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Speed of random walks, isoperimetry and compression of finitely generated groups

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Abstract

We give a solution to the inverse problem (given a prescribed function, find a corresponding group) for large classes of speed, entropy, isoperimetric profile, return probability and L_p-compression functions of finitely generated groups. For smaller classes, we give solutions among solvable groups of exponential volume growth. As corollaries, we prove a recent conjecture of Amir on joint evaluation of speed and entropy exponents and we obtain a new proof of the existence of uncountably many pairwise non-quasi-isometric solvable groups, originally due to Cornulier and Tessera. We also obtain a formula relating the L_p-compression exponent of a group and its wreath product with the cyclic group for p in $[1, 2]$.

1. Introduction

An important topic in group theory is the description of asymptotic behaviors of geometric and probabilistic quantities, such as volume growth, isoperimetric profile, Hilbert and Banach space compression on the geometric side, and speed, entropy and return probability of random walks on the probabilistic side. The study of these quantities falls into three types of questions. First given a group, compute the associated functions. Secondly the inverse problem: given a function, find a group with this asymptotic behavior. Thirdly understand the relationship between these quantities and their interactions with other topics in group theory, such as amenability, Poisson boundaries, classification up to quasi-isometry, etc. This paper contributes to solve the second question for finitely generated groups of exponential volume growth.

The first solution to an inverse problem for a large class of functions concerned compression gaps for non-amenable groups. The notion of uniform embeddings of groups into Hilbert spaces, more generally Banach spaces, was introduced by Gromov in [Gro93]. This notion has numerous geometric

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applications. Yu has proved that a finitely generated group uniformly embeddable into a Hilbert space satisfies the Novikov conjecture [Yu00]; later this result was extended by Kasparov and Yu to uniformly convex Banach spaces [KY06]. These results motivate further studies of embedding properties of groups into Banach spaces. The topic of distortions of bi-Lipschitz embeddings of finite metric spaces is originally studied in Banach space theory; it has found important applications in computer science algorithms since the work of Linial, London and Rabinovich [LLR95]. Similar to distortions of bi-Lipschitz embeddings, the $\mathfrak{X}$-compression gap measures quantitatively, for an infinite finitely generated group equipped with the word metric, the least possible distortion when one embeds it into a Banach space $\mathfrak{X}$; see definitions recalled before the statement of Theorem 1.1. Arzhantseva, Drutu and Sapir proved that essentially any sublinear function is the upper bound of a Hilbert compression gap of width $\log^{1+\epsilon}(x)$ of some non-amenable group [ADS09]. Their construction does not provide a solution to the other inverse problems, because non-amenability forces volume growth to be exponential, speed and entropy growth to be linear and return probability to decay exponentially.

In the amenable setting, a partial solution to the inverse problem is known for volume growth, entropy or speed. Bartholdi and Erschler have proved [BE14] that for any regular function $f(n)$ between $n^{0.7674\cdots}$ and n, there is a group with volume growth $e^{f(n)}$ up to multiplicative constant in front of the argument. This statement at the level of exponents was first obtained in [Bri14]. For any function between $\sqrt{n}$ and n^{γ} for $\gamma < 1$, there are a group and a finitely supported measure with this entropy up to multiplicative constant by Amir-Virag [AV17]; see also [Bri13] for a statement with precision $n^{o(1)}$. Amir and Virag also showed that for any function between $n^{\frac{3}{4}}$ and n^{γ}, $\gamma < 1$, there are a group and a finitely supported measure with this speed up to multiplicative constant. These examples are all permutational wreath products, which are extensions of groups acting on rooted trees. They generalize constructions of Grigorchuk, who provided the first uncountable family of pairwise non-quasi-isometric groups [Gri85]

In [NP08], [NP11], Naor and Peres have established quantitative connections between compression of uniform embeddings of groups into Banach spaces and speed functions of random walks. Their work has renewed interest in the longstanding question (attributed to Vershik) on what speed functions are possible for simple random walks on groups; see [AV17]. It also motivates our search for a flexible construction where the metric structure is tractable.

This paper develops the construction of new families of groups, not related to groups acting on rooted trees or their extensions, for which speed, entropy, return probability, isoperimetric profiles, Hilbert and some Banach space compression can all be computed explicitly.

Before stating our result, let us recall some necessary definitions. Let Δ be a finitely generated group equipped with a generating set T, and let μ be a probability measure on Δ. Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with distribution μ. Then $W_n = X_1 \cdots X_n$ is the random walk on Δ with step distribution μ. Its law is the n-fold convolution power μ^{*n}. Its *speed* function (rate of escape) is the expectation

$$L_\mu(n) = \mathbf{E}|W_n|_\Delta = \sum_{g\in\Delta} |g|_\Delta \, \mu^{*n}(g),$$

where $|\cdot|_\Delta$ is the word distance on the Cayley graph (Δ, T). Its *Shannon entropy* is the quantity

$$H_\mu(n) = H(W_n) = -\sum_{x\in\Delta} \mu^{*n}(x) \log \mu^{*n}(x).$$

The pair (Δ,μ) has Liouville property if the *Avez asymptotic entropy* $h_\mu = \lim_{n\to\infty} \frac{H_\mu(n)}{n}$ is 0. By the work of Avez [Ave76], Derrienic [Der76], Kaimanovich-Vershik [KV83], this is equivalent to the *Liouville property* that all bounded μ-harmonic functions are constant.

The *return probabilities* of the random walk are

$$\mathbf{P}\left[W_{2n} = e\right] = \mu^{*(2n)}(e),$$

where e is the neutral element in Δ. The ℓ^p-*isoperimetric profile* is defined as

$$\Lambda_{p,\Delta,\mu}(v) = \inf\left\{ \frac{\sum_{x,y\in\Delta} |f(xy)-f(x)|^p \, \mu(y)}{2\sum_{x\in\Delta}|f(x)|^p} : f\in\ell^p(G),\ 1 \le |\mathrm{supp}(f)| \le v \right\}.$$

The Kesten criterion for amenability [Kes59] states that Δ is non-amenable if and only if the return probability of a non-degenerate symmetric random walk on Δ decays exponentially. For amenable groups, the return probability and isoperimetric profiles provide quantitative information on the Følner sets in the groups. The asymptotic behavior of the return probability and the ℓ^2-isoperimetric profile (spectral profile) essentially determine each other, as shown by Coulhon and Grigor′yan [CG97].

The *compression* of an embedding Ψ of Δ into a Banach space $\mathfrak{X}$ is the function

$$\rho_\Psi(t) = \inf\left\{ \|\Psi(x) - \Psi(y)\|_{\mathfrak{X}} : \ \left|x^{-1}y\right|_\Delta \ge t \right\}.$$

The embedding is said to be uniform if $\rho_\Psi(t) > 0$ for all $t \ge 1$ and equivariant if Ψ is a 1-cocycle; see Section 6. The couple of functions (g_1, g_2) is an $\mathfrak{X}$–*compression gap* of Δ if any 1-Lipschitz embedding $\varphi : \Delta \to \mathfrak{X}$ satisfies $\rho_\varphi(t) \le g_2(t)$ for all $t \ge 1$ and there exists a 1-Lipschitz embedding $\Psi : \Delta \to \mathfrak{X}$ such that $\rho_\Psi(t) \ge g_1(t)$ for all $t \ge 1$. The $\mathfrak{X}$–equivariant compression gap is defined in the same manner, restricting to equivariant embeddings. Let $L_p = L_p([0,1], m)$ be the standard Lebesgue space. By [NP11, Th. 9.1], when Δ is amenable, for all

$p \geq 1$, (g_1, g_2) is an L_p-compression gap of Δ if and only if it is an equivariant L_p-compression gap of Δ.

Among these quantities, the compression gap is obviously independent of the choice of the measure μ, and up to multiplicative constants, it is invariant under quasi-isometry. The ℓ^p-isoperimetric profiles and return probability associated with symmetric probability measures of finite generating support are also known to be stable under quasi-isometry (see Pittet and Saloff-Coste [PSC00]), but the stability question regarding speed and entropy is open.

The groups we construct are diagonal products of lamplighter groups. Given a family of groups $\{\Gamma_s\}$ all generated by the union of two finite groups A and B, a factor of the diagonal product is the lamplighter group $\Gamma_s \wr \mathbb{Z} = (\oplus_{\mathbb{Z}}\Gamma_s) \rtimes \mathbb{Z}$ endowed with generating set consisting of the shift on $\mathbb{Z}$, a copy of the lamp subgroup A at position 0 and a copy of the lamp subgroup B at position $k_s \in \mathbb{Z}$. The diagonal product is the subgroup of the direct product generated by the diagonal generating set. Such a group is determined once we input a family of marked groups $\{\Gamma_s\}$, usually finite and satisfying some conditions, and a sequence (k_s) of "scaling factors" (see Section 2). When the groups $\{\Gamma_s\}$ are chosen among quotients of a residually finite group Γ that is generated by finite subgroups A and B, the choice of parameters heuristically permits one to interpolate between $\Gamma \wr \mathbb{Z}$ and the wreath product $(A \times B) \wr \mathbb{Z}$ with finite lamp group.

Constructions with expander sequences as input. The construction where the groups $\{\Gamma_s\}$ are taken to an expander sequence permits to show our main result. Denote $f(x) \simeq_C g(x)$ if $\frac{1}{C}g(x) \leq f(x) \leq Cg(x)$ for all $x \geq 1$. We write $f(x) \simeq g(x)$ and call f and g equivalent if there exists C with $f(x) \simeq_C g(x)$. Write $\log_*(x) = \log(x+1)$.

THEOREM 1.1. *There exists a universal constant $C > 1$ such that the following holds. For any non-decreasing function $f : [1, \infty) \to [1, \infty)$ such that $f(1) = 1$ and $x/f(x)$ is non-decreasing, there exists a group Δ of exponential volume growth equipped with a finite generating set T and a finitely-supported symmetric probability measure μ on Δ such that*

- *the speed and entropy functions satisfy $L_\mu(n) \simeq_C H_\mu(n) \simeq_C \sqrt{n} f(\sqrt{n})$;*
- *the ℓ^p-isoperimetric profile satisfies $\Lambda_{p,\Delta,\mu}(v) \simeq_C \left(\frac{f(\log(v))}{\log v}\right)^p$ for any $p \in [1, 2]$;*
- *the return probability satisfies $-\log\left(\mu^{*(2n)}(e)\right) \simeq_C w(n)$, where $w(n)$ is defined implicitly by $n = \int_1^{w(n)} \left(\frac{s}{f(s)}\right)^2 ds$;*
- *$\left(\frac{1}{C\epsilon}\frac{n/f(n)}{\log_*^{1+\epsilon}(n/f(n))}, C \cdot 2^p \frac{n}{f(n)}\right)$ is an equivariant L_p-compression gap for Δ for any $p > 1$, $\epsilon > 0$.*

When the function f is not asymptotically linear, i.e., $\lim_{x\to\infty} f(x)/x = 0$, the group Δ can be chosen elementary amenable with asymptotic dimension one. In this case (Δ, μ) has the Liouville property, and the equivariant L_p-compression gap is also valid for $p = 1$.

Since the constant C is universal, this result is new even when f is asymptotically linear.

The first statement asserts that any regular function between diffusive $\sqrt{n}$ and linear n is the speed and entropy function of a random walk on a group. This statement on possible speed functions should be compared with known constraints on speed functions. From subadditivity, $L_\mu(n+m) \le L_\mu(n)+L_\mu(m)$ for any convolution walk on a group Δ. By Lee-Peres [LP13], there is a universal constant $c > 0$ such that for any amenable group Δ equipped with a finite generating set T, for any symmetric probability measure μ on G, $L_\mu(n) \ge c\sqrt{p_* n}$, where $p_* = \min_{g\in T} \mu(g)$. On the other hand, we obtain that any function $g(n)$ such that $\frac{g(n)}{\sqrt{n}}$ and $\frac{n}{g(n)}$ are non-decreasing is equivalent to a speed function. It improves on Amir-Virag [AV17] by the range between diffusive and $n^{\frac{3}{4}}$ for speed, and by the range close to linear for speed and entropy. The constant in Theorem 1.1 is universal, whereas the constants in [AV17] diverge when approaching linear behavior. Moreover, if only concerning speed and entropy, we can find such a group Δ in the class of 4-step solvable groups. This is the case when the groups Γ_s are lamplighters over finite d-dimensionnal lattices with $d \ge 3$; see Theorem 3.8 and Example 3.3.

As mentioned earlier, the third statement on return probability can be derived from the ℓ^2-isoperimetric profile estimate in the second statement via the Coulhon-Grigor′yan theory; see Section 4.2. For $p \in [1, 2]$, any regular function between constant and n^{-p} is equivalent to $\Lambda_{p,\Delta,\mu} \circ \exp$ for some group Δ, and any regular function between $n^{\frac{1}{3}}$ and linear n is equivalent to $-\log \mu^{*n}(e)$. Again, this should be compared with known constraints for isoperimetric profile and return probability for groups with exponential volume growth. By Coulhon and Saloff-Coste [CSC93], for any symmetric probability measure μ on Δ, $-\log \mu^{*n}(e) \ge cn^{\frac{1}{3}}$ and $\Lambda_{p,\Delta,\mu} \circ \exp(x) \ge c' x^{-p}$, $p \in [1, 2]$, where the constants c, c' depend on the volume growth rate of (Δ, T) and $p_* = \min_{g\in T} \mu(g)$.

From the result on ℓ^1-isoperimetric profile, we derive Corollary 4.7 that any sufficiently regular function above exponential is equivalent to a Følner functions. The result extends [OO13, Cor. 1.5]. It also answers [Ers10, Question 5] positively that there exists elementary amenable groups with arbitrarily fast growing Følner function, while simple random walk on it has the Liouville property. Groups of subexponential volume growth and arbitrarily large Følner function were first constructed by Erschler [Ers06].

When f is not asymptotically linear, the fourth statement asserts that any unbounded non-decreasing sublinear function $h(n)$ is equivalent to the upper bound of an equivariant L_p-compression gap of width $\log^{1+\epsilon} h(n)$ of an amenable group. Recall that equivariant and non-equivariant compression are equivalent for amenable groups [NP11]. Regarding non-equivariant compression, it is an amenable analogue to the main result in [ADS09] and slightly improves on it as the width depends on the upper bound. It also provides other examples of amenable groups with poor compression, after [Aus11], [OO13] and [BE17]. It also follows easily that any function below $\sqrt{x}$ is the upper bound of an equivariant L_p-compression gap of a non-amenable group, simply considering the direct product of an amenable solution with the free group on two generators. The equivariant compression of a non-amenable group is at most $\sqrt{x}$, and this bound is attained for free groups by [GK04].

In order to obtain the equivariant L_p-compression gap with upper bound $x/f(x)$ bounded, we actually need to choose the family $\{\Gamma_s\}$ among quotients of a Lafforgue lattice with strong Property (T) [Laf08]. In this case, we also obtain an upper bound on the compression exponent of Δ into any uniformly convex normed space; see Corollary 6.2.

Solvable examples with finite dihedral groups as input. The large scale geometry of solvable groups of exponential growth has attracted much attention in the past decades. Algebraically, solvable groups, in particular polycyclic groups, are natural "small" amenable groups to be investigated after the Gromov polynomial growth theorem [Gro81]. Remarkable quasi-isometry rigidity results have been obtained by Farb and Mosher [FM98], [FM99] for solvable Baumslag-Solitar groups and by Eskin, Fisher and Whyte [EFW12], [EFW13] for lattices in the three dimensional solvable Lie group Sol and lamplighters. In general, solvable groups do not come naturally with actions on geometric spaces whose rich structure would facilitate an investigation of the geometry of the group. Many fundamental problems are open; see the survey [FM00].

When the input sequence $\{\Gamma_s\}$ consists of finite dihedral groups, our construction of diagonal products yields 3-step solvable groups. This collection of groups demonstrates the richness of large scale geometry in the class of solvable groups. In particular, their geometric properties differ significantly from polycyclic groups. On the other hand, these groups are closely related to lamplighters, and they might serve as candidates to be considered in the quasi-isometry classification program of solvable groups.

For random walks on these solvable groups, we have the following.

THEOREM 1.2. *Let $\epsilon > 0$. There exists a constant $C > 0$. For any function f such that $\frac{f(n)}{\log^{1+\epsilon}(n)\sqrt{n}}$ and $\frac{n^{\frac{3}{4}}}{f(n)}$ are non-decreasing, there are a 3-step solvable group Δ and a finitely-supported symmetric probability measure such that the*

Exponent of	$\mathbf{E}\,\lvert W_n\rvert_\Delta$	$H(W_n)$	$-\log\mu^{*2n}(e)$	$\Lambda_{p,\Delta,\mu}\circ\exp$	$\alpha_p^\#(\Delta)$
$\{\Gamma_s\}$ expanders	$\frac{1+\theta}{2+\theta}$	$\frac{1+\theta}{2+\theta}$	$\frac{1+\theta}{3+\theta}$	$\frac{-p}{1+\theta}$	$\frac{1}{1+\theta}$
$\{\Gamma_s\}$ dihedral	$\frac{1+3\theta}{2+4\theta}$	$\frac{1}{2}$	$\frac{1}{3}$	$-p$	$\max\left\{\frac{1}{1+\theta},\frac{2}{3}\right\}$

Figure 1. Exponents for sequences of parameters $k_s = 2^{2s}$ and $l_s = \operatorname{diam}(\Gamma_s) \simeq 2^{2\theta s}$, where $\theta \in (0,\infty)$. The isoperimetric profile and compression exponent are all valid for $p \in [1,2]$. The compression for expanders is valid for $p \in [1,\infty)$.

speed function is

$$\mathbf{E}\,\lvert W_n\rvert_\Delta \simeq_C f(n)$$

and the entropy and return probability satisfy

$$\frac{1}{C}\sqrt{n} \le H(W_n^\Delta) \le C\sqrt{n}\log^2 n \text{ and } \frac{1}{C}n^{\frac{1}{3}} \le -\log\mu^{*n}(e) \le Cn^{\frac{1}{3}}\log^{\frac{4}{3}} n.$$

The factor $\log^{1+\epsilon} n$ is only technical. It follows from Proposition 3.11 that there is no gap isolating the diffusive behavior $\sqrt{n}$, but the analysis is simplified by this mild hypothesis. In contrast, it is known that for simple random walk on a polycyclic group of exponential growth, the speed is diffusive (i.e., equivalent to $\sqrt{n}$) and the return probability decays like $e^{-n^{1/3}}$. The exponents when $\{\Gamma_s\}$ are expanders or dihedral are given in Figure 1.

Questions on the relationship between the five quantities in Figure 1 are more easily asked at the level of exponents. The *exponent* of a function f is $\lim \frac{\log f(n)}{\log n}$ when the limit exists. For a compression gap of width less than any power, the lower exponent of the upper bound coincides with the definition of the $\mathfrak{X}$-compression exponent introduced in Guentner-Kaminker [GK04],

$$\alpha^*_{\mathfrak{X}}(G) = \sup\{\alpha_{\mathfrak{X}}(\Psi) :\ \Psi \text{ is a Lipschitz map } G \to \mathfrak{X}\},$$

where the compression exponent $\alpha_{\mathfrak{X}}(\Psi)$ of the map $\Psi : \Delta \to \mathfrak{X}$ is defined as

$$\alpha_{\mathfrak{X}}(\Psi) = \sup\{\alpha \ge 0 :\ \exists c > 0 \text{ s.t. } \rho_\Psi(t) \ge ct^\alpha \ \text{ for all } t \ge 1\}.$$

When $\mathfrak{X}$ is the classical Lebesgue space L_p, we write $\alpha^*_p(G)$ for the L_p-compression exponent. The equivariant compression exponent $\alpha^\#_{\mathfrak{X}}(G)$ is defined similarly, restricting to G-equivariant embeddings Ψ. When G is amenable, $\alpha^*_p(G) = \alpha^\#_p(G)$; see [NP11, Th. 1.6].

Theorems 1.1 and 1.2 together permit to solve a recent conjecture of Amir [Ami17].

Corollary 1.3 (Joint behavior of speed and entropy exponents). *For any $\theta \in \left[\frac{1}{2},1\right]$ and $\gamma \in \left[\frac{1}{2},1\right]$ satisfying*

$$\theta \le \gamma \le \frac{1}{2}(\theta+1),$$

there exist a finitely generated group G and a symmetric probability measure μ of finite support on G, such that the random walk on G with step distribution μ has entropy exponent θ and speed exponent γ.

The case where both exponents belong to $\left[\frac{3}{4}, 1\right]$ was treated by Amir [Ami17]. Proposition 3.17 gives a more precise statement regarding functions rather than exponents.

For the diagonal products constructed with finite dihedral groups that appear in Theorem 1.2, we estimate their L_p-compression exponents for $p \in [1, \infty)$; see Theorem 8.1. Explicit evaluation of compression exponents yields the following result. It answers [NP08, Question 7.6] positively within the class of finitely generated 3-step solvable groups. Moreover, with certain choices of parameters, such groups provide the first examples of amenable groups whose L_p-compression exponent, $p > 2$, is strictly larger than the Hilbert compression exponent.

THEOREM 1.4. *For any $\frac{2}{3} \leq \alpha \leq 1$, there exists a 3-step solvable group Δ such that for any $p \in [1, 2]$,*

$$\alpha_p^*(\Delta) = \alpha_p^{\#}(\Delta) = \alpha.$$

Moreover, there exists a 3-step solvable group Δ_1 such that for all $p \in (2, \infty)$,

$$\alpha_p^*(\Delta_1) \geq \frac{3p-4}{4p-5} > \alpha_2^*(\Delta_1) = \frac{2}{3}.$$

Both speed and compression exponents depend explicitly on the parameter sequences $(k_s), (l_s)$; see Figure 1 for some concrete choices of parameters. A key metric property of the diagonal product Δ that we rely on, in the estimates on speed and compression, is that placing the two types of generators k_s apart in $\Gamma_s \wr \mathbb{Z}$ essentially has the effect of rescaling the copies of Γ_s by a factor k_s. Moreover, under suitable assumptions, the metric in the diagonal product can be understood to behave similarly to a direct product.

In [Aus11, §5], Austin remarked that compression upper bounds from classical Poincaré inequalities and Markov type inequalities could be viewed as related to random walks (i.e., obstructions can be detected efficiently by suitable random walks); it would be interesting to find examples of finitely generated amenable groups for which obstructions genuinely unrelated to inequalities concerning random walks are needed. Austin conjectured that a group with a sequence of cubes $(\mathbb{Z}_m^n, \ell^\infty)$ embedded would be a candidate for such type of obstructions. In some sense our construction of diagonal product Δ with finite dihedral groups as input realizes this idea. Because of the presence of scaled ℓ^∞-cubes of growing sizes in Δ, in the proof of Theorem 1.4 we apply deep results of Mendel and Naor [MN08] on metric cotype to estimate distortion of these embedded finite cubes.

As a corollary, we obtain a new proof of the following result of Cornulier-Tessera [CT13].

Corollary 1.5 (Cornulier-Tessera [CT13]). *There exist uncountably many pairwise non-quasi-isometric finitely generated* 3*-step solvable groups.*

The original proof used asymptotic cones. Our method is completely different, using compression as quasi-isometry invariant. Theorem 1.4 and Corollary 1.5 do not hold for 2-step solvable groups. By Baumslag [Bau72], any finitely generated metabelian group embeds into a finitely presented metabelian group, so there are countably many classes of isomorphism of metabelian groups.

A formula relating the L_p-compression exponent of $H \wr \mathbb{Z}$ to H. The method we apply to estimate compression exponent of Δ in the dihedral case fundamentally differs from the case with expanders. Along the way, to better understand the compression of Δ, which is a diagonal product of groups $\Gamma_s \wr \mathbb{Z}$, it is instructive to first evaluate compression exponent of general wreath product $H \wr \mathbb{Z}$. This has been an object of intensive study; see [AGS06], [NP08], [ANP09], [Tes11], [CSV12]. We develop a novel approach and derive the following general formula which, in particular, extends the result in [NP11] on the L_p-compression exponent of $\mathbb{Z} \wr \mathbb{Z}$.

Theorem 1.6. *Let $p \in [1,2]$, and let H be a finitely generated infinite group. Then the equivariant L_p-compression exponent of $H \wr \mathbb{Z}$ is*

$$\alpha_p^{\#}(H \wr \mathbb{Z}) = \min\left\{ \frac{\alpha_p^{\#}(H)}{\alpha_p^{\#}(H) + \left(1 - \frac{1}{p}\right)}, \alpha_p^{\#}(H) \right\}.$$

Organization of the paper. The detailed description of diagonal products is given in Section 2. The construction of diagonal products is reminiscent of the piecewise automata group of Erschler [Ers06] and the groups of Kassabov-Pak [KP13], which permit us to obtain oscillating or "close to non-amenable" behaviors, but where more precise estimates are not known.

A technical assumption on the family $\{\Gamma_s\}$ and a list of examples satisfying it appears in Section 2.1. The key estimate relating the metric of a diagonal product to that of a direct product is established in Section 2.2 under the assumption that the sequence (k_s) is strictly doubling. In Section 2.3, we describe some metric spaces naturally embedded in the diagonal product. It will be used in Sections 6 and 8 on compression gaps.

Section 3 is devoted to the evaluation of speed and entropy functions of random walks on Δ. We first treat in Section 3.1 the case, including expanders, where the groups $\{\Gamma_s\}$ have uniform linear speed up to diameter. Theorem 3.8 gives the first point of Theorem 1.1. The case of dihedral groups is studied in

Section 3.2, proving two thirds of Theorem 1.2. Evaluation of speed and entropy of diagonal products relies on estimations of traverse time of the simple random walk on $\mathbb{Z}$, which are recalled in Appendix A. The joint prescription of speed and entropy of Corollary 1.3 is obtained in Section 3.3. Section 3 is not used further in the paper, and a reader not interested in speed or entropy can omit it.

Isoperimetric profiles and return probabilities are studied in Section 4. The second point of Theorem 1.1 is derived as Theorem 4.6. It is proved together with Corollary 4.7 regarding Følner functions in Section 4.1. The third point is derived as Theorem 4.8 in Section 4.2 using Coulhon-Grigor′yan theory. Dihedral groups are treated in 4.3 finishing the proof of Theorem 1.2. A reader interested mainly in compression can formally omit Section 4, though the test functions of Proposition 4.4 will be used in Section 6.

Obstructions for embeddings into Banach spaces are reviewed in Section 5. They are based on Poincaré inequalities on finite metric spaces embedded in the group. The classical spectral version stated in Section 5.1 will be used in Sections 6 and 7. Markov type inequalities of Section 5.2 will be used in Sections 7 and 8, and the Mendel-Naor metric cotype inequalities presented in Section 5.4 will be used in Section 8.

In Section 6, we consider diagonal products where $\{\Gamma_s\}$ are quotients of a Lafforgue lattice with strong Property (T). We first establish in Section 6.1 an upper bound on compression exponent valid in any uniformly convex Banach space, and then in Section 6.2 we derive the proof of the fourth part of Theorem 1.1, in the form of Theorem 6.11. This is done after three preliminary steps: first provide an upper bound when all quotients $\{\Gamma_s\}$ are finite, and secondly an upper bound when one of them is the whole group Γ. Thirdly an explicit 1-cocycle, related to isoperimetry, is constructed to get a lower bound.

Section 7 is devoted to Theorem 1.6. It requires none of previous sections except for Poincaré and Markov type inequalities of Section 5, but it uses several facts about stable random walks on lamplighters over a segment, gathered in Appendix C. It also serves as a warm-up for Section 8.

The compression of diagonal products with dihedral lamp groups is studied in Section 8. Theorem 1.4 is proved there, as well as some explicit bounds for L_p-compression $p > 2$, stated in Theorem 8.1. As before, we first derive some upper bound using metric cotype of Section 5.4 and Markov type inequalities of Section 5.2 and then describe an explicit embedding into L_q $q \geq 2$. Section 8 formally uses only Sections 2 and 5, but is best understood reading also Sections 6 and 7.

Finally we point out a few open questions in Section 9. Appendix B explains a natural approximation of regular functions by piecewise constant and linear functions. It is used repeatedly to prove Theorem 1.1 in Sections 3, 4 and 6.

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2. **The construction and metric structure**

2.1. *The construction with diagonal product.* The wreath product of a group Γ with $\mathbb{Z}$ is the group $\Gamma \wr \mathbb{Z} = \Gamma^{(\mathbb{Z})} \rtimes \mathbb{Z}$, where $\Gamma^{(\mathbb{Z})}$ is the set of functions $f : \mathbb{Z} \to \Gamma$ with finite support$(f) = \{j \in \mathbb{Z} : \ f(j) \neq e_\Gamma\}$. An element is represented by a pair (f, i). We refer to f as the lamp configuration and i as the position of the cursor. The product rule is

$$(f, i)(g, j) = (f(\cdot)g(\cdot - i), i + j).$$

The neutral element is denoted as $(\boldsymbol{e}, 0)$, where support$(\boldsymbol{e})$ is the empty set. For $j \in \mathbb{Z}$ and $\gamma \in \Gamma$, we denote by $\gamma\delta_j$ the function taking value γ at j and e_Γ elsewhere.

Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$ be two finite groups. Let $\{\Gamma_s\}$ be a sequence of groups such that each Γ_s is marked with a generating set of the form $A(s) \cup B(s)$, where $A(s)$ and $B(s)$ are finite subgroups of Γ_s isomorphic respectively to A and B. We fix the isomorphic identification and write $A(s) = \{a_1(s), \ldots, a_{|A|}(s)\}$ and similarly for $B(s)$.

Fix a sequence $(k_s)_{s \geq 0}$ of strictly increasing integers. Take the wreath product $\Delta_s = \Gamma_s \wr \mathbb{Z}$, and mark it with generating tuple $\mathcal{T}_s$

$$\mathcal{T}_s = \big(\tau(s), \alpha_1(s), \ldots, \alpha_{|A|}(s), \beta_1(s), \ldots, \beta_{|B|}(s)\big),$$

where $\tau(s) = (\boldsymbol{e}, +1)$ and

$$\alpha_i(s) = (a_i(s)\delta_0, 0), 1 \leq i \leq |A|, \ \ \beta_i(s) = (b_i(s)\delta_{k_s}, 0), 1 \leq i \leq |B|.$$

With slight abuse of notation, we use the symbol Δ_s to denote the marked group. Alternatively, the marked group Δ_s can be viewed as the projection

$$\boldsymbol{\pi}_s : \mathbf{G} = \mathbb{Z} * A * B \to \Gamma_s \wr \mathbb{Z},$$

where $\mathbf{G}$ is the free product of

$$\langle \boldsymbol{\tau} \rangle = \mathbb{Z}, \langle \boldsymbol{\alpha}_i, 1 \leq i \leq |A| \rangle \simeq A \ \text{ and } \ \langle \boldsymbol{\beta}_i, 1 \leq i \leq |B| \rangle \simeq B$$

and the projection sends $\boldsymbol{\tau}$ to $\tau(s)$, $\boldsymbol{\alpha}_i$ to $\alpha_i(s)$ and $\boldsymbol{\beta}_i$ to $\beta_i(s)$.

The *diagonal product* Δ of the, possibly finite, sequence of marked groups $\{\Delta_s\}$ is the quotient group $\mathbf{G}/\cap_s \ker(\boldsymbol{\pi}_s)$, with the projection map $\boldsymbol{\pi} : \mathbf{G} \to \Delta$. It is marked with generating tuple $\mathcal{T} = \big(\tau, \alpha_1, \ldots, \alpha_{|A|}, \beta_1, \ldots, \beta_{|B|}\big)$. A word in $\mathcal{T}$ represents e_Δ if and only if the same word in $\mathcal{T}_s$ represents e_{Δ_s} for each s. We use $\pi_s : \Delta \to \Delta_s$ to denote the projection from Δ to the component Δ_s.

The group Δ is completely determined once the family of marked groups $\{\Gamma_s\}$ and the sequence of distances (k_s) are given. An element g of Δ is completely determined by the family of projections $\pi_s(g) = (f_s, i_s)$, and one immediately checks that the projection onto $\mathbb{Z}$ is independent of s. Therefore we write $((f_s), i)$ for a typical element of Δ, where $f_s \in \Gamma_s^{(\mathbb{Z})}$ and $i \in \mathbb{Z}$.

Assumption 2.1. *Throughout the paper, we assume the following:*

- $k_0 = 0$ *and* $\Gamma_0 = A(0) \times B(0) \simeq A \times B$.
- *We assume that*

$$\Gamma_s/[A(s), B(s)]^{\Gamma_s} \simeq A(s) \times B(s) \simeq A \times B.$$

Here $[A(s), B(s)]^{\Gamma_s}$ *is the normal closure of the subgroup generated by the commutators* $[a_i(s), b_j(s)]$. *We call the quotient group* $\Gamma_s/[A(s), B(s)]^{\Gamma_s}$ *the relative abelianization of* Γ_s.

The first assumption is mainly for convenience of notation. It follows easily from Lemma 2.7 below that the marked group $(A \times B) \wr \mathbb{Z}$ with usual generating set $(k_0 = 0)$ is a quotient of Δ as soon as (k_s) is unbounded.

The second assumption is non-trivial and restrictive. It requires that the relative abelianization, which is always a quotient of $A \times B$, is in fact isomorphic to $A \times B$. As we will see below, we can find interesting families of groups satisfying Assumption 2.1.

Notation 2.2. Take a family $\{\Gamma_s\}$ of quotients of an infinite group Γ, and parametrize the group Γ_s by its diameter $l_s = \operatorname{diam}(\Gamma_s)$ with respect to the generating set $A(s) \cup B(s)$. Taking the value $l_s = \infty$ corresponds to the choice $\Gamma_s = \Gamma$, otherwise $l_s < \infty$, Γ_s is a finite quotient group of Γ. We say that the sequences $(k_s), (l_s)$ parametrize the diagonal product Δ. Formally (k_s) can take the value ∞. We use the convention that if $k_s = \infty$, then Δ_s is the trivial group $\Delta_s = \{e_{\Delta_s}\}$.

In this paper we will take a group Γ and a family $\{\Gamma_s\}$ of quotients of Γ from the following list of specific examples.

Example 2.3. The groups $\{\Gamma_s\}$ can be taken to form a family of expanders. For example, we obtain the following sequence of finite groups from the Lafforgue super expanders. By Lafforgue [Laf08], for any local field F, the group $\mathrm{SL}(3, F)$ has Property (T) in any uniformly convex Banach space $\mathfrak{X}$. A fortiori taking $F = \mathbb{F}_p[[X^{-1}]]$, this is also the case for the non-uniform lattice $\Lambda = \mathrm{SL}(3, \mathbb{F}_p[X])$, generated by the union of the two following finite subgroups:

$$A = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \simeq \mathbb{F}_p^2,$$

$$B = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/3\mathbb{Z}.$$

There also exist positive constants c, c_1 such that the "congruence subgroups"

$$\Lambda_m = \mathrm{SL}(3, \mathbb{F}_p[X]/(X^m - 1))$$

satisfy $cm - c_1 \leq \log |\Lambda_m| \leq cm + c_1$ for all $m \geq 1$.

To make sure the second part of Assumption 2.1 holds, we take Γ (resp. Γ_m) to be the diagonal product of Λ (resp. Λ_m) with $A \times B$. Since Λ is a quotient group of Γ of index at most $|A|\,|B|$, it follows from the hereditary properties (see [BdlHV08, §1.7]) that Γ has Property (T) in any uniformly convex Banach space $\mathfrak{X}$. Since $\{\Gamma_m\}$ is a sequence of finite quotient groups of Γ, by [Laf08, Prop. 5.2], there exists a constant $\delta(\Gamma, \mathfrak{X}) > 0$ such that for any function $f : \Gamma_m \to \mathfrak{X}$, $m \in \mathbb{N}$,

$$\begin{aligned} &\frac{1}{|\Gamma_m|^2} \sum_{x,y \in \Gamma_m} \|f(x) - f(y)\|_{\mathfrak{X}}^2 \\ &\qquad \leq \frac{1}{\delta(\Gamma, \mathfrak{X})} \frac{1}{|\Gamma_m|} \sum_{x \in \Gamma_m} \sum_{u \in A(m) \cup B(m)} \|f(x) - f(xu)\|_{\mathfrak{X}}^2 . \end{aligned} \tag{1}$$

In particular, $\{(\Gamma_m, A(m) \cup B(m))\}$ form a family of expanders in ℓ^2 with spectral gap uniformly bounded from below by $\delta(\Gamma, \ell^2)$.

We will refer to this family $\{\Gamma_m\}$ as the Lafforgue super expanders. Each Γ_m is marked with generating set $A(m) \cup B(m)$. Note that by construction,

$$cm - c_1 \leq \log |\Lambda_m| \leq \log |\Gamma_m| \leq cm + c_1 + \log(3p^2).$$

From the inequality (1), by [HLW06, Th. 13.8] there exists constant $c_2 > 0$ depending only on r and $\delta(\Gamma, \ell^2)$ such that the ℓ^2-distortion satisfies

$$c_2 \log |\Gamma_m| \leq c_{\ell^2}(\Gamma_m) \leq \mathrm{diam}(\Gamma_m) \leq r \log |\Gamma_m| \,,$$

and by [ADS09, Cor. 3.5], there exists $c_3 > 0$ depending only on r and $\delta(\Gamma, \mathfrak{X})$ such that

$$c_3 \log |\Gamma_m| \leq c_{\mathfrak{X}}(\Gamma_m).$$

See Section 5 for the definition of distortion. Lafforgue super expanders are a crucial tool to study compression in arbitrary uniformly convex Banach spaces.

In most statements of this paper, it is sufficient to use the classical Property (T) rather than its strengthening in uniformly convex Banach spaces. Therefore we can also take $\Lambda = \mathrm{EL}(3, \mathbb{F}_p\langle X, Y \rangle)$, which has Property (T) by

Ershov-Jaikin-Zapirain [EJZ10] generated by the union of its finite subgroups

$$A = \left\langle \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a \in \{1, X, Y\} \right\rangle \simeq \mathbb{F}_p^3,$$

$$B = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

and with "congruence subgroups" $\mathrm{EL}\,(3, \mathrm{Mat}_{m\times m}(\mathbb{F}_p)) \simeq \mathrm{SL}(3m, \mathbb{F}_p)$ the subgroup generated by the matrices $e_{i,j}(a)$ for $a \in \mathrm{Mat}_{m\times m}(\mathbb{F}_p)$ and $i \neq j$ in $\{1, 2, 3\}$, which are identity plus the matrix with only non-zero entry a in position i, j.

Examples 2.4. Some choices of families $\{\Gamma_s\}$ permit us to obtain diagonal products in the class of solvable groups.

(1) An obvious choice is to take $\Gamma_s = D_{l_s}$ as a finite dihedral group of size $2l_s$ generated by two involutions with $A = \mathbb{Z}/2\mathbb{Z}$, $B = \mathbb{Z}/2\mathbb{Z}$ and $\Gamma = D_\infty$. Then the diagonal product Δ is 3-step solvable.

(2) Another possibility is to take $\Gamma = \mathbb{Z}/2\mathbb{Z} \wr D_\infty^k$ as the lamplighter on an ordinary k-dimensionnal lattice. We can take $A = \mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})^k$ and $B = \mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})^k$ in the obvious manner. The quotients $\Gamma_s = \mathbb{Z}/2\mathbb{Z} \wr D_{m_s}^k$ are lamplighter over an ordinary discrete k-dimensionnal torus. The diagonal product Δ is 4-step solvable.

To check that this example satisfies Assumption 2.1, we denote $\pi : D_m \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as the abelianization $\pi(x) = a^{\epsilon(x)} b^{\eta(x)}$. We set

$$\begin{aligned} \pi_A : \mathbb{Z}/2\mathbb{Z} \wr D_m^k &\to \mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})^k \simeq A \\ (f, x) &\mapsto (f_A, a_1^{\epsilon_1(x)} \cdots a_k^{\epsilon_k(x)}), \end{aligned}$$

where

$$f_A(a_1^{\epsilon_1} \cdots a_k^{\epsilon_k}) = \prod_{x:\ \epsilon_i(x) = \epsilon_i, \forall i} f(x).$$

It is clear that $[A, B]^{\Gamma_m} \subset \operatorname{Ker} \pi_A$ and that we can proceed similarly for B. Finally, one can check that $\pi_A \times \pi_B$ projects onto $A \times B$.

Remark 2.5. The requirement that Γ is a quotient of a free product of *finite* groups is crucial, but we can generalize our construction to more than two finite factors, positioned on an arithmetic progression of common difference k_s. This is straightforward for metric estimate, isoperimetry and compression, but it raises some technical questions regarding random walks, as the walk inherited on the lamps is not simple anymore. For simplicity, and by lack of relevant examples, we avoid this generality.

2.2. *Description of the metric*. We describe the metric structure of the Cayley graph $(\Delta, \mathcal{T})$.

2.2.1. *Local coincidence in relative abelianizations*. We will be able to estimate the metric in our diagonal products for sequences (k_s) growing exponentially because different factors are largely independent. But the diagonal product differs from the usual direct product. For instance, the lamp configurations of relative abelianizations behave jointly.

Notation 2.6 (Projection maps under Assumption 2.1). Let $\theta_s : \Gamma_s \to \Gamma_s/[A_s, B_s] \simeq A(s) \times B(s)$ denote the projection to the relative abelianization. The projection map extends point-wise to the lamp configuration function $f_s : \mathbb{Z} \to \Gamma_s$,

$$(\theta_s(f_s))(x) = \theta_s(f_s(x)).$$

We call $\theta_s(f_s)$ the lamp configuration of the relative abelianization.

