(\mathcal{O}^\times_{X_{\text{ét}}}) = \mathcal{M}_X$. For simplicity, we shall often write $(X, \mathcal{M}_X)$ or X, when the context is clear. Moreover, we have the notions of morphisms between log structures and between log adic spaces, of the log structure associated with a pre-log structure, and of pullbacks of log structures. These are analogous to the similar notions for schemes—see [DLLZ, Def. 2.2.2] for more details. A log adic space is *noetherian* (resp. *locally noetherian*) if its underlying adic space is.

For example, when P is a monoid such that either P is finitely generated, or that k is perfectoid and P is uniquely p-divisible, then we know that $Y := \mathrm{Spa}(k[P], k^+[P]) \cong \mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ is an étale sheafy adic space over $\mathrm{Spa}(k, k^+)$ (see [DLLZ, Lem. 2.2.13 and 2.2.15]). By abuse of notation, we shall sometimes denote by simply P the constant sheaf P_Y on Y associated with the monoid P.

Then we have the canonical log structure $P^{\log}$ on Y associated with the pre-log structure $P \to \mathcal{O}_{Y_{\text{ét}}}$ induced by $a \mapsto e^a \in k\langle P\rangle$ (see [DLLZ, Def. 2.2.16]).

Example 2.1.1. *When $P \cong \mathbb{Z}_{\geq 0}^n$ for some $n \geq 0$, we have $\mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle) \cong \mathbb{D}^n := \mathrm{Spa}(k\langle T_1, \ldots, T_n\rangle, k^+\langle T_1, \ldots, T_n\rangle)$, the n-**dimensional unit disc**, with the log structure on $\mathbb{D}^n$ induced by $\mathbb{Z}_{\geq 0}^n \to k\langle T_1, \ldots, T_n\rangle : (a_1, \ldots, a_n) \mapsto T_1^{a_1} \cdots T_n^{a_n}$.*

Given any log adic space X and any monoid P, a *chart of X modeled on P* is a morphism of sheaves of monoids $\theta : P_X \to \mathcal{M}_X$ such that $\alpha\big(\theta(P_X)\big) \subset \mathcal{O}^+_{X_{\text{ét}}}$ and such that the log structure associated with the pre-log structure $\alpha \circ \theta : P_X \to \mathcal{O}_{X_{\text{ét}}}$ is isomorphic to $\mathcal{M}_X$ (see [DLLZ, Def. 2.3.1]). When X is defined over $\mathrm{Spa}(k, k^+)$ and when $\mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ is defined as a log adic space as above, this is equivalent to having a strict morphism (see [DLLZ, Rem. 2.3.2]) from X to $\mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$. We say a log adic space is *fs* if it étale locally admits charts models on monoids that are *fs*, i.e., finitely generated, *integral*, and *saturated* (see [DLLZ, Def. 2.1.1 and 2.3.5]). We also have the notion of fs charts of morphisms between fs log adic spaces (see [DLLZ, Prop. 2.3.21 and 2.3.22]). By [DLLZ, Prop. 2.3.27], fiber products exist in the category of locally noetherian fs log adic spaces whenever the fiber products of underlying adic spaces exist (although the underlying adic spaces of the fiber products of fs log adic spaces might differ from the latter).

We say that a morphism of locally noetherian fs log adic spaces is *strictly étale* if the underlying morphism of adic spaces is étale. By using the notion of charts, we can define *log smooth* and *log étale* morphisms of locally noetherian fs adic spaces (see [DLLZ, Def. 3.1.1]). When X is log smooth over $\mathrm{Spa}(k, \mathcal{O}_k)$, where the latter is equipped with the trivial log structure, we simply say that X is log smooth over k (see [DLLZ, Def. 3.1.9]). By [DLLZ, Prop. 3.1.10], a log smooth fs log adic space X over k étale locally admits strictly étale morphisms $X \to \mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ which provide charts modeled on *toric* monoids P, i.e., fs monoids that are *sharp* in the sense that the subgroups P^* of invertible elements of P are trivial. When the underlying adic space of X is smooth, we may assume in the above that $P \cong \mathbb{Z}_{\geq 0}^n$ for some $n \geq 0$, so that $\mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle) \cong \mathbb{D}^n$ (see [DLLZ, Cor. 3.1.11]). We call any strictly étale morphism $X \to \mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ (resp. $X \to \mathbb{D}^n$) as above a *toric chart* (resp. *smooth toric chart*) (see [DLLZ, Def. 3.1.12]).

We will mostly apply the general theory to the following class of fs log adic spaces (see [DLLZ, Ex. 2.3.17 and 3.1.13] for more details):

Example 2.1.2. *Let X be a smooth rigid analytic variety over k (and so $k^+ = \mathcal{O}_k$). A* (**reduced**) **normal crossings divisor** *D of X is defined by a closed immersion $\imath : D \hookrightarrow X$ of rigid analytic varieties over k that is (analytic) locally of the form $S \times \{T_1 \cdots T_r = 0\} \hookrightarrow S \times \mathbb{D}^r$, where S is smooth over k. We equip X with the log structure $\mathcal{M}_X := \{f \in \mathcal{O}_{X_{\text{ét}}} : f \text{ is invertible on } X - D\}$, where $\alpha : \mathcal{M}_X \to \mathcal{O}_{X_{\text{ét}}}$ is the natural inclusion, which makes $(X, \mathcal{M})$ a log smooth noetherian fs adic space over k. Locally, when $X \cong S \times \mathbb{D}^r$, the log structure on X is the pullback of the one on $\mathbb{D}^r$ (see Example 2.1.1). Moreover, locally on X, we have smooth toric charts $X \to \mathbb{D}^n$ such that $\imath : D \hookrightarrow X$ is the pullback of $\{T_1 \cdots T_r = 0\} \hookrightarrow \mathbb{D}^n$, where $T_1, \ldots, T_n$ are the coordinates of $\mathbb{D}^n$, for some $0 \leq r \leq n$.*

For each locally noetherian fs log adic space X, we have the *Kummer étale site* $X_{\text{két}}$, as in [DLLZ, Sec. 4.1], whose objects are fs log adic spaces *Kummer étale* over X. A typical example of Kummer étale morphisms is, for each integer $m \geq 1$, the ramified cover $\mathbb{D}^n_m := \mathrm{Spa}(k\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}}\rangle, k^+\langle T_1^{\frac{1}{m}}, \ldots, T_n^{\frac{1}{m}}\rangle) \to$

$\mathbb{D}_n = \mathrm{Spa}(k\langle T_1, \ldots, T_n\rangle, k^+\langle T_1, \ldots, T_n\rangle)$. We also have the *pro-Kummer étale site* $X_{\text{prokét}}$, as in [DLLZ, Def. 5.1.2] (which generalizes [Sch16]). Then we have natural projections of sites $\nu_X : X_{\text{prokét}} \to X_{\text{két}}$ and $\varepsilon_{X,\text{ét}} : X_{\text{két}} \to X_{\text{ét}}$, the latter being an isomorphism when the log structure is trivial. Given any morphism $f : X \to Y$ of locally noetherian fs log adic spaces, we have canonical morphisms of sites $f_{\text{két}} : X_{\text{két}} \to Y_{\text{két}}$ and $f_{\text{prokét}} : X_{\text{prokét}} \to Y_{\text{prokét}}$ compatible with each other.

We can naturally define locally constant sheaves and torsion local systems on $X_{\text{két}}$ (see [DLLZ, Def. 4.4.14]). We define a $\mathbb{Z}_p$*-local system* (or *lisse* $\mathbb{Z}_p$*-sheaf*) on $X_{\text{két}}$ to be an inverse system of $\mathbb{Z}/p^n$-modules $\mathbb{L} = (\mathbb{L}_n)_{n\geq 1}$ on $X_{\text{két}}$ such that each $\mathbb{L}_n$ is a locally constant sheaf which are locally (on $X_{\text{két}}$) associated with finitely generated $\mathbb{Z}/p^n$-modules, and such that the inverse system is isomorphic in the pro-category to an inverse system in which $\mathbb{L}_{n+1}/p^n \cong \mathbb{L}_n$. We define a $\mathbb{Q}_p$*-local system* (or *lisse* $\mathbb{Q}_p$*-sheaf*) on $X_{\text{két}}$ to be an object of the stack associated with the fibered category of isogeny lisse $\mathbb{Z}_p$-sheaves. (See [DLLZ, Def. 6.3.1].) Let $\widehat{\mathbb{Z}}_p := \varprojlim_n (\mathbb{Z}/p^n)$ as a sheaf of rings on $X_{\text{prokét}}$, and let $\widehat{\mathbb{Q}}_p := \widehat{\mathbb{Z}}_p[\frac{1}{p}]$. A $\widehat{\mathbb{Z}}_p$*-local system* on $X_{\text{prokét}}$ is a sheaf of $\widehat{\mathbb{Z}}_p$-modules on $X_{\text{prokét}}$ that is locally isomorphic to $L \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p$ for some finitely generated $\mathbb{Z}_p$-modules L. The notion of $\widehat{\mathbb{Q}}_p$*-local systems* on $X_{\text{prokét}}$ is defined similarly. (See [DLLZ, Def. 6.3.2 and Lem. 6.3.3]).

2.2. **Definitions of period sheaves.** Let (R, R^+) be a perfectoid affinoid algebra over k, with $(R^\flat, R^{\flat+})$ its tilt. Recall that there are the period rings

$$\mathbb{A}_{\text{inf}}(R, R^+) := W(R^{\flat+}) \quad \text{and} \quad \mathbb{B}_{\text{inf}}(R, R^+) := \mathbb{A}_{\text{inf}}(R, R^+)[\tfrac{1}{p}].$$

It is well known that there is a natural surjective map

$$\theta : \mathbb{A}_{\text{inf}}(R, R^+) \to R^+, \tag{2.2.1}$$

whose kernel is a principal ideal generated by some $\xi \in \mathbb{A}_{\text{inf}}(R, R^+)$, which is not a zero divisor (see, for example, [KL15, Lem. 3.6.3]). We define

$$\mathbb{B}_{\text{dR}}^+(R, R^+) := \varprojlim_r \big(\mathbb{B}_{\text{inf}}(R, R^+)/\xi^r\big) \quad \text{and} \quad \mathbb{B}_{\text{dR}}(R, R^+) := \mathbb{B}_{\text{dR}}^+(R, R^+)[\xi^{-1}].$$

We shall equip $\mathbb{B}_{\text{dR}}(R, R^+)$ with the filtration $\mathrm{Fil}^r \mathbb{B}_{\text{dR}}(R, R^+) := \xi^r \mathbb{B}_{\text{dR}}^+(R, R^+)$, for all $r \in \mathbb{Z}$. This filtration is separated, complete, and independent of the choice of ξ. Therefore, for all $r \in \mathbb{Z}$, we have a canonical isomorphism

$$\mathrm{gr}^r\, \mathbb{B}_{\text{dR}}(R, R^+) \cong \xi^r R. \tag{2.2.2}$$

Now let X be a locally noetherian fs log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. As in [DLLZ, Def. 5.4.1] (which generalizes [Sch13, Def. 4.1 and 5.10]), we have the following sheaves and morphisms:

- $\mathcal{O}^?_{X_{\text{prokét}}} = \nu^{-1}(\mathcal{O}^?_{X_{\text{két}}})$, where $? = \emptyset$ or $+$;
- $\widehat{\mathcal{O}}^+_{X_{\text{prokét}}} = \varprojlim_n \big(\mathcal{O}^+_{X_{\text{prokét}}}/p^n\big)$ and $\widehat{\mathcal{O}}_{X_{\text{prokét}}} = \widehat{\mathcal{O}}^+_{X_{\text{prokét}}}[\frac{1}{p}]$;
- $\widehat{\mathcal{O}}^{\flat ?}_{X_{\text{prokét}}} = \varprojlim_\Phi \widehat{\mathcal{O}}^?_{X_{\text{prokét}}}$, where $? = \emptyset$ or $+$;
- $\alpha : \mathcal{M}_{X_{\text{prokét}}} := \nu^{-1}(\mathcal{M}_{X_{\text{két}}}) \to \mathcal{O}_{X_{\text{prokét}}}$; and
- $\alpha^\flat : \mathcal{M}^\flat_{X_{\text{prokét}}} := \varprojlim_{a \mapsto a^p} \mathcal{M}_{X_{\text{prokét}}} \to \widehat{\mathcal{O}}^\flat_{X_{\text{prokét}}}$.

We shall sometimes omit the subscripts "prokét" or "$X_{\text{prokét}}$".

Definition 2.2.3. We define the following sheaves on $X_{\text{prokét}}$:

(1) Let $\mathbb{A}_{\mathrm{inf},X} := W(\widehat{\mathcal{O}}^+_{X^\flat_{\mathrm{prok\acute{e}t}}})$ and $\mathbb{B}_{\mathrm{inf},X} := \mathbb{A}_{\mathrm{inf},X}[\frac{1}{p}]$, where the latter is equipped with a natural map $\theta : \mathbb{B}_{\mathrm{inf},X} \to \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}$.
(2) Let $\mathbb{B}^+_{\mathrm{dR},X} := \varprojlim_r \big(\mathbb{B}_{\mathrm{inf},X}/(\ker\theta)^r\big)$, and $\mathbb{B}_{\mathrm{dR},X} := \mathbb{B}^+_{\mathrm{dR},X}[t^{-1}]$, where t is any generator of $(\ker\theta)\mathbb{B}^+_{\mathrm{dR},X}$. (We will choice some t in (2.3.2) below.)
(3) The filtration on $\mathbb{B}^+_{\mathrm{dR},X}$ is given by $\mathrm{Fil}^r\mathbb{B}^+_{\mathrm{dR},X} := (\ker\theta)^r\mathbb{B}^+_{\mathrm{dR},X}$. It induces a filtration on $\mathbb{B}_{\mathrm{dR},X}$ given by $\mathrm{Fil}^r\mathbb{B}_{\mathrm{dR},X} := \sum_{s\geq -r} t^{-s}\mathrm{Fil}^{r+s}\mathbb{B}^+_{\mathrm{dR},X}$.

We shall omit the subscript X in the notation when the context is clear.

Proposition 2.2.4. *Suppose that $U \in X_{\mathrm{prok\acute{e}t}}$ is log affinoid perfectoid, with associated perfectoid space $\widehat{U} = \mathrm{Spa}(R, R^+)$, as in* [DLLZ, Def. 5.3.1 and Rem. 5.3.5]*.*

(1) We have a canonical isomorphism $\mathbb{A}_{\mathrm{inf}}(U) \cong \mathbb{A}_{\mathrm{inf}}(R, R^+)$, and similar isomorphisms for $\mathbb{B}_{\mathrm{inf}}$, $\mathbb{B}^+_{\mathrm{dR}}$, and $\mathbb{B}_{\mathrm{dR}}$.
(2) $H^j(U, \mathbb{B}^+_{\mathrm{dR}}) = 0$ and $H^j(U, \mathbb{B}_{\mathrm{dR}}) = 0$ for all $j > 0$.

Proof. The proof is essentially the same as in the one of [Sch13, Thm. 6.5], with the input [Sch13, Lem. 5.10] there replaced with [DLLZ, Prop. 5.4.3]. □

Remark 2.2.5. By the previous discussion and Proposition 2.2.4(1) (for $\mathbb{A}_{\mathrm{inf}}$), the element t in Definition 2.2.3(2) exists locally on $X_{\mathrm{prok\acute{e}t}}$ and is not a zero divisor. Therefore, the sheaf $\mathbb{B}_{\mathrm{dR}}$ and its filtration are indeed well defined.

Corollary 2.2.6. *If X is over a perfectoid field k (over $\mathbb{Q}_p$) containing all roots of unity, then $\mathrm{gr}^\bullet\,\mathbb{B}_{\mathrm{dR}} \cong \oplus_{r\in\mathbb{Z}}\big(\widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}(r)\big)$.*

Proof. This follows from (2.2.2) and Proposition 2.2.4, as in [Sch13, Cor. 6.4]. □

Corollary 2.2.7. *Suppose that $\imath : Z \to X$ is a strict closed immersion, as in* [DLLZ, Def. 2.2.23]*. Then $\mathbb{B}_{\mathrm{dR},X} \to \imath_{\mathrm{prok\acute{e}t},*}(\mathbb{B}_{\mathrm{dR},Z})$ is surjective. More precisely, its evaluation at every log affinoid perfectoid object U in $X_{\mathrm{prok\acute{e}t}}$ is surjective.*

Proof. This follows from [DLLZ, Prop. 5.4.5] and Proposition 2.2.4. □

Now let X be a locally noetherian fs log adic space over $\mathrm{Spa}(k, k^+)$. We shall construct $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}$, a log version of the *geometric de Rham period sheaves* $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},X}$ introduced in [Bri08, Sch13, Sch16]. As log affinoid perfectoid objects form a basis $\mathcal{B}$ of $X_{\mathrm{prok\acute{e}t}}$ (see [DLLZ, Prop. 5.3.12]), it suffices to define $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}$ as a sheaf associated with a presheaf on $\mathcal{B}$.

We adopt the notation in [DLLZ, Sec. 5.3 and 5.4]. Let $U = \varprojlim_{i\in I} U_i \in X_{\mathrm{prok\acute{e}t}}$ be a log affinoid perfectoid object, with $U_i = (\mathrm{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i)$, for each $i \in I$, and with associated perfectoid space $\widehat{U} = \mathrm{Spa}(R, R^+)$, where (R, R^+) is the p-adic completion of $\varinjlim_{i\in I}(R_i, R_i^+)$, which is perfectoid. By [DLLZ, Thm. 5.4.3], $\big(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)\big) = (R, R^+)$, and $\big(\widehat{\mathcal{O}}^\flat(U), \widehat{\mathcal{O}}^{\flat+}(U)\big)$ is its tilt $(R^\flat, R^{\flat+})$. Let us write

- $M_i := \mathcal{M}_i(U_i)$, for each $i \in I$;
- $M := \mathcal{M}_{X_{\mathrm{prok\acute{e}t}}}(U) = \varinjlim_{i\in I} M_i$; and
- $M^\flat := \mathcal{M}_{X^\flat_{\mathrm{prok\acute{e}t}}}(U) = \varprojlim_{a\mapsto a^p} M$.

Recall that we have $\alpha_i : M_i \to R_i$, for each $i \in I$, and $\alpha^\flat : M^\flat \to R^\flat$ (see Section 2.1 and [DLLZ, Def. 5.4.1]). For each $r \geq 1$, we have a multiplicative map

$R^\flat \to W(R^{\flat+})[\frac{1}{p}]/\xi^r$ induced by $R^{\flat+} \to W(R^{\flat+})$, which we still denote by $f \mapsto [f]$. Then the composition of this map with $\alpha^\flat : M^\flat \to R^\flat$ induces a map

$$\begin{aligned} \widetilde{\alpha}_{i,r} : M_i \times_M M^\flat &\to \big(R_i \widehat{\otimes}_{W(\kappa)} (W(R^{\flat+})/\xi^r)\big)[M_i \times_M M^\flat] \\ a = (a', a'') &\mapsto \big(\alpha_i(a') \otimes 1\big) - \big(1 \otimes [\alpha^\flat(a'')]\big)\, \mathbf{e}^a, \end{aligned}$$

where $\mathbf{e}^a$ denotes (in boldface, unlike in our convention) the element of the monoid algebra corresponding to $a = (a', a'') \in M_i \times_M M^\flat$. Let

$$S_{i,r} := \big(R_i \widehat{\otimes}_{W(\kappa)} (W(R^{\flat+})/\xi^r)\big)[M_i \times_M M^\flat] / \big(\widetilde{\alpha}_{i,r}(a)\big)_{a \in M_i \times_M M^\flat}. \tag{2.2.8}$$

By abuse of notation, we shall sometimes drop tensor products with 1 in the notation, and write $\alpha_i(a') = [\alpha^\flat(a'')]\, \mathbf{e}^a$ in $S_{i,r}$. There is a natural map

$$\theta_{\log} : S_{i,r} \to R \tag{2.2.9}$$

induced by the natural maps $R_i \to R$ and (2.2.1) such that $\theta_{\log}(\mathbf{e}^a) = 1$, which is well defined because $\theta([\alpha^\flat(a'')]) = \alpha_i(a')$ in R, for all $(a', a'') \in M_i \times_M M^\flat$. Let

$$\widehat{S}_i := \varprojlim_{r,s} \big(S_{i,r} / (\ker \theta_{\log})^s\big),$$

equipped with a canonically induced map $\theta_{\log} : \widehat{S}_i \to R$. Note that $\varinjlim_{i \in I} \widehat{S}_i$ depends only on $U \in X_{\text{prokét}}$, but not on the presentation.

Definition 2.2.10. (1) The geometric de Rham period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}$ on $X_{\text{prokét}}$ is the sheaf associated with the presheaf sending U to $\varinjlim_{i \in I} \widehat{S}_i$, equipped with the filtration $\mathrm{Fil}^r \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X} := (\ker \theta_{\log})^r \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}$.

(2) We define the filtration on $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}[t^{-1}]$, where t is the same as in Definition 2.2.3(2), by $\mathrm{Fil}^r(\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}[t^{-1}]) := \sum_{s \geq -r} t^{-s} \mathrm{Fil}^{r+s} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}$.

(3) Let $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X}$ be the completion of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}[t^{-1}]$ with respect to the above filtration, equipped with the induced filtration. Then we have

$$\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X} = \varprojlim_{s \geq 0} \big(\mathrm{Fil}^r(\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}[t^{-1}]) / \mathrm{Fil}^{r+s}(\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X}[t^{-1}])\big),$$

and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X} = \cup_{r \in \mathbb{Z}} \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X}$. Let $\mathcal{O}\mathbb{C}_{\log,X} := \mathrm{gr}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X}$.

We shall omit the subscript X in the notation when the context is clear.

Note that $\mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ is a sheaf of rings and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log} = (\mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})[t^{-1}]$. (However, $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \neq \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}[t^{-1}]$ in general.)

Remark 2.2.11. Even if the log structure is trivial, the definition of $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ given here is slightly different from the definitions of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ in [Bri08, Sec. 5], [Sch13, Sec. 6], and [Sch16], as we perform an additional completion with respect to the filtration. This modification is necessary because the sheaves $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ defined in *loc. cit.* are not complete with respect to the filtrations—we thank Koji Shimizu for pointing out this. We will see in Corollary 2.4.2 below that the Poincaré lemma still holds and, with the new definition of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$, all the previous arguments in *loc. cit.* (and also those in [LZ17]) remain essentially unchanged.

Remark 2.2.12. Although $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ is defined for any locally noetherian fs log adic space X over $\mathrm{Spa}(k, k^+)$, we shall only use it for log smooth ones over some finite extension k of $\mathbb{Q}_p$, or some of their closed subspaces with induced log structures.

The period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ is equipped with a natural log connection. Let $\Omega^{\log}_X$ be the sheaf of log differentials, as in [DLLZ, Def. 3.2.25]. By abuse of notation, its pullbacks to $X_{\text{ét}}$, $X_{\text{két}}$, and $X_{\text{prokét}}$ will still be denoted by the same symbols.

Note that there is a unique $\mathbb{B}^+_{\mathrm{dR}}(U)/\xi^r$-linear log connection

$$\nabla : S_{i,r} \to S_{i,r} \otimes_{R_i} \Omega^{\log}_X(U_i) \tag{2.2.13}$$

extending $d : R_i \to \Omega^{\log}_X(U_i)$ and $\delta : M_i \to \Omega^{\log}_X(U_i)$ such that

$$\nabla(\mathbf{e}^a) = \mathbf{e}^a\, \delta(a'), \tag{2.2.14}$$

for all $a = (a', a'') \in M_i \times_M M^\flat$. Essentially by definition, we have

$$\nabla\big((\ker\theta_{\log})^s\big) \subset (\ker\theta_{\log})^{s-1} \otimes_{R_i} \Omega^{\log}_X(U_i)$$

for all $s \geq 1$. By taking $\ker(\theta_{\log})$-adic completion, inverse limit over r, and direct limit over i, the above log connection (2.2.13) extends to a $\mathbb{B}^+_{\mathrm{dR}}$-linear log connection

$$\nabla : \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X. \tag{2.2.15}$$

Since $t \in \mathbb{B}^+_{\mathrm{dR}}$, (2.2.15) further extends to a $\mathbb{B}_{\mathrm{dR}}$-linear log connection

$$\nabla : \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}[t^{-1}] \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}[t^{-1}] \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X, \tag{2.2.16}$$

satisfying $\nabla\big(\mathrm{Fil}^r(\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}[t^{-1}])\big) \subset \big(\mathrm{Fil}^{r-1}(\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}[t^{-1}])\big) \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X$, for all $r \in \mathbb{Z}$. Therefore, (2.2.16) also extends to a $\mathbb{B}_{\mathrm{dR}}$-linear log connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X, \tag{2.2.17}$$

satisfying $\nabla(\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \subset (\mathrm{Fil}^{r-1}\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X$, for all $r \in \mathbb{Z}$.

2.3. **Local study of $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$.** In this subsection, we study $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ when there are good local coordinates. These results are similar to [Bri08, Sec. 5], [Sch13, Sec. 6], and [Sch16]. We assume that k is a finite extension of $\mathbb{Q}_p$, and let $\overline{k}$ be a fixed algebraic closure. For each $m \geq 1$, we denote by $\boldsymbol{\mu}_m$ (resp. $\boldsymbol{\mu}_\infty = \cup_m \boldsymbol{\mu}_m$) the group of m-th (resp. all) roots of unity in $\overline{k}$. Let $k_m = k(\boldsymbol{\mu}_m) \subset \overline{k}$, for all $m \geq 1$; let $k_\infty = k(\boldsymbol{\mu}_\infty) = \cup_m k_m$ in $\overline{k}$; and let $\widehat{k}_\infty$ be the p-adic completion of k_∞. Then $\widehat{k}_\infty$ is a perfectoid field. Let $\widehat{k}^\flat_\infty$ denote its tilt. We shall denote by k^+_m, $\widehat{k}^+_\infty$, and $\widehat{k}^{\flat+}_\infty$ the rings of integers in k_m, $\widehat{k}_\infty$, and $\widehat{k}^\flat_\infty$, respectively. Let

$$A_{\mathrm{inf}} := \mathbb{A}_{\mathrm{inf}}(\widehat{k}_\infty, \widehat{k}^+_\infty).$$

Fix an isomorphism of abelian groups

$$\zeta : \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_\infty, \tag{2.3.1}$$

and write $\zeta_n := \zeta(\frac{1}{n})$, for all $n \in \mathbb{Z}_{\geq 1}$. Then $\zeta(\frac{m}{n}) = \zeta_n^m$, for all $\frac{m}{n} \in \mathbb{Q}$. Define

$$\epsilon : \mathbb{Q} \to (\widehat{k}^{\flat+}_\infty)^\times : \; y \mapsto (\zeta(y), \zeta(\tfrac{y}{p}), \zeta(\tfrac{y}{p^2}), \ldots).$$

We shall also write $\zeta^y = \zeta(y)$ and $\epsilon^y = \epsilon(y)$. Note that $\varpi^\flat := (\epsilon - 1)/(\epsilon^{\frac{1}{p}} - 1)$ is a pseudo-uniformizer of $\widehat{k}^\flat_\infty$, and the kernel of $\theta : A_{\mathrm{inf}} \to \widehat{k}^+_\infty$ is generated by

$$\xi := ([\epsilon] - 1)/([\epsilon^{\frac{1}{p}}] - 1).$$

Consider

$$t := \log([\epsilon]) \in B^+_{\mathrm{dR}} := \mathbb{B}^+_{\mathrm{dR}}(\widehat{k}_\infty, \widehat{k}^+_\infty) = \varprojlim_r \big(A_{\mathrm{inf}}[\tfrac{1}{p}]/\xi^r\big). \tag{2.3.2}$$

Let $k \to B_{\mathrm{dR}}^+$ be the unique embedding such that the composition of $k \to B_{\mathrm{dR}}^+ \to \widehat{k}_\infty$ is the natural homomorphism $k \to \widehat{k}_\infty$.

Let P be a toric monoid which decomposes as a direct sum $P = \overline{P} \oplus Q$ of toric monoids. (Here $\overline{P}$ means P/Q as in [DLLZ, Rem. 2.1.2], rather than the sharp quotient of P as in [DLLZ, Def. 2.1.1].) Consider the closed subspace

$$\mathbb{E} := \mathrm{Spa}(k\langle \overline{P} \rangle, k^+\langle \overline{P} \rangle)$$

of $\mathrm{Spa}(k\langle P \rangle, k^+\langle P \rangle)$, which we equip with the log structure pulled back from $\mathrm{Spa}(k\langle P \rangle, k^+\langle P \rangle)$ (see [DLLZ, Rem. 2.2.16]). For each $m \in \mathbb{Z}_{\geq 1}$, let $\frac{1}{m}P$ be the toric monoid such that $P \hookrightarrow \frac{1}{m}P$ can be identified with the m-th multiple map $[m] : P \to P$, and let $\frac{1}{m}Q$ and $\frac{1}{m}\overline{P}$ be defined similarly. Let $P_{\mathbb{Q}_{\geq 0}} := \varinjlim_m \big(\frac{1}{m}P\big)$ and $P_{\mathbb{Q}}^{\mathrm{gp}} := (P_{\mathbb{Q}_{\geq 0}})^{\mathrm{gp}} \cong P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $Q_{\mathbb{Q}_{\geq 0}}$, $Q_{\mathbb{Q}}^{\mathrm{gp}}$, $\overline{P}_{\mathbb{Q}_{\geq 0}}$, and $\overline{P}_{\mathbb{Q}}^{\mathrm{gp}}$ be defined similarly. By pulling back a standard toric tower over $\mathrm{Spa}(k\langle P \rangle, k^+\langle P \rangle)$ (cf. [DLLZ, Sec. 6.1]), we obtain a pro-Kummer étale cover $\widetilde{\mathbb{E}} := \varprojlim_m \mathbb{E}_m \to \mathbb{E}$, where

$$\mathbb{E}_m := \mathrm{Spa}(k_m\langle (\tfrac{1}{m}P)/Q \rangle, k_m^+\langle (\tfrac{1}{m}P)/Q \rangle)$$

is endowed with the log structure modeled on $\frac{1}{m}P$, and where the transition maps $\mathbb{E}_{m'} \to \mathbb{E}_m$ (for $m|m'$) are induced by the natural inclusions $\frac{1}{m}P \hookrightarrow \frac{1}{m'}P$. Then $\widetilde{\mathbb{E}}$ is log affinoid perfectoid with associated perfectoid space

$$\widehat{\widetilde{\mathbb{E}}} = \mathrm{Spa}(\widehat{k}_\infty\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle, \widehat{k}_\infty^+\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle).$$

For $a \in P_{\mathbb{Q}_{\geq 0}}$, we denote by $T^a \in k^+\langle P_{\mathbb{Q}_{\geq 0}} \rangle$ the corresponding element (as opposed to e^a as usual). Let $\overline{T}^a$ denote the image of T^a in $k^+\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle$. Note that $\overline{T}^a = 0$ when $a \notin \overline{P}_{\mathbb{Q}_{\geq 0}}$. Moreover, we denote by $\frac{1}{m}a$ the unique element in $P_{\mathbb{Q}_{\geq 0}}$ such that $m(\frac{1}{m}a) = a$, so that $\big(T^{\frac{1}{m}a}\big)^m = T^a$ in $k^+\langle P_{\mathbb{Q}_{\geq 0}} \rangle$. Let $T^{a\flat} := (T^a, T^{\frac{1}{p}a}, \ldots) \in (\widehat{k}_\infty^+\langle P_{\mathbb{Q}_{\geq 0}} \rangle)^\flat$, and let $\overline{T}^{a\flat}$ denote its image in $(\widehat{k}_\infty^+\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle)^\flat$. Again, note that $\overline{T}^{a\flat} = 0$ when $a \notin \overline{P}_{\mathbb{Q}_{\geq 0}}$. The Galois group $\Gamma = \mathrm{Aut}(\widetilde{\mathbb{E}}/\mathbb{E}_{\widehat{k}_\infty})$ has a natural action on $\widehat{k}_\infty^+\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle$ given by

$$\gamma(\overline{T}^{\overline{a}}) = \gamma(\overline{a})\, \overline{T}^{\overline{a}}, \tag{2.3.3}$$

for all $\gamma \in \Gamma$ and $\overline{a} \in \overline{P}_{\mathbb{Q}_{\geq 0}} \subset \overline{P}_{\mathbb{Q}}^{\mathrm{gp}} \subset P_{\mathbb{Q}}^{\mathrm{gp}}$, where $\gamma(\overline{a})$ is the element of $\boldsymbol{\mu}_\infty$ given by $\mathrm{Aut}(\widetilde{\mathbb{E}}/\mathbb{E}_{\widehat{k}_\infty}) \cong \mathrm{Hom}(P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty)$ (cf. [DLLZ, (6.1.4)]).

For each $r \geq 1$, we view B_{dR}^+/ξ^r as a Tate k-algebra (in the sense of [Sch12, Def. 2.6]) with a ring of definition A_{inf}/ξ^r (with its p-adic topology), and view

$$(B_{\mathrm{dR}}^+/\xi^r)\langle (\tfrac{1}{m}P)/Q \rangle = (B_{\mathrm{dR}}^+/\xi^r)\widehat{\otimes}_k \big(k\langle (\tfrac{1}{m}P)/Q \rangle\big)$$

as a Tate algebra as well. The completed direct limit (over m) of these algebras is canonically isomorphic to the completed direct limit of $(B_{\mathrm{dR}}^+/\xi^r)\langle \frac{1}{m}\overline{P} \rangle$, which we denote by $(B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle$. By (2.2.2), there is a canonical isomorphism

$$(B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle \xrightarrow{\sim} \mathbb{B}_{\mathrm{dR}}^+(\widehat{k}_\infty\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle, \widehat{k}_\infty^+\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle)/\xi^r \tag{2.3.4}$$

sending $e^{\overline{a}}$ to $[\overline{T}^{\overline{a}\flat}]$, for all $\overline{a} \in \overline{P}_{\mathbb{Q}_{\geq 0}}$. Then (2.3.4) is Γ-equivariant if we equip $(B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle$ with the action of Γ defined by

$$\gamma(e^{\overline{a}}) = [(\gamma(\overline{a}), \gamma(\tfrac{1}{p}\overline{a}), \ldots)]\, e^{\overline{a}}, \tag{2.3.5}$$

for all $\gamma \in \Gamma$ and $\overline{a} \in \overline{P}_{\mathbb{Q}_{\geq 0}}$, which reduces modulo ξ to the action (2.3.3).

Now suppose that $X = \mathrm{Spa}(A, A^+)$ is an affinoid fs log adic space with a strictly étale morphism $X \to \mathbb{E}$. Let $\widetilde{X}$ be the pullback of $\widetilde{\mathbb{E}}$ under $X \to \mathbb{E}$. Then $\widetilde{X} \in X_{\text{prokét}}$ is also log affinoid perfectoid with $\widehat{\widetilde{X}} = \mathrm{Spa}(\widehat{A}_\infty, \widehat{A}_\infty^+)$ the associated perfectoid space, and $\widetilde{X}/X_{\widehat{k}_\infty}$ is a Galois cover with Galois group Γ.

Consider the sheaf of monoid algebras $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[P]$. Let $\mathfrak{M} \subset \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[P]$ denote the sheaf of ideals generated by $\{e^a - 1\}_{a \in P}$, and let

$$\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] := \varprojlim_r \big(\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[P]/\mathfrak{M}^r\big).$$

Note that $e^a \in 1 + \mathfrak{M} \subset \big(\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]]\big)^\times$, for all $a \in P$. Therefore, we have a monoid homomorphism $P_{\mathbb{Q}_{\geq 0}} \to \big(\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]]\big)^\times \subset \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]]$ (with respect to the multiplicative structure on $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]]$) defined by sending $\frac{1}{m}a$, where $m \in \mathbb{Z}_{\geq 1}$ and $a \in P$, to the formal power series of $\big(1+(e^a-1)\big)^{\frac{1}{m}}$, which we abusively still denote by $e^{\frac{1}{m}a}$. On the other hand, consider the monoid homomorphism

$$P \to \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] : \ a \mapsto \log(e^a) := \sum_{l=1}^{\infty} (-1)^{l-1} \tfrac{1}{l} (e^a - 1)^l$$

(with respect to the additive structure on $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]]$), which uniquely extends to a group homomorphism $P^{\mathrm{gp}} \to \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] : \ a \mapsto y_a$ such that

$$y_a = \log(e^{a^+}) - \log(e^{a^-})$$

when $a = a^+ - a^-$ for some $a^+, a^- \in P$. Let $P_{\mathbb{Q}}^{\mathrm{gp}} := (P_{\mathbb{Q}_{\geq 0}})^{\mathrm{gp}} \cong P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the above homomorphism further extends linearly to a $\mathbb{Q}$-vector space homomorphism $P_{\mathbb{Q}}^{\mathrm{gp}} \to \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] : \ a \mapsto y_a$. Since $y_a - (e^a - 1) \in \mathfrak{M}^2$ for all $a \in P$, if we choose a $\mathbb{Z}$-basis $\{a_1, \ldots, a_n\}$ of P^{gp}, and write $y_j = y_{a_j}$, for each $j = 1, \ldots, n$, then we have a canonical isomorphism

$$\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[y_1, \ldots, y_n]] \xrightarrow{\sim} \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] : \ y_j \mapsto y_j \tag{2.3.6}$$

of $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}$-algebras, matching the ideals $(y_1, \ldots, y_n)^r$ and $(\xi, y_1, \ldots, y_n)^r$ of the source with the ideals $\mathfrak{M}^r$ and $(\xi, \mathfrak{M})^r$ of the target, respectively, for all $r \in \mathbb{Z}_{\geq 0}$. We similarly define $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[\overline{P}-1]]$ and $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[\overline{P}-1]][[Q-1]]$, and the decomposition $P = \overline{P} \oplus Q$ induces a canonical isomorphism $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] \cong \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[\overline{P}-1]][[Q-1]]$.

Lemma 2.3.7. *There is a unique morphism of sheaves*

$$v : \mathcal{O}_{X_{\text{prokét}}}|_{\widetilde{X}} \to \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[\overline{P}-1]] \tag{2.3.8}$$

satisfying the following conditions:

1. *The composition* $\mathcal{O}_{X_{\text{prokét}}}|_{\widetilde{X}} \xrightarrow{v} \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[\overline{P}-1]] \xrightarrow{e^{\overline{a}} \mapsto 1,\, \xi \mapsto 0} \widehat{\mathcal{O}}_{X_{\text{prokét}}}|_{\widetilde{X}}$ *is the natural map.*
2. *The composition* $P_{\mathbb{Q}_{\geq 0}, X_{\text{prokét}}}|_{\widetilde{X}} \to \mathcal{O}_{X_{\text{prokét}}}|_{\widetilde{X}} \xrightarrow{v} \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[\overline{P}-1]]$ *is induced by* $a \mapsto [\overline{T}^{a\flat}]\, e^{\overline{a}}$, *where* $P_{\mathbb{Q}_{\geq 0}, X_{\text{prokét}}}$ *denotes the constant sheaf of monoids on* $X_{\text{prokét}}$ *associated with* $P_{\mathbb{Q}_{\geq 0}}$, *and where* $\overline{a}$ *denotes the image of* a *in* $\overline{P}$.

Proof. Consider any log affinoid perfectoid object $U = \varprojlim_{i \in I} U_i \in X_{\text{prokét}/\widetilde{X}}$, as in [DLLZ, Def. 5.3.1], with $U_i = (\mathrm{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i)$. We would like to construct a compatible family of maps

$$v_i : R_i \to \mathbb{B}_{\mathrm{dR}}^+(U)[[\overline{P} - 1]], \tag{2.3.9}$$

indexed by $i \in I$. For each $i \in I$, by [DLLZ, Lem. 4.2.5], $U_i \times_{\mathbb{E}} \mathbb{E}_{m_i} \to \mathbb{E}_{m_i}$ is strictly étale for some m_i. Given $U \to \widetilde{X}$, some $U_{i'} \to U_i$ factors through $U_{i'} \to U_i \times_{\mathbb{E}} \mathbb{E}_{m_i}$, so that $R_i \to R_{i'}$ factors as $R_i \to B := \mathcal{O}(U_i \times_{\mathbb{E}} \mathbb{E}_{m_i}) \to R_{i'}$. Note that we have a structure homomorphism $k_{m_i}[(\frac{1}{m_i}P)/Q] \to B$ induced by the second projection $U_i \times_{\mathbb{E}} \mathbb{E}_{m_i} \to \mathbb{E}_{m_i}$, and a canonical k_{m_i}-algebra homomorphism $k_{m_i}[(\frac{1}{m_i}P)/Q] \to \mathbb{B}_{\mathrm{dR}}^+(U)[[\overline{P} - 1]]$ defined by sending the image of e^a to $[\overline{T}^{a\flat}]\, e^{\overline{a}}$, for all $a \in \frac{1}{m_i}P$, which fit into the following commutative diagram of solid arrows

$$\begin{array}{ccccccc} & & k_{m_i}[(\frac{1}{m_i}P)/Q] & \longrightarrow & & & \mathbb{B}_{\mathrm{dR}}^+(U)[[\overline{P} - 1]] \\ & & \downarrow & & & & \downarrow {\scriptstyle e^a \mapsto 1,\, \xi \mapsto 0} \\ R_i & \longrightarrow & B & \longrightarrow & R_{i'} & \longrightarrow & \widehat{\mathcal{O}}_{X_{\text{prokét}}}(U). \end{array}$$

By [Hub96, Cor. 1.7.3(iii)], there is a finitely generated $k^+[(\frac{1}{m_i}P)/Q]$-algebra B_0^+ such that $B_0 := B_0^+[\frac{1}{p}]$ is étale over $k[(\frac{1}{m_i}P)/Q]$ and such that B is the p-adic completion of B_0. Then it follows from [Sch13, Lem. 6.11] that there is a unique continuous lifting $B \to \mathbb{B}_{\mathrm{dR}}^+(U)[[\overline{P} - 1]]$, denoted by the dotted arrow above, making the whole diagram commute. Then the composition of $R_i \to B \to \mathbb{B}_{\mathrm{dR}}^+(U)[[\overline{P} - 1]]$ gives the desired (2.3.9). It is clear that such v_i's, for all $i \in I$, are independent of the choices, compatible with each other, and define the desired (2.3.8). □

The following lemma generalizes the isomorphism (2.3.4) for $X = \mathrm{Spa}(A, A^+)$. Note that (2.3.8) induces a map $A \to (\mathbb{B}_{\mathrm{dR}}^+(\widetilde{X})/\xi^r)[[\overline{P} - 1]] \xrightarrow{e^{\overline{a}} \mapsto 1} \mathbb{B}_{\mathrm{dR}}^+(\widetilde{X})/\xi^r$. Together with (2.3.4), it induces a canonical map

$$\big(A \widehat{\otimes}_k (B_{\mathrm{dR}}^+/\xi^r)\big) \widehat{\otimes}_{(B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P} \rangle} (B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle \to \mathbb{B}_{\mathrm{dR}}^+(\widetilde{X})/\xi^r. \tag{2.3.10}$$

Lemma 2.3.11. *The map* (2.3.10) *is an isomorphism. Furthermore, the* Γ*-action on* $(B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P} \rangle$ *defined in* (2.3.5) *uniquely extends to a continuous* Γ*-action on* $A \widehat{\otimes}_k (B_{\mathrm{dR}}^+/\xi^r)$*, which is trivial modulo* ξ *and makes* (2.3.10) Γ*-equivariant.*

Proof. Since (2.3.10) is compatible with the filtrations induced by multiplication by the powers of ξ, it suffices to consider the associated graded pieces and show that $(A \widehat{\otimes}_k \widehat{k}_\infty) \widehat{\otimes}_{\widehat{k}_\infty \langle \overline{P} \rangle} \widehat{k}_\infty \langle \overline{P}_{\mathbb{Q}_{\geq 0}} \rangle \cong \widehat{A}_\infty$. Then the same argument as in the proof of [Sch13, Lem. 6.18] applies, with the input [Sch13, Lem. 4.5] there for the tower $\widetilde{\mathbb{T}}^n \to \mathbb{T}$ replaced with [DLLZ, Lem. 6.1.9] for the tower $\varprojlim_m \mathrm{Spa}(k_m\langle \frac{1}{m}\overline{P} \rangle, k_m^+\langle \frac{1}{m}\overline{P} \rangle \to \mathbb{E}$.

