The set of vertices with positive curvature in a planar graph with nonnegative curvature

Bobo Hua [a,b], Yanhui Su [c,*]

[a] *School of Mathematical Sciences, LMNS, Fudan University, Shanghai 200433, China*
[b] *Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China*
[c] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350116, China*

A R T I C L E   I N F O

A B S T R A C T

In this paper, we give the sharp upper bound for the number of vertices with positive curvature in a planar graph with nonnegative combinatorial curvature. Based on this, we show that the automorphism group of a planar—possibly infinite—graph with nonnegative combinatorial curvature and positive total curvature is a finite group, and give an upper bound estimate for the order of the group.

Contents

---

* Corresponding author.
   E-mail addresses: bobohua@fudan.edu.cn (B. Hua), suyh@fzu.edu.cn (Y. Su).

1. Introduction

The combinatorial curvature for a planar graph, embedded in the sphere or the plane, was introduced by [17,26,35,42]: Given a planar graph, one may canonically endow the ambient space with a piecewise flat metric, i.e. replacing faces by regular polygons and gluing them together along common edges. The combinatorial curvature of a planar graph is defined via the generalized Gaussian curvature of the metric surface. Many interesting geometric and combinatorial results have been obtained since then, see e.g. [9,10,12–14,16,19,20,23,27,28,30,31,37,39,43,47–49].

Let (V, E) be a (possibly infinite) locally finite, undirected simple graph with the set of vertices V and the set of edges E. It is called planar if it is topologically embedded into the sphere or the plane. We write $G = (V, E, F)$ for the combinatorial structure, or the cell complex, induced by the embedding where F is the set of faces, i.e. connected components of the complement of the embedding image of the graph (V, E) in the target. We say that a planar graph G is a *planar tessellation* if the following hold, see e.g. [28]:

 (i) Every face is homeomorphic to a disk whose boundary consists of finitely many edges of the graph.
 (ii) Every edge is contained in exactly two different faces.
(iii) For any two faces whose closures have non-empty intersection, the intersection is either a vertex or an edge.

In this paper, we only consider planar tessellations and call them planar graphs for the sake of simplicity. For a planar tessellation, we always assume that for any vertex x and face σ,

$$\deg(x) \geq 3, \ \deg(\sigma) \geq 3$$

where $\deg(\cdot)$ denotes the degree of a vertex or a face. For any planar graph $G = (V, E, F)$, we write

$$D_G := \sup_{\sigma \in F} \deg(\sigma). \tag{1}$$

For a planar graph G, the *combinatorial curvature*, the *curvature* for short, at the vertex is defined as

$$\Phi(x) = 1 - \frac{\deg(x)}{2} + \sum_{\sigma \in F : x \in \overline{\sigma}} \frac{1}{\deg(\sigma)}, \quad x \in V, \tag{2}$$

where the summation is taken over all faces σ whose closure $\bar{\sigma}$ contains x. To digest the definition, we endow the ambient space, $\mathbb{S}^2$ or $\mathbb{R}^2$, with a canonical piecewise flat metric structure and call it the (regular) *polyhedral surface*, denoted by $S(G)$: The length of each edge is set to one, each face is set to being isometric to a Euclidean regular polygon of side length one with same facial degree, and the metric is induced by gluing faces along their common edges, see [4] for the definition of gluing metrics. It is well-known that the generalized Gaussian curvature on a polyhedral surface, as a measure, concentrates on the vertices. And one easily sees that the combinatorial curvature at a vertex is in fact the mass of the generalized Gaussian curvature at that vertex up to the normalization 2π, see e.g. [2,21].

In this paper, we study planar graphs with nonnegative combinatorial curvature. We denote by

$$\mathcal{PC}_{>0} := \{G = (V, E, F) : \Phi(x) > 0, \forall x \in V\}$$

the class of planar graphs with positive curvature everywhere, and by

$$\mathcal{PC}_{\geq 0} := \{G = (V, E, F) : \Phi(x) \geq 0, \forall x \in V\}$$

the class of planar graphs with nonnegative curvature. For any finite planar graph $G \in \mathcal{PC}_{\geq 0}$, by Alexandrov's embedding theorem, see e.g. [2], its polyhedral surface $S(G)$ can be isometrically embedded into $\mathbb{R}^3$ as a boundary of a convex polyhedron. This yields many examples for the class $\mathcal{PC}_{>0}$, e.g. the 1-skeletons of 5 Platonic solids, 13 Archimedean solids, and 92 Johnson solids. Besides these, the class $\mathcal{PC}_{>0}$ contains many other examples [36,39], since in general a face of G, which is a regular polygon in $S(G)$, may split into several pieces of non-coplanar faces in the embedded image of $S(G)$ in $\mathbb{R}^3$.

We review some known results on the class $\mathcal{PC}_{>0}$. Stone [42] first obtained a Myers type theorem: A planar graph with the curvature bounded below uniformly by a positive constant is a finite graph. Higuchi [19] conjectured that it is finite even if the curvature is positive everywhere, which was proved by DeVos and Mohar [14], see [43] for the case of cubic graphs. There are two special families of graphs in $\mathcal{PC}_{>0}$ called prisms and anti-prisms, both consisting of infinite many examples, see e.g. [14]. DeVos and Mohar [14] proved that there are only finitely many other graphs in $\mathcal{PC}_{>0}$ and proposed the following problem to find the largest graph among them.

Problem 1.1 *([14])*. What is the number

$$C_{\mathbb{S}^2} := \max_{G=(V,E,F)} \sharp V,$$

where the maximum is taken over graphs in $\mathcal{PC}_{>0}$, which are not prisms or antiprisms, and $\sharp V$ denotes the cardinality of V?

On one hand, the main technique to obtain the upper bound of $C_{\mathbb{S}^2}$ is the so-called discharging method, which was used in the proof of the Four Colour Theorem, see [1,40]. DeVos and Mohar [14] used this method to show that $C_{\mathbb{S}^2} \leq 3444$, which was improved to $C_{\mathbb{S}^2} \leq 380$ by Oh [37]. By a refined argument, Ghidelli [16] proved that $C_{\mathbb{S}^2} \leq 208$. On the other hand, for the lower bound many authors [16,36,38,39] attempted to construct large examples in this class, and finally found some examples possessing 208 vertices. Hence, this completely answers the problem that $C_{\mathbb{S}^2} = 208$.