Let θ_s^A and θ_s^B denote the compositions of θ_s with the projection to $A(s)$ and $B(s)$ respectively. Then we have a decomposition of $\theta_s(f_s)$ into a commutative product of functions

$$\theta_s(f_s) = \theta_s^A(f_s)\,\theta_s^B(f_s).$$

For any element $g = ((f_s), i)$ in the diagonal product Δ, all the relative abelianization lamp configurations are determined by the first one.

Lemma 2.7. *Let $g = ((f_s), i)$ be an element in the diagonal product Δ. Then under Assumption 2.1, any one abelianized function $\theta_s(f_s)$ is determined by $\theta_0(f_0) = f_0$ and vice-versa. More precisely, for any, s*

$$\theta_s^A(f_s(x)) = \theta_0^A(f_0(x)) \text{ and } \theta_s^B(f_s(x)) = \theta_0^B(f_0(x - k_s)).$$

Proof. We proceed by induction on the word length of $g = ((f_s), i)$. Multiplying by a generator α_j, $\theta_s^A(f_s(x))$ is modified exactly at $x = i$, which also accordingly modifies $\theta_0^A(f_0(i))$. Multiplying by a generator β_j, $\theta_s^B(f_s(x))$ is modified exactly at $x = i + k_s$, which also accordingly modifies $\theta_0^B(f_0(i))$. □

2.2.2. *Local finiteness of the diagonal product*. Denote by $\pi_{\mathbb{Z}} : \mathbf{G} = \mathbb{Z} * A * B \to \mathbb{Z}$ the projection on the first factor.

Definition 2.8. The *range* $\text{Range}(w)$ of a representative word w of an element in $\mathbf{G}$ is the collection of all $\pi_{\mathbb{Z}}(w')$, where w' is a prefix of w. It is a finite subinterval of $\mathbb{Z}$, the set of sites visited by the cursor. We will also denote $\text{Range}(w)$ as its diameter.

For an element g in Δ or in Δ_s, we denote by $\text{Range}(g)$ the diameter of a minimal range interval of a word in $\mathbf{G}$ representing it and by $s_0(g)$ the maximal integer with $k_{s_0(g)} \leq \text{Range}(g)$.

Denote by $\Delta_{\leq s} = \mathbf{G}/\cap_{0\leq s'\leq s}\operatorname{Ker}(\pi_{s'})$ the diagonal product of the $s+1$ first factors and by $\pi_{\leq s} : \Delta \to \Delta_{\leq s}$ the natural projection.

Fact 2.9. *Under Assumption 2.1, for any $g \in \mathbf{G}$, the evaluation $\pi(g)$ in Δ is determined by $\pi_{\leq s_0(g)}(g)$.*

Proof. If $k_s > \operatorname{Range}(g)$, then f_s takes values in the generating set $A(s)\cup B(s)$. By Assumption 2.1, f_s is determined by $\theta_s(f_s)$ and so by $\theta_0(f_0) = f_0$ using Lemma 2.7. So all $\pi_s(g)$ for $s > s_0(g)$ can be recovered from $\pi_0(g)$. □

In particular, as the range is bounded above by the length, the above argument shows that the sequence of marked groups $(\Delta_s, \mathcal{T}_s)$ converges to $(\Delta_0, \mathcal{T}_0)$ and the sequence $(\Delta_{\leq s}, \mathcal{T}_{\leq s})$ to $(\Delta, \mathcal{T})$ in the Chabauty topology.

We also record the following.

Fact 2.10. *Assume (k_s) is unbounded.*

(i) If all the groups in the family $\{\Gamma_s\}$ are elementary amenable (e.g., finite), then the diagonal product Δ is also elementary amenable.
(ii) If all the groups in the family $\{\Gamma_s\}$ are locally embeddable into finite groups (e.g., finite), then the diagonal product Δ is also locally embeddable into finite groups.
(iii) If all the groups in the family $\{\Gamma_s\}$ are finite, then the diagonal product Δ has asymptotic dimension one.

Proof. We observe that when (k_s) is unbounded, $\ker \pi_{\mathbb{Z}}$ is locally included in a finite direct product of copies of the groups $\{\Gamma_s\}$. If G is the subgroup generated by elements $g_1, \dots, g_k$ in $\operatorname{Ker}\pi_{\mathbb{Z}}$, then by the previous fact, G is isomorphic to $\pi_{\leq S}(G)$ where $S = \max_{1\leq i\leq k} s_0(g_i)$. Moreover, in each copy Δ_s with $s \leq S$, each element g_i is described by a function $f_{i,s} : \mathbb{Z} \to \Gamma_s$ with finite support, and the group law induced is point-wise multiplication. This proves (i) and also (iii) as the asymptotic dimension of an extension is less than that of quotient plus that of kernel by [BD06].

By Fact 2.9, the ball of radius R in the Cayley graph of Δ depends only on the R-balls in Δ_s for s below the maximal s_0 with $k_{s_0} \leq R$. Each R-ball in a group Γ_s coincides with the R-ball of a finite group H_s, so the R-ball in Δ_s coincides with the R-ball in $H_s \wr \mathbb{Z}/(4R+1)\mathbb{Z}$. A fortiori, the R-ball in Δ coincides with the R-ball of the diagonal product of these s_0+1 finite groups. This proves (ii). □

2.2.3. *Metric in one copy Δ_s.* In order to estimate the metric in the diagonal product, the sequence (k_s) must grow exponentially fast. Therefore we make the

ASSUMPTION 2.11. *Throughout the paper, we assume that the sequence* (k_s) *is a sequence of strictly increasing even numbers such that* $k_{s+1} > 2k_s$ *for all* s.

Exponential growth of the sequence (k_s) is crucial for the second part of Lemma 2.13 below. We choose the factor 2 for simplicity. It could be replaced by any $m_0 > 1$, which would only modify our estimates by some multiplicative constants.

Definition 2.12. For $j \in \mathbb{Z}$, let $I_j^s = [\frac{jk_s}{2}, \frac{(j+1)k_s}{2})$. We call *essential contribution* of the function $f_s : \mathbb{Z} \to \Gamma_s$ the quantity

$$E_s(f_s) = \sum_{\left\{j\colon\, I_j^s \cap \mathrm{Range}(f_s,i) \neq \emptyset\right\}} k_s \max_{x \in I_j^s} (|f_s(x)|_{\Gamma_s} - 1)_+,$$

where $(x)_+ = \max\{x, 0\}$. In words, we partition the range into intervals of width $\frac{k_s}{2}$. Each of these intervals essentially contributes as k_s times the maximal distance $|f_s(x)|_{\Gamma_s} - 1$.

The essential contribution measures the contribution of the terms $f_s(x)$ of Γ_s-length more than 2 to the length of an element (f_s, i) of Δ_s. The range will take care of the contribution of terms of length less than 1.

LEMMA 2.13. *Let* (f_s, i) *belong to* Δ_s. *Then*

$$\max\left\{\frac{1}{8}E_s(f_s), \mathrm{Range}(f_s,i)\right\} \leq |(f_s,i)|_{\Delta_s} \leq 9\left(E_s(f_s) + \mathrm{Range}(f_s,i)\right).$$

Let Δ *be the diagonal product of* $\{\Delta_s\}$. *Under Assumptions* 2.1 *and* 2.11 *and if, moreover,* $\theta_s(f_s) = \boldsymbol{e}$, *then there is a word* $\omega(f_s, i) \in \mathbf{G}$ *of length less than the above upper bound such that*

$$\pi_{s'}(\omega(f_s,i)) = \begin{cases} (f_s, i) & \text{if } s' = s, \\ (0, i) & \text{if } s' \neq s. \end{cases}$$

Proof. The lower bound by the range is obvious.

Let $[x]$ denote the integer part of x. To get the lower bound by essential contribution, notice that in order to write $f_s(x)$, the cursor has to traverse at least $[|f_s(x)|_{\Gamma_s}/2]$ times between positions in $x - k_s$ and x because a minimal representative word alternates elements of $A(s)$ and $B(s)$. So x contributes at least $k_s\,[|f_s(x)|_{\Gamma_s}/2]$ to the length of a representative word w of (f_s, i).

If the intervals $[x - k_s, x]$ and $[x' - k_s, x']$ are disjoint, the contributions of x and x' must add up. Let x_j^s achieve the maximum of $|f_s(x)|_{\Gamma_s}$ on the interval I_j^s. The separation condition is satisfied for a family of x_j^s with the same congruence of j modulo 4. Therefore

$$\begin{aligned}|w| &\geq \max_{0\leq a\leq 3}\left(\sum_{j\equiv a \bmod 4} k_s\left[|f_s(x)|_{\Gamma_s}/2\right]\right)\\ &\geq \frac{1}{4}\sum_{j\in\mathbb{Z}} k_s \max_{I_j^s}\left[|f_s(x)|_{\Gamma_s}/2\right] \geq \frac{1}{8}E_s(f_s).\end{aligned}$$

To get the upper bound, the generic strategy is the following. We partition the convex envelope of $\operatorname{supp}(f_s)\cup\{0,i\}$, of length less than $\operatorname{Range}(f_s,i)+k_s$, into its intersections with the intervals I_j^s for $j\in\mathbb{Z}$. The elements of the partition are still denoted I_j^s. Let $J_{\max}$ (resp. $J_{\min}$) denote the maximum (resp. minimum) index of this partition, and let $w(f_s(x))$ be a fixed minimal representative word for $f_s(x)$. We produce a representative word for $(f_s,i)^{-1}$ by the following strategy.

First apply a power $p_1\leq \operatorname{Range}(f_s,i)+k_s$ of the shift τ to move the cursor from i to the rightmost point of the interval $I^s_{J_{\max}}$. Then for each integer j from $J_{\max}$ to J_{min}, produce a word ω_j that, taking the cursor from the rightmost point of I_j^s, erases all the words $w(f_s(x))$ for $x\in I_j^s$ and eventually leaves the cursor at the rightmost point of I_{j-1}^s.

The description of ω_j is the following. The first run takes the cursor to $j\frac{k_s}{2}-k_s$ and then back, so that all the last letters of $f_s(x)$ for $x\in I_j^s$ can be deleted. More precisely, while the cursor is in $[j\frac{k_s}{2},(j+1)\frac{k_s}{2})=I_j^s$, multiplying by $\alpha_l(s)$ at appropriate locations removes the last letter to those words $w(f_s(x))$ ending with $a_l(s)$ and while the cursor is in $[j\frac{k_s}{2}-k_s,(j+1)\frac{k_s}{2}-k_s)=I_{j-2}^s$ multiplying by $\beta_l(s)$ at appropriate locations removes the last letter to those words $w(f_s(x))$ ending with $b_l(s)$. One run has length $3k_s$ and cancels one letter in each of the words. The number of runs necessary to erase completely all the words is $\max_{x\in I_j^s}|f_s(x)|_{\Gamma_s}$. A last run takes the cursor from the rightmost point of I_j^s to the rightmost point of I_{j-1}^s, except for $j=J_{\min}$. Thus $|\omega_j|\leq 3k_s\max_{x\in I_j^s}|f_s(x)|_{\Gamma_s}+\max(I_j^s)-\max(I_{j-1}^s)$ without last term for $j=J_{\min}$.

Finally apply a power $p_2\leq \operatorname{Range}(f_s,i)+k_s$ of the shift τ to move the cursor from $\max(I^s_{J_{\min}})$ to 0. All in all,

$$\omega(f_s,i)=(\tau^{p_1}\omega_{J_{\max}}\cdots\omega_{J_{\min}}\tau^{p_2})^{-1}$$

is a representative word of (f_s,i) with length

$$\begin{aligned}|\omega(f_s,i)| \leq \sum_{j\in\mathbb{Z}} 3k_s\max_{I_j^s}|f_s(x)|_{\Gamma_s} + \sum_{j=J_{\min}}^{J_{\max}}\max(I_j^s)-\max(I_{j-1}^s)\\ +2\operatorname{Range}(f_s,i)+2k_s \leq 3\sum_{j\in\mathbb{Z}}k_s\max_{I_j^s}|f_s(x)|_{\Gamma_s}+3\operatorname{Range}(f_s,i)+2k_s.\end{aligned}$$

Now the number of indices j such that I_j^s intersects the range of (f_s, i) is less than $[2\operatorname{Range}(f_s,i)/k_s]+1$. Therefore

$$\sum_{j\in\mathbb{Z}} k_s \max_{I_j^s} |f_s(x)|_{\Gamma_s} \le E_s(f_s) + 2\operatorname{Range}(f_s,i) + k_s.$$

Generically, $E_s(f_s)\neq 0$ and then $k_s \le E_s(f_s)$, and the two previous inequalities give the upper bound. In the non-generic case when $E_s(f_s)=0$, then $|f_s(x)|_{\Gamma_s}\le 1$ for all x and an obvious bound is $|(f_s,i)|_{\Delta_s} \le 3\operatorname{Range}(f_s,i)$.

To get the second part of the lemma, observe first that if $E_s(f_s)=0$ and $\theta_s(f_s)=\boldsymbol{e}$, then $(f_s,i)=(\boldsymbol{e},i)$ is just a translation, and the conclusion holds trivially. In the generic case, we have to check that for each sub-word ω_j in the above procedure, the lamp function $f_{s'}^{\omega_j}$ of $\pi_{s'}(\omega_j)$ is trivial.

First assume $s'>s$. By Assumption 2.11, the cursor moves in the interval $I=[j\frac{k_s}{2}-k_s,(j+1)\frac{k_s}{2}]$ of length $\frac{3}{2}k_s<2k_s<k_{s'}$. In this condition, multiplying by α_l contributes to $f_{s'}^{\omega_j}(x)$ by $a_l(s')$ at positions $x\in I$ and multiplying by β_l contributes to $f_{s'}^{\omega_j}(x)$ by $b_l(s')$ at positions $x\in I+k_{s'}$. These intervals are disjoint, so $f_{s'}^{\omega_j}$ takes values in the generating set of Γ_s. Thus $f_{s'}^{\omega_j}(x)=\theta_{s'}(f_{s'}^{\omega_j})(x)=\theta_{s'}^{A}(f_{s'}^{\omega_j})(x)\theta_{s'}^{B}(f_{s'}^{\omega_j})(x)=e_{\Gamma_s}$, because $\theta_{s'}^{A}(f_{s'}^{\omega_j})(x)=\theta_0^{A}(f_0^{\omega_j})(x)=e$ and $\theta_{s'}^{B}(f_{s'}^{\omega_j})(x)=\theta_0^{B}(f_0^{\omega_j})(x+k_s-k_{s'})=e$ using Lemma 2.7 and our hypothesis.

Now assume $s'<s$. The generators $a_l(s')$ were applied only when the cursor i was in I_j^s. On the other hand, the generators β_l were applied only at points $i+k_s\in I_j^s$, that is, when the cursor i was in I_{j-2}^s. Then, as $k_{s'}\in[0,\frac{k_s}{2})$, by Assumption 2.11, the element $b_l(s')$ is applied only at locations $x=i+k_{s'}\in I_{j-2}^s+[0,\frac{k_s}{2})\subset I_{j-2}^s\cup I_{j-1}^s$. As the latter set is disjoint from I_j^s, the function $f_{s'}^{\omega_j}$ again takes values in the generating set of Γ_s. We conclude as above. □

2.2.4. *Description of the metric in the diagonal product.* Now we are ready to estimate metric in the diagonal product Δ.

PROPOSITION 2.14. *Suppose the sequence $\{\Gamma_s\}$ of marked groups satisfies Assumption 2.1 and the sequence (k_s) of integers satisfies Assumption 2.11. For any element $g=((f_s),i)$ in the diagonal product Δ, the word distance in $(\Delta,\mathcal{T})$ satisfies*

$$\max_{s\ge 0}|(f_s,i)|_{\Delta_s}\le |g|_{\Delta}\le 500\sum_{0\le s\le s_0(g)}|(f_s,i)|_{\Delta_s}.$$

Proof. The first inequality holds because Δ_s is a marked quotient of Δ.

For the second inequality, let $\omega(f_0,i)$ be a minimal representative word of (f_0,i). This is realized when the cursor moves across the range and at each site switches appropriately the A and B lamps. In particular, the word $\omega(f_0,i)$ has length $|(f_0,i)|_{\Delta_0} \leq 3\,\mathrm{Range}(f_0,i)$. It represents an element $((h_s),i)$ in Δ with $h_0 = f_0$ and $|h_s(x)|_{\Gamma_s} \leq 2$ for all x and s.

Then $g\omega(f_0,i)^{-1} = ((f'_s),0)$ with $f'_s = f_s h_s^{-1}$ for all s. In particular, $f'_0 = \boldsymbol{e}$, thus $\theta_s(f'_s) = \boldsymbol{e}$ for all s by Lemma 2.7. Lemma 2.13 applies and furnishes words $\omega(f'_s,0)$ such that

$$g\omega(f_0,i)^{-1}\omega(f'_1,0)^{-1}\cdots\omega(f'_{s_0(g)},0)^{-1} = e.$$

This is true in $\Delta_{\leq s_0(g)}$ and hence in Δ by Fact 2.9. This shows that

$$|g|_\Delta \leq |\omega(f_0,i)| + \sum_{1\leq s\leq s_0(g)} \left|\omega(f'_s,0)\right|.$$

We claim that $\mathrm{support}(f'_s) \subset \mathrm{support}(f_s)$. This implies that $\mathrm{Range}(f'_s,0) \leq 2\,\mathrm{Range}(f_s,i)$. Moreover, $E_s(f'_s) \leq 3E_s(f_s)$ because $|f'_s(x)|_{\Gamma_s} \leq |f_s(x)|_{\Gamma_s} + 2$ for all x and s. Therefore, using Lemma 2.13, we can conclude that

$$\begin{aligned}\left|\omega(f'_s,0)\right| &\leq 9(E_s(f'_s) + \mathrm{Range}(f'_s,0)) \leq 27(E_s(f_s) + \mathrm{Range}(f_s,i))\\ &\leq 54\max(E_s(f_s),\mathrm{Range}(f_s,i)) \leq 432\,|(f_s,i)|_{\Delta_s}.\end{aligned}$$

The claim follows from Lemma 2.7. Indeed, if $f_s(x) = e_{\Gamma_s}$, then, in particular,

$$e_A = \theta^A_s(f_s(x)) = \theta^A_0(f_0(x)) = \theta^A_0(h_0(x)) = \theta^A_s(h_s(x))$$

and

$$e_B = \theta^B_s(f_s(x)) = \theta^B_0(f_0(x-k_s)) = \theta^B_0(h_0(x-k_s)) = \theta^B_s(h_s(x))$$

so $\theta_s(h_s(x)) = e_{A\times B}$. As $|h_s(x)|_{\Gamma_s} \leq 2$, this implies that $h_s(x) = e_{\Gamma_s}$. Therefore $f'_s(x) = e_{\Gamma_s}$ as well. □

2.3. *Metric spaces embedded in* Δ. In this section, we gather some elementary facts about embeddings of some metric spaces into the diagonal product Δ. It will be used to obtain upper bounds on compression in Sections 6 and 8.

2.3.1. *Embedding a lamp group* Γ_s.

FACT 2.15. *Each group* Γ_s *embeds homothetically in the diagonal product* Δ *with ratio* k_s+1; *i.e., there is group homomorphism* $\vartheta_s : \Gamma_s \to \Delta$ *satisfying*

$$|\vartheta_s(\gamma)|_\Delta = (k_s+1)\,|\gamma|_{\Gamma_s}.$$

Proof. Let $w = a_{i_1}(s)b_{i_1}(s)\cdots a_{i_n}(s)b_{i_n}(s)$ belong to $A(s)*B(s)$. Set

$$\vartheta_s(w) = \tau^{\frac{k_s}{2}}\alpha_{i_1}\tau^{-k_s}\beta_{i_1}\tau^{k_s}\cdots\alpha_{i_n}\tau^{-k_s}\beta_{i_n}\tau^{k_s}.$$

The application $\vartheta_s : A(s)*B(s) \to \mathbf{G} = \mathbb{Z}*A*B$ induces an embedding $\vartheta_s : \Gamma_s \to \Delta$. Indeed by Lemma 2.7, we easily check that if w represents an element γ in Γ_s, then $\vartheta_s(w) = ((f_s), 0)$ with

$$f_s(x) = \begin{cases} \gamma & \text{for } x = \frac{k_s}{2}, \\ e_{\Gamma_s} & \text{for } x \neq \frac{k_s}{2} \end{cases} \text{ and } f_{s'}(x) = \begin{cases} \theta_s^A(\gamma) & \text{for } x = \frac{k_s}{2}, \\ \theta_s^B(\gamma) & \text{for } x = \frac{k_s}{2} - k_{s'} \text{ for } s' \neq s. \\ e_{\Gamma_s} & \text{for other } x \end{cases}$$

By construction, $|\vartheta_s(\gamma)|_\Delta \leq (k_s + 1)\, |\gamma|_{\Gamma_s}$. Moreover, it is clear that if w is a minimal representative of γ, then $\vartheta_s(w)$ is a minimal representative of $\pi_s(\vartheta_s(\gamma))$ in the quotient Δ_s. This proves the other inequality. $\square$

2.3.2. *Embedding products with ℓ^∞-norm.* We denote $\Gamma'_s = [A(s), B(s)]^{\Gamma_s}$. By Assumption 2.1, Γ'_s is the same as $\ker(\Gamma_s \to A(s) \times B(s))$; it is a subgroup of Γ_s of finite index $|A|\,|B|$.

Given an integer $t \geq 0$, we consider

$$\Pi_s^t = \left\{ ((f_s), 0) : \begin{array}{ll} f_s(x) \in \Gamma'_s & \text{for } x \in [0, t) \\ f_s(x) = e_{\Gamma_s} & \text{for } x \notin [0, t) \\ f_{s'} = \boldsymbol{e} & \text{for } s' \neq s \end{array} \right\}. \tag{2}$$

This is a subset of Δ. Indeed, $\theta_s(f_s) = \boldsymbol{e}$ by choice of Γ'_s, so $\theta_0(f_0) = \boldsymbol{e}$ by Lemma 2.7. Thus all such elements $((f_s), 0)$ actually belong to Δ by Lemma 2.13. Clearly, Π_s^t is a subgroup of Δ isomorphic to a direct product of t copies of Γ'_s:

$$\Pi_s^t \simeq \prod_{t \in [0,t)} \Gamma'_s.$$

By abuse of notation, we denote the elements of Π_s^t simply by functions $f_s : [0, t) \to \Gamma'_s$. The metric induced by Δ on Π_s^t can be estimated via Lemma 2.13.

LEMMA 2.16. *For any f_s in Π_s^t,*

$$\frac{1}{2} k_s \max_{[0,t)} |f_s(x)|_{\Gamma_s} \leq |f_s|_\Delta \leq 36t \max_{[0,t)} |f_s(x)|_{\Gamma_s}.$$

In particular, $\operatorname{diam}_\Delta \Pi_s^t \leq 36t \operatorname{diam}_{\Gamma_s}(\Gamma'_s) \leq 36t l_s$, *where* $l_s = \operatorname{diam}(\Gamma_s)$. *Moreover,*

$$\left| \left\{ f_s \in \Pi_s^t : \; |f_s|_\Delta \geq \frac{1}{72} \operatorname{diam}_\Delta \left(\Pi_s^t\right) \right\} \right| \geq \frac{1}{2} \left|\Pi_s^t\right|.$$

The last statements imply that Π_s^t satisfies the $(p; \frac{\operatorname{diam}_\Delta(\Pi_s^t)}{72}, \frac{1}{2})$-mass distribution condition (21); see Section 5.

Proof. This follows from Lemma 2.13. To get the lower bound, notice that as $f_s(x)$ belongs to Γ'_s, we necessarily have $|f_s(x)|_{\Gamma_s} \geq 2$ when $f_s(x) \neq e_{\Gamma_s}$. To get the upper bound, observe that there are $[2t/k_s] + 1$ intervals I_j^s intersecting

$[0,t)$, so the essential contribution is at most

$$E_s(f_s) \leq k_s\left(\left[\frac{2t}{k_s}\right]+1\right)\max_{[0,t)}|f_s(x)|_{\Gamma_s} \leq (2t+1)\max_{[0,t)}|f_s(x)|_{\Gamma_s}$$

and the range is bounded by t.

To get the second part, observe that for more than half of functions $I_j^s \to \Gamma'_s$, there exists $x \in I_j^s$ with $|f_s(x)|_{\Gamma_s} \geq \mathrm{diam}_{\Gamma_s}(\Gamma'_s)/2$. This holds for each j. Therefore there exists a subset A of Π_s^t of size $|A| \geq |\Pi_s^t|/2$ such that for each $f_s \in A$, more than half of the $[2t/k_s]+1$ intervals I_j^s intersecting $[0,t)$ satisfy $\max_{I_j^s}|f_s(x)|_{\Gamma_s} \geq \mathrm{diam}_{\Gamma_s}(\Gamma'_s)/2$. This implies that for any $f_s \in A$,

$$\begin{aligned}|f_s|_\Delta \geq E_s(f_s) &\geq \frac{1}{4}\left(\left[\frac{2t}{k_s}\right]+1\right)k_s\,\mathrm{diam}_{\Gamma_s}(\Gamma'_s)\\ &\geq \frac{t}{2}\,\mathrm{diam}_{\Gamma_s}(\Gamma'_s) \geq \frac{1}{72}\,\mathrm{diam}_\Delta(\Pi_s^t).\end{aligned}$$

□

Example 2.17. When $\Gamma_s = D_{2l_s}$ is a dihedral group of size $2l_s$, then $\Gamma'_s \simeq \mathbb{Z}_{l_s/2}$ is a cyclic group. Edges of $\mathbb{Z}_{l_s/2}$ have length 4 in the D_{2l_s} metric. For $t = \frac{k_s}{2}$, Lemma 2.16 gives

$$|f_s|_\Delta \simeq_{72} k_s \max_{[0,\frac{k_s}{2})}|f_s(x)|_{\mathbb{Z}_{l_s/2}}.$$

In particular, Π_s^t is then a copy of the discrete torus $\mathbb{Z}_{l_s/2}^{k_s/2}$ with l^∞-metric rescaled by k_s, embedded with bounded distortion in Δ.

We fix a generating set for Γ'_s using the following classical lemma.

LEMMA 2.18 (Reidemeister-Schreier algorithm). *We let (Γ, S) be a group marked with a finite generating set and $\pi : \Gamma \to F$ be a surjective mapping to a finite group F. Then*

(1) *there exists a set $C = \{a_1, \dots, a_{|F|}\}$ of coset representatives*

$$\Gamma = \bigcup_{i=1}^{|F|}(\mathrm{Ker}\,\pi)\,a_i$$

of length $|a_i|_S \leq \mathrm{diam}_{\pi(S)}(F)$;

(2) *the set $R = CSC^{-1} \cap \mathrm{Ker}\,\pi$ is a finite symmetric generating set of* $\mathrm{Ker}\,\pi$;

(3) *for any $\gamma \in \mathrm{Ker}\,\pi$,*

$$|\gamma|_R \leq |\gamma|_S \leq \left(2\,\mathrm{diam}_{\pi(S)}(F)+1\right)|\gamma|_R.$$

For Γ_s that satisfies Assumption 2.1 and $F = A(s) \times B(s)$, we fix a generating set $R(s)$ for $\Gamma'_s = \ker(\Gamma_s \to A(s) \times B(s))$ provided by the Reidemeister-Schreier algorithm. In this case $\mathrm{diam}(A(s) \times B(s)) = 2$. It follows from Lemma 2.18 that the inclusion map from $(\Gamma'_s, R(s))$ into $(\Gamma_s, A(s) \cup B(s))$ is bi-Lipschitz $|\gamma|_{R(s)} \leq |\gamma|_{\Gamma_s} \leq 5|\gamma|_{R(s)}$ for all $\gamma \in \Gamma'_s$ and that $|R(s)| \leq (|A||B|)^5$.

Consider the direct product

$$H = \prod_{s\in\mathbb{N}} \left(\Gamma_s'\right)^{k_s/2} = \prod_{s\geq 1} \Pi_s^{k_s/2},$$

and denote elements of H as $\mathbf{h} = (h_s)$, where h_s is a vector

$$h_s = \left(h_s(0), \ldots h_s\left(\frac{k_s}{2} - 1\right)\right) \in \left(\Gamma_s'\right)^{k_s/2}.$$

Equip H with a left invariant metric l,

$$l_s(\mathbf{h}) = \frac{k_s}{2} \max_{0\leq j\leq k_s/2-1} |h_s(j)|_{R_s}\,,\ l(\mathbf{h}) = \sum_{s\in\mathbb{N}} l_s(\mathbf{h}).$$

PROPOSITION 2.19. *Suppose $\{\Gamma_s\}$ is a sequence of finite groups satisfying Assumption 2.1 and (k_s) satisfies the growth assumption 2.11. Let Δ be the diagonal product constructed with $\{\Gamma_s\}$ and parameters (k_s). Then Δ is elementary amenable, and there exists an embedding $\theta : H \to \Delta$ such that for every $\mathbf{h} \in H$,*

$$\max_{s\in\mathbb{N}} l_s(\mathbf{h}) \leq |\theta(\mathbf{h})|_\Delta \leq 45000\ l(\mathbf{h}).$$

Proof. The group Δ is elementary amenable by Fact 2.10. By Proposition 2.14 and Lemma 2.16, we have for each $s \geq 0$,

$$|\theta(\mathbf{h})|_\Delta \geq |\pi_s\theta(\mathbf{h})|_{\Delta_s} \geq \frac{k_s}{2} \max_{[0,k_s/2)} |h_s(j)|_{\Gamma_s} \geq \frac{k_s}{2} \max_{[0,k_s/2)} |h_s(j)|_{R_s} = l_s(\mathbf{h})$$

and similarly

$$\begin{aligned}|\theta(\mathbf{h})|_\Delta \leq& 500 \sum_{s\geq 0} |\pi_s\left(\theta(\mathbf{h})\right)|_{\Delta_s} \leq 500 \sum_{s\geq 0} 10k_s \max_{[0,k_s/2)} |h_s(j)|_{\Gamma_s} \\ \leq& 500 \cdot 10 \cdot 5 \sum_{s\geq 0} k_s \max_{[0,k_s/2)} |h_s(j)|_{R_s} = 25000\ l(\mathbf{h}).\end{aligned}$$

□

Since (k_s) satisfies the growth assumption 2.11, it is clear that H has at most exponential volume growth with respect to the length function l. By the general theorem of Olshanskii-Osin [OO13], there exists an elementary amenable group G equipped with a finite generating set S such that H embeds as a subgroup of G, and there exists a constant $c > 0$ such that $c\,|h|_S \leq l(h) \leq |h|_S$ for all h. In general the group G provided by the Olshanskii-Osin embedding is rather large compared to H. In the current setting, although the embedding $\theta : H \to \Delta$ is not necessarily bi-Lipschitz, the geometry of group Δ is in some sense controlled by H. In particular, we will show in Sections 6 and 8 that if $\{\Gamma_s\}$ is taken to be an expander family or finite dihedral groups and if the sequences (k_s), $(\mathrm{diam}_{R(s)}(\Gamma_s'))$ satisfy certain growth conditions, then the Hilbert compression exponent of (Δ, d_Δ) is the same as (H, l),

$$\alpha_2^*\left((\Delta, d_\Delta)\right) = \alpha_2^*((H, l)).$$

3. Speed and entropy of random walk on Δ

Recall that Δ denotes the diagonal product of the sequence of marked groups $\{\Delta_s\}$. It is marked with generating tuple

$$\mathcal{T} = \left(\tau, \alpha_1, \ldots, \alpha_{|A|}, \beta_1, \ldots, \beta_{|B|}\right).$$

Let U_α and U_β denote the uniform measure on the subgroups $A = \left\{\alpha_1, \ldots, \alpha_{|A|}\right\}$ and $B = \left\{\beta_1, \ldots, \beta_{|B|}\right\}$ respectively. Let μ denote the uniform measure on $\left\{\tau^{\pm 1}\right\}$. For the convenience of speed calculation, we take the following specific "switch-walk-switch" step distribution on Δ:

$$q = (U_\alpha * U_\beta) * \mu * (U_\alpha * U_\beta).$$

Note that in the construction of Δ, since $\Gamma_0 = A(0) \times B(0)$ and $k_s > 0$ for all $s > 0$, it follows that α_i commutes with β_j, and therefore $U_\alpha * U_\beta$ is a symmetric probability measure on Δ. Let $A_{1,s}, A_{2,s}, \ldots$ (resp. $B_{1,s}, B_{2,s}, \ldots$) be a sequence of independent random variables with uniform distribution on $A(s)$ (resp. $B(s)$). We will refer to $A_{1,s}B_{1,s} \cdots A_{t,s}B_{t,s}$ as a random alternating word of length t in $A(s)$ and $B(s)$ starting with A.

We first describe what the random walk trajectory with step distribution q looks like. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables uniform on $\{\pm 1\}$, $S_n = X_1 + \cdots + X_n$; $\mathcal{A}_1, \mathcal{A}_2, \ldots$ (resp. $\mathcal{B}_1, \mathcal{B}_2, \ldots$) a sequence of independent and identically distributed random variables with distribution U_α (resp. U_β). Let W_n denote the random variable on Δ given by

$$W_n = \mathcal{A}_1 \mathcal{B}_1 \tau^{X_1} \mathcal{A}_2 \mathcal{B}_2 \cdots \mathcal{A}_{2n-1} \mathcal{B}_{2n-1} \tau^{X_n} \mathcal{A}_{2n} \mathcal{B}_{2n}.$$

Then W_n has distribution q^n. Each letter $\mathcal{A}$ can be written as $\mathcal{A} = \left(\left(f_s^{\mathcal{A}}\right), 0\right)$, where $f_s^{\mathcal{A}}(0) = \mathcal{A}_s$, with $\mathcal{A}_s = a_j(s)$ if $\mathcal{A} = \alpha_j$, and $f_s^{\mathcal{A}}(x) = e_{\Gamma_s}$ for all $x \neq 0$; similarly, $\mathcal{B} = \left(\left(f_s^{\mathcal{B}}\right), 0\right)$ where $f_s^{\mathcal{B}}(k_s) = \mathcal{B}_s$, with $\mathcal{B}_s = b_j(s)$ if $\mathcal{B} = \beta_j$, and $f_s^{\mathcal{B}}(x) = e_{\Gamma_s}$ for all $x \neq k_s$.

Now we rewrite W_n into the standard form $((f_s), z)$. Consider the projection to the copy Δ_s, from the definition of generators $\alpha_i(s)$ and $\beta_i(s)$,

$$\begin{aligned} f_s^{W_n}(y) = {} & \mathcal{A}_{1,s}^{\mathbf{1}_{\{S_0 = y\}}} \mathcal{B}_{1,s}^{\mathbf{1}_{\{S_0 = y - k_s\}}} \\ & \cdot \left(\prod_{j=1}^{n-1} \left(\mathcal{A}_{2j,s} \mathcal{A}_{2j+1,s}\right)^{\mathbf{1}_{\{S_j = y\}}} \left(\mathcal{B}_{2j,s} \mathcal{B}_{2j+1,s}\right)^{\mathbf{1}_{\{S_j = y - k_s\}}} \right) \\ & \cdot \mathcal{A}_{2n,s}^{\mathbf{1}_{\{S_n = y\}}} \mathcal{B}_{2n}^{\mathbf{1}_{\{S_n = y - k_s\}}}. \end{aligned}$$

For $x \in \mathbb{Z}$, let $T(k, x, m)$ be the number of excursions of the simple random walk $\{S_n\}$ away from x that cross $x - k$ and are completed before time m. Then conditioned on $\{S_k\}_{0 \leq k \leq n}$, the distribution of $f_s^{W_n}(y)$ is the same as a random alternating word in $A(s)$ and $B(s)$ of length $T(k_s, y, n)$ with an appropriate random letter added as prefix/suffix.

3.1. *The case with linear speed in* $\{\Gamma_s\}$. In this subsection we consider the case where speed of a simple random walk on Γ_s grows linearly up to some time comparable to the diameter l_s.

Definition 3.1. Let $\{\Gamma_s\}$ be a sequence of finite groups where each Γ_s is marked with a generating set $A(s) \cup B(s)$. Let $\eta_s = U_{A(s)} * U_{B(s)} * U_{A(s)}$, where $U_{A(s)}, U_{B(s)}$ are uniform distributions on $A(s), B(s)$. We say $\{\Gamma_s\}$ satisfies the (σ, T_s)-linear speed assumption if in each Γ_s,

$$L_{\eta_s}(t) = \mathbf{E}\left|X_t^{(s)}\right|_{\Gamma_s} \geq \sigma t \text{ for all } t \leq T_s,$$

where $X_t^{(s)}$ has distribution η_s^{*t}.

Recall that since $X_t^{(s)}$ is a random walk on a transitive graph, by [AV17, Prop. 8],

$$\max\left\{H(X_t^{(s)}), 1\right\} \geq \frac{1}{t}\left(\frac{1}{4}\mathbf{E}\left|X_t^{(s)}\right|_{\Gamma_s}\right)^2.$$

Note that if $\{\Gamma_s\}$ satisfies Assumption 2.1, η_s projects onto the uniform distribution on $A(s) \times B(s)$, and thus $H(X_t^{(s)}) \geq H(X_1^{(s)}) \geq \log(|A|\,||B||) \geq \log 4$. Therefore, in this case, the (σ, T_s)-linear speed assumption implies that

$$H(X_t^{(s)}) \geq \sigma' t \text{ for all } t \leq T_s, \text{ where } \sigma' = \left(\frac{\sigma}{4}\right)^2. \tag{3}$$

One important class of examples that satisfies the linear speed assumption consists of expander families.

Example 3.2. On $\Gamma_s = \langle A(s), B(s)\rangle$, $A(s) \simeq A, B(s) \simeq B$, let $d = |A(s)| + |B(s)| - 2$, υ_s be the uniform probability measure on $A(s) \cup B(s)$. Suppose there exists $\delta > 0$ such that the spectral gap $\lambda(\Gamma_s, \upsilon_s)$ satisfies

$$\lambda(\Gamma_s, \upsilon_s) = \inf_{f:\Gamma_s \to \mathbb{R}, f \neq c}\left\{\frac{\sum_{u,v\in\Gamma_s}|f(u)-f(uv)|^2\,\nu_s(v)}{\frac{1}{|\Gamma_s|}\sum_{u,v\in\Gamma_s}|f(u)-f(v)|^2}\right\} \geq \delta$$

for all s; that is, $\{\Gamma_s\}$ forms a family of d-regular δ-expanders in ℓ^2. Then $\{\Gamma_s\}$ satisfies the $(\sigma, c_0 \log|\Gamma_s|)$-linear speed assumption with constants $\sigma, c_0 > 0$ only depending on δ and $|A|, |B|$. We reproduce the proof of this fact for completeness; see [HLW06, Th. 3.6].

By standard comparison of Dirichlet forms,

$$\lambda(\Gamma_s, \eta_s) = \hat{\delta} \geq \frac{\delta}{|A||B|}.$$

From the spectral gap we have

$$\left|\mathbf{P}(X_t^{(s)}=x)-\frac{1}{|\Gamma_s|}\right|\leq e^{-\hat{\delta}t}.$$

Then for $t<\frac{1}{\hat{\delta}}\log|\Gamma_s|$, $\gamma=\frac{\hat{\delta}}{2\log d}$,

$$\mathbf{P}\left(X_t^{(s)}\in B(e,\gamma t)\right)\leq d^{\gamma t}\left(e^{-\hat{\delta}t}+\frac{1}{|\Gamma_s|}\right)\leq 2\exp\left(\left(\gamma\log d-\hat{\delta}\right)t\right)=2e^{-\hat{\delta}t/2}.$$

Therefore $\mathbf{E}\left|X_t^{(s)}\right|_{\Gamma_s}\geq\gamma t\left(1-2e^{-\hat{\delta}t/2}\right)$. We conclude that $\{\Gamma_s\}$ satisfies the $(\sigma,c_0\log|\Gamma_s|)$-linear speed assumption with $\sigma=\min\left\{\frac{\hat{\delta}}{4\log d},\frac{\hat{\delta}}{2\log 4}\right\}$, $c_0=1/\hat{\delta}$.

The lamplighter groups over $\mathbb{Z}^d$, $d\geq 3$, are the first examples of solvable groups where simple random walk has linear speed; see Kaimanovich-Vershik [KV83]. The following examples satisfying the linear speed assumption are analogues of finite quotients of $\mathbb{Z}_2\wr\mathbb{Z}^d$.

Example 3.3. Let $\Gamma=\mathbb{Z}_2\wr D_\infty^d$, $d\geq 3$ as in the second item of Example 2.4, marked with generating subgroups $A=\mathbb{Z}_2\wr\langle a_j,1\leq j\leq d\rangle$, $B=\mathbb{Z}_2\wr\langle b_j,1\leq j\leq d\rangle$. Fix an increasing sequence $n_s\in\mathbb{N}$, and let $\Gamma_s=\mathbb{Z}_2\wr D_{2n_s}^d$. Then Γ_s is a finite quotient of Γ. Let $A(s)$, $B(s)$ denote the projection of A and B to Γ_s. There exists constant $\sigma_d>0$ only depending on d such that $\{\Gamma_s\}$ satisfies the $\left(\sigma_d,(2n_s)^d\right)$-linear speed assumption. A proof of this fact is included in Lemma C.4 in the appendix.