The unique existence of a continuous Γ-action on $A \widehat{\otimes}_k (B_{\mathrm{dR}}^+/\xi^r)$ extending the Γ-action on $(B_{\mathrm{dR}}^+/\xi^r)\langle \overline{P} \rangle$ follows from an argument similar to the one in the proof of Lemma 2.3.7. Concretely, since the map $k\langle \overline{P} \rangle \to A$ arises as the p-adic completion of an étale morphism $k[\overline{P}] \to A_0$ of finite type, the Γ-action on $(B_{\mathrm{dR}}^+/\xi^r)[\overline{P}]$ uniquely extends to a continuous Γ-action on $A_0 \otimes_k (B_{\mathrm{dR}}^+/\xi^r)$, and further uniquely extends to a continuous Γ-action after p-adic completion, with desired properties. □

Lemma 2.3.12. *There exists a map of sheaves of monoids*

$$\beta : \mathcal{M}^\flat|_{\widetilde{X}} \to (\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P - 1]])^\times \tag{2.3.13}$$

satisfying the following conditions:

(1) *For all* $a \in \mathcal{M}^\flat$, *we have* $v\big(\alpha(a^\sharp)\big) = [\alpha^\flat(a)]\,\beta(a)$, *where we denote by* $a \mapsto a^\sharp$ *the natural projection* $\mathcal{M}^\flat \to \mathcal{M}$.

(2) *The composition of* β *with the canonical map* $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] \xrightarrow{e^a \mapsto 1,\, \xi \mapsto 0} \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}|_{\widetilde{X}}$ *is the constant* 1.

(3) *The restriction of* β *to* $Q_{\mathbb{Q}_{\geq 0}}$ *(as a constant sheaf) is given by* $a \mapsto e^a$.

Proof. The sheaf $\mathcal{M}^\flat|_{\widetilde{X}}$ is generated by $P_{\mathbb{Q}_{\geq 0}} \cong \overline{P}_{\mathbb{Q}_{\geq 0}} \oplus Q_{\mathbb{Q}_{\geq 0}}$ (as constant sheaves) and $\varprojlim_{f \mapsto f^p} \mathcal{O}^\times_{X_{\mathrm{prok\acute{e}t}}}|_{\widetilde{X}}$. We need to define the map for $\overline{a} \in \overline{P}_{\mathbb{Q}_{\geq 0}}$. If we write (locally) $\alpha^\flat(\overline{a}) = h\,\overline{T}^{\overline{a}_0 \flat}$, for some section h of $\varprojlim_{f \mapsto f^p} \mathcal{O}^\times_{X_{\mathrm{prok\acute{e}t}}}|_{\widetilde{X}} \subset \widehat{\mathcal{O}}^\times_{X^\flat_{\mathrm{prok\acute{e}t}}}$ and $\overline{a}_0 \in \overline{P}_{\mathbb{Q}_{\geq 0}}$, then $\alpha(\overline{a}^\sharp) = \big(\alpha^\flat(\overline{a})\big)^\sharp = h^\sharp\,\overline{T}^{\overline{a}_0}$, and the conditions of the lemma are satisfied by the local section $\beta(\overline{a}) := \frac{v(h^\sharp)}{[h]} e^{\overline{a}_0}$ of $(\mathbb{B}_{\mathrm{dR}}^+[[P-1]])^\times|_{\widetilde{X}}$. This expression is independent of the local choices, and hence globalizes and defines the desired map β. □

Next, we give an explicit description of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ on the localized site $X_{\mathrm{prok\acute{e}t}/\widetilde{X}}$. Let $U = \varprojlim_{i \in I} U_i \in X_{\mathrm{prok\acute{e}t}/\widetilde{X}}$ be a log affinoid perfectoid object, with $U_i = (\mathrm{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i)$, as in the proof of Lemma 2.3.7. Let $S_{i,r}$ be as in (2.2.8). Note that, for $a = (a', a'')$ with $a'' \in Q_{\mathbb{Q}_{\geq 0}}$, we have $\alpha^\flat(a'') = 0$ and hence the relation $\alpha_i(a') = [\alpha^\flat(a'')]\,\mathbf{e}^a$ reduces to simply $\alpha_i(a') = 0$ in $S_{i,r}$, with no constraint on $\mathbf{e}^a$, in which case we can view $\mathbf{e}^a$ as a free variable. Consider the map

$$(\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)[P] \to S_{i,r} :\ e^a \mapsto \mathbf{e}^{(a,a)},\ \text{for all } a \in P,$$

which sends $(\xi, \mathfrak{M})$ to $\ker\theta_{\log}$, and therefore induces a map $\mathbb{B}_{\mathrm{dR}}^+(U)[[P-1]] \to \widehat{S}_i$. By taking completion and sheafification, we obtain a map

$$\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}|_{\widetilde{X}} \tag{2.3.14}$$

on $X_{\mathrm{prok\acute{e}t}/\widetilde{X}}$. If we define $\mathrm{Fil}^r\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] := (\xi, \mathfrak{M})^r\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]]$, then (2.3.14) is compatible with the filtrations on both sides (see Definition 2.2.10).

The following is a log analogue of [Bri08, Prop. 5.2.2] and [Sch13, Prop. 6.10], formulated in terms of charts and monoids:

Proposition 2.3.15. *The map* (2.3.14) *is an isomorphism of filtered sheaves.*

Proof. Let $U = \varprojlim_{i \in I} U_i \in X_{\mathrm{prok\acute{e}t}/\widetilde{X}}$ be a log affinoid perfectoid object, with $U_i = (\mathrm{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i)$, as in Lemma 2.3.7. For $i \in I$ and $r \geq 1$, the map (2.3.8) induces a natural map $R_i \to (\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)[[\overline{P}-1]]$. Together with the map (2.3.13), these maps induce a ring homomorphism

$$\big(R_i \widehat{\otimes}_{W(\kappa)} (\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)\big)[M_i \times_M M^\flat] \to (\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)[[P-1]], \tag{2.3.16}$$

sending $\mathbf{e}^a$ to $\beta(a'')$, for all $a = (a', a'') \in M_i \times_M M^\flat$. By Lemmas 2.3.7 and 2.3.12, this map factors through $S_{i,r} \to (\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)[[P-1]]$, and its composition with $(\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)[[P-1]] \xrightarrow{e^a \mapsto 1,\, \xi \mapsto 0} \widehat{R}_\infty$ is the map $\theta_{\log}$, where $S_{i,r}$ and $\theta_{\log}$ are as in (2.2.8) and (2.2.9), respectively. Therefore, this map sends $\ker\theta_{\log}$ to $(\xi, \mathfrak{M})$. By taking $\ker\theta_{\log}$-adic completion, inverse limit over $r \geq 1$, and direct limit over $i \in I$, we obtain a map $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}(U) \to \mathbb{B}_{\mathrm{dR}}^+(U)[[P-1]]$, whose pre-composition with the map $\mathbb{B}_{\mathrm{dR}}^+(U)[[P-1]] \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}(U)$ given by (2.3.14) is the identity map, because it is $\mathbb{B}_{\mathrm{dR}}^+(U)$-linear and sends e^a to e^a, for all $a \in P$. On the other hand, the post-composition of (2.3.9) with $\mathbb{B}_{\mathrm{dR}}^+(U)[[P-1]] \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+(U)$ is the natural map $R_i \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+(U)$, because $k[P_{\mathbb{Q}_{\geq 0}}] \to \mathbb{B}_{\mathrm{dR}}^+(U)[[\overline{P}-1]] \to \widehat{S}_i$ sends $e^a \mapsto [\overline{T}^{a\flat}]\, e^{\overline{a}} \mapsto [\overline{T}^{a\flat}]\, \mathbf{e}^{(\overline{a}, \overline{a})} = \overline{T}^a$ and the map $k[(\frac{1}{m_i}P)/Q] \to B_0$ in the proof of Lemma 2.3.7 is étale. Consequently, the map $S_{i,r} \to \widehat{S}_i/\xi^r$ induced by the composition of (2.3.16) with $(\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r)[[P-1]] \to \widehat{S}_i/\xi^r$ coincides with the natural map, because both maps send the image of $\mathbf{e}^a$ in $S_{i,r}$ to the same further image in $\widehat{S}_i/\xi^r$, for all $a \in M_i \times_M M^\flat$. Thus, the composition of $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+(U) \to \mathbb{B}_{\mathrm{dR}}^+(U)[[P-1]] \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+(U)$ is also the identity map, as desired. □

Now, as before, let us fix a $\mathbb{Z}$-basis $\{a_1, \ldots, a_n\}$ of P^{gp}, and write $y_j = y_{a_j}$, for each $j = 1, \ldots, n$, so that we have $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[P-1]] \cong \mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[y_1, \ldots, y_n]]$ as in (2.3.6).

Corollary 2.3.17. *The isomorphism* (2.3.14) *induces isomorphisms*

$$\begin{aligned} \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} &\cong t^r \mathbb{B}_{\mathrm{dR}}^+\{W_1, \ldots, W_n\} \\ &:= \Big\{ t^r \sum_{\Lambda \in \mathbb{Z}_{\geq 0}^n} b_\Lambda\, W^\Lambda \in \mathbb{B}_{\mathrm{dR}}^+[[W_1, \ldots, W_n]] : b_\Lambda \to 0,\ t\text{-adically, as } \Lambda \to \infty \Big\} \end{aligned}$$

over $X_{\mathrm{prok\acute{e}t}/\widetilde{X}}$, for all $r \in \mathbb{Z}$, where we have the variable

$$W_j := t^{-1} y_j, \tag{2.3.18}$$

for each $1 \leq j \leq n$, and the monomial

$$W^\Lambda := W_1^{\Lambda_1} \cdots W_n^{\Lambda_n}, \tag{2.3.19}$$

for each exponent $\Lambda = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{Z}_{\geq 0}^n$. (Here we denote by $\{W_1, \ldots, W_n\}$ the ring of power series that are t-adically convergent, which is similar to the notation $\langle W_1, \ldots, W_n \rangle$ for the ring of power series that are p-adically convergent.) Therefore,

$$\mathrm{gr}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \cong t^r \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}[W_1, \ldots, W_n],$$

for all $r \in \mathbb{Z}$, and so $\mathrm{gr}^\bullet \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \cong \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}[t^\pm, W_1, \ldots, W_n]$.

By comparing the constructions, we obtain the following:

Corollary 2.3.20. *Suppose that $\imath : Z \to X$ is a strict closed immersion of log adic spaces such that the underlying morphism of adic spaces is the pullback of $\mathrm{Spa}(k\langle \overline{P}/Q' \rangle, k^+\langle \overline{P}/Q' \rangle) \hookrightarrow \mathbb{E} = \mathrm{Spa}(k\langle \overline{P} \rangle, k^+\langle \overline{P} \rangle)$ for some direct summand Q' of $\overline{P}$. Let $\widetilde{Z} := \widetilde{X} \times_X Z$. Suppose that $U \in X_{\mathrm{prok\acute{e}t}/\widetilde{X}}$ is log affinoid perfectoid, whose pullback $V := U \times_X Z$ is log affinoid perfectoid in $Z_{\mathrm{prok\acute{e}t}/\widetilde{Z}}$. Then, for each $r \geq 1$, the canonical surjection $\mathbb{B}_{\mathrm{dR},X}^+(U)/\xi^r \twoheadrightarrow \mathbb{B}_{\mathrm{dR},Z}^+(V)/\xi^r$ (cf. Corollary 2.2.7) induces*

$$\big(\mathbb{B}_{\mathrm{dR},X}^+(U)/\xi^r\big)/([T^{sa\flat}])^\wedge_{s \in \mathbb{Q}_{>0}, a \in Q'-\{0\}} \xrightarrow{\sim} \big(\mathbb{B}_{\mathrm{dR},Z}^+(V)/\xi^r\big), \tag{2.3.21}$$

where $([T^{sa\flat}])^\wedge_{s \in \mathbb{Q}_{>0}, a \in Q'-\{0\}}$ denotes the p-adic completion of the ideal generated by $\{[T^{sa\flat}]\}_{s \in \mathbb{Q}_{>0}, a \in Q'-\{0\}}$. In addition, the canonical isomorphisms $\mathbb{B}_{\mathrm{dR},Z}^+|_{\widetilde{Z}}[[P-1]] \cong \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,Z}^+|_{\widetilde{Z}}$ and $\mathbb{B}_{\mathrm{dR},X}^+|_{\widetilde{X}}[[P-1]] \cong \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X}^+|_{\widetilde{X}}$ given by Proposition 2.3.15 are compatible with each other via pullback and pushforward.

2.4. **Consequences.**

Remark 2.4.1. Let k be a finite extension of $\mathbb{Q}_p$. We may apply the calculations in Section 2.3 in the following two cases:

(1) When X is log smooth over k, étale locally there are *toric charts* $X \to \mathbb{E} = \mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ (with $Q = 0$), as in [DLLZ, Def. 3.1.12].
(2) Let Y be smooth over k, with log structure defined by a normal crossings divisor $E \hookrightarrow Y$ as in Example 2.1.2, and let X be a smooth intersection of irreducible components of E, equipped with the log structure pulled back from Y, as in [DLLZ, Ex. 2.3.18]. Then, étale locally, there is a toric chart of Y as above inducing a strictly étale morphism $X \to \mathbb{E} = \mathrm{Spa}(k\langle P/Q\rangle, k^+\langle P/Q\rangle)$ (for some direct summand Q of P).

In both cases, the sheaves of log differentials $\Omega_X^{\log}$ and $\Omega_X^{\log,\bullet} = \wedge^\bullet \Omega_X^{\log}$ are defined as in [DLLZ, Def. 3.2.25 and 3.2.28], and are known to be vector bundles on X, by [DLLZ, Thm. 3.2.26 and Cor. 3.2.27].

In particular, we have the following *Poincaré lemma* for $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$:

Corollary 2.4.2. *Let X be as in Remark* 2.4.1.

(1) We have an exact complex $0 \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \otimes \Omega_X^{\log,1} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \otimes \Omega_X^{\log,2} \to \cdots$.

(2) The above statement holds with $\mathbb{B}_{\mathrm{dR}}^+$ *and* $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$ *replaced with* $\mathbb{B}_{\mathrm{dR}}$ *and* $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$*, respectively.*

(3) The subcomplex $0 \to \mathrm{Fil}^r \mathbb{B}_{\mathrm{dR}} \to \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \xrightarrow{\nabla} (\mathrm{Fil}^{r-1} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \otimes \Omega_X^{\log,1} \xrightarrow{\nabla} (\mathrm{Fil}^{r-2} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \otimes \Omega_X^{\log,2} \cdots$ *of the complex for* $\mathbb{B}_{\mathrm{dR}}$ *and* $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ *is also exact, for each* $r \in \mathbb{Z}$.

(4) For each $r \in \mathbb{Z}$, *the quotient complex* $0 \to \mathrm{gr}^r \mathbb{B}_{\mathrm{dR}} \to \mathrm{gr}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \xrightarrow{\nabla} (\mathrm{gr}^{r-1} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \otimes \Omega_X^{\log,1} \xrightarrow{\nabla} (\mathrm{gr}^{r-2} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \otimes \Omega_X^{\log,2} \cdots$ *of the previous complex is exact, and can be identified with the complex* $0 \to \mathcal{O}_{X_{\mathrm{prok\acute{e}t}}}(r) \to \mathcal{O}\mathbb{C}_{\log}(r) \xrightarrow{\nabla} (\mathcal{O}\mathbb{C}_{\log}(r)) \otimes \Omega_X^{\log,1}(-1) \xrightarrow{\nabla} (\mathcal{O}\mathbb{C}_{\log}(r)) \otimes_{\mathcal{O}_{X_{\mathrm{prok\acute{e}t}}}} \Omega_X^{\log,2}(-2) \cdots$.

(*All the above tensor products are over* $\mathcal{O}_{X_{\mathrm{prok\acute{e}t}}}$, *which we omitted for simplicity.*)

Proof. In both cases of Remark 2.4.1, up to étale localization on X, we may assume that there exists a strictly étale morphism $X \to \mathbb{E} = \mathrm{Spa}(k\langle \overline{P}\rangle, k^+\langle \overline{P}\rangle)$, and then pass to $\widetilde{X}$ pro-Kummer étale locally, as in Section 2.3. Choose a $\mathbb{Z}$-basis $\{a_1, \ldots, a_n\}$ of P^{gp}, and write $a_j = a_j^+ - a_j^-$ for some $a_j^+, a_j^- \in P$, for each $j = 1, \ldots, n$. By Proposition 2.3.15 and Corollary 2.3.17, it suffices to prove the exactness of the complexes by using $\mathbb{B}_{\mathrm{dR}}^+|_{\widetilde{X}}[[y_1, \ldots, y_n]]$ and $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}}\{W_1, \ldots, W_n\}$ in place of $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$, respectively. Note that the isomorphism (2.3.14) matches y_j with $\log(\mathbf{e}^{(a_j^+, a_j^+)}) - \log(\mathbf{e}^{(a_j^-, a_j^-)})$. By [DLLZ, Thm. 3.2.26, Cor. 3.2.27, Prop. 3.2.15, and Cor. 3.2.19], $\Omega_X^{\log} = \oplus_{j=1}^n \big(\mathcal{O}_{X_{\mathrm{prok\acute{e}t}}}\, \delta(a_j)\big)$, and hence (because of (2.2.14))

$$\nabla(y_j) = \nabla\big(\log(\mathbf{e}^{(a_j^+, a_j^+)})\big) - \nabla\big(\log(\mathbf{e}^{(a_j^-, a_j^-)})\big) = \delta(a_j^+) - \delta(a_j^-) = \delta(a_j) \tag{2.4.3}$$

and (because of (2.3.18))

$$\nabla(W_j) = t^{-1}\delta(a_j). \tag{2.4.4}$$

The exactness then follows from a straightforward calculation. (Note that the t-adic convergence condition on power series is not affected by taking anti-derivatives.) □

By combining Corollaries 2.2.6 and 2.4.2, we obtain the *log Faltings's extension*:

Corollary 2.4.5. *We have a short exact sequence of sheaves of $\widehat{\mathcal{O}}_{X_{\text{prokét}}}$-modules*

$$0 \to \widehat{\mathcal{O}}_{X_{\text{prokét}}}(1) \to \mathrm{gr}^1\, \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log} \to \widehat{\mathcal{O}}_{X_{\text{prokét}}} \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X \to 0.$$

3. Log Riemann–Hilbert Correspondences

In this section, we establish our log p-adic Riemann–Hilbert and Simpson correspondences. Let k be a finite extension of $\mathbb{Q}_p$, with a fixed algebraic closure $\overline{k}$. Let K be a perfectoid field containing $k_\infty = k(\boldsymbol{\mu}_\infty) \subset \overline{k}$, and let $\mathrm{Gal}(K/k)$ abusively denote the group of continuous field automorphisms of K over k.

3.1. Filtered log connections "relative to B_{dR}".

Let us begin with a few definitions and constructions for a general locally noetherian adic space X over k.

Definition 3.1.1. (1) As in [LZ17, Sec. 3.1], let

$$B^+_{\mathrm{dR}} = \mathbb{B}^+_{\mathrm{dR}}(K, \mathcal{O}_K) \quad \text{and} \quad B_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}(K, \mathcal{O}_K) \tag{3.1.2}$$

(the first replacing (2.3.2) from now on). Let $t = \log([\epsilon]) \in B^+_{\mathrm{dR}}$ as in (2.3.2). Then the homomorphism $k \to K$ lifts uniquely to $k \to B^+_{\mathrm{dR}}$.

(2) For each integer $r \geq 1$, we define $\mathcal{O}_X \widehat{\otimes} (B^+_{\mathrm{dR}}/t^r)$ to be the sheaf on X_{an} associated with the presheaf which assigns to each affinoid open subset $U = \mathrm{Spa}(A, A^+) \subset X$ the ring $A \widehat{\otimes}_k (B^+_{\mathrm{dR}}/t^r)$. Then we define

$$\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}} = \varprojlim_r \big(\mathcal{O}_X \widehat{\otimes} (B^+_{\mathrm{dR}}/t^r)\big) \quad \text{and} \quad \mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}} = (\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}})[t^{-1}].$$

(3) The filtrations on $\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}$ and $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}$ are defined by setting

$$\mathrm{Fil}^r(\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}) = t^r(\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}) \quad \text{and} \quad \mathrm{Fil}^r(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}) = t^{-s}\mathrm{Fil}^{r+s}(\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}})$$

for some (and hence every) $s \geq -r$. Then we define

$$(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}})^{[a,b]} = \mathrm{Fil}^a(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}) / \mathrm{Fil}^{b+1}(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}),$$

for any $-\infty \leq a \leq b \leq \infty$. In particular, $\mathrm{gr}^r(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}) = (\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}})^{[r,r]}$.

(4) By replacing affinoid open subsets $U \subset X$ in (2) with general étale morphisms $U \to X$ from affinoid adic spaces, we similarly define the sheaves $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} (B^+_{\mathrm{dR}}/t^r)$, for all integers $r \geq 1$; $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B^+_{\mathrm{dR}}$; and $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B_{\mathrm{dR}}$ on $X_{\text{ét}}$. They are equipped with similarly defined filtrations.

Remark 3.1.3. These sheaves were introduced slightly differently in [LZ17, Sec. 3.1] as sheaves on $X_{K,\mathrm{an}}$ and $X_{K,\text{ét}}$. But since the base changes of objects of X_{an} and $X_{\text{ét}}$ generate $X_{K,\mathrm{an}}$ and $X_{K,\text{ét}}$, respectively (see, e.g., [LZ17, Lem. 2.5]), the categories of finite locally free $\mathcal{O}_X \widehat{\otimes} (B^+_{\mathrm{dR}}/t^r)$-modules (resp. $\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}$-modules) and finite locally free $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} (B^+_{\mathrm{dR}}/t^r)$-modules (resp. $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B^+_{\mathrm{dR}}$-modules) are naturally equivalent to the categories introduced in [LZ17, Def. 3.5]. For example, the category of finite locally free $\mathcal{O}_X \widehat{\otimes}_k K$-modules (i.e., $\mathrm{gr}^0(\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}})$-modules) on X_{an} is equivalent to the category of vector bundles on $X_{K,\mathrm{an}}$.

Thanks to Remark 3.1.3, the arguments in the proofs of [LZ17, Lem. 3.1 and 3.2, Prop. 3.3, and Cor. 3.4] also apply in the current setting and give the following:

Lemma 3.1.4. *Recall that $\lambda : X_{\text{ét}} \to X_{\mathrm{an}}$ denotes the natural projection of sites.*

(1) If $X = \mathrm{Spa}(A, A^+)$ is affinoid, then

$$H^i\big(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} (B_{\mathrm{dR}}^+/t^r)\big) = \begin{cases} A \widehat{\otimes}_k (B_{\mathrm{dR}}^+/t^r), & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

(2) There is a canonical isomorphism $\mathrm{gr}^r(\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B_{\mathrm{dR}}) \cong \mathcal{O}_{X_{\text{ét}}} \widehat{\otimes}_k K(r)$.

(3) There are canonical isomorphisms

$$\mathcal{O}_X \widehat{\otimes} (B_{\mathrm{dR}}^+/t^r) \cong \lambda_*\big(\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} (B_{\mathrm{dR}}^+/t^r)\big) \cong R\lambda_*\big(\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} (B_{\mathrm{dR}}^+/t^r)\big),$$

which in turn induce, for $? = \emptyset$ and $+$, isomorphisms

$$\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^? \cong \lambda_*(\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B_{\mathrm{dR}}^?) \cong R\lambda_*(\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B_{\mathrm{dR}}^?).$$

(4) If $X = \mathrm{Spa}(A, A^+)$ is affinoid, then the categories
- *of finite projective $A \widehat{\otimes}_k B_{\mathrm{dR}}^+$-modules;*
- *of finite locally free $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$-modules; and*
- *of finite locally free $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B_{\mathrm{dR}}^+$-modules*

are all equivalent to each other.

(5) The pushforward λ_* induces an equivalence from the category of finite locally free $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} (B_{\mathrm{dR}}^+/t^r)$-modules (resp. $\mathcal{O}_{X_{\text{ét}}} \widehat{\otimes} B_{\mathrm{dR}}^+$-modules) to the category of finite locally free $\mathcal{O}_X \widehat{\otimes} (B_{\mathrm{dR}}^+/t^r)$-modules (resp. $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$-modules).

As in [LZ17, Sec. 3.1], for $? = \emptyset$ or $+$, we can define the ringed space

$$\mathcal{X}^? = (X_{\mathrm{an}}, \mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^?), \tag{3.1.5}$$

where $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$ and $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}$ are as in Definition 3.1.1(2). They should be interpreted as the (not-yet-defined) base changes of X under $k \to B_{\mathrm{dR}}^+$ and $k \to B_{\mathrm{dR}}$, respectively. Then we have $\mathcal{O}_{\mathcal{X}^+} = \mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$ and $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}$.

Following [LZ17, Def. 3.5], we call a finite locally free $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$-module a *vector bundle* on $\mathcal{X}^+$. By varying the open subspaces of X, these objects form a stack on X_{an}. By passing to the t-isogeny category, we obtain the stack of vector bundles on open subspaces of $\mathcal{X}$. Then the category of vector bundles on $\mathcal{X}$ is the groupoid of global sections of the stack. Note that, unlike in [LZ17, Def. 3.5], we do not require that a vector bundle on $\mathcal{X}$ comes from a vector bundle on $\mathcal{X}^+$ via a global extension of scalars (although this extra generality will not be needed in the following). Clearly, there is a faithful functor from the category of vector bundles on $\mathcal{X}$ to the category of $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}$-modules.

Hence, for each vector bundle $\mathcal{E}$ on X_{an}, the sheaf $\mathcal{E} \widehat{\otimes}_k B_{\mathrm{dR}}^+$ (resp. $\mathcal{E} \widehat{\otimes}_k B_{\mathrm{dR}}$) (with its obvious meaning) is a vector bundle on $\mathcal{X}^+$ (resp. $\mathcal{X}$). More generally, if $\mathcal{E}$ is a vector bundle on X_{an}, and if $\mathcal{M}$ is a vector bundle on $\mathcal{X}^+$ (resp. $\mathcal{X}$), then we may regard $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M}$ as a vector bundle on $\mathcal{X}^+$ (resp. $\mathcal{X}$).

Now let X be a log smooth fs log adic space over k. Let $\Omega_X^{\log}$ and $\Omega_X^{\log,\bullet} = \wedge^\bullet \Omega_X^{\log}$ be the sheaves of log differentials on X_{an}, as in [DLLZ, Def. 3.2.25 and 3.2.28].

Definition 3.1.6. For $? = \emptyset$ or $+$, let $\Omega^{\log}_{\mathcal{X}^?/B_{\mathrm{dR}}^?} := \Omega_X^{\log} \widehat{\otimes}_k B_{\mathrm{dR}}^?$ and $\Omega^{\log,\bullet}_{\mathcal{X}^?/B_{\mathrm{dR}}^?} := \Omega_X^{\log,\bullet} \widehat{\otimes}_k B_{\mathrm{dR}}^?$, called the *sheaves of relative log differentials* on $\mathcal{X}^?$ over $B_{\mathrm{dR}}^?$.

For $? = \emptyset$ or $+$, there is a natural $B_{\mathrm{dR}}^?$-linear differential map $d : \mathcal{O}_{\mathcal{X}^?} \to \Omega^{\log}_{\mathcal{X}^?/B_{\mathrm{dR}}^?}$ inducing differential maps on $\Omega^{\log,\bullet}_{\mathcal{X}^?/B_{\mathrm{dR}}^?}$, extending the ones on $\mathcal{O}_X$ and $\Omega_X^{\log,\bullet}$.

Definition 3.1.7. (1) A *log connection* on a vector bundle $\mathcal{E}$ on $\mathcal{X}$ is a B_{dR}-linear map of sheaves $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\log}_{\mathcal{X}/B_{\mathrm{dR}}}$ satisfying the usual Leibniz rule. We say that ∇ is *integrable* if $\nabla^2 = 0$, in which case we have the *log de Rham complex* $DR_{\log}(\mathcal{E}) = (\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\log,\bullet}_{\mathcal{X}/B_{\mathrm{dR}}}, \nabla)$ and the *log de Rham cohomology* $H^i_{\log \mathrm{dR}}(\mathcal{X}, \mathcal{E}) := H^i(\mathcal{X}, DR_{\log}(\mathcal{E}))$.

(2) Let $t = \log([\epsilon]) \in B^+_{\mathrm{dR}}$ be as in (2.3.2). A *log t-connection* on a vector bundle $\mathcal{E}^+$ on $\mathcal{X}^+$ is a B^+_{dR}-linear map of sheaves $\nabla^+ : \mathcal{E}^+ \to \mathcal{E}^+ \otimes_{\mathcal{O}_{\mathcal{X}^+}} \Omega^{\log}_{\mathcal{X}^+/B^+_{\mathrm{dR}}}$ satisfying the (modified) Leibniz rule $\nabla^+(fe) = (te) \otimes df + f\nabla^+(e)$, for all $f \in \mathcal{O}_{\mathcal{X}^+}$ and $e \in \mathcal{E}^+$. We say ∇^+ is *integrable* if $(\nabla^+)^2 = 0$, in which case we have a similar log de Rham complex (as above).

(3) A log Higgs bundle on X_K is a vector bundle E on $X_{K,\mathrm{an}}$ equipped with an $\mathcal{O}_{X_K}$-linear map of sheaves $\theta : E \to E \otimes_{\mathcal{O}_{X_K}} \Omega^{\log}_{X_K}(-1)$ such that $\theta \wedge \theta = 0$. (We shall often omit the subscript "an" in the following, when there is no risk of confusion.) Then we have the *log Higgs complex* $\mathit{Higgs}_{\log}(E) = (E \otimes_{\mathcal{O}_{X_K}} \Omega^{\log,\bullet}_{\mathcal{X}/B_{\mathrm{dR}}}(-\bullet), \theta)$ (where the two $\bullet$ are equal to each other) and the *log Higgs cohomology* $H^i_{\log \mathrm{Higgs}}(X_K, E) := H^i(X_K, \mathit{Higgs}_{\log}(\mathcal{E}))$.

(4) A *log connection* on a coherent sheaf E on X is a k-linear map of sheaves $\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^{\log}_X$ satisfying the usual Leibniz rule. We say that ∇ is *integrable* if $\nabla^2 = 0$, in which case we have the *log de Rham complex* $DR_{\log}(E) = (E \otimes_{\mathcal{O}_X} \Omega^{\log,\bullet}_X, \nabla)$ and the *log de Rham cohomology* $H^i_{\log \mathrm{dR}}(X, E) := H^i(X, DR_{\log}(E))$.

Suppose that E is equipped with a decreasing filtration by coherent subsheaves $\mathrm{Fil}^\bullet E$ satisfying the (usual) Griffiths transversality condition $\nabla(\mathrm{Fil}^r E) \subset (\mathrm{Fil}^{r-1} E) \otimes_{\mathcal{O}_X} \Omega^{\log}_X$, for all r. Then the complex $DR_{\log}(E)$ admits a filtration defined by $\mathrm{Fil}^r DR_{\log}(E) := \big((\mathrm{Fil}^{r-\bullet} E) \otimes_{\mathcal{O}_X} \Omega^{\log,\bullet}_X, \nabla\big)$, with the two $\bullet$ equal to each other, and with ∇ respecting the filtration and inducing $\mathcal{O}_X$-linear morphisms on the graded pieces. The graded pieces form a complex $\mathrm{gr}\, DR_{\log}(E)$ with $\mathcal{O}_X$-linear differentials, and we also have the *log Hodge cohomology* $H^{a,b}_{\log \mathrm{Hodge}}\big(X, E\big) := H^{a+b}\big(X, \mathrm{gr}^a DR_{\log}(E)\big)$.

The log de Rham cohomology and the log Hodge cohomology are related by the (*log*) *Hodge–de Rham spectral sequence* (associated with the filtration $\mathrm{Fil}^\bullet DR_{\log}(E)$ above) $E^{a,b}_1 = H^{a,b}_{\log \mathrm{Hodge}}\big(X, E\big) \Rightarrow H^{a+b}_{\log \mathrm{dR}}\big(X, E\big)$.

The following two lemmas are clear.

Lemma 3.1.8. *The functor*

$$(\mathcal{E}^+, \nabla^+) \mapsto (\mathcal{E}, \nabla, \{\mathrm{Fil}^r\}_{r \geq 0}) := (\mathcal{E}^+ \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}, t^{-1}\nabla^+, \{t^r \mathcal{E}^+\}_{r \geq 0})$$

is an equivalence of categories from the category of integrable log t-connections on $\mathcal{X}^+$ to the category of integrable log connections $(\mathcal{E}, \nabla)$ on $\mathcal{X}$ that are equipped with filtrations $\{\mathrm{Fil}^r\}_{r \geq 0}$ by locally free $\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}$-modules satisfying the Griffiths transversality condition $\nabla(\mathrm{Fil}^r \mathcal{E}) \subset (\mathrm{Fil}^{r-1} \mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{X}^+}} \Omega^{\log}_{\mathcal{X}^+/B^+_{\mathrm{dR}}}$.

Lemma 3.1.9. *The functor $(\mathcal{E}^+, \nabla^+) \mapsto (E, \theta) := (\mathcal{E}^+/t, \nabla^+)$, where ∇^+ abusively also denotes its induced map on $\mathcal{E}^+/t$, is a functor from the category of integrable log t-connections to the category of log Higgs bundles on X_K.*

3.2. **Statements of theorems.** Let us now state the main theorems of this section. Let X be any log adic space over k as in Example 2.1.2, with its log structure induced by a normal crossings divisor D. Let $U := X - D$. Let

$$\mu' : X_{\text{prokét}/X_K} \to X_{\text{an}}. \tag{3.2.1}$$

be the natural projection of sites. For a $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{két}}$, let $\widehat{\mathbb{L}}$ be the corresponding $\widehat{\mathbb{Q}}_p$-local system on $X_{\text{prokét}}$, as in [DLLZ, Lem. 6.3.3], and consider

$$\mathcal{RH}_{\log}(\mathbb{L}) := R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\text{dR},\log}). \tag{3.2.2}$$

Theorem 3.2.3. *(1) The assignment $\mathbb{L} \mapsto \mathcal{RH}_{\log}(\mathbb{L})$ is an exact functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{két}}$ to the category of $\mathrm{Gal}(K/k)$-equivariant vector bundles on $\mathcal{X}$ equipped with an integrable log connection $\nabla : \mathcal{RH}_{\log}(\mathbb{L}) \to \mathcal{RH}_{\log}(\mathbb{L}) \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\log}_{\mathcal{X}/B_{\text{dR}}}$ and a decreasing filtration (by locally free $\mathcal{O}_X \widehat{\otimes} B^+_{\text{dR}}$-submodules) satisfying the Griffiths transversality.*

(2) For each irreducible component Z (defined as in [Con99]) of the normal crossings divisor D, let $\mathrm{Res}_Z(\nabla)$ denote the residue of the log connection ∇ along Z (see Section 3.4 below for details on the definition of residues). Then all the eigenvalues of $\mathrm{Res}_Z(\nabla)$ are in $\mathbb{Q} \cap [0, 1)$.

(3) Assume that X is proper over k, and that $K = \widehat{\overline{k}}$. Let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{két}}$. Then there is a canonical $\mathrm{Gal}(K/k)$-equivariant isomorphism

$$H^i\big(X_{K,\text{két}}, \mathbb{L}\big) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H^i_{\log \text{dR}}\big(\mathcal{X}, \mathcal{RH}_{\log}(\mathbb{L})\big),$$

for each $i \geq 0$, where $H^i_{\log \text{dR}}\big(\mathcal{X}, \mathcal{RH}_{\log}(\mathbb{L})\big)$ is as in Definition 3.1.7(1), compatible with the filtrations on both sides.

(4) Suppose that Y is another log adic space whose log structure is defined by some normal crossings divisor E as in Example 2.1.2, and that $h : Y \to X$ is a morphism of log adic spaces. Suppose that, for each irreducible component W of E, there is at most one irreducible component Z of D which contains $f(W)$ and along which the geometric monodromy of $\mathbb{L}|_{U_{\text{ét}}}$ is **not** *unipotent, in which case we also assume that W has multiplicity one in the pullback divisor $f^{-1}(Z)$. Then there is a canonical $\mathrm{Gal}(K/k)$-equivariant isomorphism $h^*\big(\mathcal{RH}_{\log}(\mathbb{L}), \nabla_{\mathbb{L}}\big) \xrightarrow{\sim} \big(\mathcal{RH}_{\log}(h^*(\mathbb{L})), \nabla_{h^*(\mathbb{L})}\big)$, compatible with the filtrations on both sides.*

As a byproduct, we obtain the log p-adic Simpson functor in our setting. We refer to [Fal05, AGT16] for more general and thorough treatments.

Theorem 3.2.4. *(1) There is a natural functor $\mathcal{H}_{\log}$ from the category of $\mathbb{Q}_p$-local systems $\mathbb{L}$ on $X_{\text{két}}$ to the category of $\mathrm{Gal}(K/k)$-equivariant log Higgs bundles $\theta : \mathcal{H}_{\log}(\mathbb{L}) \to \mathcal{H}_{\log}(\mathbb{L}) \otimes_{\mathcal{O}_{X_K}} \Omega^{\log}_{X_K}(-1)$ on $X_{K,\text{an}}$. Concretely, by Lemma 3.1.8, $\mathcal{RH}^+_{\log} := \mathrm{Fil}^0 \mathcal{RH}_{\log}$ is a functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{két}}$ to the category of $\mathrm{Gal}(K/k)$-equivariant integrable log t-connections on $\mathcal{X}^+$. Then, by Lemma 3.1.9, $\mathcal{H}_{\log} := \mathrm{gr}^0 \mathcal{RH}_{\log} = \mathcal{RH}^+_{\log}/t$ is the desired functor.*

(2) Under the same assumption as in Theorem 3.2.3(3), there is a canonical $\mathrm{Gal}(K/k)$-equivariant isomorphism

$$H^i\big(X_{K,\text{két}}, \mathbb{L}\big) \otimes_{\mathbb{Z}_p} K \cong H^i_{\log \text{Higgs}}\big(X_{K,\text{an}}, \mathcal{H}_{\log}(\mathbb{L})\big),$$

for each $i \geq 0$, where $H^i_{\log \text{Higgs}}\big(X_{K,\text{an}}, \mathcal{H}_{\log}(\mathbb{L})\big)$ is as in Definition 3.1.7(3).

(3) *Under the same assumption as in Theorem* 3.2.3(4)*, there is a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism* $h^*\big(\mathcal{H}_{\log}(\mathbb{L}), \theta_{\mathbb{L}}\big) \xrightarrow{\sim} \big(\mathcal{H}_{\log}(h^*(\mathbb{L})), \theta_{h^*(\mathbb{L})}\big)$.

We also have an arithmetic log p-adic Riemann–Hilbert functor. Consider the natural projection of sites

$$\mu : X_{\text{prokét}} \to X_{\text{an}}. \tag{3.2.5}$$

For any $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{két}}$, consider

$$D_{\mathrm{dR},\log}(\mathbb{L}) := \mu_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}). \tag{3.2.6}$$

Theorem 3.2.7. (1) *The assignment* $\mathbb{L} \mapsto D_{\mathrm{dR},\log}(\mathbb{L})$ *defines a functor from the category of* $\mathbb{Q}_p$*-local systems on* $X_{\text{két}}$ *to the category of vector bundles on* X_{an} *with integrable log connections* $\nabla_{\mathbb{L}} : D_{\mathrm{dR},\log}(\mathbb{L}) \to D_{\mathrm{dR},\log}(\mathbb{L}) \otimes_{\mathcal{O}_X} \Omega_X^{\log}$ *and decreasing filtrations* $\mathrm{Fil}^\bullet D_{\mathrm{dR},\log}(\mathbb{L})$ (*by coherent subsheaves*) *satisfying the* (*usual*) *Griffiths transversality.*

(2) *For each irreducible component* Z (*defined as in* [Con99]) *of the normal crossings divisor* D*, all eigenvalues of the residue* $\mathrm{Res}_Z(\nabla)$ *are in* $\mathbb{Q} \cap [0, 1)$*. If the restriction of* $\mathbb{L}$ *to* $U_{\text{két}} \cong U_{\text{ét}}$ *is* **de Rham** (*as reviewed in the introduction*)*, then* $\mathrm{gr}\, D_{\mathrm{dR},\log}(\mathbb{L})$ *is a vector bundle on* X *of rank* $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$.