In this paper, we study the class of planar graphs with nonnegative curvature $\mathcal{PC}_{\geq 0}$. It turns out the class $\mathcal{PC}_{\geq 0}$ is much larger than $\mathcal{PC}_{>0}$ and contains many interesting examples. Among the finite planar graphs in $\mathcal{PC}_{\geq 0}$ is a family of so-called fullerenes. A fullerene is a finite cubic planar graph whose faces are either pentagon or hexagon. There are plenty of examples of fullerenes which are important in the real-world applications, to cite a few [5–8,29,45]. Among the infinite planar graphs in $\mathcal{PC}_{\geq 0}$ is a family of all planar tilings with regular polygons as tiles, see e.g. [15,18]. These motivate our investigations of the structure of the class $\mathcal{PC}_{\geq 0}$. For a planar graph G, we denote by

$$\Phi(G) := \sum_{x \in V} \Phi(x)$$

the total curvature of G. For a finite planar graph G, Gauss–Bonnet theorem states that $\Phi(G) = 2$. For an infinite planar graph $G \in \mathcal{PC}_{\geq 0}$, the Cohn-Vossen type theorem, see [12,14], yields that

$$\Phi(G) \leq 1. \tag{3}$$

In [24], the authors proved that the total curvature for a planar graph with nonnegative curvature is an integral multiple of $\frac{1}{12}$.

For any $G = (V, E, F) \in \mathcal{PC}_{\geq 0}$, we denote by

$$T_G := \{v \in V : \Phi(x) > 0\}$$

the set of vertices with non-vanishing curvature. For any infinite planar graph in $\mathcal{PC}_{\geq 0}$, Chen and Chen [12,13] obtained the interesting result that T_G is a finite set. By Alexandrov's embedding theorem [2], the polyhedral surface $S(G)$ can be isometrically embedded into $\mathbb{R}^3$ as a boundary of a noncompact convex polyhedron. The set T_G serves as the set of the vertices/corners of the convex polyhedron, so that much geometric information of the polyhedron is contained in T_G. We are interested in the structure of the set T_G.

Analogous to the prisms and antiprisms in $\mathcal{PC}_{>0}$, we define some similar families of planar graphs in $\mathcal{PC}_{\geq 0}$.

Definition 1.2. We call a planar graph $G = (V, E, F) \in \mathcal{PC}_{\geq 0}$ a prism-like graph if either

(1) G is an infinite graph and $D_G \geq 43$, where D_G is defined in Equation (1), or
(2) G is a finite graph and there are at least two faces with degree at least 43.

The name of "prism-like" graph is chosen since the structure of these graphs is simple in some sense, analogous to a prism or an antiprism, and can be completely determined, see Theorem 2.3 and Theorem 2.4. One may ask the following problem analogous to that of DeVos and Mohar.

Problem 1.3. What are the numbers

$$K_{\mathbb{S}^2} := \max_{\text{finite } G} \sharp T_G, \quad K_{\mathbb{R}^2} := \max_{\text{infinite } G} \sharp T_G,$$

where the maxima are taken over finite and infinite graphs in $\mathcal{PC}_{\geq 0}$ which are not prism-like graphs respectively?

The second part of the problem was proposed in [22] and an elementary result was obtained therein,

$$K_{\mathbb{R}^2} \leq 1722.$$

In this paper, we give the answer to the second part of the problem.

Theorem 1.4.

$$K_{\mathbb{R}^2} = 132.$$

Moreover, a graph in this class attains the maximum if and only if its polyhedral surface contains 12 disjoint hendecagons.

On one hand, we give the upper bound $K_{\mathbb{R}^2} \leq 132$ by the discharging method in Section 3. On the other hand, we construct an example possessing 132 vertices with non-vanishing curvature as in Fig. 1, see the construction in Section 4.

For the first part of Problem 1.3, we give a family of infinitely many examples with arbitrarily large number of vertices of non-vanishing curvature, see Example 3.3, which are not prism-like graphs, and hence

$$K_{\mathbb{S}^2} = \infty.$$

By modifying the definition of $K_{\mathbb{S}^2}$, we get the following result.

Theorem 1.5. *Let*

$$\widetilde{K}_{\mathbb{S}^2} := \max_{\text{finite } G} \sharp T_G,$$

where the maximum is taken over finite graphs in $\mathcal{PC}_{\geq 0}$ whose maximal facial degree is less than 132. Then
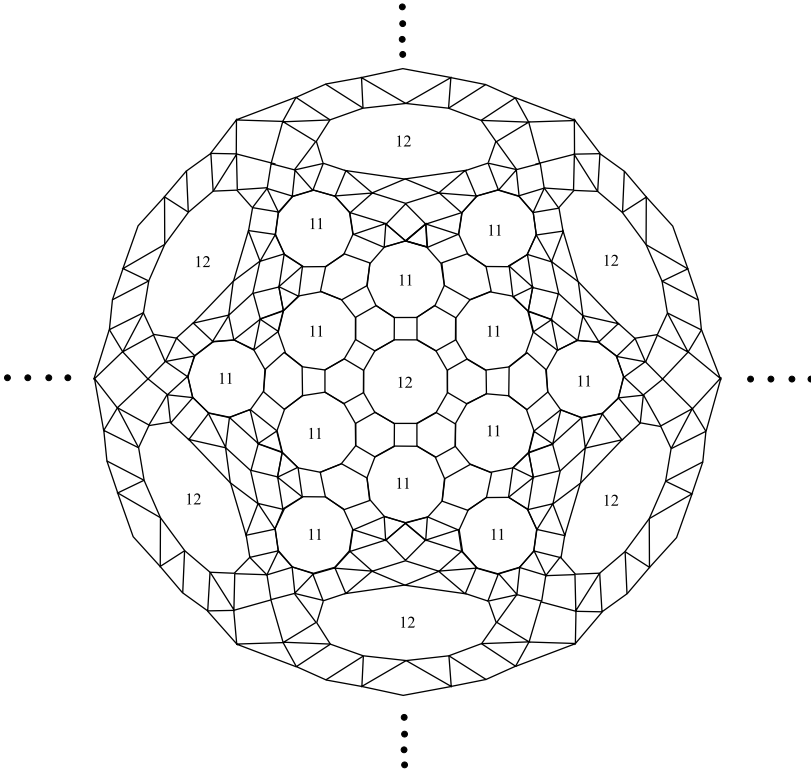
Fig. 1. A planar graph $G \in \mathcal{PC}_{\geq 0}$ with $\sharp T_G = 132$.

$$\widetilde{K}_{\mathbb{S}^2} = 264.$$

Moreover, a graph in this class attains the maximum if and only if its polyhedral surface contains 24 *disjoint hendecagons.*

The proof of the theorem follows from the same argument as in Theorem 1.4, see Section 3 and Section 4. The upper bound of maximal facial degree, 132, in the above theorem is chosen by considering both the discharging argument and concrete examples with many vertices of non-vanishing curvature in Example 3.3, see the explanations above that example.