3.1.1. *Bounds on speed and entropy in one copy.* In the upper bound direction, we will use the trivial bound that in each lamp group,

$$\left|f_s^{W_n}(x)\right|_{\Gamma_s}\leq\min\left\{2T\left(k_s,x,n\right)+\mathbf{1}_{\{L(x,n)>0\}}+\mathbf{1}_{\{L(x-k_s,n)>0\}},\operatorname{diam}\left(\Gamma_s\right)\right\},\tag{4}$$

where $L(x,n)=\#\{0<k\leq n:S_k=x\}$ is the local time at x. Recall that we set the parameter $l_s=\operatorname{diam}(\Gamma_s)$.

Lemma 3.4. *There exists an absolute constant $C>0$ such that for all $s\geq 0$,*

$$\mathbf{E}\left[\left|\left(f_s^{W_n},S_n\right)\right|_{\Delta_s}\mathbf{1}_{\{s\leq s_0(W_n)\}}\right]\leq\begin{cases}Cn^{\frac{1}{2}}\min\left\{\frac{n^{\frac{1}{2}}}{k_s},l_s\right\} & \text{if } k_s^2\leq n,\\ C\left(n^{\frac{1}{2}}+k_s\right)e^{-\frac{k_s^2}{8n}} & \text{if } k_s^2>n.\end{cases}$$

Proof. From the metric upper estimate in Lemma 2.13,

$$\left|\left(f_s^{W_n},S_n\right)\right|_{\Delta_s}\leq 9\left(\sum_{j\in\mathbb{Z}}k_s\max_{x\in I_j^s}\left|f_s^{W_n}(x)\right|_{\Gamma_s}+\mathcal{R}_n\right),$$

where $\mathcal{R}_n = \#\{S_k : 0 \le k \le n\}$ is the size of the range of simple random walk on $\mathbb{Z}$ and $I_j^s = \left[j\frac{k_s}{2}, (j+1)\frac{k_s}{2}\right)$. Observe that for each $x \in I_j^s$,

$$T(k_s, x, n) \le T\left(\frac{k_s}{2}, j\frac{k_s}{2}, n\right), \tag{5}$$

because each excursion from x to the left that crosses $x - k_s$ must contain an excursion from $j\frac{k_s}{2}$ to the left that crosses $(j-1)\frac{k_s}{2}$. Apply (4):

$$\begin{aligned}\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s} &\le 9\sum_{j\in\mathbb{Z}} k_s \max_{x\in I_j^s}\{2T(k_s,x,n)\} + 11(\mathcal{R}_n)\\ &\le 11\left(\sum_{j\in\mathbb{Z}} k_s T\left(\frac{k_s}{2}, j\frac{k_s}{2}, n\right) + \mathcal{R}_n\right).\end{aligned}$$

By Lemma A.1,

$$\mathbf{E}T\left(\frac{k_s}{2}, j\frac{k_s}{2}, n\right) \le \frac{2Cn^{\frac{1}{2}}}{k_s}\exp\left(-\frac{(jk_s/2)^2}{2n}\right).$$

The size of the range of the simple random walk on $\mathbb{Z}$ satisfies

$$\mathbf{P}(\mathcal{R}_n \ge x) \le \mathbf{P}\left(\max_{0\le k\le n}|S_k| \ge \frac{x}{2}\right) \le 4\exp\left(-\frac{x^2}{8n}\right).$$

Recall that by definition of $s_0(g)$ in Section 2.2.2,

$$\{s \le s_0(W_n)\} \subseteq \{\mathcal{R}_n \ge k_s\}.$$

Summing up,

$$\begin{aligned}&\mathbf{E}\left[\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s}\mathbf{1}_{\{s\le s_0(W_n)\}}\right]\\ &\le 11k_s\sum_{j\in\mathbb{Z}}\mathbf{E}T\left(\frac{k_s}{2}, j\frac{k_s}{2}, n\right) + 11\mathbf{E}\left[(\mathcal{R}_n)\mathbf{1}_{\{\mathcal{R}_n\ge k_s\}}\right]\\ &\le 11k_s\sum_{j\in\mathbb{Z}}\frac{Cn^{\frac{1}{2}}}{k_s}\exp\left(-\frac{(jk_s/2)^2}{2n}\right) + C'n^{\frac{1}{2}}e^{-\frac{k_s^2}{8n}}\\ &\le \begin{cases} C''\left(\frac{n}{k_s} + n^{\frac{1}{2}}\right) & \text{if } k_s^2 \le n,\\ C''\left(n^{\frac{1}{2}} + k_s\right)e^{-\frac{k_s^2}{8n}} & \text{if } k_s^2 > n.\end{cases}\end{aligned}$$

For $k_s^2 \le n$, since $\left|f_s^{W_n}(y)\right|_{\Gamma_s}$ cannot exceed the diameter of Γ_s, together with Lemma 2.13, we have a second upper bound

$$\mathbf{E}|(f_s^{W_n}, S_n)|_{\Delta_s} \le 10l_s\mathbf{E}(\mathcal{R}_n + k_s) \le Cl_sn^{\frac{1}{2}}.$$

Combining these bounds, we obtain the statement. $\square$

Now we turn to the lower bound direction.

LEMMA 3.5. *Suppose that $\{\Gamma_s\}$ satisfies the $(\sigma, c_0 l_s)$-linear speed assumption. Then there exists an absolute constant $C > 0$ such that for s with $k_s^2 \leq n$,*

$$\mathbf{E}\left(\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s}\right) \geq \frac{\sigma}{C} \min\left\{\frac{n}{k_s}, c_0 n^{\frac{1}{2}} l_s\right\}$$

and

$$H(f_s^{W_n}) \geq \frac{\sigma'}{C} \min\left\{\frac{n}{k_s}, c_0 n^{\frac{1}{2}} l_s\right\}.$$

Proof. We use a weaker lower bound for the metric,

$$|(f_s, z)|_{\Delta_s} \geq \sum_{y \in \mathbb{Z}} |f_s(y)|_{\Gamma_s}.$$

Applying the $(\sigma, c_0 l_s)$-linear speed assumption of Definition 3.1, we have

$$\mathbf{E}\left(\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s}\right) \geq \mathbf{E}\left(\sum_{y \in \mathbb{Z}} \left|f_s^{W_n}(y)\right|\right) \geq \sum_{y \in \mathbb{Z}} \mathbf{E}\left[\sigma \min\left\{T(k_s, y, n), c_0 l_s\right\}\right].$$

Then by Lemma A.2, there exists a constant $c > 0$ for $k_s \leq c^2 n^{\frac{1}{2}}$:

$$\mathbf{E}\left[\min\left\{T(k_s, y, n), c_0 l_s\right\}\right] \geq \frac{1}{2} \min\left\{\frac{c\sqrt{n}}{4k}, c_0 l_s\right\} \mathbf{P}\left(L\left(y, \frac{n}{2}\right) > 0\right).$$

Summing up over y,

$$\sum_{y \in \mathbb{Z}} \mathbf{E}\left[\min\left\{T(k_s, y, n), c_0 l_s\right\}\right] \geq \frac{1}{2} \min\left\{\frac{c\sqrt{n}}{4k_s}, c_0 l_s\right\} \mathbf{E}\mathcal{R}_{n/2},$$

where $\mathcal{R}_{n/2}$ is the size of the range of simple random walk on $\mathbb{Z}$ up to $n/2$. Since $\mathbf{E}\mathcal{R}_n \simeq n^{\frac{1}{2}}$, it follows that there exists a constant $C > 1$ such that for s with $k_s^2 \leq n$,

$$\mathbf{E}\left(\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s}\right) \geq \frac{\sigma}{C} \min\left\{\frac{n}{k_s}, c_0 n^{\frac{1}{2}} l_s\right\}.$$

Concerning entropy, we condition by the traverse time function (see, for instance, [AV17] for the basic properties of entropy),

$$\begin{aligned} H(f_s^{W_n}) &\geq H(f_s^{W_n} | T(k_s, \cdot, n)) = \sum_{z \in \mathbb{Z}} \mathbf{E} H(X^{(s)}_{T(k_s, z, n)}) \\ &\geq \sum_{z \in \mathbb{Z}} \mathbf{E}\left[\sigma' \min\{T(k_s, z, b), c_0 l_s\}\right] \geq \frac{\sigma'}{C} \min\left\{\frac{n}{k_s}, c_0 n^{\frac{1}{2}} l_s\right\}, \end{aligned}$$

using (3) and the same computation. □

3.1.2. *Speed and entropy estimates in the diagonal product* Δ. Recall from Assumption 2.11 that (k_s) grows exponentially. In the diagonal product Δ, by the metric upper estimate in Proposition 2.14 and speed upper estimates in Lemma 3.4, we have

$$\begin{aligned}\mathbf{E}(|W_n|_\Delta) &\le \mathbf{E}\left(500\sum_{s\le s_0(W_n)}\left|\left(f_s^{W_n},S_n\right)\right|_{\Delta_s}\right)\\ &\le \sum_{s\le s_0(n)} C\left(\min\left\{\frac{n}{k_s},n^{\frac12}l_s\right\}+n^{\frac12}\right),\end{aligned}\tag{6}$$

where

$$s_0(n)=\min\left\{s:\ k_s^2\ge n\right\}.$$

Indeed, denote $x_s=\frac{k_s}{\sqrt{n}}$ growing exponentially. Then

$$\sum_{s>s_0(n)}(n^{\frac12}+k_s)e^{-\frac{k_s^2}{8n}}\le n^{\frac12}\sum_{x_s\ge1}(1+x_s)e^{-\frac{x_s^2}{8}}\le Cn^{\frac12}.$$

In the lower bound direction, suppose $\{\Gamma_s\}$ satisfies the (σ,c_0)-linear speed Assumption 3.1. Then by the metric lower estimate in Proposition 2.14 and Lemma 3.5,

$$\mathbf{E}\left(|W_n|_\Delta\right)\ge\max_s\mathbf{E}\left(\left|\left(f_s^{W_n},S_n\right)\right|_{\Delta_s}\right)\ge\frac{\sigma}{C}\max_{s\le s_0(n)}\min\left\{\frac{n}{k_s},n^{\frac12}c_0l_s\right\}.\tag{7}$$

To understand the upper bound (6), divide the collection of Δ_s with $s\le s_0(n)$ into two subsets:

(i) Let $s_1(n)$ denote the index

$$s_1(n)=\max\left\{s\ge0:\ n^{\frac12}\ge k_sl_s\right\}.$$

Then the contribution of these $s\le s_1(n)$ to the sum is bounded by

$$\sum_{s\le s_1(n)}\mathbf{E}\left|\left(f_s^{W_n},S_n\right)\right|_{\Delta_s}\le Cn^{\frac12}\sum_{s\le s_1(n)}(l_s+1).$$

(ii) The contribution of $s\in(s_1(n)+1,s_0(n)]$ to the sum is bounded by

$$\sum_{s=s_1(n)+1}^{s_0(n)}\mathbf{E}\left|\left(f_s^{W_n},S_n\right)\right|_{\Delta_s}\le C\sum_{s=s_1(n)+1}^{s_0(n)}\left(\frac{n}{k_s}+n^{\frac12}\right).$$

Combining these parts, we have

$$\mathbf{E}(|W_n|_\Delta)\le 2C\left(n^{\frac12}\sum_{s=0}^{s_1(n)}l_s+\sum_{s=s_1(n)+1}^{s_0(n)}\frac{n}{k_s}\right).\tag{8}$$

Proposition 3.6. *Suppose that $\{\Gamma_s\}$ satisfies the $(\sigma, c_0 l_s)$-linear speed assumption and $\operatorname{diam}(\Gamma_s) \le C_0 l_s$. Suppose there exists a constant $m_0 > 1$ such that*

$$k_{s+1} > 2k_s, l_{s+1} \ge m_0 l_s \text{ for all } s.$$

Let

$$s_0(n) = \min\left\{s : \ k_s^2 \ge n\right\}, \ s_1(n) = \max\left\{s \ge 0 : \ n^{\frac{1}{2}} \ge k_s l_s\right\}.$$

Then

$$\frac{\sigma c_0}{2C}\left(n^{\frac{1}{2}} l_{s_1(n)} + \frac{n}{k_{s_1(n)+1}}\right) \le \mathbf{E}\left|W_n\right|_\Delta \le \frac{4C}{1-1/m_0}\left(n^{\frac{1}{2}} l_{s_1(n)} + \frac{n}{k_{s_1(n)+1}}\right).$$

The same bounds hold for the entropy $H(W_n^\Delta)$ with σ replaced by $\sigma' = \left(\frac{\sigma}{4}\right)^2$ and C replaced by a constant $C' > 0$ that only depends on the size of generating sets $|A| + |B|$.

Proof. The lower bound is a direct consequence of (7). For the upper bound, apply (8) and note that because of the assumption on growth of k_s, l_s, the sums satisfy

$$\sum_{s \le s_1(n)} l_s \le l_{s_1(n)} \frac{1}{1-1/m_0},$$

$$\sum_{s \ge s_1(n)+1} \frac{1}{k_s} \le \frac{2}{k_{s_1(n)+1}}.$$

For the entropy, by [Ers03, Lemma 6], there is C' depending only on the exponential rate of volume growth in Γ, which cannot exceed $\log(|A| + |B|)$, such that $H(W_n^\Delta) \le C' \mathbf{E}\left|W_n\right|_\Delta$, giving the upper bound. Lemma 3.5 gives the lower bound (7). ☐

3.1.3. *Possible speed and entropy functions.*

Example 3.7. If Γ_s is an family as in Examples 3.2 or 3.3 and $k_s = 2^{\beta s + o(s)}$ and $l_s = 2^{\iota s + o(s)}$, with $\beta, \iota \in [1, \infty)$, a direct application of Proposition 3.6 shows that the speed and entropy exponents are

$$\lim \frac{\log \mathbf{E}|W_n|_\Delta}{\log n} = \lim \frac{\log H(W_n^\Delta)}{\log n} = \frac{\beta + 2\iota}{2\beta + 2\iota} = 1 - \frac{1}{2(1 + \frac{\iota}{\beta})},$$

which can take any value in $(\frac{1}{2}, 1)$.

Theorem 3.8. *There exist universal constants $c, C > 0$ such that the following holds. For any function $\varrho : [1, \infty) \to [1, \infty)$ such that $\frac{\varrho(x)}{\sqrt{x}}$ and $\frac{x}{\varrho(x)}$ are non-decreasing, there exist a group Δ and a symmetric probability measure q of finite support on Δ such that*

$$c\varrho(n) \le \mathbf{E}\left|W_n\right|_\Delta \le C\varrho(n) \text{ and } c\varrho(n) \le H(W_n^\Delta) \le C\varrho(n).$$

Moreover, the group Δ can be chosen to be 4-step solvable.

Proof. The choice of a family of Γ_s as in Examples 3.2 or 3.3 guarantees the existence of $C_1 > 1$ such that for all $x \geq 1$, there is Γ_s of diameter l_s with $\frac{x}{C_1} \leq l_s \leq C_1 x$.

As ϱ belongs to $\mathcal{C}_{\frac{1}{2},1}$, Corollary B.3 provides two sequences (k_s) of integers and (l_s) among diameters of Γ_s with $k_{s+1} \geq m_0 k_s$ and $l_{s+1} \geq m_0 l_s$ for all s such that the function

$$\bar{\varrho}(x) = x^{\frac{1}{2}} l_s + \frac{x}{k_{s+1}} \text{ for } (k_s l_s)^2 \leq x \leq (k_{s+1} l_{s+1})^2$$

satisfies $\bar{\varrho}(x) \simeq_{2m_0 C_1^5} \varrho(x)$ for all x.

Combining with Proposition 3.6, the diagonal product Δ associated to these sequences has a speed and entropy satisfying

$$\frac{\sigma c_0}{4m_0 C C_1^5} \varrho(x) \leq \mathbf{E}|W_n|_\Delta \leq \frac{4m_0 C C_1^5}{1 - \frac{1}{m_0}} \varrho(x).$$

For $\{\Gamma_s\}$ as in Example 3.3, the group Δ is 4-step solvable. □

Note that when the speed is linear, the last term of the sequence (l_s) is infinite, and thus the last quotient Γ_s is in fact the whole group Γ. In our examples, Γ is either $\mathbb{Z}_2 \wr D_\infty^d$ for $d \geq 3$ or a lattice in $\mathrm{SL}(3, F)$. In the latter case, the finite diagonal product Δ is non-amenable. When the speed is diffusive, the last term of the sequence (k_s) is infinite, and the group Δ is a diagonal product of finitely many groups Δ_s where the lamp groups Γ_s are finite.

3.2. *The case of Δ with dihedral groups*. In this subsection we focus on the case where $\{\Gamma_s\}$ is taken to be a sequence of finite dihedral groups. Since the unlabelled Cayley graph of D_{2l} is the same as a cycle of size $2l$, the simple random walk on D_{2l} can be identified with the simple random walk on the cycle of size $2l$. Consider a simple random walk on the cycle as the projection of the simple random walk on $\mathbb{Z}$. Then the classical Gaussian bounds on the simple random walk on $\mathbb{Z}$ imply that there exist constants $c, C > 0$ such that for all $1 \leq t \leq l_s^2, 1 \leq x \leq t^{\frac{1}{2}}$,

$$c \exp\left(-Cx^2\right) \leq \mathbf{P}\left(\left|X_t^{(s)}\right|_{D_{2l_s}} \geq x t^{\frac{1}{2}}\right) \leq C \exp\left(-cx^2\right), \tag{9}$$

where $X_t^{(s)}$ is a random alternating word in $\{e, a(s)\}$ and $\{e, b(s)\}$.

Lemma 3.9. *Suppose $\Gamma_s = D_{2l_s}$. There exists an absolute constant $C > 0$ such that in each Δ_s,*

$$\begin{aligned} &\mathbf{E}\left[\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s} \mathbf{1}_{\{s \leq s_0(W_n)\}}\right] \\ &\quad \leq \begin{cases} C \min\left\{n^{\frac{3}{4}} k_s^{-\frac{1}{2}} \log^{\frac{1}{2}} k_s, n k_s^{-1}, n^{\frac{1}{2}} l_s\right\} & \textit{if } k_s^2 < n, \\ C\left(n^{\frac{1}{2}} + k_s\right) e^{-\frac{k_s^2}{8n}} & \textit{if } k_s^2 \geq n. \end{cases} \end{aligned}$$

Proof. From the upper bounds in Lemma 3.4 that are valid for any choice of $\{\Gamma_s\}$, the only bound we need to show here is that if $k_s \leq n^{\frac{1}{2}}$, then

$$\mathbf{E}\left[\left|\left(f_s^{W_n}, S_n\right)\right|_{\Delta_s} \mathbf{1}_{\{s\leq s_0(W_n)\}}\right] \leq Cn^{\frac{3}{4}} k_s^{-\frac{1}{2}} \log^{\frac{1}{2}} k_s. \tag{10}$$

To prove this, note that the collection $\left(\left|f_s^{W_n}(z)\right|_{D_\infty}\right)_{z\in I_j^s}$ as a vector is stochastically dominated by the random vector

$$\left(\left|X_{T(k_s/2, jk_s/2, n)}(z)\right|_{D_\infty} + \mathbf{1}_{\{L(n,z)>0\}} + \mathbf{1}_{\{L(n,z-k_s)>0\}}\right)_{z\in I_j^s},$$

where $\{X_t(z)\}$ is a sequence of independent random alternating words in $A(s)$ and $B(s)$ of length t. Then $\max_{z\in I_j^s}\left|f_s^{W_n}(z)\right|_{D_\infty}$ is stochastically dominated by

$$\max_{z\in I_j^s}\left|X_{T(k_s/2, jk_s/2, n)}(z)\right|_{D_\infty} + \mathbf{1}_{\{L(n,I_j^s)>0\}} + \mathbf{1}_{\{L(n,I_{j-2}^s)>0)\}}.$$

Plug in the metric estimate in Lemma 2.13,

$$\begin{aligned}
&\mathbf{E}\left|\left(f_s^{W_n}, S_n\right)\right| \mathbf{1}_{\{s\leq s_0(W_n)\}} \\
&\quad \leq 9k_s \left(\sum_{j\in\mathbb{Z}} \mathbf{E} \max_{z\in I_j^s}\left|X_{T(k_s/2, jk_s/2, n)}(z)\right|_{D_\infty}\right) + 11\mathbf{E}\left[\mathcal{R}_n \mathbf{1}_{\{\mathcal{R}_n\geq k_s\}}\right].
\end{aligned} \tag{11}$$

From the upper bound in (9), since $\max_{z\in I_j^s}|X_t(z)|$ is maximum of $k_s/2$ independent and identically distributed random variables,

$$\mathbf{P}\left(\max_{z\in I_j^s}|X_t(z)| \leq xt^{\frac{1}{2}}\right) \geq \left(1 - c_1\exp\left(-c_2x^2\right)\right)^{\left|I_j^s\right|}.$$

Then

$$\begin{aligned}
&\mathbf{E}\left[\max_{z\in I_j^s}\left|X_{T(k_s/2, jk_s/2, n)}(z)\right| \Big| \left|T(k_s/2, jk_s/2, n)\right.\right] \\
&\qquad \leq C_1 T(k_s/2, jk_s/2, n)^{\frac{1}{2}} \log^{\frac{1}{2}} k_s,
\end{aligned}$$

where C_1 depends on c_1, c_2. Applying Lemma A.1,

$$\mathbf{E}\left[\max_{z\in I_j^s}\left|X_{T(k_s/2, jk_s/2, n)}(z)\right|\right] \leq C_2\left(\frac{n^{\frac{1}{2}}}{k_s}\exp\left(-\frac{(jk_s/2)^2}{2n}\right)\right)^{\frac{1}{2}} \log^{\frac{1}{2}} k_s.$$

Plugging in the estimates in (11) and summing up over j, we obtain (10), because of the Gaussian tail. The main contribution comes from $1 \leq j \leq n^{\frac{1}{2}}/k_s$. ◻

LEMMA 3.10. *Suppose $\Gamma_s = D_{2l_s}$. There exists an absolute constant $c > 0$ such that in each Δ_s, for $n \geq ck_s^2$,*

$$\mathbf{E}\left|(f_s^{W_n}, S_n)\right|_{\Delta_s} \geq c\min\left\{n^{\frac{3}{4}} k_s^{-\frac{1}{2}} \log^{\frac{1}{2}} k_s, nk_s^{-1}, n^{\frac{1}{2}} l_s\right\}.$$

Proof. For each $z \in I_j^s = [\frac{j}{2}k_s, \frac{j+1}{2}k_s)$, the traverse time satisfies $T(k_s, z, n) \geq T(2k_s, \frac{j+1}{2}k_s, n)$. So

$$\mathbf{E} \max_{z \in I_j^s} \left| f_s^{W_n}(z) \right|_{D_{2l_s}} \geq \mathbf{E} \max_{z \in I_j^s} \left| X^s_{T(2k_s, \frac{j+1}{2}k_s, n)}(z) \right|_{D_{2l_s}},$$

where $\{X_t(z)\}$ is a sequence of independent random alternating words in $A(s)$ and $B(s)$ of length t.

For any $t \geq 1$ and $1 \leq x \leq t^{\frac{1}{2}}$,

$$\mathbf{P}\left(|X_t(z)|_{D_{2l_s}} \geq \min\left\{ x t^{\frac{1}{2}}, \frac{l_s}{2} \right\} \right) \geq c_1 e^{-c_2 x^2}.$$

By independence, this implies the existence of $c > 0$ with

$$\mathbf{P}\left(\max_{z \in I_j^s} |X_t(z)|_{D_{2l_s}} \geq \min\left\{ c \log^{\frac{1}{2}} k_s t^{\frac{1}{2}}, ct, \frac{l_s}{2} \right\} \right) \geq c.$$

Using Lemmas 2.13 and A.2, for some $c > 0$ and for all $n \geq c k_s^2$,

$$\begin{aligned}
\mathbf{E}|(f_s^{W_n}, S_n)|_{\Delta_s} &\geq \frac{k_s}{4} \sum_{j \in \mathbb{Z}} \mathbf{E} \max_{z \in I_j^s} \left| X^s_{T(2k_s, \frac{j+1}{2}k_s, n)}(z) \right|_{D_{2l_s}} \\
&\geq c k_s \sum_{j \in \mathbb{Z}} \min\left\{ n^{\frac{1}{4}} k_s^{-\frac{1}{2}} \log^{\frac{1}{2}} k_s, n^{\frac{1}{2}} k_s^{-1}, \frac{l_s}{2} \right\} \\
&\quad \cdot P_0\left(L\left(\frac{j+1}{2} k_s, \frac{n}{2} \right) > 0 \right).
\end{aligned}$$

Finally $\sum_{j \in \mathbb{Z}} P_0\left(L(\frac{j+1}{2}k_s, \frac{n}{2}) > 0 \right) \geq \mathbf{E}\mathcal{R}_{\frac{n}{2}} / k_s \geq c \frac{n^{\frac{1}{2}}}{k_s}$. □

PROPOSITION 3.11. *Suppose that $\Gamma_s = D_{2l_s}$ and that there exists $m_0 > 1$ such that $k_{s+1} > 2k_s$, $l_{s+1} > m_0 l_s$ for all s. Define*

$$t_1(n) = \max\left\{ s : \frac{l_s^2 k_s}{\log k_s} < n^{\frac{1}{2}} \text{ and } l_s k_s < n^{\frac{1}{2}} \right\}.$$

Then in the diagonal product Δ,

$$\begin{aligned}
\frac{n^{\frac{1}{2}}}{C} \left(l_{t_1(n)} + \min\left\{ n^{\frac{1}{4}} \left(\frac{\log k_{t_1(n)+1}}{k_{t_1(n)+1}} \right)^{\frac{1}{2}}, \frac{n^{\frac{1}{2}}}{k_{t_1(n)+1}} \right\} \right) &\leq \mathbf{E}(|W_n|_\Delta) \\
&\leq \frac{2Cn^{\frac{1}{2}}}{1 - 1/m_0} \left(l_{t_1(n)} + \min\left\{ n^{\frac{1}{4}} \left(\frac{\log k_{t_1(n)+1}}{k_{t_1(n)+1}} \right)^{\frac{1}{2}}, \frac{n^{\frac{1}{2}}}{k_{t_1(n)+1}} \right\} \right).
\end{aligned}$$

Remark 3.12. The bounds here are more complicated than the linear case because in Lemmas 3.9 and 3.10 we have to consider minimum over three quantities. If we further assume that $l_s \geq \log k_s$ for all s, then the bounds simplify to

$$\mathbf{E}(|W_n|_\Delta) \simeq_{Cm_0} n^{\frac{1}{2}} l_{t_1(n)} + n^{\frac{3}{4}} \left(\frac{\log k_{t_1(n)+1}}{k_{t_1(n)+1}} \right)^{\frac{1}{2}}.$$

Proof. By Proposition 2.14 and the third line of Lemma 3.9,

$$c \max_{s \leq s_0(n)} \mathbf{E}\,|W_n|_{\Delta_s} \leq \mathbf{E}\,|W_n|_\Delta \leq C \sum_{s \leq s_0(n)} \mathbf{E}\,|W_n|_{\Delta_s}.$$

The choice of $t_1(n)$ implies that

$$\min\left\{ n^{\frac{3}{4}} k_s^{-\frac{1}{2}} \log^{\frac{1}{2}} k_s, n k_s^{-1}, n^{\frac{1}{2}} l_s \right\} = \begin{cases} n^{\frac{1}{2}} l_s & \forall s \leq t_1(n), \\ \min\left\{ n^{\frac{3}{4}} k_s^{-\frac{1}{2}} \log^{\frac{1}{2}} k_s, n k_s^{-1} \right\} & \forall s > t_1(n). \end{cases}$$

Using Lemmas 3.9 and 3.10 and the exponential growth of $(k_s), (l_s)$ gives the proposition. □

Example 3.13. Let $k_s = 2^{\beta s}$ and $l_s = 2^{\iota s}$, with $\beta > 1$, $\iota > 0$, and $\Gamma_s = D_{2l_s}$. Proposition 3.11 implies with this choice of parameters that

$$\mathbf{E}\left(|W_n|_\Delta\right) \simeq n^{\frac{3\iota+\beta}{4\iota+2\beta}} (\log n)^{\frac{\iota}{2\iota+\beta}}.$$

THEOREM 3.14. *There exist universal constants $c, C > 0$ such that the following holds. For any continuous function $\varrho : [1,\infty) \to [1,\infty)$ satisfying $\varrho(1) = 1$ and $\frac{\varrho(x)}{x^{\frac{1}{2}} \log^{1+\epsilon} x}$, $\frac{x^{\frac{3}{4}}}{\varrho(x)}$ non-decreasing for some $\epsilon > 0$, there exists a 3-step solvable group Δ with dihedral groups Γ_s and a symmetric probability measure q of finite support on Δ such that*

$$c\varrho(n) \leq \mathbf{E}\,|W_n|_\Delta \leq C\varrho(n).$$

Remark 3.15. The lower condition on $\varrho(x)$ is only technical. There is no gap that would isolate diffusive behaviors among 3-step solvable groups. Indeed, it easily follows from Lemmas 3.9 and 3.10 that for any $\varrho(x)$ such that $\frac{\varrho(x)}{\sqrt{x}}$ tends to infinity, there is a group Δ with dihedral Γ_s such that $cn^{\frac{1}{2}} \leq \mathbf{E}|W_n|_\Delta \leq C\varrho(n)$ for all n.

Proof. By Corollary B.3, we can find two sequences $(\kappa_s), (l_s)$ satisfying $\log \kappa_s \leq l_s$ such that $\bar{\varrho}(x)$ and $\varrho(x)$ agree up to multiplicative constants.

Let us set $\kappa_s = \left(\frac{k_s}{\log k_s}\right)^{\frac{1}{2}}$. Then $\log k_s \simeq \log \kappa_s \leq l_s^{\frac{1}{1+\varepsilon}}$. By Corollary 3.11, the diagonal product Δ with dihedral groups $\Gamma_s = D_{2l_s}$ and sequence (k_s) satisfies, for any $(l_s\kappa_s)^4 \leq n \leq (l_{s+1}\kappa_{s+1})^4$,

$$c\bar{\varrho}(x) = c\left(n^{\frac{1}{2}} l_s + \frac{n^{\frac{3}{4}}}{\kappa_{s+1}} \right) \leq \mathbf{E}\,|W_n|_\Delta \leq C\left(n^{\frac{1}{2}} l_s + \frac{n^{\frac{3}{4}}}{\kappa_{s+1}} \right) = C\bar{\varrho}(x).$$

The group Δ clearly has a trivial third derived subgroup. □

PROPOSITION 3.16. *There exist two constants $c, C > 0$ such that on any diagonal product Δ with dihedral Γ_s satisfying Assumption 2.11, the entropy of the switch-walk-switch random walk satisfies*

$$c\sqrt{n} \leq H(W_n^{\Delta}) \leq C\sqrt{n}\log^2 n.$$

Proof. By Proposition 2.14, the entropy of the random walk on Δ is related to the entropy on the factors Δ_s by the following:

$$\max_{s\geq 0} H(f_s^{W_n}, S_n) \leq H((f_s^{W_n}), S_n) \leq H(S_n) + \sum_{s\leq s_2(n)} H(f_s^{W_n}),$$

where $s_0(W_n) \leq s_2(n) = \max\{s : \ k_s \leq n\}$ and $H(S_n) \simeq \log n$. The lower bound comes from the first factor $H(f_0^{W_n}, S_n)$, which is the usual random walk on the lamplighter group with finite lamps.

Denote by $T_s^n = T(k_s, \cdot, n)$ the traverse time function and by $[\operatorname{supp} T_s^n]$ the convex envelope of its support. Using conditional entropy (see, for instance, [KV83] or [AV17] for standard properties of conditional entropy), we have

$$\begin{aligned} H(f_s^{W_n}) &\leq H(f_s^{W_n}|T_s^n) + H(T_s^n) \\ &\leq H(f_s^{W_n}|T_s^n) + H(T_s^n|[\operatorname{supp} T_s^n]) + H([\operatorname{supp} T_s^n]). \end{aligned} \tag{12}$$

The convex envelope $[\operatorname{supp} T_s^n]$ is included in $\mathrm{Range}(W_n)$, and therefore $H([\operatorname{supp} T_s^n]) \simeq \log n$ and $\mathbf{E}|[\operatorname{supp} T_s^n]| \leq C\sqrt{n}$. As for all z, $0 \leq T_s^n(z) \leq n$, we deduce that

$$H(T_s^n|[\operatorname{supp} T_s^n]) \leq \mathbf{E}\left[|[\operatorname{supp} T_s^n]|\right]\log n \leq C\sqrt{n}\log n.$$

As each $f_s^{W_n}(z)$ is distributed as an independent sample of a random walk on Γ_s of length $T(k_s, z, n)$, we have

$$H(f_s^{W_n}|T_s^n) = \sum_{z\in\mathbb{Z}} \mathbf{E} H(X^{(s)}_{T(k_s,z,n)}).$$

The groups Γ_s being dihedral $H(X_t^{(s)}) \leq \log t$, we therefore have

$$H(f_s^{W_n}|T_s^n) \leq \sum_{z\in\mathbb{Z}} \mathbf{E}\log T(k_s, z, n) \leq C\sqrt{n}\log n.$$

Obviously $n \geq k_{s_0(W_n)}$, so finally piling up the inequalities,

$$H((f_s^{W_n}), S_n) \leq C\log(n)(1 + 2\sqrt{n}\log n + \log n). \qquad \square$$

3.3. *Joint evaluation of speed and entropy*. Using the idea of Amir [Ami17] of taking the direct product of two groups, we can combine the speed and entropy estimates on Δ together with the results of Amir-Virag [AV17] to show the following result concerning the joint behavior of growth of entropy and speed.

Recall that for symmetric probability measure μ on G with finite support, entropy and speed satisfy

$$\frac{1}{n}\left(\frac{1}{4}L_\mu(n)\right)^2 - 1 \le H_\mu(n) \le (v+\varepsilon)L_\mu(n) + \log n + C,$$

where v is the volume growth rate of $(G, \mathrm{supp}\mu)$ and $C > 0$ is an absolute constant ([Ers03], [AV17]).

Proposition 3.17. *Let $f, h : \mathbb{N} \to \mathbb{N}$ be two functions such that $h(1) = 1$ and*

- *either $\frac{f(n)}{n^{\frac{3}{4}}}$ and $\frac{n^{1-\epsilon}}{f(n)}$ are non-decreasing for some $\epsilon > 0$ and*
$$h(n) \le f(n) \le \sqrt{\frac{nh(n)}{\log n}},$$
- *or $\frac{h(n)}{n^{\frac{1}{2}}\log^2 n}$ and $\frac{n^{\frac{3}{4}}}{f(n)}$ are non-decreasing and*
$$h(n) \le f(n) \le \sqrt{nh(n)}.$$

Then there exist a constant $C > 0$ depending only on $\epsilon > 0$, a finitely generated group G and a symmetric probability measure μ of finite support on G such that

$$L_\mu(n) \simeq_C f(n) \text{ and } H_\mu(n) \simeq_C h(n).$$

As a corollary, we derive the Corollary 1.3 regarding joint entropy and speed exponents (Conjecture 3 in [Ami17]).

Proof of Corollary 1.3. If $\theta \in \left[\frac{1}{2}, 1\right)$, then take functions

$$h(n) = \max\left\{n^{\frac{1}{2}}\log^2 n, n^\theta\right\}, \quad f(n) = \max\left\{n^\gamma, h(n)\right\}.$$

Note that in the case $\gamma < 1$, the pair of functions f and h is covered by one of the cases in Proposition 3.17; the statement follows.

If $\theta = 1$, in this case $\gamma = 1$ as well. We can take G to be any finitely generated group that admits a symmetric probability measure μ of finite support such that (G, μ) has linear entropy growth. $\square$

Proof. We follow Amir's approach in [Ami17] to take the direct product of two appropriate groups such that one would control the speed function and the other would control the entropy function.

In the first case where both $f(n)/n^{\frac{3}{4}}$ and $n^{1-\epsilon}/f(n)$ are non-decreasing, consider the direct product of the following two groups. By [AV17], there exist a group $G_1 = \mathbb{Z} \wr_{\mathcal{S}} \mathcal{M}_m$ and step distribution q_1 on G_1 such that

$$L_{q_1}(n) \simeq f(n) \text{ and } H_{q_1}(n) \simeq \frac{f(n)^2}{n}\log(n+1).$$

By Theorem 3.8, there exist a group Δ and step distribution q_2 on Δ such that

$$L_{q_2}(n) \simeq H_{q_2}(n) \simeq h(n).$$

Then on the direct product $G_1 \times \Delta$ with step distribution $q_1 \otimes q_2$,

$$L_{q_1 \otimes q_2}(n) \simeq \max\{L_{q_1}(n), L_{q_2}(n)\} \simeq f(n),$$
$$H_{q_1 \otimes q_2}(n) \simeq \max\{H_{q_1}(n), H_{q_2}(n)\} \simeq \max\left\{\frac{f(n)^2}{n}\log(n+1), h(n)\right\} = h(n).$$

In the second case, where $f(n)/\left(n^{\frac{1}{2}}\log^2 n\right)$ and $n^{\frac{3}{4}}/f(n)$ are non-decreasing, by Theorem 3.14 and Proposition 3.16, there exist a group Δ_1 and step distribution q_1' on Δ_1 such that

$$L_{q_1'}(n) \simeq f(n) \text{ and } \frac{1}{C}n^{\frac{1}{2}} \leq H_{q_1'}(n) \leq Cn^{\frac{1}{2}}\log^2 n.$$

By Theorem 3.8, there exist a group Δ_2 and step distribution q_2' such that

$$L_{q_2'}(n) \simeq H_{q_2'}(n) \simeq h(n).$$

Then on the direct product $\Delta_1 \times \Delta_2$ with step distribution $q_1' \otimes q_2'$,

$$L_{q_1' \otimes q_2'}(n) \simeq \max\left\{L_{q_1'}(n), L_{q_2'}(n)\right\} \simeq f(n),$$
$$H_{q_1' \otimes q_2'}(n) \simeq \max\left\{H_{q_1'}(n), H_{q_2'}(n)\right\} \simeq h(n). \qquad \square$$

Remark 3.18. This method permits us to prove Corollary 1.3 but cannot handle functions that oscillate cross the $n^{\frac{3}{4}}$ borderline. The problem can be reduced to finding extremal examples where the speed follows a prescribed function while entropy growth is as slow as possible. The Amir-Virag result covers the case where speed grows at least like $n^{\frac{3}{4}}$. The construction of the diagonal product Δ is not designed to achieve such a goal.

4. **Isoperimetric profiles and return probabilities**

In this section we consider the isoperimetric profiles and return probabilities of Δ when $\{\Gamma_s\}$ are taken to be expanders or dihedral groups. For convenience of calculation, we take the switch-or-walk measure $\mathfrak{q} = \frac{1}{2}(\mu+\nu)$ on Δ where μ is the simple random walk measure on the base $\mathbb{Z}$, $\mu\left(\tau^{\pm 1}\right) = \frac{1}{2}$, and ν is uniform on $\{\alpha_i, \beta_j : 1 \leq i \leq |A|, 1 \leq j \leq |B|\}$. Let $\mathfrak{q}_s$ be the projection of $\mathfrak{q}$ to the quotient Δ_s.

4.1. *Isoperimetric profiles.* We first recall some background information. Given a symmetric probability measure ϕ on G, the p-Dirichlet form associated with (G, ϕ) is

$$\mathcal{E}_{p,\phi}(f) = \frac{1}{2}\sum_{x,y\in G}|f(xy) - f(x)|^p \phi(y)$$

and the ℓ^p-isoperimetric profile $\Lambda_{p,G,\phi} : [1,\infty) \to \mathbb{R}$ is defined as

$$\Lambda_{p,G,\phi}(v) = \inf \{\mathcal{E}_{p,\phi}(f) : |\mathrm{support}(f)| \leq v, \|f\|_p = 1\}. \tag{13}$$

The most important ones are the ℓ^1 and ℓ^2-isoperimetric profiles. Using an appropriate discrete co-area formula, $\Lambda_{1,\phi}$ can equivalently be defined by

$$\Lambda_{1,G,\phi}(v) = \inf \left\{ |\Omega|^{-1} \sum_{x,y} \mathbf{1}_{\Omega}(x)\mathbf{1}_{G\setminus\Omega}(xy)\phi(y) : |\Omega| \leq v \right\}.$$

If we define the boundary of Ω to be the set

$$\partial\Omega = \{(x,y) \in G \times G : x \in \Omega, y \in G \setminus \Omega\}$$

and set $\phi(\partial\Omega) = \sum_{x\in\Omega, xy\in G\setminus\Omega} \phi(y)$, then

$$\Lambda_{1,G,\phi}(v) = \inf\{\phi(\partial\Omega)/|\Omega| : |\Omega| \leq v\}.$$

When ϕ is a symmetric measure supported by a finite generating set S, then $\Lambda_{1,G,\phi}(v)$ is closely related to the Følner function : $(0,\infty) \to \mathbb{N}$ defined as

$$\mathrm{Fol}_{G,S}(r) = \min \left\{ |\Omega| : \ \Omega \subset G, \ \frac{|\partial_S \Omega|}{|\Omega|} < \frac{1}{r} \right\},$$

where $|\partial_S\Omega| = \{x \in \Omega : \ \exists u \in S, \ xu \notin \Omega\}$. Namely, let $p_* = \min\{\phi(u) : \ u \in S, u \neq \mathrm{id}\}$; then

$$\Lambda^{-1}_{1,G,\phi}(1/r) \leq \mathrm{Fol}_{G,S}(r) \leq \Lambda^{-1}_{1,G,\phi}(p_*/r),$$

where $\Lambda^{-1}_{1,G,\phi}$ is the generalized inverse of $\Lambda_{1,G,\phi}$.