(3) *Assume that* X *is proper over* k*, that* $K = \widehat{\overline{k}}$*, and that* $\mathbb{L}$ *is a* $\mathbb{Z}_p$*-local system on* $X_{\text{két}}$ *whose restriction to* $U_{\text{ét}}$ *is* **de Rham***. Then, for each* $i \geq 0$*, there is a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism*

$$H^i\big(X_{K,\text{két}}, \mathbb{L}\big) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i_{\log \mathrm{dR}}\big(X_{\text{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_k B_{\mathrm{dR}} \tag{3.2.8}$$

compatible with the filtrations on both sides. Moreover, the (*log*) *Hodge–de Rham spectral sequence for* $D_{\mathrm{dR},\log}(\mathbb{L})$ *degenerates on the* E_1 *page, and there is also a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism*

$$H^i\big(X_{K,\text{két}}, \mathbb{L}\big) \otimes_{\mathbb{Z}_p} K \cong \oplus_{a+b=i} \Big(H^{a,b}_{\log \mathrm{Hodge}}\big(X_{\text{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_k K(-a)\Big), \tag{3.2.9}$$

for each $i \geq 0$*, which can be identified with the* 0*-th graded piece of the isomorphism* (3.2.8)*, giving the* (**log**) **Hodge–Tate decomposition**.

(4) *Under the same assumption as in Theorem* 3.2.3(4)*, there is a canonical isomorphism* $h^*\big(D_{\mathrm{dR},\log}(\mathbb{L}), \nabla_{\mathbb{L}}\big) \xrightarrow{\sim} \big(D_{\mathrm{dR},\log}(h^*(\mathbb{L})), \nabla_{h^*(\mathbb{L})}\big)$*, compatible with the filtrations on both sides.*

(5) *Suppose that* Y *is another log adic space with its log structure defined by a normal crossings divisor* $E \hookrightarrow Y$ *as in Example* 2.1.2*. Let* $V = Y - E$*. Let* $f : X \to Y$ *be a proper log smooth morphism that restricts to a proper smooth morphism* $f|_U : U \to V$*. Let* $\mathbb{L}$ *be a* $\mathbb{Z}_p$*-local system on* $X_{\text{két}}$ *that is de Rham when restricted to* $U_{\text{két}} \cong U_{\text{ét}}$*. Then* $R^i f_{\text{két},*}(\mathbb{L})$ *is a* $\mathbb{Z}_p$*-local system on* $Y_{\text{két}}$ *that is de Rham when restricted to* $V_{\text{két}} \cong V_{\text{ét}}$*, and we have a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism*

$$\big(D_{\mathrm{dR},\log}(R^i f_{\text{két},*}(\mathbb{L})), \nabla_{R^i f_{\text{két},*}(\mathbb{L})}\big) \cong \big(R^i f_{\log \mathrm{dR},*}(D_{\mathrm{dR},\log}(\mathbb{L}), \nabla_{\mathbb{L}})\big)_{\text{free}},$$

compatible with the filtrations on both sides, where $R^i f_{\log \mathrm{dR},*}\big(D_{\mathrm{dR},\log}(\mathbb{L}), \nabla\big)$ *is the usual relative analogue of the log de Rham cohomology, and where the subscript "free" denotes the* $\mathcal{O}_Y$*-torsion-free quotient.*

By Theorem 3.2.7(3) and [DLLZ, Cor. 6.3.4], we obtain the following:

Corollary 3.2.10. *Let Y be a smooth rigid analytic variety over k, and let $K = \widehat{\overline{k}}$. Assume that Y admits a proper smooth compactification $Y \hookrightarrow \overline{Y}$ such that $\overline{Y} - Y$ is a normal crossings divisor. Let $\mathbb{L}$ be a* **de Rham** *$\mathbb{Z}_p$-local system on $Y_{\text{ét}}$, with its extension $\overline{\mathbb{L}} := \jmath_{\text{két},*}(\mathbb{L})$ to a $\mathbb{Z}_p$-local system on $\overline{Y}_{\text{két}}$. Then $H^i\big(Y_{K,\text{ét}}, \mathbb{L}\big)$ is a* **finite** *$\mathbb{Z}_p$-module, and there is a canonical $\mathrm{Gal}(K/k)$-equivariant isomorphism*

$$H^i\big(Y_{K,\text{ét}}, \mathbb{L}\big) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i_{\log \mathrm{dR}}\big(\overline{Y}_{\mathrm{an}}, D_{\mathrm{dR},\log}(\overline{\mathbb{L}}_{\mathbb{Q}_p})\big) \otimes_k B_{\mathrm{dR}}, \tag{3.2.11}$$

compatible with the filtrations on both sides. Moreover, the (log) Hodge–de Rham spectral sequence for $D_{\mathrm{dR},\log}(\overline{\mathbb{L}}_{\mathbb{Q}_p})$ degenerates on the E_1 page, and the 0-th graded piece of (3.2.11) *is also a canonical $\mathrm{Gal}(K/k)$-equivariant isomorphism*

$$H^i\big(Y_{K,\text{ét}}, \mathbb{L}\big) \otimes_{\mathbb{Z}_p} K \cong \oplus_{a+b=i} \Big(H^{a,b}_{\log \mathrm{Hodge}}\big(\overline{Y}_{\mathrm{an}}, D_{\mathrm{dR},\log}(\overline{\mathbb{L}}_{\mathbb{Q}_p})\big) \otimes_k K(-a)\Big).$$

Note that, as explained in [DLLZ, Rem. 6.2.2], the finiteness of $H^i\big(Y_{K,\text{ét}}, \mathbb{L}\big)$ as a $\mathbb{Z}_p$-module cannot hold for an arbitrary smooth rigid analytic variety Y.

As mentioned in the introduction, due to the failure of the surjectivity of (1.8), $D_{\mathrm{dR},\log}$ is not a tensor functor in general, and we have similar failures for $\mathcal{RH}_{\log}$ and $\mathcal{H}_{\log}$. Nevertheless, we have the following:

Theorem 3.2.12.
1. *The functor $\mathcal{RH}_{\log}$ (resp. $\mathcal{H}_{\log}$) restricts to a* **tensor functor** *from the category of $\mathbb{Q}_p$-local systems on $X_{\text{két}}$ whose restrictions to $U_{\text{ét}}$ have* **unipotent** *geometric monodromy along D to the category of filtered $\mathrm{Gal}(K/k)$-equivariant vector bundles on $\mathcal{X}$ equipped with integrable log connections with* **nilpotent** *residues along D (resp. the category of $\mathrm{Gal}(K/k)$-equivariant log Higgs bundles on $X_{K,\mathrm{an}}$).*
2. *The functor $D_{\mathrm{dR},\log}$ restricts to a tensor functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{két}}$ whose restrictions to $U_{\text{ét}}$ are* **de Rham** *and have unipotent geometric monodromy along D to the category of filtered vector bundles on X_{an} equipped with integrable log connections with nilpotent residues along D.*

3.3. **Coherence.** In this subsection, we prove Theorems 3.2.3(1) and 3.2.4(1), and show that $D_{\mathrm{dR},\log}(\mathbb{L})$ is a torsion-free reflexive coherent sheaf on X_{an}.

By factoring μ' as $X_{\text{prokét}/X_K} \cong X_{K,\text{prokét}} \to X_{K,\text{ét}} \to X_{K,\mathrm{an}} \to X_{\mathrm{an}}$, we see that $\mathcal{RH}_{\log}(\mathbb{L})$ admits a natural $\mathrm{Gal}(K/k)$-action. We need to show that $R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})$ is a locally free $\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}$-module of rank $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$, for every r. Assuming this, it follows that

$$\mathcal{RH}_{\log}(\mathbb{L}) = R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \cong R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})[t^{-1}]$$

is a vector bundle on $\mathcal{X}$, equipped with the filtration

$$\mathrm{Fil}^r \mathcal{RH}_{\log}(\mathbb{L}) := \mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}).$$

by locally free $\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}$-submodules. Consider the integrable log connection

$$\nabla : \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log}_X$$

formed by tensoring the one of $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ with $\widehat{\mathbb{L}}$. By the projection formula

$$R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega^{\log,j}_X) \cong R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) \otimes_{\mathcal{O}_X} \Omega^{\log,j}_X, \tag{3.3.1}$$

we obtain a log connection $\nabla_{\mathbb{L}} : \mathcal{RH}_{\log}(\mathbb{L}) \to \mathcal{RH}_{\log}(\mathbb{L}) \otimes_{\mathcal{O}_X} \Omega_X^{\log}$. The integrability of $\nabla_{\mathbb{L}}$ and the Griffiths transversality with respect to the filtration $\mathrm{Fil}^\bullet \mathcal{RH}_{\log}(\mathbb{L})$ follow from the corresponding properties of the connection (2.2.17).

In what follows, we shall denote by Z either the whole X or an open subspace of a smooth intersection of irreducible components of D, equipped with the log structure pulled back from X, which fits into the second case of Remark 2.4.1.

Lemma 3.3.2. *Let Z be as above. For any $-\infty \leq a < b \leq \infty$, there is a natural isomorphism $(\mathcal{O}_Z \widehat{\otimes} B_{\mathrm{dR}})^{[a,b]} \cong R\mu'_{Z,*}(\mathcal{O}\mathbb{B}_{\mathrm{dR},\log,Z}^{[a,b]})$.*

Proof. By Lemma 3.1.4(3), it suffices to prove the analogue for the morphism $\nu'_Z : Z_{\text{prokét}/Z_K} \to Z_{\text{ét}}$ (instead of μ'_Z). By using Corollary 2.3.17, the argument is similar to the ones in the proofs of [Sch13, Prop. 6.16(i)] and [LZ17, Lem. 3.7]. □

By the same arguments as in the proofs of [LZ17, Thm. 2.1(i) and 3.8(i)], in order to show that $R\mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})$ is a locally free $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$-module of rank $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$, for every r, it suffices to prove the following:

Proposition 3.3.3. *Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{két}}$. Let Z be as above, and let $\widehat{\mathbb{L}}_Z$ denote the pullback of $\widehat{\mathbb{L}}$ under $Z_{\text{prokét}} \to X_{\text{prokét}}$.*

(1) $R^i\mu'_{Z,*}(\widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z}) = 0$, for all $i > 0$.

(2) $\mu'_{Z,*}(\widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z})$ is a locally free $\mathrm{gr}^0(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}})$-module, whose rank is equal to $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$ when $Z = X$.

For simplicity, we may assume that $K = \widehat{k}_\infty$, so that $\mathrm{Gal}(K/k)$ is identified with an open subgroup of $\widehat{\mathbb{Z}}^\times$ via the cyclotomic character χ. (The assertions for larger perfectoid fields then follow by base change.) By Lemma 3.1.4(5), it suffices to prove similar statements for the projection of sites $\nu'_Z : Z_{\text{prokét}/Z_K} \to Z_{\text{ét}}$ (instead of $\mu'_Z : Z_{\text{prokét}/Z_K} \to Z_{\text{an}}$). Since such statements are local in nature, we may assume that $X = \mathrm{Spa}(R, R^+)$ is an affinoid log adic space over $\mathrm{Spa}(k, k^+)$, where $k^+ = \mathcal{O}_k$; and that X admits a smooth toric chart $X \to \mathbb{E} := \mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ (see [DLLZ, Cor. 3.1.11 and Def. 3.1.12]), where $P = \mathbb{Z}_{\geq 0}^n = \oplus_{j=1}^n (\mathbb{Z}_{\geq 0}\, a_j)$. We shall write $T_j = e^{a_j}$, for each j. Note that this fits into the setup in Section 2.3, with $Q = 0$ there, and we may assume that Z is defined by $T_1 = \cdots = T_l = 0$, for some $l \leq n$. Therefore, we have a log affinoid perfectoid object $\widetilde{X}$ in $X_{\text{prokét}}$ (resp. $\widetilde{Z}$ in $Z_{\text{prokét}}$) obtained by pulling back $\widetilde{\mathbb{E}} := \varprojlim_m \mathbb{E}_m \to \mathbb{E}$, where $\mathbb{E}_m := \mathrm{Spa}(k_m\langle \frac{1}{m}P\rangle, k_m^+\langle \frac{1}{m}P\rangle)$, and we shall write $T_j^{\frac{1}{m}} = e^{\frac{1}{m}a_j}$, for each j. Then $\widetilde{X} \to X_{k_\infty}$ is a Galois pro-Kummer étale cover with Galois group $\Gamma_{\mathrm{geom}} \cong (\widehat{\mathbb{Z}}(1))^n$, and $\widetilde{X} \to X$ is also a Galois pro-Kummer étale cover, whose Galois group Γ fits into a short exact sequence

$$1 \to \Gamma_{\mathrm{geom}} \to \Gamma \to \mathrm{Gal}(k_\infty/k) \to 1, \tag{3.3.4}$$

with $\mathrm{Gal}(k_\infty/k)$ acting on $\Gamma_{\mathrm{geom}} \cong (\widehat{\mathbb{Z}}(1))^n$ via the cyclotomic character $\chi : \mathrm{Gal}(k_\infty/k) \to \widehat{\mathbb{Z}}^\times$. The same is true for the pullbacks $\widetilde{Z} \to Z_{k_\infty}$ and $\widetilde{Z} \to Z$.

Let $R_K := R\widehat{\otimes}_k K$. Also, let $\overline{R} := R/(T_1, \ldots, T_l)$ and $\overline{R}_K := \overline{R}\widehat{\otimes}_k K$. By Corollary 2.3.17, we have $\mathcal{O}\mathbb{C}_{\log,Z}|_{\widetilde{Z}} \cong \widehat{\mathcal{O}}_{Z_{\text{prokét}}}|_{\widetilde{Z}}[W_1, \ldots, W_n]$, where $W_j = t^{-1}y_j = t^{-1}\log(e^{a_j})$ in the notation there, for all $j = 1, \ldots, n$. Let

$$\mathcal{L}_Z := \widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_{Z_{\text{prokét}}},$$

which is a locally free $\widehat{\mathcal{O}}_{Z_{\text{prok\'et}}}$-module of rank $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$. Then

$$(\widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z})|_{\widetilde{Z}} \cong \mathcal{L}_Z|_{\widetilde{Z}}[W_1, \ldots, W_n].$$

Note that $R^i\nu'_{Z,*}(\widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z})$ is the sheaf on $Z_{\text{\'et}}$ associated with the presheaf

$$Y \mapsto H^i(Z_{\text{prok\'et}}{}_{/Y_K}, \widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z}).$$

In order to prove Proposition 3.3.3, it suffices to prove the following two statements:

(a) $H^0(Z_{\text{prok\'et}}{}_{/Z_K}, \widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z})$ is a finite projective $\overline{R}_K$-module, of rank $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$ when $Z = X$; and $H^i(Z_{\text{prok\'et}}{}_{/Z_K}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z}) = 0$, for all $i > 0$.

(b) If $Y = \mathrm{Spa}(S, S^+) \to Z$ is a composition of rational embeddings and finite étale morphisms, then we have a canonical isomorphism of S_K-modules

$$H^0(Z_{\text{prok\'et}}{}_{/Z_K}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z}) \otimes_{\overline{R}_K} S_K \xrightarrow{\sim} H^0(Y_{\text{prok\'et}}{}_{/Y_K}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Y}).$$

Our approach to proving (a) and (b) is similar to the one in the proof of [LZ17, Thm. 2.1]. We will only explain the new ingredients here, and refer to [LZ17] for more details. For any Y as in (b), we endow it with the induced log structure. Then $\widetilde{Y} := Y \times_Z \widetilde{Z} \in Z_{\text{prok\'et}}$, where $\widetilde{Z} \to Z$ is as above, is log affinoid perfectoid; and $\widetilde{Y} \to Y_{k_\infty}$ is also a Galois pro-Kummer étale cover with Galois group Γ_{geom}.

By Corollary 2.3.17 and [DLLZ, Thm. 5.4.4], and by the same arguments as in the proofs of [LZ17, Cor. 2.4, and Lem. 2.7], we obtain the following lemma:

Lemma 3.3.5. *Let $\mathbb{M}$ be a $\mathbb{Q}_p$-local system on $X_{\text{k\'et}}$.*

(1) Let U be log affinoid perfectoid object in $Z_{\text{prok\'et}}{}_{/\widetilde{Z}_K}$. For any $-\infty \leq a \leq b \leq \infty$, and for each $i > 0$, we have $H^i(Z_{\text{prok\'et}}{}_{/U}, \widehat{\mathbb{M}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}^{[a,b]}_{\mathrm{dR},\log,Z}) = 0$.

(2) $H^i\big(\Gamma_{\text{geom}}, (\widehat{\mathbb{M}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z})(\widetilde{Y})\big) \cong H^i\big(Z_{\text{prok\'et}}{}_{/Y_K}, \widehat{\mathbb{M}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z}\big)$, for all $i \geq 0$.

Let $\Gamma_0 := \Gamma_{\text{geom}} \cong (\widehat{\mathbb{Z}}(1))^n$. Then (2.3.1) induces $(\widehat{\mathbb{Z}}(1))^n \xrightarrow{\sim} \widehat{\mathbb{Z}}^n$. Consider the topological basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\Gamma_{\text{geom}} \cong (\widehat{\mathbb{Z}}(1))^n$ given by pulling back the image in $\widehat{\mathbb{Z}}^n$ of the standard basis $\{a_1, \ldots, a_n\}$ of $\mathbb{Z}^n$, characterized by the property that

$$\gamma_j \, T_{j'}^{\frac{1}{m}} = \zeta_m^{\delta_{jj'}} T_{j'}^{\frac{1}{m}}, \tag{3.3.6}$$

for all $1 \leq j, j' \leq n$ and $m \geq 1$ (cf. [DLLZ, (6.1.4)] and (2.3.3)).

For each $m \geq 1$, we write $X_{K,m} = \mathrm{Spa}(R_{K,m}, R^+_{K,m}) := X_K \times_{\mathbb{E}_K} (\mathbb{E}_m)_K$, and write $Z_{K,m} := Z_K \times_{\mathbb{E}_K} (\mathbb{E}_m)_K$ and $(Z_{K,m})_{\text{red}} = \mathrm{Spa}(\overline{R}_{K,m}, \overline{R}^+_{K,m})$, where the subscript "red" denotes the reduced subspace. Note that $\overline{R}_{K,m} \cong R_{K,m}/(T_1^{\frac{1}{m}}, \ldots, T_l^{\frac{1}{m}})$. Let $\widehat{R}_{K,\infty}$ and $\widehat{\overline{R}}_{K,\infty}$ be the p-adic completions of $\varinjlim_m R_{K,m}$ and $\varinjlim_m \overline{R}_{K,m}$, respectively (cf. [DLLZ, Rem. 5.3.2]). By Theorem A.2.2.2, $(\{R_{K,m}\}_{m\geq 1}, \Gamma_0)$ is a decompletion system, so that $L_\infty = \mathcal{L}_X(\widetilde{X})$ has a *model* over R_{K,m_0}, for some $m_0 \geq 1$, as in Definition A.1.2(1), i.e., a finite projective R_{K,m_0}-module $L_{m_0}(X_K)$ of rank $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$ with an R_{K,m_0}-semilinear continuous action of Γ_{geom} such that $L_{m_0}(X_K) \otimes_{R_{K,m_0}} \widehat{R}_{K,\infty} \cong \mathcal{L}_X(\widetilde{X})$, and we may assume that it is *good* as in Definition A.1.2(2), i.e., $H^i\big(\Gamma_{\text{geom}}, L_{m_0}(X_K)\big) \xrightarrow{\sim} H^i\big(\Gamma_{\text{geom}}, \mathcal{L}_X(\widetilde{X})\big)$, for all $i \geq 0$. Note that $\Gamma_0 \cong (\widehat{\mathbb{Z}}(1))^n$ acts on $\overline{R}_{K,m}$ via the last $n-l$ factors $\overline{\Gamma}_0 \cong (\widehat{\mathbb{Z}}(1))^{n-l}$

(see (3.3.6)). By Theorem A.2.2.2 again, $(\{\overline{R}_{K,m}\}_{m\geq 1}, \overline{\Gamma}_0)$ is also a decompletion system. Since $\mathcal{L}_Z(\widetilde{Z}) \cong \mathcal{L}_X(\widetilde{X}) \otimes_{\widehat{R}_{K,\infty}} \widehat{\overline{R}}_{K,\infty}$ by [DLLZ, Lem. 6.3.6],

$$L_{m_0}(Z_K) := L_{m_0}(X_K) \otimes_{R_{K,m}} \overline{R}_{K,m} \cong L_{m_0}(X_K)/(T_1^{\frac{1}{m_0}}, \ldots, T_l^{\frac{1}{m_0}}) \tag{3.3.7}$$

is a model of $\mathcal{L}_Z(\widetilde{Z})$, and it inherited an action of the whole Γ_0 from $L_{m_0}(X_K)$. Up to replacing $L_{m_0}(X_K)$ with $L_{m_0}(X_K) \otimes_{R_{K,m_0}} R_{K,m}$ for some $m \geq m_0$, and replacing m_0 with m accordingly, we may assume that the resulted model $L_{m_0}(Z_K)$ is good, as in Definition A.1.2(2), for the action of the whole Γ_0.

Lemma 3.3.8. *The R_K-linear representation of Γ_0 on $L_{m_0}(X_K)$ is quasi-unipotent; i.e., there is a finite index subgroup Γ'_{geom} of Γ_{geom} acting unipotently on $L_{m_0}(X_K)$. By base change, the same is true for the $\overline{R}_K$-linear representation of Γ_0 on $L_{m_0}(Z_K)$.*

Proof. Let $A_m := R\widehat{\otimes}_k k_m\langle T_1^{\frac{1}{m_0}}, \ldots, T_n^{\frac{1}{m_0}}\rangle$ and $\Gamma_m := \mathrm{Gal}(K/k_m)$, for each $m \geq m_0$. By Theorem A.2.1.2, for sufficiently large l_0, the pair $(\{A_{p^{l_0}m}\}_{m\geq m_0}, \Gamma_{p^{l_0}m_0})$ is a decompletion system, with R_{K,m_0} equal to the completion of $\varinjlim_m A_{p^{l_0}m}$. By Definition A.1.2 (with $L_\infty = L_{m_0}(X_K)$), up to increasing m_0, there exists an A_{m_0}-submodule $L_{k_{m_0}}$ equipped with a continuous Γ-action and a canonical isomorphism

$$L_{k_{m_0}} \otimes_{A_{m_0}} R_{K,m_0} \xrightarrow{\sim} L_{m_0}(X_K). \tag{3.3.9}$$

Then the same argument as in the proof of [LZ17, Lem. 2.15] works here. □

By Lemma 3.3.8, we obtain decompositions

$$L_{m_0}(X_K) = \oplus_\tau L_{m_0,\tau}(X_K) \quad \text{and} \quad L_{m_0}(Z_K) = \oplus_\tau L_{m_0,\tau}(Z_K), \tag{3.3.10}$$

where τ are characters of Γ_{geom} of finite order and the subscript "τ" denotes the maximal K-subspaces on which $\gamma - \tau(\gamma)$ acts nilpotently, for all $\gamma \in \Gamma$. Then each $L_{m_0,\tau}(X_K)$ (resp. $L_{m_0,\tau}(Z_K)$) is a finite projective R_K-module (resp. $\overline{R}_K$-module) stable under the action of Γ_{geom}. Consider, in particular, the unipotent parts

$$L(X_K) := L_{m_0,1}(X_K) \quad \text{and} \quad L(Z_K) := L_{m_0,1}(Z_K).$$

Up to increasing m_0 as before, we may assume that the order of every τ in (3.3.10) divides m_0. For each such τ, there exists some monomial T^{a_τ} in R_{K,m_0}, with a_τ in $\frac{1}{m_0}\mathbb{Z}^n_{\geq 0}$, on which Γ_0 acts via τ. Since all monomials T^a with $a \in \frac{1}{m_0}\mathbb{Z}^n$ are in $R_{K,m_0}[T_1^{-1}, \ldots, T_n^{-1}]$, it follows that

$$L(X_K) \otimes_{R_K} R_{K,m_0}[T_1^{-1}, \ldots, T_n^{-1}] \cong L_{m_0}(X_K)[T_1^{-1}, \ldots, T_n^{-1}], \tag{3.3.11}$$

and that the rank of $L(X_K)$ as a finite projective R_K-module is $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$.

Remark 3.3.12. However, the natural map

$$L(X_K) \otimes_{R_K} R_{K,m_0} \to L_{m_0}(X_K) \tag{3.3.13}$$

might not be an isomorphism in general. This is the source of the failure of the surjectivity of (1.8) mentioned in the introduction.

Remark 3.3.14. In general, the two decompositions in (3.3.10) are not compatible via base change from R_K to $\overline{R}_K \cong R_K/(T_1, \ldots, T_l)$. Nevertheless, since the induced morphisms $L_{m_0,\tau}(X_K)/(T_1, \ldots, T_l) \to L_{m_0,\tau'}(Z_K)$ are zero whenever $\tau \neq \tau'$, we have a canonical surjection $L(X_K)/(T_1, \ldots, T_l) \to L(Z_K)$.

For each $\tau \neq 1$, there exists some j such that the action of $\gamma_j - 1$ on $L_{m_0,\tau}(Z_K)$ is invertible, and so $H^i\bigl(\Gamma_0, L_{m_0,\tau}(Z_K)\bigr) = 0$, for all $i \geq 0$. Hence, we have a canonical isomorphism $H^i\bigl(\Gamma_0, L(Z_K)\bigr) \xrightarrow{\sim} H^i\bigl(\Gamma_0, L_{m_0}(Z_K)\bigr)$, and the following lemma follows from essentially the same argument as in the proof of [LZ17, Lem. 2.9]:

Lemma 3.3.15. *There is a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism*

$$H^i(Z_{\text{prokét}/Z_K}, \widehat{\mathbb{L}}_Z \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log,Z}) \cong \begin{cases} L(Z_K), & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

By Definition A.1.2, up to increasing m_0, the formation of $L_{m_0}(Z_K)$ is compatible with base changes under compositions of rational embeddings and finite étale morphisms $Y \to Z$. The same is true for the formation of the direct summands $L_{m_0,\tau}(Z_K)$ in the decomposition (3.3.10). These yield the following:

Lemma 3.3.16. *The formation of the finite projective* $\overline{R}_K$*-module* $L(Z_K)$*, which is of rank* $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$ *when* $Z = X$*, is compatible with base changes under compositions of rational embeddings and finite étale morphisms* $Y \to Z$*.*

Thus, we have established the statements (a) and (b) above, and completed the proofs of Proposition 3.3.3 and hence also of Theorems 3.2.3(1) and 3.2.4(1). (The cases where $Z \neq X$ will be also useful in Section 3.7 and in [LLZ].)

Next, we move to the arithmetic situation. We will only consider $Z = X$.

Lemma 3.3.17. *The sheaf* $D_{\mathrm{dR},\log}(\mathbb{L})$ *is a coherent sheaf on* X_{an}*.*

Proof. For simplicity, we may still assume that $K = \widehat{k}_\infty$. Again, to show the coherence of $D_{\mathrm{dR},\log}(\mathbb{L})$, we shall consider the projection $\nu : X_{\text{prokét}} \to X_{\text{ét}}$ instead, and we may assume that $X = \mathrm{Spa}(R, R^+)$ admits a smooth toric chart. Note that this modified $D_{\mathrm{dR},\log}(\mathbb{L})$ is the sheaf on $X_{\text{ét}}$ associated with the presheaf

$$Y \mapsto H^0\bigl(X_{\text{prokét}/Y}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}\bigr) = H^0\bigl(\mathrm{Gal}(K/k), \mathcal{RH}_{\log}(\mathbb{L})(Y)\bigr).$$

From the proof of Theorem 3.2.3(1), we know that

$$\mathrm{gr}^r \mathcal{RH}_{\log}(\mathbb{L}) \cong \mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log})(r).$$

It suffices to prove the following two statements (parallel to (a) and (b) above):

(a′) The R-module $H^0\bigl(\mathrm{Gal}(K/k), \mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log})(r)(X)\bigr)$ is finitely generated.

(b′) If $Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$ is a composition of rational localizations and finite étale morphisms, then we have a canonical isomorphism

$$H^0\bigl(\mathrm{Gal}(K/k), \mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log})(r)(X)\bigr) \otimes_R S \xrightarrow{\sim} H^0\bigl(\mathrm{Gal}(K/k), \mu'_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log})(r)(Y)\bigr).$$

As for the statement (a′), by Theorem A.2.1.2 (as in the proof of Lemma 3.3.8), there exists some $m_0 \geq 1$ such that, for $R_{k_{m_0}} := R \otimes_k k_{m_0}$, the finite projective R_K-module $L(X_K)$ descends to some finite projective $R_{k_{m_0}}$-module $L := L_{k_{m_0}}(X)$ that is a *good model* (see Definition A.1.2(2)) in the sense that $H^i\bigl(\mathrm{Gal}(K/k), L\bigr) \to H^i\bigl(\mathrm{Gal}(K/k), L(X_K)\bigr)$ is an isomorphism, for all $i \geq 0$. Hence, by Lemma 3.3.15,

$$H^0\bigl(\mathrm{Gal}(K/k), \nu_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}_{\log})(r)(X)\bigr) \cong H^0\bigl(\mathrm{Gal}(K/k), L\bigr),$$

which is clearly a finitely generated R-module, and vanishes when $|r| \gg 0$.

As for the statement (b′), up to increasing m_0, we may assume in addition that $L_{k_{m_0}}(X) \otimes_{R_{k_{m_0}}} S_{k_{m_0}}$ is a good model of $L(Y_K)$. Hence, it suffices to show that

$$H^0\big(\mathrm{Gal}(K/k), L\big) \otimes_R S \xrightarrow{\sim} H^0\big(\mathrm{Gal}(K/k), L \otimes_R S\big).$$

Since the kernel of the projection from $\mathrm{Gal}(K/k)$ to its maximal pro-p-quotient (which is isomorphic to $\mathbb{Z}_p$) acts on the finite projective R-module L via a finite quotient, $\mathrm{Gal}(K/k)$ acts via a quotient that is topologically generated by finitely many $\delta_1, \ldots, \delta_s$. Thus, the desired base change property follows from the exactness of $0 \to H^0(\mathrm{Gal}(K/k), L) \to L \xrightarrow{(\delta_1 - 1, \ldots, \delta_s - 1)} L^s$ and the flatness of $R \to S$. □

Lemma 3.3.18. *The coherent sheaf $D_{\mathrm{dR},\log}(\mathbb{L})$ on X_{an} is reflexive.*

Proof. Being torsion-free, $D_{\mathrm{dR},\log}(\mathbb{L})$ is locally free outside some locus X_0 of codimension at least two in X. Let $\jmath : X - X_0 \to X$ denote the canonical open immersion. We claim that $\mathcal{RH}_{\log}(\mathbb{L}) \cong \jmath_* \jmath^*\big(\mathcal{RH}_{\log}(\mathbb{L})\big)$. Since $\mathcal{RH}_{\log}(\mathbb{L})$ is locally free, we may work locally and assume that it is isomorphic to $(\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}})^n$ for some $n \geq 0$. By using the filtration on $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}$ in Definition 3.1.1, it suffices to treat the case of $\mathcal{O}_{X_K}$, which follows from [Kis99, Cor. 2.2.4]. By taking $\mathrm{Gal}(K/k)$-invariants, we obtain a similar canonical isomorphism $D_{\mathrm{dR},\log}(\mathbb{L}) \cong \jmath_* \jmath^*\big(D_{\mathrm{dR},\log}(\mathbb{L})\big)$. Since $D_{\mathrm{dR},\log}(\mathbb{L})$ is coherent and $\jmath^* D_{\mathrm{dR},\log}(\mathbb{L})$ is locally free, it follows that $D_{\mathrm{dR},\log}(\mathbb{L})$ is reflexive, by the same argument as in the proof of [Ser66, Prop. 7]. □

3.4. **Calculation of residues.** The main goal of this subsection is to prove Theorems 3.2.3(2), 3.2.7(1)–(2), and 3.2.12.

Let us first review the definition of residues for log connections and some basic properties. We shall only consider the case when X is as Example 2.1.2, although the definition can be made more generally. We first suppose that F is vector bundle on X_{an} equipped with an integrable log connection $\nabla : F \to F \otimes_{\mathcal{O}_X} \Omega_X^{\log}$.

Let $Z \subset D$ be an *irreducible component* (i.e., the image of a connected component of the normalization of D, as in [Con99]). To define the residue $\mathrm{Res}_Z(\nabla)$ of ∇ along Z, we may shrink X and assume that Z is smooth and connected. Locally on X, we may assume that there is a smooth toric chart $X \to \mathbb{D}^n$ such that $Z = \{T_1 = 0\}$. Let $\imath : Z \hookrightarrow X$ denote the closed immersion, and let $F|_Z$ denote the $\mathcal{O}$-module pullback $\imath^*(F)$. Then there is an $\mathcal{O}_Z$-linear endomorphism

$$\mathrm{Res}_Z(\nabla) := \nabla(T_1 \tfrac{\partial}{\partial T_1}) \bmod T_1 : F|_Z \to F|_Z, \tag{3.4.1}$$

where $T_1 \frac{\partial}{\partial T_1}$ denotes the dual of $\frac{dT_1}{T_1}$. As in the classical situation, this operator does not depend on the choice of the coordinate T_1. Also, its formation is compatible with rational localizations, and hence is a well-defined endomorphism of $F|_Z$.

Consider Z as a smooth rigid analytic variety by itself, which is equipped with the normal crossings divisor $D' = \cup_j (D_j \cap Z)$, where the D_j's are irreducible components of D other than Z. Then Z admits the structure of a log adic space, defined by D', as in Example 2.1.2. Again as in the classical situation, the pullback $F|_Z$ is equipped with a log connection $\nabla' : F|_Z \to F|_Z \otimes_{\mathcal{O}_Z} \Omega_Z^{\log}$, and the residue $\mathrm{Res}_Z(\nabla)$ is horizontal with respect to ∇'. It follows that the characteristic polynomial $P_Z(x)$ of $\mathrm{Res}_Z(\nabla)$ has coefficients in the algebraic closure k_Z of k in $\Gamma(Z, \mathcal{O}_Z)$, so that the *eigenvalues* of $\mathrm{Res}_Z(\nabla)$ (i.e., the *roots* of $P_Z(x)$) are in $\overline{k}$. For each root α of $P_Z(x)$ defined over a finite extension k' of k, let

$$F|_{Z_{k'}}^{\alpha} \subset (F|_Z) \otimes_k k' \tag{3.4.2}$$

be the corresponding generalized eigenspace of $\mathrm{Res}_Z(\nabla)$. This is a direct summand (and hence a quotient) of $(F|_Z) \otimes_k k'$, which is preserved by the log connection ∇'.

Given any integrable log connection $(\mathcal{F}, \nabla)$ on $\mathcal{X} = (X_{\mathrm{an}}, \mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}})$, by using Lemma 3.1.4, the above discussions carry through. Specifically, the coefficients of the characteristic polynomial $P_Z(x)$ are in the integral closure $\overline{B}_{\mathrm{dR}}^+$ of B_{dR}^+ in an algebraic closure $\overline{B}_{\mathrm{dR}}$ of B_{dR}, and we can also define $(\mathcal{F}|_Z^\alpha, \nabla')$ as above.

Now suppose that F is a torsion-free coherent $\mathcal{O}_X$-module equipped with an integrable log connection ∇. Let U be the maximal open subset of X such that $F|_U$ is a vector bundle, which is the complement of an analytic closed subvariety X_0 of X of codimension at least two. In particular, X_0 cannot contain any irreducible component of D. Hence, by replacing X with U, we can proceed as above and attach a polynomial $P_Z(x) \in k_Z[x]$ to each irreducible component $Z \subset D$.

Now let us begin the proof of Theorem 3.2.3(2). The question is local and we may assume that $X = \mathrm{Spa}(R, R^+)$ is affinoid and admits a smooth toric chart $X \to \mathbb{D}^n$, and that $\mathcal{RH}_{\log}(\mathbb{L})$ is free. By Proposition 2.3.15 and Corollary 2.3.17,

$$\mathcal{RH}_{\log}(\mathbb{L})(X) \cong H^0\big(\Gamma_{\mathrm{geom}}, (\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathbb{B}_{\mathrm{dR}})(\widetilde{X})\{W_1, \dots, W_n\}\big),$$

where each $W_j = t^{-1} y_j$ is defined as in (2.3.18). Let $N_\infty := (\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathbb{B}_{\mathrm{dR}}^+)(\widetilde{X})$, which is a module over $\mathbb{B}_{\mathrm{dR}}^+(\widetilde{X}) \cong \mathbb{B}_{\mathrm{dR}}^+(\widehat{R}_\infty, \widehat{R}_\infty^+)$ (see Proposition 2.2.4). Then we have

$$\begin{aligned}
&\mathrm{Fil}^0(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})(\widetilde{X}) \cong N_\infty\{W_1, \dots, W_n\} \\
&= \Big\{ \sum_{\Lambda \in \mathbb{Z}_{\geq 0}^n} b_\Lambda W^\Lambda : b_\Lambda \in N_\infty,\, b_\Lambda \to 0,\, t\text{-adically, as } \Lambda \to \infty \Big\} \\
&= \Big\{ \sum_{\Lambda \in \mathbb{Z}_{\geq 0}^n} c_\Lambda \tbinom{W}{\Lambda} : c_\Lambda \in N_\infty,\, c_\Lambda \to 0,\, t\text{-adically, as } \Lambda \to \infty \Big\},
\end{aligned}$$

where W^Λ is as in (2.3.19), and $\binom{W}{\Lambda} := \binom{W_1}{\Lambda_1} \cdots \binom{W_n}{\Lambda_n}$, for each $\Lambda = (\Lambda_1, \dots, \Lambda_n)$.

Recall that we have chosen the topological basis $\{\gamma_1, \dots, \gamma_n\}$ of $\Gamma_{\mathrm{geom}} \cong (\widehat{\mathbb{Z}}(1))^n$ satisfying the characterizing property (3.3.6). For each $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \mathbb{Z}_{\geq 0}^n$, let us write $(\gamma - 1)^\Lambda$ for $(\gamma_1 - 1)^{\Lambda_1} \cdots (\gamma_n - 1)^{\Lambda_n}$.

Lemma 3.4.3. *(1) If $\sum c_\Lambda \binom{W}{\Lambda} \in N_\infty\{W_1, \dots, W_n\}$ is Γ_{geom}-invariant, then $(\gamma - 1)^\Lambda c_0 \to 0$, t-adically, as $\Lambda \to \infty$, and $c_\Lambda = (\gamma - 1)^\Lambda c_0$ for all $\Lambda \in \mathbb{Z}_{\geq 0}^n$.*

(2) Let

$$N^+ := \big\{ c \in N_\infty : (\gamma - 1)^\Lambda c \to 0,\, t\text{-adically, as } \Lambda \to \infty \big\}. \tag{3.4.4}$$

Then the map $N_\infty\{W_1, \dots, W_n\} \to N_\infty$ sending all $W_1, \dots, W_n$ to zero induces a canonical isomorphism

$$\eta : \mathcal{RH}_{\log}^+(\mathbb{L})(X) \cong (N_\infty\{W_1, \dots, W_n\})^{\Gamma_{\mathrm{geom}}} \cong N^+, \tag{3.4.5}$$

with the inverse map given by $c \mapsto \sum_{\Lambda \in \mathbb{Z}_{\geq 0}^n} (\gamma - 1)^\Lambda c \binom{W}{\Lambda}$.

(3) Let $N := N^+ \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \cong N^+[t^{-1}]$. Then the above isomorphism η induces a canonical isomorphism $\mathcal{RH}_{\log}(\mathbb{L})(X) \cong N$, which we still denote by η.

Proof. We have $\gamma_i^{-1} W_j = W_j + \delta_{ij}$. (Note that the W_j defined in Corollary 2.3.17 differs from the V_j defined in the proof of [Sch13, Prop. 6.16] by a sign, and therefore we need γ_i^{-1} in our formula rather than the γ_i as in [Sch13, Lem. 6.17].) This implies that $\gamma_i^{-1}\binom{W_i}{j} = \binom{W_i+1}{j} = \binom{W_i}{j} + \binom{W_i}{j-1}$, and so $(\gamma_i^{-1} - 1)\big(\sum_\Lambda c_\Lambda \binom{W}{\Lambda}\big) = \sum_\Lambda \big(\gamma_i^{-1} c_{\Lambda+e_i} + \gamma_i^{-1} c_\Lambda - c_\Lambda\big)\binom{W}{\Lambda}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ has only the i-th entry equal to 1. Therefore, $c_\Lambda - \gamma_i^{-1} c_\Lambda = \gamma_i^{-1} c_{\Lambda+e_i}$, or, equivalently, $\gamma_i c_\Lambda - c_\Lambda = c_{\Lambda+e_i}$, for all i and Λ. In particular, this implies that $(\gamma - 1)^\Lambda c_0 = c_\Lambda$, which goes to 0 as $\Lambda \to \infty$. This proves (1). Then (2) and (3) also follow easily. □

By the proof of Theorem 3.2.4(1) in Section 3.3, $\mathcal{RH}^+_{\log}(\mathbb{L})(X)$ is a finite projective $R\widehat{\otimes}_k B^+_{\mathrm{dR}}$-module. Also, by Lemma 2.3.11, we have a natural Γ_{geom}-equivariant embedding $R\widehat{\otimes}_k B^+_{\mathrm{dR}} \to \mathbb{B}^+_{\mathrm{dR}}(\widehat{R}_\infty, \widehat{R}^+_\infty)$ sending T_i to $[T_i^\flat]$. Via this embedding, we may regard N_∞ as an $R\widehat{\otimes}_k B^+_{\mathrm{dR}}$-module, and N^+ as an $R\widehat{\otimes}_k B^+_{\mathrm{dR}}$-submodule of N_∞.

Lemma 3.4.6. *The isomorphism* (3.4.5) *is an isomorphism of* $R\widehat{\otimes}_k B^+_{\mathrm{dR}}$*-modules.*

Proof. This follows from the fact that the map (2.3.10) (which is an isomorphism by Lemma 2.3.11) is obtained from the map (2.3.8) via $e^a \mapsto 1$, for all $a \in P$. □

Note that the natural action of Γ_{geom} on N_∞ preserves N^+, and by transport of structure gives an action of Γ_{geom} on $\mathcal{RH}_{\log}(\mathbb{L})(X)$. This action is closely related to the residues, as we shall see. Recall from Lemma 2.3.11 that, if we define the Γ_{geom} action of $R\widehat{\otimes}_k B_{\mathrm{dR}}$ by requiring that $\gamma_i(T_j) = [\epsilon]^{\delta_{ij}} T_j$ and that the action becomes trivial after reduction mod ξ, then the embedding $R\widehat{\otimes}_k B_{\mathrm{dR}} \to \mathbb{B}_{\mathrm{dR}}(\widehat{R}_\infty, \widehat{R}^+_\infty)$ is Γ_{geom}-equivariant. Hence, the action of γ_i on $N/T_i N$ is $(A/T_i)\widehat{\otimes}_k B_{\mathrm{dR}}$-linear.