As an application, we may estimate the order of automorphism groups of planar graphs with nonnegative curvature. The automorphism groups of planar graphs have been extensively studied in the literature, to cite a few [3,11,32,41]. Let $G = (V, E, F)$ be a planar graph. A bijection $R : V \to V$ is called a graph automorphism if it preserves the graph structure of (V, E). A triple (H_V, H_E, H_F) with bijections on V, E and F respectively is called a cellular automorphism of $G = (V, E, F)$ if they preserve the incidence structure of the cell complex G. We denote by $\mathrm{Aut}(G)$ the graph automorphism group of the graph (V, E), and by $\widetilde{\mathrm{Aut}}(G)$ the cellular automorphism group of the planar

graph G, see Section 5 for definitions. We prove for any graph $G \in \mathcal{PC}_{\geq 0}$ with positive total curvature, the cellular automorphism group is finite, and give the estimate for the order of the group.

Theorem 1.6. *Let $G = (V, E, F)$ be a planar graph with nonnegative combinatorial curvature and $\Phi(G) > 0$. Then we have the following:*

(1) *If G is infinite, then*

$$\sharp \widetilde{\mathrm{Aut}}(G) \leq \begin{cases} 132! \times 5!, & \text{for } D_G \leq 42, \\ 2D_G, & \text{for } D_G > 42. \end{cases}$$

(2) *If G is finite, then*

$$\sharp \widetilde{\mathrm{Aut}}(G) \leq \begin{cases} 264! \times 5!, & \text{for } D_G \leq 42, \\ 4D_G, & \text{for } D_G > 42. \end{cases}$$

Note that Whitney [46] proved a well-known theorem that any finite 3-connected planar graph, i.e. remaining connected after deleting any two vertices, can be uniquely embedded into $\mathbb{S}^2$. This has been generalized to infinite graphs by Mohar [33] that a locally finite, 3-connected planar graph, whose faces are bounded by cycles of finite size, has a unique embedding in the plane, see also [25,44]. These results imply that for any 3-connected planar graph $G = (V, E, F)$ in our setting,

$$\widetilde{\mathrm{Aut}}(G) \cong \mathrm{Aut}(G).$$

Hence all results in Theorem 1.6 apply to the graph automorphism group if the graph is 3-connected.

The paper is organized as follows: In the next section, we recall some basic facts on the combinatorial curvature of planar graphs. Section 3 is devoted to the upper bound estimates for Theorem 1.4 and Theorem 1.5. In Section 4, we construct examples to show the lower bound estimates for Theorem 1.4 and Theorem 1.5. The last section contains the proof of Theorem 1.6.

2. Preliminaries

Let $G = (V, E, F)$ be a planar graph induced by an embedding of a graph (V, E) into $\mathbb{S}^2$ or $\mathbb{R}^2$. We only consider the appropriate embedding such that G is a tessellation of S, see the definition in the introduction. Hence G is a finite graph if and only if it embeds into $\mathbb{S}^2$, and G is an infinite graph if and only if it embeds into $\mathbb{R}^2$.

We say that a vertex x is incident to an edge e, denoted by $x \prec e$, (similarly, an edge e is incident to a face σ, denoted by $e \prec \sigma$; or a vertex x is incident to a face σ, denoted

by $x \prec \sigma$) if the former is a subset of the closure of the latter. For any face σ, we denote by

$$\partial \sigma := \{x \in V : x \prec \sigma\}$$

the vertex boundary of σ. Two vertices are called neighbors if there is an edge connecting them. We denote by $\deg(x)$ the degree of a vertex x, i.e. the number of neighbors of a vertex x, and by $\deg(\sigma)$ the degree of a face σ, i.e. the number of edges incident to a face σ (equivalently, the number of vertices incident to σ). Two faces σ and τ are called adjacent, denoted by $\sigma \sim \tau$, if there is an edge incident to both of them, i.e. they share a common edge. Note that by the tessellation properties they share at most one edge.

For a planar graph $G = (V, E, F)$, let $S(G)$ denote the polyhedral surface with piecewise flat metric defined in the introduction. For $S(G)$, it is locally isometric to a flat domain in $\mathbb{R}^2$ near any interior point of an edge or a face, while it might be non-smooth near some vertices. As a metric surface, the generalized Gaussian curvature K of $S(G)$ vanishes at smooth points and can be regarded as a measure concentrated on the isolated singularities, i.e. on vertices. One can show that the mass of the generalized Gaussian curvature at each vertex x is given by $K(x) = 2\pi - \Sigma_x$, where Σ_x denotes the total angle at x in the metric space $S(G)$, see [2]. Moreover, by direct computation one has $K(x) = 2\pi\Phi(x)$, where the combinatorial curvature $\Phi(x)$ is defined in (2). Hence one can show that a planar graph G has nonnegative combinatorial curvature if and only if the polyhedral surface $S(G)$ is a generalized convex surface.

In a planar graph, the pattern of a vertex x is defined as a vector

$$(\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_N)),$$

where $N = \deg(x)$, $\{\sigma_i\}_{i=1}^N$ are the faces to which x is incident, and $\deg(\sigma_1) \le \deg(\sigma_2) \le \cdots \le \deg(\sigma_N)$.

Table 1 is the list of all possible patterns of a vertex with positive curvature (see [12,14]); Table 2 is the list of all possible patterns of a vertex with vanishing curvature (see [12,18]).

The following lemma is useful for our purposes, see [12, Lemma 2.5].