We will repeatedly use the following two facts.

For any $1 \leq p \leq q \leq 2$, the isoperimetric profiles $\Lambda_{p,G,\phi}$ and $\Lambda_{q,G,\phi}$ are related by the Cheeger type inequality

$$c_0 \Lambda^{q/p}_{p,G,\phi} \leq \Lambda_{q,G,\phi} \leq C_0 \Lambda_{p,G,\phi}, \tag{14}$$

where c_0, C_0 are absolute constants; see [LS88], [SCZ16, Prop. 2.8].

Let H be a quotient group of G, and let $\bar{\phi}$ be the projection of ϕ on H. Then by [Tes13, Prop. 4.5], for all $1 \leq p < \infty$,

$$\Lambda_{p,G,\phi} \geq \Lambda_{p,H,\bar{\phi}}.$$

4.1.1. *Isoperimetric profiles of one factor*. Let $\{\Gamma_s\}$ be a family of expanders, as in Example 2.3. Let $\nu = \nu_s$ be the uniform distribution on $A(s) \cup B(s)$ in Γ_s. Denote by $h(\Gamma_s, \nu)$ the Cheeger constant

$$h(\Gamma_s, \nu) = \inf \left\{ \Lambda_{1,\Gamma_s,\nu}(v) : \ v \leq \frac{|\Gamma_s|}{2}, \ v < \infty \right\}.$$

In one copy $\Delta_s = \Gamma_s \wr \mathbb{Z}$, we establish the following lower bound of ℓ^p-isoperimetric profile of Δ_s.

LEMMA 4.1. *Let $p \in [1,2]$, Γ_s be a finite group marked with generating subgroups $A(s), B(s)$. Let $\mathfrak{q}_s$ be the uniform distribution on $\{\tau^{\pm 1}\} \cup A(s) \cup B(s)$ in Δ_s. Then there exists an absolute constant $C > 1$ such that the following is true*:

- (*Slow phase*) *for $1 \leq v \leq 2^{k_s/2}$,*

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{(\log_2 v)^{-p}}{C\left(|A(s)| + |B(s)|\right)}.$$

- (*Fast, then slow down*) *for $|\Gamma_s|^r \leq v \leq |\Gamma_s|^{r+1}$, $r \geq k_s$,*

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{h(\Gamma_s,\nu)^p}{C\left(|A(s)| + |B(s)|\right) r^p}.$$

Remark 4.2. By monotonicity of the profile function Λ_{p,Δ_s,q_s}, we have that

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{h(\Gamma_s,\nu)^p}{C\left(|A(s)| + |B(s)|\right) k_s^p} \quad \text{for } v \in \left(2^{k_s/2}, |\Gamma_s|^{k_s}\right).$$

Proof. We prove a lower bound for the ℓ^1-isoperimetric profile $\Lambda_{1,\Delta_s,\mathfrak{q}_s}$ and use the Cheeger inequality (14) to derive a lower bound on $\Lambda_{p,\Delta_s,\mathfrak{q}_s}$. Given $r \geq 1$, consider the product of Γ_s in the zero-section over the segment $[0,r]$, namely,

$$\Pi_r = \prod_{x\in[0,r]} \left(\Gamma_s\right)_x.$$

We now construct a product kernel ζ_r on Π_r and discuss its ℓ^1-isoperimetry.

Phase I. In the first phase, $r < k_s/2$. Let η_x denote the uniform measure on the finite subgroup $A_s \simeq \mathbb{Z}/2\mathbb{Z}$ in the copy $\left(\Gamma_s\right)_x$. Let ζ_r denote the product kernel

$$\zeta_r = \eta_0 \otimes \cdots \otimes \eta_r.$$

Then ζ_r is indeed the uniform measure on the subgroup $\prod_{x\in[0,r]} \left(A_s\right)_x$ in the zero section of Δ_s, it follows that

$$\Lambda_{1,\Delta_s,\zeta_r}(v) \geq \frac{1}{2} \quad \text{for all } v \leq \frac{1}{2}\left(2^{r+1}\right).$$

Phase II. In the second phase, $r > k_s$. Let ν_x denote the uniform measure on the generating set $A(s) \cup B(s)$ in the copy $\left(\Gamma_s\right)_x$. Let ζ_r denote the product kernel

$$\zeta_r = \nu_0 \otimes \cdots \otimes \nu_r.$$

As a transition kernel, ζ_r changes every copy $\left(\Gamma_s\right)_x$ independently according to ν_x. By [BH97, Th. 1.1], the Cheeger constant of ζ_r on Π_r satisfies

$$h(\Pi_r,\zeta_r) \geq \frac{1}{2\sqrt{6}} h(\Gamma_s,\nu).$$

In other words, for $r > k_s$,

$$\Lambda_{1,\Delta_s,\zeta_r}(v) \geq \frac{1}{2\sqrt{6}} h(\Gamma_s, \nu) \text{ for all } v \leq \frac{1}{2} |\Gamma_s|^{r+1}.$$

By the Cheeger inequality (14), for $p \in [1, 2]$,

$$\Lambda_{p,\Delta_s,\zeta_r}(v) \geq c_0 \Lambda_{1,\Delta_s,\zeta_r}(v)^p.$$

Now we go back to the simple random walk kernel $\mathfrak{q}$. By construction of the transition kernel ζ_r in both cases, the metric estimate in Lemma 2.13 implies every element g in the support of ζ_r satisfies

$$|g|_{\Delta_s} \leq 40r.$$

By the standard path length argument (see [PSC00, Lemma 2.1]), we have by comparison of Dirichlet forms on Δ_s that

$$2\left(|A| + |B|\right) \mathcal{E}_{p,\Delta_s,\mathfrak{q}_s} \geq \frac{1}{(40r)^p} \mathcal{E}_{p,\Delta_s,\zeta_r}.$$

It follows that

- for $r < \frac{k_s}{2}$,

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{c_0}{2\left(|A| + |B|\right)(40r)^p} \text{ for all } v \leq 2^r;$$

- for $r > k_s$,

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{c_0 h(\Gamma_s, \nu)^p}{2\left(|A| + |B|\right)\left(80\sqrt{6}r\right)^p} \text{ for all } v \leq \frac{1}{2} |\Gamma_s|^{r+1}. \qquad \square$$

When $\Gamma_s = \Gamma$ is an infinite group, we have the following bound.

LEMMA 4.3. *Let $p \in [1, 2]$, $\Gamma_s = \Gamma$ be an infinite group marked with generating subgroups A, B. Let ν be the uniform distribution on $A \cup B$. Then there exists an absolute constant $C > 1$ such that the following is true*:

- *for $1 \leq v \leq 2^{k_s/2}$*,

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{(\log_2 v)^{-p}}{C\left(|A(s)| + |B(s)|\right)};$$

- *for $v > 2^{k_s/2}$*,

$$\Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \geq \frac{h(\Gamma, \nu)^p}{C\left(|A(s)| + |B(s)|\right) k_s^p}.$$

Proof. The first item is the same as in proof of Lemma 4.1. For the second item, consider the copy of Γ at 0 and regard ν as a measure supported on the subgroup $(\Gamma)_0$. Then

$$C\left(|A(s)| + |B(s)|\right) k_s^p \mathcal{E}_{p,\Delta_s,\mathfrak{q}_s} \geq \mathcal{E}_{p,\Gamma,\nu},$$

and the result follows. $\square$

4.1.2. *Isoperimetric profile of the diagonal product*. First we put together isoperimetric estimates on the copies Δ_s to describe the isoperimetric profile of the diagonal product Δ. Let us denote $\ell_s = \log |\Gamma_s|$. Mind the difference with the diameter l_s of Γ_s. For a family of expanders, these two quantities differ only by multiplicative constants depending only on the volume growth and the spectral gap of $\{\Gamma_s\}$.

Proposition 4.4. *Suppose $\{(\Gamma_s, A(s) \cup B(s))\}$ with $A(s) \simeq A, B(s) \simeq B$ is a family of groups with the Cheeger constant $h(\Gamma_s, \nu_s) \geq \delta > 0$, where ν_s is uniform on $A(s) \cup B(s)$. Suppose $\{k_s\}$ satisfies the growth assumption 2.11. Let Δ be the diagonal product constructed with $\{\Gamma_s\}$ and parameters $\{k_s\}$, and let $\mathfrak{q}$ be the uniform measure on $\{\tau^{\pm 1}\} \cup A \cup B$ in Δ.*

There exists an absolute constant $C > 1$ such that the following estimates hold for any $s \geq 0$, $p \in [1, 2]$:

(1) *for volume $v \in \left[e^{k_s \ell_s}, e^{k_{s+1}\ell_s}\right)$,*

$$\Lambda_{p,\Delta,\mathfrak{q}}(v) \geq \frac{1}{|A| + |B|} \left(\frac{\delta \ell_s}{C \log v}\right)^p,$$

$$\Lambda_{p,\Delta,\mathfrak{q}}\left(v^{\frac{\sum_{j\leq s}\ell_j}{\ell_s}}\right) \leq \left(\frac{C\ell_s}{\log v}\right)^p;$$

(2) *for volume $v \in \left[e^{k_{s+1}\ell_s}, e^{k_{s+1}\ell_{s+1}}\right]$,*

$$\Lambda_{p,\Delta,\mathfrak{q}}(v) \geq \frac{1}{|A| + |B|} \left(\frac{\delta}{C k_{s+1}}\right)^p,$$

$$\Lambda_{p,\Delta,\mathfrak{q}}(v) \leq \left(\frac{C}{k_{s+1}}\right)^p \text{ if } v \geq \exp\left(\left(\sum_{j \leq s} \ell_j\right) k_{s+1}\right).$$

The upper bounds are valid without the requirement of a positive Cheeger constant.

Proof. Let $U_r^\Delta = \{((f_s), z) : \ \text{Range}(f_s, z) \subset [-r, r]\}$, and take a function supported on the subset U_r^Δ,

$$\varphi_r((f_s), z) = \left(1 - \frac{|z|}{r}\right) \mathbf{1}_{U_r^\Delta}(((f_s), z)).$$

Let $U_r^\Delta(0) = \{((f_s), 0) : \ \text{Range}(f_s, 0) \subset [-r, r]\}$. Then U_r^Δ can be viewed as the product of $U_r^\Delta(0)$ and the interval $[-r, r]$. To compute the Rayleigh quotient of the function φ_r, first note that $\varphi_r(Z\alpha_i) = \varphi_r(Z\beta_j) = \varphi_r(Z)$ for all

$Z \in \Delta$ and $\alpha_i \in A$, $\beta_j \in B$. For the generator τ,

$$\sum_{((f_s),z)\in U_r^\Delta} |\varphi_r((f_s), z+1) - \varphi_r((f_s), z)|^p = \frac{1}{r^p}(2r)\left|U_r^\Delta(0)\right|,$$

$$\sum_{((f_s),z)\in U_r^\Delta} \varphi_r((f_s), z)^p = \sum_{z\in[-r,r]} \left(1 - \frac{|z|}{r}\right)^p \left|U_r^\Delta(0)\right|.$$

Therefore

$$\frac{\mathcal{E}_{p,\Delta,\mathfrak{q}}(\varphi_r)}{\|\varphi_r\|_p^p} \sim \frac{1+p}{2r^p}.$$

For the size of support of φ_r,

$$|\mathrm{supp}\varphi_r| \le \prod_{k_s \le 2r} |\mathrm{supp}\varphi_r^s| \le \prod_{k_s\le 2r} |\Gamma_s|^r = e^{r\sum_{k_s\le 2r}\ell_s}.$$

In the first interval $v \in \left[e^{k_s\ell_s}, e^{k_{s+1}\ell_s}\right)$, let

$$r = \frac{\log v}{\ell_s}.$$

The test function φ_r^Δ gives the upper bound on $\Lambda_{p,\Delta,\mathfrak{q}}$ stated. For the lower bound on $\Lambda_{p,\Delta,\mathfrak{q}}$, consider the projection to the quotient Δ_s. Then from the first item in Lemma 4.1, for $v \in \left[e^{k_s\ell_s}, e^{k_{s+1}\ell_s}\right)$, we have

$$\Lambda_{p,\Delta,\mathfrak{q}}(v) \ge \Lambda_{p,\Delta_s,\mathfrak{q}_s}(v) \ge \frac{1}{|A|+|B|}\left(\frac{\delta}{Ck_s}\right)^p.$$

In the second interval $v \in \left[e^{k_{s+1}\ell_s}, e^{k_{s+1}\ell_{s+1}}\right]$, first consider the projection to the quotient Δ_{s+1}. The second item in Lemma 4.1 provides

$$\Lambda_{p,\Delta,\mathfrak{q}}\left(\frac{1}{2}|\Gamma_{s+1}|^{k_{s+1}}\right) \ge \Lambda_{p,\Delta_s,\mathfrak{q}_s}\left(\frac{1}{2}|\Gamma_{s+1}|^{k_{s+1}}\right) \ge \frac{1}{|A|+|B|}\left(\frac{\delta}{Ck_{s+1}}\right)^p.$$

In the upper bound direction, note that the right end point in the first interval gives

$$\Lambda_{p,\Delta,\mathfrak{q}}\left(\exp\left(\left(\sum_{j\le s}\ell_j\right)k_{s+1}\right)\right) \le \left(\frac{C}{k_{s+1}}\right)^p.$$

The statement follows from monotonicity of $\Lambda_{p,\Delta,\mathfrak{q}}$. □

Example 4.5. A direct application of Proposition 4.4 shows that when $k_s = 2^{\beta s}$ and $\ell_s = 2^{\iota s}$ with $\beta, \iota > 0$, then for $p \in [1,2]$,

$$\Lambda_{p,\Delta,\mathfrak{q}}(v) \simeq (\log v)^{-\frac{p}{1+\frac{\iota}{\beta}}},$$

and the exponent $\frac{p}{1+\frac{\iota}{\beta}}$ can take any value in $(0,p)$.

We allow the sequence (k_s), (l_s) to take the value ∞; the bounds are still valid. In our convention, $k_{s+1}=\infty$ means Δ_{s+1} is trivial. In this case we only use the first item in Proposition 4.4, which covers $v\in\left[e^{k_s\ell_s},\infty\right)$. The bounds in Proposition 4.4 are good when $\{\ell_s\}$ grows at least exponentially. In particular, from these estimates of isoperimetric profiles we deduce that $\Lambda_{p,\Delta,\mathfrak{q}}\circ\exp$ can follow a prescribed function satisfying some log-Lipschitz condition.

THEOREM 4.6. *There exist universal constants $c,C>0$ such that for any $p\in[1,2]$ and for any non-decreasing function $\varrho(x)$ such that $\frac{x^p}{\varrho(n)}$ is non-decreasing, there is a group Δ such that*

$$\forall v\geq 3,\ \ \frac{c}{\varrho(\log v)}\leq \Lambda_{p,\Delta,\mathfrak{q}}(v)\leq \frac{C}{\varrho(\log v)}.$$

Proof. We write $\varrho(x)=\left(\frac{x}{f(x)}\right)^p$ with $f(x)$ between 1 and x. The sets $K=\mathbb{Z}_+\cup\{\infty\}$ and $L=\{\log|\Gamma_m|,m\geq 1\}\cup\{\infty\}$, where $\{\Gamma_s\}$ are groups in the family of Examples 2.3, satisfy the assumptions of Proposition B.2. So we can find sequences $(k_s),(l_s)$ taking values in K and L such that the function defined by $\tilde f(x)=l_s$ on $[k_sl_s,k_{s+1}l_s]$ and $\tilde f(x)=\frac{x}{k_{s+1}}$ on $[k_{s+1}l_s,k_{s+1}l_{s+1}]$ satisfies $\tilde f(x)\simeq_{m_0C_1^5} f(x)$. Since the infinite group Γ in Example 2.3 has Property (T), there exists a constant $\delta>0$ such that the Cheeger constants satisfy $h(\Gamma_s,\nu_s)\geq\delta$ for all $s\geq 1$.

We use Proposition 4.4 to evaluate the profile of the group Δ associated to these sequences. The lower bounds show that for all $x\geq 1$,

$$\Lambda_{p,\Delta,\mathfrak{q}}\circ\exp(x)\geq\left(\frac{\delta\tilde f(x)}{Cx}\right)^p\geq\frac{c\delta^p}{\varrho(x)}.$$

As $\sum_{j\leq s}\ell_j\leq\frac{1}{1-\frac{1}{m_0}}\ell_s$, making the change of variable $x=\left(1-\frac{1}{m_0}\right)y=\log v$, the first upper bound shows that

$$\Lambda_{p,\Delta,\mathfrak{q}}\circ\exp(y)\leq\left(\frac{Cl_s}{\log v}\right)^p=\left(\frac{C\tilde f\left((1-\frac{1}{m_0})y\right)}{(1-\frac{1}{m_0})y}\right)^p\leq\left(\frac{C\tilde f(y)}{y}\right)^p\leq\frac{C'}{\varrho(x)}$$

for $\frac{1}{1-\frac{1}{m_0}}k_s\ell_s\leq y\leq\frac{1}{1-\frac{1}{m_0}}k_{s+1}\ell_s$. The second upper bound shows that

$$\Lambda_{p,\Delta,\mathfrak{q}}\circ\exp(y)\leq\left(\frac{C}{k_{s+1}}\right)^p=\left(\frac{C\tilde f(k_{s+1}l_s)}{k_{s+1}l_s}\right)^p=\left(\frac{C\tilde f\left((1-\frac{1}{m_0})y\right)}{(1-\frac{1}{m_0})y}\right)^p\leq\frac{C''}{\varrho(y)}$$

for $\frac{1}{1-\frac{1}{m_0}}k_{s+1}\ell_s\leq y\leq\frac{1}{1-\frac{1}{m_0}}k_{s+1}\ell_{s+1}$. We used the fact that $\tilde\varrho(x)$ is constant on the interval $[k_{s+1}\ell_s,k_{s+1}\ell_{s+1}]$. □

We derive the following corollary regarding Følner functions from Theorem 1.1. The definition of a Følner function is recalled in the beginning of Section 4. We use the convention that on a non-amenable group G, if $1/r \leq \inf \frac{|\partial S|}{|S|}$, then $\mathrm{Fol}_{G,S}(r) = \infty$.

Corollary 4.7. *There exists an universal constant $C > 1$. Let $g : [1, \infty) \to [1, \infty]$ be any non-decreasing function with $g(1) = 1$ and $\frac{\log(g(x))}{x}$ non-decreasing. Then there exists a group Δ marked with finite generating set T such that*

$$g(r/C) \leq \mathrm{Fol}_{\Delta,T}(r) \leq g(Cr).$$

Further, when range of g is contained in $[1, \infty)$, the group Δ constructed is elementary amenable and there exists a symmetric probability measure q with finite generating support on Δ such that (Δ, q) is Liouville.

Proof. Let $p_* = \min\{q(u) : \ u \in \mathcal{T}, u \neq \mathrm{id}\}$; then $p_* \geq \frac{1}{2(|A|+|B|)}$. Recall that by definition of the Følner function

$$\Lambda^{-1}_{1,\Delta,q}(1/r) \leq \mathrm{Fol}_{\Delta,\mathcal{T}}(r) \leq \Lambda^{-1}_{1,\Delta,q}(p_*/r).$$

By Proposition 4.6, there exist universal constants $C > 0$ such that for any function $\varrho(x)$ between 1 and x, there is a group Δ such that

$$\forall v \geq 3,\ \frac{1}{C\varrho(\log v)} \leq \Lambda_{1,\Delta,\mathfrak{q}}(v) \leq \frac{C}{\varrho(\log v)}.$$

In particular, in the construction of Δ we can choose $\{\Gamma_s\}$ from Lafforgue's expanders as in Example 2.3, where $|A| = 2$, $|B| = r_0$ for some fixed r_0. Therefore

$$\exp\left(\varrho^{-1}(r/C)\right) \leq \mathrm{Fol}_{\Delta,\mathcal{T}}(r) \leq \exp(\varrho^{-1}(Cr_0 r)).$$

Since ϱ is any function between 1 and x, the statement about the Følner function follows.

When the range of g is in $[1, \infty)$, the group Δ in the proof of Proposition 4.6 is constructed with an infinite sequence of finite groups $\{\Gamma_s\}$ and $\{k_s\}$ satisfying the growth assumption (2.11). By Fact 2.10, Δ is elementary amenable. Apply Theorem 3.6, we have that $L_q(\mu)$ and $H_q(\mu)$ have sub-linear growth, thus (Δ, q) is Liouville. □

4.2. *Return probabilities of simple random walk on Δ.* By Theorem 4.6 with $p = 2$, we have that $\Lambda_{2,\Delta,\mathfrak{q}} \circ \exp$ can follow a prescribed function satisfying some log-Lipschitz condition. Now we turn the ℓ^2-isoperimetric profile estimates into return probability bounds using the Coulhon-Grigor′yan theory. Let μ be a symmetric probability measure on a group G. Between discrete time random walk and continuous time random walk, we have (see [PSC00, §3.2])

$$\mu^{(2n+2)}(e) \leq 2h^{\mu}_{2n}(e) \text{ and } h^{\mu}_{4n}(e) \leq e^{-2n} + \mu^{(2n)}(e),$$

where

$$h_t^{\phi} = e^{-t} \sum_0^{\infty} \frac{t^k}{k!} \phi^{(k)}. \tag{15}$$

Define the function $\psi : [0, +\infty) \to [1, +\infty)$ implicitly by

$$t = \int_1^{\psi(t)} \frac{dv}{v \Lambda_{2,G,\mu}(v)}. \tag{16}$$

Then by [Cou96, Prop. II.1], we have

$$\mu^{(2n+2)}(e) \leq \frac{8}{\psi(8n)}.$$

In the current context, it is convenient to do a change of variable in (16). Setting $v = \exp(s)$,

$$t = \int_1^{w(t)} \frac{ds}{\Lambda_{2,\Delta,\mathfrak{q}} \circ \exp(s)}. \tag{17}$$

If, in addition, $\Lambda_{2,G,\mu} \circ \exp$ is doubling, namely, $\Lambda_{2,G,\mu} \circ \exp(2s) \geq c \Lambda_{2,G,\mu} \circ \exp(s)$ for all $s > 1$, then by [BPS12, Prop. 2.3], $w'(t)$ is doubling with the same constant. Applying [CG97, Th. 3.2],

$$\mu^{(2n)}(e) \geq \frac{1}{\exp \circ \psi(8n/c)} - e^{-2n}.$$

Combining the upper and lower bounds, if $\Lambda_{2,G,\mu} \circ \exp(2s) \geq c \Lambda_{2,G,\mu} \circ \exp(s)$, we have

$$-\log \mu^{(2n)}(e) \simeq_C w(2n) \tag{18}$$

with constant $C > 0$ only depending on the doubling constant c.

THEOREM 4.8. *There exist universal constants $c, C > 0$ such that the following is true. Let $\gamma : [1, \infty) \to [1, \infty)$ be any function such that $\frac{\gamma(n)}{n^{\frac{1}{3}}}$ and $\frac{n}{\gamma(n)}$ are non-decreasing. Then there is a group Δ such that*

$$\forall t \geq 1, \; c\gamma(t) \leq -\log\left(\mathfrak{q}^{(2t)}(e_\Delta)\right) \leq C\gamma(t).$$

Proof. Given such a function $\gamma : [1, \infty) \to [1, \infty)$, which is strictly increasing and continuous, define $\varrho : [1, \infty) \to [1, \infty)$ by

$$\varrho(x) = \frac{1}{x} \gamma^{-1}(x).$$

From the assumption on γ we have $\gamma(1) = 1$ and $a^{\frac{1}{3}} \gamma(x) \leq \gamma(ax) \leq a\gamma(x)$ for any $a, x \geq 1$. Thus

$$a\gamma^{-1}(x) \leq \gamma^{-1}(ax) \leq a^3 \gamma^{-1}(x),$$

and therefore

$$\varrho(x) \leq \varrho(ax) \leq a^2 \varrho(x),$$

which satisfies the assumption of Proposition 4.6 with $p = 2$.

By Proposition 4.6, there exist universal constants $c, C > 0$ such that there is a group Δ, for all $v \geq 3$:

$$\frac{c}{\varrho(\log v)} \leq \Lambda_{\Delta,\mathfrak{q}}(v) \leq \frac{C}{\varrho(\log v)}.$$

Note that since $\varrho(2x) \leq 4\varrho(x)$, it follows that for all $s > 0$,

$$\Lambda_{\Delta,\mathfrak{q}} \circ \exp(2s) \geq \frac{c}{4C} \Lambda_{\Delta,\mathfrak{q}} \circ \exp(s).$$

In particular, the function $\Lambda_{\Delta,\mathfrak{q}} \circ \exp : (0, \infty) \to \mathbb{R}$ is doubling at infinity. Then by [BPS12, Lemma 2.5], the solution $w(t)$ to (17) satisfies

$$\Lambda_{\Delta,\mathfrak{q}} \circ \exp \circ w(t) \leq \frac{w(t)}{t} \leq D \Lambda_{\Delta,\mathfrak{q}} \circ \exp \circ w(t),$$

where D is a constant that only depends on the doubling constant $c/4C$. Plugging in the estimate of $\Lambda_{\Delta,\mathfrak{q}}$, we have

$$\frac{c}{\varrho(w(t))} \leq \frac{w(t)}{t} \leq \frac{DC}{\varrho(w(t))}.$$

By the definition of ϱ,

$$\gamma(t) \varrho(\gamma(t)) = t,$$

note that $x\varrho(x)$ is strictly increasing, and therefore

$$c^{\frac{1}{3}} \gamma(t) \leq \gamma(ct) \leq w(t) \leq \gamma(DCt) \leq DC\gamma(t).$$

Since the constants c, C, D are universal, from (18) provided by the Coulhon-Grigor′yan theory, we conclude that

$$\forall t \geq 1, \ c'\gamma(t) \leq -\log\left(\mathfrak{q}^{(2t)}(e_\Delta)\right) \leq C'\gamma(t),$$

where $c', C' > 0$ are universal constants. □

Example 4.9. When $k_s = 2^{\beta s}$ and $l_s = 2^{\iota s}$ with $\beta > 1$, $\iota > 0$, the L^2-profile given in Example 4.5 turns to return probability

$$-\log\left(\mathfrak{q}^{(2t)}(e_\Delta)\right) \simeq t^{\frac{\beta+\iota}{3\beta+\iota}},$$

where the exponent can take any value in $(\frac{1}{3}, 1)$.

4.3. *The case of dihedral groups.* In this subsection we estimate decay of return probability of simple random walk on Δ where $\Gamma_s = D_{2l_s}$ are dihedral groups. We show that in this case the return exponent of simple random walks is $\frac{1}{3}$. Obtaining more precise estimates requires further work.

PROPOSITION 4.10. *There exists an absolute constant $C > 0$ such that the following holds. Let Δ be the diagonal product constructed with $\Gamma_s = D_{2l_s}$ and parameters $\{k_s\}$ satisfying Assumption (2.11). Then*

$$\frac{1}{C}n^{\frac{1}{3}} \le -\log \mathfrak{q}^{(2n)}\left(e_\Delta\right) \le Cn^{\frac{1}{3}}\log^{\frac{4}{3}} n.$$

Proof. Since Δ projects onto $\Delta_0 \simeq (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, we have

$$\mathfrak{q}^{(2n)}\left(e_\Delta\right) \le \mathfrak{q}_0^{(2n)}\left(e_{\Delta_0}\right).$$

The lower bound on $-\log \mathfrak{q}^{(2n)}\left(e_\Delta\right)$ follows from the decay of return probability on Δ_0 (see [PSC02]),

$$\mathfrak{q}_0^{(2n)}\left(e_{\Delta_0}\right) \le \exp\left(-\frac{1}{C}n^{\frac{1}{3}}\right).$$

In the other direction we construct a test function on Δ. First take a test function ψ_r on D_{2l_s}:

$$\psi_r(g) = \max\left\{1 - \frac{|g|_{D_{2l_s}}}{r}, 0\right\} \quad \text{for } 1 \le r \le l_s.$$

Recall that the set U_r^Δ is defined as $U_r^\Delta = \{((f_s), z) : \text{ Range}(f_s, z) \subset [-r, r]\}$. Let $\mathcal{S}(r) = \left\{s : \; k_s \le r, \; l_s \ge r^2\right\}$, and take

$$\Psi_r^\Delta\left((f_s), z\right) = \left(1 - \frac{|z|}{r}\right)\mathbf{1}_{U_r^\Delta}\left(((f_s), z)\right)\prod_{s\in\mathcal{S}(r)}\prod_{x\in[-r+k_s,r]}\psi_{r^2}(f_s(x)).$$

Depending on the sequences $(k_s), (l_s)$, the set $\mathcal{S}(r)$ might be empty, in which case we recover the test function of Proposition 4.4. (Recall the notation $\ell_s = \log|D_{2l_s}| = \log 2l_s$.) As in the proof of Proposition 4.4, we have

$$\frac{\sum_{Z\in\Delta}\left(\Psi_r^\Delta(Z) - \Psi_r^\Delta(Z\tau)\right)^2}{\|\Psi_r^\Delta\|_2^2} \le \frac{C_1}{r^2}.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{\sum_{Z\in\Delta}\left(\Psi_r^\Delta(Z) - \Psi_r^\Delta(Z\alpha)\right)^2}{\|\Psi_r^\Delta\|_2^2} &\le |\mathcal{S}(r)|\sum_{s\in\mathcal{S}(r)}\frac{\sum_{g\in D_{2l_s}}\left(\psi_{r^2}(ga(s)) - \psi_{r^2}(g)\right)^2}{\|\psi_{r^2}\|^2_{l^2(D_{2l_s})}}\\ &\le C_1\left|\mathcal{S}(r)\right|^2 r^{-4}.\end{aligned}$$

The same estimates holds for β with $a(s)$ replaced by $b(s)$. Since $\{k_s\}$ satisfies the growth assumption (2.11), we have $|\mathcal{S}(r)| \leq \log_2 r$, and therefore

$$\frac{\mathcal{E}_{\Delta,\mathfrak{q}}\left(\Psi_r^\Delta\right)}{\|\Psi_r^\Delta\|_2^2} \leq C_1 r^{-2}.$$

The support of function Ψ_r^Δ is bounded by

$$\begin{aligned}\left|\mathrm{supp}\Psi_r^\Delta\right| &\leq (2r+1)\left(\prod_{s:k_s\leq r, l_s<r^2}(2l_s)^{4r}\right)\left(\prod_{s:k_s\leq r, l_s>r^2}\left(2r^2\right)^{4r}\right)\\ &\leq (2r+1)\left(2r^2\right)^{4r\log_2 r}.\end{aligned}$$

From these test functions we have

$$\Lambda_{2,\Delta,\mathfrak{q}}(v) \leq C_1' \frac{(\log\log v)^4}{\log^2 v}. \tag{19}$$

By the Coulhon-Grigor′yan theory, we conclude that there exists an absolute constant C,

$$\mathfrak{q}^{(2n)}\left(e_\Delta\right) \geq \exp\left(-Cn^{\frac{1}{3}}\log^{\frac{4}{3}}(2n)\right). \qquad \square$$

Remark 4.11. To get an estimate for $\Lambda_{p,\Delta,\mathfrak{q}}$, $p \in [1,2]$, note that by projecting onto Δ_0, we have

$$\Lambda_{1,\Delta,\mathfrak{q}}(v) \geq \Lambda_{1,\Delta,\mathfrak{q}_0}(v) \geq \frac{1}{C\log v},$$

and in the proof above we have an upper bound (19) for $\Lambda_{2,\Delta,\mathfrak{q}}$. By the Cheeger inequality (14), we have

$$\frac{1}{C_p}\frac{1}{\log^p v} \leq \Lambda_{p,\Delta,\mathfrak{q}}(v) \leq C_p \frac{(\log\log v)^{2p}}{\log^p v}.$$

5. **Review: distortions of metric embeddings**

We first recall the standard definition of distortion of a map between metric spaces. Given an injective map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y), the distortion of f measures quantitatively how far away f is from being a homothety,

$$\mathrm{distortion}(f) = \left(\sup_{u,v\in X, u\neq v}\frac{d_Y(f(u),f(v))}{d_X(u,v)}\right)\left(\sup_{u,v\in X, u\neq v}\frac{d_X(u,v)}{d_Y(f(u),f(v))}\right).$$

When f is C-Lipschitz, the first sup is bounded by C. In this case, the distortion is comparable to the second factor. It is the inverse of the expansion ratio, defined as

$$\mathrm{ratio}(f) = \inf_{u,v\in X, u\neq v}\frac{d_Y(f(u),f(v))}{d_X(u,v)}.$$

The smallest distortion with which X can be embedded in Y is denoted by $c_Y(X)$,

$$c_Y(X) = \inf\{\mathrm{distortion}(f) : \ f : X \hookrightarrow Y\}.$$

To connect with uniform embedding of an infinite group G, it is well known that a sequence of finite metric spaces (X_k, d_k) embedded in the group G can provide obstruction for good embedding of the whole space; see, e.g., Arzhantseva-Drutu-Sapir [ADS09] and Austin [Aus11]. We quote a special case of a lemma in [Aus11].

LEMMA 5.1 (The Austin Lemma [Aus11]). *Let $\mathfrak{X}$ be a metric space. Let Γ be a finitely generated infinite group equipped with a finite generating set S, and let d denote the word distance on the Cayley graph (Γ, S). Suppose that we can find a sequence of finite graphs (X_n, σ_n), where σ_n is a 1-discrete metric on X_n, and embeddings $\vartheta_n : X_n \hookrightarrow \Gamma$ such that there are constants $C, L \geq 1$, $\delta > 0$ that are independent of n:*

- $\mathrm{diam}(X_n, \sigma_n) \to \infty$ *as* $n \to \infty$;
- *there exists a sequence of positive reals* $(r_n)_{n\geq 1}$ *such that*

$$\frac{1}{L} r_n \sigma_n(u, v) \leq d\left(\vartheta_n(u), \vartheta_n(v)\right) \leq L r_n \sigma_n(u, v) \text{ for all } u, v \in X_n, n \geq 1$$

  and moreover, $r_n \leq C \,\mathrm{diam}(X_n, \sigma_n)^{\beta}$ *for all* $n \geq 1$;
- *distortion of* (X_n, σ_n) *into* $\mathfrak{X}$ *is large in the sense that*

$$c_{\mathfrak{X}}(X_n, \sigma_n) \geq \delta \,\mathrm{diam}(X_n, \sigma_n)^{\eta},$$

  and then

$$\alpha^*_{\mathfrak{X}}(\Gamma, d) \leq 1 - \frac{\eta}{1+\beta}.$$

The second assumption in Lemma 5.1 requires that under the embedding ϑ_k, the induced metric $d_{(G,S)}$ only dilates d_k with uniformly bounded distortion. This point-wise assumption is rather restrictive. In what follows we will present some bounds that are more flexible.

The term "Poincaré inequalities" in the context of metric embeddings was first systematically used in Linial-Magen-Naor [LMN02]. It is a key ingredient for many existing lower bounds for distortion of finite metric spaces. We review the basic idea now. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a finite metric space, $\mathbf{a} = (a_{u,v})$, $\mathbf{b} = (b_{u,v})$, where $u, v \in \mathcal{M}$ are two non-zero arrays of non-negative real numbers. A p-Poincaré type inequality for $f : \mathcal{M} \to \mathfrak{X}$ is an inequality of the form

$$\sum_{u,v\in\mathcal{M}} a_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p \leq C \sum_{u,v\in\mathcal{M}} b_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p. \tag{20}$$

The infimum of the constant C such that the inequality holds for all non-trivial $f : \mathcal{M} \to \mathfrak{X}$ is known as the $\mathfrak{X}$-valued Poincaré constant associated with $\mathbf{a}, \mathbf{b}$,

$$P_{\mathbf{a},\mathbf{b},p}(\mathcal{M}, \mathfrak{X}) = \sup \frac{\sum_{u,v} a_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p}{\sum_{u,v} b_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p},$$

where the sup is taken over all $f : \mathcal{M} \to \mathfrak{X}$ such that $\sum_{u,v} a_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p \neq 0$. It follows from definition of the Poincaré constant that

$$\begin{aligned}\left(\inf_{u,v\in\mathcal{M},u\neq v} \frac{d_{\mathfrak{X}}(f(u), f(v))^p}{d_{\mathcal{M}}(u,v)^p}\right)&\left(\sum_{u,v\in\mathcal{M}} a_{u,v} d_{\mathcal{M}}(u,v)^p\right) \\ &\leq P_{\mathbf{a},\mathbf{b},p}(\mathcal{M}, \mathfrak{X}) \left(\sum_{u,v} b_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p\right);\end{aligned}$$

that is, the expansion ratio of f satisfies

$$\inf_{u,v\in\mathcal{M},u\neq v} \frac{d_{\mathfrak{X}}(f(u), f(v))^p}{d_{\mathcal{M}}(u,v)^p} \leq P_{\mathbf{a},\mathbf{b},p}(\mathcal{M}, \mathfrak{X}) \left(\frac{\sum_{u,v} b_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p}{\sum_{u,v} a_{u,v} d_{\mathcal{M}}(u,v)^p}\right).$$

To relate to compression function, we need an extra ingredient that resembles a mass distribution assumption. We say that the array $\mathbf{a}$ satisfies the $(p; l, c)$-mass distribution condition if

$$\frac{\sum_{d_{\mathcal{M}}(u,v)\geq l} a_{u,v} d_{\mathcal{M}}(u,v)^p}{\sum_{d_{\mathcal{M}}(u,v)} a_{u,v} d_{\mathcal{M}}(u,v)^p} \geq c; \tag{21}$$

in other words, the c-fraction of the total $\mathbf{a}$ array sum is from vertices at least l apart. Under this additional assumption, for any $f : \mathcal{M} \to \mathfrak{X}$, there exists $u, v \in \mathcal{M}$ with $d_{\mathcal{M}}(u,v) \geq l$ such that

$$\begin{aligned}\rho_f(l) &\leq d_{\mathfrak{X}}(f(u), f(v)) \\ &\leq \operatorname{diam}(\mathcal{M}) P_{\mathbf{a},\mathbf{b},p}(\mathcal{M}, \mathfrak{X})^{\frac{1}{p}} \left(\frac{\sum_{u,v} b_{u,v} d_{\mathfrak{X}}(f(u), f(v))^p}{c \sum_{u,v} a_{u,v} d_{\mathcal{M}}(u,v)^p}\right)^{\frac{1}{p}}.\end{aligned} \tag{22}$$

This compression upper bound is very useful. In practice, to apply this we need to choose the arrays $\mathbf{a}$, $\mathbf{b}$ and obtain a good Poincaré inequality of the form (20). This is not an easy task in general. In what follows we review some special cases. These settings have been investigated extensively in the literature, thus established results are available for application to metric embeddings.

5.1. *Poincaré inequalities in the classical form.* Pioneered by work of Enflo [Enf69], it is well known that the spectral gap of certain Markov operators on a finite metric space (X, d) can be used to show the lower bound for distortion of the embedding of (X, d) into Hilbert spaces. This method appeared in Linial-Magen [LM00], Newman-Rabinovich [NR03] and was extended in Grigorchuk-Nowak [GN12], Jolissaint-Valette [JV14] and Mimura [Mim15].

Interested readers may also consult Chapter 13.5 in the book [LP16] for a nice introduction to this topic.

Let $(\mathcal{M}, d_{\mathcal{M}})$ be a finite metric space, and let $K : \mathcal{M} \times \mathcal{M} \to [0,1]$ be a Markov transition kernel kernel on $\mathcal{M}$. Suppose K is reversible with respect to stationary distribution π. The most familiar Poincaré inequality for such a finite Markov chain takes the following form: for $f : \mathcal{M} \to \mathbb{R}$,

$$\sum_{u,v} |f(u) - f(v)|^2 \pi(u)\pi(v) \leq C \sum_{u,v} |f(u) - f(v)|^2 K(u,v)\pi(u).$$

The reciprocal of the Poincaré constant is known as the spectral gap,

$$\lambda(K) = \inf_{f:\mathcal{M}\to\mathbb{R}, f\neq c} \left\{ \frac{\sum_{u,v\in\mathcal{M}} |f(u) - f(v)|^2 K(u,v)\pi(v)}{\sum_{u,v\in\mathcal{M}} |f(u) - f(v)|^2 \pi(u)\pi(v)} \right\}. \tag{23}$$

In this case the Poincaré constant is often referred to as the relaxation time of K. Mixing times of finite Markov chains have been a very active research area in the past decades. For a great variety of Markov chains, good estimates of their spectral gaps are known; examples can be found in [SC97], [LPW09]. Note that the same Poincaré inequality holds for Hilbert space valued functions $f : \mathcal{M} \to \mathcal{H}$. This fact can be checked by eigenbasis expansion. In some examples, based on the ℓ^2-Poincaré inequality, one can apply Matoušek extrapolation (see [Mat97] and the version in [NS11]) to obtain useful Poincaré inequalities for ℓ^p-valued functions.