Lemma 3.4.7. *Under the isomorphism* $\mathcal{RH}_{\log}(\mathbb{L})(X) \cong N$ *given by Lemma* 3.4.3*, the residue of the connection* $\nabla_{\mathbb{L}}$ *of* $\mathcal{RH}_{\log}(\mathbb{L})$ *along* $Z_i = \{T_i = 0\}$ *corresponds to the endomorphism* $t^{-1}\log(\gamma_i)$ *of* $N/T_i N$*.*

Proof. Let us expand elements of $N_\infty\{W_1, \ldots, W_n\}$ in the basis $\{W^\Lambda\}_\Lambda$ instead of $\{\binom{W}{\Lambda}\}_\Lambda$. Suppose that $c_0 \in N$ and $\eta^{-1}(c_0) = \sum_\Lambda c_\Lambda \binom{W}{\Lambda} = \sum_\Lambda b_\Lambda W^\Lambda$. Then, by the definition of residues as in (3.4.1), by Lemma 3.4.3, and by (2.4.3) and (2.4.4), we obtain the identities $\eta\big((\mathrm{Res}_{Z_i}(\nabla_{\mathbb{L}}))(\eta^{-1}(c_0))\big) = t^{-1} b_{e_i} = t^{-1} \sum_{a=1}^\infty (-1)^{a-1} \frac{1}{a} c_{a e_i} = t^{-1} \sum_{a=1}^\infty (-1)^{a-1} \frac{1}{a} (\gamma_i - 1)^a c_0 = t^{-1}\log(\gamma_i)(c_0)$, as desired. □

Remark 3.4.8. The definitions of both t and γ_i (in (2.3.2) and (3.3.6)) depend on the choice of $\zeta : \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_\infty$ in (2.3.1), but $t^{-1}\log(\gamma_i)$ does not.

To proceed further, we need the following lemma, which follows from [DLLZ, Lem. 6.3.6] by induction on r.

Lemma 3.4.9. *Let* $\imath : Z \to Y$ *be a strict closed immersion of locally noetherian fs log adic spaces over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$*. Let* $\widehat{\mathbb{L}}$ *be a* $\widehat{\mathbb{Q}}_p$*-local system on* $Y_{\text{prokét}}$*. Then*

$$\big(\imath^{-1}_{\text{prokét}}(\widehat{\mathbb{L}}) \otimes_{\widehat{\mathbb{Q}}_p} (\mathbb{B}_{\mathrm{dR},Z}/\xi^r)\big)(U \times_Y Z)$$
$$\cong \big(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} (\mathbb{B}_{\mathrm{dR},Y}/\xi^r)\big)(U) \otimes_{(\mathbb{B}_{\mathrm{dR},X}/\xi^r)(U)} (\mathbb{B}_{\mathrm{dR},Z}/\xi^r)(U \times_Y Z).$$

for every $r \geq 1$ *and every log affinoid perfectoid object* U *of* $Y_{\text{prokét}}$*.*

Now let $\imath : Z_i \hookrightarrow X$ be the (possibly empty) smooth divisor defined by $T_i = 0$. Equip Z_i with the pullback of the log structure under $\imath$, and denote the resulted log adic space by Z_i^∂. Consider the log affinoid perfectoid object $\widetilde{Z}_i^\partial := Z_i^\partial \times_X \widetilde{X} \cong Z_i \times_{\mathbb{D}^n} \widetilde{\mathbb{D}}^n$ in $(Z_i^\partial)_{\text{prokét}}$ (as in Corollary 2.3.20), with associated perfectoid space $\widehat{\widetilde{Z}}{}_i^\partial$. By (2.3.21) and Lemma 3.4.9, we have a canonical isomorphism

$$\big(N_\infty/\xi^r\big)/([T_i^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}} \cong \big(\imath^{-1}(\widehat{\mathbb{L}}) \otimes_{\widehat{\mathbb{Q}}_p} \mathbb{B}^+_{\mathrm{dR},Z_i^\partial}\big)(\widetilde{Z}_i^\partial)/\xi^r. \tag{3.4.10}$$

Lemma 3.4.11. *If $\gamma_i v = xv$ for some nonzero $v \in \big(N_\infty/\xi^r\big)/([T_i^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}}$ and some $x \in \overline{B}^+_{\mathrm{dR}}/\xi^r$, then $x = \zeta^y$ for some $y \in \mathbb{Q}$.*

Proof. Up to replacing k with a finite extension, we may assume that Z_i contains a k-point z. Let z^∂ denote z equipped with the log structure pulled back from X. Let $\widetilde{z}^\partial := z^\partial \times_X \widetilde{X}$, with associated perfectoid space $\widehat{\widetilde{z}}^\partial$. Then $\gamma_i v = xv$ still holds in the base change of $\big(N_\infty/\xi^r\big)/([T_i^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}}$ along $(\mathbb{B}^+_{\mathrm{dR},Z_i^\partial}/\xi^r)(\widetilde{Z}_i^\partial) \to (\mathbb{B}^+_{\mathrm{dR},z^\partial}/\xi^r)(\widetilde{z}^\partial)$, which is isomorphic to $\big(\widehat{\mathbb{L}}|_z \otimes_{\widehat{\mathbb{Q}}_p} (\mathbb{B}^+_{\mathrm{dR},z^\partial}/\xi^r)\big)(\widetilde{z}^\partial)$, by Lemma 3.4.9. Note that γ_i acts trivially on $\widehat{\widetilde{z}}^\partial$. Hence, $\widehat{\mathbb{L}}|_{\widehat{\widetilde{z}}^\partial}$ is equipped with an action of γ_i, which is quasi-unipotent because it extends to a continuous one of $\widehat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}(k_\infty/k)$ (see (3.3.4)), and the same argument as in the proof of [LZ17, Lem. 2.15] also works here. $\square$

Next we use the decompletion of $\mathbb{B}^+_{\mathrm{dR}}/\xi^r$ established in Section A.2.3 to descend $N_\infty/\xi^r = (\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathbb{B}^+_{\mathrm{dR}})(\widetilde{X})/\xi^r$ to some finite level.

Lemma 3.4.12. *For each $r \geq 1$, there exist some sufficiently large $m \geq 1$ and a finite projective $\mathbb{B}_{r,m}$-module $N_{r,m}$, equipped with a semilinear Γ-action, such that*

$$N_\infty/\xi^r \cong N_{r,m} \otimes_{\mathbb{B}_{r,m}} \big(\mathbb{B}^+_{\mathrm{dR}}(\widetilde{X})/\xi^r\big)$$

as $\mathbb{B}^+_{\mathrm{dR}}(\widetilde{X})/\xi^r$-modules with semilinear Γ-actions. In addition, up to replacing $N_{r,m}$ with $N_{r,m} \otimes_{\mathbb{B}_{r,m}} \mathbb{B}_{r,m'}$ for some sufficiently large multiple m' of m and replacing m with m', we may assume that $N^+/\xi^r \subset N_{r,m}$ (as submodules of N_∞/ξ^r).

Proof. The first statement follows from Lemma 2.3.11 and Theorem A.2.3.4. As for the second statement, we may assume that $H^i(\Gamma, N_{r,m}) \to H^i(\Gamma, N_\infty/\xi^r)$ is an isomorphism for $i = 0, 1$ (by Definition A.1.2(2)), so that $H^0(\Gamma, (N_\infty/\xi^r)/N_{r,m}) = 0$. Then the whole Γ acts unipotently on each element of N^+/ξ^r (by (3.4.4)), while each nonzero element of $(N_\infty/\xi^r)/N_{r,m}$ lies outside the kernel of $\gamma - 1$ for some $\gamma \in \Gamma$. It follows that $N^+/\xi^r \subset N_{r,m} \subset N_\infty/\xi^r$, as desired. $\square$

Lemma 3.4.13. *If $\gamma_i v = xv$ for some nonzero $v \in (N^+/\xi^r)/(T_i)$ and some $x \in \overline{B}^+_{\mathrm{dR}}/\xi^r$, then $x = \zeta^y[\epsilon^z]$ for some $y \in \mathbb{Q}$ and $z \in \mathbb{Q} \cap [0, 1)$.*

Proof. By Lemma 3.4.12, we may assume that $v \in (N^+/\xi^r)/(T_i) \subset N_{r,m}/T_i N_{r,m}$ for some m and $N_{r,m}$. Consider the filtration $T_i^{\frac{a}{m}} N_{r,m}/T_i N_{r,m} \subset N_{r,m}/T_i N_{r,m}$, with $0 \leq a \leq m$. Since $v \neq 0$, there exists some $0 \leq a < m$ such that the image $\overline{v}$ of v in $T_i^{\frac{a}{m}} N_{r,m}/T_i^{\frac{a+1}{m}} N_{r,m}$ is nonzero, which also satisfies $\gamma_i \overline{v} = x\overline{v}$. By Lemma 3.4.12 again, the natural embedding $N_{r,m} \hookrightarrow N_\infty/\xi^r$ induces by restriction to $T_i^{\frac{a}{m}} N_{r,m}$ and by factoring out a multiplication by $T_i^{\frac{a}{m}}$ a well-defined embedding $T_i^{\frac{a}{m}} N_{r,m}/T_i^{\frac{a+1}{m}} N_{r,m} \hookrightarrow (N_\infty/\xi^r)/([T_i^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}}$, and the nonzero image w of $\overline{v}$ in $(N_\infty/\xi^r)/([T_i^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}}$ satisfies the twisted relation $\gamma_i w = [\epsilon^{-\frac{a}{m}}]xw$ because $\gamma_i T_i^{\frac{1}{m}} = [\epsilon^{\frac{1}{m}}] T_i^{\frac{1}{m}}$ (cf. (2.3.5)). Thus, by Lemma 3.4.11, we have $x = \zeta^y[\epsilon^z]$ with $y \in \mathbb{Q}$ and $z := \frac{a}{m} \in \frac{1}{m}\mathbb{Z} \cap [0, 1) \subset \mathbb{Q} \cap [0, 1)$, as desired. $\square$

Finally, let us finish the proof of Theorem 3.2.3(2). Let $v \in N/T_i N$ be an eigenvector of $t^{-1}\log(\gamma_i)$ with eigenvalue $\widetilde{x} \in \overline{B}_{\mathrm{dR}}$. Up to multiplying v by some power of t, we may assume that $v \in N^+/T_i N^+$. By Lemma 3.4.13, and by the assumption that $(\gamma_i - 1)^l\, v \to 0$, t-adically, as $l \to \infty$, it is easy to see that $\exp(\widetilde{x}t) =$

$\zeta^y[\epsilon^z]$, with $y+z \in \mathbb{Z}$ and $z \in \mathbb{Q} \cap [0,1)$. Therefore, we may assume that $-y = z \in \mathbb{Q} \cap [0,1)$. By Lemma 3.4.7, the eigenvalues of the residue along $\{T_i = 0\}$ are of the form $t^{-1}\log(\zeta^{-z}[\epsilon^z]) = z \in \mathbb{Q} \cap [0,1)$, which verifies Theorem 3.2.3(2), as desired.

Remark 3.4.14. By the proof Lemma 3.4.13, the surjection $L(X_K)/(T_1) \twoheadrightarrow L(Z_K)$ in Remark 3.3.14 (when $l = 1$ there) is the evaluation on $\widetilde{X}$ of $\big(\mathrm{gr}^0\, \mathcal{RH}_{\log}(\mathbb{L})\big)|_{D_1} \to \mathrm{gr}^0\big(\mathcal{RH}_{\log}(\mathbb{L})|^0_{D_1}\big)$ (cf. (3.4.2)), where $\mathcal{RH}_{\log}(\mathbb{L})|^0_{D_1}$ is interpreted as a quotient of $\mathcal{RH}_{\log}(\mathbb{L})|_{D_1}$ and equipped with the canonically induced filtration.

Proposition 3.4.15. *For the connection $\nabla_{\mathbb{L}} : D_{\mathrm{dR},\log}(\mathbb{L}) \to D_{\mathrm{dR},\log}(\mathbb{L}) \otimes_{\mathcal{O}_X} \Omega^{\log}_X$, all eigenvalues of $\mathrm{Res}_{\{T_i=0\}}(\nabla_{\mathbb{L}})$ are in $\mathbb{Q} \cap [0,1)$.*

Proof. Note that $D_{\mathrm{dR},\log}(\mathbb{L})(X) \cong \mathcal{RH}_{\log}(\mathbb{L})(\widetilde{X})^{\mathrm{Gal}(K/k)}$, and the isomorphism η in (3.4.5) is $\mathrm{Gal}(K/k)$-equivariant. Therefore, $D_{\mathrm{dR},\log}(\mathbb{L})(X) \cong N^{\mathrm{Gal}(K/k)}$. In addition, the residue $\mathrm{Res}_{\{T_i=0\}}(\nabla_{\mathbb{L}})$ is still given by $t^{-1}\log(\gamma_i)$ as in Lemma 3.4.7. Then the proposition follows from the arguments just explained above. □

In order to complete the proof of Theorem 3.2.7(1), it remains to apply the following proposition to conclude that $D_{\mathrm{dR},\log}(\mathbb{L})$ is a vector bundle.

Proposition 3.4.16. *A torsion-free coherent $\mathcal{O}_X$-module F with an integrable log connection $\nabla : F \to F \otimes_{\mathcal{O}_X} \Omega^{\log}_X$ is locally free when the following conditions hold:*

1. *F is reflexive (i.e., isomorphic to its bidual).*
2. *For every i, all eigenvalues of $\mathrm{Res}_{\{T_i=0\}}(\nabla)$ are in $\mathbb{Q} \cap [0,1)$.*

Proof. This follows from the same argument as in the proof of [AB01, Ch. 1, Prop. 4.5]. More precisely, it suffices to note that, under the assumptions, the completion of the stalk of $\mathcal{E}$ at each classical point of X is free, by [AB01, Ch. 1, Lem. 4.6.1]. □

To apply Proposition 3.4.16 to $D_{\mathrm{dR},\log}(\mathbb{L})$, it suffices to note that the condition (1) is satisfied by Lemma 3.3.18, and the condition (2) is satisfied by Proposition 3.4.15. The proof of Theorem 3.2.7(1) is now complete.

Proposition 3.4.17. *Suppose that F (resp. F') is a locally free (resp. torsion-free) coherent $\mathcal{O}_X$-module, with an integrable log connection ∇ (resp. ∇') as in Definition 3.1.7(4), whose residues along the irreducible components of D all have eigenvalues in $\mathbb{Q} \cap [0,1)$. Then any morphism $(F, \nabla) \to (F', \nabla')$ whose restriction to U is an isomorphism is necessarily an isomorphism over the whole X. The same is true if we replace $\mathcal{O}_X$-modules with $\mathcal{O}_X \widehat{\otimes} B^+_{\mathrm{dR}}$-modules.*

Proof. Let $(F'', \nabla'') := ((F', \nabla')^\vee)^\vee$, where F'' is the double $\mathcal{O}_X$-dual of F', which is by definition a reflexive coherent sheaf over X, and where ∇'' is the induced log integrable connection, whose residues along the irreducible components of D also have eigenvalues in $\mathbb{Q} \cap [0,1)$. Hence, by Proposition 3.4.16, F'' is *locally free*. Since the restriction of the given morphism $(F, \nabla) \to (F', \nabla')$ to the dense subspace $X - D$ is an isomorphism, we have injective morphisms $(F, \nabla) \to (F', \nabla') \to (F'', \nabla'')$, and it suffices to show that their composition is an isomorphism over X. Therefore, we can replace F' with F'', and assume that both F and F' are locally free. Thus, by working locally, we may replace X with its affinoid open subspaces which admit strictly étale morphisms to $\mathbb{D}^n$ as in Example 2.1.2, and assume that both F and F' are free of rank d. Then, with respect to the chosen bases, the map $F \to F'$ is represented by a matrix A in $M_d\big(\mathcal{O}_X(X)\big)$, which is invertible outside D. In order to show that it is invertible over X, it suffices to show that the entries of A^{-1}, which are a priori analytic functions on X *meromorphic* along D, are everywhere regular analytic functions on X. But this is classical—see, for example, the proof of [AB01, Ch. 1, Prop. 4.7]. Moreover, by Lemma 3.1.4, the above arguments also apply to integrable log connections on $\mathcal{X}$ (as in Definition 3.1.7(1)). □

As usual, we define a decreasing filtration on $D_{\mathrm{dR},\log}(\mathbb{L})$ by setting

$$\mathrm{Fil}^\bullet D_{\mathrm{dR},\log}(\mathbb{L}) := \big(\mathrm{Fil}^\bullet \mathcal{RH}_{\log}(\mathbb{L})\big)^{\mathrm{Gal}(K/k)}.$$

Lemma 3.4.18. *We endow $D_{\mathrm{dR},\log}(\mathbb{L})\widehat{\otimes}_k B_{\mathrm{dR}}$ with the usual product filtration. Then the canonical morphism*

$$D_{\mathrm{dR},\log}(\mathbb{L})\widehat{\otimes}_k B_{\mathrm{dR}} \to \mathcal{RH}_{\log}(\mathbb{L}) \tag{3.4.19}$$

defined by adjunction is injective (by definition) and strictly compatible with the filtrations on both sides. That is, for each r, (3.4.19) *induces an injective morphism*

$$\mathrm{gr}^r\big(D_{\mathrm{dR},\log}(\mathbb{L})\widehat{\otimes}_k B_{\mathrm{dR}}\big) \to \mathrm{gr}^r\,\mathcal{RH}_{\log}(\mathbb{L}). \tag{3.4.20}$$

Proof. Since $D_{\mathrm{dR},\log}(\mathbb{L}) \cong (\mathcal{RH}_{\log}(\mathbb{L}))^{\mathrm{Gal}(K/k)}$, the left-hand side of (3.4.20) can be identified with $\oplus_{a+b=r}\Big(\big(\mathrm{gr}^a\big((\mathcal{RH}_{\log}(\mathbb{L}))^{\mathrm{Gal}(K/k)}\big)\big)\otimes_k K(b)\Big)$, while the right-hand side of (3.4.20) contains $\oplus_{a+b=r}\Big(\big(\mathrm{gr}^a\,\mathcal{RH}_{\log}(\mathbb{L})\big)^{\mathrm{Gal}(K/k)}\otimes_k K(b)\Big)$ as a subspace, where we have direct sums in such forms because of the $\mathrm{Gal}(K/k)$-actions. Thus, it suffices to note that the canonical morphism $\mathrm{gr}^a\big((\mathcal{RH}_{\log}(\mathbb{L}))^{\mathrm{Gal}(K/k)}\big) \to \big(\mathrm{gr}^a\,\mathcal{RH}_{\log}(\mathbb{L})\big)^{\mathrm{Gal}(K/k)}$ is injective, for each a, essentially by definition. □

Corollary 3.4.21. *If $\mathbb{L}|_{U_{\mathrm{\acute{e}t}}}$ is de Rham, then* (3.4.19) *is an isomorphism of vector bundles on $\mathcal{X}$, compatible with the log connections and filtrations on both sides.*

Proof. Since $\mathbb{L}|_{(X-D)_{\mathrm{\acute{e}t}}}$ is de Rham, by [LZ17, Cor. 3.12(ii)], the restriction of (3.4.19) to $X-D$ is an isomorphism. By Proposition 3.4.17 and Theorems 3.2.3(2) and 3.2.7(1), the morphism (3.4.19) is an isomorphism, compatible with the log connections. By Lemma 3.4.18, it is also compatible with the filtrations. □

Corollary 3.4.22. *If $\mathbb{L}|_{U_{\mathrm{\acute{e}t}}}$ is de Rham, then $\mathrm{gr}\,D_{\mathrm{dR},\log}(\mathbb{L})$ is a vector bundle of rank $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$.*

Proof. By Corollary 3.4.21, $\oplus_a\Big(\big(\mathrm{gr}^a\,D_{\mathrm{dR},\log}(\mathbb{L})\big)\widehat{\otimes}_k K(-a)\Big) \xrightarrow{\sim} \mathrm{gr}^0\,\mathcal{RH}_{\log}(\mathbb{L}) \cong \mathcal{H}_{\log}(\mathbb{L})$. Since $\mathcal{H}_{\log}(\mathbb{L})$ is a vector bundle on X_K by Theorem 3.2.4(1), this shows that $\mathrm{gr}\,D_{\mathrm{dR},\log}(\mathbb{L})$ is a vector bundles on X of rank equal to that of $\mathcal{H}_{\log}(\mathbb{L})$, which is in turn equal to $\mathrm{rk}_{\mathbb{Q}_p}(\mathbb{L})$ by the proof of Theorem 3.2.4(1) in Section 3.3. □

Thus, by Proposition 3.4.15 and Corollary 3.4.22, the proof of Theorem 3.2.7(2) is also complete. We conclude this subsection with the following:

Proof of Theorem 3.2.12. Given any $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\mathrm{k\acute{e}t}}$ such that $\mathbb{L}|_{U_{\mathrm{\acute{e}t}}}$ has unipotent geometric monodromy along D, the action of γ_i as in Lemma 3.4.7 on any stalk of $\mathbb{L}|_{\widehat{\widetilde{Z}}_i^\partial}$ is *unipotent*. Consequently, $x = 1$ in Lemmas 3.4.11 and 3.4.13, and the residues of $\mathcal{RH}_{\log}(\mathbb{L})$ (by Lemma 3.4.7) are all *nilpotent* (i.e., have zero eigenvalues). For such $\mathbb{L}$, in the decomposition (3.3.10), we only need the characters τ such that $\tau(\gamma_i) = 1$ whenever $Z_i \neq \emptyset$. Hence, the morphism (3.3.13) is an isomorphism, and the canonical morphisms $\mu'^*\big(\mathcal{H}_{\log}(\mathbb{L})\big) \otimes_{\mathcal{O}_{X_{\mathrm{prok\acute{e}t}}}} \mathcal{O}\mathbb{C}_{\log} \to$

$\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{C}_{\log}$ and $\mu'^*\big(\mathcal{RH}_{\log}(\mathbb{L})\big) \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ are also isomorphisms (cf. [LZ17, Thm. 2.1(ii) and 3.8(iii)]). Therefore, we can argue as in the proofs of [LZ17, Thm. 2.1(iv) and 3.8(i)] that $\mathcal{H}_{\log}$ and $\mathcal{RH}_{\log}$ restrict to natural tensor functors. This proves part (1) of the theorem.

As for part (2), suppose moreover that $\mathbb{L}|_{U_{\text{ét}}}$ is *de Rham*. Then the residues of $D_{\mathrm{dR},\log}(\mathbb{L})$ are nilpotent by Proposition 3.4.15, and (3.4.19) is an isomorphism by Corollary 3.4.21. Hence, the canonical morphism $\mu^*\big(D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ (cf. (1.8)) is also an isomorphism, and we can conclude as in [LZ17, Thm. 3.9(v)] that $D_{\mathrm{dR},\log}$ also restricts to a natural tensor functor. □

3.5. **Compatibility with pullbacks and pushforwards.** In this subsection, we prove Theorems 3.2.3(4), 3.2.4(3), and 3.2.7(4)(5). We shall omit the explicit verifications of the $\mathrm{Gal}(K/k)$-equivariance of the adjunction morphisms, because they are obvious from the constructions of the functors $\mathcal{RH}_{\log}$ and $\mathcal{H}_{\log}$ (cf. Section 3.3).

Let us begin with pullbacks. Let $\mathcal{Y}$ be defined by Y as in (3.1.5). Let $h : Y \to X$ be as in the statements of the theorems. Let E be the normal crossings divisor defining the log structure on Y, as in Example 2.1.2, and let $V := Y - E$.

Remark 3.5.1. Theorems 3.2.3(4), 3.2.4(3), and 3.2.7(4) are obvious when $h : Y \to X$ is an open immersion. Moreover, when the log structure is trivial, the functors $\mathcal{RH}_{\log}$, $\mathcal{H}_{\log}$, and $D_{\mathrm{dR},\log}$ coincide with their analogues $\mathcal{RH}$, $\mathcal{H}$, and D_{dR} in [LZ17, Thm. 3.8, 2.1, and 3.9] (modulo the correction in Remark 2.2.11).

Lemma 3.5.2. *In the above setting, we have $h^{-1}(D) \subset E$ set-theoretically.*

Proof. By the definition of the log structures $\mathcal{M}_X$ and $\mathcal{M}_Y$ of X and Y, respectively, as in Example 2.1.2, the map $h^\sharp : h^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y$ between log structures is defined only when $h^{-1}(D) \subset E$ set-theoretically. Hence, the lemma follows. □

Lemma 3.5.3. *The canonical morphism*

$$h^*\big(\mathcal{H}_{\log}(\mathbb{L})\big) \to \mathcal{H}_{\log}\big(h^*(\mathbb{L})\big), \tag{3.5.4}$$

defined by adjunction is injective. The morphisms

$$h^*\big(\mathcal{RH}_{\log}(\mathbb{L})\big) \to \mathcal{RH}_{\log}\big(h^*(\mathbb{L})\big), \tag{3.5.5}$$

and

$$h^*\big(D_{\mathrm{dR},\log}(\mathbb{L})\big) \to D_{\mathrm{dR},\log}\big(h^*(\mathbb{L})\big) \tag{3.5.6}$$

are injective and strictly compatible with the filtrations on their both sides.

Proof. Since V_K is dense in Y_K, and since $h^*\big(\mathcal{H}_{\log}(\mathbb{L})\big)$ is a vector bundle on Y_K by Theorem 3.2.4(1), the morphism (3.5.4) is injective because the corresponding morphisms for $h|_V : V \to U$ is an isomorphism by [LZ17, Thm. 2.1(iii)]. Since $\mathrm{gr}^r\, \mathcal{RH}_{\log}(\mathbb{L}) \cong \mathcal{H}_{\log}(\mathbb{L})(r)$ and $\mathrm{gr}^r\, \mathcal{RH}_{\log}\big(h^*(\mathbb{L})\big) \cong \mathcal{H}_{\log}\big(h^*(\mathbb{L})\big)(r)$, for all r, the statement for (3.5.5) follows from that of (3.5.4). Also, the statement for (3.5.6) follows from those for (3.5.4) and (3.4.20), by Lemma 3.4.18 (and its proof). □

Corollary 3.5.7. *Under the same assumption as in Theorem* 3.2.3(4)*, the canonical morphisms* (3.5.5) *and* (3.5.6) *are isomorphisms compatible with the log connections and filtrations on both sides. In this case, the canonical morphism* (3.5.4)*, which can be identified with the* 0*-th graded piece of* (3.5.5)*, is an isomorphism compatible with the log Higgs fields on both sides.*

Proof. By assumption and Theorems 3.2.3(2), 3.2.7(2), and 3.2.12, all the eigenvalues of residues of both sides of (3.5.5) and (3.5.6) belong to $\mathbb{Q} \cap [0, 1)$. Hence, by Proposition 3.4.17 and Theorems 3.2.3(1) and 3.2.7(1), the assertions for (3.5.5) and (3.5.6) follow from the corresponding ones for $h|_V : V \to U$ in [LZ17, Thm. 3.8(iv) and 3.9(ii)], and the assertion for (3.5.4) follows from the one for (3.5.5). □

Thus, we have finished the proofs of Theorems 3.2.3(4), 3.2.4(3), and 3.2.7(4).

Next, let us turn to Theorem 3.2.7(5). Let $f : X \to Y$ be as in the statement of the theorem. Let $\mathcal{Y}$ be defined by Y as in (3.1.5). Since $f^{-1}(E) \subset D$ by the same argument as in the proof of Lemma 3.5.2, and since $f|_U : U \to V$ is proper smooth, we must have $D = f^{-1}(E)$ because U is dense in X. Let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{két}}$. By [DLLZ, Cor. 6.3.5] $R^i f_{\text{két},*}(\mathbb{L})$ is a $\mathbb{Z}_p$-local system on $Y_{\text{két}}$.

Lemma 3.5.8. *Under the assumption that $\mathbb{L}|_{U_{\text{ét}}}$ is de Rham, $R^i(f|_U)_{\text{ét},*}(\mathbb{L}|_{U_{\text{ét}}})$ is also de Rham, and the morphism*

$$D_{\mathrm{dR,log}}\big(R^i f_*(\mathbb{L})_{\mathbb{Q}_p}\big) \to \Big(R^i f_{\log \mathrm{dR},*}\big(D_{\mathrm{dR,log}}(\mathbb{L}_{\mathbb{Q}_p})\big)\Big)_{\mathrm{free}} \tag{3.5.9}$$

defined by adjunction is injective and strictly compatible with the filtrations on the both sides. That is, for each r, (3.5.9) *induces an injective morphism*

$$\mathrm{gr}^r\, D_{\mathrm{dR,log}}\big(R^i f_{\text{két},*}(\mathbb{L})_{\mathbb{Q}_p}\big) \to \mathrm{gr}^r\Big(\Big(R^i f_{\log \mathrm{dR},*}\big(D_{\mathrm{dR,log}}(\mathbb{L}_{\mathbb{Q}_p})\big)\Big)_{\mathrm{free}}\Big). \tag{3.5.10}$$

Proof. Since $\mathbb{L}|_{U_{\text{ét}}}$ is de Rham, $\big(R^i f_{\text{két},*}(\mathbb{L})\big)|_{V_{\text{ét}}} \cong R^i(f|_U)_{\text{ét},*}(\mathbb{L}|_{U_{\text{ét}}})$ is also de Rham, by [Sch13, Thm. 8.8] and [LZ17, Thm. 3.8(iv)]. Therefore, by Corollary 3.4.22, $\mathrm{gr}\, D_{\mathrm{dR,log}}\big(R^i f_{\text{két},*}(\mathbb{L})_{\mathbb{Q}_p}\big)$ is a vector bundles on Y. Since V is dense in Y, the morphism (3.5.10) is injective because the corresponding morphism for $f|_U : U \to V$ is an isomorphism, by [Sch13, Thm. 8.8]. □

Corollary 3.5.11. *Under the assumption that $\mathbb{L}|_{U_{\text{ét}}}$ is de Rham,* (3.5.9) *is an isomorphism compatible with the log connections and filtrations on both sides.*

Proof. By Lemma 3.5.8, it suffices to show that (3.5.9) is an isomorphism compatible with the log connections on both sides. By the same argument as in [Kat71, Sec. VII], the eigenvalues of the residues of $\Big(R^i f_{\log \mathrm{dR},*}\big(D_{\mathrm{dR,log}}(\mathbb{L}), \nabla_{\mathbb{L}}\big)\Big)_{\mathrm{free}}$ are still in $\mathbb{Q} \cap [0, 1)$. Hence, by Proposition 3.4.17 and Theorem 3.2.7(1), the assertion follows from the corresponding one for $f|_U : U \to V$ in [Sch13, Thm. 8.8]. □

The proof of Theorem 3.2.7(5) is now complete.

3.6. **Comparison of cohomology.** In this subsection, we prove the remaining Theorems 3.2.3(3), 3.2.4(2), and 3.2.7(3). We shall assume that X is proper over k, and that $K = \widehat{\overline{k}}$. (In this case, B_{dR}^+ and B_{dR} are the usual Fontaine's rings.)

Lemma 3.6.1. *For each $\mathbb{Z}_p$-local system $\mathbb{L}$ on $X_{K,\text{két}}$, and for each $i \geq 0$, we have a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism of B_{dR}^+-modules*

$$H^i(X_{K,\text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \cong H^i(X_{K,\text{prokét}}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{dR}}^+),$$

compatible with the filtrations on both sides, and also (*by taking* 0*-th graded pieces*) *a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism of K-modules*

$$H^i(X_{K,\text{két}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} K \cong H^i(X_{K,\text{prokét}}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}_{X_{K,\text{prokét}}}).$$

Proof. The proof is the same as [Sch13, Thm. 8.4], with the input [Sch13, Thm. 5.1] there replaced with [DLLZ, Thm. 6.2.1]. □

Lemma 3.6.2. *Let $\mathbb{L}$ be any $\mathbb{Z}_p$-local system on $X_{\text{két}}$. For each $i \geq 0$, we have a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism of* B_{dR}*-modules*

$$H^i\big(X_{K,\text{proként}}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{dR}}\big) \cong H^i_{\log \mathrm{dR}}\big(\mathcal{X}, \mathcal{RH}_{\log}(\mathbb{L})\big)$$

and also a canonical $\mathrm{Gal}(K/k)$*-equivariant isomorphism of* K*-modules*

$$H^i\big(X_{K,\text{proként}}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}_{X_{K,\text{proként}}}\big) \cong H^i_{\log \mathrm{Higgs}}\big(X_{K,\mathrm{an}}, \mathcal{H}_{\log}(\mathbb{L})\big).$$

Proof. Let us simply denote the complexes $(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{X_{\text{proként}}}} \Omega_X^{\log,\bullet}, \nabla)$ and $(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{C}_{\log} \otimes_{\mathcal{O}_{X_{\text{proként}}}} \Omega_X^{\log,\bullet}(-\bullet), \theta)$ (where the two $\bullet$ in the latter complex are equal to each other) by $DR_{\log}(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})$ and $Higgs_{\log}(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{C}_{\log})$, respectively. By Corollary 2.4.2, we have quasi-isomorphisms $\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{dR}} \xrightarrow{\sim} DR_{\log}(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})$ and (by taking the 0-th graded pieces) $\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}_{X_{K,\text{proként}}} \xrightarrow{\sim} Higgs_{\log}(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{C}_{\log})$ over $X_{K,\text{proként}}$. By Theorem 3.2.3(1) and Proposition 3.3.3, and by the projection formula (cf. (3.3.1)), $R\mu'_*\big(DR_{\log}(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})\big) \cong DR_{\log}(\mathcal{RH}_{\log}(\mathbb{L}))$ and $R\mu'_*\big(Higgs_{\log}(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{C}_{\log})\big) \cong Higgs_{\log}(\mathcal{H}_{\log}(\mathbb{L}))$, and the lemma follows. □

Thus, Theorems 3.2.3(3) and 3.2.4(2) follow from Lemmas 3.6.1 and 3.6.2.

It remains to complete the proof of Theorem 3.2.7(3). In the remainder of this subsection, we shall assume in addition that $\mathbb{L}|_{(X-D)_{\text{ét}}}$ is a de Rham local system. Firstly, the isomorphism (3.2.8) is given by Theorem 3.2.3(3) and the following:

Lemma 3.6.3. *With assumptions as above, there is a canonical isomorphism*

$$H^i_{\log \mathrm{dR}}\big(\mathcal{X}, \mathcal{RH}_{\log}(\mathbb{L})\big) \cong H^i_{\log \mathrm{dR}}\big(X_{\mathrm{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_k B_{\mathrm{dR}}.$$

Proof. Since $\mathbb{L}|_{(X-D)_{\text{ét}}}$ is de Rham, by Corollary 3.4.21, we have

$$R\Gamma\big(X_{\mathrm{an}}, DR_{\log}(\mathcal{RH}_{\log}(\mathbb{L}))\big) \cong R\Gamma\big(X_{\mathrm{an}}, DR_{\log}(D_{\mathrm{dR},\log}(\mathbb{L}))\widehat{\otimes}_k B_{\mathrm{dR}}\big).$$

We claim that the last term coincides with $R\Gamma\big(X_{\mathrm{an}}, DR_{\log}(D_{\mathrm{dR},\log}(\mathbb{L}))\big) \otimes_k B_{\mathrm{dR}}$. It suffices to check that we have isomorphisms on the grade pieces. Concretely, it suffices to note that, for any finite locally free $\mathcal{O}_X$-module $\mathcal{F}$, we have

$$R\Gamma(X_{\mathrm{an}}, \mathcal{F}) \otimes_k \mathrm{gr}^r B_{\mathrm{dR}} \xrightarrow{\sim} R\Gamma(X_{K,\mathrm{an}}, \mathcal{F} \otimes_k \mathrm{gr}^r B_{\mathrm{dR}}), \tag{3.6.4}$$

where $\mathrm{gr}^r B_{\mathrm{dR}} \cong K(r)$, as in [Sch13, the proof of Lem. 7.13]. □

Secondly, $\mathrm{gr}\, D_{\mathrm{dR},\log}(\mathbb{L})$ is a vector bundle of rank $\mathrm{rk}_{\mathbb{Z}_p}(\mathbb{L})$ by Corollary 3.4.22, and the isomorphism (3.2.9) is given by Theorem 3.2.4(2) and the following:

Lemma 3.6.5. *With assumptions as above, there is a canonical isomorphism*

$$H^i_{\log \mathrm{Higgs}}\big(X_{K,\mathrm{an}}, \mathcal{H}_{\log}(\mathbb{L})\big) \cong \oplus_{a+b=i} \Big(H^{a,b}_{\log \mathrm{Hodge}}\big(X_{\mathrm{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_k K(-a)\Big).$$

Proof. Since $\mathbb{L}|_{(X-D)_{\text{ét}}}$ is de Rham, by Corollary 3.4.21 and (3.6.4), we have

$$\begin{aligned} &H^i_{\log \mathrm{Higgs}}\big(X_{K,\mathrm{an}}, \mathcal{H}_{\log}(\mathbb{L})\big) = H^i\big(X_{K,\mathrm{an}}, Higgs_{\log}(\mathcal{H}_{\log}(\mathbb{L}))\big) \\ &\cong H^i\big(X_{K,\mathrm{an}}, \mathrm{gr}^0\big(DR_{\log}(D_{\mathrm{dR},\log}(\mathbb{L}))\widehat{\otimes}_k B_{\mathrm{dR}}\big)\big) \\ &\cong \oplus_a \Big(H^i\big(X_{\mathrm{an}}, \mathrm{gr}^a\, DR_{\log}(D_{\mathrm{dR},\log}(\mathbb{L}))\big) \otimes_k K(-a)\Big) \\ &\cong \oplus_{a+b=i} \Big(H^{a,b}_{\log \mathrm{Hodge}}\big(X_{\mathrm{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big) \otimes_k K(-a)\Big). \end{aligned}$$

□

Finally, the (log) Hodge–de Rham spectral sequence for $D_{\mathrm{dR},\log}(\mathbb{L})$ degenerates on the E_1 page because, by (3.2.8) and (3.2.9), we have

$$\dim_k H^i_{\log \mathrm{dR}}\big(X_{\mathrm{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big) = \sum_{a+b=i} \dim_k H^{a,b}_{\log \mathrm{Hodge}}\big(X_{\mathrm{an}}, D_{\mathrm{dR},\log}(\mathbb{L})\big).$$

The proof of Theorem 3.2.7(3) is now complete.

3.7. **Compatibility with nearby cycles.** Let $f : X \to \mathbb{D} = \mathrm{Spa}(k\langle T\rangle, k^+\langle T\rangle)$ be a morphism of smooth rigid analytic varieties such that $D := f^{-1}(0)$ is a normal crossings divisor. We endow X with the log structure defined by $\imath : D_{\mathrm{red}} \hookrightarrow X$ as in Example 2.1.2. Let $U := X - D$. Recall that we have introduced in [DLLZ, Def. 6.4.1] the functors of unipotent and quasi-unipotent nearby cycles $R\Psi^u_f(\mathbb{L}|_U)$ and $R\Psi^{qu}_f(\mathbb{L}|_U)$, respectively, for $\mathbb{Q}_p$-local systems $\mathbb{L}$ on $X_{\text{két}}$. In this subsection, we show that their formation is compatible with the log Riemann–Hilbert functors, in the simplest situation to which the methods of this paper are directly applicable.

As usual, for any $\mathcal{O}_Y$-module $\mathcal{F}$ on a locally ringed space Y and any closed immersion $\imath : Z \hookrightarrow Y$ such that $\mathcal{I}_Z := \ker\big(\mathcal{O}_Y \to \imath_*(\mathcal{O}_Z)\big)$ is an invertible $\mathcal{O}_Y$-ideal, let $\mathcal{F}(nZ) := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{I}_Z^{\otimes(-n)}$, for each $n \in \mathbb{Z}$. Also, if we have compatible inclusions $\mathcal{F}(nZ) \hookrightarrow \mathcal{F}(mZ)$ extending the identity morphism on $\mathcal{F}|_U$, for all $m \geq n$, then we let $F(*Z) := \varinjlim_{n\in\mathbb{Z}} F(nZ) = \cup_{n\in\mathbb{Z}} F(nZ)$. The following lemma is elementary:

Lemma 3.7.1. *Let (F, ∇) be any vector bundle with an integrable log connection on X_{an}. Let $Z \subset D_{\mathrm{red}}$ be an irreducible component, and suppose that all the eigenvalues of the residue* (3.4.1) *belong to $\mathbb{Q} \cap [0, 1)$. Then $F(*Z)$ is defined, and there is a unique decreasing $\mathbb{Q}$-filtration $V^\bullet$ on $F(*Z)$ by locally free $\mathcal{O}_X$-submodules equipped with compatible integrable log connections, characterized by the following properties:*

1. *We have $V^0 F(*Z) = F$ and $V^{i+1} F(*Z) = \big(V^i F(*Z)\big)(-Z)$.*
2. *The isomorphism $\big(V^0 F(*Z)\big)/\big(V^1 F(*Z)\big) \cong F|_Z$ canonically induces, for each $\alpha \in [0, 1)$, an isomorphism*

$$\mathrm{gr}^\alpha_V F(*Z) := \big(V^\alpha F(*Z)\big)/\big(V^{>\alpha} F(*Z)\big) \cong F|^\alpha_Z, \tag{3.7.2}$$

where $V^{>\alpha} F(*Z) := \cup_{\beta>\alpha} V^\beta F(*Z)$ and where $F|^\alpha_Z$ is as in (3.4.2).

By using Lemma 3.1.4*, we have analogues of the above for any integrable log connection $(\mathcal{F}, \nabla)$ on $\mathcal{X}$ such that all the eigenvalues of $\mathrm{Res}_Z(\nabla)$ belong to $\mathbb{Q} \cap [0, 1)$.*

Remark 3.7.3. Since $\nabla : F \to F \otimes_{\mathcal{O}_X} \Omega^{\log}_X$ induces a connection $\nabla : F(*Z) \to F(*Z) \otimes_{\mathcal{O}_X} \Omega_X$ satisfying $\nabla^2 = 0$, we can view $F(*Z)$ as a D-module, and view the filtration in Lemma 3.7.1 as a special case of the Kashiwara–Malgrange V-filtrations (cf. [Sai88, Sec. 3.1], with $V^\bullet$ here corresponding to $V_{-1-\bullet}$ there). (Note that there are different conventions of indices in the literature.)