Lemma 2.1. *If there is a face σ such that $\deg(\sigma) \ge 43$ and $\Phi(x) \ge 0$ for any vertex x incident to σ, then*

$$\sum_{x \in V : x \prec \sigma} \Phi(x) \ge 1.$$

For an infinite planar graph G with nonnegative curvature if there is a face of degree at least 43, then the graph has rather special structure, see [21, Theorem 2.10]. As in the introduction, we call it an infinite prism-like graph.

The following observation is useful for our purposes.

Table 1
The patterns of a vertex with positive curvature.

Patterns		$\Phi(x)$
$(3, 3, k)$	$3 \leq k$	$1/6 + 1/k$
$(3, 4, k)$	$4 \leq k$	$1/12 + 1/k$
$(3, 5, k)$	$5 \leq k$	$1/30 + 1/k$
$(3, 6, k)$	$6 \leq k$	$1/k$
$(3, 7, k)$	$7 \leq k \leq 41$	$1/k - 1/42$
$(3, 8, k)$	$8 \leq k \leq 23$	$1/k - 1/24$
$(3, 9, k)$	$9 \leq k \leq 17$	$1/k - 1/18$
$(3, 10, k)$	$10 \leq k \leq 14$	$1/k - 1/15$
$(3, 11, k)$	$11 \leq k \leq 13$	$1/k - 5/66$
$(4, 4, k)$	$4 \leq k$	$1/k$
$(4, 5, k)$	$5 \leq k \leq 19$	$1/k - 1/20$
$(4, 6, k)$	$6 \leq k \leq 11$	$1/k - 1/12$
$(4, 7, k)$	$7 \leq k \leq 9$	$1/k - 3/28$
$(5, 5, k)$	$5 \leq k \leq 9$	$1/k - 1/10$
$(5, 6, k)$	$6 \leq k \leq 7$	$1/k - 2/15$
$(3, 3, 3, k)$	$3 \leq k$	$1/k$
$(3, 3, 4, k)$	$4 \leq k \leq 11$	$1/k - 1/12$
$(3, 3, 5, k)$	$5 \leq k \leq 7$	$1/k - 2/15$
$(3, 4, 4, k)$	$4 \leq k \leq 5$	$1/k - 1/6$
$(3, 3, 3, 3, k)$	$3 \leq k \leq 5$	$1/k - 1/6$

Table 2
The patterns of a vertex with vanishing curvature.

$(3, 7, 42)$,	$(3, 8, 24)$,	$(3, 9, 18)$,	$(3, 10, 15)$,	$(3, 12, 12)$,
$(4, 5, 20)$,	$(4, 6, 12)$,	$(4, 8, 8)$,	$(5, 5, 10)$,	$(6, 6, 6)$,
$(3, 3, 4, 12)$,	$(3, 3, 6, 6)$,	$(3, 4, 4, 6)$,	$(4, 4, 4, 4)$,	$(3, 3, 3, 3, 6)$,
$(3, 3, 3, 4, 4)$,	$(3, 3, 3, 3, 3, 3)$.			

Observation 2.2. *For any planar graph G with nonnegative curvature, we divide hexagons into triangles, i.e. each hexagon is divided into 6 triangles, such that the modified graph possesses no hexagons and still has nonnegative curvature. So that we may assume that G has no hexagons.*

We have the following theorem.

Theorem 2.3 *([21]). Let $G = (V, E, F)$ be an infinite planar graph with nonnegative curvature and $D_G \geq 43$. Then there is only one face σ of degree at least 43. Suppose that there is no hexagonal faces, e.g. by Observation 2.2. Then the set of faces F consists of σ, and triangles and/or squares. Moreover,*

$$S(G) = \sigma \cup (\cup_{i=1}^{\infty} L_i),$$

where L_i, $i \geq 1$, is a set of faces of the same type (triangle or square) which composite a band, i.e. an annulus, such that

$$\min_{\substack{x \in \cup_{\tau \in L_i} \partial \tau, \\ y \in \partial \sigma}} d(x, y) = i - 1,$$
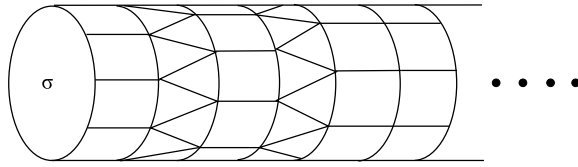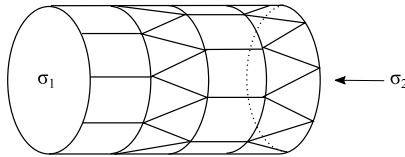
Fig. 2. A half flat-cylinder in $\mathbb{R}^3$.

Fig. 3. A cylinder barrel in $\mathbb{R}^3$.

where d is the graph distance in (V, E). And $S(G)$ is isometric to the boundary of a half flat-cylinder in $\mathbb{R}^3$, see Fig. 2.

We collect some properties of an infinite prism-like graph:

- There is only one face σ of degree at least 43, and $\deg(\sigma) = D_G$;
- T_G consists of all vertices incident to the largest face σ;
- Any face, which is not σ, is either a triangle, a square or a hexagon;
- The polygonal surface $S(G)$ is isometric to the boundary of a half flat-cylinder in $\mathbb{R}^3$;
- $\Phi(G) = 1$.

Next, we study finite prism-like graphs. Recall that a finite graph G with nonnegative curvature is called prism-like if the number of faces with degree at least 43 is at least two.

Theorem 2.4. *Let $G = (V, E, F)$ be a finite prism-like graph. Then there are exactly two disjoint faces σ_1 and σ_2 of same facial degree at least 43. Suppose that there is no hexagonal faces, e.g. by Observation 2.2. Then the set of faces F consists of σ_1 and σ_2, triangles or squares. Moreover,*

$$S(G) = \sigma_1 \cup \left(\cup_{i=1}^M L_i \right) \cup \sigma_2,$$

where $M \geq 1$, L_i, $1 \leq i \leq M$, is a set of faces of the same type (triangle or square) which composite a band, i.e. an annulus, such that

$$\min_{\substack{x \in \cup_{\tau \in L_i} \partial \tau, \\ y \in \partial \sigma_1}} d(x, y) = i - 1.$$

And $S(G)$ is isometric to the boundary of a cylinder barrel in $\mathbb{R}^3$, see Fig. 3.