In the setting of inequality (20), having variance of f on the left side of the inequality and Dirichlet form on the right side corresponds to taking

$$a_{u,v} = \pi(u)\pi(v) \text{ and } b_{u,v} = \pi(u)K(u,v). \tag{24}$$

Define $\lambda_p(\mathcal{M}, K, \mathfrak{X})$ of the Markov operator K on Y to be

$$\lambda_p(\mathcal{M}, K, \mathfrak{X}) = \frac{1}{P_{\mathbf{a},\mathbf{b},p}(\mathcal{M}, \mathfrak{X})}, \tag{25}$$

where $\mathbf{a}, \mathbf{b}$ are specified by (24). When $\mathfrak{X}$ is a Hilbert space and $p = 2$, this definition agrees with the standard variational formula of the spectral gap.

We now formulate an analogue of Lemma 5.1. Since the bound relies crucially on the $\mathfrak{X}$-valued Poincaré constants $1/\lambda_p(X_n, K_n, \mathfrak{X})$ of the Markov operator K_n on X_n, we refer to it as the spectral method for bounding compression functions.

Lemma 5.2. *Let G be an infinite group equipped with a metric d and $p \in [1, \infty)$. Let X_n be a sequence of finite subsets in G and K_n be reversible Markov kernels on X_n with stationary distribution π_n. Suppose there exist a constant $c \in (0,1)$ and an increasing sequence $\{l_n\}$such that the array $\mathbf{a}_n$ defined as $\mathbf{a}_n(u,v) = \pi(u)\pi(v)$ satisfies the $(p; l_n, c)$-mass distribution condition (21).*

Let $f : G \to \mathfrak{X}$ be a 1-Lipschitz uniform embedding. Then the compression function of f satisfies

$$\rho_f(l_n) \leq \mathrm{diam}_d(X_n) \left(\frac{1}{\lambda_p(X_n, K_n, \mathfrak{X})} \left(\frac{\sum_{u,v \in X_n} d_{\mathfrak{X}}(f(u), f(v))^p K_n(u,v)\pi_n(v)}{c \sum_{u,v\in X_n} d(u,v)^p \pi(u)\pi(v)} \right) \right)^{\frac{1}{p}}.$$

Proof. Equip X_n with the metric induced by the metric d on G; the inequality follows from (22). □

Example 5.3. Consider the special case where X_n is a sequence of finite subgroups in G and d is a left invariant metric on G, e.g., the word metric. Take μ_n to be a symmetric probability measure on X_n and $K_n(u,v) = \mu_n\left(u^{-1}v\right)$. Then the Markov chain with transition kernel K_n is the random walk on X_n with step distribution μ_n. It is reversible with respect to the uniform distribution U_n on X_n. In this case, because of transitivity, the mass distribution condition is easily satisfied, namely,

$$\sum_{v:\ d(u,v) \geq \frac{1}{2}\mathrm{diam}_d(X_n)} U_n(v) \geq \frac{1}{2} \text{ for every } u \in X_n.$$

It follows that $\mathbf{a}_n = (U_n(u)U_n(v))$ satisfies the $\left(p, \frac{1}{2}\mathrm{diam}_d(X_n), \frac{1}{2}\right)$-mass distribution condition, and the bound in Lemma 5.2 simplifies to

$$\rho_f\left(\frac{\mathrm{diam}_d(X_n)}{2}\right) \leq \left(\frac{2^{p+2}\sum_{u,v\in X_n} d_{\mathfrak{X}}(f(u), f(v))^p K_n(u,v)\pi_n(v)}{\lambda_p(X_n, K_n, \mathfrak{X})} \right)^{\frac{1}{p}}.$$

5.2. *Markov type inequalities* . The notion of the Markov type of a metric space was introduced by K. Ball in [Bal92]. It has found important applications in metric geometry. In [LMN02], Linal, Magen and Naor pointed out that the basic assumption of this concept can be viewed as Poincaré inequalities. The Markov type method for bounding a compression exponent was first introduced by Naor and Peres in [NP08] and later significantly extended in [NP11].

Definition 5.4 (K. Ball [Bal92]). Given a metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ and $p \in [1, \infty)$, we say that $\mathfrak{X}$ has Markov-type p if there exists a constant $C > 0$ such that for every stationary reversible Markov chain $\{Z_t\}_{t=0}^{\infty}$ on $\{1, \dots, n\}$, every mapping $f : \{1, \dots, n\} \to \mathfrak{X}$ and every time $t \in \mathbb{N}$,

$$\mathbf{E}d_{\mathfrak{X}}(f(Z_t), f(Z_0))^p \leq C^p t \mathbf{E} d_{\mathfrak{X}}(f(Z_1), f(Z_0))^p. \tag{26}$$

The least such constant C is called the Markov-type p constant of $\mathfrak{X}$ and is denoted by $M_p(\mathfrak{X})$.

Theorem 2.3 in Naor-Peres-Sheffield-Schramm [NPSS06] implies the following results for the classical Lebesgue spaces L_p. For $p \in (1, 2]$, the space L_p has Markov type p and $M_p(L_p) \leq \frac{8}{(2^{p+1}-4)^{1/p}}$; and for every $p \in [2, \infty)$, L_p has

Markov type 2 and $M_2(L_p) \leq 4(p-1)^{\frac{1}{2}}$. See [NPSS06] for more examples of metric spaces of known Markov type.

In the setting of (20), the inequality (26) in the definition of Markov type p can be viewed as a Poincaré inequality with

$$a_{u,v} = K^t(u,v)\pi(u) \text{ and } b_{u,v} = K(u,v)\pi(u),$$

where K is the transition kernel of a reversible Markov chain on state space $\mathcal{M}$ of n points, and π is its stationary distribution. The Poincaré inequality provided by (26) reads

$$\sum_{u,v\in\mathcal{M}} d_{\mathfrak{X}}(f(u),f(v))^p K^t(u,v)\pi(u) \leq M_p^p(\mathfrak{X})\, t \sum_{u,v\in\mathcal{M}} d_{\mathfrak{X}}(f(u),f(v))K(u,v)\pi(u)$$

for all functions $f : \mathcal{M} \to \mathfrak{X}$. Note that the notion of Markov type is very powerful. If $\mathfrak{X}$ has Markov type p, then the inequality above is valid for any finite state space $\mathcal{M}$ and any reversible Markov transition kernel K on $\mathcal{M}$.

Now we examine the mass distribution condition. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a finite metric space, and let K be a reversible Markov kernel on $\mathcal{M}$ with stationary distribution π. Let $\{Z_t\}_{t=0}^{\infty}$ be a stationary Markov chain on $\mathcal{M}$ with transition kernel K. At time t, set

$$\gamma(t)^p = \frac{1}{2}\mathbf{E}_\pi\left[d_{\mathcal{M}}(Z_t,Z_0)^p\right].$$

Then

$$\begin{aligned}
&\mathbf{E}_\pi[d_{\mathcal{M}}(Z_t,Z_0)^p \mathbf{1}_{\{d_{\mathcal{M}}(Z_t,Z_0)>\gamma(t)\}}] \\
&\quad= \mathbf{E}_\pi\left[d_{\mathcal{M}}(Z_t,Z_0)^p\right] - \mathbf{E}_\pi\left[d_{\mathcal{M}}(Z_t,Z_0)^p \mathbf{1}_{\{d_Y(Z_t,Z_0)\leq\gamma(t)\}}\right] \\
&\quad\geq \mathbf{E}_\pi\left[d_{\mathcal{M}}(Z_t,Z_0)^p\right] - \gamma(t)^p = \frac{1}{2}\mathbf{E}_\pi\left[d_{\mathcal{M}}(Z_t,Z_0)^p\right].
\end{aligned}$$

That is, the array $\mathbf{a}$ satisfies the $\left(p; \left(\frac{1}{2}\mathbf{E}_\pi\left[d_{\mathcal{M}}(Z_t,Z_0)^p\right]\right)^{\frac{1}{p}}, \frac{1}{2}\right)$-mass distribution condition, where $\mathbf{a}$ is defined by $a_{u,v} = K^t(u,v)\pi(u)$. From the inequality (22) we derive the following upper bound on compression function.

Lemma 5.5. *Let G be an infinite group equipped with a metric d. Let $f : G \to \mathfrak{X}$ be a 1-Lipschitz uniform embedding. Assume that $\mathfrak{X}$ has Markov type p.*

Let X_n be a sequence of finite sets of G, and let K_n be a reversible Markov kernel on X_n with stationary distribution π_n. Let $\left\{Z_t^{(n)}\right\}_{t=0}^{\infty}$ be a stationary Markov chain on X_n with transition kernel K_n. Then for any $t_n \in \mathbb{N}$, the *compression function of f satisfies*

$$\rho_f\left(\left(\frac{1}{2}\mathbf{E}_{\pi_n}\left[d\left(Z_{t_n}^{(n)},Z_0^{(n)}\right)^p\right]\right)^{\frac{1}{p}}\right)$$
$$\leq\left(2M_p^p(\mathfrak{X})t_n\operatorname{diam}_{(G,d)}(X_n)^p\frac{\mathbf{E}_{\pi_n}\left[d_{\mathfrak{X}}\left(f\left(Z_1^{(n)}\right),f\left(Z_0^{(n)}\right)\right)^p\right]}{\mathbf{E}_{\pi_n}\left[d\left(Z_{t_n}^{(n)},Z_0^{(n)}\right)^p\right]}\right)^{\frac{1}{p}}.$$

Remark 5.6. This upper bound on the compression function is in the same spirit as the argument of Naor and Peres in Section 5 of [NP11]. The difference is that in [NP11] the authors considered random walks on the infinite group G starting at identity and f is taken to be a 1-cocycle on G. Then the Markov type inequality for 1-cocycles was applied to bound the compression function. One restriction for such an approach is that the step distribution of the random walk needs to have finite p-moment. While in the finite subsets, in principle one can experiment with any reversible transition kernel and choose the best one available. Examples that illustrate this point can be found in Section 7.1.

5.3. *Comparing spectral and Markov type methods*. It is interesting to compare the classical Poincaré inequalities and the ones from Markov type method. Suppose in the infinite group G that we have chosen a sequence of subsets $\{X_n\}$ and reversible Markov kernels K_n on X_n. With this sequence $\{(X_n,K_n)\}$ we compare the results given by the two methods. Let $\mathfrak{X}$ be a metric space of Markov type p and $f:G\to\mathfrak{X}$ be a 1-Lipschitz embedding from (G,d) to $(\mathfrak{X},d_{\mathfrak{X}})$. To compare terms in the bounds of Lemmas 5.2 and 5.5, first note that

$$\mathbf{E}d_{\mathfrak{X}}\left(f\left(Z_1^{(n)}\right),f\left(Z_0^{(n)}\right)\right)^p=\sum_{u,v\in X_n}d_{\mathfrak{X}}(f(u),f(v))^pK_n(u,v)\pi_n(u).$$

Now we choose t_n to be comparable to the Poincaré constant $P_p(X_n,K_n,\mathfrak{X})$. (It corresponds to relaxation time when $\mathfrak{X}$ is a Hilbert space and $p=2$.) Suppose in addition that π_n satisfies the $(p;\theta\operatorname{diam}(X_n),c)$-mass distribution condition. Then essentially the difference in the two bounds comes from the ratio

$$\frac{\operatorname{diam}_{(G,d)}(X_n)^p}{\mathbf{E}d\left(Z_{t_n}^{(n)},Z_0^{(n)}\right)^p}.$$

Thus if there is a constant $c_1>0$ such that for $t_n\simeq P_p(X_n,K_n,\mathfrak{X})$,

$$\mathbf{E}d_{X_n}\left(Z_{t_n}^{(n)},Z_0^{(n)}\right)^p\geq c_1^p\operatorname{diam}_{(G,d)}(X_n)^p,$$

then up to some multiplicative constants, the two methods give the same compression upper bound.

It is important in applications that the choice of the sequence of finite subsets X_n and Markov kernels K_n is flexible. For example, in order to use Poincaré inequalities to obtain an upper bound on the compression function of uniform embedding f from G into a Hilbert space, the subsets X_n should be chosen to capture some worst distorted elements in the group under f, and the Markov kernel K_n on X_n should be chosen so that

$$\frac{1}{\lambda(K_n)}\left(\sum_{u,v\in X_n} d_{\mathcal{H}}(f(u),f(v))^2 K_n(u,v)\pi_n(v)\right)$$

is as small as possible. That is, K_n needs to achieve a balance between spectral gap and Dirichlet form $\mathcal{E}_{K_n}(f)$. This point will be the guideline for the choice of (X_n, K_n) in the examples we treat.

5.4. *Metric cotype inequalities*. The notion of type and cotype plays a central role in the local theory of Banach spaces. The classical linear notion of type and cotype is defined as follows. A Banach space $\mathfrak{X}$ is said to have (Rademacher) type $p > 0$ if there exists a constant $T > 0$ such that for every n and every $x_1, \ldots, x_n \in \mathfrak{X}$,

$$\mathbb{E}\left\|\sum_{j=1}^n \varepsilon_j x_j\right\|_{\mathfrak{X}}^p \le T^p \sum_{j=1}^n \|x_j\|_{\mathfrak{X}}^p,$$

where $\mathbb{E}$ is the expectation with respect to uniform distribution on $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$. A Banach space $\mathfrak{X}$ is said to have (Rademacher) cotype $q > 0$ if there exists a constant $C > 0$ such that for every n and every $x_1, \ldots, x_n \in \mathfrak{X}$,

$$\mathbb{E}\left\|\sum_{j=1}^n \varepsilon_j x_j\right\|_{\mathfrak{X}}^q \ge \frac{1}{C^q} \sum_{j=1}^n \|x_j\|_{\mathfrak{X}}^q.$$

Given a Banach space $\mathfrak{X}$, define

$$p_{\mathfrak{X}} = \sup\{p : \ \mathfrak{X} \text{ has type } p\}, \ q_{\mathfrak{X}} = \inf\{q : \ \mathfrak{X} \text{ has cotype } q\}.$$

The space $\mathfrak{X}$ is said to be of non-trivial type if $p_{\mathfrak{X}} > 1$, and it is of non-trivial cotype if $q_{\mathfrak{X}} < \infty$.

Mendel and Naor [MN08] introduced the non-linear notion of metric cotype. By [MN08, Def. 1.1], $(\mathfrak{X}, d_{\mathfrak{X}})$ has *metric cotype* q with constant Γ if for every integer $n \in \mathbb{N}$, there exists an even integer m, such that for every $f : \mathbb{Z}_m^n \to \mathfrak{X}$,

$$\sum_{j=1}^n \sum_{u\in\mathbb{Z}_m^n} d_{\mathfrak{X}}\left(f\left(u+\frac{m}{2}\mathbf{e}_j\right), f(u)\right)^q \pi(u)$$
$$\le \Gamma^q m^q \sum_{u\in\mathbb{Z}_m^n} \mathbb{E}\left[d_{\mathfrak{X}}\left(f\left(u+\boldsymbol{\varepsilon}\right), f(u)\right)^q\right]\pi(u),$$

where π is the uniform distribution on $\mathbb{Z}_m^n$ and $\mathbb{E}$ is the expectation taken with respect to uniform distribution on $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,0,1\}^n$, and $\{\mathbf{e}_j\}$ is the standard basis of $\mathbb{R}^n$. Mendel and Naor proved in [MN08] that for a Banach space $\mathfrak{X}$ and $q \in [2,\infty)$, $\mathfrak{X}$ has metric cotype q if and only if it has Rademacher cotype q. As a key step, they established the following sharp estimate, which we will refer to as the metric cotype inequality.

THEOREM 5.7 ([MN08, Th. 4.2]). *Let π be the uniform distribution on $\mathbb{Z}_m^n$ and σ be the uniform distribution on $\{-1,0,1\}^n$. Let $\mathfrak{X}$ be a Banach space of Rademacher type $p>1$ and cotype $q\in[2,\infty)$. Then for every $f:\mathbb{Z}_m^n\to\mathfrak{X}$,*

$$\sum_{u\in\mathbb{Z}_m^n}\sum_{j=1}^{n} d_{\mathfrak{X}}\left(f\left(u+\frac{m}{2}\mathbf{e}_j\right),f(u)\right)^q\pi(u)$$
$$\leq\left(5\max\left\{C(\mathfrak{X})m,n^{\frac{1}{q}}\right\}\right)^q\sum_{u\in\mathbb{Z}_m^n}\sum_{\boldsymbol{\varepsilon}\in\{-1,0,1\}^n} d_{\mathfrak{X}}\left(f(u+\boldsymbol{\varepsilon}),f(u)\right)^q\sigma(\boldsymbol{\varepsilon})\pi(u),$$

where $C(\mathfrak{X})>0$ is a constant that only depends on the cotype constant and K_q-convexity constant of $\mathfrak{X}$.

For our purposes, the Mendal-Naor metric cotype inequality can be viewed as a Poincaré inequality with a rather unusual choice of arrays $\mathbf{a},\mathbf{b}$ on $\mathbb{Z}_m^n$, namely,

$$a_{u,v}=\sum_{j=1}^{n}\pi(u)\mathbf{1}_{\left\{v=u+\frac{m}{2}\mathbf{e}_j\right\}}\text{ and }b_{u,v}=\sum_{\boldsymbol{\varepsilon}\in\{-1,0,1\}^n}\pi(u)\mathbf{1}_{\{v=u+\boldsymbol{\varepsilon}\}}\sigma(\boldsymbol{\varepsilon}).$$

Then the Poincaré constant is bounded by

$$P_{\mathbf{a},\mathbf{b},2}\left(\mathbb{Z}_m^n,\mathfrak{X}\right)\leq\left(5\max\left\{C(\mathfrak{X})mn^{\frac{1}{2}-\frac{1}{q}},n^{\frac{1}{2}}\right\}\right)^2.$$

It captures a subtle comparison between a transition kernel that moves far in one fiber and another kernel that moves by ±1 across the whole product.

Among many other applications of metric cotype, such inequalities provide sharp lower bound for the distortion of embeddings of the ℓ^∞-integer lattice $[m]_\infty^n$ into Banach spaces of non-trivial type and cotype q; see Theorem 1.12 in [MN08]. In Section 8 we will apply these metric cotype inequalities in the study of compression of diagonal product Δ constructed with dihedral groups, exactly because of the presence of l^∞-lattices of growing side length in the group.

6. Compression of Δ with embedded expanders

In this section we consider compression of the diagonal product Δ constructed with $\{\Gamma_s\}$ chosen to be certain families of expanders. Let $\mathfrak{X}$ be a Banach space. A map $\Psi:G\to\mathfrak{X}$ is called G-equivariant if there exists an action τ of G on $\mathfrak{X}$ by affine isometries and a vector $v\in\mathfrak{X}$ such that $\Psi(g)=\tau(g)v$ for all $x\in G$. Such a map is called a 1-cocycle; see [dCTV07].

A couple of functions (g_1,g_2) is an *equivariant-$\mathfrak{X}$–compression gap* of G if any 1-Lipschitz G-equivariant embedding $\varphi:G\to\mathfrak{X}$ satisfies $\rho_\varphi(t)\leq g_2(t)$ for all $t \geq 1$ and there exists a 1-Lipschitz G-equivariant embedding $\Psi : G \to \mathfrak{X}$ such that $\rho_\Psi(t) \geq g_1(t)$ for all $t \geq 1$.

We address the question regarding possible L_p-compression exponents of finitely generated amenable groups.

PROPOSITION 6.1. *For any $\gamma \in [0,1]$, there exists a finitely generated elementary amenable group Δ such that for all $p \geq 1$,*

$$\alpha_p^{\#}(\Delta) = \gamma.$$

This result follows from a more precise result about the equivariant compression gap of the diagonal product group Δ; see Theorem 6.11. We will see that when the lamp groups $\{\Gamma_s\}$ are chosen to be expanders, single copies of these lamp groups provide sufficient obstruction for embedding. In some sense this case can be viewed as an amenable analogue of [ADS09].

6.1. *An upper bound in any uniformly convex Banach spaces.* In this subsection we take $\{\Gamma_s\}$ as a subsequence in the Lafforgue super expanders $\{\Gamma_m\}$ described in Example 2.3. By Fact 2.15, each group Γ_s embeds homothetically in the diagonal product Δ with ratio $k_s + 1$, i.e., there is group homomorphism $\vartheta_s : \Gamma_s \to \Delta$ satisfying

$$|\vartheta_s(\gamma)|_\Delta = (k_s+1)\,|\gamma|_{\Gamma_s}.$$

From these distortion estimates and the embeddings $\vartheta_s : \Gamma_s \hookrightarrow \Delta$, we immediately derive an upper bound on the compression function of Δ into $\mathfrak{X}$ by Lemma 5.2.

LEMMA 6.2. *Let Δ be the diagonal product with parameters (k_s) and lamp groups $\{\Gamma_s\}$ chosen as a subsequence of Lafforgue super expanders in Example 2.3, $\operatorname{diam}(\Gamma_s) = l_s$. Then for any uniformly convex Banach space $\mathfrak{X}$, there exists a constant $\delta = \delta(\Gamma, \mathfrak{X}, |A| + |B|) > 0$ such that the compression function of any 1-Lipschitz embedding $\Psi : \Delta \to \mathfrak{X}$ satisfies*

$$\rho_\Psi\left(\frac{1}{2}(k_s+1)l_s\right) \leq 4\delta^{-\frac{1}{2}}(k_s+1).$$

Proof. Take $X_s = \vartheta_s(\Gamma_s)$ and $K_s(u,v) = \nu_s\left(\vartheta_s^{-1}(u^{-1}v)\right)$, where ν_s is uniform on the generating set $A(s) \cup B(s)$. To apply Lemma 5.2, note that $\operatorname{diam}_{d_\Delta}(X_s) = (k_s+1)l_s$, the Poincaré constant $P_2(X_s, K_s, \mathfrak{X}) \leq 1/\delta$ by Lafforgue's result (1), where δ is a constant only depending on $\Gamma, \mathfrak{X}$ and $|A| + |B|$. Since Ψ is 1-Lipschitz with respect to $|\cdot|_\Delta$,

$$\sum_{u,v \in X_s} d_{\mathfrak{X}}(\Psi(u),\Psi(v))^2 K_s(u,v)\pi_s(u) \leq (k_s+1)^2.$$

Since π_s is the uniform distribution on the subgroup X_s, the upper bound on ρ_Ψ then follows from the Poincaré inequalities (1) in Example 5.3 with $p = 2$. $\square$

6.2. *Compression gaps of embeddings of Δ into L_p.* In this subsection we focus on the case where L_p, $p \geq 1$, are target spaces for embedding.

6.2.1. *Upper bound when $\{\Gamma_s\}$ are expanders.* When the target space is L_p, $p \geq 1$, a more precise piecewise upper bound of the compression gap can be obtained. Recall that a symmetric probability measure μ on a group G defines a Markov transition kernel $K(u,v) = \mu(u^{-1}v)$ that is reversible with respect to the uniform distribution on G. Its ℓ^2-spectral gap $\lambda(G,\mu) = \lambda(G,K)$ is defined as in (23).

PROPOSITION 6.3. *Let Δ be the diagonal product with parameters (k_s) and lamp groups $\{\Gamma_s\}$ expanders where $\operatorname{diam}(\Gamma_s) = l_s < \infty$. Suppose $\{\Gamma_s\}$ satisfies Assumption 2.1 and*

$$\lambda(\Gamma_s, \nu_s) \geq \delta > 0 \quad \textit{for all } s \textit{ with } l_s < \infty,$$

where ν_s is uniform on $A(s) \cup B(s)$. Then there exists a constant C_0 depending only on $|A|, |B|$ such that for any 1-Lipschitz embedding $\Psi : \Delta \to L_p$, the compression function of Ψ satisfies, for all $s \geq 1$ with $k_s, l_s < \infty$,

$$\rho_\Psi\left(\frac{1}{2}x\right) \leq C(\delta,p)\frac{x}{l_s} \ \textit{if } x \in [k_s l_s, k_{s+1} l_s],$$

$$\textit{where } C(\delta,p) = \begin{cases} C_0\delta^{-\frac{1}{p}} & \textit{if } 1 \leq p \leq 2, \\ C_0 p\delta^{-\frac{1}{2}} & \textit{if } p > 2. \end{cases} \tag{27}$$

Remark 6.4. If $k_{s+1}, l_{s+1} < \infty$, then by monotonicity of the compression function, the bound extends to the interval $[k_{s+1}l_s, k_{s+1}l_{s+1}]$; namely, for $x \in [k_{s+1}l_s, k_{s+1}l_{s+1}]$,

$$\rho_\Psi\left(\frac{1}{2}x\right) \leq \rho_\Psi\left(\frac{1}{2}k_{s+1}l_{s+1}\right) \leq C(\delta,p)k_{s+1}.$$

If $l_{s+1} = \infty$, the situation is different. We need to have information regarding compression of the infinite group Γ; see Lemma 6.6.

Proof. Consider the subgroup

$$\Gamma_s' = [A(s), B(s)]^{\Gamma_s} = \ker\left(\Gamma_s \to A(s) \times B(s)\right).$$

Take the symmetric generating set $R(s)$ for Γ_s' using the Reidemeister-Schreier algorithm in Lemma 2.18, where $F = A(s) \times B(s)$, $S = A(s) \cup B(s)$. Then the inclusion map from $(\Gamma_s', R(s))$ into $(\Gamma_s, A(s) \cup B(s))$ is bi-Lipschitz, $|\gamma|_{R(s)} \leq |\gamma|_{\Gamma_s} \leq 5|\gamma|_R$ for all $\gamma \in \Gamma_s'$. Let μ_s be the uniform distribution on $R(s)$. It is known that if there is a (C,C)-quasi isometric map $\psi : (G,S) \to (H,T)$ and if the image $\psi(G)$ is R dense in H, then the Poincaré constant of (H,ν) is comparable to the Poincaré constant of (G,μ) with constants only depending on $C, R, |S|, |T|$, where μ (resp. ν) is the uniform distribution on $S \cup S^{-1}$ (resp.

$T \cup T^{-1}$); see the proof of [CSC95, Prop. 4.2] or [PSC00, Th. 1.2]. In the current situation, since the inclusion map $(\Gamma'_s, R(s))$ into $(\Gamma_s, A(s) \cup B(s))$ is a $(5,5)$-quasi-isometry and Γ'_s is 2-dense in Γ_s, there exists a constant c_0 only depending on $|A|$ and $|B|$ such that the spectral gap of μ_s satisfies

$$\lambda(\Gamma'_s, \mu_s) = \tilde{\delta} \geq c_0 \delta.$$

Let $t \in [k_s, k_{s+1}]$. Consider the direct product Π_s^t of t copies of Γ'_s in the factor Δ_s at site $0, 1, \ldots, t-1$. By Section 2.3, Π_s^t is an embedded subgroup of Δ. Denote such an embedding by $\theta_s : \Pi_s^t \hookrightarrow \Delta$. On Π_s^t, take the product kernel $\zeta_t = (\mu_s)_0 \otimes \cdots \otimes (\mu_s)_{t-1}$. By the tensorizing property of classical Poincaré inequalities, we have that for any function $f : \Pi_t \to \mathbb{R}$,

$$\sum_{u,v \in \Pi_t} |f(u) - f(v)|^2 \, \pi_{\Pi_s^t}(u) \pi_{\Pi_s^t}(v) \leq \tilde{\delta}^{-1} \sum_{u,v \in \Pi_t} |f(u) - f(v)|^2 \, \pi_{\Pi_t}(u) \zeta_t \left(u^{-1} v\right).$$

By Matoušek's extrapolation lemma for Poincaré inequalities [Mat97] (see the version in [NS11, Lemma 4.4]), it follows that for any $f : \Pi_t \to \ell^p$,

- if $1 \leq p \leq 2$, then
$$\sum_{u,v \in \Pi_s^t} \|f(u) - f(v)\|_p^p \, \pi_{\Pi_s^t}(u) \pi_{\Pi_s^t}(v) \leq \tilde{\delta}^{-1} \sum_{u,v \in \Pi_s^t} \|f(u) - f(v)\|_p^p \, \pi_{\Pi_s^t}(u) \zeta_t \left(u^{-1} v\right);$$
- if $p > 2$, then
$$\sum_{u,v \in \Pi_s^t} \|f(u) - f(v)\|_p^p \, \pi_{\Pi_s^t}(u) \pi_{\Pi_s^t}(v) \leq (2p)^p \tilde{\delta}^{-p/2} \sum_{u,v \in \Pi_s^t} \|f(u) - f(v)\|_p^p \, \pi_{\Pi_s^t}(u) \zeta_t \left(u^{-1} v\right).$$

Let $\varphi : \Delta \to \ell^p$ be a 1-Lipschitz uniform embedding of Δ with respect to the word metric $|\cdot|_\Delta$. Apply Lemma 5.2 to the subset $\theta_s(\Pi_t)$ equipped with the kernel $\zeta_t \circ \theta_s^{-1}$. With the mass distribution condition satisfied by Lemma 2.16, we have

$$\rho_\varphi\left(\frac{1}{2} t l_s\right) \leq c(\tilde{\delta}, p) t,$$

where the constant $c(\tilde{\delta}, p)$ is given by

$$c(\tilde{\delta}, p) = \begin{cases} 4(2/\tilde{\delta})^{\frac{1}{p}} & \text{if } 1 \leq p \leq 2, \\ 4 \cdot 2^{1+\frac{1}{p}} \tilde{\delta}^{-\frac{1}{2}} p & \text{if } p > 2. \end{cases} \tag{28}$$

From the standard fact that L_p is $(1+\varepsilon)$-finitely presentable in l_p (see, for example, the proof of [JV14, Th. 1.1]), we conclude that for for any 1-Lipschitz uniform embedding $\Psi : \Delta \to L_p$, the same bound holds:

$$\rho_\Psi \left(\frac{1}{2} t l_s \right) \leq c(\tilde{\delta}, p) t. \qquad \square$$

6.2.2. *Upper bound with an infinite group Γ_s having strong property (T).* Next we consider the case where $\Gamma_s = \Gamma$ is an infinite group. (It corresponds to $l_s = \infty$.) Let Γ be a discrete group equipped with a finite generating set S, and let $\mathfrak{X}$ be a Banach space. A linear isometric Γ-representation on $\mathfrak{X}$ is a homomorphism $\varrho : \Gamma \to O(\mathfrak{X})$, where $O(\mathfrak{X})$ denotes the groups of all invertible linear isometries of $\mathfrak{X}$. Denote by $\mathfrak{X}^{\varrho(\Gamma)}$ the closed subspace of Γ-fixed vectors. When $\mathfrak{X}$ is uniformly convex, by [BFGM07, Prop. 2.6] the subspace of $\mathfrak{X}^{\varrho(\Gamma)}$ is complemented in $\mathfrak{X}$, $\mathfrak{X} = \mathfrak{X}^{\varrho(\Gamma)} \oplus \mathfrak{X}'(\varrho)$, and the decomposition is canonical.

Definition 6.5. Let Γ be a discrete group equipped with a finite generating set S, and let $\mathfrak{X}$ be a uniformly convex Banach space.

- Following [BFGM07], we say that Γ has Property $(F_{\mathfrak{X}})$ if any action of Γ on $\mathfrak{X}$ by affine isometries has a Γ-fixed point.
- We say Γ has Property $(T_{\mathfrak{X}})$ if there exists a constant $\varepsilon > 0$ such that for any representation $\varrho : \Gamma \to O(\mathfrak{X})$,

  $$\max_{s \in S} \|\varrho(s)v - v\|_{\mathfrak{X}} \geq \varepsilon \|v\|_{\mathfrak{X}} \quad \text{for all } v \in \mathfrak{X}'(\varrho).$$

  The maximal ε with this property is called the $\mathfrak{X}$-Kazhdan constant of Γ with respect to S and is denoted by $\kappa_{\mathfrak{X}}(\Gamma, S)$.

By [BFGM07, Th. 1.3 and Rem. 2.28], Property $(F_{\mathfrak{X}})$ implies Property $(T_{\mathfrak{X}})$ in any uniformly convex Banach space $\mathfrak{X}$ with uniformly convex dual.

Lemma 6.6. *Let Δ be the diagonal product with parameters $(k_s)_{s \leq \mathfrak{s}_0}$ and lamp groups $\{\Gamma_s\}_{s \leq \mathfrak{s}_0}$, where $\Gamma_{\mathfrak{s}_0} = \Gamma$ is an infinite group marked with generating subgroups A, B. Suppose $\mathfrak{X}$ is a uniformly convex Banach space and Γ has Property $(F_{\mathfrak{X}})$. Then for any equivariant 1-Lipschitz embedding $\Psi : \Delta \to \mathfrak{X}$, the compression function of Ψ satisfies*

$$\rho_\Psi(x) \leq \frac{2}{\kappa_{\mathfrak{X}}(\Gamma, A \cup B)} (k_{\mathfrak{s}_0} + 1) \quad \textit{for all } x \in [k_{\mathfrak{s}_0} + 1, \infty].$$

Proof. Since the embedding $\vartheta_{\mathfrak{s}_0} : \Gamma \hookrightarrow \Delta$ is a homothety with $|\vartheta_{\mathfrak{s}_0}(\gamma)|_\Delta = (k_{\mathfrak{s}_0}+1)|\gamma|_\Gamma$, $\psi = \Psi \circ \vartheta_{\mathfrak{s}_0} : \Gamma \to \mathfrak{X}$ is a $(k_{\mathfrak{s}_0}+1)$-Lipschitz equivariant embedding. Consider $\tilde{\psi} = \frac{\psi}{k_{\mathfrak{s}_0}+1}$. Since $\tilde{\psi}$ equivariant, it is a 1-cocycle with respect to some representation $\varrho : \Gamma \to O(\mathfrak{X})$. Since Γ has Property $(F_{\mathfrak{X}})$, $H^1(\Gamma, \varrho) = Z^1(\Gamma, \varrho)/B^1(\Gamma, \varrho)$ vanishes. It follows that $\tilde{\psi}$ is a 1-coboundary; that is, there exists $v \in \mathfrak{X}$ such that

$$\tilde{\psi}(g) = \varrho(g)v - v.$$

We may take v in the complement $\mathfrak{X}'(\varrho)$. Then by Property $(T_{\mathfrak{X}})$,

$$\max_{s\in A\cup B}\|\varrho(s)v-v\|_{\mathfrak{X}}\geq\kappa\,\|v\|_{\mathfrak{X}}\,,\ \text{ where }\kappa=\kappa_{\mathfrak{X}}(\Gamma,A\cup B).$$

Since $\tilde{\psi}$ is 1-Lipschitz, we have $\kappa\,\|v\|_{\mathfrak{X}}\leq\max_{s\in A\cup B}\left\|\tilde{\psi}(s)\right\|_{\mathfrak{X}}\leq 1$. It follows that for any $g\in\Gamma$,

$$\left\|\tilde{\psi}(g)\right\|_{\mathfrak{X}}=\|\varrho(g)v-v\|_{\mathfrak{X}}\leq\|\varrho(g)v\|_{\mathfrak{X}}+\|v\|_{\mathfrak{X}}=2\,\|v\|_{\mathfrak{X}}\leq 2/\kappa.$$

Now we get back to Ψ. Since $\Psi\circ\vartheta_{\mathfrak{s}_0}=(k_{\mathfrak{s}_0}+1)\tilde{\psi}$ and $|\vartheta_{\mathfrak{s}_0}(\gamma)|_{\Delta}=(k_{\mathfrak{s}_0}+1)\,|\gamma|_{\Gamma}$, we deduce from $\left\|\tilde{\psi}(g)\right\|_{\mathfrak{X}}\leq 2/\kappa$ that

$$\rho_{\Psi}\left(x\right)\leq\frac{2}{\kappa_{\mathfrak{X}}(\Gamma,A\cup B)}\left(k_{\mathfrak{s}_0}+1\right)\quad\text{for all }x\in[k_{\mathfrak{s}_0}+1,\infty].\qquad\square$$

Remark 6.7. In practice, we use the bound in Lemma 6.6 for the interval $[(k_{\mathfrak{s}_0}+1)\,l_{\mathfrak{s}_0-1},\infty]$, because for smaller length x, the copies Γ_s with $s\leq\mathfrak{s}_0-1$ provide better upper bounds.

Property $(F_{\mathfrak{X}})$ is very strong. By Bader-Furman-Gelander-Monod (see [BFGM07, Th. B]) and standard Hereditary properties ([BdlHV08, §2.5]), the lattice Γ in Example 2.3 has Property F_{L_p} for all $1<p<\infty$.

When we specialize to Lebesgue spaces L_p, $p\in(1,\infty)$, the p-Kazhdan constant can be estimated in terms of the Kazhdan constant in Hilbert space, via the explicit Mazur map.

Fact 6.8 (Follows from [BFGM07], [Maz29]). *Let Γ be a discrete group equipped with finite generating set S, and suppose that Γ has the Kazhdan property (T_{L_2}). Then for $p>2$, $\kappa_{L_p}(\Gamma,S)\geq\frac{1}{p2^{p/2}}\kappa_{L_2}(\Gamma,S)$; for $1<p<2$, $\kappa_{L_p}(\Gamma,S)\geq 2^{-\frac{p+2}{p}}\kappa_{L_2}^{2/p}(\Gamma,S)$.*

Proof. Let $\varrho:\Gamma\to O(L_p)$ be a Γ-representation in L_p, and take any unit vector f in the complement $\mathfrak{X}'(\varrho)$. Let $M_{p,q}:L_p\to L_q$ be the Mazur map

$$M_{p,q}(f)=\operatorname{sign}(f)|f|^{p/q}.$$

By [BFGM07, Lemma 4.2], the conjugation $U\mapsto M_{p,2}\circ U\circ M_{2,p}$ sends $O(L_p)$ to $O(L_2)$. Define $\pi:\Gamma\to O(L_2)$ by $\pi(g)=M_{p,2}\circ\varrho(g)\circ M_{2,p}$. By definition of the Mazur map, we have $\|M_{p,2}(f)\|_2^2=\|f\|_p^p=1$.

Consider first the case $p>2$. From [Maz29],

$$\left||a|^{\frac{2}{p}}\operatorname{sign}(a)-|b|^{\frac{2}{p}}\operatorname{sign}(b)\right|\leq 2|a-b|^{\frac{2}{p}},\tag{29}$$

and so we have $\|M_{2,p}(u)-M_{2,p}(v)\|_p^p\leq 2^p\,\|u-v\|_2^2$ for $u,v\in L_2$. Note that $M_{p,2}$ maps $L_p^{\varrho(\Gamma)}$ onto $L_2^{\pi(\Gamma)}$, and therefore for any unit vector $f\in\mathfrak{X}'(\varrho)$, we have

$$\inf_{v\in L_2^{\pi(\Gamma)}} \|M_{p,2}(f)-v\|_2^2 \geq \frac{1}{2^p}\inf_{h\in L_p^{\varrho(\Gamma)}}\|f-h\|_p^p \geq \frac{1}{2^p};$$

that is, the projection of $M_{p,2}(f)$ to $\left(L_2^{\pi(\Gamma)}\right)^{\perp}$ has L_2-norm at least $2^{-p/2}$. By [Maz29],

$$\left||a|^{p/2}\mathrm{sign}(a)-|b|^{p/2}\mathrm{sign}(b)\right| \leq \frac{p}{2}|a-b|\left(|a|^{\frac{p}{2}-1}+|b|^{\frac{p}{2}-1}\right), \tag{30}$$

and so we have

(31)

$$\begin{aligned}\|\pi(s)u-u\|_2^2 &= \int \left||\varrho(s)f|^{p/2}\mathrm{sign}(\varrho(s)f)-|f|^{p/2}\mathrm{sign}(f)\right|^2 dm\\ &\leq \int \left(\frac{p}{2}\right)^2|\varrho(s)f-f|^2\left(|\varrho(s)f|^{\frac{p}{2}-1}+|f|^{\frac{p}{2}-1}\right)^2 dm\\ &\leq \left(\frac{p}{2}\right)^2\left(\int|\varrho(s)f-f|^p\,dm\right)^{2/p}\left(\int\left(|\varrho(s)f|^{\frac{p}{2}-1}+|f|^{\frac{p}{2}-1}\right)^{\frac{2p}{p-2}}dm\right)^{\frac{p-2}{p}}.\end{aligned}$$

The last step uses Hölder inequality. By triangle inequality in $L_{\frac{p}{p-2}}$ and $\varrho(s)\in O(L_p)$,

$$\left(\int\left(|\varrho(s)f|^{\frac{p}{2}-1}+|f|^{\frac{p}{2}-1}\right)^{\frac{2p}{p-2}}dm\right)^{\frac{p-2}{p}} \leq 4\,\|f\|_p^{p-2}=4.$$

Let u' be the projection of u to $L_2'(\pi)$. Then

$$\max_{s\in S}\|\pi(s)u-u\|_2=\max_{s\in S}\left\|\pi(s)u'-u'\right\|_2\geq\kappa_{L_2}(\Gamma,S)\left\|u'\right\|_2\geq 2^{-p/2}\kappa_{L_2}(\Gamma,S).$$

Combining with (31), we have

$$1\leq\frac{2^p}{\kappa_{L_2}(\Gamma,S)^2}\max_{s\in S}\|\pi(s)u-u\|_2^2\leq\left(\frac{p2^{p/2}}{\kappa_{L_2}(\Gamma,S)}\right)^2\max_{s\in S}\|\varrho(s)f-f\|_p^2.$$

We conclude that $\kappa_{L_p}(\Gamma,S)\geq\frac{1}{p2^{p/2}}\kappa_{L_2}(\Gamma,S)$.