We apply this construction to $\mathcal{RH}_{\log}(\mathbb{L})$, which is justified by Theorem 3.2.3(2). Recall that there is a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{RH}_{\log}(\mathbb{L})$ on $\mathcal{RH}_{\log}(\mathbb{L})$ by locally free $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$-submodules. This induces, for each $\alpha > -1$, the filtration

$$\mathrm{Fil}^i V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*Z) := \big(\mathrm{Fil}^i \mathcal{RH}_{\log}(\mathbb{L})\big)(Z) \cap \big(V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*Z)\big)$$

on $V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*Z)$ by $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}^+$-submodules. By construction, for $\alpha \geq \beta > -1$, the inclusion $V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*Z) \hookrightarrow V^\beta \mathcal{RH}_{\log}(\mathbb{L})(*Z)$ is strictly compatible with the filtrations. For each $\alpha \geq -1$, we similarly define the filtration on $V^{>\alpha} \mathcal{RH}_{\log}(\mathbb{L})(*Z)$. Then we have an induced quotient filtration on $\mathrm{gr}_V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*Z)$, for each $\alpha > -1$.

Remark 3.7.4. For each $\alpha \in [0, 1)$, in general, the isomorphism (3.7.2) is compatible with filtrations only if we view $\mathcal{RH}_{\log}(\mathbb{L})|_Z^\alpha$ as a quotient (rather than a subsheaf) of $\mathcal{RH}_{\log}(\mathbb{L})|_Z$ with its induced filtration. We emphasize that it is the quotient filtration on $\mathrm{gr}_V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*Z)$ that will be important in the following.

Theorem 3.7.5. *Assume that D is smooth. Then, for each $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\mathrm{k\acute{e}t}}$, there is a canonical* $\mathrm{Gal}(K/k)$*-equivariant isomorphism of $\mathcal{O}_D \widehat{\otimes}_k B_{\mathrm{dR}}$-modules*

$$\mathcal{RH}\big(R\Psi_f^{qu}(\mathbb{L}|_U)\big) \cong \oplus_{\alpha \in (-1,0]} \big(\mathrm{gr}_V^\alpha \mathcal{RH}_{\log}(\mathbb{L})(*)\big), \tag{3.7.6}$$

which restricts to an isomorphism

$$\mathcal{RH}\big(R\Psi_f^{u}(\mathbb{L}|_U)\big) \cong \mathrm{gr}_V^0 \mathcal{RH}_{\log}(\mathbb{L})(*), \tag{3.7.7}$$

compatible with filtrations and integrable connections. Here $\mathcal{RH}$ is the functor defined in [LZ17, Thm. 3.8] (*see Remark* 3.5.1)*, and we write $\mathcal{RH}_{\log}(\mathbb{L})(*)$ instead of $\mathcal{RH}_{\log}(\mathbb{L})(*D)$ for simplicity.*

Proof. Let us first prove (3.7.7). Besides the trivial log structure, there is another natural log structure on D given by the pullback of the log structure on X. Let D^∂ denote the corresponding log adic space. Then we have a correspondence of log adic spaces $D \stackrel{\varepsilon^\partial}{\longleftarrow} D^\partial \stackrel{\imath}{\to} X$. Let $\widehat{\mathbb{L}}^\partial := \imath_{\mathrm{prok\acute{e}t}}^{-1}(\widehat{\mathbb{L}})$, which is associated with $\mathbb{L}^\partial := \imath_{\mathrm{k\acute{e}t}}^{-1}(\mathbb{L})$ by [DLLZ, Lem. 6.3.3]. Let $\widehat{\mathbb{J}}_r^\partial$ denote the limit of the pullbacks of the similarly denoted torsion local systems in [DLLZ, Ex. 4.4.25].

By Corollary 2.3.20 and Lemma 3.3.5, we have canonical morphisms of sheaves $\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,X} \to \imath_{\mathrm{prok\acute{e}t},*}\big(\widehat{\mathbb{L}}^\partial \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,D^\partial}\big) \cong R\imath_{\mathrm{prok\acute{e}t},*}\big(\widehat{\mathbb{L}}^\partial \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,D^\partial}\big)$ on $X_{\mathrm{prok\acute{e}t}}$. By applying $R\mu'_{X,*}$, we obtain a morphism of $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{dR}}$-modules

$$\mathcal{RH}_{\log}(\mathbb{L}) \to \imath_* \, R\mu'_{D^\partial,*}(\widehat{\mathbb{L}}^\partial \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,D^\partial}). \tag{3.7.8}$$

Also by Corollary 2.3.20, and by matching a basis of $\mathbb{J}_r$ with binomial monomials up to degree $r-1$ in W as in the proof of [DLLZ, Lem. 6.4.2], for any $-\infty \leq a \leq b \leq \infty$, there is a natural isomorphism $\varepsilon_{\mathrm{prok\acute{e}t}}^{\partial,-1}(\mathcal{O}\mathbb{B}_{\mathrm{dR},\log,D}^{[a,b]}) \otimes_{\widehat{\mathbb{Z}}_p} \varinjlim_r (\widehat{\mathbb{J}}_r^\partial) \cong \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,D^\partial}^{[a,b]}$. Since $R\mu'_{D^\partial,*} \cong R\mu'_{D,*} \circ R\varepsilon_{\mathrm{prok\acute{e}t},*}^\partial$, by adjunction (and taking colimit and limit), and by [DLLZ, Def. 6.4.1], we obtain a canonical morphism of sheaves

$$\mathcal{RH}\big(R\Psi_f^u(\mathbb{L}|_U)\big) \to R\mu'_{D^\partial,*}(\widehat{\mathbb{L}}^\partial \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},\log,D^\partial}) \tag{3.7.9}$$

We claim that (3.7.9) is an isomorphism, and that the combination of (3.7.8) and (3.7.9) induce a canonical isomorphism $\mathrm{gr}_V^0 \mathcal{RH}_{\log}(\mathbb{L})(*) \cong \mathcal{RH}\big(R\Psi_f^u(\mathbb{L}|_U)\big)$.

Since the question is local, we may assume that X is affinoid, and that f factors as $X \to \mathbb{D}^n \cong \mathbb{D} \times \mathbb{D}^{n-1} \to \mathbb{D}$, where the first map is a smooth toric chart, and where the second map is the first projection. Accordingly, we have $\Gamma_{\mathrm{geom}} \cong$

$(\widehat{\mathbb{Z}}\gamma_1) \times \widehat{\mathbb{Z}}(1)^{n-1}$. By Lemma 3.4.3, Corollary 2.3.20 and (3.4.10), in the notation there, the evaluation of (3.7.8) at X can be identified with the natural map

$$\text{(3.7.10)} \qquad \big(\varprojlim_r (N_\infty/\xi^r)^{\mathrm{unip}}\big)[\tfrac{1}{t}] \to \big(\varprojlim_r \big((N_\infty/\xi^r)/([T_1^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}}\big)^{\mathrm{unip}}\big)[\tfrac{1}{t}],$$

where $(\,\cdot\,)^{\mathrm{unip}}$ denotes the maximal quotient spaces on which Γ_{geom} acts unipotently. By the arguments in the proof of Lemma 3.4.13, the right-hand side of (3.7.10) can be identified with the quotient of N/T_1 on which $t^{-1}\log(\gamma_1)$ acts nilpotently, which is $\imath_*\big(\mathrm{gr}_V^0\,\mathcal{RH}_{\log}(\mathbb{L})(*)\big)(X)$ by Lemma 3.4.7 and (3.7.2). This also gives the left-hand side of the evaluation of (3.7.9) at D, by [DLLZ, Prop. 6.4.4], Lemma 3.3.15, and Remark 3.4.14. Since $(N_\infty/\xi)^{\mathrm{unip}} \to \big((N_\infty/\xi)/([T_1^{s\flat}])^\wedge_{s\in\mathbb{Q}_{>0}}\big)^{\mathrm{unip}}$ is surjective by Lemma 3.3.15 again, so is (3.7.10). Thus, the claim and (3.7.7) follow.

Next, we reduce (3.7.6) to (3.7.7). Since γ_1 acts quasi-unipotently on $\mathbb{L}|_{D^\partial}$ (as in the proof of Lemma 3.4.11), there is some degree m standard Kummer étale cover of $\mathbb{D} \to \mathbb{D}$, with base change $g : X_m \to X$, such that $g^{-1}(\mathbb{L}|_U)$ has purely unipotent monodromy along D. By [DLLZ, Lem. 6.4.3], $R\Psi_f^{qu}(\mathbb{L}|_U) \cong R\Psi_{g\circ f}^{u}\big(g^{-1}(\mathbb{L}|_U)\big)$. By Lemma 3.5.3, we have a canonical morphism $g^{-1}\big(\mathcal{RH}_{\log}(\mathbb{L})\big) \to \mathcal{RH}_{\log}\big(g^{-1}(\mathbb{L})\big)$, strictly compatible with filtrations, which restricts to an isomorphism over $U_m := X_m - D$. By Theorem 3.2.12, the residue of $\mathcal{RH}_{\log}\big(g^{-1}(\mathbb{L})\big)$ along D is nilpotent, and therefore induces an injective map $\mathcal{RH}_{\log}\big(g^{-1}(\mathbb{L})\big) \to g^{-1}\big(\mathcal{RH}_{\log}(\mathbb{L})(D)\big)$, also strictly compatible with the filtrations, by the same argument as in the proof of Lemma 3.5.3. By pushing down to X, we obtain the following inclusions of sheaves

$$\mathcal{RH}_{\log}(\mathbb{L}) \otimes_{\mathcal{O}_X} g_*(\mathcal{O}_{X_m}) \hookrightarrow g_*\mathcal{RH}_{\log}\big(g^{-1}(\mathbb{L})\big) \hookrightarrow \mathcal{RH}_{\log}(\mathbb{L})(D) \otimes_{\mathcal{O}_X} g_*(\mathcal{O}_{X_m}),$$

which are strictly compatible with filtrations, which can be identified with

$$\begin{aligned} \oplus_{i=0}^{m-1} T_1^{\frac{i}{m}} \mathcal{RH}_{\log}(\mathbb{L}) &\hookrightarrow \oplus_{i=0}^{m-1} \big(T_1^{\frac{i}{m}} V^{-\frac{i}{m}} \mathcal{RH}_{\log}(\mathbb{L})(*)\big) \\ &\hookrightarrow \oplus_{i=0}^{m-1} \big(T_1^{\frac{i}{m}} V^{-1} \mathcal{RH}_{\log}(\mathbb{L})(*)\big), \end{aligned}$$

by considering residues. Thus, we obtain the desired isomorphisms

$$\begin{aligned} \oplus_{i=0}^{m-1} \big(\mathrm{gr}_V^{-\frac{i}{m}}\, \mathcal{RH}_{\log}(\mathbb{L})(*)\big) &\cong \mathrm{gr}_V^0 \Big(g_*\, \mathcal{RH}_{\log}\big(g^{-1}(\mathbb{L})\big)\Big)(*) \\ &\cong \mathrm{gr}_V^0\, \mathcal{RH}_{\log}\big(g^{-1}(\mathbb{L})\big)(*), \end{aligned}$$

which are compatible with filtrations. □

Now suppose moreover that $\mathbb{L}|_{U_{\text{ét}}}$ is de Rham. By Theorem 3.2.7(2) and Lemma 3.7.1, we also have the V-filtration $V^\bullet D_{\mathrm{dR},\log}(\mathbb{L})(*)$ on $D_{\mathrm{dR},\log}(\mathbb{L})(*)$, and the filtration $\mathrm{Fil}^\bullet D_{\mathrm{dR},\log}(\mathbb{L})$ on $D_{\mathrm{dR},\log}(\mathbb{L})$ induces, for each $\alpha > -1$, the filtration

$$\mathrm{Fil}^i V^\alpha D_{\mathrm{dR},\log}(\mathbb{L})(*) := \big(\mathrm{Fil}^i D_{\mathrm{dR},\log}(\mathbb{L})\big)(D) \cap \big(V^\alpha D_{\mathrm{dR},\log}(\mathbb{L})(*)\big)$$

on $V^\alpha D_{\mathrm{dR},\log}(\mathbb{L})(*)$ by $\mathcal{O}_X$-submodules, and similar filtrations on $V^{>\alpha} D_{\mathrm{dR},\log}(\mathbb{L})(*)$ and $\mathrm{gr}_V^\alpha\, D_{\mathrm{dR},\log}(\mathbb{L})(*)$.

By Corollary 3.4.21 and by taking $\mathrm{Gal}(K/k)$-invariants, we obtain the following:

Theorem 3.7.11. *In Theorem* 3.7.5*, suppose moreover that* $\mathbb{L}|_{U_{\text{ét}}}$ *is* **de Rham**. *Then there is a canonical isomorphism*

$$\text{(3.7.12)} \qquad D_{\mathrm{dR}}\big(R\Psi_f^{qu}(\mathbb{L}|_U)\big) \cong \oplus_{\alpha\in(-1,0]} \big(\mathrm{gr}_V^\alpha\, D_{\mathrm{dR},\log}(\mathbb{L})(*)\big),$$

which restricts to an isomorphism

$$\text{(3.7.13)} \qquad D_{\mathrm{dR}}\big(R\Psi_f^{u}(\mathbb{L}|_U)\big) \cong \mathrm{gr}_V^0\, D_{\mathrm{dR},\log}(\mathbb{L})(*),$$

compatible with filtrations and integrable connections. Here D_{dR} *is the functor defined in* [LZ17, Thm. 3.9] (*see Remark* 3.5.1). *In particular,* $R\Psi_f^u(\mathbb{L}|_U)$ *is a de Rham* $\mathbb{Q}_p$*-local system on* $D_{\text{ét}}$.

4. Riemann–Hilbert functor for p-adic algebraic varieties

4.1. **The functor $D_{\mathrm{dR}}^{\mathrm{alg}}$.** In this subsection, we shall prove Theorem 1.1 and record some byproducts. Let X be a smooth algebraic variety over a finite extension k of $\mathbb{Q}_p$. By [Nag62, Hir64a, Hir64b], there is a smooth compactification $\jmath : X \hookrightarrow \overline{X}$ such that the boundary $D = \overline{X} - X$ (with its reduced subscheme structure) is a normal crossings divisor. Let X^{an}, $\overline{X}^{\mathrm{an}}$, $\jmath^{\mathrm{an}}$, and D^{an} denote the analytifications (realized in the category of adic spaces over $\mathrm{Spa}(k, k^+)$, where $k^+ = \mathcal{O}_k$). We shall equip $\overline{X}^{\mathrm{an}}$ with the log structure defined by D^{an}, as in Example 2.1.2.

To simplify the terminology, we shall use the term *filtered connection* (resp. *filtered regular connection*) to mean a filtered vector bundle on X equipped with an integrable connection (resp. an integrable connection with regular singularities) satisfying the Griffiths transversality. Likewise, we shall use the term *filtered log connection* to mean a filtered vector bundle on $\overline{X}$ (resp. $\overline{X}^{\mathrm{an}}$) equipped with an integrable connection satisfying the Griffiths transversality. In addition, by abuse of terminology, we say that a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$ is de Rham if its analytification is, and that a $\mathbb{Z}_p$-local system on $\overline{X}^{\mathrm{an}}_{\text{két}}$ is de Rham if its restriction to X^{an} is.

Let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$, with analytification $\mathbb{L}^{\mathrm{an}}$. Let $\overline{\mathbb{L}}^{\mathrm{an}} := \jmath^{\mathrm{an}}_{\text{két},*}(\mathbb{L}^{\mathrm{an}})$ be its extension to a $\mathbb{Z}_p$-local system on $\overline{X}^{\mathrm{an}}_{\text{két}}$ (by [DLLZ, Cor. 6.3.4]). By Theorem 3.2.7, we obtain a filtered log connection $D_{\mathrm{dR,log}}(\overline{\mathbb{L}}^{\mathrm{an}})$ on $\overline{X}^{\mathrm{an}}$, which is the analytification of an algebraic one, by GAGA (see [Köp74]), which we abusively denote by $D^{\mathrm{alg}}_{\mathrm{dR,log}}(\mathbb{L})$. Then its restriction to X is a filtered regular connection

$$D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L}) := \big(D^{\mathrm{alg}}_{\mathrm{dR,log}}(\mathbb{L})\big)\big|_X. \tag{4.1.1}$$

Let us summarize the constructions in the following commutative diagram:

$$\begin{array}{ccc}
\big\{\text{de Rham } \mathbb{Z}_p\text{-local systems on } X_{\text{ét}}\big\} & \xrightarrow{D^{\mathrm{alg}}_{\mathrm{dR}}} & \big\{\text{filtered regular connections on } X\big\} \\
\downarrow {\scriptstyle (\,\cdot\,)^{\mathrm{an}}} & & \uparrow {\scriptstyle \jmath^*} \\
\big\{\text{de Rham } \mathbb{Z}_p\text{-local systems on } X^{\mathrm{an}}_{\text{ét}}\big\} & & \big\{\text{filtered log connections on } \overline{X}\big\} \\
\downarrow {\scriptstyle \jmath^{\mathrm{an}}_{\text{két},*}} & & \downarrow {\scriptstyle (\,\cdot\,)^{\mathrm{an}}\ \cong \text{ by GAGA}} \\
\big\{\text{de Rham } \mathbb{Z}_p\text{-local systems on } \overline{X}^{\mathrm{an}}_{\text{két}}\big\} & \xrightarrow{D_{\mathrm{dR,log}}} & \big\{\text{filtered log connections on } \overline{X}^{\mathrm{an}}\big\}
\end{array}$$

Note that the de Rham assumptions on the local systems ensure that the associated regular connections or log connections are of the right ranks, and are filtered by vector subbundles rather than coherent subsheaves.

Lemma 4.1.2. *The functor* $D^{\mathrm{alg}}_{\mathrm{dR}}$ *is a tensor functor, and is independent of the choice of the compactification* $\overline{X}$.

Proof. By Proposition 3.4.15 and [AB01, Ch. 1, Prop. 6.2.2], for all $\mathbb{L}$, the exponents of the integrable connection $D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L})$ consist of only rational numbers, which are not Liouville numbers. Then the lemma follows from the following two facts:

(1) By [AB01, Ch. 4, Thm. 4.1] or [Bal88] and (a slight variant of) [AB01, Ch. 4, Cor. 3.6], the analytification functor from the category of algebraic integrable connections on X with rational exponents to the category of analytic integrable connections on X^{an} is fully faithful.
(2) The composition of $D_{\mathrm{dR}}^{\mathrm{alg}}$ with the analytification functor is the functor D_{dR} in [LZ17, Thm. 3.9(v)], a tensor functor independent of the choice of $\overline{X}$. □

It remains to establish the comparison isomorphism in Theorem 1.1. As in Section 3.6, let $K = \widehat{\overline{k}}$, so that the rings B_{dR}^{+} and B_{dR} in Definition 3.1.1(1) have their usual meaning as Fontaine's rings. By [Hub96, Prop. 2.1.4 and Thm. 3.8.1], if $\mathbb{L}$ is an étale local $\mathbb{Z}_p$-local systems on X, and if $\mathbb{L}^{\mathrm{an}}$ is its analytification on X^{an}, then we have a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$H^i_{\text{ét}}\big(X_{\overline{k}}, \mathbb{L}\big) \cong H^i_{\text{ét}}\big(X^{\mathrm{an}}_{\overline{k}}, \mathbb{L}^{\mathrm{an}}\big). \tag{4.1.3}$$

By [DLLZ, Cor. 6.3.4] and Theorem 3.2.7(3), we have a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $H^i_{\text{ét}}\big(X^{\mathrm{an}}_{\overline{k}}, \mathbb{L}^{\mathrm{an}}\big) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i_{\log \mathrm{dR}}\big(\overline{X}^{\mathrm{an}}, D_{\mathrm{dR},\log}(\overline{\mathbb{L}}^{\mathrm{an}})\big) \otimes_k B_{\mathrm{dR}}$, compatible with the filtrations on both sides. Finally, by GAGA again (see [Köp74]) and by Deligne's comparison result in [Del70, II, 6], we have

$$H^i_{\log \mathrm{dR}}\big(\overline{X}^{\mathrm{an}}, D_{\mathrm{dR},\log}(\overline{\mathbb{L}}^{\mathrm{an}})\big) \cong H^i_{\log \mathrm{dR}}\big(\overline{X}, D^{\mathrm{alg}}_{\mathrm{dR},\log}(\overline{\mathbb{L}}^{\mathrm{an}})\big) \cong H^i_{\mathrm{dR}}\big(X, D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L})\big).$$

This completes the proof of Theorem 1.1.

By combining (4.1.3), [DLLZ, Cor. 6.3.4], and GAGA (see [Köp74]) with the other assertions in Theorem 3.2.7(3), we also obtain the following:

Theorem 4.1.4. *In the above setting, the (log) Hodge–de Rham spectral sequence*

$$E_1^{a,b} = H^{a,b}_{\log \mathrm{Hodge}}\big(\overline{X}, D^{\mathrm{alg}}_{\mathrm{dR},\log}(\overline{\mathbb{L}}^{\mathrm{an}})\big) \Rightarrow H^{a+b}_{\log \mathrm{dR}}\big(\overline{X}, D^{\mathrm{alg}}_{\mathrm{dR},\log}(\overline{\mathbb{L}}^{\mathrm{an}})\big)$$

degenerates on the E_1 page, and the 0-th graded piece of (1.2) can be identified with a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant comparison isomorphism

$$H^i_{\text{ét}}\big(X_{\overline{k}}, \mathbb{L}\big) \otimes_{\mathbb{Q}_p} \widehat{\overline{k}} \cong \oplus_{a+b=i} \Big(H^{a,b}_{\log \mathrm{Hodge}}\big(\overline{X}, D^{\mathrm{alg}}_{\mathrm{dR},\log}(\overline{\mathbb{L}}^{\mathrm{an}})\big) \otimes_k \widehat{\overline{k}}(-a)\Big).$$

4.2. **Generalizations of Kodaira–Akizuki–Nakano vanishing.**

This subsection will be devoted to the proof of the following theorem:

Theorem 4.2.1. *Let X be a proper smooth algebraic variety over a finite extension k of $\mathbb{Q}_p$, and let D be a normal crossings divisor on X. Let $U := X - D$. Let $\mathbb{L}$ be a de Rham $\mathbb{Q}_p$-local system on $U_{\text{ét}}$. Let $\overline{\mathcal{E}} = D^{\mathrm{alg}}_{\mathrm{dR},\log}(\mathbb{L})$ be as in Section 4.1 (with the X and $\overline{X}$ there given by the U and X here, respectively), and let $DR_{\log}(\overline{\mathcal{E}})$ and $\mathrm{gr}\, DR_{\log}(\overline{\mathcal{E}})$ be as in Definition 3.1.7. Let $\mathcal{L}$ be an invertible sheaf on X such that there exists an effective divisor D' supported on D such that*

$$\mathcal{L}^N(-D') \text{ is ample for all sufficiently large } N. \tag{4.2.2}$$

Then we have

$$H^i\big(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log}(\overline{\mathcal{E}})\big) = 0, \text{ for all } i < d; \tag{4.2.3}$$

$$H^i\big(X, \mathcal{L}(-D) \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log}(\overline{\mathcal{E}}^\vee)\big) = 0, \text{ for all } i > d. \tag{4.2.4}$$

If $\mathbb{L}$ has **unipotent** *geometry monodromy along D, then we also have*

$$H^i\big(X, \mathcal{L}(-D) \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log}(\overline{\mathcal{E}})\big) = 0, \text{ for all } i > d. \tag{4.2.5}$$

Remark 4.2.6. The condition (4.2.2) implies that $\mathcal{L}$ is nef and big—see [EV92, Rem. 11.6 a)]. In fact, it is equivalent to being nef and big up to applying embedded resolution of singularities as in [Hir64a, Hir64b]—see [Suh18, footnote 1].

Remark 4.2.7. When $\mathbb{L}$ is trivial, in which case $\overline{\mathcal{E}} = \mathcal{O}_X$, our p-adic Hodge-theoretic proof of Theorem 4.2.1 provides new proofs for the classical vanishing theorems (in characteristic zero) due to Kodaira, Akizuki, and Nakano [Kod53, AN54] (when $D = \emptyset$) ; Deligne, Illusie, and Raynaud [DI87] (when $D' = \emptyset$); and Esnault and Viehweg [EV92]. Also, when $\mathbb{L}$ is of the form $R^a f_*(\mathbb{Q}_p)$ for some a and some proper smooth morphism $f : V \to U$, Theorem 4.2.1 provides a p-adic Hodge-theoretic generalization (as opposed to the complex analytic one in [Suh18]) of the characteristic-zero consequences in [Ill90] and [LS13, Sec. 3], without having to assume that f extends to a morphism over X with very good properties.

Proof of Theorem 4.2.1. It suffices to prove (4.2.3), since (4.2.4) follows by Serre duality, and since (4.2.5) follows because $\overline{\mathcal{E}} \cong D^{\mathrm{alg}}_{\mathrm{dR},\log}(\mathbb{L}^\vee)^\vee$ under the unipotency assumption, by Theorem 3.2.12 and GAGA [Köp74]. We will closely follow the first strategy in [Suh18, Sec. 2], but with the input from Saito's direct image theorem (see [Sai90, Thm. 2.14]) replaced with our p-adic Hodge-theoretic results.

Up to replacing D' with a positive multiple, we may assume that there exists some N_0 such that $\mathcal{L}^N(-D')$ is very ample for all $N \geq N_0$. We may enlarge N_0 and assume that, for each irreducible component Z of D along which D' has multiplicity e_Z, the eigenvalues of the residue of $\overline{\mathcal{E}}$ are contained in $\mathbb{Q} \cap [0, 1 - \frac{e_Z}{N_0})$. By the same Bertini-type argument as in [LS13, Sec. 2.1], there exist some $N \geq N_0$ and some section $s \in H^0\big(X, \mathcal{L}^N(-D')\big)$ such that the corresponding hyperplane section $H \subset X$ is smooth and meets D transversally, so that $D + H$ and $D|_H$ are normal crossings divisors on X and H, respectively. Up to replacing k with a finite extension, we may assume that k contains all the N-th roots of unity in $\overline{k}$.

Let $\imath : H \to X$ denote the canonical closed immersion. For the sake of clarity, we shall denote by $DR_{\log D}(\,\cdot\,)$ the log de Rham complex associated with $\Omega^\bullet_X(\log D)$, and similarly denote complexes associated with log structures defined by other normal crossings divisors. In order to prove (4.2.3), by considering the long exact sequence associated with the following twist of the adjunction exact sequence

$$\begin{aligned} 0 \to \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log D}(\overline{\mathcal{E}}) &\to \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log(D+H)}(\overline{\mathcal{E}}) \\ &\to \imath_*\big(\mathcal{L}|_H^{-1} \otimes_{\mathcal{O}_H} \mathrm{gr}\, DR_{\log(D|_H)}(\overline{\mathcal{E}}|_H)(-1)\big)[-1] \to 0, \end{aligned}$$

and by induction on the dimension of X (since $\overline{\mathcal{E}}|_H \cong D^{\mathrm{alg}}_{\mathrm{dR},\log}(\mathbb{L}|_{U\cap H})$ by Theorem 3.2.7(4), and since pullback by $\imath$ preserved ampleness), it suffices to prove that

$$H^i\big(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log(D+H)}(\overline{\mathcal{E}})\big) = 0, \text{ for all } i < d. \tag{4.2.8}$$

As in [EV92, Sec. 3], consider $\mathcal{L}^{(a)^{-1}} := \mathcal{L}^{-a}(\lfloor \frac{a(D'+H)}{N} \rfloor)$, which is equipped with an integrable log connection $\nabla^{(a)}$ such that the eigenvalues of the residue of $\nabla^{(a)}$ along H (resp. each irreducible component Z of D) are $\frac{a}{N}$ (resp. $\frac{a e_Z}{N} - \lfloor \frac{a e_Z}{N} \rfloor$). Let Y denote the relative spectrum of the $\mathcal{O}_X$-algebra $\oplus_{a=0}^{N-1} \mathcal{L}^{(a)^{-1}}$, whose multiplicative structure is induced by the dual of $\mathcal{O}_X \xrightarrow{s} \mathcal{L}^{\otimes N}(-D') \subset \mathcal{L}^{\otimes N}$. Then the structure morphism of the "cyclic cover" $\pi : Y \to X$ is finite flat, and the pullback of π to $W := X - (U + H)$ is a finite étale Galois cover $\pi_W : V \to W$ with Galois group $\mathrm{Hom}\big(\mathbb{Z}/N\mathbb{Z}, k^\times\big)$.

By construction, $\mathcal{L}^{(a)^{-1}}|_W \cong \pi_{W,*}(\mathcal{O}_V)[\chi_a]$, where $[\chi_a]$ denotes the isotypical component for the character $\chi_a : \mathrm{Hom}(\mathbb{Z}/N\mathbb{Z}, k^\times) \to k^\times$ defined by evaluation at the image of a, which is compatible with the connections (and trivial filtrations) on both sides. Consider $\mathbb{M}_a := \pi_{W,\text{ét},*}(k)[\chi_a]$, where k denotes the constant k-local system over V of rank one, i.e., a constant $\mathbb{Q}_p$-local system of rank $[k : \mathbb{Q}_p]$ equipped with the canonical action of k. Then $\mathbb{M}_a$ is a k-local system of rank one over W. Let $\tau : k \otimes_{\mathbb{Q}_p} k \to k$ be the multiplication map, and let us denote by τM the pushout under τ of any $k \otimes_{\mathbb{Q}_p} k$-module M. Since $\pi_{W,*}(\mathcal{O}_V) \cong \tau D_{\mathrm{dR}}^{\mathrm{alg}}(\pi_{W,\text{ét},*}(k))$, by Theorem 3.2.7(5), we obtain $\mathcal{L}^{(a)^{-1}}|_W \cong \tau D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{M}_a)$, which uniquely extends to $\mathcal{L}^{(a)^{-1}} \cong \tau D_{\mathrm{dR},\log}^{\mathrm{alg}}(\mathbb{M}_a)$ by [AB01, Ch. 1, Prop. 4.7], because both sides have eigenvalues of residues in $\mathbb{Q} \cap [0, 1)$, by the above and Theorem 3.2.7(2).

Since $\frac{e_Z}{N} \leq \frac{e_Z}{N_0} < 1$ for each irreducible component Z of D, we have $\mathcal{L}^{(1)^{-1}} = \mathcal{L}^{-1}$. Since $\overline{\mathcal{E}}|_U \cong D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})$, the residue of $\overline{\mathcal{E}}$ along H is zero. By Lemma 4.1.2, we have $\mathcal{L}^{-1}|_W \otimes_{\mathcal{O}_W} \overline{\mathcal{E}}|_W \cong \tau D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{M}_1 \otimes_{\mathbb{Q}_p} \mathbb{L}|_W)$, which uniquely extends to $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \overline{\mathcal{E}} \cong \tau D_{\mathrm{dR},\log}^{\mathrm{alg}}(\mathbb{M}_1 \otimes_{\mathbb{Q}_p} \mathbb{L}|_W)$, again because both sides have eigenvalues of residues in $\mathbb{Q} \cap [0, 1)$. Thus, the Hodge–de Rham spectral sequences for $D_{\mathrm{dR},\log}^{\mathrm{alg}}(\mathbb{M}_1 \otimes_{\mathbb{Q}_p} \mathbb{L}|_W)$ and $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \overline{\mathcal{E}}$ degenerate by Theorem 4.1.4, and (4.2.8) is equivalent to

$$H^i\big(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} DR_{\log(D+H)}(\overline{\mathcal{E}})\big) = 0, \text{ for all } i < d. \tag{4.2.9}$$

Since the eigenvalue of the residue of $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \overline{\mathcal{E}}$ along H is positive, by [EV92, Lem. 2.10], for any $b \geq 0$, the statement (4.2.9) is in turn equivalent to

$$H^i\big(X, \mathcal{L}^{-1}(-bH) \otimes_{\mathcal{O}_X} DR_{\log(D+H)}(\overline{\mathcal{E}})\big) = 0, \text{ for all } i < d. \tag{4.2.10}$$

Finally, by considering the filtration spectral sequence, it suffices to show that

$$H^i\big(X, \mathcal{L}^{-1}(-bH) \otimes_{\mathcal{O}_X} \mathrm{gr}\, DR_{\log(D+H)}(\overline{\mathcal{E}})\big) = 0, \text{ for all } i < d. \tag{4.2.11}$$

Since the divisor H is ample, and since $\mathrm{gr}\, DR_{\log(D+H)}(\overline{\mathcal{E}})$ is a complex of finite locally free $\mathcal{O}_X$-modules concentrated in degrees $[0, d]$, by considering the spectral sequence associated with the stupid ("bête") filtration, the last statement (4.2.11) follows from Serre vanishing and Serre duality, as desired. □

4.3. **De Rham local systems at the boundary.** In this subsection, we apply the results in Section 3.7 to study nearby cycles in some simple cases. We will leave a more general treatment to a future work.

Let X be an algebraic variety with a divisor D over k. Suppose that there exist an étale neighborhood $D \to W \to X$ and a morphism $f : W \to \mathbb{A}^1$ over k such that $f^{-1}(0) = D$. In this case, there is the notion of (quasi-unipotent) nearby cycles due to Beilinson (see [Bei87]; cf. [Rei10]). Let us briefly recall the definition. Let $\mathbb{G}_m := \mathbb{A}^1 - \{0\}$ be the multiplicative group scheme over k. We have a canonical isomorphism $\pi_1(\mathbb{G}_m, 1) \cong \pi_1(\mathbb{G}_{m,\overline{k}}, 1) \rtimes \mathrm{Gal}(\overline{k}/k)$, and $\pi_1(\mathbb{G}_{m,\overline{k}}, 1) \cong \widehat{\mathbb{Z}}(1)$ as $\mathrm{Gal}(\overline{k}/k)$-modules. For each $r \geq 1$, let $\mathbb{J}_r$ denote the rank r unipotent étale $\mathbb{Q}_p$-local system on $\mathbb{G}_m$ such that a topological generator $\gamma \in \pi_1(\mathbb{G}_{m,\overline{k}}, 1)$ acts as a principal unipotent matrix J_r and such that $\mathrm{Gal}(\overline{k}/k)$ acts on $\ker(J_r - 1)$ trivially. There is an obvious inclusion $\mathbb{J}_r \hookrightarrow \mathbb{J}_{r+1}$, and a projection $\mathbb{J}_{r+1} \to \mathbb{J}_r(-1)$ such that the composition $\mathbb{J}_r \to \mathbb{J}_r(-1)$ is given by the monodromy action. For each $m \geq 1$, let $[m]$ denote the m-th power homomorphism of $\mathbb{G}_m$, and let $\mathbb{K}_m := [m]_*(\mathbb{Q}_p)$. If $m \mid m'$, there is a natural inclusion $\mathbb{K}_m \hookrightarrow \mathbb{K}_{m'}$ (defined by adjunction). Let

$U := W - D$, and let $\imath : D \to W$ and $\jmath : U \to W$ denote the canonical morphisms. We shall also denote by $\mathbb{J}_r$ and $\mathbb{K}_m$ their pullbacks to U. Then for each $\mathbb{Q}_p$-perverse sheaf $\mathcal{F}$ on $U_{\text{ét}}$, its unipotent and quasi-unipotent nearby cycles are

$$R\Psi_f^u(\mathcal{F}) := \varinjlim_r \imath^{-1} R\jmath_*(\mathcal{F} \otimes_{\mathbb{Q}_p} \mathbb{J}_r) \quad \text{and} \quad R\Psi_f^{qu}(\mathcal{F}) := \varinjlim_m R\Psi_f^u(\mathcal{F} \otimes_{\mathbb{Q}_p} \mathbb{K}_m),$$

respectively, where the limits are taken in the category of perverse sheaves on $D_{\text{ét}}$.

Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $U_{\text{ét}}$. Let $f^{\text{an}} : W^{\text{an}} \to \mathbb{A}^{1,\text{an}}$ denote the analytification of f, whose pullback under $\mathbb{D} \hookrightarrow \mathbb{A}^{1,\text{an}}$ we denote by $f_{\mathbb{D}}^{\text{an}}$. If the reduced subspace of D^{an} is a normal crossings divisor in X^{an}, the quasi-unipotent nearby cycles $R\Psi_{f_{\mathbb{D}}^{\text{an}}}^{qu}(\mathbb{L}^{\text{an}})$ has been introduced in [DLLZ, Def. 6.4.1].

Lemma 4.3.1. *In the above setting, we have* $\big(R\Psi_f^{qu}(\mathbb{L})\big)^{\text{an}} \cong R\Psi_{f_{\mathbb{D}}^{\text{an}}}^{qu}(\mathbb{L}^{\text{an}})$.

Proof. This follows from [Hub96, Prop. 2.1.4 and Thm. 3.8.1] and [DLLZ, Lem. 4.5.4 and Thm. 4.6.1]. □

Suppose moreover that f is smooth, and that (F, ∇) is a vector bundle with an integrable connection on $U = W - D$ that (necessarily uniquely) extends to a vector bundle $\overline{F}$ on W with a log connection $\overline{\nabla}$ whose eigenvalues of residues along $\overline{D}$ belong to $\mathbb{Q} \cap [0, 1)$. Then we can define the $\mathbb{Q}$-filtration $V^\bullet$ on $F(*D) := \cup_n F(nD)$, as in Lemma 3.7.1; and $R\Psi_f^u(E, \nabla)$ and $R\Psi_f^{qu}(E, \nabla)$ (which are defined much more generally using the theory of holonomic algebraic D-modules) are canonically isomorphic to $\operatorname{gr}_V^0 F(*D)$ (resp. $\oplus_{\alpha \in (-1,0]} \big(\operatorname{gr}_V^\alpha F(*D)\big)$), with canonically induced integrable connections and filtrations (cf. [Sai88, (5.1.3.3)]). By [Nag62, Hir64a, Hir64b] again, we can compactify (W, D) to some $(\overline{W}, \overline{D})$, where $\overline{W}$ is proper, and where $\overline{D}$ is a simple normal crossings divisor such that $D = W \cap \overline{D}$ and the closure of D in $\overline{D}$ is a union of smooth irreducible components of $\overline{D}$. Since we have a log connection $D_{\text{dR,log}}^{\text{alg}}(\mathbb{L})$ over $\overline{W}$ as in Section 4.1, its restriction to W gives an extension of $D_{\text{dR}}^{\text{alg}}(\mathbb{L})$ as in the last paragraph, and hence we have $R\Psi_f^u\big(D_{\text{dR}}^{\text{alg}}(\mathbb{L})\big) \cong \operatorname{gr}_V^0 D_{\text{dR}}^{\text{alg}}(\mathbb{L})(*D)$ and $R\Psi_f^{qu}\big(D_{\text{dR}}^{\text{alg}}(\mathbb{L})\big) \cong \oplus_{\alpha \in (-1,0]} \big(\operatorname{gr}_V^\alpha D_{\text{dR}}^{\text{alg}}(\mathbb{L})(*D)\big)$.

Theorem 4.3.2. *Assume that f is smooth. Let $\mathbb{L}$ be a de Rham $\mathbb{Q}_p$-local system on $U_{\text{ét}}$. Then $R\Psi_f^{qu}(\mathbb{L})$ is a de Rham $\mathbb{Q}_p$-local system on $D_{\text{ét}}$, and there is a canonical isomorphism $D_{\text{dR}}^{\text{alg}}\big(R\Psi_f^{qu}(\mathbb{L})\big) \cong R\Psi_f^{qu}\big(D_{\text{dR}}^{\text{alg}}(\mathbb{L})\big)$ which restricts to an isomorphism $D_{\text{dR}}^{\text{alg}}\big(R\Psi_f^u(\mathbb{L})\big) \cong R\Psi_f^u\big(D_{\text{dR}}^{\text{alg}}(\mathbb{L})\big)$, as filtered (integrable) connections.*

Proof. As explained in Lemma 4.1.2, all the exponents of $D_{\text{dR}}^{\text{alg}}\big(R\Psi_f^{qu}(\mathbb{L})\big)$ are non-Liouville numbers. Moreover, since the eigenvalues of the residues of the log connection $D_{\text{dR,log}}^{\text{alg}}(\mathbb{L})$ on $\overline{W}$ along the irreducible divisors of $\overline{D}$ are all in $\mathbb{Q} \cap [0, 1)$, the exponents of the connection $D_{\text{dR,log}}^{\text{alg}}(\mathbb{L})|_D^0$ are also non-Liouville numbers. Thus, the theorem follows from the algebraization of the canonical isomorphisms in Theorem 3.7.11, by using Lemma 4.3.1 and the fact (1) in the proof of Lemma 4.1.2. □

Remark 4.3.3. As the monodromy of $\mathbb{L}$ along D is quasi-unipotent, and as the eigenvalues of the residue of $D_{\text{dR,log}}^{\text{alg}}(\mathbb{L})$ along D are in $\mathbb{Q} \cap [0, 1)$, the quasi-unipotent nearby cycles of $\mathbb{L}$ and $D_{\text{dR}}^{\text{alg}}(\mathbb{L})$ coincide with their respective full nearby cycles.

When X is a smooth curve over k, Theorem 4.3.2 has the following concrete interpretation. In this case, $D = x$ is a k-point, and $f = z$ is an étale local coordinate of X at x. We can identify $R\Psi_z(\mathbb{L})$ with the finite-dimensional $\mathbb{Q}_p$-representation $\mathbb{L}_{\overline{\eta}_x}$ of $\mathrm{Gal}(\overline{K}_x/K_x)$, where K_x is the local field around x, and $\overline{\eta}_x$ is a geometric point above $\eta_x = \mathrm{Spec}(K_x)$, which specializes to a geometric point $\overline{x} = \mathrm{Spec}(\overline{k})$ above x. The coordinate z splits the natural projection $\mathrm{Gal}(\overline{K}_x/K_x) \to \mathrm{Gal}(\overline{k}/k)$, and so we may regard $R\Psi_z(\mathbb{L})$ as a representation of $\mathrm{Gal}(\overline{k}/k)$.

Corollary 4.3.4. *If* $\mathbb{L}$ *is a de Rham* $\mathbb{Q}_p$*-local system on* $(X - x)_{\text{ét}}$*, then* $R\Psi_z(\mathbb{L})$ *is a de Rham representation of* $\mathrm{Gal}(\overline{k}/k)$ (*with the choice of coordinate* z).

5. Application to Shimura Varieties

In this section, we shall prove Theorem 1.5, which serves as an evidence of Conjecture 1.4, and also Corollary 1.6. Unless otherwise specified, the symbol K will be reserved for levels (rather than fields).