Proof. We adopt the same argument as in the proof of Theorem 2.10 in [21]. Since the curvature is nonnegative, one easily sees that the boundaries of any two faces of degree at least 43 are disjoint. By Lemma 2.1, for any face σ of degree at least 43,

$$\sum_{x \in V : x \prec \sigma} \Phi(x) \geq 1.$$

Since the graph is finite, by Gauss–Bonnet theorem, $\Phi(G) = 2$. Hence there are exactly two faces of degree at least 43, denoted by σ_1 and σ_2. Moreover, we have

$$\sum_{x \in V : x \prec \sigma_i} \Phi(x) = 1, \quad i = 1, 2,$$

and $\Phi(x) = 0$, for any $x \in V \setminus (\partial \sigma_1 \cup \partial \sigma_2)$. For any $x \in \partial \sigma_1 \cup \partial \sigma_2$, the pattern of x is $(4, 4, k_i)$ or $(3, 3, 3, k_i)$, where $k_i = \deg(\sigma_i)$ for $i = 1, 2$. Suppose that there is a vertex $x \in \partial \sigma_1$ whose pattern is $(4, 4, k_1)$ ($(3, 3, 3, k_1)$ resp.), then the patterns of all vertices in $\partial \sigma_1$ are $(4, 4, k_1)$ ($(3, 3, 3, k_1)$ resp.). Same results hold for $\partial \sigma_2$. We denote by L_1 the set of all faces incident to a vertex of σ_1, and different from σ_1, which composite a band. Inductively, for any $i \geq 1$ define L_{i+1} as the set of faces incident to a vertex of a face in L_i, which are not in $\sigma_1 \cup (\cup_{j=1}^{i-1} L_j)$. Note that L_i, $i \geq 1$, consists of faces of same degree, triangles or squares. Since the graph is finite, there is some M such that σ_2 is incident to some face in L_M. Considering the properties of faces incident to σ_2 similarly, we have all faces in L_M are incident to σ_2. This yields that $\deg(\sigma_1) = \deg(\sigma_2)$ and proves the theorem. $\quad\square$

A finite prism-like graph G has the following properties:

- There are exactly two faces σ_1 and σ_2 of degree at least 43;
- $\deg(\sigma_1) = \deg(\sigma_2)$;
- T_G consists of all vertices in $\partial \sigma_1 \cup \partial \sigma_2$;
- Any face, which is not σ_1 and σ_2, is either a triangle, a square or a hexagon;
- The polygonal surface $S(G)$ is isometric to the boundary of a cylinder barrel in $\mathbb{R}^3$.

3. Upper bound estimates for the size of T_G

In this section, we prove the upper bound estimates for the number of vertices in T_G for planar graph G with nonnegative curvature.

Definition 3.1. We say that a vertex $x \in T_G$ is bad if $0 < \Phi(x) < \frac{1}{132}$, and good if $\Phi(x) \geq \frac{1}{132}$.

Let $G \in \mathcal{PC}_{\geq 0}$ be either an infinite graph or a finite graph with $D_G < 132$, which is not a prism-like graph. Then direct computation shows that all patterns of bad vertices of G are given by

$$(3,7,k), \ 32 \leq k \leq 41, \quad (3,8,k), \ 21 \leq k \leq 23, \quad (3,9,k), \ 16 \leq k \leq 17$$
$$(3,10,14), \quad (3,11,13), \quad (4,5,k), \ 18 \leq k \leq 19, \quad (4,7,9).$$

Our main tool to prove the results is the discharging method, see e.g. [1,14,16,34,37,40]. The curvature at vertices of a planar graph can be regarded as the charge concentrated on vertices. The discharging method is to redistribute the charge on vertices, via transferring the charge on good vertices to bad vertices, such that the final/terminal charge on involved vertices is uniformly bounded below. In the following, we don't distinguish the charge with the curvature. We need to show that for each bad vertex in T_G, one can find some nearby vertices with a fair amount of curvature, from which the bad vertex will receive some amount of curvature. By the discharging process, we get the final curvature uniformly bounded below on T_G by the constant $\frac{1}{132}$, so that it yields the upper bound of the cardinality of T_G by the upper bound of total curvature. Generally speaking, since the discharging method is divided into several steps, one shall check that there remains enough amount of curvature at the vertices which are involved in two or more steps to contribute the curvature. However, in our discharging method, we distribute the curvature at each vertex only once.

Proof of Theorem 1.4 (upper bound). In the following, we prove the upper bound estimate

$$K_{\mathbb{R}^2} \leq 132.$$

Let $G \in \mathcal{PC}_{\geq 0}$ be an infinite graph which is not a prism-like graph. We will introduce a discharging process to distribute some amount of the curvature at good vertices to bad vertices via the discharging rules. We denoted by Φ the curvature at vertices of G and by $\widetilde{\Phi}$ the final curvature at vertices after carrying out the following discharging rules. We will show that the final curvature satisfies $\sum_{x \in V} \widetilde{\Phi}(x) = \sum_{x \in V} \Phi(x)$, and

$$\widetilde{\Phi}(x) \geq \frac{1}{132}, \quad \forall x \in T_G.$$

This will imply the upper bound estimate $\sharp T_G \leq 132$ since the total curvature $\Phi(G) \leq 1$.

The proof consists of three steps: In the first step, we consider the cases for bad vertices, $(3,8,k), 21 \leq k \leq 23, (3,9,k), 16 \leq k \leq 17, (3,10,14)$ and $(3,11,13)$; In the second step, we deal with the cases, $(3,7,k), 32 \leq k \leq 41$ and $(4,7,9)$; The last step is devoted to the cases, $(4,5,k), 18 \leq k \leq 19$.