In the case $1<p<2$, rewrite (29) as

$$|a-b|^p\leq 2^p\left||a|^{p/2}\mathrm{sign}(a)-|b|^{p/2}\mathrm{sign}(b)\right|^2.$$

We deduce that the projection of $M_{p,2}(f)$ to $\left(L_2^{\pi(\Gamma)}\right)^{\perp}$ has L_2-norm at least $2^{-p/2}$. Applying (29), we get

$$\begin{aligned}\|\pi(s)u-u\|_2^2 &= \int\left||\varrho(s)f|^{p/2}\mathrm{sign}(\varrho(s)f)-|f|^{p/2}\mathrm{sign}(f)\right|^2dm\\ &\leq\int 4|\varrho(s)f-f|^p dm=4\,\|\varrho(s)f-f\|_p^p.\end{aligned}$$

It follows that $\kappa_{L_p}(\Gamma,S)\geq 2^{-\frac{p+2}{p}}\kappa_{L_2}^{2/p}(\Gamma,S)$. ∎

6.2.3. *Basic test functions and* 1*-cocycles on* Δ. The discussion in this subsection is valid for any choice of $\{\Gamma_s\}$ that satisfies Assumption 2.1. The 1-cocycle constructed using the basic test functions will be useful in later sections as well.

First recall some basic test functions on Δ constructed in Section 4.1.2. They capture the feature that in each copy of Γ_s in Δ_s, the generators $a_i(s)$ and $b_i(s)$ are kept distance k_s apart. In the group Δ, for $r \geq 2$, define the subset U_r as

$$U_r = \{Z \in \Delta : \ \mathrm{Range}(Z) \subseteq [-r, r]\}.$$

Recall that $\mathrm{Range}(Z)$ is defined in Section 2.2.2; it is the minimal interval of $\mathbb{Z}$ visited by the cursor of a path representing Z. Take a function supported on the subset U_r,

$$\varphi_r\left((f_s), z\right) = \max\left\{0, 1 - \frac{|z|}{r}\right\} \mathbf{1}_{U_r}\left((f_s), z\right). \tag{32}$$

Let $\mathfrak{q}$ be the switch-or-walk measure $\mathfrak{q} = \frac{1}{2}(\mu + \nu)$ on Δ, where ν is the uniform measure on $\{\alpha_i, \beta_j : \ 1 \leq i \leq |A|, 1 \leq j \leq |B|\}$ and μ is the simple random walk measure on the base $\mathbb{Z}$, $\mu\left(\tau^{\pm 1}\right) = \frac{1}{2}$. We have seen in the proof of Proposition 4.4 that for $p = 2$,

$$\frac{\mathcal{E}_{\Delta,\mathfrak{q}}\left(\varphi_r\right)}{\|\varphi_r\|_2^2} \sim \frac{3}{2r^2}.$$

Define φ_1 to be the indicator function of the identity e_Δ,

$$\varphi_1 = \mathbf{1}_{e_\Delta}.$$

Motivated by Tessera's embedding in [Tes11, §3], given a non-decreasing function $\gamma : \mathbb{Z}_{\geq 0} \to \mathbb{R}_+$ with $\gamma(0) = 1$ and

$$C(\gamma) = \sum_{t=1}^{\infty} \left(\frac{1}{\gamma(t)}\right)^2 < \infty, \tag{33}$$

we define a 1-cocycle $b_\gamma : \Delta \to \oplus_{j=0}^{\infty} \ell^2(\Delta) \subset L_2$,

$$b_\gamma(Z) = \bigoplus_{j=0}^{\infty} \left(\frac{1}{\gamma(j)} \frac{\varphi_{2^j} - \tau_Z \varphi_{2^j}}{\mathcal{E}_{\Delta,\mathfrak{q}}\left(\varphi_{2^j}\right)^{\frac{1}{2}}}\right), \tag{34}$$

where τ_g denote right translation of functions, $\tau_g \varphi(h) = \varphi\left(hg^{-1}\right)$.

Lemma 6.9. *The* 1*-cocycle* $b_\gamma : \Delta \to L_2$ *defined in* (34) *is* $\sqrt{2C(\gamma)}$*-Lipschitz. Suppose in addition that there exists* $m_0 \geq 2$ *such that* $k_{s+1} \geq m_0 k_s$*,* $l_{s+1} \geq m_0 l_s$ *for any* $s \geq 1$*. Then there is a constant* $C(m_0) > 0$ *depending only* *on m_0 such that*

$$\rho_{b_\gamma}(x) \geq C(m_0)\frac{x/l_s}{\gamma(\log_2(x/l_s))} \quad \textit{if } x \in [k_s l_s, k_{s+1} l_s),$$
$$\rho_{b_\gamma}(x) \geq C(m_0)\frac{k_{s+1}}{\gamma\left(\log_2 k_{s+1}\right)} \quad \textit{if } x \in [k_{s+1} l_s, k_{s+1} l_{s+1}).$$

Proof. Clearly $\|b_\gamma(Z)\| = 1$ when Z is a generator in $A \cup B$ and $\|b_\gamma(Z)\| = \sqrt{2C(\gamma)}$ when $Z = \tau$ is the generator of the cyclic base $\mathbb{Z}$.

The 1-cocycle b_γ captures the size of Range(Z). Denote $Z = ((f_s), i)$ and observe that if Range$(Z) > 2^{j+2}$, then

$$(\mathrm{supp}\varphi_{2^j}) \cap (\mathrm{supp}\tau_Z\varphi_{2^j}) = \emptyset.$$

Indeed, either there exist $\iota \notin [-2^{j+1}, 2^{j+1}]$ and $s \leq s_0(Z)$ with $f_s(\iota) \neq e_{\Delta_s}$ and then this also holds for all elements of $\mathrm{supp}\tau_Z\varphi_{2^j}$ and none of $\mathrm{supp}\varphi_{2^j}$, or $|i| > 2^{j+1}$ and then the projections on the base of the two supports are disjoint. Therefore

$$\frac{\|\varphi_{2^j} - \tau_Z\varphi_{2^j}\|_2^2}{\mathcal{E}_{\Delta,\mathfrak{q}}\left(\varphi_{2^j}\right)} = \frac{2\left\|\varphi_{2^j}\right\|_2^2}{\mathcal{E}_{\Delta,\mathfrak{q}}\left(\varphi_{2^j}\right)} \sim \frac{4 \cdot 2^{2j}}{3}.$$

By construction of b_γ, this implies that if Range$(Z) > 2^{j+2}$, then

$$\|b_\gamma(Z)\|_2 \geq \frac{2^j}{\sqrt{3}\gamma(j)}.$$

By Definition 2.8, for Z with Range$(Z) = r \in [k_s, k_{s+1})$, we have $s_0(Z) \leq s$.

Denote $Z = ((f_s), i)$. Then by Lemma 2.13,

$$|(f_s, i)|_{\Delta_s} = |\pi_s(Z)|_{\Delta_s} \leq 18(r+1)l_s$$

because at most $2r/k_s + 1$ intervals contribute to the essential contribution. By Proposition 2.14, the word distance of Z to e_Δ is bounded:

$$|Z|_\Delta \leq 500 \cdot 18(r+1)\left(l_0 + \cdots + l_s\right) \leq \frac{9000(r+1)l_s}{1 - 1/m_0}.$$

It follows that

$$\rho_{b_\gamma}\left(\frac{9000(r+1)l_s}{1-1/m_0}\right) \geq \frac{r}{8\gamma(\log_2 r)}.$$

To write it into the first inequality stated, note that since b_γ is equivariant, ρ_{b_γ} is subadditive.

The second bound follows from the first bound evaluated at $x = k_{s+1}l_s$ and the monotonicity of the compression function ρ_{b_γ}. □

Remark 6.10. The function $t/\gamma \circ \log(t)$ with γ satisfying (33) does not satisfy any a priori majoration by a sublinear function. More precisely, [Tes11, Prop. 8] implies that for any increasing sublinear function $h : \mathbb{R}_+ \to \mathbb{R}_+$, there exist a non-decreasing function γ satisfying (33), a constant $c > 0$ and an increasing subsequence of integers n_i, such that

$$\frac{k_{n_i}}{\gamma \circ \log_2 (k_{n_i})} \geq ch\,(k_{n_i}) \quad \text{for all } i.$$

It follows that the upper bound in Proposition 6.3 cannot be improved.

6.2.4. *Possible compression gaps of embeddings into L_p.* The 1-cocycle b_γ defined in (34) is almost optimal in the sense that the lower bounds of Lemma 6.9 match with the upper bounds of Proposition 6.3 up to the factor sequence $1/\gamma \circ \log_2(k_s)$. The following result is an analogue of [ADS09, Th. 5.5(II)] in our setting.

THEOREM 6.11. *There exist absolute constants $\delta > 0, C > 0$ such that the following holds. Let $\rho(x)$ be any non-decreasing function such that $\frac{x}{\rho(x)}$ is non-decreasing. Then there exists a finitely generated group Δ such that $\left(\frac{1}{C\varepsilon}\frac{\rho}{\log(1+\rho)^{(1+\varepsilon)/2}}, Cp2^{p/2}\rho\right)$ is an equivariant L_p-compression gap of Δ for any $p > 1$.*

Furthermore, if $\lim_{x\to\infty} \rho(x) = \infty$, then the group Δ constructed is elementary amenable, and $\left(\frac{1}{C\varepsilon}\frac{\rho}{\log(1+\rho)^{(1+\varepsilon)/2}}, C\rho\right)$ is an equivariant L_1-compression gap of Δ.

Proof. We write $\rho(x) = \frac{x}{f(x)}$ with $f(x)$ between 1 and x. The sets $K = \mathbb{Z}_+ \cup \{\infty\}$ and $L = \{\operatorname{diam} \Gamma_m, m \geq 1\} \cup \{\infty\}$ of diameters of Lafforgue expanders from Example 2.3 satisfy the assumptions of Proposition B.2. So we can find sequences $(k_s), (l_s)$ taking values in K and L such that the function defined by $\tilde{f}(x) = l_s$ on $[k_s l_s, k_{s+1} l_s]$ and $\tilde{f}(x) = \frac{x}{k_{s+1}}$ on $[k_{s+1} l_s, k_{s+1} l_{s+1}]$ satisfies $\tilde{f}(x) \simeq_{m_0 C_1^5} f(x)$. Then $\tilde{\rho}(x) = \frac{x}{\tilde{f}(x)} \simeq_{m_0 C_1^5} \rho(x)$. Let Δ be the diagonal product associated to these sequences.

By Proposition 6.3 and Lemma 6.6, any 1-Lipschitz equivariant embedding $\Psi : \Delta \to L_p$ satisfies, for all x,

$$\rho_\Psi(x) \leq C\tilde{\rho}(2x) \leq 2m_0 C_1^5 C\rho(x),$$

where by Fact 6.8,

$$C \leq \max\{C(\delta, p), \frac{2}{\kappa_{L_p}(\Gamma)})\} \leq \max\{C'p, C''p2^{p/2}\}.$$

This gives the upper bound of the compression gap. For $p = 1$, Lemma 6.6 does not hold, so the upper bound is valid only on the condition that all the diameters l_s are finite, which is satisfied when ρ is unbounded, or equivalently, f is not asymptotically linear.

The lower bound is given by the 1-cocyle $b_\gamma : \Delta \to \ell^2$ of Lemma 6.9 with $\gamma(x) = C_\epsilon x^{\frac{1+\epsilon}{2}}$, where $C_\epsilon \sim \frac{1}{\epsilon}$ is such that $\sqrt{2C(\gamma)} = 1$. For all x,

$$\rho_{b_\gamma}(x) \geq C_\epsilon C(m_0) \frac{\tilde{\rho}(x)}{\log_2(\tilde{\rho}(x))^{\frac{1+\epsilon}{2}}} \geq \frac{1}{C\epsilon} \frac{\rho(x)}{\log(1+\rho(x))^{\frac{1+\epsilon}{2}}}.$$

Since ℓ^2 embeds isometrically in L_p for all $p \geq 1$ (see Lemma 2.3 in [NP08]), it is also an L_p-compression lower bound. □

7. L_p-compression of the wreath product $H \wr \mathbb{Z}$

In general, for the diagonal products constructed with $\{\Gamma_s\}$ chosen to be finite groups other than expanders, the analysis of L_p-compression is more involved. Since our main object Δ is a diagonal product of a sequence of wreath products, in this section we first understand compression of uniform embedding of a single copy of the wreath product $H \wr \mathbb{Z}$.

In [NP08], Naor and Peres prove that if $\alpha_2^{\#}(H) = \frac{1}{2\beta^*(H)}$, where $\beta^*(H)$ is the supremum of upper speed exponent of symmetric random walk of bounded support on H, then ([NP08, Cor. 1.3])

$$\alpha_2^{\#}(H \wr \mathbb{Z}) = \frac{2\alpha_2^{\#}(H)}{2\alpha_2^{\#}(H)+1}.$$

In their subsequent work [NP11], which significantly extends the method in [NP08], the L_p-compression exponent of $\mathbb{Z} \wr \mathbb{Z}$ is determined ([NP11, Th. 1.2]): for every $p \in [1, \infty)$,

$$\alpha_p^{\#}(\mathbb{Z} \wr \mathbb{Z}) = \max\left\{\frac{p}{2p-1}, \frac{2}{3}\right\}.$$

In [NP11] Naor and Peres also prove the following result when the base group is of polynomial volume growth at least quadratic. Let H be a non-trivial finitely generated amenable group and Γ a group of polynomial volume growth. Suppose the volume growth rate of Γ is at least quadratic. Then [NP11, Th. 3.1] states that for every $p \in [1, 2]$,

$$\alpha_p^{\#}(H \wr \Gamma) = \min\left\{\frac{1}{p}, \alpha_p^{\#}(H)\right\}.$$

One central idea in these works is the Markov type method that connects compression exponents of G to the speed exponent of certain random walks on G. The aim of this section is to obtain the following generalization of the aforementioned results of Naor and Peres.

THEOREM 7.1. *Let $p \in [1,2]$, and let H be a finitely generated infinite group. Then the equivariant L_p-compression exponent of $H \wr \mathbb{Z}$ is*

$$\alpha_p^{\#}(H \wr \mathbb{Z}) = \min\left\{ \frac{\alpha_p^{\#}(H)}{\alpha_p^{\#}(H) + \left(1 - \frac{1}{p}\right)}, \alpha_p^{\#}(H) \right\}.$$

Remark 7.2. It will be transparent in the proof of Theorem 7.1 that there is a dichotomy of the kind of obstruction that $H \wr \mathbb{Z}$ observes, depending on $\alpha_p^{\#}(H)$ as indicated in the formula. Namely, when $\alpha_p^{\#}(H) > \frac{1}{p}$, then the sequence of subsets X_n as defined in the proof of Proposition 7.3 captures distorted elements under the embedding; when $\alpha_p^{\#}(H) \leq \frac{1}{p}$, then one single copy of H already provides sufficient distortion, and in the latter case we have $\alpha_p^{\#}(H \wr \mathbb{Z}) = \alpha_p^{\#}(H)$.

7.1. *Upper bound of compression function of $H \wr \mathbb{Z}$.* Let $\Psi : H \wr \mathbb{Z} \to \mathfrak{X}$ be an equivariant embedding of $G = H \wr \mathbb{Z}$ into metric space $(\mathfrak{X}, d_{\mathfrak{X}})$. Recall that the group H is naturally identified with the lamp group over site 0,

$$\begin{aligned} i : H &\hookrightarrow G, \\ h &\to (h\delta_0, 0). \end{aligned}$$

Then the embedding Ψ induces an equivariant embedding of H into $\mathfrak{X}$. We denote it by ψ_H

$$\psi_H(h) = \Psi \circ i(h) = \Psi\left((h\delta_0, 0)\right). \tag{35}$$

Denote the compression function of $\psi_H : H \to \mathfrak{X}$ by ρ_{ψ_H}. Since distortion of the inclusion map i is 1, $|h|_H = |i(h)|_G$, it follows that

$$\rho_\Psi(t) \leq \rho_{\psi_H}(t) \quad \text{for all } t \geq 1. \tag{36}$$

We now explain how to apply the spectral method to derive a second upper bound on ρ_Ψ when Ψ is a uniform embedding of G into L_p, $p \in (1,2]$. The novelty here is in the choice of Markov kernels on lamplighter graphs.

PROPOSITION 7.3. *There exists a constant $C > 0$ such that for any $p \in (1,2]$ and any 1-Lipschitz equivariant embedding $\Psi : G \to L_p$ of $G = H \wr \mathbb{Z}$ into L_p, the compression function ρ_Ψ of Ψ satisfies*

$$\rho_\Psi(t) \leq C \left(\frac{p}{p-1}\right)^{\frac{1}{p}} \left(2\rho_{\psi_H} \circ \tau^{-1}(t)\right)^{\frac{p}{p-1}} \log^{\frac{1}{p}} \left(2\rho_{\psi_H} \circ \tau^{-1}(t)\right), \tag{37}$$

where ψ_H is the induced embedding of the subgroup H into L_p as in (35) and the function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$\tau(x) = 2^{\frac{1}{p-1}} x \left(\rho_{\psi_H}(x)\right)^{\frac{p}{p-1}}.$$

Proof. By definition of the compression function ρ_{ψ_H}, for every $n \geq 1$, there exists an element $h_n \in H$ such that $|h_n|_H \geq n$ and

$$\left\|\psi_H\left(h_n\right)-\psi_H\left(e_H\right)\right\|_p \leq 2 \rho_{\psi_H}(n).$$

Take X_n to be a finite subset in G defined by

$$X_n=\left\{(f, z): \ z \in\left[0, m_n-1\right], \quad \begin{array}{cl} f(x) \in\left\{e_H, h_n\right\} & \text { for } x \in\left[0, m_n-1\right] \\ f(x)=e_H & \text { for } x \notin\left[0, m_n-1\right] \end{array}\right\}.$$

The length m_n will be determined later. Note that X_n has the structure of a lamplighter graph; namely, let $\mathcal{L}_{m_n}$ be lamplighter graph over segment $[0, m_n-1]$ defined in Appendix C. Then there is a bijection

$$\begin{aligned} & \sigma_n: \mathcal{L}_{m_n} \rightarrow X_n, \\ & \sigma_n(f, x)=(\tilde{f}, x), \text { where } \tilde{f}(z)=h_n^{f(z)}. \end{aligned}$$

In Appendix C we define a Markov transition kernel $\mathfrak{p}_{m_n}$ on $\mathcal{L}_{m_n}$ that moves on the base segment with a Cauchy-like step distribution ζ_{m_n}. On X_n, take the Markov kernel K_n to be $\mathfrak{p}_{m_n} \circ \sigma_n^{-1}$; that is,

$$K_n(u, v)=\mathfrak{p}_{m_n}\left(\sigma_n^{-1}(u), \sigma_n^{-1}(v)\right).$$

Denote by π_n the stationary distribution of K_n on X_n, $\pi_n=U_{m_n} \circ \sigma_n^{-1}$. Under the bijection σ_n, the Poincaré inequality that $\Psi \circ \sigma_n: \mathcal{L}_{m_n} \rightarrow L_p$ satisfies as in Lemma C.1 implies

$$\begin{aligned} & \sum_{u, v \in X_n}\|\Psi(u)-\Psi(v)\|_p^p \pi_n(u) \pi_n(v) \\ & \quad \leq C m_n \log m_n \sum_{u, v \in X_n}\|\Psi(u)-\Psi(v)\|_p^p K_n(u, v) \pi_n(u). \end{aligned} \tag{38}$$

Now we deduce an upper bound on ρ_Ψ from Poincaré inequalities (38) by applying Lemma 5.2. Because of equivariance of Ψ, for any $u \in G$,

$$\left\|\Psi\left(u \cdot\left(h \delta_0, 0\right)\right)-\Psi(u)\right\|_p=\left\|\Psi\left(\left(h \delta_0, 0\right)\right)-\Psi\left(e_G\right)\right\|_p=\left\|\psi_H(h)-\psi_H\left(e_H\right)\right\|_p.$$

Recall that K_n moves as a "switch-walk-switch" transition kernel. By the Hölder inequality,

$$\begin{aligned} & \sum_{u, v \in X_n}\|\Psi(u)-\Psi(v)\|_p^p K_n(u, v) \pi_n(u) \\ & \leq 2 \cdot 3^{p-1}\left\|\psi_H\left(e_H\right)-\psi_H\left(h_n\right)\right\|_p^p \\ & +3^{p-1} \sum_{(f, z) \in X_n} \sum_{y \in\left[0, m_n-1\right]}\left\|\Psi((f, z))-\Psi\left((f, z) \cdot\left(\mathbf{e}_H, y\right)\right)\right\|_p^p \zeta_{m_n}(z, y) \pi_n((f, z)) \\ & \leq 3^p\left[\left(2 \rho_{\psi_H}(n)\right)^p+\sum_{z, y \in\left[0, m_n-1\right]}|z-y|^p \zeta_{m_n}(z, y) \mathcal{C}_{m_n}(z)\right], \end{aligned}$$

where $\mathcal{C}_{m_n}(z)$ denotes the stationary distribution of ζ_{m_n}. The last step used the choice that $\|\psi_H(h_n) - \psi_H(e_H)\|_p \le 2\rho_{\psi_H}(n)$ and the assumption that Ψ is 1-Lipschitz. From the explicit formula that defines ζ_{m_n}, for $p > 1$, we have

$$\sum_{z,y\in[0,m_n-1]} |z-y|^p \, \zeta_{m_n}(z,y)\mathcal{C}_{m_n}(z) \le C m_n^{p-1}.$$

Set

$$m_n = \left\lceil (2\rho_{\psi_H}(n))^{\frac{p}{p-1}} \right\rceil,$$

so that the two terms in the L_p-energy upper bound are comparable. Then

$$\sum_{u,v\in X_n} \|\Psi(u) - \Psi(v))\|_p^p \, K_n(u,v)\pi_n(u) \le 3^p(1+C)m_n^{p-1}. \tag{39}$$

From the explicit lamplighter structure, one checks that the set X_n satisfies $\mathrm{diam}_G(X_n) = (2 + |h_n|_H)\, m_n$, and that for any $u \in X_n$,

$$\sum_{v\in X_n} \mathbf{1}_{\left\{d_G(u,v)\ge \frac{1}{2}\left(|h_n|_H+2\right)m_n\right\}} \pi_n(v) \ge \frac{1}{5}. \tag{40}$$

Applying (38), (39), and (40) in Lemma 5.2, we have

$$\rho_\Psi\left(\frac{(2+|h_n|_H)\, m_n}{2}\right) \le \left(C' m_n^{p-1} \cdot m_n \log m_n\right)^{\frac{1}{p}}.$$

Since $|h_n|_H \ge n$, it follows that there exist $u, v \in X_n$, $d_G(u,v) \ge \frac{1}{2}(n+1)\, m_n$, and

$$\|\Psi(u) - \Psi(v)\|_p \le C' m_n \log^{\frac{1}{p}} m_n.$$

Note that by definition of τ, $\tau(n) = \frac{1}{2} n \left(2\rho_{\psi_H}(n)\right)^{\frac{p}{p-1}}$, with the choice of $m_n = \left\lceil (2\rho_{\psi_H}(n))^{\frac{p}{p-1}} \right\rceil$, the statement follows from rewriting the inequality

$$\rho_\Psi\left(\frac{1}{2} n m_n\right) \le C' m_n \log^{\frac{1}{p}} m_n. \qquad \square$$

The Markov type method can also be applied to this situation; it actually yields more general results. We presented the proof for L_p-compression of G with $p \in (1,2]$ using the Poincaré inequalities because spectral gap considerations motivate the choice of the α-stable walk on the base with $\alpha = 1$. Now we explain how to apply the Markov type method. Let $(\mathfrak{X}, d_{\mathfrak{X}})$ be a metric space of Markov type p, $p > 1$, and let $\Psi : H \wr \mathbb{Z} \to \mathfrak{X}$ be a 1-Lipschitz equivariant embedding. Make the same choice of a sequence of finite subsets X_n and K_n as in the proof of Proposition 7.3 with

$$m_n = \left\lceil (2\rho_{\psi_H}(n))^{\frac{p}{p-1}} \right\rceil.$$

Let Z_t denote a Markov chain on $\mathcal{L}_{m_n}$ with transition kernel $\mathfrak{p}_{m_n}$. Then $\widetilde{Z}_t = \sigma_n(Z_t)$ is a Markov chain on X_n with transition kernel K_n. Run the Markov chain $\widetilde{Z}_t$ up to time $t_n = m_n \log m_n$. To apply Lemma 5.5, we need a lower bound for $\mathbf{E}_{\pi_n}\left[d_G\left(\widetilde{Z}_{t_n},\widetilde{Z}_0\right)^p\right]$. The bijection $\sigma_n:\mathcal{L}_{m_n}\to X_n$ induces a metric on $\mathcal{L}_{m_n}$ by

$$d_{\sigma_n}(u,v)=d_G\left(\sigma_n(u),\sigma_n(v)\right).$$

Direct inspection shows that this metric d_{σ_n} coincides with the metric $d_{\mathbf{w}_n}$ with $\mathbf{w}_n=(1,|h_n|_H)$ introduced in Appendix C. By Lemma C.2, we have that for $t_n=m_n\log m_n$,

$$\begin{aligned}\mathbf{E}_{\pi_n}\left[d_G\left(\widetilde{Z}_{t_n},\widetilde{Z}_0\right)^p\right]&=\mathbf{E}_{U_{\alpha,m_n}}\left[d_{\mathbf{w}_n}\left(Z_{t_n},Z_0\right)^p\right]\\&\geq\mathbf{E}_{U_{\alpha,m_n}}\left[d_{\mathbf{w}_n}\left(Z_{t_n},Z_0\right)\right]^p\\&\geq\left(c\left(1+|h_n|_H\right)m_n\right)^p.\end{aligned}$$

In the proof of Proposition 7.3 we checked that

$$\begin{aligned}\mathbf{E}_{\pi_n}\left[d_{\mathfrak{X}}\left(\Psi\left(\widetilde{X}_1\right),\Psi\left(\widetilde{X}_0\right)\right)^p\right]&=\sum_{u,v\in X_n}d_{\mathfrak{X}}\left(\Psi(u),\Psi(v)\right)^pK_n(u,v)\pi_n(v)\\&\leq 3^p\left[\left(2\rho_{\psi_H}(n)\right)^p+Cm_n^{p-1}\right].\end{aligned}$$

Choose $m_n=\left\lceil\left(2\rho_{\psi_H}(n)\right)^{\frac{p}{p-1}}\right\rceil$, and plug these estimates into Lemma 5.5. Then we have

$$\rho_\Psi\left(\frac{c}{2}\left(1+|h_n|_H\right)m_n\right)\leq C'M_p\left(\mathfrak{X}\right)m_n\log^{\frac{1}{p}}m_n.$$

Recall that $|h_n|_H\geq n$, and therefore

$$\rho_\Psi\left(\frac{c}{2}nm_n\right)\leq C'M_p\left(\mathfrak{X}\right)m_n\log^{\frac{1}{p}}m_n.$$

The result given by the Markov type method is recorded in the following proposition. By [NPSS06], L_p space with $p>2$ has Markov type 2. Proposition 7.4 applies to L_p with $p>2$ as well and therefore is more general than Proposition 7.3.

Proposition 7.4. *There exists a constant $C>0$ such that the following holds. Let $(\mathfrak{X},d_{\mathfrak{X}})$ be a metric space of Markov type p, $p>1$, and let $\Psi: H\wr\mathbb{Z}\to\mathfrak{X}$ be a 1-Lipschitz equivariant embedding. The compression function ρ_Ψ satisfies*

$$\rho_\Psi(t)\leq C\left(\frac{p}{p-1}\right)^{\frac{1}{p}}M_p\left(\mathfrak{X}\right)\left(2\rho_{\psi_H}\circ\tau^{-1}(t)\right)^{\frac{p}{p-1}}\log^{\frac{1}{p}}\left(2\rho_{\psi_H}\circ\tau^{-1}(t)\right),\tag{41}$$

where ψ_H is the induced embedding of the subgroup H into $\mathfrak{X}$ as in (35), $M_p\left(\mathfrak{X}\right)$ is the Markov-type p constant of $\mathfrak{X}$, and the function $\tau:\mathbb{R}_+\to\mathbb{R}_+$ is defined as

$$\tau(x)=\frac{1}{C}x\left(2\rho_{\psi_H}(x)\right)^{\frac{p}{p-1}}.$$

7.2. *L_p-compression exponent of $H \wr \mathbb{Z}$.* For the lower bound in the L_p-compression gap of $H \wr \mathbb{Z}$, $p \in [1,2]$, we use the embedding constructed in [NP08, Th. 3.3]. An explicit description of an equivariant embedding is included here as a warm-up for Section 8. Given a good equivariant embedding φ of the group H into L_p ,

$$\|\varphi(h_1) - \varphi(h_2)\|_p \geq \rho_-\left(d_H\left(h_1, h_2\right)\right),$$

we exhibit an embedding of $G = H \wr \mathbb{Z}$ into L_p with φ as building blocks. Recall that for an element $(f, z) \in G$, the word distance is given by

$$|(f,z)|_G = |\omega| + \sum_{x \in \mathbb{Z}} |f(x)|_H\,,$$

where ω is a path of shortest length that starts at 0, visits every point x in the support of f, and ends at z. We refer to such a path as a traveling salesman path for (f, z). The embedding of G into L_p consists of two parts: Part I captures the length of ω, and Part II embeds the lamp configurations $\{f(x)\}_{x \in \mathbb{Z}}$ using the embedding of H into L_p.

Part I. This part is essentially the same as an embedding of $\mathbb{Z}_2 \wr \mathbb{Z}$ into L_p. Consider the following sequence of functions on $G = H \wr \mathbb{Z}$:

$$\phi_n\left((f,z)\right) = \mathbf{1}_{\{\mathrm{supp} f \subseteq [-2^n, 2^n]\}} \max\left\{1 - \frac{|z|}{2^n}, 0\right\}. \tag{42}$$

Note that if $g \in G$ is in the support of ϕ_n, then

$$|\omega_g| \leq 4 \cdot 2^n,$$

where ω_g denotes a traveling salesman path for g. Also,

$$\left(\mathrm{supp}\phi_n\right)\left(\mathrm{supp}\phi_n\right)^{-1} \subseteq \mathrm{supp}\phi_{n+1}.$$

As in [Tes11], given a non-decreasing function $\gamma : \mathbb{N} \to \mathbb{R}_+$ with $\gamma(1) = 1$ and $\sum_{t=0}^{\infty}\left(\frac{1}{\gamma(t)}\right)^p = C_p(\gamma) < \infty$, take a cocycle $b_\gamma : G \to L_p$ defined as

$$b_\gamma\left(g\right) = \bigoplus_{n=1}^{\infty} \frac{1}{\gamma(n)} \left(\frac{\tau_g \phi_n - \phi_n}{\mathcal{E}_p\left(\phi_n\right)^{1/p}}\right), \tag{43}$$

where τ_g denotes the right translation of a function $\phi : G \to \mathbb{R}$,

$$\tau_g \phi(u) = \phi\left(ug^{-1}\right).$$

Because of the normalization in the definition of b_γ, one readily checks that b_γ is $(2C_p(\gamma))^{\frac{1}{p}}$-Lipschitz.

Given $g = (f, x) \in G$, we say a path ω on $\mathbb{Z}$ is a traveling salesman path for g if it starts at 0, visits every $z \in \mathbb{Z}$ where $f(z) \neq e_H$, and ends at x. Let

ω_g be a shortest traveling salesman path for g. Suppose $|\omega_g| > 2^{n+3}$. Then $g \notin (\mathrm{supp}\phi_n)(\mathrm{supp}\phi_n)^{-1}$, and it follows that in this case,

$$\frac{\|\tau_g\phi_n - \phi_n\|_p^p}{\mathcal{E}_p(\phi_n)} = \frac{2\|\phi_n\|_p^p}{\mathcal{E}_p(\phi_n)} \geq (2^n)^p.$$

Putting the components together, we have that

$$\|b_\gamma(g)\|_p \geq \frac{|\omega_g|}{8\gamma(\log_2|\omega_g|)}.$$

Part II. Let $\varphi : H \to L_p$ be a 1-Lipschitz equivariant embedding of H into L_p. Define a map $\Xi_\varphi : G \to L_p$ as

$$\Xi_\varphi((f,z)) = \bigoplus_{x\in\mathbb{Z}} \varphi(f(x)).$$

Since the map Ξ_φ factors through the projection $G \to \oplus_{x\in\mathbb{Z}}H$ and φ is equivariant, it follows that Ξ_φ is a cocycle. By construction, Ξ_φ is 1-Lipschitz and

$$\|\Xi_\varphi((f,z))\|_p^p = \sum_{x\in\mathbb{Z}} \|\varphi(f(x))\|_p^p.$$

Combining the two parts, we define

$$\begin{aligned} &\Phi : G \to L_p, \\ &\Phi(g) = b_\gamma(g) \bigoplus \Xi_\varphi(g). \end{aligned} \tag{44}$$

Then we have that Ψ is $(2C_p(\gamma))^{1/p}$-Lipschitz and

$$\begin{aligned} \|\Phi(g)\|_p^p &= \|b_\gamma(g)\|_p^p + \sum_{x\in\mathbb{Z}} \|\varphi(f(x))\|_p^p \\ &\geq \frac{|\omega_g|^p}{8\gamma(\log_2|\omega_g|)} + c\sum_{x\in\mathbb{Z}} \rho_-(|f(x)|_H)^p. \end{aligned}$$

Let $p \in (1,2]$. The bounds (35), (36) and the embedding constructed above provide rather detailed information about the L_p-compression gap of $G = H \wr \mathbb{Z}$ in terms of L_p-compression gap of H. We now derive the formula relating the L_p-compression exponents of G and H stated in Theorem 7.1.

Proof of Theorem 7.1. We first treat the case $p \in (1,2]$. Let $\Psi : G \to L_p$ be a 1-Lipschitz equivariant embedding of $G = H \wr \mathbb{Z}$ into L_p and ψ_H its induced embedding $H \hookrightarrow L_p$. From (36), it is always true that $\alpha_p^\#(H \wr \mathbb{Z}) \leq \alpha_p^\#(H)$. Now we apply Proposition 7.3 to prove the upper bound. By definition of the compression exponent, there exists a constant $C = C(\psi_H) > 0$ and an increasing sequence $n_i \in \mathbb{N}$ with $n_i \to \infty$ such that

$$\rho_{\psi_H}(n_i) \leq C n_i^{\alpha_p^\#(H)}.$$

Along the sequence $\{n_i\}$, by Proposition 7.3,

$$\rho_\Psi(n_i) \leq C_1(C,p) n_i^{\frac{p\alpha}{p-1+\alpha p}} \log^{\frac{1}{p}}\left(n_i^{\frac{p\alpha}{p-1+\alpha p}}\right),$$

where $\alpha = \alpha_p^{\#}(H)$ and $C_1(C,p)$ is a constant depending on C and p. Therefore

$$\alpha_p^{\#}(H \wr \mathbb{Z}) \leq \frac{\alpha_p^{\#}(H)}{\alpha_p^{\#}(H) + \left(1 - \frac{1}{p}\right)}.$$

We have proved the upper bound.

Note that if $\alpha_p^{\#}(H) = 0$, then $\alpha_p^{\#}(G) = 0$. Thus in the lower bound direction, we consider the case $\alpha_p^{\#}(H) > 0$. For any $0 < \varepsilon < \alpha_p^{\#}(H)$, let $\varphi : H \to L_p$ be a 1-Lipschitz equivariant embedding such that

$$\rho_\varphi(t) \geq (ct)^{\alpha_p^{\#}(H) - \varepsilon},$$

and set $\gamma(n) = \log^{\frac{1+\varepsilon}{p}}(1+n)$. Take the embedding $\Phi : G \to L_p$ as defined in (44). Then

$$\begin{aligned}
\|\Phi(g)\|_p^p &= \|b_\gamma(g)\|_p^p + \sum_{x\in\mathbb{Z}} \|\varphi(f(x))\|_p^p \\
&\geq \frac{|\omega_g|^p}{8\log^{1+\varepsilon}(1+|\omega_g|)} + \sum_{x\in\mathbb{Z}} \left(c\,|f(x)|_H\right)^{\left(\alpha_p^{\#}(H)-\varepsilon\right)p}.
\end{aligned}$$

Such an embedding Φ is analyzed in [NP08]. As ε is arbitrarily small, the proof of [NP08, Th. 3.3] applies to show that if $\alpha_p^{\#}(H) \leq \frac{1}{p}$, then $\alpha_p(\Phi) \geq \alpha_p^{\#}(H)$, and if $\alpha_p^{\#}(H) > \frac{1}{p}$, then

$$\alpha_p^{\#}(\Phi) \geq \frac{\alpha_p^{\#}(H)}{\alpha_p^{\#}(H) + 1 - 1/p}.$$

Finally when $p=1$, from $\rho_\Psi(t) \leq \rho_{\psi_H}(t)$ and the explicit embedding $\Phi : G \to L_1$ given a good embedding $\varphi : H \to L_1$, we deduce that

$$\alpha_1^{\#}(G) = \alpha_1^{\#}(H). \qquad \square$$

8. Compression of Δ constructed with dihedral groups

Throughout this section Δ denotes a diagonal product with input $\{\Gamma_s = D_{2l_s}\}$. The main result of this section is the following.

THEOREM 8.1. *Let Δ be the diagonal product with $\{\Gamma_s = D_{2l_s}\}$ and parameters (k_s), and set*

$$\theta := \limsup_{s\to\infty} \frac{\log l_s}{\log k_s}.$$

Assume that (k_s) satisfies the growth assumption 2.11. Then

(i) *for* $p \in [1,2]$,

$$\alpha_p^*(\Delta) = \max\left\{\frac{1}{1+\theta}, \frac{2}{3}\right\};$$

(ii) *for* $q \in (2,\infty)$,

$$\alpha_q^*(\Delta) = \frac{1}{1+\theta} \quad \text{if } 0 \le \theta \le \frac{1}{q},$$

$$\max\left\{\frac{\theta + 1 - \frac{2}{q}}{\left(2-\frac{1}{q}\right)\theta + 1 - \frac{2}{q}}, \frac{2}{3}\right\} \le \alpha_q^*(\Delta) \le \frac{2\theta + 1 - \frac{2}{q}}{3\theta + 1 - \frac{2}{q}} \quad \text{if } \theta > \frac{1}{q}.$$

When $p \in [1,2]$, the upper bound on $\alpha_p^*(\Delta)$ is a consequence of the Mendel-Naor metric cotype inequality cited in Section 5.4; in the lower bound direction we construct an explicit embedding $\Delta \to \ell^2$. The case of $p \in (2,\infty)$ is more involved. The proof of Theorem 8.1 is completed in Section 8.4.

Since Δ is 3-step solvable, in particular it is amenable. By [NP11, Th. 1.6], we have

$$\alpha_p^*(\Delta) = \alpha_p^{\#}(\Delta) \text{ for all } p \in [1,\infty).$$

8.1. *Upper bounds on compression functions*. We first explain why it is necessary to examine the distortion in a block of side length k_s in Δ_s. For notational convenience, assume that (k_s), (l_s) are multiples of 4. As in Section 2.3, consider the subset $\Pi_s^{k_s/2}$ of Δ defined as in (2). Note that $\Pi_s^{k_s/2}$ is isomorphic to the direct product of $k_s/2$ copies of $D'_{2l_s} \simeq \mathbb{Z}_{l_s/2}$. Denote by $\vartheta_s : \mathbb{Z}_{l_s/2}^{k_s/2} \to \Pi_s^{k_s/2}$ the isomorphism. Write elements of $\Pi_s^{k_s/2}$ as vectors $u = (u(0),\ldots,u(k_s-1))$, $u(j) \in \mathbb{Z}_{l_s/2}$.

Now consider the induced metric on $\Pi_s^{k_s/2}$ of the word metric d_Δ on Δ. Then by Lemma 2.16, we have that for $u \in \Pi_s^{k_s/2}$,

$$|u|_\Delta \simeq_{72} k_s \max_{0\le j\le k_s/2-1} |u(j)|_{\mathbb{Z}/l_s\mathbb{Z}}.$$

Therefore the induced metric $|\cdot|_\Delta$ on $\Pi_s^{k_s/2}$ can be viewed as the ℓ^∞ metric being dilated by k_s.