5.1. **The setup.** Let (G, X) be any Shimura datum. That is, G is a connected reductive $\mathbb{Q}$-group, and X is a hermitian symmetric domain parameterizing a conjugacy class of homomorphisms

$$h : \mathbf{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{\mathbb{R}}, \tag{5.1.1}$$

satisfying a list of axioms (see [Del79, 2.1.1] and [Mil05, Def. 5.5]). For each *neat* (see [Pin89, 0.6]) open compact subgroup K of $\mathrm{G}(\mathbb{A}_f)$, we denote by $\mathrm{Sh}_K = \mathrm{Sh}_K(\mathrm{G}, \mathsf{X})$ the canonical model of the associated Shimura variety at level K, which is a smooth quasi-projective algebraic variety over a number field $E \subset \mathbb{C}$, called the *reflex field* E of (G, X). Recall that, essentially by definition, the analytification of its base change $\mathrm{Sh}_{K,\mathbb{C}}$ from E to $\mathbb{C}$ is the *complex manifold*

$$\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}} \cong \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}_f)\big)/K, \tag{5.1.2}$$

where $\mathrm{G}(\mathbb{Q})$ acts diagonally on $\mathsf{X} \times \mathrm{G}(\mathbb{A}_f)$ from the left, and where K acts trivially on X and canonically on $\mathrm{G}(\mathbb{A}_f)$ from the right. Note that right multiplication by $g \in \mathrm{G}(\mathbb{A}_f)$ induces an isomorphism $[g] : \mathrm{Sh}_{gKg^{-1},\mathbb{C}}^{\mathrm{an}} \xrightarrow{\sim} \mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$, which algebraizes and descends to an isomorphism $\mathrm{Sh}_{gKg^{-1}} \xrightarrow{\sim} \mathrm{Sh}_K$, still denoted by $[g]$. (See [Mil05, Lan17] and the references there for basic facts concerning Shimura varieties.)

As in [Mil90, Ch. III], let G^c denote the quotient of G by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of G (as algebraic groups over $\mathbb{Q}$). For any subgroup of $\mathrm{G}(\mathbb{A})$ (including those of $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}_f)$, etc), we shall denote its image in $\mathrm{G}^c(\mathbb{A})$ with an additional superscript "c". In particular, given any open compact subgroup K of $\mathrm{G}(\mathbb{A}_f)$, we have an open compact subgroup K^c of $\mathrm{G}^c(\mathbb{A}_f)$. Given neat open compact subgroups K_1 and K_2 such that K_1 is a normal subgroup of K_2, we obtain a Galois finite étale cover $\mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$ with Galois group K_2^c/K_1^c. It will be convenient to consider the projective system $\{\mathrm{Sh}_K\}_K$, which can be viewed as the scheme $\mathrm{Sh} := \varprojlim_K \mathrm{Sh}_K$ over E, which admits the canonical right action of $\mathrm{G}(\mathbb{A}_f)$ described above. We call these actions (and their various extensions to other objects) *Hecke actions* of $\mathrm{G}(\mathbb{A}_f)$ (sometimes with $\mathrm{G}(\mathbb{A}_f)$ omitted).

Let $\mathrm{G}^{\mathrm{der}}$ denote the derived group of G, and let $\mathrm{G}^{\mathrm{der},c}$ denote the image of $\mathrm{G}^{\mathrm{der}}$ in G^c. Let G^{ad} denote the adjoint quotient of G. We have the canonical central isogenies $\mathrm{G}^{\mathrm{der}} \to \mathrm{G}^{\mathrm{der},c} \to \mathrm{G}^{\mathrm{ad}}$ of connected semisimple $\mathbb{Q}$-algebraic groups.

For any field F, let $\mathrm{Rep}_F(\mathrm{G}^c)$ denote the category of finite-dimensional algebraic representations of G^c over F, which we also view as an algebraic representation of G by pullback. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, together with a fixed isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, which induces an injective field homomorphism $\iota^{-1}|_{\overline{\mathbb{Q}}} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

5.2. **Local systems on Shimura varieties.** Let us begin with the complex analytic constructions. For any $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$, we define the (*Betti*) $\overline{\mathbb{Q}}$*-local system*

$$_{\mathrm{B}}\underline{V} := \mathrm{G}(\mathbb{Q})\backslash\big((\mathsf{X} \times V) \times \mathrm{G}(\mathbb{A}_f)\big)/K$$

over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$. (See Proposition 5.2.10 below for some formal properties.)

Let us also explain the construction of $_{\mathrm{B}}\underline{V}$ more concretely via the representation of the fundamental groups of (connected components) of $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$, under the classical correspondence between local systems and fundamental group representations.

Suppose that we have a connected component of $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ (see (5.1.2)) given by

$$\Gamma^+_{K,g_0}\backslash\mathsf{X}^+ \cong \mathrm{G}(\mathbb{Q})_+\backslash\big(\mathsf{X}^+ \times (\mathrm{G}(\mathbb{Q})_+ g_0 K)\big)/K \tag{5.2.1}$$

(cf. [Del79, 2.1.2] or [Mil05, Lem. 5.13]), where X^+ is a fixed connected component of X and $g_0 \in \mathrm{G}(\mathbb{A}_f)$, and where $\mathrm{G}(\mathbb{Q})_+$ is the stabilizer of X^+ in $\mathrm{G}(\mathbb{Q})$ and

$$\Gamma^+_{K,g_0} := \mathrm{G}(\mathbb{Q})_+ \cap (g_0 K g_0^{-1})$$

is a *neat* (see [Bor69, 17.1]) arithmetic subgroup of $\mathrm{G}(\mathbb{Q})$. It follows from the definitions that Γ^+_{K,g_0} is neat when K is. Let $\Gamma^{+,c}_{K,g_0}$ and $\Gamma^{+,\mathrm{ad}}_{K,g_0}$ denote the images of Γ^+_{K,g_0} in $\mathrm{G}^c(\mathbb{Q})$ and $\mathrm{G}^{\mathrm{ad}}(\mathbb{Q})$, respectively, so that we have surjective homomorphisms

$$\Gamma^+_{K,g_0} \twoheadrightarrow \Gamma^{+,c}_{K,g_0} \twoheadrightarrow \Gamma^{+,\mathrm{ad}}_{K,g_0}. \tag{5.2.2}$$

Lemma 5.2.3. *The subgroup* $\Gamma^{+,c}_{K,g_0}$ *of* $\mathrm{G}^c(\mathbb{Q})$ *is contained in* $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$*, and the second homomorphism in* (5.2.2) *is an isomorphism* $\Gamma^{+,c}_{K,g_0} \xrightarrow{\sim} \Gamma^{+,\mathrm{ad}}_{K,g_0}$.

Proof. Since $\ker(\mathrm{G} \to \mathrm{G}^c)$ is the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of G, the quotient $\mathrm{G}^c/\mathrm{G}^{\mathrm{der},c}$ is a torus isogenous to a product of a split torus and a torus of compact type (i.e., $\mathbb{R}$-anisotropic) over $\mathbb{Q}$. Since all neat arithmetic subgroups of such a torus are trivial, the neat image $\Gamma^{+,c}_{K,g_0}$ of Γ^+_{K,g_0} in $\mathrm{G}^c(\mathbb{Q})$ is contained in $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$. Consequently, the second homomorphism in (5.2.2) is an isomorphism, because its kernel, being both neat and *finite*, is trivial. □

Corollary 5.2.4. *The connected component* $\Gamma^+_{K,g_0}\backslash\mathsf{X}^+$ *is a smooth manifold whose fundamental group* (*with any base point of* X^+) *is canonically isomorphic to* $\Gamma^{+,c}_{K,g_0}$.

Proof. As Γ^+_{K,g_0} acts on X^+ via $\Gamma^{+,\mathrm{ad}}_{K,g_0} \subset \mathrm{G}^{\mathrm{ad}}(\mathbb{Q})$, this follows from Lemma 5.2.3. □

Remark 5.2.5. We shall not write $\Gamma^{+,\mathrm{ad}}_{K,g_0}$ again in what follows.

By taking X^+ as a universal cover of $\Gamma^+_{K,g_0}\backslash\mathsf{X}^+$, and by fixing the choice of a base point on X^+, the pullback of $_{\mathrm{B}}\underline{V}$ to $\Gamma^+_{K,g_0}\backslash\mathsf{X}^+$ determines and is determined by the fundamental group representation

$$\rho^+_{K,g_0}(V) : \Gamma^{+,c}_{K,g_0} \to \mathrm{GL}_{\overline{\mathbb{Q}}}(V), \tag{5.2.6}$$

which coincides with the restriction of the representation of G^c on V. In particular, it is compatible with the change of levels $K' \subset K$.

Moreover, given $g \in g_0^{-1}\mathrm{G}(\mathbb{Q})_+ g_0$, so that $g_0 g = \gamma g_0$ for some $\gamma \in \mathrm{G}(\mathbb{Q})_+$, we have $\Gamma^{+,c}_{gKg^{-1},g_0} = \gamma \Gamma^{+,c}_{K,g_0} \gamma^{-1}$, and the Hecke action $[g]$ induces a morphism

$$\Gamma^{+}_{gKg^{-1},g_0} \backslash \mathsf{X}^+ \xrightarrow{\sim} \Gamma^{+}_{K,g_0} \backslash \mathsf{X}^+, \tag{5.2.7}$$

which is nothing but the isomorphism defined by left multiplication by γ^{-1}. It follows that the canonical isomorphism $[g]^*({}_{\mathrm{B}}\underline{V}) \xrightarrow{\sim} {}_{\mathrm{B}}\underline{V}$ of local systems corresponds to the following equality of fundamental group representations

$$\rho^{+}_{gKg^{-1},g_0}(V) = \gamma\big(\rho^{+}_{K,g_0}(V)\big), \tag{5.2.8}$$

where $\gamma\big(\rho^{+}_{K,g_0}(V)\big)$ means the representation of $\Gamma^{+,c}_{gKg^{-1},g_0} = \gamma \Gamma^{+,c}_{K,g_0} \gamma^{-1}$ defined by conjugating the values of $\rho^{+}_{K,g_0}(V)$ by γ in $\mathrm{GL}(V)$.

Now, by base change along $\overline{\mathbb{Q}} \subset \mathbb{C}$ via the canonical homomorphism, we obtain the object $V_{\mathbb{C}} := V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{G}^c)$, as well as the $\mathbb{C}$-local system

$${}_{\mathrm{B}}\underline{V}_{\mathbb{C}} := {}_{\mathrm{B}}\underline{V} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

over $\mathrm{Sh}_K(\mathbb{C})$, which via the classical Riemann–Hilbert correspondence (as reviewed in the introduction) corresponds to the (complex analytic) integrable connection

$$({}_{\mathrm{dR}}\underline{V}^{\mathrm{an}}_{\mathbb{C}}, \nabla) := ({}_{\mathrm{B}}\underline{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}}, 1 \otimes d).$$

Moreover, any $h \in \mathsf{X}$ (as in (5.1.1)) induces a homomorphism $h_{\mathbb{C}} : \mathbf{G}_{\mathrm{m},\mathbb{C}} \times \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{\mathbb{C}}$, whose restriction to the first factor defines the so-called *Hodge cocharacter*

$$\mu_h : \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{\mathbb{C}}, \tag{5.2.9}$$

inducing a (decreasing) filtration $\mathrm{Fil}^\bullet$ on ${}_{\mathrm{dR}}\underline{V}^{\mathrm{an}}_{\mathbb{C}}$ satisfying the Griffiths transversality condition. Then we obtain a *filtered integrable connection*$({}_{\mathrm{dR}}\underline{V}^{\mathrm{an}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$.

Let $\mathrm{Sh}^{\mathrm{tor}}_K$ be a toroidal compactification of Sh_K (as in [Pin89]), which we assume to be projective and smooth, with the boundary divisor $D := \mathrm{Sh}^{\mathrm{tor}}_K - \mathrm{Sh}_K$ (with its reduced subscheme structure) a normal crossings divisor, whose base change from E to $\mathbb{C}$ and whose further complex analytification are denoted by $\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}$ and $\mathrm{Sh}^{\mathrm{tor,an}}_{K,\mathbb{C}}$, respectively. As explained in [LS13, Sec. 6.1], ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ has *unipotent monodromy* along $D^{\mathrm{an}}_{\mathbb{C}}$. Therefore, by [Del70, II, 5] and [Kat71, Sec. VI and VII], $({}_{\mathrm{dR}}\underline{V}^{\mathrm{an}}_{\mathbb{C}}, \nabla)$ uniquely extends to an integrable log connection $({}_{\mathrm{dR}}\underline{V}^{\mathrm{can,an}}_{\mathbb{C}}, \nabla)$, with *nilpotent residues* along $D^{\mathrm{an}}_{\mathbb{C}}$. By [Del70, II, 5.2(d)], $V \mapsto ({}_{\mathrm{dR}}\underline{V}^{\mathrm{can,an}}_{\mathbb{C}}, \nabla)$ defines a tensor functor from $\mathrm{Rep}_{\mathbb{C}}(\mathrm{G}^c)$ to the category of integrable log connections on $\mathrm{Sh}^{\mathrm{tor,an}}_{K,\mathbb{C}}$. Moreover, by [Sch73] (see also [CKS87]), the filtration $\mathrm{Fil}^\bullet$ on ${}_{\mathrm{dR}}\underline{V}^{\mathrm{an}}_{\mathbb{C}}$ uniquely extends to a filtration on ${}_{\mathrm{dR}}\underline{V}^{\mathrm{can,an}}_{\mathbb{C}}$ (by subbundles), still denote by $\mathrm{Fil}^\bullet$. The extended ∇ and $\mathrm{Fil}^\bullet$ still satisfy the Griffiths transversality, and therefore $({}_{\mathrm{dR}}\underline{V}^{\mathrm{can,an}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ is an analytic *filtered log connection*. By GAGA (see the proof of [Del70, II, 5.9]), this triple canonically algebraizes to an algebraic filtered log connection

$$({}_{\mathrm{dR}}\underline{V}^{\mathrm{can}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet).$$

(These ${}_{\mathrm{dR}}\underline{V}^{\mathrm{can,an}}_{\mathbb{C}}$ and ${}_{\mathrm{dR}}\underline{V}^{\mathrm{can}}_{\mathbb{C}}$ agree with the canonical extensions defined differently in [Har89, Sec. 4], and also [Har90] and [Mil90].) The restriction of $({}_{\mathrm{dR}}\underline{V}^{\mathrm{can}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ then defines an algebraic filtered regular connection

$$({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$$

over $\mathrm{Sh}_{K,\mathbb{C}}$, whose complex analytification is isomorphic to $({}_{\mathrm{dR}}\underline{V}^{\mathrm{an}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$. We call $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla)$ the *automorphic vector bundle* associated with $V_{\mathbb{C}}$. We summarize the above discussions as the following:

Proposition 5.2.10. *The assignment of* ${}_{\mathrm{B}}\underline{V}$ *(resp.* $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$*) to* V *defines a tensor functor from* $\mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$ *to the category of* $\mathrm{G}(\mathbb{A}_f)$*-equivariant* $\overline{\mathbb{Q}}$*-local systems (resp. filtered regular connections) on* $\{\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}\}_K$ *(resp.* $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$*), which is functorial with respect to pullbacks under morphisms between Shimura varieties induced by morphisms between Shimura data. Hence, the assignment of* $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla)$ *(resp.* $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$*) to* V *defines a* $\mathrm{G}(\mathbb{A}_f)$*-equivariant* G^c*-bundle with an integrable connection* $(\mathcal{E}_{\mathbb{C}}, \nabla)$ *(resp. a* $\mathrm{P}^c_{\mathbb{C}}$*-bundle* $\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}}$*) over* $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$*, where* $\mathrm{P}^c_{\mathbb{C}}$ *is the parabolic subgroup of* $\mathrm{G}^c_{\mathbb{C}}$ *defined by some* μ_h *as in* (5.2.9) *(cf.* [LZ17, Rem. 4.1(i)]*). By forgetting filtrations, we obtain a* $\mathrm{G}(\mathbb{A}_f)$*-equivariant morphism* $\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}} \to \mathcal{E}_{\mathbb{C}}$*.*

Remark 5.2.11. As explained in [LZ17, Rem. 4.1(i)], the conjugacy class of μ_h as in (5.2.9) defines a partial flag variety $\mathcal{F}\ell_{\mathbb{C}} \cong \mathrm{G}^c_{\mathbb{C}}/\mathrm{P}^c_{\mathbb{C}}$ parameterizing the associated conjugacy class of parabolic subgroups, which depends only on the Shimura datum (G, X) and descends to a partial flag variety $\mathcal{F}\ell$ of G^c over the reflex field E. Let $\mathcal{E}_{\mathcal{F}\ell_{\mathbb{C}}} := \mathcal{E}_{\mathbb{C}} \times^{\mathrm{G}^c_{\mathbb{C}}} \mathcal{F}\ell_{\mathbb{C}}$. Then any filtrations as in Proposition 5.2.10 define a section of $\mathcal{E}_{\mathcal{F}\ell_{\mathbb{C}}}$ over $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$. For any particular choice of $\mathrm{P}^c_{\mathbb{C}}$ in $\mathcal{F}\ell(\mathbb{C})$, this section amounts to the reduction of $\mathcal{E}_{\mathbb{C}}$ to a $\mathrm{P}^c_{\mathbb{C}}$-bundle $\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}}$ as in Proposition 5.2.10. Moreover, given the canonical model $(\mathcal{E}, \nabla)$ of $(\mathcal{E}_{\mathbb{C}}, \nabla)$ as in [Mil90, Ch. III, Thm. 4.3], we have the canonical model $\mathcal{E}_{\mathcal{F}\ell} := \mathcal{E} \times^{\mathrm{G}^c} \mathcal{F}\ell$ of $\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}} \cong \mathcal{E}_{\mathcal{F}\ell_{\mathbb{C}}}$, over Sh_K.

Next, let us turn to the p-adic analytic constructions. Given any $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$ as above, by base change via $\iota^{-1}|_{\overline{\mathbb{Q}}} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we obtain the object $V_{\overline{\mathbb{Q}}_p} := V \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p$ in $\mathrm{Rep}_{\overline{\mathbb{Q}}_p}(\mathrm{G}^c)$. As explained in [LS18b, Sec. 3] (see also [LZ17, Sec. 4.2]), given such a finite-dimensional representation $V_{\overline{\mathbb{Q}}_p}$ of G^c over $\overline{\mathbb{Q}}_p$, there is a canonical *automorphic* $\overline{\mathbb{Q}}_p$*-étale local system* ${}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$ (i.e., lisse $\overline{\mathbb{Q}}_p$-étale sheaf) over Sh_K (with stalks isomorphic to $V_{\overline{\mathbb{Q}}_p}$). In fact, by the very construction of ${}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$, for each finite extension L of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ such that $V_{\overline{\mathbb{Q}}_p}$ has a model V_L over L, we have an L-étale local system ${}_{\text{ét}}\underline{V}_L$ over Sh_K (with stalks isomorphic to V_L) such that

$$
{}_{\text{ét}}\underline{V}_L \otimes_L \overline{\mathbb{Q}}_p \cong {}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}. \tag{5.2.12}
$$

In addition, by [AGV73, XI, 4.4] (or by using the canonical homomorphism from the fundamental group to the étale fundamental group), its pullback to $\mathrm{Sh}_{K,\mathbb{C}}$ induces a $\overline{\mathbb{Q}}_p$-local system ${}_{\mathrm{B}}\underline{V}_{\overline{\mathbb{Q}}_p}$, together with a canonical isomorphism

$$
{}_{\mathrm{B}}\underline{V}_{\overline{\mathbb{Q}}_p} \otimes_{\overline{\mathbb{Q}}_p, \iota} \mathbb{C} \cong {}_{\mathrm{B}}\underline{V}_{\mathbb{C}}. \tag{5.2.13}
$$

Note that this implies that ${}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$ has unipotent geometric monodromy along $D_{\overline{\mathbb{Q}}}$.

Suppose that $V_{\overline{\mathbb{Q}}_p}$ has a model V_L over a finite extension L of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. Let k be a finite extension of the composite of L and the image of $E \overset{\text{can.}}{\hookrightarrow} \overline{\mathbb{Q}} \overset{\iota^{-1}}{\hookrightarrow} \overline{\mathbb{Q}}_p$ in $\overline{\mathbb{Q}}_p$. Let us denote with an additional subscript "k" (resp. "$\overline{\mathbb{Q}}_p$") the base changes of Sh_K etc from E to k (resp. $\overline{\mathbb{Q}}_p$) via the above composition. We will adopt a similar notation for sheaves. We can view the L-étale local system ${}_{\text{ét}}\underline{V}_L$ as a $\mathbb{Q}_p$-étale local system with compatible L-actions. By [LZ17, Thm. 1.2], the pullback of ${}_{\text{ét}}\underline{V}_L$ to $\mathrm{Sh}_{K,k}$, which we still denote by the same symbols, is de Rham. By working as in Section 4.1, and by pushing out via the multiplication homomorphism

$$
\tau : L \otimes_{\mathbb{Q}_p} k \to k : \ a \otimes b \mapsto ab, \tag{5.2.14}
$$

we obtain a filtered log connection $({}_{p\text{-dR}}\underline{V}_k^{\mathrm{can}} := D^{\mathrm{alg}}_{\mathrm{dR},\log}({}_{\text{ét}}\underline{V}_L) \otimes_{(L\otimes_{\mathbb{Q}_p} k),\tau} k, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}^{\mathrm{tor}}_{K,k}$, which has nilpotent residues along D_k by [DLLZ, Cor. 6.4.4], Theorem 3.2.12, and GAGA (see [Köp74]); and also a filtered regular connection $({}_{p\text{-dR}}\underline{V}_k := D^{\mathrm{alg}}_{\mathrm{dR}}({}_{\text{ét}}\underline{V}_L) \otimes_{(L\otimes_{\mathbb{Q}_p} k),\tau} k, \nabla, \mathrm{Fil}^\bullet)$. These constructions are compatible with replacements of L and k with extension fields satisfying the same conditions. Thus, we can canonically assign to each ${}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$ as above the filtered log connection

$$({}_{p\text{-dR}}\underline{V}^{\mathrm{can}}_{\overline{\mathbb{Q}}_p}, \nabla, \mathrm{Fil}^\bullet) := ({}_{p\text{-dR}}\underline{V}^{\mathrm{can}}_k, \nabla, \mathrm{Fil}^\bullet) \otimes_k \overline{\mathbb{Q}}_p \tag{5.2.15}$$

over $\mathrm{Sh}^{\mathrm{tor}}_{K,\overline{\mathbb{Q}}_p}$, whose restriction to $\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}$ is the filtered regular connection

$$({}_{p\text{-dR}}\underline{V}_{\overline{\mathbb{Q}}_p}, \nabla, \mathrm{Fil}^\bullet) := ({}_{p\text{-dR}}\underline{V}_k, \nabla, \mathrm{Fil}^\bullet) \otimes_k \overline{\mathbb{Q}}_p. \tag{5.2.16}$$

Both (5.2.15) and (5.2.16) are independent of the choices of L and k for a given V.

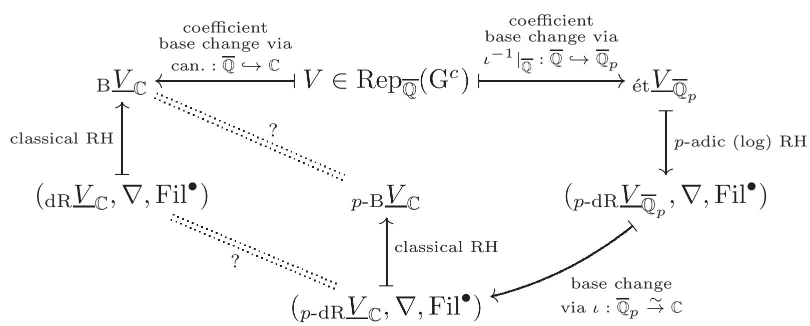
Since $({}_{p\text{-dR}}\underline{V}_{\overline{\mathbb{Q}}_p}, \nabla, \mathrm{Fil}^\bullet)$ is *algebraic*, its base change under $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ is a filtered regular connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{C}}$, the horizontal sections of whose *complex analytification* defines a $\mathbb{C}$-local system ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$. Since $D^{\mathrm{alg}}_{\mathrm{dR}}$ is a tensor functor uniquely determined by D_{dR} via the analytification functor (see Lemma 4.1.2 and its proof), by [LZ17, Thm. 3.9(ii)], we obtain the following:

Proposition 5.2.17. *The analogue of Proposition* 5.2.10 *holds for the assignments of* ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$*,* $({}_{p\text{-dR}}\underline{V}_{\overline{\mathbb{Q}}_p}, \nabla, \mathrm{Fil}^\bullet)$*, and* $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$*. In particular, they define a* $\mathrm{G}(\mathbb{A}_f)$*-equivariant* G^c*-bundle with an integrable connection* $({}_p\mathcal{E}_{\mathbb{C}}, {}_p\nabla)$ *(resp. a* $\mathrm{P}^c_{\mathbb{C}}$*-bundle* ${}_p\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}}$*) over* $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$*, with a* $\mathrm{G}(\mathbb{A}_f)$*-equivariant morphism* ${}_p\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}} \to {}_p\mathcal{E}_{\mathbb{C}}$*.*

Likewise, the base change of $({}_{p\text{-dR}}\underline{V}^{\mathrm{can}}_{\overline{\mathbb{Q}}_p}, \nabla, \mathrm{Fil}^\bullet)$ under ι is a filtered log connection $({}_{p\text{-dR}}\underline{V}^{\mathrm{can}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}$, with nilpotent residues along $D^{\mathrm{an}}_{\mathbb{C}}$. The analogues of Proposition 5.2.17 for $({}_{p\text{-dR}}\underline{V}^{\mathrm{can}}_{\overline{\mathbb{Q}}_p}, \nabla, \mathrm{Fil}^\bullet)$ and $({}_{p\text{-dR}}\underline{V}^{\mathrm{can}}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ also hold.

Remark 5.2.18 (cf. Remark 5.2.11)**.** By construction (based on (5.2.16)), $({}_p\mathcal{E}_{\mathbb{C}}, {}_p\nabla)$ (resp. ${}_p\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}} \cong {}_p\mathcal{E}_{\mathcal{F}\ell_{\mathbb{C}}}$) canonically admits a model $({}_p\mathcal{E}_k, {}_p\nabla)$ (resp. ${}_p\mathcal{E}_{\mathcal{F}\ell_k}$) over $\mathrm{Sh}_{K,k}$, where k is the completion of E at the place determined by ι.

5.3. **Statements of theorems.** It is natural to ask whether the Betti local systems ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ (resp. the filtered connections $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ and $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$) over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ (resp. $\mathrm{Sh}_{K,\mathbb{C}}$) are canonically isomorphic to each other, as in the following summarizing diagram:



The following theorem provides affirmative (and finer) answers:

Theorem 5.3.1. *We have canonical isomorphisms* ${}_{p\text{-B}}\underline{V}_{\mathbb{C}} \cong {}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ *over* $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$ *and* $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet) \cong ({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ *over* $\mathrm{Sh}_{K,\mathbb{C}}$*, compatible with each other under the complex Riemann–Hilbert correspondence. Furthermore, we have canonical* $\mathrm{G}(\mathbb{A}_f)$*-equivariant isomorphisms between the relevant pairs of tensor functors in Propositions* 5.2.17 *and* 5.2.10*, compatible with pullbacks under morphisms between Shimura varieties induced by morphisms of Shimura data, inducing compatible canonical* $\mathrm{G}(\mathbb{A}_f)$*-equivariant isomorphisms* $({}_p\mathcal{E}_{\mathbb{C}}, {}_p\nabla) \cong (\mathcal{E}_{\mathbb{C}}, \nabla)$ *and* ${}_p\mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}} \cong \mathcal{E}_{\mathrm{P}^c_{\mathbb{C}}}$.

These isomorphisms are compatible with the formation of canonical models in the sense that they descend to canonical $\mathrm{G}(\mathbb{A}_f)$*-equivariant isomorphisms* $({}_p\mathcal{E}_k, {}_p\nabla) \cong (\mathcal{E}_E, \nabla) \otimes_E k$ *and* ${}_p\mathcal{E}_{\mathcal{F}\ell_k} \cong \mathcal{E}_{\mathcal{F}\ell} \otimes_E k$*, respectively* (*see Remarks* 5.2.11 *and* 5.2.18).

The analogous assertions hold for the filtered log connections $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}}, \nabla, \mathrm{Fil}^\bullet)$ *and* $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}}, \nabla, \mathrm{Fil}^\bullet)$ (*and the associated torsors*).

The proofs of Theorem 5.3.1 will be given in the remaining subsections. Note that it verifies, in particular, the conjecture in [LZ17, Rem. 4.1(ii)].

Assuming this theorem for the moment, since every irreducible algebraic representation of G^c over $\overline{\mathbb{Q}}_p$ has a model over $\overline{\mathbb{Q}}$, we obtain the following:

Corollary 5.3.2. *Theorem* 1.5 *also holds.*

Next, we turn to Corollary 1.6. Consider the (p-adic) analytic $\mathrm{G}_k^{c,\mathrm{an}}$-torsor with the integrable connection $({}_p\mathcal{E}_k^{\mathrm{an}}, {}_p\nabla)$ over $\mathrm{Sh}_{K,k}^{\mathrm{an}}$ defined by the assignment of the p-adic analytification ${}_{p\text{-dR}}\underline{V}_k^{\mathrm{an}}$ of ${}_{p\text{-dR}}\underline{V}_k$ to $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G}^c)$, where k is the completion of E with respect to the p-adic place determined by ι. As in Remark 5.2.11, the filtrations on ${}_{p\text{-dR}}\underline{V}_k^{\mathrm{an}}$, for all V, define a section of ${}_p\mathcal{E}_{\mathcal{F}\ell_k}^{\mathrm{an}} := {}_p\mathcal{E}_k^{\mathrm{an}} \times^{\mathrm{G}_k^{c,\mathrm{an}}} \mathcal{F}\ell_k^{\mathrm{an}}$ over $\mathrm{Sh}_{K,k}^{\mathrm{an}}$. Now let $x \in \mathrm{Sh}_K(k')$, where k' is a finite extension of k in $\overline{\mathbb{Q}}_p$. Then there is an analytic neighborhood U of x in $\mathrm{Sh}_{K,k'}^{\mathrm{an}}$ trivializing (the pullback of) $({}_p\mathcal{E}_k^{\mathrm{an}}, {}_p\nabla)$ as a $\mathrm{G}_{k'}^{c,\mathrm{an}}$-torsor with an integrable connection. Then the above section of ${}_p\mathcal{E}_{\mathcal{F}\ell_k}^{\mathrm{an}}$ over $\mathrm{Sh}_{K,k}^{\mathrm{an}}$ defines the so-called *Grothendieck–Messing period map* $\pi_{\mathrm{GM}} : U \to \mathcal{F}\ell_{k'}^{\mathrm{an}}$.

Corollary 5.3.3 (restatement of Corollary 1.6)**.** *This morphism* π_{GM} *is étale.*

Proof. Let $\mathcal{O}_{U,x}^\wedge$ (resp. $\mathcal{O}_{\mathcal{F}\ell,\pi_{\mathrm{GM}}(x)}^\wedge$) denote the completion of the local ring $\mathcal{O}_{U,x}$ (resp. $\mathcal{O}_{\mathcal{F}\ell_{k'}^{\mathrm{an}},\pi_{\mathrm{GM}}(x)}$). It suffices to show that $\mathcal{O}_{\mathcal{F}\ell,\pi_{\mathrm{GM}}(x)}^\wedge \to \mathcal{O}_{U,x}^\wedge$ is an isomorphism (cf. [Hub96, Prop. 1.7.5]). By Theorem 5.3.1, its pullback via $k' \overset{\mathrm{can.}}{\hookrightarrow} \overline{\mathbb{Q}}_p \overset{\iota}{\underset{\sim}{\to}} \mathbb{C}$ can be identified (via the algebraic construction) with the corresponding homomorphism for the usual complex analytic period map, which is an isomorphism because the complex analytic period map is locally given by the Borel embedding $\mathsf{X}^+ \hookrightarrow \mathcal{F}\ell_{\mathbb{C}}^{\mathrm{an}}$ (see [Mil90, Ch. III, Sec. 1] and [Hel01, Ch. VIII, Sec. 7]), as desired. □

Remark 5.3.4. Theorem 5.3.1 and Corollary 5.3.3 are not surprising when there are families of motives whose relative Betti, de Rham, and p-adic étale realizations define the local systems ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$, $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$, and ${}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$, respectively. This is the case, for example, when Sh_K is a Shimura variety of Hodge type. (We will take advantage of this in Section 5.5 below.) But Theorem 5.3.1 and Corollary 5.3.3 also apply to Shimura varieties associated with exceptional groups, over which there are (as yet) no known families of motives defining our local systems as above.

Remark 5.3.5. By Theorem 5.3.1 and Deligne's comparison result in [Del70, II, 6], and by Theorem 4.1.4, the spectral sequence

$$E_1^{a,b} = H_{\log \mathrm{Hodge}}^{a,b}(\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{tor}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}}) \Rightarrow H_{\log \mathrm{dR}}^{a+b}(\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{tor}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}^{\mathrm{can}}) \cong H_{\mathrm{dR}}^{a+b}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}})$$

degenerates on the E_1 page. While this degeneration was already known thanks to Saito's direct image theorem (see [Sai90, Thm. 2.14] and [Suh18, Sec. 4]), we have a new proof here based on p-adic Hodge theory. Also, we can determine the Hodge–Tate weights of $H^i(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p})$ in terms of $\dim_{\mathbb{C}} H^{a,i-a}_{\log \mathrm{Hodge}}(\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}^{\mathrm{can}}_{\mathbb{C}})$, for all $a \in \mathbb{Z}$, which can be computed using the *dual BGG decomposition* and relative Lie algebra cohomology. (We will explain these in more detail in [LLZ].)

Remark 5.3.6. The comparison isomorphisms in Theorem 5.3.1 over Shimura varieties induce similar isomorphisms on general locally symmetric varieties, by pullback and by finite étale descent. Consequently, the analogue of the statements in Remark 5.3.5 for general locally symmetric varieties also hold.

Remark 5.3.7. By replacing the input [Suh18] in the proof of [Lan16b, Thm. 4.3] with Theorem 4.2.1, and by Theorem 5.3.1 and Remark 5.3.6, we obtain new p-adic Hodge-theoretic proofs of the vanishing results for the coherent and de Rham cohomology in [Lan16b, Thm. 4.1, 4.4, 4.7, and 4.10], generalizing the characteristic-zero cases of previous results in [LS12, LS13, LS14, Lan16a].

5.4. **Proofs of theorems: preliminary reductions.** Let us fix a connected component $\Gamma^+_{K,g_0}\backslash \mathsf{X}^+$ of $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ as in (5.2.1), which is the analytification of a quasi-projective variety defined over some finite extension E^+ of E in $\overline{\mathbb{Q}}$. Let $h \in \mathsf{X}^+$ be a *special point* such that (5.1.1) factors through $\mathrm{T}_{\mathbb{R}}$ for some maximal torus T of G (over $\mathbb{Q}$). (Recall that special points are dense in X^+—see the proof of [Mil05, Lem. 13.5].) Up to replacing E^+ with a finite extension in $\overline{\mathbb{Q}}$, we may assume that the image of $h \in \mathsf{X}^+$ in $\Gamma^+_{K,g_0}\backslash \mathsf{X}^+$ is defined over E^+ (see [Mil05, Lem. 13.4]).

The pullbacks of ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ to $h \in \mathsf{X}^+$ can be canonically identified with $V_{\mathbb{C}}$ by its very construction. On the other hand, the pullback of ${}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}}$ can also be canonically identified with $V_{\mathbb{C}}$. In fact, in both cases, we have slightly more:

Proposition 5.4.1. *The pullbacks of* ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ *and* ${}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}}$ *to* $(\mathrm{G}(\mathbb{Q})h) \times \mathrm{G}(\mathbb{A}_f)$ *are canonically and* $\mathrm{G}(\mathbb{Q}) \times \mathrm{G}(\mathbb{A}_f)$*-equivariantly isomorphic to the trivial local system* $(\mathrm{G}(\mathbb{Q})h) \times V_{\mathbb{C}} \times \mathrm{G}(\mathbb{A}_f)$ *(on which* $\mathrm{G}(\mathbb{Q})$ *acts by diagonal left multiplication on all three factors, and* $\mathrm{G}(\mathbb{A}_f)$ *acts by right multiplication on the last factor.)*

Proof. The assertion for ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ follows from its very construction. As for the assertion for ${}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}}$, let us first identify the pullback of ${}_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$ to the images of (h, g), for $g \in \mathrm{G}(\mathbb{A}_f)$, by recalling the arguments in the proof of [LZ17, Lem. 4.8]. We shall write $\Gamma^+_{K,g} := \mathrm{G}(\mathbb{Q})_+ \cap (gKg^{-1})$ (cf. (5.2.1)), so that $\Gamma^+_{K,g}\backslash \mathsf{X}^+$ gives the connected component of $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ containing the image of (h, g).

By assumption, $h : \mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{\mathbb{R}}$ (as in (5.1.1)) factors through $\mathrm{T}_{\mathbb{R}}$, and the Hodge cocharacter (as in (5.2.9)) induces a cocharacter $\mu_h : \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{T}_{\mathbb{C}}$, which is the base change of some cocharacter $\mu : \mathbf{G}_{\mathrm{m},F} \to \mathrm{T}_F$ defined over some number field F in $\overline{\mathbb{Q}}$. Then the composition of μ with the norm map from T_F to T defines a homomorphism $\mathrm{N}\,\mu : \mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{G}_{\mathrm{m},F} \to \mathrm{T}$ of tori over $\mathbb{Q}$, and we have a composition of homomorphisms $F^\times\backslash \mathbb{A}_F^\times \stackrel{\mathrm{N}\,\mu}{\to} \mathrm{T}(\mathbb{Q})\backslash \mathrm{T}(\mathbb{A}) \to \overline{\mathrm{T}(\mathbb{Q})}\backslash \mathrm{T}(\mathbb{A}_f)$, where $\overline{\mathrm{T}(\mathbb{Q})}$ denotes the closure of $\mathrm{T}(\mathbb{Q})$ in $\mathrm{T}(\mathbb{A}_f)$, which factors through

$$F^\times\backslash \mathbb{A}_F^\times \stackrel{\mathrm{Art}_F}{\to} \mathrm{Gal}(F^{\mathrm{ab}}/F) \stackrel{r(\mu)}{\to} \overline{\mathrm{T}(\mathbb{Q})}\backslash \mathrm{T}(\mathbb{A}_f), \tag{5.4.2}$$

where F^{ab} is the maximal abelian extension of F in $\overline{\mathbb{Q}}$. If $F_{K,g}$ is the subfield of F^{ab} such that $\mathrm{Gal}(F^{\mathrm{ab}}/F_K)$ is the preimage of $(gKg^{-1} \cap \overline{\mathrm{T}(\mathbb{Q})})\backslash(gKg^{-1} \cap \mathrm{T}(\mathbb{A}_f))$

under (5.4.2), then we have an induced Galois representation

$$r(\mu)^+_{K,g} : \mathrm{Gal}(\overline{\mathbb{Q}}/F_{K,g}) \to (gKg^{-1} \cap \overline{\mathrm{T}(\mathbb{Q})}) \backslash (gKg^{-1} \cap \mathrm{T}(\mathbb{A}_f)). \tag{5.4.3}$$

(If $g = g_0$, then $F_{K,g} \subset E^+$, since the image of h in $\Gamma^+_{K,g_0} \backslash \mathsf{X}^+$ is defined over E^+.)

Since $\mathrm{T}_\mathbb{R}$ stabilizes the special point h, it is $\mathbb{R}$-anisotropic modulo the center of G, and hence its maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus is the same as that of the center of G. Therefore, as explained in the proof of [LZ17, Lem. 4.5], the pullback of V to T satisfies the requirement that its restriction to $gKg^{-1} \cap \overline{\mathrm{T}(\mathbb{Q})}$ is trivial as in [LZ17, (4.4)] (with the neatness of gKg^{-1} here implying that the open compact subgroup K there is sufficiently small). Thus, the composition $\mathrm{T}(\mathbb{A}_f) \to \mathrm{T}(\mathbb{Q}_p) \to \mathrm{G}(\mathbb{Q}_p) \to \mathrm{GL}_{\overline{\mathbb{Q}}_p}(V_{\overline{\mathbb{Q}}_p})$ factors through $(gKg^{-1} \cap \overline{\mathrm{T}(\mathbb{Q})}) \backslash \mathrm{T}(\mathbb{A}_f)$ and induces, by composition with (5.4.3), a Galois representation $r(\mu, V)^+_{K,g,p}$: $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{K,g}) \to \mathrm{GL}_{\overline{\mathbb{Q}}_p}(V_{\overline{\mathbb{Q}}_p})$ describing the pullback of $_{\text{ét}}\underline{V}_{\overline{\mathbb{Q}}_p}$ to the geometric point above the image of h in $\Gamma^+_{K,g} \backslash \mathsf{X}^+$ given by the composition of $F_{K,g} \stackrel{\text{can.}}{\hookrightarrow} \overline{\mathbb{Q}} \stackrel{\iota^{-1}}{\hookrightarrow} \overline{\mathbb{Q}}_p$.

Let L, V_L, k, and τ be as in Section 5.2. Without loss of generality, we may assume that k also contains the image of $F_{K,g}$ in $\overline{\mathbb{Q}}_p$ (via the above composition). Then the image of $r(\mu, V)^+_{K,g,p}$ is contained in the subgroup $\mathrm{GL}_L(V_L)$ of $\mathrm{GL}_{\overline{\mathbb{Q}}_p}(V_{\overline{\mathbb{Q}}_p})$, and we can view this representation over L as a representation over $\mathbb{Q}_p$ with an additional action of L, as usual. By [LZ17, Lem. 4.4], this representation is potentially crystalline. Since this representation factors through the abelian group $\mathrm{T}(\mathbb{Q}_p)$, by [FM97, §6, Prop.] (which is based on [Ser68, Ch. III, Sec. A.7, Thm. 3] and [DMOS82, Ch. IV, Prop. D.1]; see also [Hen82]), it is the p-adic étale realization of an object in the Tannakian subcategory of motives over $F_{K,g}$ generated by Artin motives and potentially CM abelian varieties. By construction, the pullback of $_{p\text{-dR}}\underline{V}_\mathbb{C}$ to h is the pushout of the corresponding de Rham realization via $L \otimes_{\mathbb{Q}_p} k \xrightarrow{\tau} k \stackrel{\text{can.}}{\hookrightarrow} \overline{\mathbb{Q}}_p \stackrel{\iota}{\xrightarrow{\sim}} \mathbb{C}$. Hence, by comparison with the Betti realization, we can canonically identify the pullback of $_{p\text{-B}}\underline{V}_\mathbb{C}$ to h with $V_\mathbb{C}$.