Step 1. In this step, we divide it into the following three cases:

Case 1.1: There is at least one bad vertex on some k-gon with $21 \leq k \leq 23$. In this case, the pattern of those bad vertices must be $(3,8,k)$. Fix a bad vertex a of the k-gon, see Fig. 4. Then we will show that the vertices x, y of the octagon in Fig. 4 and the good vertices on the k-gon have enough curvature to be distributed to all
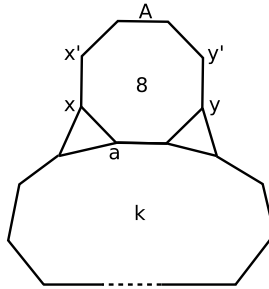
Fig. 4. The case for bad vertices of the pattern $(3, 8, k)$ for $21 \leq k \leq 23$.

bad vertices of k-gon such that the final curvature $\widetilde{\Phi}$ at vertices involved is greater than $\frac{1}{132}$. In fact, all the possible patterns of x and y are

$$(3, m, 8), 3 \leq m \leq 8, (3, 3, 3, 8) \text{ and } (3, 3, 4, 8). \tag{4}$$

Note that the curvature of vertices on the k-gon is at least $\frac{1}{k} - \frac{1}{24}$, and for $21 \leq k \leq 23$,

$$2 \times \frac{13}{176} + k \left(\frac{1}{k} - \frac{1}{24} \right) \geq \frac{k+2}{132}. \tag{5}$$

By the above equation (5), if we can find sets of good vertices on the k-gon, A_x and A_y, satisfying that

- $A_x \cap A_y = \emptyset$,
- $\Phi(x) + \sum\limits_{z \in A_x} \Phi(z) - \sharp A_x \cdot \left(\frac{1}{k} - \frac{1}{24} \right) > \frac{13}{176}$ and
- $\Phi(y) + \sum\limits_{z \in A_y} \Phi(z) - \sharp A_y \cdot \left(\frac{1}{k} - \frac{1}{24} \right) > \frac{13}{176}$,

then x, y, A_x and A_y have enough curvature to be distributed to all bad vertices of the k-gon. In the following, we try to find the sets A_x and A_y case by case.

Note that all possible patterns listed in (4) except $(3, 3, 4, 8)$ have curvature at least $\frac{1}{12}$ which is strictly greater than $\frac{13}{176}$. First, we consider the pattern of the vertex x. It splits into the following two subcases:

Subcase 1.1.1: x is not of the pattern $(3, 3, 4, 8)$. We take $A_x = \emptyset$.

Subcase 1.1.2: x is of the pattern $(3, 3, 4, 8)$. In this situation, we have three cases, as depicted in Fig. 5. In either case, $\Phi(z) \geq \frac{1}{k}$, i.e. z has a fair amount of curvature, and

$$\Phi(x) + \Phi(z) - \left(\frac{1}{k} - \frac{1}{24} \right) \geq \frac{1}{12} > \frac{13}{176}.$$
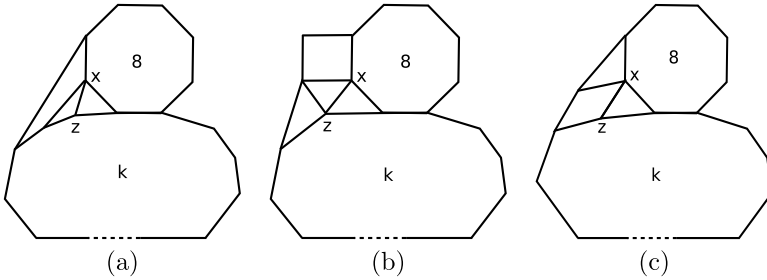
We take $A_x = \{z\}$.

Fig. 5. Three situations in Subcase 1.1.2.

Considering the pattern of the vertex y, we may choose the set A_y similarly as A_x above. It is easy to check that A_x and A_y satisfy the conditions as required. This finishes the proof for this case.

Remark 3.2. Note that the only edge of the octagon which could be incident to another l-gon with $21 \leq l \leq 23$ is the edge A, see Fig. 4. Hence we can distribute the curvature of vertices x' and y' to bad vertices on the l-gon. This means that the curvature at every vertex on the octagon is transferred to other bad vertices in the discharging process no more than once.

Case 1.2: There is at least one bad vertex on some k-gon with $16 \leq k \leq 17$. In this case, the pattern of bad vertices must be $(3, 9, k)$. The situation is similar to Case 1.1. We can argue verbatim as above to conclude the results, and hence omit the proof here.

Case 1.3: There is at least one bad vertex on some 14-gon. In this case, the pattern of bad vertices must be $(3, 10, 14)$. We can use a similar argument as in Case 1.1 and omit the proof here.

Case 1.4: There is at least one bad vertex on some tridecagon, denoted by σ. In this case, the pattern of bad vertices must be $(3, 11, 13)$, see Fig. 6. Note that all possible patterns of x and y are

$$(3, m, 11), 3 \leq m \leq 11, (3, 3, 3, 11) \text{ and } (3, 3, 4, 11).$$

Note that the curvature of vertices on the tridecagon σ is at least $\frac{1}{858}$ and the following holds,

$$2 \times \frac{13}{264} + 13 \times \frac{1}{858} = \frac{13 + 2}{132}.$$

Suppose that we can find sets of good vertices, A_x, A_y, B_x and B_y, satisfying that
- $(A_x \cup A_y) \subset \partial\sigma, (B_x \cup B_y) \cap (\partial\sigma \cup \{x, y\}) = \emptyset$,
- $(A_x \cup B_x) \cap (A_y \cup B_y) = \emptyset$,
- $\Phi(x) + \sum\limits_{z \in A_x} \Phi(z) - \frac{\sharp A_x}{858} + \sum\limits_{z \in B_x} \Phi(z) - \frac{\sharp B_x}{132} > \frac{13}{264}$ and
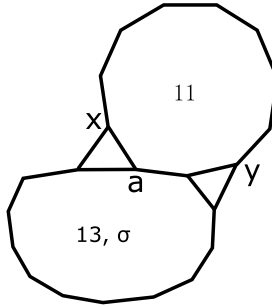
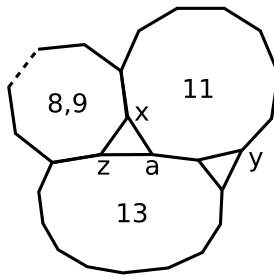Fig. 6. The case for bad vertices of the pattern $(3, 11, 13)$.

Fig. 7. The case that the pattern of x is $(3, 8, 11)$ or $(3, 9, 11)$.

- $\Phi(y) + \sum\limits_{z \in A_y} \Phi(z) - \frac{\sharp A_y}{858} + \sum\limits_{z \in B_y} \Phi(z) - \frac{\sharp B_y}{132} > \frac{13}{264}$.