Let $[m]_\infty^k$ denote the set $\{0,1,\ldots,m\}^k$ equipped with the metric induced by ℓ^∞,

$$d_\infty(x,x') := \max_{0\le j\le k-1} \left|x_j - x'_j\right|, \ x = (x_0,\ldots,x_{k-1}).$$

Let $\mathfrak{X}$ be a Banach space of non-trivial type and cotype q. Then by [MN08, Th. 1.12], there exists a constant $c(\mathfrak{X},q)$ depending only on $\mathfrak{X}$ and q such that the distortion of embedding of $[m]_\infty^k$ into $\mathfrak{X}$ satisfies

$$c_{\mathfrak{X}}\left([m]_\infty^k\right) \ge c(\mathfrak{X},q)\left(\min\left\{k^{\frac{1}{q}}, m\right\}\right).$$

We now explain how to apply this distortion lower bound and the Austin lemma to derive an upper bound on $\alpha^*_{\mathfrak{X}}(\Delta)$. Define

$$m_s = \left\lfloor \min\left\{k_s^{\frac{1}{q}}, \frac{1}{4}l_s\right\}\right\rfloor,$$

and consider $\{0, 1, \ldots, m_s\}$ as elements in $\mathbb{Z}_{l_s}$. The grid $[m_s]_\infty^{k_s}$ is embedded in the group Δ via the map ϑ_s . Let $\theta := \limsup_{s\to\infty} \frac{\log l_s}{\log k_s}$, and suppose $\theta \in (0, \infty)$. For any $\epsilon > 0$, select a subsequence s_n such that $l_{s_n} \geq C_\epsilon k_{s_n}^{\theta-\epsilon}$ along this subsequence. To apply Lemma 5.1 using the sequence of finite metric spaces $\left([m_{s_n}]_\infty^{k_{s_n}}, d_\infty\right)$, we check that

- $\mathrm{diam}_{d_\infty}\left([m_{s_n}]_\infty^{k_{s_n}}\right) = m_{s_n} + 1$;
- the word distance on Δ relates to the metric d_∞ on $[m_{s_n}]_\infty^{k_{s_n}}$ by

  $$d_\Delta(\vartheta_{s_n}(u, o), \vartheta_{s_n}(u', 0)) \simeq_{72} k_{s_n} d_\infty(u, u')$$

  and

  $$k_{s_n} \leq C'_\epsilon \left(\mathrm{diam}_{d_\infty}\left([m_{s_n}]_\infty^{k_{s_n}}\right);\right)^{\min\left\{q, \frac{1}{\theta-\epsilon}\right\}};$$
- the distortion of $[m_{s_n}]_\infty^{k_{s_n}}$ satisfies

  $$c_{\mathfrak{X}}\left([m_{s_n}]_\infty^{k_{s_n}}\right) \geq c(\mathfrak{X}, q) m_{s_n} \geq \frac{c(\mathfrak{X}, q)}{2} \mathrm{diam}_{d_\infty}\left([m_{s_n}]_\infty^{k_{s_n}}\right).$$

Then by Lemma 5.1, we have that if $\mathfrak{X}$ is a Banach space of non-trivial type and cotype q, then

$$\alpha^*_{\mathfrak{X}}(\Delta) \leq 1 - \frac{1}{1 + \min\left\{q, \frac{1}{\theta}\right\}} = \max\left\{\frac{1}{1+\theta}, \frac{q}{1+q}\right\}. \tag{45}$$

Note that the $\mathfrak{X}$-distortion of the grid $[m_{s_n}]_\infty^{k_{s_n}}$ selected is comparable to its d_∞-diameter. Unlike the case with $\{\Gamma_s\}$ taken to be expanders, the size of m_s is constrained by $k_s^{\frac{1}{q}}$. We will see in Section 8.2 that when $q_{\mathfrak{X}} > 2$ and $\mathfrak{X}$ is of Markov type $p < 2$, this upper bound on $\alpha^*_{\mathfrak{X}}(\Delta)$ can be improved. In the rest of this subsection we give a more detailed description of distorted elements and derive an upper bound for the compression functions.

8.1.1. *A first upper bound by the metric cotype inequality.*

PROPOSITION 8.2. *Let ϑ_s and $\Pi_s^{k_s/2}$ be introduced as above, and suppose $\mathfrak{X}$ is a Banach space of non-trivial type and cotype q. Then there exists a constant $C = C(\mathfrak{X}, q)$ such that for any 1-Lipschitz equivariant embedding $\varphi : \Delta \to \mathfrak{X}$,*

$$\rho_\varphi\left(\frac{1}{8}k_s \min\left\{l_s, k_s^{\frac{1}{q}}\right\}\right) \leq C(\mathfrak{X}, q) k_s.$$

This proposition improves on (45) as it applies to functions. Its proof will also be useful to "locate" the obstructions and derive a better upper bound in the next subsection.

Proof. Take $m \leq l_s/4$, and consider $0, \ldots, m+1$ as elements of $\mathbb{Z}_{l_s/2}$. By [MN08, Lemma 6.12], for each $\epsilon > 0$, $\mathbb{Z}_{2m}^n$ equipped with an ℓ^∞ metric embeds with distortion $1+6\epsilon$ into $[m+1]_\infty^{(\lceil 1/\epsilon \rceil+1)n}$. Take $\epsilon = 1$, and fix a 1-Lipschitz embedding

$$\psi_s : \mathbb{Z}_{2m}^{k_s/2} \to [m+1]_\infty^{k_s}$$

with distortion $c(\psi_s) \leq 8$. Let $\tilde{d}_\Delta$ be the induced metric by d_Δ on $\mathbb{Z}_{2m}^{k_s/2}$

$$\tilde{d}_\Delta(u,v) = d_\Delta\left(\vartheta_s \circ \psi_s(u), \vartheta_s \circ \psi_s(v)\right),$$

and let $\tilde{\varphi}$ be the induced embedding

$$\tilde{\varphi} = \varphi \circ \vartheta_s \circ \psi_s : \mathbb{Z}_{2m}^{k_s/2} \to \mathfrak{X}.$$

Let U_s be the uniform measure on $\mathbb{Z}_{2m}^{k_s/2}$ and σ_s be the uniform measure on $\{-1,0,1\}^{k_s/2}$. Let $\{\mathbf{e}_j\}_{j=0}^{k_s/2-1}$ be the standard basis of $\mathbb{R}^{k_s/2}$. Since $\mathfrak{X}$ is a K-convex Banach space of cotype q, then by the metric cotype inequality in [MN08, Th. 4.2] (cited in Section 5.4),

(46)
$$\sum_{j=0}^{k_s/2-1} \sum_{u \in \mathbb{Z}_{2m}^{k_s/2}} \left\|\tilde{\varphi}(u) - \tilde{\varphi}\left(u + m\mathbf{e}_j\right)\right\|_{\mathfrak{X}}^q U_s(u)$$
$$\leq \Omega^p \sum_{u \in \mathbb{Z}_{2m}^{k_s/2}} \sum_{\varepsilon \in \{-1,0,1\}^{k_s/2}} \left\|\tilde{\varphi}(u) - \tilde{\varphi}\left(u + \sum_{j=0}^{k_s/2-1} \varepsilon_j \mathbf{e}_j\right)\right\|_{\mathfrak{X}}^q \sigma(\varepsilon) U_s(u),$$

where

$$\Omega = 5 \max\left\{C\left(\mathfrak{X}, q\right) m, \left(\frac{k_s}{2}\right)^{\frac{1}{q}}\right\}$$

and $C\left(\mathfrak{X}, q\right)$ is a constant that only depends on the cotype constant and K_q-convexity constant of $\mathfrak{X}$.

Since φ is 1-Lipschitz, by Lemma 2.16 we have

$$\left\|\tilde{\varphi}(u,0) - \tilde{\varphi}\left(u + \sum_{j=0}^{k_s/2-1} \varepsilon_j \mathbf{e}_j, 0\right)\right\|_{\mathfrak{X}} \leq \tilde{d}_\Delta\left(u + \sum_{i=0}^{k_s/2-1} \varepsilon_i \mathbf{e}_i, u\right) \leq 72 k_s.$$

Plug in (46):

$$\sum_{j=0}^{k_s/2-1} \sum_{u\in(\mathbb{Z}/2m\mathbb{Z})^{k_s/2}} \|\tilde\varphi(u)-\tilde\varphi(u+m\mathbf{e}_j)\|_{\mathfrak{X}}^q U_s(u)$$
$$\leq \left(5\max\left\{C(\mathfrak{X})m, k_s^{\frac{1}{q}}\right\}\right)^q (72k_s)^q .$$

It follows that there exist $u \in (\mathbb{Z}/2m\mathbb{Z})^{k_s/2}$ and $j_0 \in \{0,\dots,k_s/2-1\}$ such that

$$\|\tilde\varphi(u)-\tilde\varphi(u+m\mathbf{e}_{j_0})\|_{\mathfrak{X}} \leq 360\max\left\{C(\mathfrak{X})mk_s^{1-\frac{1}{q}}, k_s\right\}.$$

To obtain the upper bound on ρ_φ, choose $m = \left\lfloor \frac{1}{4}\min\left\{l_s, k_s^{\frac{1}{q}}\right\}\right\rfloor$. By Lemma 2.16,

$$\tilde d_\Delta(u, u+m\mathbf{e}_{j_0}) \geq \frac{1}{2}mk_s,$$

and it follows that

$$\rho_\varphi\left(\frac{1}{2}k_s m\right) \leq 360C(\mathfrak{X})k_s. \qquad \square$$

Remark 8.3. Since L_1 has trivial type, embeddings $\varphi : \Delta \hookrightarrow L_1$ are not covered by the lemma. However it is true that there exists constant $C>0$ such that for $\varphi : \Delta \to L_1$ a 1-Lipschitz embedding,

$$\rho_\varphi\left(\frac{1}{C}k_s\min\left\{l_s, k_s^{\frac{1}{2}}\right\}\right) \leq Ck_s.$$

To see this, as pointed out in [MN08, Rem. 7.5], since L_1 equipped with the metric $\sqrt{\|x-y\|_1}$ is isomorphic to a subset of Hilbert space, [MN08, Th. 4.2] applied to the Hilbert space gives

$$\sum_{j=0}^{k_s/2-1} \sum_{u\in u\in(\mathbb{Z}/2m\mathbb{Z})^{k_s/2}} \|\tilde\varphi(u)-\tilde\varphi(u+m\mathbf{e}_j)\|_{L_1} U_s(u) \leq C^2\max\left\{m^2, k_s\right\}$$
$$\cdot \sum_{u\in u\in(\mathbb{Z}/2m\mathbb{Z})^{k_s/2}} \sum_{\varepsilon\in\{-1,0,1\}^{k_s}} \left\|\tilde\varphi(u)-\tilde\varphi\left(u+\sum_{j=0}^{k_s-1}\varepsilon_j\mathbf{e}_j\right)\right\|_{L_1} \sigma(\varepsilon)U_s(u),$$

which implies the stated bound.

8.2. *A more refined upper bound when $\mathfrak{X}$ has cotype* >2 . In this subsection we develop an improvement of the compression upper bound in Proposition 8.2. The idea is that when $q_{\mathfrak{X}} > 2$, we can further apply the Markov type method to find obstruction in lamplighter graphs with elements in blocks of side length k_s considered as lamp configurations. The argument is similar to the one for the wreath product $H \wr \mathbb{Z}$ in Section 7.1.

We continue to use notation introduced at the beginning of Section 8.1. Let $\varphi : \Delta \to \mathfrak{X}$ be an equivariant 1-Lipschitz embedding of the group Δ into $\mathfrak{X}$, and assume that $\mathfrak{X}$ is a Banach space of cotype q and non-trivial type $p > 1$. From the proof of Proposition 8.2, we have that there exists an element $h_0^s = \vartheta_s\left(\frac{l_s}{2}\mathbf{e}_{j_0}, 0\right)$ satisfying $|h_0^s|_\Delta \geq \frac{1}{4}k_s l_s$:

$$\|\varphi(h_0^s) - \varphi(e_\Delta)\|_{\mathfrak{X}} \leq C(\mathfrak{X}, q)\max\left\{l_s k_s^{1-\frac{1}{q}}, k_s\right\}. \tag{47}$$

The element h_0^s is in the zero section of Δ_s, and it is supported at site j_0 in the interval $[0, k_s - 1)$. Let h_j^s denote the translation of h_0^s to the block $[jk_s, (j+1)k_s)$,

$$h_j^s(x) = h_0^s(x - jk_s).$$

Consider the following subset (not a subgroup) in Δ_s:

$$L_m^s = \left\{ (f_s, z) : \begin{array}{c} f_s \restriction_{[jk_s,(j+1)k_s)} \in \left\{\mathbf{0}, h_j^s \restriction_{[jk_s,(j+1)k_s)}\right\},\ 0 \leq j \leq m-1 \\ \operatorname{supp} f_s \subseteq [0, mk_s) \\ z \in \{0, k_s, \ldots, (m-1)k_s\}. \end{array} \right\}.$$

Again L_m^s is naturally embedded in Δ, and we identify it with its embedded image and consider L_m^s as a subset of Δ. The subset L_m^s has the structure of a lamplighter graph over a segment, and the lamp configuration is divided into blocks of side length k_s. In each block it is either identically zero or it coincides with h_j^s. As explained in Section 7.1, we can apply the Markov type method to derive a lower bound for distortion of L_m^s.

Proposition 8.4. *Let $\mathfrak{X}$ be a Banach space of cotype q and Markov type p such that $2 < q < \infty$, $p > 1$. Then there exists a constant $C > 0$ such that for any 1-Lipschitz equivariant embedding $\varphi : \Delta \to \mathfrak{X}$, for each $s \in \mathbb{N}$, we have*

- *if $l_s \leq k_s^{\frac{1}{q}}$, then*
$$\rho_\varphi\left(\frac{1}{4}k_s l_s\right) \leq C(\mathfrak{X}, q)k_s;$$
- *if $l_s > k_s^{\frac{1}{q}}$, then*
$$\rho_\varphi\left(\frac{1}{2C}k_s l_s \left(l_s k_s^{-\frac{1}{q}}\right)^{\frac{p}{p-1}}\right) \leq C\left(\frac{p}{p-1}\right)^{\frac{1}{p}} M_p(\mathfrak{X})\, C(\mathfrak{X}, q)k_s \left(l_s k_s^{-\frac{1}{q}}\right)^{\frac{p}{p-1}} \log^{\frac{1}{p}}\left(l_s k_s^{-\frac{1}{q}}\right).$$

Proof. The case where $l_s \leq k_s^{\frac{1}{q}}$ is covered by Proposition 8.2.

In the case where $l_s > k_s^{\frac{1}{q}}$, we apply the Markov type method. Let L_m^s be defined as above. There is a natural bijection σ_m^s between the lamplighter graph $\mathcal{L}_m$ over the segment $\{0,\ldots,m-1\}$ and L_m^s; explicitly,

$$\sigma_m^s : \mathcal{L}_m \to L_m^s,$$
$$\sigma_m^s(u,x) = (f^u, mx) \text{ where } f^u \restriction_{[jk_s,(j+1)k_s)} = \left(h_j^s\right)^{u(j)}.$$

Let $\mathfrak{p}_m$ be the lamplighter kernel on $\mathcal{L}_m$ defined in Appendix C. Under the bijection σ_m^s, let $K_m^s = \mathfrak{p}_m \circ (\sigma_m^s)^{-1}$ be the corresponding Markov kernel on L_m^s. Now we run the Markov chain with transition kernel K_m^s up to time $t = m\log m$. Lemma 5.5 implies

$$\rho_\varphi\left(\left(\frac{1}{2}\mathbf{E}_\pi d_\Delta\left(Z_t, Z_0\right)^p\right)^{\frac{1}{p}}\right) \le \left(2M_p^p(\mathfrak{X}) t \operatorname{diam}_\Delta\left(L_m^s\right)^p \frac{\mathbf{E}_\pi d_{\mathfrak{X}}\left(\varphi\left(Z_1\right), f\left(Z_0\right)\right)^p}{\mathbf{E}_\pi d_\Delta\left(Z_t, Z_0\right)^p}\right)^{\frac{1}{p}}, \tag{48}$$

where π is the stationary distribution of K_m^s and Z_t is a stationary Markov chain on L_m^s with transition kernel K_m^s.

Now we estimate the quantities that appear in the inequality. Let d_Δ be the metric on $\mathcal{L}_m$ induced by word metric on Δ,

$$d_\Delta(u,v) = d_\Delta\left(\sigma_m^s(u), \sigma_m^s(v)\right).$$

Then direct inspection shows that there exists an absolute constant $C > 0$ such that

$$\frac{1}{C} d_{\mathbf{w}}(u,v) \le d_\Delta(u,v) \le C d_{\mathbf{w}}(u,v), \ \mathbf{w} = (k_s, k_s l_s).$$

It follows that $\operatorname{diam}_\Delta\left(L_m^s\right) \le 2C\left(k_s + k_s l_s\right) m$. By Lemma C.3, we have that for $t = m\log m$,

$$\mathbf{E}_{\pi_n}\left[d_\Delta\left(Z_t, Z_0\right)^p\right] \ge \left(c\left(k_s + k_s l_s\right) m\right)^p.$$

For the other term, using Lemma C.2 and (47) when $l_s^q > k_s$, we have

$$\begin{aligned}\mathbf{E}_\pi\left[d_{\mathfrak{X}}\left(\varphi\left(Z_1\right), \Psi\left(Z_0\right)\right)^p\right] &= \sum_{u,v\in X_n} d_{\mathfrak{X}}\left(\varphi(u), \varphi(v)\right)^p K_n(u,v)\pi_n(v) \\ &\le 3^p\left[Ck_s^p m^{p-1} + \left(C(\mathfrak{X}) l_s k_s^{1-\frac{1}{q}}\right)^p\right].\end{aligned}$$

With the choice

$$m = \left\lceil \left(l_s k_s^{-\frac{1}{q}}\right)^{\frac{p}{p-1}} \right\rceil,$$

(48) implies

$$\rho_\Psi\left(\frac{1}{2C} k_s l_s m\right) \le C\left(\frac{p}{p-1}\right)^{\frac{1}{p}} M_p\left(\mathfrak{X}\right) C(\mathfrak{X}, q) k_s \left(l_s k_s^{-\frac{1}{q}}\right)^{\frac{p}{p-1}} \log^{\frac{1}{p}}\left(l_s k_s^{-\frac{1}{q}}\right).$$

$\square$

The upper bound on the compression function immediately yields the following upper bound on compression exponent.

COROLLARY 8.5. *Let $\mathfrak{X}$ be a Banach space of cotype q and Markov type p with $2 < q < \infty$ and $p > 1$. Let Δ be the diagonal product with $\Gamma_s = D_{2l_s}$,*

$$\theta := \limsup_{s\to\infty} \frac{\log l_s}{\log k_s}.$$

Then

$$\alpha^{\#}_{\mathfrak{X}}(\Delta) \leq \begin{cases} \frac{1}{1+\theta} & \text{if } \theta \leq \frac{1}{q}, \\ \frac{p\theta+p-1-\frac{p}{q}}{(2p-1)\theta+p-1-\frac{p}{q}} & \text{if } \theta > \frac{1}{q}. \end{cases}$$

8.3. *An explicit embedding of Δ into L_q, $q \geq 2$.* We construct an embedding of Δ into L_q in two parts, similar to the embedding of wreath products in Section 7.2.

We first recall a standard embedding of finite dihedral groups into the Euclidean plane $\mathbb{R}^2$. The (unlabelled) Cayley graph of D_{2l} is the same as a cycle of size $2l$. One can embed it as vertices of a regular $2l$-gon in the plane. For each element $\gamma \in D_{2l}$, fix a word of minimal length in a and b such that the word represents γ and starts with the letter a. Let $k(\gamma)$ be the length of such a chosen word, and let $k_a(\gamma)$ (resp. $k_b(\gamma)$) be the number of occurrence of a (resp. b) in this word. Take $\theta_l : D_{2l} \to \mathbb{R}^2$ as

$$\theta_l(\gamma) = \frac{1}{2\sin(\pi/2l)}\left(\cos\left(\frac{\pi k(\gamma)}{l}\right), \sin\left(\frac{\pi k(\gamma)}{l}\right)\right).$$

It is clear that θ_l is 1-Lipschitz and equivariant. We also consider maps to vertices of l-gons. Let $\theta_l^{(a)} : D_{2l} \to \mathbb{R}^2$ be the map given by

$$\theta_l^{(a)}(\gamma) = \frac{1}{2\sin(\pi/l)}\left(\cos\left(\frac{2\pi k_a(\gamma)}{l}\right), \sin\left(\frac{2\pi k_a(\gamma)}{l}\right)\right).$$

The map $\theta_l^{(b)} : D_{2l} \to \mathbb{R}^2$ is defined in the same way with $k_a(\gamma)$ replaced by $k_b(\gamma)$. Since $|k_a(\gamma) - k_b(\gamma)| \leq 1$ for any element $\gamma \in D_{2l}$, by definition of $\theta_l^{(a)}$, $\theta_l^{(b)}$ we have

$$\left\|\theta_l^{(a)}(\gamma) - \theta_l^{(b)}(\gamma)\right\|_2 \leq 1.$$

Recall the classical fact that ℓ^2 embeds isometrically in L_q for all $q \geq 1$; see [AK06, Prop. 6.4.2]. For each $q > 2$, fix an isometric embedding $i_q : \ell^2 \to L_q$, and set $b_{\gamma,q} = i_q \circ b_\gamma$. Similarly, write $\theta_{l,q}^{(a)} = i_q \circ \theta_l^{(a)}$, $\theta_{l,q}^{(b)} = i_q \circ \theta_l^{(b)}$.

Direct inspection on $\theta_l^{(a)}, \theta_l^{(b)}$ shows the following.

FACT 8.6. *For all $\gamma, \gamma' \in D_{2l_s}$,*

$$\left\|\theta_{l_s,2}^{(a)}(\gamma\gamma') - \theta_{l_s,2}^{(a)}(\gamma)\right\|_2 = \left\|\theta_{l_s,2}^{(a)}(\gamma') - \theta_{l_s,2}^{(a)}\left(e_{D_{2l_s}}\right)\right\|_2.$$

The same equality holds with a replaced by b.

Now we introduce a weight function. For $s \in \mathbb{N}$, let $w_s : \mathbb{Z} \to [0,1]$ be the function defined as

$$w_s(y) = \begin{cases} \frac{1}{2} & \text{for } y \leq -\frac{1}{2}k_s \text{ or } y \geq \frac{3}{2}k_s, \\ \frac{|y|}{k_s} & \text{for } -\frac{1}{2}k_s < y < k_s, \\ 1 - \frac{y-k_s}{k_s} & \text{for } k_s \leq y < \frac{3}{2}k_s. \end{cases}$$

It is a piecewise linear function taking value 1/2 outside $[-k_s/2, 3k_s/2]$, 0 at 0 and 1 at k_s, and its slope is in $\left\{\pm\frac{1}{k}, 0\right\}$. For $x \in \mathbb{Z}$, write $\tau_x w_s$ for the translation of w_s by x,

$$\tau_x w_s(y) = w_s(y-x).$$

Define the map $\Phi_{s,p} : \Delta_s \to L_q = \left(\bigoplus_{y\in\mathbb{Z}}(L_q)_y\right)_q$ by setting for each $y \in \mathbb{Z}$,

(49)
$$\left[\Phi_{s,q}\left(f_s, z\right)\right](y) = k_s^{1-\frac{1}{q}}\left((\tau_z w_s)(y)\theta^{(a)}_{l_s,q}\left(f_s(y)\right) + (1-(\tau_z w_s)(y))\theta^{(b)}_{l_s,q}(f_s(y))\right).$$

In other words, at each coordinate y, the image of $f_s(y)$ is a linear combination of $\theta^{(a)}_{l_s,q}$ and $\theta^{(b)}_{l_s,q}$ with the weights depending on the relative position between y and the cursor z.

Recall the 1-cocycle constructed in Section 6.2.3. Fix a choice of an increasing function $\gamma : \mathbb{N} \to \mathbb{R}_+$ such that $\gamma(1) = 1$, $C(\gamma) = \sum_{n=1}^{\infty}\gamma(n)^{-2} < \infty$. Let $b_\gamma : \Delta \hookrightarrow L_2$ be the 1-cocycle defined by (34) with $p = 2$ using the basic test functions. Finally define an embedding $\Phi_{\gamma,q} : \Delta \to L_q$ by

$$\Phi_{\gamma,q}\left((f_s), z\right) = \left(\bigoplus_{s=0}^{\infty}\left(\frac{1}{\gamma(s)}\Phi_{s,q}\left(f_s, z\right)\right)\right)\bigoplus b_{\gamma,q}\left((f_s), z\right), \tag{50}$$

where $\bigoplus$ is direct sum in L_q.

We now check some basic properties of the map $\Phi_{\gamma,q}$.

Lemma 8.7. *Let $\gamma : \mathbb{N} \to \mathbb{R}_+$ be a function such that $\gamma(1) = 1$, $C(\gamma) = \sum_{n=1}^{\infty}\gamma(n)^{-2} < \infty$. Assume $q \geq 2$. The map $\Phi_{\gamma,q} : \Delta \to L^q$ defined in (50) is C-Lipschitz with C only depending on $C(\gamma)$.*

Proof. It suffices to check that for any $u = ((f_s), z) \in \Delta$ and $s \in \{\tau, \alpha, \beta\}$ a generator, the increment $\left\|\Phi_{\gamma,p}(us) - \Phi_{\gamma,p}(u)\right\|_q$ is bounded by C.

For the generator α, $((f_s), z)\,\alpha = ((f'_s), z)$, where $f'_s(y) = f'_s(y)$ for all $y \neq z$ and $f'_s(z) = f_s(z)a(s)$. Recall that by definition, the weight function w_s satisfies $\tau_z w_s(z) = 0$, and the map $\theta^{(b)}_{l_s,q}$ satisfies $\theta^{(b)}_{l_s,q}(\gamma a(s)) = \theta^{(b)}_{l_s,q}(\gamma)$ for all $\gamma \in D_{2l_s}$. Then by (49),

$$\Phi_{s,q}\left(\left(f_s, z\right)\alpha\right) = \Phi_{s,q}\left(\left(f_s, z\right)\right).$$

Therefore in the embedding (50),

$$\left\|\Phi_{\gamma,q}\left(u\alpha\right) - \Phi_{\gamma,q}(u)\right\|_q = \left\|b_\gamma\left(u\alpha\right) - b_\gamma(u)\right\|_2 \leq \sqrt{2C(\gamma)}.$$

The last inequality uses the fact that the $1-$cocycle b_γ is Lipschitz; see Section 6.2.3. Similarly, since $\tau_z w_s(z+k_s)=1$ and $\theta^{(a)}_{l_s,q}(\gamma b(s))=\theta^{(a)}_{l_s,q}(\gamma)$ for all $\gamma\in D_{2l_s}$, we have $\|\Phi_{\gamma,q}(u\beta)-\Phi_{\gamma,q}(u)\|_q=\|b_\gamma(u\beta)-b_\gamma(u)\|_2$ as well. For the generator τ,

$$\begin{aligned}&\Phi_{s,q}(u\tau)\\&=\left(k_s^{1-\frac{1}{q}}\left(w_s(y-z-1)\theta^{(a)}_{l_s,q}(f_s(y))+(1-w_s(y-z-1))\theta^{(b)}_{l_s,q}(f_s(y))\right)\right)_{y\in\mathbb{Z}}.\end{aligned}$$

Then

$$\begin{aligned}&\|\Phi_{s,q}(u\tau)-\Phi_{s,q}(u)\|_q^q\\&=\sum_{y\in\mathbb{Z}}k_s^{q-1}\left|w_s(y-z-1)-w_s(y-z)\right|^q\left\|\theta^{(a)}_{l_s,q}(f_s(y))-\theta^{(b)}_{l_s,q}(f_s(y))\right\|_q^q.\end{aligned}$$

Recall that $\left\|\theta^{(a)}_{l_s}(\gamma)-\theta^{(b)}_{l_s}(\gamma)\right\|_2\le 1$ for all $\gamma\in D_{2l_s}$, $w_s(y-z)\neq w_s(y-z-1)$ only if $y-z\in\left[-\frac{k_s}{2},\frac{3k_s}{2}\right]$, and in this interval $|w_s(y-z)-w_s(y-z-1)|=\frac{1}{k_s}$. Therefore

$$\|\Phi_{s,q}(u\tau)-\Phi_{s,q}(u)\|_q^q\le 2k_sk_s^{q-1}\left(\frac{1}{k_s}\right)^q=2.$$

Summing up in the embedding (50),

$$\begin{aligned}\|\Phi_{\gamma,q}(u\tau)-\Phi_{\gamma,q}(u)\|_q^q&=\sum_s\frac{1}{\gamma(s)^q}\|\Phi_s(u\tau)-\Phi_s(u)\|_q^q+\|b_\gamma(u\tau)-b_\gamma(u)\|_2^q\\&\le 2C(\gamma)+(2C(\gamma))^{\frac{q}{2}}.\end{aligned}$$ □

Because of the presence of the weight function w_s, the embedding $\Phi_{\gamma,q}:\Delta\to L_q$ fails to be equivariant. But the increment $\|\Phi_{\gamma,q}(uv)-\Phi_{\gamma,q}(u)\|_q$ is actually comparable to $\|\Phi_{\gamma,q}(v)\|_q$.

Lemma 8.8. *There exists a constant $c>0$ depending only on $C(\gamma)$ such that for $q\ge 2$,*

$$\|\Phi_\gamma(uv)-\Phi_\gamma(u)\|_q\ge c\,\|\Phi_\gamma(v)\|_q.$$

Proof. By formula (49), which defines $\Phi_{s,q}$, $\Phi_{s,q}(uv)$ is

$$\begin{aligned}&\left(k_s^{1-\frac{1}{q}}\left(w_s(y-z-z')\theta^{(a)}_{l_s,q}\left(f_s(y)f'_s(y-z)\right)\right.\right.\\&\qquad\left.\left.+(1-w_s(y-z-z'))\theta^{(b)}_{l_s,q}\left(f_s(y)f'_s(y-z)\right)\right)\right)_{y\in\mathbb{Z}}.\end{aligned}$$

Then by the triangle inequality and Fact 8.6,

$$\begin{aligned}\left\|\Phi_{s,q}(uv)-\Phi_{s,q}(u)\right\|_q &\geq \left\|\Phi_{s,q}(v)\right\|_q \\ &-\left(k_s^{q-1}\sum_{y\in\mathbb{Z}}\left|w_s(y-z)-w_s(y-z-z')\right|^q\left\|\theta^{(a)}_{l_s}(f_s(y))-\theta^{(b)}_{l_s}(f_s(y))\right\|_2^q\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left\|\theta^{(a)}_{l_s}(\gamma)-\theta^{(b)}_{l_s}(\gamma)\right\|_2\leq 1$ for all γ, and

$$\sum_{y\in\mathbb{Z}}\left|w_s(y-z)-w_s(y-z-z')\right|^q\leq\left(\frac{1}{k_s}\right)^q\min\left\{\left|z'\right|,2k_s\right\},$$

we have $\left\|\Phi_{s,q}(uv)-\Phi_{s,q}(u)\right\|_2\geq\left\|\Phi_{s,q}(v)\right\|_2-2^{\frac{1}{q}}$. Using $(a-b)_+^q\geq(a/2)^q-b^q$ for $a,b\geq 0$,

$$\begin{aligned}\left\|\Phi_{\gamma,q}(uv)-\Phi_{\gamma,q}(u)\right\|_q^q &\geq \sum_s\frac{1}{\gamma(s)^q}\left(\left\|\Phi_{s,q}(v)\right\|_q-2^{\frac{1}{q}}\right)_+^q+\left\|b_{\gamma,q}(v)\right\|_q^q \\ &\geq \frac{1}{4C(\gamma)}\left(\sum_s\frac{1}{\gamma(s)^q}\left(\frac{1}{2^q}\left\|\Phi_{s,q}(v)\right\|_q^q-2\right)\right)+\left\|b_{\gamma,q}(v)\right\|_q^q \\ &\geq \frac{1}{2^{2+q}C(\gamma)}\sum_s\frac{1}{\gamma(s)^q}\left\|\Phi_{s,q}(v)\right\|_q^q+\left\|b_{\gamma,q}(v)\right\|_q^q-\frac{1}{2} \\ &\geq \frac{1}{2^{2+q}C(\gamma)}\left\|\Phi_{\gamma,q}(v)\right\|_q^q.\end{aligned}$$

∎

8.4. *The compression exponent* $\alpha_p^*(\Delta)$. In this subsection we estimate the L_p-compression exponent of Δ. Recall that for $p\in[1,2]$, L_p has Markov type p and cotype 2; and for $p\in(2,\infty)$, L_p has Markov type 2 and cotype p; see [NPSS06] and references therein.

Proof of Theorem 8.1. *Upper bound in* (i). For the upper bound on $\alpha_p^*(\Delta)$, $p\in[1,2]$, when $\theta=0$, the bound is trivially true. Assume now that $\theta\in(0,\infty)$. For any $\varepsilon>0$ sufficiently small, take a subsequence $\{s_i\}_{i\in\mathbb{N}}$ such that

$$\frac{\log l_{s_i}}{\log k_{s_i}}>\theta-\varepsilon.$$

Recall that the cotype of L_p is 2 for $p\in[1,2]$. We apply Proposition 8.2, or the remark after it for L_1, along this subsequence of (k_{s_i},l_{s_i}). We obtain the required upper bound after sending $\varepsilon\to 0$. The argument also extends to $\theta=\infty$.

Upper bound in (ii). The upper bound on $\alpha_q^*(\Delta)=\alpha_q^{\#}(\Delta)$ is covered by Corollary 8.5.

Lower bound in (ii). In the case $\theta\leq\frac{1}{q}$, $q\geq 2$, simply take the 1-cocycle $b_\gamma:\Delta\to\ell^2$ defined in (34) using the basic test functions (32), with $\gamma(n)=n^{-\frac{1}{2}-\epsilon}$.

Then Lemma 6.9 implies that the compression exponent of Δ satisfies

$$\alpha_q^*(\Delta) \geq \alpha_2^*(\Delta) \geq \liminf_{s\to\infty} \frac{\log k_s}{\log (k_s l_s)} \geq \frac{1}{1+\theta}.$$

Now we focus on the case $\theta > \frac{1}{q}$. Consider the explicit embedding $\Phi_{\gamma,q} : \Delta \to L_q$ defined in (50) with γ taken to be $\gamma(n) = (1+n)^{-\frac{1+\epsilon}{2}}$. By Lemma 8.7, $\Phi_{\gamma,q}$ is Lipschitz. By Lemma 8.8, there exists a constant $c = c\,(C(\gamma))$ such that for any $u, v \in \Delta$,

$$\left\|\Phi_{\gamma,q}(uv) - \Phi_{\gamma,q}(u)\right\|_q \geq c\left\|\Phi_{\gamma,q}(v)\right\|_q.$$

Let $v = ((f_s), z)$ be an element of Δ. At each site y, from the definition of $\theta^{(a)}_{l_s,q}$, $\theta^{(b)}_{l_s,q}$,

$$\begin{aligned}
&\left\|(\tau_z w_s)(y)\theta^{(a)}_{l_s,q}(f_s(y)) + (1-(\tau_z w_s)(y))\theta^{(b)}_{l_s,q}(f_s(y))\right\|_q \\
&\quad\geq \frac{1}{\sin(\pi/l_s)} \sin\left(\frac{\pi}{2l_s}\left(|f_s(y)|_{D_{2l_s}} - 1\right)_+\right) \\
&\quad\geq \frac{1}{\pi}\left(|f_s(y)|_{D_{2l_s}} - 1\right)_+.
\end{aligned}$$

From the explicit formula (49), which defines $\Phi_{s,q}$,

$$\|\Phi_{s,q}(f_s,z)\|_q^q \geq \sum_{j\in\mathbb{Z}}\left(\frac{1}{\pi}k_s^{1-\frac{1}{q}}\max_{y\in I_j^s}\left(|f_s(y)|_{D_{2l_s}} - 1\right)_+\right)^q.$$

In what follows we write $R = |\mathrm{Range}(v)|$, and $\tilde{f}_s(y) = \left(|f_s(y)|_{D_{2l_s}} - 1\right)_+$. We have

$$\begin{aligned}
\|\Phi_{\gamma,q}(v)\|_q^q &\geq \sum_{s\leq s_0(v)} \frac{1}{\gamma(s)^q}\sum_{j\in\mathbb{Z}}\left(\frac{1}{\pi}k_s^{1-\frac{1}{q}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)^q + \|b_\gamma(v)\|_2^q \\
&\geq \sum_{s\leq s_0(v)} \frac{1}{\gamma(s)^q}\sum_{j\in\mathbb{Z}}\left(\frac{1}{\pi}k_s^{1-\frac{1}{q}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)^q + \left(\frac{cR}{\gamma\circ\log_2(R)}\right)^q.
\end{aligned}$$

The last step used Lemma 6.9 and the fact that since $k_{s+1} \geq 2k_s$ for all s, $s_0(v) \leq \log_2 R$. Note that in the factor Δ_s, the number of intervals I_j^s with $\max_{x\in I_j^s}\tilde{f}_s(y) \neq 0$ is bounded from above by $2R/k_s$. Therefore by the Hölder inequality,

$$\sum_{j\in\mathbb{Z}}\max_{y\in I_j^s}\tilde{f}_s(y)^q \geq \left(\frac{2R}{k_s}\right)^{1-q}\left(\sum_{j\in\mathbb{Z}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)^q = 2Rk_s^{-1}\ell(s)^q,$$

where

$$\ell(s) = \frac{k_s\sum_{j\in\mathbb{Z}}\max_{y\in I_j^s}\tilde{f}_s(y)}{2R}.$$

Consider the following three cases:

- If $0 \leq \ell(s) \leq R^{\frac{1}{q}}$, then

$$\sum_{j\in\mathbb{Z}}\left(\frac{1}{2}k_s^{1-\frac{1}{q}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)^q+(cR)^q \geq (cR)^q \geq (c/3)^q\left(R+2R\ell(s)\right)^{\frac{q^2}{q+1}}.$$

- If $R^{\frac{1}{q}} \leq \ell(s) \leq R^{1-\frac{1}{q}}k_s^{\frac{2}{q}-1}$, then it is necessary that $R \leq k_s$. From the metric description in Section 2.2.3, $R \leq k_s$ implies $\sum_{j\in\mathbb{Z}}\max_{y\in I_j^s}\tilde{f}_s(y) \leq 1$. Then in this case,

$$\sum_{j\in\mathbb{Z}}\left(\frac{1}{2}k_s^{1-\frac{1}{q}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)^q+(cR)^q \geq (cR)^q \geq (c/3)^q\left(R+2R\ell(s)\right)^q.$$

- If $\ell(s) > R^{1-\frac{1}{q}}k_s^{\frac{2}{q}-1}$, from the second item we only need to consider $R > k_s$. Recall that $\ell(s) \leq l_s \leq C_\varepsilon k_s^{\theta+\varepsilon}$. It follows that $R \leq \left(C_\varepsilon k_s^{\theta+\varepsilon+1-\frac{2}{q}}\right)^{\frac{q}{q-1}}$. Then in this case,

$$\sum_{j\in\mathbb{Z}}\left(\frac{1}{2}k_s^{1-\frac{1}{q}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)^q+(cR)^q \geq \frac{1}{2^{q-1}}k_s^{q-1}Rk_s^{-1}\ell(s)^q \geq c'_\varepsilon\left(R\ell(s)\right)^{q\gamma(\varepsilon)},$$

  where

$$\gamma(\varepsilon)=\frac{\theta+\varepsilon+1-\frac{2}{q}}{\left(2-\frac{1}{q}\right)(\theta+\varepsilon)+1-\frac{2}{q}}.$$

  In the last inequality, we used $\ell(s) > R^{1-\frac{1}{q}}k_s^{\frac{2}{q}-1}$ and $R \leq \left(C_\varepsilon k_s^{\theta+\varepsilon+1-\frac{2}{q}}\right)^{\frac{q}{q-1}}$.

Note that since $\theta > \frac{1}{q}$, when ε is sufficiently small, $\gamma(\varepsilon) < \frac{q}{q+1}$. Thus the worst case is represented by the third item, and we have

$$\|\Phi_{\gamma,q}(v)\|_q^q \geq \sum_{s\leq s_0(v)}\frac{c'_\varepsilon}{\gamma(s)^q}\left(R+R\ell(s)\right)^{q\gamma(\varepsilon)}.$$

Recall that by the metric estimate Proposition 2.14,

$$|v|_\Delta \leq 500\left(R+\sum_{s\leq s_0(v)}k_s\sum_{j\in\mathbb{Z}}\max_{y\in I_j^s}\tilde{f}_s(y)\right)=500\left(R+\sum_{s\leq s_0(v)}2R\ell(s)\right),$$

and $s_0(v) \leq \log_2 R$ by Assumption 2.11. Combining with Lemma 8.8, we have that for any $u \in \Delta$,

$$\begin{aligned}\|\Phi_{\gamma,q}(u) - \Phi_{\gamma,q}(uv)\|_2 &\geq \frac{c'}{\log_2^{\frac{1}{2}+\epsilon}(R)} \max_{s \leq s_0(v)} (R + R\ell(s))^{\gamma(\varepsilon)} \\ &\geq \frac{c'}{\log_2^{\frac{1}{2}+\epsilon}(R)} \left(\frac{|v|_\Delta}{1000 \log_2(R)}\right)^{\gamma(\varepsilon)},\end{aligned}$$

where $c' > 0$ is a constant depending on θ and ϵ. Sending $\varepsilon \to 0$, we conclude that when $\theta \in \left(\frac{1}{q}, \infty\right]$,

$$\alpha_q^*(\Delta) \geq \frac{\theta + 1 - \frac{2}{q}}{\left(2 - \frac{1}{q}\right)\theta + 1 - \frac{2}{q}}.$$

Note that the formula is simplified when $q = 2$, namely, $\alpha_2^*(\Delta) \geq \frac{2}{3}$. Combining with the fact that $\alpha_q^*(\Delta) \geq \alpha_2^*(\Delta)$, we obtain the statement.

Lower bound in (i). Since ℓ^2 embeds isometrically in all L_p, $p \geq 1$, it follows that

$$\alpha_p^*(\Delta) \geq \alpha_2^*(\Delta) \geq \max\left\{\frac{1}{1+\theta}, \frac{2}{3}\right\}.$$

This completes the proof of Theorem 8.1. □

Example 8.9 (Proof of Theorem 1.4). Considering the construction of Δ with $\Gamma_s = D_{2l_s}$, the parameters $\{k_s\}$, $\{l_s\}$ are chosen to be $k_s = 2^{\beta s}$, $l_s = 2^{\iota s}$ with $\beta > 1, \iota \geq 0$. Then $\theta = \iota/\beta$.