By the same argument as above, we can also canonically identify the pullback of $_{p\text{-B}}\underline{V}_\mathbb{C}$ to γh with $V_\mathbb{C}$, for each $\gamma \in \mathrm{G}(\mathbb{Q})$. When put together as a canonical identification over the whole $(\mathrm{G}(\mathbb{Q})h) \times \mathrm{G}(\mathbb{A}_f)$, it is $\mathrm{G}(\mathbb{Q})$-equivariant because the pointwise comparisons are canonically made via the Betti realizations, and is $\mathrm{G}(\mathbb{A}_f)$-equivariant because it does not involve the second factor $\mathrm{G}(\mathbb{A}_f)$ at all. □

Proposition 5.4.4. *Suppose that the assertions for Betti local systems in Theorem* 5.3.1 *hold. Then the remaining assertions in Theorem* 5.3.1 *also hold.*

Proof. Since $_{p\text{-B}}\underline{V}_\mathbb{C} \cong {}_{\mathrm{B}}\underline{V}_\mathbb{C}$ over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$ by assumption, we have $(_{p\text{-dR}}\underline{V}_\mathbb{C}, \nabla) \cong (_{\mathrm{dR}}\underline{V}_\mathbb{C}, \nabla)$ over $\mathrm{Sh}_{K,\mathbb{C}}$, because both sides admit extensions $(_{p\text{-dR}}\underline{V}^{\mathrm{can}}_\mathbb{C}, \nabla)$ and $(_{\mathrm{dR}}\underline{V}^{\mathrm{can}}_\mathbb{C}, \nabla)$ over $\mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}}$ (and hence have regular singularities along $D_\mathbb{C} = \mathrm{Sh}^{\mathrm{tor}}_{K,\mathbb{C}} - \mathrm{Sh}_{K,\mathbb{C}}$). Since both $(_{p\text{-dR}}\underline{V}^{\mathrm{can}}_\mathbb{C}, \nabla)$ and $(_{\mathrm{dR}}\underline{V}^{\mathrm{can}}_\mathbb{C}, \nabla)$ have nilpotent residues along $D_\mathbb{C}$ (by the explanations in Section 5.2), these two extensions are also canonically isomorphic to each other (by [AB01, Ch. 1, Thm. 4.9]). To verify that the filtrations are respected by such isomorphisms, it suffices to do so at the special points, or just at the arbitrary special point h we have chosen, because special points are dense in the complex analytic topology (see the proof of [Mil05, Lem. 13.5]).

Consider the Galois representation $r(\mu, V)^+_{K,g,p}$ in the proof of Proposition 5.4.1. By decomposing the representation $V_{\overline{\mathbb{Q}}_p}$ of $\mathrm{T}(\overline{\mathbb{Q}}_p)$ into a direct sum of characters of $\mathrm{T}(\overline{\mathbb{Q}}_p)$, we obtain a corresponding decomposition of $r(\mu, V)^+_{K,g,p}$ into a direct sum of characters $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{K,g}) \to \overline{\mathbb{Q}}_p^\times$. By construction, the restrictions of these characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{K,g})$ to the decomposition group at the place v of $F_{K,g}$ given by the composition $F_{K,g} \stackrel{\text{can.}}{\hookrightarrow} \overline{\mathbb{Q}} \stackrel{\iota^{-1}}{\hookrightarrow} \overline{\mathbb{Q}}_p$ are locally algebraic (in the sense of [Ser68, Ch. III, Sec. 1.1, Def.]), because they are induced (up to a sign convention) by the composition of the local Artin map, the cocharacter $\overline{\mathbb{Q}}_p^\times \to T(\overline{\mathbb{Q}}_p)$ given by the base change of μ under the same $F_{K,g} \hookrightarrow \overline{\mathbb{Q}}_p$, and the corresponding characters of $\mathrm{T}(\overline{\mathbb{Q}}_p)$. Thus, the Hodge filtrations on the pullbacks of ${}_{p\text{-dR}}\underline{V}_{\mathbb{C}}$ and ${}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}$ to h are both determined by the Hodge cocharacter $\mu_h : \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{\mathbb{C}}$.

The remainder of Theorem 5.3.1 follows from the fact that the formations of $(\mathcal{E}_E, \nabla)$ (resp. $\mathcal{E}_{\mathcal{F}\ell}$) and $({}_p\mathcal{E}_k, \nabla)$ (resp. ${}_p\mathcal{E}_{\mathcal{F}\ell_k}$) are determined by their pullbacks to special points, which are compatible with each other by the arguments in the proof of Proposition 5.4.1 (resp. of this proposition); and that the descent data for such torsors extend to their canonical extensions as in [Mil90, Ch. V, Sec. 6]. □

By Proposition 5.4.4, it remains to prove the assertions for Betti local systems in Theorem 5.3.1. As explained before (cf. (5.2.6)), the pullbacks of ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ to $\Gamma^+_{K,g_0}\backslash \mathsf{X}^+$ determine and are determined by the fundamental group representations

$$\rho^{+,(p)}_{K,g_0}(V) : \Gamma^{+,c}_{K,g_0} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}}) \quad \text{and} \quad \rho^{+}_{K,g_0}(V) : \Gamma^{+,c}_{K,g_0} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}}),$$

respectively, by canonically identifying the pullbacks of the local systems ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ to the image of $h \in \mathsf{X}^+$ in $\Gamma^+_{K,g_0}\backslash \mathsf{X}^+$ using Proposition 5.4.1. Then it suffices to show that $\rho^{+,(p)}_{K,g_0}(V)$ and $\rho^{+}_{K,g_0}(V)$ *coincide* as representations of Γ^+_{K,g_0}. In this case, they are isomorphic via the identity morphism on $V_{\mathbb{C}}$, and therefore the choice of such an isomorphism is functorial in V and compatible with tensor products and duals because the assignment of $\rho^{+}_{K,g_0}(V)$ to V is, with Hecke actions because of Proposition 5.4.1, and with morphisms between Shimura varieties induced by morphisms of Shimura data because all constructions involved are.

Since $\Gamma^{+,c}_{K,g_0}$ is contained in $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$, and since $\rho^{+}_{K,g_0}(V)$ depends only on the restriction $V_{\mathbb{C}}|_{\mathrm{G}^{\mathrm{der},c}}$ of $V_{\mathbb{C}}$ to $\mathrm{G}^{\mathrm{der},c}$, it remains to prove the following proposition.

Proposition 5.4.5. *The representation $\rho^{+,(p)}_{K,g_0}(V)$ extends to an algebraic representation of $\mathrm{G}^{\mathrm{der},c}$ that coincides with the representation $V_{\mathbb{C}}|_{\mathrm{G}^{\mathrm{der},c}}$ of $\mathrm{G}^{\mathrm{der},c}$ on $V_{\mathbb{C}}$.*

Remark 5.4.6. Such an extension is necessarily unique, as arithmetic subgroups of semisimple groups without compact $\mathbb{Q}$-simple factors are Zariski dense, by the *Borel density theorem* (see [Bor60, Lem. 1.4 and Cor. 4.3] and [BHC62, Thm. 7.8]).

Lemma 5.4.7. *It suffices to prove Proposition 5.4.5 in the special case where $\mathrm{G}^{\mathrm{der}}$ and $\mathrm{G}^{\mathrm{der},c}$ are $\mathbb{Q}$-simple and simply-connected as algebraic groups over $\mathbb{Q}$.*

Proof. Let us begin with an arbitrary $\mathrm{G}^{\mathrm{der}}$ that is not necessarily simply-connected. By [Del79, Lem. 2.5.5], there exists a connected Shimura datum with the semisimple algebraic group over $\mathbb{Q}$ being the simply-connected cover $\widetilde{\mathrm{G}}$ of $\mathrm{G}^{\mathrm{der}}$. Moreover, there is a decomposition $\widetilde{\mathrm{G}} \cong \prod_{i \in I} \widetilde{\mathrm{G}}_i$ of $\widetilde{\mathrm{G}}$ into a product of its $\mathbb{Q}$-simple factors such that each $\widetilde{\mathrm{G}}_i$ is part of a connected Shimura datum. Suppose that the analogue of Proposition 5.4.5 for $\widetilde{\mathrm{G}}_i$ is known for each $i \in I$, and suppose that $\widetilde{\Gamma}$ is any arithmetic subgroup of $\widetilde{\mathrm{G}}(\mathbb{Q})$ of the form $\widetilde{\Gamma} = \prod_{i \in I} \widetilde{\Gamma}_i$ for some neat arithmetic subgroups $\widetilde{\Gamma}_i$

of $\widetilde{\mathrm{G}}_i(\mathbb{Q})$ such that its image $\overline{\Gamma}$ in $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$ is a normal subgroup of $\Gamma^{+,c}_{K,g_0}$ (of finite index). Since $\widetilde{\Gamma}$ is neat, it maps isomorphically into $\mathrm{G}^{\mathrm{der}}(\mathbb{Q})$, $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$, and $\mathrm{G}^{\mathrm{ad}}(\mathbb{Q})$. Then the restriction of $\rho^{+,(p)}_{K,g_0}(V)$ to $\overline{\Gamma}$ lifts to a representation of $\widetilde{\Gamma}$, which (by assumption) extends to an algebraic representation of $\widetilde{\mathrm{G}}$ that coincides with the pullback of $V_{\mathbb{C}}$. This algebraic representation is trivial on $\ker(\widetilde{\mathrm{G}} \to \mathrm{G}^{\mathrm{der},c})$ and hence descends back to the algebraic representation $V_{\mathbb{C}}|_{\mathrm{G}^{\mathrm{der},c}}$ of $\mathrm{G}^{\mathrm{der},c}$.

For simplicity of notation, let $\sigma : \Gamma^{+,c}_{K,g_0} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ denote the homomorphism given by the restriction of $V_{\mathbb{C}}|_{\mathrm{G}^{\mathrm{der},c}}$, let $\tau := \rho^{+,(p)}_{K,g_0}(V)$, and let $\delta : \Gamma^{+,c}_{K,g_0} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ be the map defined by $\delta(\gamma) := \sigma(\gamma)^{-1}\tau(\gamma)$, for all γ. For any $\gamma \in \Gamma^{+,c}_{K,g_0}$ and $\gamma' \in \overline{\Gamma}$, we have $\gamma\gamma'\gamma^{-1} \in \overline{\Gamma}$, and hence $\delta(\gamma)\sigma(\gamma')\delta(\gamma)^{-1} = \sigma(\gamma)^{-1}\tau(\gamma)\tau(\gamma')\tau(\gamma^{-1})\sigma(\gamma) = \sigma(\gamma)^{-1}\tau(\gamma\gamma'\gamma^{-1})\sigma(\gamma) = \sigma(\gamma)^{-1}\sigma(\gamma\gamma'\gamma^{-1})\sigma(\gamma) = \sigma(\gamma')$, which shows that $\delta(\Gamma^{+,c}_{K,g_0})$ commutes with $\sigma(\overline{\Gamma})$. But since σ is the restriction of the algebraic representation $V_{\mathbb{C}}|_{\mathrm{G}^{\mathrm{der},c}}$, and since $\overline{\Gamma}$ is Zariski dense in $\mathrm{G}^{\mathrm{der},c}$ by the Borel density theorem, as in Remark 5.4.6, it follows that $\delta(\Gamma^{+,c}_{K,g_0})$ commutes with the whole $\sigma(\Gamma^{+,c}_{K,g_0})$. Then $\delta : \Gamma^{+,c}_{K,g_0} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ is a group homomorphism, because $\delta(\gamma\gamma') = \sigma(\gamma\gamma')^{-1}\tau(\gamma\gamma') = \sigma(\gamma')^{-1}\sigma(\gamma)^{-1}\tau(\gamma)\tau(\gamma') = \sigma(\gamma')^{-1}\delta(\gamma)\sigma(\gamma')\delta(\gamma') = \delta(\gamma)\delta(\gamma')$, for all $\gamma, \gamma' \in \Gamma^{+,c}_{K,g_0}$, which factors through the finite quotient $\Gamma^{+,c}_{K,g_0}/\overline{\Gamma}$ by construction. (We learned these arguments from the proof of [Mar91, Ch. VII, Lem. 5.1], although we have slightly modified them to avoid some extra assumptions there.)

By Proposition 5.2.17, without loss of generality, we may assume that V and hence $V_{\mathbb{C}}$ are irreducible. By Schur's lemma, the finite image of δ lies in the roots of unity in $\mathbb{C}^\times$, which is then trivial because Γ^{+}_{K,g_0} and hence $\Gamma^{+,c}_{K,g_0}$ are neat. Thus, Proposition 5.4.5 also holds for $\mathrm{G}^{\mathrm{der},c}$, as desired. □

Consequently, in what follows, we may and we shall assume that $\mathrm{G}^{\mathrm{der},c}$ is simply-connected as an algebraic group over $\mathbb{Q}$, so that $\mathrm{G}^{\mathrm{der}} \cong \mathrm{G}^{\mathrm{der},c}$ also is.

5.5. **Cases of real rank one, or of abelian type.** In this subsection, we assume that $\mathrm{G}^{\mathrm{der},c}$ is $\mathbb{Q}$-simple and simply-connected as an algebraic group over $\mathbb{Q}$ (so that $\mathrm{G}^{\mathrm{der}} \cong \mathrm{G}^{\mathrm{der},c}$), and that $\mathrm{rk}_{\mathbb{R}}(\mathrm{G}^{\mathrm{der}}_{\mathbb{R}}) \leq 1$.

Lemma 5.5.1. *Under the above assumptions, the Shimura datum* (G, X) *is necessarily of abelian type* (*see, for example,* [Lan17, Def. 5.2.2.1]). *Up to replacing* G *with another group with the same derived group* $\mathrm{G}^{\mathrm{der}}$*, we may assume that* (G, X) *is of Hodge type* (*see, for example,* [Lan17, Def. 5.2.1.1]).

Proof. Let $\mathrm{G}^{\mathrm{der}}(\mathbb{R})_{\mathrm{nc}}$ denote the product of all noncompact simple factors of $\mathrm{G}^{\mathrm{der}}(\mathbb{R})$ (as a real Lie group). By the classification of Hermitian symmetric domains (see [Hel01, Ch. X, Sec. 6]), any $\mathrm{G}^{\mathrm{der}}(\mathbb{R})_{\mathrm{nc}}$ here must be isomorphic to $\mathrm{SU}_{n,1}$, for some $n \geq 1$. By the classification in [Del79, 2.3], (G, X) is necessarily of abelian type. The last statement then also follows, essentially by definition. □

Consequently, for our purpose, we may assume that the Shimura datum (G, X) is of Hodge type. Note that $\mathrm{G} \cong \mathrm{G}^c$ in this case. Moreover, there exists some faithful representation V_0 of $\mathrm{G} \cong \mathrm{G}^c$ over $\overline{\mathbb{Q}}$, together with a perfect alternating pairing

$$V_0 \times V_0 \to \overline{\mathbb{Q}}(-1), \tag{5.5.2}$$

where (-1) denotes the formal Tate twist (induced by the pullback of the symplectic similitude character), which are defined by some Siegel embedding in the definition of a Shimura datum of Hodge type, together with an abelian scheme $\mathrm{f} : \mathrm{A} \to \mathrm{Sh}_K$ with a polarization λ, whose m-fold self-fiber product we denote by $\mathrm{f}^m : \mathrm{A}^m \to \mathrm{Sh}_K$, such that, for all $i \geq 0$, we have $R^i\mathrm{f}^m_{\mathbb{C},*}(\overline{\mathbb{Q}}) \cong \wedge^i({}_{\mathrm{B}}\underline{V}_0^{\oplus m})$ over $\mathrm{Sh}_{K,\mathbb{C}}$; $R^i\mathrm{f}^m_{\text{ét},*}(\overline{\mathbb{Q}}_p) \cong \wedge^i({}_{\text{ét}}\underline{V}^{\oplus m}_{0,\overline{\mathbb{Q}}_p})$ over Sh_K, where $V_{0,\overline{\mathbb{Q}}_p} := V_0 \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p$; and $\big(R^i\mathrm{f}^m_*(\Omega^\bullet_{\mathrm{A}^m/\mathrm{Sh}_K}) \otimes_E \mathbb{C}, \nabla, \mathrm{Fil}^\bullet\big) \cong \big(\wedge^i({}_{\mathrm{dR}}\underline{V}^{\oplus m}_{0,\mathbb{C}}), \nabla, \mathrm{Fil}^\bullet\big)$ over $\mathrm{Sh}_{K,\mathbb{C}}$, where $V_{0,\mathbb{C}} := V_0 \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ and the ∇ and $\mathrm{Fil}^\bullet$ at the left-hand side are the Gauss–Manin connection and the relative Hodge filtration, respectively. The polarization λ then compatibly induces (as in [DP94, 1.5]) the pairings ${}_{\mathrm{B}}\underline{V}_0 \times {}_{\mathrm{B}}\underline{V}_0 \to {}_{\mathrm{B}}\underline{\overline{\mathbb{Q}}}(-1)$, ${}_{\text{ét}}\underline{V}_{0,\overline{\mathbb{Q}}_p} \times {}_{\text{ét}}\underline{V}_{0,\overline{\mathbb{Q}}_p} \to {}_{\text{ét}}\underline{\overline{\mathbb{Q}}_p}(-1)$, and ${}_{\mathrm{dR}}\underline{V}_{0,\mathbb{C}} \times {}_{\mathrm{dR}}\underline{V}_{0,\mathbb{C}} \to {}_{\mathrm{dR}}\underline{\mathbb{C}}(-1)$ defined by (5.5.2), with (-1) denoting the Tate twists in the respective categories.

Lemma 5.5.3. *We have a canonical isomorphism*

$$\big({}_{\mathrm{dR}}\underline{V}^{\otimes m}_{0,\mathbb{C}}(-t), \nabla, \mathrm{Fil}^\bullet\big) \cong \big({}_{p\text{-}\mathrm{dR}}\underline{V}^{\otimes m}_{0,\mathbb{C}}(-t), \nabla, \mathrm{Fil}^\bullet\big) \tag{5.5.4}$$

for all $i \geq 0$, $m \geq 0$, and $t \in \mathbb{Z}$. Accordingly, we have a canonical isomorphism

$$ {}_{\mathrm{B}}\underline{V}^{\otimes m}_{0,\mathbb{C}}(-t) \cong {}_{p\text{-}\mathrm{B}}\underline{V}^{\otimes m}_{0,\mathbb{C}}(-t). \tag{5.5.5}$$

Moreover, the pullback of (5.5.5) *to the image of the special point $h \in \mathsf{X}^+$ in $\Gamma^+_{K,g_0}\backslash\mathsf{X}^+$ (see the beginning of Section* 5.4*), which is defined over a finite extension E^+ of E in $\overline{\mathbb{Q}}$, is given by the identity morphism of $V^{\otimes m}_{0,\mathbb{C}}(-t)$.*

Proof. Since the p-adic analytification functor from the category of algebraic filtered connections to the category of p-adic analytic ones is fully faithful, by [AB01, Ch. 4, Cor. 6.8.2] (which is applicable because Sh_K is defined over E); since the p-adic analytification of $\big(R^i\mathrm{f}^m_*(\Omega^\bullet_{\mathrm{A}^m/\mathrm{Sh}_K}) \otimes_E k, \nabla, \mathrm{Fil}^\bullet\big) \cong \big(R^i\mathrm{f}^m_{k,*}(\Omega^\bullet_{\mathrm{A}^m_k/\mathrm{Sh}_{K,k}}), \nabla, \mathrm{Fil}^\bullet\big)$ is canonically isomorphic to $\big(D_{\mathrm{dR}}(R^i\mathrm{f}^m_{k,\text{ét},*}(\overline{\mathbb{Q}}_p)), \nabla, \mathrm{Fil}^\bullet\big)$, for any finite extension k of the composite of E and $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, by [Sch13, Thm. 8.8 and 9.1]; and since such an isomorphism is functorial, we have (5.5.4), from which (5.5.5) follows by taking horizontal sections, because both sides of (5.5.4) can be canonically identified (up to the same Tate twist $(-t)$) with the image of $\big(R^m\mathrm{f}^m_*(\Omega^\bullet_{\mathrm{A}^m/\mathrm{Sh}_K}) \otimes_E \mathbb{C}, \nabla, \mathrm{Fil}^\bullet\big)$ under ε_m^*, for some endomorphism ε_m of the abelian scheme $\mathrm{f}^m : \mathrm{A}^m \to \mathrm{Sh}_K$, by Lieberman's trick (see, e.g., [LS12, Sec. 3.2]). Since the comparison isomorphisms among the Betti, étale, and de Rham cohomology of an abelian variety defined over E^+ are all compatible with each other, the second assertion also follows. □

Lemma 5.5.6. *For each irreducible representation V of G over $\overline{\mathbb{Q}}$, there exist integers $m_V \geq 0$ and t_V (depending noncanonically on V) such that V is a direct summand of $V_0^{\otimes m_V}(-t_V)$, where $(-t_V)$ denotes the formal Tate twist (which has no effect when restricted to the subgroup $\mathrm{G}^{\mathrm{der}}$ of G), so that $V = s_V\big(V_0^{\otimes m_V}(-t_V)\big)$ for some Hodge tensor $s_V \in V_0^{\otimes}$ (i.e., a tensor of weight $(0,0)$ with respect to the induced Hodge structure on $V_0^{\otimes}$; cf.* [DMOS82, Ch. I, Prop. 3.4]).

Proof. See [LS18a, Prop. 3.2], which is based on [DMOS82, Ch. I, Prop. 3.1(a)]. □

By combining Lemmas 5.5.3 and 5.5.6, we obtain the following:

Corollary 5.5.7. *For each irreducible $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G})$, there exist some integers $m_V \geq 0$ and t_V such that the local systems ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ and ${}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}}$ are direct summands of ${}_{\mathrm{B}}\underline{V}^{\otimes m_V}_{0,\mathbb{C}}(-t_V)$ and ${}_{p\text{-}\mathrm{B}}\underline{V}^{\otimes m_V}_{0,\mathbb{C}}(-t_V)$, respectively. Consequently, there is a morphism*

$$ {}_{\mathrm{B}}\underline{V}_{\mathbb{C}} \to {}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}} \tag{5.5.8}$$

defined by composing ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}} \overset{\text{can.}}{\hookrightarrow} {}_{\mathrm{B}}\underline{V}_{0,\mathbb{C}}^{\otimes m_V}(-t_V) \overset{(5.5.5)}{\overset{\sim}{\to}} {}_{p\text{-}\mathrm{B}}\underline{V}_{0,\mathbb{C}}^{\otimes m_V}(-t_V) \overset{\text{can.}}{\twoheadrightarrow} {}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}}$.

Proposition 5.5.9. *The above morphism* (5.5.8) *is an isomorphism over the connected component* $\Gamma_{K,g_0}^{+}\backslash \mathsf{X}^{+}$ *of* $\mathrm{Sh}_{K,\mathbb{C}}$ *that induces the identity morphism between the two representations* $\rho_{K,g_0}^{+,(p)}(V)$ *and* $\rho_{K,g_0}^{+}(V)$. *In particular, Proposition* 5.4.5 *holds under the assumptions of this subsection.*

Proof. It suffices to show that the pullback of (5.5.8) to the image of h, as in Lemma 5.5.3, is given by the identity morphism of a subspace of $V_{0,\mathbb{C}}^{\otimes m}(-t)$. Since the comparison isomorphisms thus far are functorial and compatible with pullbacks to special points, the pullback of (5.5.8) is induced by the simpler comparison isomorphisms for the cohomology of an abelian variety, and it suffices to note that the Hodge tensor s_V in Lemma 5.5.6 are respected by such comparison isomorphisms, because Hodge cycles on abelian varieties over number fields are *absolute Hodge* (see [DMOS82, Ch. I, Main Thm. 2.11]) and *de Rham* (see [Bla94, Thm. 0.3]). □

5.6. **Cases of real rank at least two.** In this subsection, we assume that $\mathrm{G}^{\mathrm{der},c}$ is $\mathbb{Q}$-simple and simply-connected as an algebraic group over $\mathbb{Q}$ (so that $\mathrm{G}^{\mathrm{der}} \cong \mathrm{G}^{\mathrm{der},c}$), and that $\mathrm{rk}_{\mathbb{R}}(\mathrm{G}_{\mathbb{R}}^{\mathrm{der}}) \geq 2$. We shall make use of the following special case of Margulis's *superrigidity theorem*:

Theorem 5.6.1. *Let* H *be a* $\mathbb{Q}$*-simple simply-connected connected algebraic group over* $\mathbb{Q}$, *and let* Γ *be an arithmetic subgroup of* $\mathrm{H}(\mathbb{Q})$. *Suppose that* $\mathrm{rk}_{\mathbb{R}}(\mathrm{H}_{\mathbb{R}}) \geq 2$. *Then, given any representation* $\rho : \Gamma \to \mathrm{GL}_m(\mathbb{C})$, *there exists a finite index normal subgroup* Γ_0 *of* Γ *such that* $\rho|_{\Gamma_0}$ *extends to a* (*unique*) *group homomorphism* $\widetilde{\rho} : \mathrm{H}(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ *that is induced by an algebraic group homomorphism* $\mathrm{H}_{\mathbb{C}} \to \mathrm{GL}_{m,\mathbb{C}}$, *and such that* $\rho(\gamma) = \delta(\gamma)\widetilde{\rho}(\gamma)$, *for all* $\gamma \in \Gamma$, *for some representation* $\delta : \Gamma/\Gamma_0 \to \mathrm{GL}_m(\mathbb{C})$ *whose image commutes with* $\widetilde{\rho}(\mathrm{H}(\mathbb{C}))$.

Proof. This follows from [Mar91, Ch. VIII, Thm. (B), part (iii)] with $S = \{\infty\}$, $\Lambda = \Gamma$, $K = \mathbb{Q}$, and $\ell = \mathbb{C}$ (in the notation there). □

By applying Theorem 5.6.1 with $\mathrm{H} = \mathrm{G}^{\mathrm{der},c}$, $\Gamma = \Gamma_{K,g_0}^{+,c}$, and $\rho = \rho_{K,g_0}^{+,(p)}(V)$ as in Section 5.4, we see that $\rho_{K,g_0}^{+,(p)}(V) : \Gamma_{K,g_0}^{+,c} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ extends to an algebraic representation $\widetilde{\rho}_{K,g_0}^{+,(p)}(V) : \mathrm{G}_{\mathbb{C}}^{\mathrm{der},c} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ up to a finite error given by a representation $\delta_{K,g_0}(V) : \Gamma_{K,g_0}^{+,c}/\Gamma_{K,g_0,0}^{+,c} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ whose image in $\mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ commutes with $\widetilde{\rho}_{K,g_0}^{+,(p)}(V)(\mathrm{G}^{\mathrm{der},c}(\mathbb{C}))$. Let us record what we have obtained as a triple

$$\big(\widetilde{\rho}_{K,g_0}^{+,(p)}(V), \Gamma_{K,g_0,0}^{+,c}, \delta_{K,g_0}(V)\big), \tag{5.6.2}$$

which depends on K and g_0, or rather the arithmetic subgroup $\Gamma_{K,g_0}^{+,c}$ they define.

If $K_1 \subset K_2$ are neat open compact subgroups of $\mathrm{G}(\mathbb{A}_f)$, then $\Gamma_{K_1,g_0}^{+,c} \subset \Gamma_{K_2,g_0}^{+,c}$ and $\rho_{K_1,g_0}^{+,(p)}(V) = \rho_{K_2,g_0}^{+,(p)}(V)|_{\Gamma_{K_1,g_0}^{+,c}}$, and therefore

$$\rho_{K_1,g_0}^{+,(p)}(V)|_{\Gamma_{K_1,g_0,0}^{+,c}\cap\Gamma_{K_2,g_0,0}^{+,c}} = \rho_{K_2,g_0}^{+,(p)}(V)|_{\Gamma_{K_1,g_0,0}^{+,c}\cap\Gamma_{K_2,g_0,0}^{+,c}}, \tag{5.6.3}$$

where $\Gamma_{K_1,g_0,0}^{+,c}\cap\Gamma_{K_2,g_0,0}^{+,c}$ is a neat arithmetic subgroup of $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$ because it is of finite index in $\Gamma_{K_1,g_0,0}^{+,c}$ (because $\Gamma_{K_1,g_0,0}^{+,c}/(\Gamma_{K_1,g_0,0}^{+,c}\cap\Gamma_{K_2,g_0,0}^{+,c}) \subset \Gamma_{K_2,g_0}^{+,c}/\Gamma_{K_2,g_0,0}^{+,c}$). For $i = 1, 2$, let $\big(\widetilde{\rho}_{K_i,g_0}^{+,(p)}(V), \Gamma_{K_i,g_0,0}^{+,c}, \delta_{K_i,g_0}(V)\big)$ be the triples associated with $\rho_{K_i,g_0}^{+,(p)}(V)$, respectively, as in (5.6.2). Then it follows from (5.6.3) and the Borel density theorem (see Remark 5.4.6) that $\widetilde{\rho}^{+,(p)}_{K_1,g_0}(V) = \widetilde{\rho}^{+,(p)}_{K_2,g_0}(V)$. Since K_1 and K_2 are arbitrary, there is a well-defined assignment (to V) of an algebraic representation

$$\widetilde{\rho}^{+,(p)}_{g_0}(V) : \mathrm{G}^{\mathrm{der},c}_{\mathbb{C}} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$$

such that $\widetilde{\rho}^{+,(p)}_{K,g_0}(V) = \widetilde{\rho}^{+,(p)}_{g_0}(V)$ for all neat open compact subgroups K of $\mathrm{G}(\mathbb{A}_f)$.

By the above construction, and by Proposition 5.2.17, the assignment of $\widetilde{\rho}^{+,(p)}_{g_0}(V)$ to $V \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{G}^c)$ defines a tensor functor from $\mathrm{Rep}_{\mathbb{C}}(\mathrm{G}^c)$ to $\mathrm{Rep}_{\mathbb{C}}(\mathrm{G}^{\mathrm{der},c})$, and hence induces (as in [DMOS82, Ch. II, Cor. 2.9]) a group homomorphism $\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}} \to \mathrm{G}^c_{\mathbb{C}}$. Since $\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}}$ is semisimple, this homomorphism factors through

$$\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}} \to \mathrm{G}^{\mathrm{der},c}_{\mathbb{C}}. \tag{5.6.4}$$

For each $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$, let us denote by

$$\pi(V) : \mathrm{G}^{\mathrm{der},c}_{\mathbb{C}} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$$

the algebraic representation given by the restriction $V_{\mathbb{C}}|_{\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}}}$.

For each $\gamma \in \mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$, the Hecke action of $g = g_0^{-1}\gamma g_0 \in \mathrm{G}(\mathbb{A}_f)$ induces an isomorphism $\Gamma^+_{gKg^{-1},g_0}\backslash \mathsf{X}^+ \xrightarrow{\sim} \Gamma^+_{K,g_0}\backslash \mathsf{X}^+$ (see (5.2.7)) defined by left multiplication by γ^{-1}, compatible with the isomorphism $\Gamma^+_{gKg^{-1},g_0} \xrightarrow{\sim} \Gamma^+_{K,g_0}$ induced by conjugation by γ^{-1}. By Proposition 5.4.1 and (5.2.8), we have the compatibility

$$\begin{aligned} \widetilde{\rho}^{+,(p)}_{g_0}(V)(\gamma\gamma'\gamma^{-1}) &= \big(\widetilde{\rho}^{+,(p)}_{g_0}(V)(\gamma)\big)\big(\widetilde{\rho}^{+,(p)}_{g_0}(V)(\gamma')\big)\big(\widetilde{\rho}^{+,(p)}_{g_0}(V)(\gamma)\big)^{-1} \\ &= \big(\pi(V)(\gamma)\big)\big(\widetilde{\rho}^{+,(p)}_{g_0}(V)(\gamma')\big)\big(\pi(V)(\gamma)\big)^{-1}, \end{aligned} \tag{5.6.5}$$

for all $\gamma' \in \Gamma^{+,c}_{K,g_0,0} \cap \gamma^{-1}\Gamma^{+,c}_{gKg,g_0,0}\gamma$, where $g = g_0^{-1}\gamma g_0$.

Lemma 5.6.6. *Suppose that the representation $\widetilde{\rho}^{+,(p)}_{g_0}(V)$ is irreducible when V is. Then the following are true for all $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$:*

1. *We have $\widetilde{\rho}^{+,(p)}_{g_0}(V) = \pi(V)$ as algebraic representations of $\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}}$.*
2. *The representation $\delta_{K,g_0}(V) : \Gamma^{+,c}_{K,g_0}/\Gamma^{+,c}_{K,g_0,0} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ is trivial.*

Proof. By Proposition 5.2.17, we may assume that $\pi := \pi(V)$ and hence $\widetilde{\rho} := \widetilde{\rho}^{+,(p)}_{g_0}(V)$ are irreducible. Let us measure their difference by the algebraic morphism $\epsilon : \mathrm{G}^{\mathrm{der},c}_{\mathbb{C}} \to \mathrm{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ (which is not shown to be a group homomorphism yet) defined by $\epsilon(g) = \widetilde{\rho}(g)^{-1}\pi(g)$, for all $g \in \mathrm{G}^{\mathrm{der},c}(\mathbb{C})$. By (5.6.5), we have

$$\widetilde{\rho}(\gamma') = \epsilon(\gamma)\widetilde{\rho}(\gamma')\epsilon(\gamma)^{-1} \tag{5.6.7}$$

for all $\gamma' \in \Gamma' := \Gamma^{+,c}_{K,g_0,0} \cap \gamma^{-1}\Gamma^{+,c}_{gKg,g_0,0}\gamma$. Since Γ' is a neat arithmetic subgroup of $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$, by the Borel density theorem (see Remark 5.4.6), we also have (5.6.7) for all $\gamma' \in \mathrm{G}^{\mathrm{der},c}(\mathbb{C})$. Then the morphism ϵ is a group homomorphism, because $\epsilon(\gamma\gamma') = \widetilde{\rho}(\gamma\gamma')^{-1}\pi(\gamma\gamma') = \widetilde{\rho}(\gamma')^{-1}\widetilde{\rho}(\gamma)^{-1}\pi(\gamma)\pi(\gamma') = \widetilde{\rho}(\gamma')^{-1}\epsilon(\gamma)\pi(\gamma') = \epsilon(\gamma)\widetilde{\rho}(\gamma')^{-1}\pi(\gamma') = \epsilon(\gamma)\epsilon(\gamma')$, for all $\gamma, \gamma' \in \mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$, and because $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$ is Zariski dense in $\mathrm{G}^{\mathrm{der},c}$ (by [Spr98, Cor. 13.3.10], or still by the Borel density theorem). By Schur's lemma, ϵ factors through an algebraic group homomorphism $\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}} \to \mathbf{G}_{\mathrm{m},\mathbb{C}}$, which must be trivial because $\mathrm{G}^{\mathrm{der},c}_{\mathbb{C}}$ is semisimple. Thus, part (1) follows. As for part (2), we can prove it by using Schur's lemma and the neatness of Γ^+_{K,g_0}, as in the last paragraph of the proof of Lemma 5.4.7. □

Lemma 5.6.8. *The above homomorphism* (5.6.4) *is an automorphism. In particular, the representation* $\widetilde{\rho}_{g_0}^{+,(p)}(V)$ *is indeed irreducible when* V *is.*

Proof. Since $\mathrm{G}_{\mathbb{C}}^{\mathrm{der},c}$ is semisimple and simply-connected, it suffices to show that the Lie algebra of the kernel of (5.6.4), which is a priori a product of $\mathbb{C}$-simple factors of the Lie algebra of $\mathrm{G}_{\mathbb{C}}^{\mathrm{der},c}$, is trivial. Therefore, it suffices to show that (5.6.4) has nontrivial restrictions to all $\mathbb{C}$-simple factors of $\mathrm{G}_{\mathbb{C}}^{\mathrm{der},c}$, and it suffices to find some $V \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{G}^c)$ such that $\widetilde{\rho}_{g_0}^{+,(p)}(V)$ is nontrivial on all simple $\mathbb{C}$-simple factors.

As explained in [Bor84, Mil83], based on a construction due to Piatetski-Shapiro, there exist morphisms $\varphi_1 : (\mathrm{G}, \mathsf{X}) \hookrightarrow (\mathrm{G}_1, \mathsf{X}_1)$ and $\varphi_2 : (\mathrm{G}_2, \mathsf{X}_2) \hookrightarrow (\mathrm{G}_1, \mathsf{X}_1)$ between Shimura data such that the following hold:

- $\mathrm{G}_1^{\mathrm{der},c}$ is $\mathbb{Q}$-simple, and we have $\mathrm{G}^{\mathrm{der},c} \stackrel{\varphi_1}{\hookrightarrow} \mathrm{G}_1^{\mathrm{der},c} \hookrightarrow \mathrm{Res}_{F/\mathbb{Q}}\, \mathrm{G}_F^{\mathrm{der},c}$ for some totally real number field F, identifying $\mathrm{G}_{\mathbb{C}}^{\mathrm{der},c}$ as a direct factor of $\mathrm{G}_{1,\mathbb{C}}^{\mathrm{der},c}$.
- $\mathrm{G}_2^{\mathrm{der},c}$ is also $\mathbb{Q}$-simple, and all $\mathbb{C}$-simple factors of $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c}$ are of type A_1. In this case, as in the proof of Lemma 5.5.1, the Shimura datum $(\mathrm{G}_2, \mathsf{X}_2)$ is of abelian type, for which Proposition 5.5.9 and hence Theorem 5.3.1 hold. Moreover, the homomorphism $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c} \to \mathrm{G}_{1,\mathbb{C}}^{\mathrm{der},c}$ induced by φ_2 embeds distinct $\mathbb{C}$-simple factors of $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c}$ into distinct $\mathbb{C}$-simple factors of $\mathrm{G}_{1,\mathbb{C}}^{\mathrm{der},c}$, and every simple factor of $\mathrm{G}_{1,\mathbb{C}}^{\mathrm{der},c}$ meets $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c}$ nontrivially.

Therefore, there exists some $V_1 \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{G}_1^c)$ such that its pullback $V_2 \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{G}_2^c)$ is nontrivial on all $\mathbb{C}$-simple factors of $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c}$. By Proposition 5.2.17, ${}_{p\text{-}\mathrm{B}}\underline{V}_{\mathbb{C}}$ and ${}_{p\text{-}\mathrm{B}}\underline{V}_{2,\mathbb{C}}$ are canonically isomorphic to pullbacks of ${}_{p\text{-}\mathrm{B}}\underline{V}_{1,\mathbb{C}}$, and we already know that the fundamental group representations associated with ${}_{p\text{-}\mathrm{B}}\underline{V}_{2,\mathbb{C}} \cong {}_{\mathrm{B}}\underline{V}_{2,\mathbb{C}}$ are given by the restrictions of $V_{2,\mathbb{C}}$. Let $\widetilde{\rho}_{g_0}^{+,(p)}(V_1)$ be associated with ${}_{p\text{-}\mathrm{B}}\underline{V}_{1,\mathbb{C}}$ as in the case of $\widetilde{\rho}_{g_0}^{+,(p)}(V)$. By [Mar91, Ch. I, Sec. 3, Lem. 3.13], the pullbacks of arithmetic subgroups of $\mathrm{G}_1^{\mathrm{der},c}(\mathbb{Q})$ to $\mathrm{G}^{\mathrm{der},c}(\mathbb{Q})$ and $\mathrm{G}_2^{\mathrm{der},c}(\mathbb{Q})$ contain arithmetic subgroups. Therefore, by the Borel density theorem (see Remark 5.4.6), the pullback of $\widetilde{\rho}_{g_0}^{+,(p)}(V_1)$ to $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c}$ is nontrivial on all $\mathbb{C}$-simple factors of $\mathrm{G}_{2,\mathbb{C}}^{\mathrm{der},c}$, and hence $\widetilde{\rho}_{g_0}^{+,(p)}(V_1)$ is nontrivial on all $\mathbb{C}$-simple factors of $\mathrm{G}_{1,\mathbb{C}}^{\mathrm{der},c}$. By the Borel density theorem again, $\widetilde{\rho}_{g_0}^{+,(p)}(V)$ is isomorphic to the pullback of $\widetilde{\rho}_{g_0}^{+,(p)}(V_1)$, which is then nontrivial on all simple $\mathbb{C}$-simple factors of $\mathrm{G}_{\mathbb{C}}^{\mathrm{der},c}$, as desired. □

Thus, Proposition 5.4.5 also holds under the assumptions of this subsection, by Lemmas 5.6.6 and 5.6.8. The proof of Theorem 5.3.1 is now complete.

Appendix A. A formalism of decompletion

In this appendix, we generalize the formalism of decompletion developed in [KL16, Sec. 5], in order to treat the general Kummer towers.

A.1. **Results.** It will be convenient to work with Banach rings. We refer to [KL15, Sec. 2.2] for general background and terminology. We will always assume Banach rings to be commutative, and (for simplicity) contain topological nilpotent units. Note that, by [KL15, Lem. 2.2.12], a finite projective module over a Banach ring A admits a canonical structure of Banach A-modules (up to equivalence). Thus, in what follows, we do not distinguish between finite projective modules and finite projective Banach modules when the base ring is a Banach ring. If $M \to N$ is a homomorphism of Banach A-modules, we always equip N/M with the quotient seminorm, which becomes a norm if the image is closed.

For a Banach ring A equipped with an action by a profinite group Γ, let $\mathrm{Proj}_A(\Gamma)$ (resp. $\mathrm{Rep}_A(\Gamma)$) denote the category of finite projective (resp. finite free) A-modules equipped with a semilinear continuous Γ-action. For simplicity, they are also called finite projective (resp. finite free) Γ-modules over A. Note that, given a finite free Γ-module L of rank l, after choosing a basis of L, the action of Γ on L can be represented by a 1-cocycle $f \in C^1\big(\Gamma, \mathrm{GL}_l(A)\big)$, and any change of basis only results in a change of the cocycle by a coboundary. It follows that the isomorphism classes of finite free Γ-modules of rank l over A are classified by the cohomology set $H^1\big(\Gamma, \mathrm{GL}_l(A)\big)$. In addition, for $L_1, L_2 \in \mathrm{Proj}_A(\Gamma)$, we have

$$\mathrm{Hom}_{\mathrm{Proj}_A(\Gamma)}(L_1, L_2) \cong H^0(\Gamma, L_1^\vee \otimes_A L_2). \tag{A.1.1}$$

In the remainder of this subsection, let $\{A_i\}_{i\in I}$ be a direct system of Banach rings with submetric transition morphisms $\varphi_{ij} : A_i \to A_j$, i.e., $|\varphi_{ij}(a)|_{A_j} \leq |a|_{A_i}$, where I is a small filtered index category. Let $A_\infty := \varinjlim_i A_i$, and let $\widehat{A}_\infty$ denote its completion (with respect to the infimum seminorm on A_∞). Let Γ be a profinite group continuously acting on the direct system $\{A_i\}_{i\in I}$, and therefore on $\widehat{A}_\infty$.