Then x, y, A_x, A_y, B_x and B_y have enough curvature to be distributed to all bad vertices of the tridecagon.

For our purposes, we first consider the case that the pattern of x is not of the pattern $(3, 11, 11)$. Then we have the following cases, **A–D**.

A: The pattern of x is one of $(3, 3, 11)$, $(3, 4, 11)$, $(3, 5, 11)$, $(3, 6, 11)$, $(3, 7, 11)$ and $(3, 3, 3, 11)$. In this subcase, since $\Phi(x) > \frac{13}{264}$, we take $A_x = \emptyset$ and $B_x = \emptyset$.

B: x is of the pattern $(3, 8, 11)$ or $(3, 9, 11)$, see Fig. 7. In this subcase, noting that $\Phi(x) + \Phi(z) - \frac{1}{858} > \frac{13}{264}$, we can take $A_x = \{z\}$ and $B_x = \emptyset$.

C: x is of the pattern $(3, 10, 11)$, see Fig. 8. In this subcase, noting that $\Phi(w') \geq \frac{1}{60}$, we have

$$\Phi(x) + \Phi(z) + \Phi(w) + \left(\Phi(w') - \frac{1}{132} \right) - \frac{2}{858} > \frac{13}{264}.$$

We can take $A_x = \{z, w\}$ and $B_x = \{w'\}$. Note that in this situation, the curvature at the vertex w' is not distributed to the vertices of some 14-gon or other tridecagon, since the edge A cannot be incident to any 14-gon or tridecagon. That is, we only distribute the curvature of w' to other vertices once.
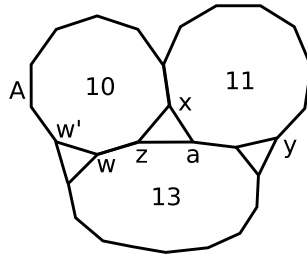
Fig. 8. The case that the pattern of x is $(3, 10, 11)$.

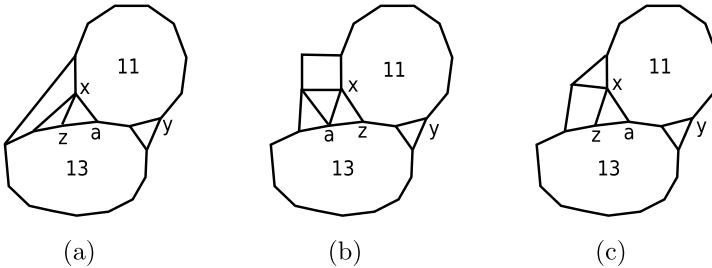(a)   (b)   (c)

Fig. 9. The case that the pattern of x is $(3, 3, 4, 11)$.

D: x is of the pattern $(3, 3, 4, 11)$. There are three possible situations, as depicted in Fig. 9. In either case, $\Phi(z) \geq \frac{1}{13}$, hence

$$\Phi(x) + \Phi(z) - \left(\frac{1}{13} - \frac{5}{66} \right) \geq \frac{13}{264}.$$

We take $A_x = \{z\}$ and $B_x = \emptyset$.

In case that y is not of the pattern $(3, 11, 11)$, we can take A_y and B_y according to the above cases, **A**–**D**, similarly.

Considering the patterns of x and y together, we have the following subcases for Case 1.4:

Subcase 1.4.1: Both of x and y are not of the pattern $(3, 11, 11)$. In this subcase, we choose A_x, B_x, A_y and B_y according to the cases **A**–**D** mentioned above.

Subcase 1.4.2: One of x and y is of the pattern $(3, 11, 11)$ and the other is not. Without loss of generality, we may assume that x is of the pattern $(3, 11, 11)$ and y is not. We can take A_y and B_y according to the cases **A**–**D** above, since the pattern of y is not $(3, 11, 11)$. Consider the patterns of vertices which have a neighbor on the boundary of the tridecagon, e.g. x_1, x_2 etc. as depicted in Fig. 10. If x_1 is not of the pattern $(3, 11, 11)$, then we can take A_{x_1} and B_{x_1} for the vertex x_1 according to the cases **A**–**D** above. This yields four sets of good vertices A_{x_1}, B_{x_1}, A_y and B_y satisfying the properties as required and proves the result. So that it suffices to consider the case that x_1 is of the pattern $(3, 11, 11)$. If we further consider the vertices x_2 and x_3 by the same argument
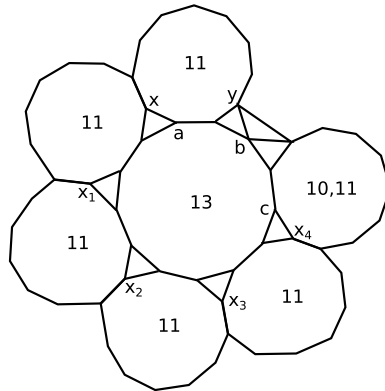
Fig. 10. The case that the pattern of x is $(3, 11, 11)$.

as above consecutively, as depicted in Fig. 10, then it suffices to consider that x_2 and x_3 are of the pattern $(3, 11, 11)$. Now we consider the vertex x_4. If it is not of the pattern $(3, 10, 11)$ or $(3, 11, 11)$, then we can take A_{x_4} and B_{x_4} according to the cases **A**–**D**. Note that in this case $A_{x_4} = \emptyset$ or $\{c\}$ and $B_{x_4} = \emptyset$. Hence A_{x_4}, B_{x_4}, A_y and B_y satisfy the properties as required. Suppose that x_4 is of the pattern $(3, 10, 11)$ or $(3, 11, 11)$, then we get that the pattern of b is $(3, 3, 3, 13)$ and $\Phi(b) = \frac{1}{13}$. The vertices x, x_1 and b have enough curvature to be distributed to all bad vertices on the tridecagon, since

$$\frac{2}{66} + \frac{1}{13} + \frac{12}{858} > \frac{13 + 2}{132}.$$

Subcase 1.4.3: Both of x and y are of the pattern $(3, 11, 11)$. In this subcase, noting that 13 is an odd number, it is impossible that all vertices, which have a neighbor on the boundary of the tridecagon, are of the pattern $(3, 11, 11)$, see e.g. Fig. 10. Hence, we can find two vertices x', y', which have a neighbor on the boundary of the tridecagon and are incident to a common hendecagon, such that one of them is of the pattern $(3, 11, 11)$ and the other is not. This reduces this subcase to Subcase 1.4.2 by renaming x' and y' to x and y.