For $p \in [1, 2]$, Theorem 8.1 implies that

$$\alpha_p^*(\Delta) = \max\left\{\frac{1}{1+\theta}, \frac{2}{3}\right\},$$

which can take any value in $\left[\frac{2}{3}, 1\right]$.

For $q > 2$, $\theta > \frac{1}{q}$, the upper and lower bound in Theorem 8.1 do not match up. But in some region of parameters we can still compare it to the Hilbert compression exponent. For $\theta \in \left(\frac{1}{q}, 1\right)$, we have

$$\alpha_q^*(\Delta) \geq \frac{\theta + 1 - \frac{2}{q}}{\left(2 - \frac{1}{q}\right)\theta + 1 - \frac{2}{q}} > \alpha_2^*(\Delta) = \max\left\{\frac{1}{1+\theta}, \frac{2}{3}\right\}.$$

In particular, we can take $\theta = \frac{1}{2}$. Then the corresponding diagonal product Δ_1 satisfies

$$\alpha_q^*(\Delta_1) \geq \frac{3q-4}{4q-5} > \alpha_2^*(\Delta_1) = \frac{2}{3}.$$

9. **Discussion and some open problems**

The groups of Theorem 1.1 are diagonal products of lamplighter groups. In particular, they contain many torsion elements and admit many quotients.

Problem 9.1. Find solutions to the inverse problems for speed, entropy, return probability, isoperimetric profile or compression in the class of torsion-free groups or in the class of simple groups.

In Theorem 3.8, we imposed the regularity assumption on ϱ that $\varrho(n)/\sqrt{n}$ is non-decreasing. This is not always satisfied; it is possible to construct examples of groups where the speed function is roughly constant over certain long time intervals. The following question asks if this regularity assumption can be dropped.

Problem 9.2. Let $\varrho : [1,\infty) \to [1,\infty)$ be a non-decreasing subadditive function satisfying $\varrho(x) \geq \sqrt{x}$ for all x. Are there a group G and a symmetric probability measure μ of finite support on G such that

$$L_\mu(n) \simeq \varrho(n)?$$

Proposition 3.17 only partially answers the question of what joint behavior of speed and entropy can occur. Further, the question of possible joint behavior of speed, entropy and return probability, even restricting to group of exponential volume growth, is wide open; see [Ami17, Question 6]. Solving the following problem would be a step in this direction.

Problem 9.3. Find an open set $\mathcal{O}$ in $(0,1)^3$ such that for any point $(\alpha,\beta,\gamma) \in \mathcal{O}$, there exist a finitely generated group G and a symmetric probability measure of finite support on G, such that (α,β,γ) is the exponent of $\left(L_\mu(n), H_\mu(n), -\log \mu^{(2n)}(e)\right)$.

In [Gou16], Gournay showed that if a simple random walk on G satisfies that for some $C > 0$, $\gamma \in (0,1)$, $\phi^{(2n)}(e) \geq \exp(-Cn^\gamma)$, and an off-diagonal decay bound

$$\phi^{(2n)}(g) \leq C\phi^{(2n)}(e) \exp\left(-\frac{C|g|^2}{n}\right) \text{ for all } g \in G,\ n \in \mathbb{N}, \tag{51}$$

then we have the Hilbert compression exponent $\alpha_2^{\#}(G) \geq 1-\gamma$. The off-diagonal decay assumption (51) is difficult to check in general. We illustrate a family of examples where it is not valid. In Figure 1, take the diagonal product Δ with parameters $\left(k_s = 2^{2s}\right)$ and $\{\Gamma_s\}$ expanders with $\operatorname{diam}(\Gamma_s) \simeq 2^{2\theta}$. When $\theta > 1$, we have

$$-\log q^{(2n)}(e) \simeq n^{\frac{1+\theta}{3+\theta}},\ \alpha_2^*(\Delta) = \frac{1}{1+\theta} < 1 - \frac{1+\theta}{3+\theta}.$$

Therefore we deduce from [Gou16, Th. 1.4] that in this case the simple random walk on Δ fails the off-diagonal upper bound (51). On the other hand, we have the strict inequality

$$\alpha_2^*(\Delta) = \frac{1}{1+\theta} > \frac{1}{2+\theta} = \frac{1-\gamma}{1+\gamma},$$

showing the gap is far from the lower bound of [Gou16, Th. 1.1]. A better understanding of the relation between return probability and compression remains open.

Problem 9.4. Let G be a finitely generated infinite group such that for some $\gamma \in (0,1)$, the simple random walk satisfies $\phi^{(2n)}(e) \geq \exp(-Cn^{\gamma})$ for some $C > 0$. Find the sharp lower bound for $\alpha_2^{\#}(G)$ in terms of γ and explicit examples where the bound is sharp.

In Theorem 7.1 we give an explicit formula that relates equivariant L_p-compression exponents of $H \wr \mathbb{Z}$ and H when $p \in [1,2]$. Less is known about compression exponent of embeddings into L_p with $p > 2$. In particular, the following problem is open.

Problem 9.5. For $p > 2$, is there an explicit formula that connects equivariant compression exponents $\alpha_p^{\#}(H \wr \mathbb{Z})$ and $\alpha_p^{\#}(H)$?

The problem of determining $\alpha_p^{\#}(\Delta)$ for Δ constructed with dihedral groups as discussed in Section 8 is related to the previous problem.

Problem 9.6. Determine the equivariant compression exponent $\alpha_p^{\#}(\Delta)$, $p > 2$, where Δ is the diagonal product constructed with dihedral groups $\{D_{2l_s}\}$.

Appendix A. Some auxiliary facts about excursions

In this section we recall some classical facts about local time and excursions of standard simple random walk on $\mathbb{Z}$. Let $\{S_k\}$ denote the standard simple random walk on $\mathbb{Z}$, starting at $S_0 = 0$. Let $L(x,n)$ denote the local time of the random walk at site x,

$$L(x,n) = \#\{k:\ 0 < k \leq n,\ S_k = x\}.$$

The distribution of $L(x,n)$ is known explicitly ([Rao12, Th. 9.4]) for $x = 0, 1, 2, \ldots$:

$$P_0\left(L(x,n) = m\right) = \begin{cases} \frac{1}{2^{n-m+1}} \begin{pmatrix} n-m+1 \\ (n+x)/2 \end{pmatrix} & \text{if } n+x \text{ if even,} \\ \frac{1}{2^{n-m}} \begin{pmatrix} n-m \\ (n+x-1)/2 \end{pmatrix} & \text{if } n+x \text{ if odd.} \end{cases}$$

Let $\rho_0 = 0$ and $\rho_m = \min\{j > \rho_{m-1} : \ S_j = 0\}$. Then $\rho_1, \rho_2 - \rho_1, \ldots$ record the time duration of the excursions from 0, and they form a sequence of independent and identically distributed random variables with distribution

$$P_0(\rho_1 > 2n) = P_0(S_{2n} = 0) \sim \frac{1}{\sqrt{4\pi n}}.$$

The chance that an excursion from 0 crosses k is ([Rao12, Th. 9.6])

$$P_0\left(\max_{0\le i\le\rho_1} S_i \ge k\right) = \frac{1}{2k}, \ \ k = 1, 2, \ldots \ .$$

For $x \in \mathbb{Z}$, let $\rho_1(x) = \min\{j > 0 : \ S_j = x\}$ denote the first time the random walk visits x, and let $T(k, x, n)$ be the number of excursions away from x that cross $x-k$ and are completed before time n. We need estimates on the moments $E_0[T(k, x, n)^q]$, $0 < q \le 1$.

Lemma A.1. *There exists constant $C > 0$ such that for all $k, n \in \mathbb{N}, x \in \mathbb{Z}$,*

$$E_0[T(k, x, n)] \le \frac{C\sqrt{n}}{k} \exp\left(-\frac{x^2}{2n}\right).$$

Proof. Let $\rho_m(x) = \min\{j > \rho_{m-1}(x) : \ S_j = x\}$ be the m-th time the random walk visits x. Then

$$\begin{aligned} E_0[T(k, x, n)] &\le \sum_{j\ge0} E_0\left[T(k, x, \rho_{2^j+1}(x))\, \mathbf{1}_{\{\rho_{2^j}(x)\le n<\rho_{2^j+1}(x)\}}\right] \\ &\le \sum_{j\ge0} E_0\left[T(k, x, \rho_{2^j+1}(x))\, \mathbf{1}_{\{\rho_{2^j}(x)\le n\}}\right]. \end{aligned}$$

Conditioned on the event $\{\rho_{2^j}(x) \le n\}$, the random variable $T(k, x, \rho_{2^j+1}(x))$ is stochastically dominated by a binomial random variable with parameter 2^{j+1} and $\frac{1}{2k}$. Therefore

$$\begin{aligned} &\sum_{j\ge0} E_0\left[T(k, x, \rho_{2^j+1}(x))\, \mathbf{1}_{\{\rho_{2^j}(x)\le n\}}\right] \\ &\qquad \le \sum_{j\ge0} \frac{2^{j+1}}{2k} P_0(\rho_{2^j}(x) \le n) \\ &\qquad \le \sum_{j\ge0} \frac{2^{j+1}}{2k} P_0(\rho_1(x) \le n)\, P_0\left(L(n, 0) \ge 2^j\right) \\ &\qquad \le \frac{1}{2k} P_0(\rho_1(x) \le n) E_0(4L(n, 0)). \end{aligned}$$

Plugging in the estimates

$$P_0(\rho_1(x) \leq n) = P_0\left(\max_{0\leq t\leq n} S_t \geq |x|\right) \leq 2\exp\left(-\frac{x^2}{2n}\right),$$

$$E_0(L(n,0)) = \sum_{t=0}^{n} P_0(S_t = 0) \leq Cn^{\frac{1}{2}},$$

we obtain the statement. □

LEMMA A.2. *There exists a constant $c > 0$ such that for all $x \in \mathbb{Z}$, $k, n \in \mathbb{N}$ satisfying $k \leq c^2 n^{\frac{1}{2}}$,*

$$P_0\left(T(k,x,n) \geq \frac{c\sqrt{n}}{4k}\right) \geq \frac{1}{2} P_0\left(L(x,n/2) \geq 1\right).$$

Proof. Conditioned on the event $\{\rho_1(x) = t\}$, $0 \leq t < n$, the distribution of $T(k,x,n)$ is the same as $T(k,0,n-t)$. Therefore for any $m > 0$,

$$P_0\left(T(k,x,n) \geq m\right) \geq P_0\left(T(k,0,n/2) \geq m\right) P_0\left(L(x,n/2) \geq 1\right).$$

Now we show that there exists a constant $c > 0$ such that for $k \leq c^2 n^{\frac{1}{2}}$,

$$P_0\left(T(k,0,n/2) \geq \frac{c\sqrt{n}}{4k}\right) \geq \frac{1}{2}.$$

Note that

$$\{T(k,0,n/2) \geq m\} \supset \left\{\rho_{\left\lfloor cn^{\frac{1}{2}}\right\rfloor}(0) \leq \frac{n}{2}\right\} \cap \left\{T\left(k,0,\rho_{\left\lfloor cn^{\frac{1}{2}}\right\rfloor}\right) \geq m\right\}.$$

For ease of notation, in what follows write $l = \left\lfloor cn^{\frac{1}{2}}\right\rfloor$. Since $\rho_l(0)$ is sum of l independent and identically distributed random variables with distribution

$$P_0\left(\rho_1 > 2t\right) = P_0(S_{2t} = 0) \sim \frac{1}{\sqrt{4\pi t}},$$

by classical theory of sum of independent and identically distributed α-stable variables (here $\alpha = \frac{1}{2}$), there exists constant $C = C(\alpha)$ such that

$$P_0\left(\rho_l(0) \geq t\right) \leq C\frac{l}{t^{\frac{1}{2}}}.$$

Therefore

$$P_0\left(\rho_l(0) \geq \frac{n}{2}\right) \leq \sqrt{2}Cc.$$

For the other term, $T\left(k,0,\rho_l\right)$ is binomial with parameters l and $1/2k$, and therefore by Bernstein inequality (see, for example, [Rao12, Th. 2.3]),

$$P_0\left(T\left(k,0,\rho_l\right) \leq \frac{cn^{\frac{1}{2}}}{4k}\right) \leq 2\exp\left(-\frac{l}{4^3 k}\right).$$

Then

$$P_0\left(T(k,0,n/2) \geq \frac{c\sqrt{n}}{4k}\right) \geq 1-\sqrt{2}Cc - 2\exp\left(-\frac{cn^{\frac{1}{2}}}{4^3k}\right).$$

Choosing c sufficiently small so that $\sqrt{2}Cc \leq 1/4$, $\exp\left(-\frac{1}{4^3c}\right) \leq 1/8$, we obtain the statement. ☐

Appendix B. Approximation of functions

For $p_1, p_2 \geq 0$, consider the following space $\mathcal{C}_{p_1,p_2}$ of continuous functions between x^{p_1} and x^{p_2},

$$\mathcal{C}_{p_1,p_2} = \left\{ f:[1,\infty)\to[1,\infty) : \begin{array}{l} f \text{ is continuous, } f(1)=1 \\ \frac{f(x)}{x^{p_1}} \text{ is non-decreasing} \\ \frac{x^{p_2}}{f(x)} \text{ is non-increasing} \end{array} \right\}.$$

Equivalently, $\mathcal{C}_{p_1,p_2}$ is the set of functions with $f(1)=1$ satisfying

$$a^{p_1}f(x) \leq f(ax) \leq a^{p_2}f(x) \text{ for all } a,x \geq 1.$$

We aim to approximate functions in $\mathcal{C}_{p_1,p_2}$ up to multiplicative constants by piecewise extremal functions.

Given two unbounded non-decreasing sequences $(k_s),(l_s)$ of real numbers, possibly finite with last value infinity, define the function

$$\tilde{f}(x) = \tilde{f}_{(k_s),(l_s)}(x) = \begin{cases} l_s & \text{for } k_sl_s \leq x \leq k_{s+1}l_s, \\ \frac{x}{k_{s+1}} & \text{for } k_{s+1}l_s \leq x \leq k_{s+1}l_{s+1}. \end{cases} \tag{52}$$

Similarly, define

$$\bar{f}(x) = \bar{f}_{(k_s),(l_s)}(x) = l_s + \frac{x}{k_{s+1}} \text{ for } k_sl_s \leq x \leq k_{s+1}l_{s+1}. \tag{53}$$

LEMMA B.1. *For any f in $\mathcal{C}_{0,1}$ and for any $m_0 > 1$, there exist two sequences $(k_s),(l_s)$ of real numbers, possibly finite with last value infinity, such that $k_{s+1} \geq m_0k_s$ and $l_{s+1} \geq m_0l_s$ for all s. The functions defined above satisfy*

$$\tilde{f}(x) \simeq_{m_0} f(x) \text{ and } \bar{f}(x) \simeq_{2m_0} f(x).$$

Moreover, if for some $\alpha > \alpha_0 > 0$ the function $\frac{f(x)}{\log^\alpha(x)}$ is non-decreasing, it is possible to find such functions with sequences $(k_s),(l_s)$ satisfying $\log k_s \leq l_s^{\frac{1}{\alpha_0}}$ for all s.

Proof. The proof is best understood looking at Figure 2. Observe that by construction, $\tilde{f}(x)$ is continuous and non-decreasing, which is not necessarily true for $\bar{f}(x)$. By induction, assume k_s, l_s already known with $\tilde{f}(k_sl_s) = l_s = f(k_sl_s)$. The hypothesis on f gives that $l_s \leq f(x) \leq \frac{x}{k_s}$ for all $x \geq k_sl_s$.

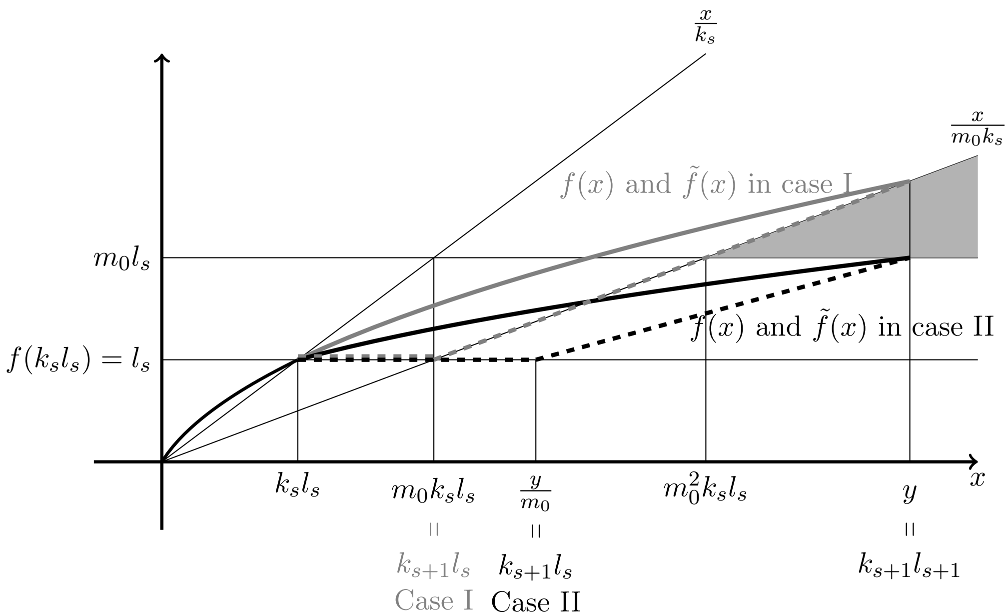

Figure 2. Approximation of functions in $\mathcal{C}_{0,1}$ by functions piecewise constant and linear, as in the proof of Lemma B.1.

We consider the minimal $y \geq m_0^2 k_s l_s$ such that $m_0 l_s \leq f(y) \leq \frac{y}{m_0 k_s}$. Assume first that such a y exists. By continuity of f, there are two cases.

Case I. If $f(y) = \frac{y}{m_0 k_s}$, set $k_{s+1} = m_0 k_s$ and $l_{s+1} = \frac{y}{m_0 k_s} \geq m_0 l_s$. Then for all $k_s l_s \leq x \leq k_{s+1} l_{s+1} = y$,

$$\frac{x}{m_0 k_s} \leq \tilde{f}(x) \leq \frac{x}{k_s},$$

and the same inequalities hold for f by sublinearity.

Case II. If $f(y) = m_0 l_s$, set $l_{s+1} = m_0 l_s$ and $k_{s+1} = \frac{y}{m_0 l_s} \geq m_0 k_s$. Then for all $k_s l_s \leq x \leq k_{s+1} l_{s+1} = y$,

$$l_s \leq \tilde{f}(x) \leq m_0 l_s,$$

and the same inequalities hold for f (non-decreasing).

If such a y does not exist, there are again two cases:

- either $f(x) \geq \frac{x}{m_0 k_s}$ for all $x \geq k_s l_s$ — then set $k_{s+1} = m_0 k_s$ and $l_{s+1} = \infty$ so $\tilde{f}(x) = \frac{x}{m_0 k_s}$ for all $x \geq k_s l_s$, generalizing Case I;
- or $f(x) \leq m_0 l_s$ for all $x \geq k_s l_s$ — then set $k_{s+1} = \infty$ so $\tilde{f}(x) = l_s$ for all $x \geq k_s l_s$, generalizing Case II.

Concerning the function $\bar{f}$, a routine inspection (of the intervals $[k_s l_s, k_{s+1} l_s]$ and $[k_{s+1} l_s, k_{s+1} l_{s+1}]$ in both cases) shows that $\tilde{f}(x) \leq \bar{f}(x) \leq 2\tilde{f}(x)$ for any x.

To check the last statement, it is sufficient to check that

$$f(m_0 l_s \exp((m_0 l_s)^{\frac{1}{\alpha_0}})) \geq m_0 l_s.$$

Thus in Case II we take $l_{s+1} = m_0 l_s$ and get $y \leq l_{s+1} \exp(l_{s+1}^{\frac{1}{\alpha_0}})$ and $k_{s+1} = \frac{y}{l_{s+1}} \leq \exp(l_{s+1}^{\frac{1}{\alpha_0}})$. Our assumptions give

$$\begin{aligned} f(l_{s+1} \exp(l_{s+1}^{\frac{1}{\alpha_0}})) &\geq f(k_s l_s) \left(\frac{\log(l_{s+1} \exp(l_{s+1}^{\frac{1}{\alpha_0}}))}{\log(k_s l_s)} \right)^{\alpha} \\ &\geq l_s \left(\frac{(m_0 l_s)^{\frac{1}{\alpha_0}}}{\log k_s + \log l_s} \right)^{\alpha} \\ &\geq l_s m_0^{\frac{\alpha}{\alpha_0}} \left(\frac{l_s^{\frac{1}{\alpha_0}}}{l_s^{\frac{1}{\alpha_0}} + \log l_s} \right)^{\alpha}, \end{aligned}$$

and the last parenthesis tends to 1 as l_s tends to infinity. □

Proposition B.2. *Let $C_1 > 0$ and $K, L \subset [1, \infty]$ such that for any x in $[1, \infty]$, there exist $k \in K$ and $l \in L$ with $k \simeq_{C_1} x$ and $l \simeq_{C_1} x$. For any f in $\mathcal{C}_{0,1}$ and for any $m_0 > 1$, there exist two sequences $(k_s), (l_s)$ taking values in K and L respectively such that $k_{s+1} \geq m_0 k_s$ and $l_{s+1} \geq m_0 l_s$ for all s. The functions defined in (52) and (53) satisfy*

$$\tilde{f}(x) \simeq_{m_0 C_1^5} f(x) \text{ and } \bar{f}(x) \simeq_{2 m_0 C_1^5} f(x).$$

Moreover, if for some $\alpha > \alpha_0 > 0$ the function $\frac{f(x)}{\log^{\alpha}(x)}$ is non-decreasing, it is possible to find such functions with sequences $(k_s), (l_s)$ satisfying $\log k_s \leq l_s^{\frac{1}{\alpha_0}}$ for all s.

Proof. We apply Lemma B.1 with $m_0' > C_1^2 m_0$, and obtain two sequences $(k_s), (l_s)$ of real numbers satisfying $k_{s+1} \geq m_0 C_1^2 k_s$ and $l_{s+1} \geq m_0 C_1^2 l_s$. The hypothesis on K, L permits us to find two sequences $(k_s'), (l_s')$ with $k_s' \simeq_{C_1} k_s$ and $l_s' \simeq_{C_1} l_s$. The choice of m_0' guarantees that $k_{s+1}' \geq m_0 k_s'$ and $l_{s+1}' \geq m_0 l_s'$.

Denote by $\tilde{f}'$ and $\bar{f}'$ the functions defined by (52) and (53) with the sequences $(k_s'), (l_s')$. It is sufficient to check that $\tilde{f}'(x) \simeq_{C_1^3} \tilde{f}(x)$.

When $k_s' l_s' \leq x \leq k_{s+1}' l_s'$, then $\tilde{f}'(x) = l_s' \simeq_{C_1} l_s$. On the other hand, $\frac{k_s l_s}{C_1^2} \leq x \leq C_1^2 k_{s+1} l_s$ so

$$\frac{l_s}{C_1^2} = \frac{\tilde{f}(k_s l_s)}{C_1^2} \leq \tilde{f}(\frac{k_s l_s}{C_1^2}) \leq \tilde{f}(x) \leq \tilde{f}(C_1^2 k_{s+1} l_s) \leq C_1^2 \tilde{f}(k_{s+1} l_s) = C_1^2 l_s.$$

Thus $\tilde{f}'(x) \simeq_{C_1^3} \tilde{f}(x)$.

When $k'_{s+1}l'_s \leq x \leq k'_{s+1}l'_{s+1}$, set $x = \lambda k'_{s+1}l'_s + (1-\lambda)k'_{s+1}l'_{s+1}$. Then

$$\tilde{f}'(x) = \lambda l'_s + (1-\lambda)l'_{s+1} \simeq_{C_1} \lambda l_s + (1-\lambda)l_{s+1}.$$

On the other hand, $x \simeq_{C_1^2} \lambda k_{s+1}l_s + (1-\lambda)k_{s+1}l_{s+1}$ so $\tilde{f}(x) \simeq_{C_1^2} \tilde{f}(\lambda k_{s+1}l_s + (1-\lambda)k_{s+1}l_{s+1}) = \lambda l_s + (1-\lambda)l_{s+1}$. Thus $\tilde{f}'(x) \simeq_{C_1^3} \tilde{f}(x)$. $\square$

Corollary B.3. *Let $C_1 > 0$ and $K, L \subset [1,\infty]$ such that for any x in $[1,\infty]$, there exist $k \in K$ and $l \in L$ with $k \simeq_{C_1} x$ and $l \simeq_{C_1} x$. For any ϱ in $\mathcal{C}_{p_1,p_2}$ and for any $m_0 > 1$, there exist two sequences $(k_s), (l_s)$ taking values in K and L respectively such that $k_{s+1} \geq m_0 k_s$ and $l_{s+1} \geq m_0 l_s$ for all s. The function defined by*

$$\bar{\varrho}(x) = x^{p_1} l_s + \frac{x^{p_2}}{k_{s+1}} \text{ for } (k_s l_s)^{\frac{1}{p_2-p_1}} \leq x \leq (k_{s+1}l_{s+1})^{\frac{1}{p_2-p_1}}$$

satisfies

$$\bar{\varrho}(x) \simeq_{2m_0C_1^5} \varrho(x).$$

Moreover, if for any $\alpha > \alpha_0 > 0$ the function $\frac{\varrho(x)}{x^{p_1}\log^{\alpha}(x)}$ is non-decreasing, it is possible to find such functions with sequences $(k_s), (l_s)$ satisfying $\log k_s \leq l_s^{\frac{1}{\alpha_0}}$ for all s.

Proof. The application $T_{p_1,p_2} : \mathcal{C}_{0,1} \to \mathcal{C}_{p_1,p_2}$ given by

$$T_{p_1,p_2} f(x) = x^{p_1} f(x^{p_2-p_1})$$

is a bijection. Take $f = T^{-1}_{p_1,p_2}\varrho$, apply Proposition B.2, and set $\bar{\varrho} = T_{p_1,p_2}\bar{f}$. $\square$

Appendix C. Symmetric α-stable like walk on lamplighter over a segment

For $m > 1$, let I_m be a subgraph of one dimensional lattice $\mathbb{Z}$ with vertex set $\{0, 1, \ldots, m-1\}$. Consider the lamplighter graph $\mathcal{L}_m$ over the segment I_m. Formally, $\{(f, x) : f : I_m \to \{0,1\}, x \in I_m\}$ is the vertex set of $\mathcal{L}_m$, and an edge connects (f, x) and (f', x') if $f \equiv f'$ and $x \sim x'$ in I_m, or $x = x'$, $f \neq f'$ and f differs from f' only at site x. Random walks on lamplighter graphs have been studied in [Ers03], [PR04]; see also references therein.

Random walk on $\mathbb{Z}$ driven by step distribution

$$\nu_\alpha(x) \simeq \frac{1}{(1+|x|)^{1+\alpha}}, \ \alpha \in (0,2)$$

is often referred to as a symmetric α-stable like walk on $\mathbb{Z}$. By abuse of terminology, we call it an α-stable walk. General theory regarding heat kernel estimates of α-stable walks on volume doubling graphs is available; see [CK08] and references therein. The connection between α-stable walks and the Markov type method for a bounding compression exponent was first introduced in [NP11], where the p-stable walk on $\mathbb{Z}$ is used to determine the L_p-compression exponent of $\mathbb{Z} \wr \mathbb{Z}$, $p > 1$. For our purpose, we focus on the case of the stable walk of the index $\alpha = 1$ on the base graph.

On I_m, define transition kernel

$$\zeta_m(x, x') = \frac{c_{x,x'}}{\sum_{x' \in I_m} c_{x,x'}}, \ x, x' \in I_m,$$

where

$$c_{x,x'} = \frac{1}{1 + |x - x'|^2}.$$

Then ζ_m is a Markov transition kernel on I_m with stationary distribution $\mathcal{C}_m(x) = \frac{\sum_{x' \in I_m} c_{x,x'}}{\sum_{x,x' \in I_m} c_{x,x'}}$. One readily checks that $\frac{1}{5m} \leq \mathcal{C}_m(x) \leq \frac{5}{m}$ for all $x \in I_m$.

Now consider a random walk on the lamplighter graph $\mathcal{L}_m$ with transition ζ_m on the base. Let $\mathfrak{p}_m$ be the transition kernel in $\mathcal{L}_m$ such that for $x \neq x'$, $\mathfrak{p}_m((f, x), (f', x')) = \frac{1}{4}\zeta_m(x, x')$ if $f(z) = f'(z)$ for all $z \notin \{x, x'\}$; for $x = x'$, $\mathfrak{p}_m((f, x), (f', x')) = \frac{1}{2}\zeta_m(x, x)$ if $f(z) = f'(z)$ for all $z \neq x$. In other words, in each step the walker first randomizes the lamp configuration at the current location, then moves according to the transition kernel ζ_m and randomizes the lamp at the arrival location. This Markov chain is reversible with stationary distribution

$$U_m(f, x) = 2^{-m}\mathcal{C}_m(x).$$

From the upper bound on relaxation time in [PR04, Th. 1.2], we have

$$T_{rel}(\mathcal{L}_m, \mathfrak{p}_m) = \frac{1}{\lambda_2(\mathcal{L}_m, \mathfrak{p}_m)} \leq \max_{x,y \in I_m} \mathbf{E}_x \tau_y,$$

where $\tau_y = \min\{t : \ X_t = y\}$, X_t denotes the Markov chain on $\mathcal{L}_m$ with transition kernel $\mathfrak{p}_m$. Note that although in the statement of [PR04, Th. 1.2] it is assumed that the Markov chain on the base is a lazy simple random walk on a transitive graph, the coupling argument that proves the relaxation time upper bound is completely general, and it applies to any reversible Markov chain on the base graph. The quantity $\max_{x,y \in I_m} \mathbf{E}_x \tau_y$ is known as the maximal hitting time of the chain $\mathfrak{p}_m$. By [AF02, Lemma 4.1],

$$\max_{x,y \in I_m} \mathbf{E}_x \tau_y = \frac{1}{2}\left(\sum_{z,z' \in I_m} c_{z,z'}\right) \max_{x,y} R_{x,x'},$$

where $R_{x,x'}$ denotes the effective resistance between vertices x and x' in the electric network on I_m with edge conductances $c_{z,z'}$ between the pair of vertices z, z'. Estimates of effective resistance of α-stable walks follow from classical methods. For the particular transition kernel ζ_m with $\alpha = 1$, there exists a constant $C > 0$

$$R_{x,x'} \leq C \log |x - x'| ;$$

see, for example, [CFG09, App. B.2]. We conclude that for the Markov chain with transition kernel $\mathfrak{p}_m$ on $\mathcal{L}_m$,

$$\lambda_2(\mathcal{L}_m, \mathfrak{p}_m) \geq \frac{c}{m \log m}.$$

Equivalently, we have the following Poincaré inequality: for any function $f : \mathcal{L}_m \to \mathbb{R}$,

$$\sum_{u,v\in\mathcal{L}_m} (f(u) - f(v))^2 \, U_m(u)U_m(v) \leq \frac{2m\log m}{c} \sum_{u,v\in\mathcal{L}_m} (f(u) - f(v))^2 \, \mathfrak{p}_m(u,v)U_m(u).$$

By Matoušek's extrapolation lemma for Poincaré inequalities (see [Mat97] and [NS11, Lemma 4.4]), we deduce the following l_p-Poincaré inequalities.

LEMMA C.1. *In the setting introduced above, there exists an absolute constant $C > 0$ such that for any $f : \mathcal{L}_m \to \ell^p$,*

- *if $1 \leq p \leq 2$, then*

$$\sum_{u,v\in\mathcal{L}_m} \|f(u) - f(v)\|_p^p \, U_m(u)U_m(v) \leq Cm\log m \sum_{u,v\in\mathcal{L}_m} \|f(u) - f(v)\|_p^p \, U_m(u)\mathfrak{p}_m(u,v)\,;$$

- *if $p > 2$, then*

$$\sum_{u,v\in\mathcal{L}_m} \|f(u) - f(v)\|_p^p \, U_m(u)U_m(v) \leq (Cm\log m)^{\frac{p}{2}} (2p)^p \sum_{u,v\in\mathcal{L}_m} \|f(u) - f(v)\|_p^p \, U_m(u)\mathfrak{p}_m(u,v).$$

Now we introduce a distance function on $\mathcal{L}_m$. Let $\mathbf{w} = (w_1, w_2) \in \mathbb{R}_2^+$. In the lamplighter graph $\mathcal{L}_m$, let $w(e) = w_1$ if the edge e connects (f, x) and (f, x'), where $x \sim x'$ (edges of first type); let $w(e) = w_2$ if the edge e connects (f, x) and (f', x), where f, f' only differs at site x (edges of second type). Define $d_{\mathbf{w}}$ to be distance on $\mathcal{L}_m$

$$d_{\mathbf{w}}(u,v) = \min\left\{\sum_{e\in P} w(e) : \ P \text{ is a path in } \mathcal{L}_m \text{ connecting } u, v\right\}.$$

From definition of $\mathfrak{p}_m$, it is straightforward to check that the following.

LEMMA C.2. *There exists a constant $C>0$ such that for all $p>1$,*

$$\sum_{u,v\in\mathcal{L}_m} d_{\mathbf{w}}(u,v)^p U_m(u)\mathfrak{p}_m(u,v)\le C\left(w_2^p+\frac{1}{p-1}w_1^p m^{p-1}\right),$$

$$\sum_{d_{\mathbf{w}}(u,v)\ge\frac14(w_2+w_1)m} d_{\mathbf{w}}(u,v)^p U_m(u)U_m(v)\ge\frac{1}{C}(w_2+w_1)m.$$

These ingredients allow us to carry out the Poincaré inequality method to the upper bound L_p-compression function of $H\wr\mathbb{Z}$. It can also be used in the study of the diagonal product Δ with dihedral groups. Alternatively, we may apply the Markov type method. To this end, the following speed lower estimate is needed.

LEMMA C.3. *Let X_t be a stationary Markov chain on $\mathcal{L}_m$ with transition kernel $\mathfrak{p}_m$ reversible with stationary measure U_m. Then there exists $c>0$ such that*

$$\mathbf{E}_{U_m}\left[d_{\mathbf{w}}\left(X_t,X_0\right)\right]\ge c\left(w_1+w_2\right)\frac{t}{\log_* t}\text{ for all }1\le t\le m\log m.$$

Proof. Let $S_{[0,t]}=\{S_n,0\le n\le t\}$ denote the sites visited by the induced random walk $\{S_t\}$ on I_m. Since in each step the chain randomizes the lamp at the current and new locations, and any path in the graph $\mathcal{L}_m$ that connects X_0 to X_t must visit all the sites where the lamp configurations of X_0 and X_t differ, we have that for any $u\in\mathcal{L}_m$,

$$\mathbf{E}_u\left[d_{\mathbf{w}}\left(X_t,X_0\right)\right]\ge\frac12(w_1+w_2)\mathbf{E}_u\left[\left|S_{[0,t]}\right|\right].$$

Thus the question is reduced to the range of the ζ_m-random walk on the base I_m. Methods to estimate the expected size of range of the random walk go back to Dvoretzky and Erdos [DE51]. Here we include a straightforward adaptation of the argument in [NP08, Lemma 6.3] for completeness.

In what follows $\mathbb{E}$ means taking expectation with the law of random walk S_n on I_m with step distribution ζ_m. For any $k\in\{1,\ldots,m\}$, denote by $V_1,\ldots,V_k$ the first k elements of I_m that are visited by the random walk S_n. Let

$$Y_k(t)=\left|\{0\le n\le t:S_n\in\{V_1,\ldots V_k\}\}\right|.$$

Note that $\{Y_k(t)<t+1\}=\left\{\left|S_{[0,t]}\right|>k\right\}$. For any starting point $z\in I_m$,

$$\mathbb{E}_z\left[Y_k(t)\right]=\sum_{l=1}^k\mathbb{E}_z\left[\left|\{0\le n\le t:S_n=V_l\}\right|\right]\le k\sum_{n=0}^t\max_{x\in I_m}\mathbb{P}_x\left(S_n=x\right).$$

Therefore

$$\begin{aligned}\mathbb{E}_z\left(\left|S_{[0,t]}\right|\right) &\geq k\mathbb{P}_z\left(\left|S_{[0,t]}\right| > k\right) \geq k\left(1-\frac{\mathbb{E}_z\left[Y_k(t)\right]}{t+1}\right)\\ &\geq k\left(1-\frac{k\sum_{n=0}^{t}\max_{x\in I_m}\mathbb{P}_x\left(S_n=x\right)}{t+1}\right).\end{aligned}\tag{54}$$

The argument in [CK08, Th. 3.1] implies that the chain (I_m,ζ_m) satisfies a Nash inequality that there exists an absolute constant $C>0$,

$$\theta\left(\|u\|_2^2\right)\leq C\mathcal{E}_{\zeta_m}\left(u,u\right)\quad\text{for all } u: I_m\to\mathbb{R},\ \text{where } \theta(r)=r^2.$$

This Nash inequality implies on-diagonal decay upper bound (see [DSC96])

$$\mathbb{P}_x\left(S_l=x\right)\leq\frac{c_2}{l}\text{ for all } l\leq m,\ x\in I_m.$$

For $l>m$, by monotonicity, $\mathbf{P}_x\left(S_l=x\right)\leq\mathbf{P}_x\left(S_m=x\right)\leq c_2 l^{-1}$. It follows that for $k\in\{1,\dots,m\}$,

$$\sum_{l=0}^{k\log k}\max_{x\in I_m}\mathbb{P}_x\left[S_l=x\right]\leq c_3\log k.$$

Together these estimates imply that

$$\mathbf{E}_u\left[d_{\mathbf{w}}\left(X_{k\log_* k},X_0\right)\right]\geq c\left(w_1+w_2\right)k\text{ for any } k\in\{1,\dots,m\}, u\in\mathcal{L}_m.\quad\square$$

The same method can be used to estimate the speed of random walks on lamplighter graphs over other choices of the base graph.

LEMMA C.4. *Let $\Gamma=\mathbb{Z}_2\wr D_\infty^d$, $d\geq 3$ as in the second item of Example 2.4, marked with generating subgroups $A=\mathbb{Z}_2\wr\langle a_j,1\leq j\leq d\rangle$, $B=\mathbb{Z}_2\wr\langle b_j,1\leq j\leq d\rangle$. Fix an increasing sequence $n_s\in\mathbb{N}$, and let $\Gamma_s=\mathbb{Z}_2\wr D_{2n_s}^d$ be a finite quotient of Γ. Let $A(s)$, $B(s)$ denote the projection of A and B to Γ_s. There exists a constant $\sigma_d>0$ only depending on d such that $\{\Gamma_s\}$ satisfies the $\left(\sigma_d,(2n_s)^d\right)$-linear speed assumption.*

Proof. Let $\bar{A}(s)=\langle a_j,1\leq j\leq d\rangle$, $\bar{B}(s)=\langle b_j,1\leq j\leq d\rangle$. Consider a random alternating word $X_t^{(s)}$ in $A(s)$ and $B(s)$ of length t, and let $\bar{X}_t^{(s)}$ be its projection to $D_{2n_s}^d$. In other words, if the last letter in $X_t^{(s)}$ is a random element in $A(s)$, then to get to $X_{t+1}^{(s)}$, the lamp configuration in the neighborhood $\bar{X}_t^{(s)}\bar{B}(s)$ is randomized, and the walker on the base $D_{2n_s}^d$ is multiplied by a random element in $\bar{B}(s)$. Similarly if $X_t^{(s)}$ ends with $B(s)$, then the next move is uniform in $A(s)$. From this description, we have that the lamps over the sites visited by $\bar{X}_t^{(s)}$ are randomized,

$$\left|X_{2t}^{(s)}\right|_{\Gamma_s}\geq\frac{1}{8}\left|\mathcal{R}_{[[0,2t]]}^{(s)}\right|,$$

where $\mathcal{R}^{(s)}_{[[0,2t]]} = \left\{x \in D^d_{2n_s} : \bar{X}^{(s)}_{2l} = x \text{ for some } 0 \le l \le t\right\}$. Comparing $\left\{\bar{X}^{(s)}_{2l}\right\}$ to the standard simple random walk on $D^d_{2n_s}$, we have that there exists a constant $C_d > 0$ such that

$$\mathbf{P}\left(\bar{X}^{(s)}_{2l} = e\right) \le C_d (2l)^{-\frac{d}{2}} \text{ for all } 1 \le l \le 2n_s^2.$$

It follows that

$$\begin{aligned}\sum_{l=0}^{(2n_s)^d} \mathbf{P}\left(\bar{X}^{(s)}_{2l} = e\right) &\le 1 + \sum_{l=1}^{2n_s^2} C_d (2l)^{-\frac{d}{2}} + (2n_s)^d \left(4n_s^2\right)^{-d/2} \\ &\le 2C_d + 2.\end{aligned}$$

To estimate $\mathbf{E}\left|\mathcal{R}^{(s)}_{[[0,2t]]}\right|$, we apply the argument as in Lemma C.3. For $t \le (2n_s)^d$, in the inequality (54) choose $k = \frac{t}{4C_d+4}$. We conclude that there exists a constant $\sigma_d > 0$,

$$\mathbf{E}\left|X^{(s)}_{2t}\right|_{\Gamma_s} \ge \sigma_d t. \qquad \square$$

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