Definition A.1.2. We call the pair $(\{A_i\}_{i\in I}, \Gamma)$ a *decompletion system* (resp. *weak decompletion system*) if the following two conditions hold:

(1) For any finite projective (resp. finite free) Γ-module L_∞ over $\widehat{A}_\infty$, there exists some $i \in I$ and a finite projective (resp. finite free) Γ-module L_i over A_i together with a Γ-equivariant continuous A_i-linear morphism $\iota_i : L_i \to L_\infty$ inducing an isomorphism $\iota_i \otimes 1 : L_i \otimes_{A_i} \widehat{A}_\infty \xrightarrow{\sim} L_\infty$. We shall call such a pair (L_i, ι_i) a *model* of L_∞ over A_i.
(2) For any model (L_i, ι_i) over A_i, there exists some $i_0 \geq i$ such that for any $i' \geq i_0$, the model $(L_{i'}, \iota_{i'}) := (L_i \otimes_{A_i} A_{i'}, \iota_i \otimes 1)$ is *good* in the sense that the natural map $H^\bullet(\Gamma, L_{i'}) \to H^\bullet(\Gamma, L_\infty)$ is an isomorphism.

Remark A.1.3. Note that, if $\{A_i\}_{i\in I}$ is a decompletion system (resp. weak decompletion system), then the natural functor $\varinjlim_{i\in I} \mathrm{Proj}_{A_i}(\Gamma) \to \mathrm{Proj}_{\widehat{A}_\infty}(\Gamma)$ (resp. $\varinjlim_{i\in I} \mathrm{Rep}_{A_i}(\Gamma) \to \mathrm{Rep}_{\widehat{A}_\infty}(\Gamma)$) is an equivalence of categories. Indeed, the condition (1) implies that the functor is essentially surjective, and the condition (2) and the isomorphism (A.1.1) imply that the functor is fully faithful. Similarly, the condition (2) implies that, for any two models $(L_{i,1}, \iota_{i,1})$ and $(L_{i,2}, \iota_{i,2})$ of L_∞ over A_i, there exists some $i' \geq i$ such that the two models $(L_{i,1} \otimes_{A_i} A_{i'}, \iota_{i,1} \otimes 1)$ and $(L_{i,2} \otimes_{A_i} A_{i'}, \iota_{i,2} \otimes 1)$ over $A_{i'}$ are isomorphic.

Definition A.1.4. Let $(\{A_i\}_{i\in I}, \Gamma)$ be as before. We further assume that there is an initial object $0 \in I$, and that all transition morphisms $A_i \to A_{i'}$ are isometric (instead of merely submetric). Let $\{\Gamma_i\}_{i\in I^{\mathrm{op}}}$ be an inverse system of open normal subgroups of $\Gamma = \Gamma_0$ forming a basis of the open neighborhoods of the identity. The pair $(\{A_i\}_{i\in I}, \{\Gamma_i\}_{i\in I^{\mathrm{op}}})$ is called *decompleting* if it satisfies the following conditions:

(1) For each $i \in I$, the Γ-action on A_i is isometric. For sufficiently large i, the image of $\Gamma_i \to \mathrm{Aut}(A_i)$ is a central subgroup in the image of $\Gamma \to \mathrm{Aut}(A_i)$.
(2) (*Splitting.*) For each $i \in I$, the natural projection $\widehat{A}_\infty \to \widehat{A}_\infty/A_i$ admits an isometric section as Banach modules over A_i.

(3) (*Uniform strict exactness.*) There exists $c > 0$ such that, for each $i \in I$ and each cocycle $f \in C^\bullet(\Gamma_i, \widehat{A}_\infty / A_i)$, there exists a cochain g with $\|g\| \leq c\|f\|$ such that $dg = f$, where $\|\cdot\|$ denotes the supremum norm among the images. In particular, $\widehat{A}_\infty / A_i$ has totally trivial Γ_i-cohomology (and hence totally trivial Γ-cohomology, by the Hochschild–Serre spectral sequence).

The pair $(\{A_i\}_{i \in I}, \{\Gamma_i\}_{i \in I^{\mathrm{op}}})$ is called *stably decompleting* if it satisfies the following conditions:

(a) A_i's and $\widehat{A}_\infty$ are all stably uniform. That is, (A_i, A_i°) and $(\widehat{A}_\infty, \widehat{A}_\infty^\circ)$ are stably uniform Banach adic rings (in the sense of [KL15, Rem. 2.8.5]).

(b) For each $i \in I$, the action of Γ_i on A_i is trivial and the base change of the pair $(\{A_j\}, \{\Gamma_j\})_{j \geq i}$ under every rational localization of A_i is decompleting.

Remark A.1.5. Note that Definition A.1.4(2) is equivalent to the existence of a normalized trace map $\mathrm{pr}_i : \widehat{A}_\infty \to A_i$ that is *submetric*. Indeed, Theorem A.1.6 holds under the weaker condition that all the normalized trace maps pr_i are continuous and $\sup_{i \in I}\big\{\| \mathrm{pr}_i \|\big\} < \infty$. On the other hand, unlike the classical Tate trace maps, the maps pr_i here are not required to be Galois equivariant.

The main result of this appendix is the following:

Theorem A.1.6. *(1) A decompleting pair is a weak decompletion system.*
(2) A stably decompleting pair is a decompletion system.

We first establish a few lemmas, which will imply Theorem A.1.6(1).

Lemma A.1.7. *Let $(C^\bullet, d)$ be a uniformly strict exact complex of Banach modules over a Banach ring A. That is, there exists some constant $c \geq 0$ such that, for each degree s and each cocycle $f \in C^s$, there exists some $g \in C^{s-1}$ such that $f = dg$ and $|g| \leq c|f|$. Then, for each cochain $f \in C^s$, there exists some $h \in C^{s-1}$ such that $|h| \leq \max\big\{c|f|, c^2|df|\big\}$ and $|f - dh| \leq c|df|$.*

Proof. Since df is a cocycle, there exists $g \in C^s$ such that $df = dg$ and $|g| \leq c|df|$. Since $d(f - g) = 0$, there exists $h \in C^{s-1}$ such that $dh = f - g$ and $|h| \leq c|f - g| \leq \max\big\{c|f|, c^2|df|\big\}$. □

Let A be a Banach ring with a continuous action by a profinite group Γ, and let M be a Banach A-module with a semilinear Γ-action by isometry. We equip the A-module $C^s(\Gamma, M)$ of continuous maps from Γ^s to M with the supremum norm given by $\|f\| = \sup_{\gamma \in \Gamma^s} |f(\gamma)|$. Then $C^\bullet(\Gamma, M)$ is a complex of Banach A-modules.

Lemma A.1.8. *Let (A, Γ, M) be as above. Assume that the complex $C^\bullet(\Gamma, M)$ is uniformly strict exact with respect to some constant c (see Lemma A.1.7). Let $L = \oplus_{i=1}^l A e_i$ be an object of $\mathrm{Rep}_A(\Gamma)$, equipped with the supremum norm. Suppose that, for each i and each $\gamma \in \Gamma$, there exists some $r > 1$ such that $|\gamma(e_i) - e_i| < \frac{1}{rc}$. Then $C^\bullet(\Gamma, L \otimes_A M)$ is uniformly strict exact with respect to the same constant c.*

Proof. Let $f = \sum_{j=1}^l (e_j \otimes f_j)$ be a cocycle with $f_j \in C^s(\Gamma, M)$ for all $1 \leq j \leq l$. Note that the norm of $\sum_{j=1}^l (e_j \otimes df_j) = \sum_{j=1}^l (e_j \otimes df_j) - df = \sum_{j=1}^l (e_j \otimes df_j) - d\big(\sum_{j=1}^l (e_j \otimes f_j)\big)$ is bounded by $\frac{\|f\|}{rc}$. That is, for each j, we have $\|df_j\| \leq \frac{\|f\|}{rc}$. By Lemma A.1.7, there exist $h_j \in C^{s-1}(\Gamma, M)$ with $\|f_j - dh_j\| \leq \frac{\|f\|}{r}$ and $\|h_j\| \leq c\|f\|$, for all $j = 1, \ldots l$. Put $h = \sum_{j=1}^l (e_j \otimes h_j)$. Then $\|h\| \leq c\|f\|$, and the norm of

$f-dh=\sum_{j=1}^{l}(e_j\otimes(f_j-dh_j))+\Big(\sum_{j=1}^{l}(e_j\otimes dh_j)-d\big(\sum_{j=1}^{l}(e_j\otimes h_j)\big)\Big)$ is bounded by $\frac{\|f\|}{r}$. By iterating this process, we obtain cochains $H_1,H_2,\ldots\in C^{s-1}(\Gamma,L\otimes_A M)$ satisfying $\|H_n\|\leq\frac{c\|f\|}{r^{n-1}}$ and $\|f-dH_1-\cdots-dH_n\|\leq\frac{\|f\|}{r^n}$, for all $n\geq 1$. Then $f=dH$ for $H=\sum_{i=1}^{\infty}H_i\in C^{s-1}\big(\Gamma,L\otimes_A M\big)$, and $\|H\|\leq c\|f\|$, as desired. □

Lemma A.1.9. *Let $A\to B$ be an isometric homomorphism of Banach rings. If the projection $\pi:B\to B/A$ admits an isometric section $s:B/A\to B$ as Banach A-modules, then $|\pi(b_1b_2)|\leq\max\big\{|b_2||\pi(b_1)|,|b_1||\pi(b_2)|\big\}$, for all $b_1,b_2\in B$.*

Proof. Note that the isomorphism $\mathrm{Id}\oplus s:A\oplus B/A\to B$ of Banach A-modules is isometric, where the left-hand side is equipped with the supremum norm. Indeed, for $e=a+s(b)\in B$, we have $|e|\leq\max\big\{|a|,|s(b)|\big\}=\max\big\{|a|_A,|b|_{B/A}\big\}$, where $|\cdot|_A$ and $|\cdot|_{B/A}$ denote the norms on A and B/A, respectively. Next, note that $|s(b)|=|b|_{B/A}\leq|e|$ because $\pi(e)=b$, and so $|a|=|e-s(b)|\leq|e|$. Thus, $|e|=\max\big\{|a|,|s(b)|\big\}$. Now write $b_i=a_i+s(\pi(b_i))$, for $i=1,2$, so that

$$\pi(b_1b_2)=a_1\pi(b_2)+a_2\pi(b_1)+\pi(s(\pi(b_1))s(\pi(b_2))).$$

Then $|a_1\pi(b_2)|\leq|a_1||\pi(b_2)|\leq|b_1||\pi(b_2)|$. Similarly, $|a_2\pi(b_1)|\leq|b_2||\pi(b_1)|$. Finally,

$$|\pi(s(\pi(b_1))s(\pi(b_2)))|\leq|s(\pi(b_1))s(\pi(b_2))|\leq|(s(\pi(b_1))||s(\pi(b_2)))|=|\pi(b_1)||\pi(b_2)|.$$

Since $|\pi(b_1)||\pi(b_2)|\leq\min\big\{|b_2||\pi(b_1)|,|b_1||\pi(b_2)|\big\}$, the lemma follows. □

Lemma A.1.10. *Let $A\to B$ be a Γ-equivariant isometric homomorphism of Banach rings. Assume that the projection $\pi:B\to B/A$ of Banach A-modules admits an isometric section s, and that $C^\bullet(\Gamma,B/A)$ is uniformly strict exact with respect to some $c\geq 1$ (see Lemma* A.1.7*). Let f be a cocycle in $C^1\big(\Gamma,\mathrm{GL}_l(B)\big)$. Suppose that there exists $r>1$ such that $|f(\gamma)-1|\leq\frac{1}{rc}$ for all $\gamma\in\Gamma$ and that $\|\overline{f}\|\leq\frac{1}{rc^2}$, where $\overline{f}$ is the image of f in $C^1\big(\Gamma,\mathrm{M}_l(B/A)\big)$ (which is merely a cochain), where $\mathrm{M}_l(B/A)$ abusively denotes the direct sum of l^2 copies of B/A (with a canonical map of Banach A-modules from $\mathrm{M}_l(B)$). (We shall also denote similar images by overlines in the proof below.) Then f is equivalent to a cocycle in $C^1\big(\Gamma,\mathrm{GL}_l(A)\big)$.*

Proof. We claim that there exists some $\varsigma\in\mathrm{M}_l(B)$ with $|\varsigma|\leq c\|\overline{f}\|$ such that the cocycle $f':\gamma\mapsto\gamma(1+\varsigma)f(\gamma)(1+\varsigma)^{-1}$ satisfies $|f'(\gamma)-1|\leq\frac{1}{rc}$ for all $\gamma\in\Gamma$ and $\|\overline{f'}\|\leq\frac{\|\overline{f}\|}{r}$ in $C^1\big(\Gamma,\mathrm{M}_l(B/A)\big)$. Granting the claim, by iterating this process, we can find a sequence $\varsigma_1,\varsigma_2,\ldots$ in $\mathrm{M}_l(B)$ with $|\varsigma_n|\leq\frac{c\|\overline{f}\|}{r^{n-1}}\leq\frac{1}{r^n}$ such that $\Big|\overline{\gamma\big(\prod_{i=1}^{n}(1+\varsigma_i)\big)f(\gamma)\big(\prod_{i=1}^{n}(1+\varsigma_i)\big)^{-1}}\Big|\leq\frac{\|\overline{f}\|}{r^n}$. Put $\varsigma_\infty=\prod_{i=1}^{\infty}(1+\varsigma_i)\in\mathrm{GL}_l(B)$. It follows that the cocycle $\widetilde{f}:\gamma\mapsto\gamma(\varsigma_\infty)f(\gamma)\varsigma_\infty^{-1}$ takes values in $\mathrm{M}_l(A)\cap\mathrm{GL}_l(B)$ and satisfies $|\widetilde{f}(\gamma)-1|\leq\frac{1}{rc}\leq\frac{1}{r}$ for $\gamma\in\Gamma$. This implies that $\widetilde{f}$ is a cocycle in $C^1\big(\Gamma,\mathrm{GL}_l(A)\big)$, and the lemma follows.

It remains to prove the claim. Note that $f(\gamma_1\gamma_2)=\gamma_1(f(\gamma_2))f(\gamma_1)$ because f is cocycle in $C^1\big(\Gamma,\mathrm{GL}_l(B)\big)$. By Lemma A.1.9, we have

$$|d\overline{f}(\gamma_1,\gamma_2)|=|\overline{\gamma_1f(\gamma_2)+f(\gamma_1)-f(\gamma_1\gamma_2)}|=|\overline{(\gamma_1f(\gamma_2)-1)(f(\gamma_1)-1)}|\leq\frac{\|\overline{f}\|}{rc}.$$

By Lemma A.1.7, there exists $\overline{h}\in\mathrm{M}_l(B/A)$ such that

$$|\overline{h}|\leq\max\big\{c\|\overline{f}\|,c^2\|d\overline{f}\|\big\}=c\|\overline{f}\|\leq\frac{1}{rc}\leq\frac{1}{r}\tag{A.1.11}$$

and

$$\|\overline{f} - d\overline{h}\| \leq c\|d\overline{f}\| \leq \tfrac{\|\overline{f}\|}{r}. \tag{A.1.12}$$

By assumption, we can lift $\overline{h}$ to some $h \in \mathrm{M}_l(B)$ such that $|h| = |\overline{h}| \leq c\|\overline{f}\|$. For $\gamma \in \Gamma$, by (A.1.11), we have $|\gamma(1+h)f(\gamma)(1+h)^{-1} - f(\gamma)| \leq |h| \leq \frac{1}{rc}$, and therefore $|\gamma(1+h)f(\gamma)(1+h)^{-1} - 1| \leq \frac{1}{rc}$. Also, again by (A.1.11), we have $|\overline{\gamma(1+h)f(\gamma)(1+h)^{-1} - \gamma(1+h)f(\gamma)(1-h)}| \leq |h|^2 \leq \frac{1}{rc}\big(c\|\overline{f}\|\big) = \frac{\|\overline{f}\|}{r}$. Also, by Lemma A.1.9, we get $|\overline{\gamma(1+h)f(\gamma)(1-h)} - \overline{f}(\gamma) - \gamma(\overline{h}) + \overline{h}| = |\overline{\gamma(h)(f(\gamma)-1)} - \overline{(f(\gamma)-1)h}) - \overline{\gamma(h)f(\gamma)h}| \leq \frac{\|\overline{f}\|}{r}$. By combining these and (A.1.12), we obtain the desired estimate $|\overline{\gamma(1+h)f(\gamma)(1+h)^{-1}}| \leq \frac{\|\overline{f}\|}{r}$, and the claim follows. □

Proof of Theorem A.1.6(1). First, let us verify the condition (2) in Definition A.1.2. Let $\{e_1, \ldots, e_l\}$ be a basis of L_i. Since $\{\Gamma_i\}_{i \in I^{\mathrm{op}}}$ form a basis of the open neighborhoods of 1 in Γ, there exists some $i_0 \geq i$ such that $|\gamma(e_j) - e_j| \leq \frac{1}{2c}$, for all $\gamma \in \Gamma_{i_0}$. Therefore, for each $i' \geq i_0$, by applying Lemma A.1.8 to the triple $(A_{i'}, \Gamma_{i'}, M = \widehat{A}_\infty / A_{i'})$, we see that $L_i \otimes_{A_i} (\widehat{A}_\infty / A_{i'})$ has totally trivial $\Gamma_{i'}$-cohomology, and therefore has totally trivial Γ-cohomology by the Hochschild–Serre spectral sequence.

Next, let us verify the condition (1) in Definition A.1.2. Let L_∞ be a finite free Γ-module over $\widehat{A}_\infty$. As mentioned earlier, by choosing an $\widehat{A}_\infty$-basis of L_∞, the Γ-module structure of L_∞ amounts to a cocycle $f \in C^1\big(\Gamma, \mathrm{GL}_l(\widehat{A}_\infty)\big)$. Moreover, we can choose i sufficiently large such that, when restricting f to a 1-cocycle of Γ_i, the assumptions of Lemma A.1.10 hold. Then L_∞ admits a free model (L_i, ι_i) over A_i, when regarded as a Γ_i-module over $\widehat{A}_\infty$. In addition, up to enlarging i, we may assume that the image of $\Gamma_i \to \mathrm{Aut}(A_i)$ is central in the image of $\Gamma \to \mathrm{Aut}(A_i)$. Now, for each $\gamma \in \Gamma$, let γL_i be the Γ_i-module over A_i defined as follows: the underlying A_i-module is the same L_i, and the new action of Γ_i on L_i is given by $\gamma_i \cdot^{\mathrm{new}} m = (\gamma \gamma_i \gamma^{-1}) m$. Note that there is a composition of canonical isomorphisms

$$L_\infty \cong L_i \otimes_{A_i} \widehat{A}_\infty \cong (\gamma L_i) \otimes_{A_i} \widehat{A}_\infty,$$

of Γ_i-modules over $\widehat{A}_\infty$, sending $m \otimes 1$ to $\gamma(m \otimes 1)$. Therefore, by applying the previous paragraph to Γ_i, there exists $i_\gamma \geq i$ such that there is a unique Γ_i-equivariant isomorphism $L_i \otimes_{A_i} A_{i_\gamma} \cong (\gamma L_i) \otimes_{A_i} A_{i_\gamma}$ inducing the above canonical one after base change from A_{i_γ} to $\widehat{A}_\infty$. Since Γ / Γ_i is finite, we can find $i' \geq i$ such that $L_i \otimes_{A_i} A_{i'}$ is stable under the action of Γ, which gives a model of L_∞, as desired. □

In the remainder of this subsection, we assume that the pair $(\{A_i\}_{i \in I}, \{\Gamma_i\}_{i \in I^{\mathrm{op}}})$ is stably decompleting.

Lemma A.1.13. *Let L be a finite projective Γ-module over A_i for some $i \in I$. Then there exists some $i_0 \geq i$ such that $L \otimes_{A_i} (\widehat{A}_\infty / A_{i'})$ has totally trivial $\Gamma_{i'}$-cohomology, for each $i' \geq i_0$, and therefore has totally trivial Γ-cohomology by the Hochschild–Serre spectral sequence.*

Proof. We may choose a finite covering $\mathfrak{B}$ of $\mathrm{Spa}(A_i, A_i^\circ)$ by rational subsets over which the restrictions of L are free. Since $(\{A_i\}_{i \in I}, \{\Gamma_i\}_{i \in I^{\mathrm{op}}})$ is stably decompleting, by Lemma A.1.8, for some sufficiently large $i_0 \geq i$, the restrictions of $L \otimes_{A_i} (\widehat{A}_\infty / A_{i'})$ to all the rational subsets as well as their intersections have totally trivial $\Gamma_{i'}$-cohomology, for every $i' \geq i_0$. By the Tate sheaf property for the structure sheaf of a stably uniform adic Banach ring (see [KL15, Thm. 2.7.7 and

2.8.10]), $\widehat{A}_\infty/A_{i'}$ satisfies the Tate acyclic condition with respect to $\mathfrak{B}$. Since L is finite projective over A_i, the same property holds for $L \otimes_{A_i} (\widehat{A}_\infty/A_{i'})$. We therefore conclude that $L \otimes_{A_i} (\widehat{A}_\infty/A_{i'})$ has totally trivial $\Gamma_{i'}$-cohomology, as desired. □

Proof of Theorem A.1.6(2). Let L_∞ be a finite projective Γ-module over $\widehat{A}_\infty$. Given Lemma A.1.13, we may proceed as in the proof of Theorem A.1.6(1) and verify the condition (2) and the second half of the condition (1) in Definition A.1.2. It remains to show the existence of a model of L_∞ over some A_i. By [KL16, Lem. 5.6.8], L_∞ is the base change to $\widehat{A}_\infty$ of a finite projective A_i-module $\widetilde{L}_i$ (without considering the Γ-action). Choose a finite covering $\widetilde{\mathfrak{B}}$ of $\mathrm{Spa}(A_i, A_i^\circ)$ by rational subsets such that the restriction of $\widetilde{L}_i$ to each rational subset in $\widetilde{\mathfrak{B}}$ is free. Then the restriction of L_∞ to each rational subset in $\widetilde{\mathfrak{B}}$ is also free. By Theorem A.1.6(1), for some $i' \geq i$, the restriction of L_∞ (as a Γ_i-module over $\widehat{A}_\infty$) to each rational subset in $\widetilde{\mathfrak{B}}$ admits a model (as a Γ_i-module) over $A_{i'}$, and these models coincide on intersections of rational subsets in $\widetilde{\mathfrak{B}}$. Thus, they glue to a model $L_{i'}$ of L_∞ (as a Γ_i-module) over $A_{i'}$, by the Kiehl gluing property for stably uniform adic Banach rings (see, again, [KL15, Thm. 2.7.7 and 2.8.10]). Finally, as in the proof of Theorem A.1.6(1), there exists $i'' \geq i'$ such that the semilinear Γ_i-action on $L_{i'} \otimes_{A_{i'}} A_{i''}$ extends to a semilinear Γ-action, which gives a model $L_{i''}$ of L_∞ as a Γ-module over $A_{i''}$. □

A.2. **Examples.** We present three examples of decompletion systems.

A.2.1. *Arithmetic towers.* We will assume that k is a finite extension of $\mathbb{Q}_p$, equipped with the standard p-adic norm. Let k_m, k_∞, $\widehat{k}_\infty$, k_m^+, $\widehat{k}_\infty^+$, etc be as in Section 2.3, for all $m \geq 1$. Let A be a reduced Tate algebra topologically of finite type over k, equipped with the supremum norm. For each $m \geq 1$, let $A_m := A \otimes_k k_m$. Then $\widehat{A}_\infty \cong A \widehat{\otimes}_k \widehat{k}_\infty$. Let $\Gamma_m := \mathrm{Gal}(k_\infty/k_m)$.

Proposition A.2.1.1. *There exists some sufficiently large l_0 (independent of A) such that the pair $(\{A_{p^{l_0}m}\}_{m\geq 1}, \{\Gamma_{p^{l_0}m}\}_{m\geq 1})$ is stably decompleting.*

Proof. It suffices to show that the pair is decompleting. For any $C > 1$, by using [BGR84, Sec. 2.7.1, Prop. 3], we can find a Schauder basis $\{e_j\}_{j\in J}$ of A over k such that $\max_{j\in J}\{|a_j e_j|\} \leq C\big|\sum_{j\in J} a_j e_j\big|$, for every convergent sum $\sum_{j\in J} a_j e_j$. Then $\{e_j\}_{j\in J}$ is also a Schauder basis of A_m over k_m, satisfying the same condition. Hence, it suffices to show that the pair $(\{k_{p^{l_0}m}\}_{m\geq 1}, \{\Gamma_{p^{l_0}m}\}_{m\geq 1})$ is decompleting for some sufficiently large l_0. We need to verify the splitting and uniform strict exactness conditions in Definition A.1.4.

As for the splitting condition, it suffices to know that each k_m admits an orthogonal complement in $\widehat{k}_\infty$ as k_m-Banach spaces; or rather that $k_{n'}$ admits an orthogonal complement in $k_{n''}$ as normed k_m-vector spaces, whenever $m|n'|n''$, by [BGR84, Sec. 2.4.2, Prop. 3, and Sec. 2.4.1, Prop. 5].

As for the uniform strict exactness condition, we shall apply the Tate–Sen formalism. By [BC08, Prop. 4.1.1], for sufficiently large l_0, and for all $l_2 \geq l_1 \geq l_0$, the cohomology $H^i(\mathrm{Gal}(k_{p^{l_2}}/k_{p^{l_1}}), k_{p^{l_2}}^+/k_{p^{l_1}}^+)$ is annihilated by p^2. Therefore, $H^i(\mathrm{Gal}(k_{p^{l_2}}/k_{p^{l_1}}), (k_{p^{l_2}}^+/k_{p^{l_1}}^+)/p^3)$ is annihilated by p^2 as well. Up to enlarging l_0, we may also assume that $k_{p^{l_1}}$ is totally ramified over $k_{p^{l_0}}$, for all $l_1 \geq l_0$. In this situation, we claim that the uniform strict exactness condition holds with $c = p^2$.

Let f be a cocycle in $C^i(\Gamma_{p^{l_1}m'}, \widehat{k}_\infty/k_{p^{l_1}m'})$, with $(p, m') = 1$. Up to replacing f with a scalar multiple, we may suppose that f lies in $C^i(\Gamma_{p^{l_1}m'}, \widehat{k}^+_\infty/k^+_{p^{l_1}m'})$. Choose any sufficiently large m'' divisible by m' and prime to p, and any $l_2 \geq l_1$ such that the image of f in $C^i(\Gamma_{p^{l_1}m'}, (\widehat{k}^+_\infty/k^+_{p^{l_1}m'})/p^3)$ is induced by a cocycle $\overline{f} \in C^i(\mathrm{Gal}(k_{p^{l_2}m''}/k_{p^{l_1}m'}), (k^+_{p^{l_2}m''}/k^+_{p^{l_1}m'})/p^3)$. From the identifications $k^+_{p^{l_2}m''} \cong k^+_{p^{l_2}} \otimes_{k^+_{p^{l_1}}} k^+_{p^{l_1}m''/m'}$ and $\mathrm{Gal}(k_{p^{l_2}m''}/k_{p^{l_1}m''}) \cong \mathrm{Gal}(k_{p^{l_2}}/k_{p^{l_1}})$, we deduce that $H^i(\mathrm{Gal}(k_{p^{l_2}m''}/k_{p^{l_1}m''}), (k^+_{p^{l_2}m''}/k^+_{p^{l_1}m'})/p^3)$ is annihilated by p^2, and so is $H^i(\mathrm{Gal}(k_{p^{l_2}m''}/k_{p^{l_1}m'}), (k^+_{p^{l_2}m''}/k^+_{p^{l_1}m'})/p^3)$, by the Hochschild–Serre spectral sequence. Hence, by using the short exact sequence

$$0 \to p^3(k^+_{p^{l_2}m''}/k^+_{p^{l_1}m'}) \to (k^+_{p^{l_2}m''}/k^+_{p^{l_1}m'}) \to (k^+_{p^{l_2}m''}/k^+_{p^{l_1}m'})/p^3 \to 0,$$

there exists $g \in C^{i-1}(\Gamma_{p^{l_1}m'}, \widehat{k}_\infty/k_{p^{l_1}m'})$ such that $\|g\| \leq p^2\|f\|$ and $f - dg \in C^i\big(\Gamma_{p^{l_1}m'}, p(\widehat{k}^+_\infty/k^+_{p^{l_1}m'})\big)$. By iterating this process, the claim follows. □

Theorem A.2.1.2. *For some sufficiently large l_0, the pair $(\{A_{p^{l_0}m}\}_{m\geq 1}, \Gamma_{p^{l_0}})$ is a decompletion system.*

Proof. Combine Proposition A.2.1.1 and Theorem A.1.6. □

A.2.2. *Geometric towers.* We consider the geometric tower as in [DLLZ, Sec. 6.1] and Section 2.3. Let k be a perfectoid field over $\mathbb{Q}_p$ containing all roots of unity, with $k^+ = \mathcal{O}_k$. Let $X = \mathrm{Spa}(A, A^+) \to \mathbb{E} := \mathrm{Spa}(k\langle P\rangle, k^+\langle P\rangle)$ be a toric chart (as in [DLLZ, Def. 3.1.12]). For $m \geq 1$, let $X_m := \mathrm{Spa}(A_m, A^+_m) := X \times_{\mathbb{E}} \mathbb{E}_m$. As in Section 2.3, for each $a \in P_{\mathbb{Q}_{\geq 0}}$, we denote $e^a \in k^+[P_{\mathbb{Q}_{\geq 0}}]$ by T^a. Let $(\widehat{A}_\infty, \widehat{A}^+_\infty)$ be the completed direct limit of (A_m, A^+_m) over the integers $m \geq 1$, which is a perfectoid algebra over k. Let $\Gamma = \Gamma_0 := \mathrm{Hom}(P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty) \cong \mathrm{Hom}\big(P^{\mathrm{gp}}, \widehat{\mathbb{Z}}(1)\big)$, as in [DLLZ, (6.1.4)], and let $\Gamma_m := \mathrm{Hom}(P^{\mathrm{gp}}_{\mathbb{Q}}/\frac{1}{m}P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty) \subset \Gamma_0$, which acts on the algebras A_m's and $\widehat{A}_\infty$ by $\gamma T^a = \gamma(a)\, T^a$, for all $\gamma \in \Gamma_0$ and $a \in P_{\mathbb{Q}_{\geq 0}}$ (and by letting γ acting trivially on A and k). We shall equip $k\langle \frac{1}{m}P\rangle$ with the supremum norm, and equip $A_m \cong A\widehat{\otimes}_{k\langle P\rangle} k\langle\frac{1}{m}P\rangle$ with the product norm, for all $m \geq 1$.

Proposition A.2.2.1. *The pair $(\{A_m\}_{m\geq 1}, \{\Gamma_m\}_{m\geq 1})$ is stably decompleting.*

Proof. Clearly, the X_m's and $\widehat{\widetilde{X}} = \mathrm{Spa}(\widehat{A}_\infty, \widehat{A}^+_\infty)$ are stably uniform. Since a rational subset of X_m is strictly étale over $\mathbb{E}_m$, it reduces to showing that the pair $(\{A_m\}_{m\geq 1}, \{\Gamma_m\}_{m\geq 1})$ is decompleting. To begin with, the condition (1) of Definition A.1.4 is clearly satisfied. Moreover, we have a direct sum decomposition $k^+[P_{\mathbb{Q}_{\geq 0}}] = k^+[P] \oplus \big(\oplus_{\chi\neq 1}(k^+[P_{\mathbb{Q}_{\geq 0}}]_\chi)\big)$, which respects the norm on both sides, by [DLLZ, (6.1.5) and Lem. 6.1.6]. By tensoring this direct sum with A_0 and completing, we obtain the desired splitting $\widehat{A}_\infty \cong A_0 \oplus (\widehat{A}_\infty/A_0)$, verifying the condition (2) of Definition A.1.4. Finally, note that [DLLZ, (6.1.5)] also induces a Γ_m-equivariant isomorphism $\widehat{A}_\infty/A_m \cong \big(\varprojlim_n (\oplus_{\chi\neq 1} M_{n,\chi})\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, where $M_{n,\chi} := (A^+_m/p^n) \otimes_{(k^+/p^n)[\frac{1}{m}P]} \big((k^+/p^n)[P_{\mathbb{Q}_{\geq 0}}]_\chi\big)$ and the direct sum is over all nontrivial finite-order characters χ of Γ_m. Thus, by using [DLLZ, Lem. 6.1.7] and proceeding as in the proof of Proposition A.2.1.1, we see that the condition (3) of Definition A.1.4 holds with $c = \max_{m>1}\big\{|\zeta_m - 1|^{-1}\big\} = p^{\frac{1}{p-1}}$, as desired. □

Theorem A.2.2.2. *The pair $(\{A_m\}_{m\geq 1}, \Gamma_0)$ is a decompletion system.*

Proof. Combine Proposition A.2.2.1 and Theorem A.1.6. □

A.2.3. *Deformation of geometric towers.* In this example, we shall follow the setup in Section 2.3. In particular, k is a finite extension of $\mathbb{Q}_p$. Let

$$\mathbb{B}_{r,m} := \big(A\widehat{\otimes}_k(B_{\mathrm{dR}}^+/\xi^r)\big) \otimes_{(B_{\mathrm{dR}}^+/\xi^r)\langle P\rangle} (B_{\mathrm{dR}}^+/\xi^r)\langle \tfrac{1}{m}P\rangle \tag{A.2.3.1}$$

be a direct system of Banach k-algebras. Recall that there is a natural action of Γ on $\mathbb{B}_{r,m}$ (see (2.3.5)). Let us fix the choice of a uniformizer ϖ of k, and equip B_{dR}^+/ξ^r with the norm $|x| := \inf\big\{|\varpi|^n : n \in \mathbb{Z},\, \varpi^{-n}x \in A_{\mathrm{inf}}/\xi^r\big\}$. This norm on B_{dR}^+/ξ^r extends the norm on k and makes B_{dR}^+/ξ^r a Banach k-algebra.

Lemma A.2.3.2. *The natural projection $\theta : B_{\mathrm{dR}}^+/\xi^r \to \widehat{k}_\infty$ admits a section s in the category of k-Banach spaces whose operator seminorm satisfies $|s| \leq 2|\varpi|^{-1}$.*

Proof. By using [BGR84, Sec. 2.7.1, Prop. 3], we can find a Schauder basis $\{e_j\}_{j\in J}$ of $\widehat{k}_\infty$ over k such that $\max_{j\in J}\big\{|b_je_j|\big\} \leq 2\big|\sum_{j\in J} b_je_j\big|$, for every convergent sum $\sum_{j\in J} b_je_j$. Moreover, we may rescale e_j such that $|\varpi| < |e_j| \leq 1$ for all $j \in J$, and lift each e_j to some element $\widetilde{e}_j$ in A_{inf}. Then we define the desired section s by mapping each convergent sum $\sum_{j\in J} b_je_j$ to $\sum_{j\in J} b_j\widetilde{e}_j$. □

Proposition A.2.3.3. *For $r \geq 1$, the pair $(\{\mathbb{B}_{r,m}\}_{m\geq1}, \{\Gamma_m\}_{m\geq1})$ is decompleting.*

Proof. The only nontrivial part is the uniform strict exactness. Let

$$\widehat{\mathbb{B}}_{r,\infty} := \big(A\widehat{\otimes}_k(B_{\mathrm{dR}}^+/\xi^r)\big)\widehat{\otimes}_{(B_{\mathrm{dR}}^+/\xi^r)\langle P\rangle}(B_{\mathrm{dR}}^+/\xi^r)\langle P_{\mathbb{Q}_{\geq0}}\rangle$$

be the completed direct limit of $\{\mathbb{B}_{r,m}\}_{m\geq1}$ (which is canonically isomorphic to $\mathbb{B}_{\mathrm{dR}}^+(\widetilde{X})/\xi^r$, by Lemma 2.3.11). Note that $(\{\mathbb{B}_{1,m}\}_{m\geq1}, \{\Gamma_m\}_{m\geq1})$ is the geometric tower considered in Section A.2.2, with $Y = X_{\widehat{k}_\infty}$, which is stably decompleting by Proposition A.2.2.1. Let $c \geq 1$ be a constant for the uniform strict exactness there. We shall show by induction that $(\{\mathbb{B}_{r,m}\}_{m\geq1}, \{\Gamma_m\}_{m\geq1})$ satisfies the uniform strict exactness for the constant $(2|\varpi|^{-2})^{r-1}c^r$. The case $r = 1$ has already been verified.

For each $r > 1$, let f be a cocycle in $C^\bullet(\Gamma_m, \widehat{\mathbb{B}}_{r,\infty}/\mathbb{B}_{r,m})$. Then its image $\overline{f}$ in $C^\bullet(\Gamma_m, \widehat{\mathbb{B}}_{1,\infty}/\mathbb{B}_{1,m})$ satisfies $\|\overline{f}\| \leq |\varpi|^{-1}\|f\|$. Let $\overline{g}$ be a cochain satisfying $d\overline{g} = \overline{f}$ with $\|\overline{g}\| \leq c\|\overline{f}\| \leq c|\varpi|^{-1}\|f\|$. By Lemma A.2.3.2, we can lift $\overline{g}$ to a cochain $\widetilde{g} \in C^\bullet(\Gamma_m, \widehat{\mathbb{B}}_{r,\infty}/\mathbb{B}_{r,m})$ with $\|\widetilde{g}\| \leq 2|\varpi|^{-1}\|\overline{g}\| \leq 2|\varpi|^{-2}c\|f\|$. Now it is straightforward to see that there is a cochain $f_1 \in C^\bullet(\Gamma_m, \widehat{\mathbb{B}}_{r-1,\infty}/\mathbb{B}_{r-1,m})$ such that $f - d\widetilde{g} = \xi f_1$ via the isometry $B_{\mathrm{dR}}^+/\xi^{r-1} \cong \xi B_{\mathrm{dR}}^+/\xi^r$ induced by multiplication by ξ, and $\|f_1\| = \|\xi f_1\| = \|f - d\widetilde{g}\| \leq 2|\varpi|^{-2}c\|f\|$. By the inductive hypothesis, we can find a cochain $g_1 \in C^\bullet(\Gamma_m, \widehat{\mathbb{B}}_{r-1,\infty}/\mathbb{B}_{r-1,m})$ satisfying $dg_1 = f_1$ with $\|g_1\| \leq (2|\varpi|^{-2})^{r-2}c^{r-1}\|f_1\| \leq (2|\varpi|^{-2})^{r-1}c^r\|f\|$. Now put $g = \widetilde{g} + \xi g_1$; here again ξg_1 is a cochain in $C^\bullet(\Gamma_m, \widehat{\mathbb{B}}_{r,\infty}/\mathbb{B}_{r,m})$ via the isometry $B_{\mathrm{dR}}^+/\xi^{r-1} \cong \xi B_{\mathrm{dR}}^+/\xi^r$. Then it is clear that $dg = f$ and $\|g\| \leq \max\big\{\|\widetilde{g}\|, \|g_1\|\big\} \leq (2|\varpi|^{-2})^{r-1}c^r\|f\|$. □

Theorem A.2.3.4. *The pair $(\{\mathbb{B}_{r,m}\}_{m\geq1}, \Gamma)$ forms a decompletion system.*

Proof. Let L_∞ be a finite projective Γ-module over $\widehat{\mathbb{B}}_{r,\infty}$. Note that if (L_m, ι_m) is a model of L_∞ over $\mathbb{B}_{r,m}$, then $(\xi^{s-1}L_m/\xi^sL_m, \overline{\iota}_m)$ is a model of $L_\infty/\xi L_\infty$ over $\mathbb{B}_{1,m}$, for each $1 \leq s \leq r$. Since $(\{\mathbb{B}_{1,m}\}_{m\geq1}, \Gamma)$ is a decompletion system, we can find an $m_0 \geq 1$ such that, for each $m' \geq m_0$ and each $1 \leq s \leq r$, the base change of $(\xi^{s-1}L_m/\xi^sL_m, \overline{\iota}_m)$ to $\mathbb{B}_{1,m'}$ is a good model, yielding that the base change of

(L_m, ι_m) to $\mathbb{B}_{r,m'}$ is a good model. Thus, we have verified the condition (2) in Definition A.1.2. Moreover, by the same argument as in the proof of Theorem A.1.6(1), this property further ensures that any two models over $\mathbb{B}_{r,m}$ becomes identical in L_∞ after base change to $\mathbb{B}_{r,m'}$ for some sufficiently large $m' \geq m$.

It remains to show the existence of a model of L_∞. Firstly, by the same argument as in the proof of Theorem A.1.6(2), for some m, we can find a finite covering of $(X_m)_{\widehat{k}_\infty}$ by rational subsets over which the restrictions of $L_\infty/\xi L_\infty$ are free. Note that, for an affinoid space Y over k, the analytic topology of $Y_{\widehat{k}_\infty}$ is generated by the base change of rational subsets of $Y_{k'}$ with $[k' : k] < \infty$. Hence, by replacing X with X_m and replacing k with a finite extension, we may assume that there exists a finite covering $X = \cup_{i \in I} \operatorname{Spa}(R_i, R_i^+)$ by rational subsets such that the restriction of $L_\infty/\xi L_\infty$ to each $\operatorname{Spa}(R_i, R_i^+)_{\widehat{k}_\infty}$ is free. Then the base change of L_∞ to $R_i \widehat{\otimes}_k (B_{\mathrm{dR}}^+/\xi^r)$, denote by $L_{\infty,i}$, is also free, because ξ is a nilpotent element. By Proposition A.2.3.3 and Theorem A.1.6(1), for some sufficiently large m, each $L_{\infty,i}$ admits a free model $(L_{m,i}, \iota_{m,i})$ over $\big(R_i \widehat{\otimes}_k (B_{\mathrm{dR}}^+/\xi^r)\big) \otimes_{(B_{\mathrm{dR}}^+/\xi^r)\langle P \rangle} (B_{\mathrm{dR}}^+/\xi^r)\langle \frac{1}{m} P \rangle$. Moreover, up to further enlarging m, we may assume that these models coincide on the intersections of rational subsets in the covering. Thus, by [LZ17, Prop. 3.3], these models glue to a model of L_∞, as desired. □

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Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China
E-mail address: `hdiao@mail.tsinghua.edu.cn`

University of Minnesota, 127 Vincent Hall, 206 Church Street SE, Minneapolis, MN 55455, USA
E-mail address: `kwlan@math.umn.edu`

Beijing International Center for Mathematical Research, Peking University, 5 Yi He Yuan Road, Beijing 100871, China
E-mail address: `liuruochuan@math.pku.edu.cn`

California Institute of Technology, 1200 East California Boulevard, Pasadena, CA 91125, USA
E-mail address: `xzhu@caltech.edu`