Step 2. In this step, we consider bad vertices on the heptagons, i.e. those of the patterns $(3, 7, k)$, $32 \leq k \leq 41$, and $(4, 7, 9)$. Note that for any two heptagons whose boundaries have non-empty intersection, since the curvature is nonnegative, they must share a common edge, i.e. they are neighbors in the dual graph G^*. We consider connected components of the induced graph on vertices of degree seven in G^*, which corresponds to the heptagons in G. Let $\mathcal{S}$ be such a connected component. We divide it into the following cases.
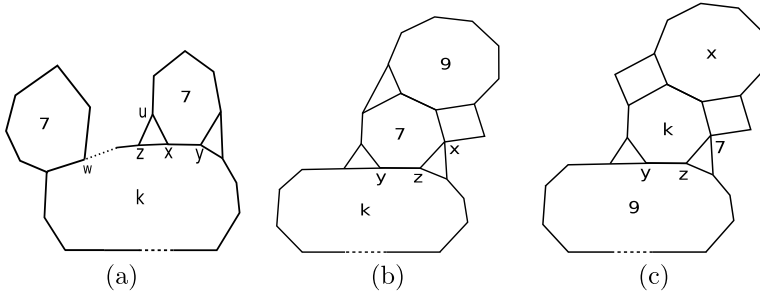
Fig. 11. The heptagon is adjacent to a k-gon for $32 \leq k \leq 41$.

Case 2.1: $\sharp \mathcal{S} = n \geq 2$. We denote by $\{\sigma_i\}_{i=1}^{n}$ the set of vertices in $\mathcal{S}$, where σ_i, $1 \leq i \leq n$, are heptagons in G. Let m be the number of edges in $\mathcal{S}$, which corresponds to the number of edges shared by the heptagons $\{\sigma_i\}_{i=1}^{n}$. We denote by $\{e_i\}_{i=1}^{m}$ the set of edges shared by the heptagons $\{\sigma_i\}_{i=1}^{n}$ in G. Hence, the number of vertices in G incident to $\cup_{i=1}^{n}\sigma_i$ is $7n - 2m$, since any vertex in G is incident to at most two heptagons. The two vertices of any edge in $\{e_i\}_{i=1}^{m}$ are of the pattern $(3,7,7)$ or $(4,7,7)$, hence the curvature at these vertices is at least $\frac{1}{28}$. This means that there are $2m$ vertices with the curvature at least $\frac{1}{28}$. Note that for any heptagon, the faces adjacent to it consists of at most one k-gon for $32 \leq k \leq 42$. Suppose that there are s heptagons in $\{\sigma_i\}_{i=1}^{n}$ adjacent to a k-gon with $32 \leq k \leq 41$ and there are t heptagons in $\{\sigma_i\}_{i=1}^{n}$ adjacent to a 42-gon. Clearly $s + t \leq n$. Hence, there are $2s$ vertices with the curvature at least $\frac{1}{1722}$ and $2t$ vertices with vanishing curvature. The curvature of other vertices is at least $\frac{1}{252}$. Since $m \geq n - 1 > \frac{46}{109}n$ for $n \geq 2$, we have

$$\frac{2m}{28} + \frac{2s}{1722} + \frac{7n - 2m - 2(s+t)}{252} \geq \frac{2m}{28} + \frac{5n - 2m}{252} > \frac{7n - 2m}{132}.$$

This means that good vertices on these heptagons have enough curvature to be distributed to bad vertices on them.

Case 2.2: $\sharp \mathcal{S} = 1$. For the heptagon in $\mathcal{S}$, the intersection of its boundary and the boundary of any other heptagon in G is empty. We consider this isolated heptagon and divide it into the following three subcases:

Subcase 2.2.1: The heptagon is adjacent to a k-gon for $32 \leq k \leq 41$, and there are no vertices on the heptagon of the pattern $(4,7,9)$, see Fig. 11(a). In this subcase, all possible patterns of vertices on the k-gon are $(3,3,k)$, $(3,4,k)$, $(3,5,k)$, $(3,6,k)$, $(3,7,k)$, $(4,4,k)$ and $(3,3,3,k)$. Except the pattern $(3,7,k)$, all patterns have the curvature at least $\frac{1}{k}$. Note that the pattern of z cannot be $(3,7,k)$. Otherwise, the vertex u, as depicted in Fig. 11(a), would be of pattern $(3,7,7)$ which will imply that two heptagons are adjacent to each other and yields a contradiction to the assumption. Hence $\Phi(z) \geq \frac{1}{k}$. Denote by w the first vertex on the boundary of the k-gon in the direction from y to x which is of the pattern $(3,7,k)$. If the heptagon which contains w belongs to Case 2.1,
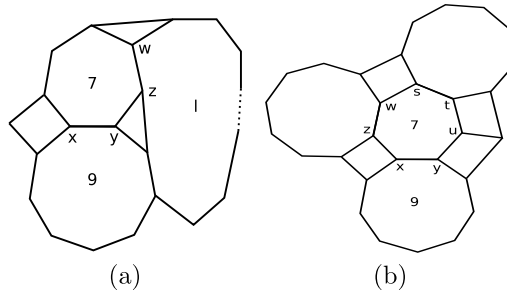
Fig. 12. The heptagon is not adjacent to any k-gon for $32 \leq k \leq 41$.

i.e. it is incident to another heptagon, then we distribute the curvature at z only to x. Otherwise, we distribute the curvature at z to both x and w. For the vertex y, we can use a similar argument to derive the result.

Subcase 2.2.2: The heptagon is adjacent to a k-gon for $32 \leq k \leq 41$, and there exists at least one vertex on the heptagon of the pattern $(4, 7, 9)$. In this subcase, there are two situations, see Fig. 11(b, c). In either case, note that the pattern of x must be $(3, 3, 4, 7)$ and $\Phi(x) = \frac{5}{84}$, which is enough to be distributed to bad vertices of the pattern $(4, 7, 9)$, since there are at most two such bad vertices on the heptagon. And bad vertices of the pattern $(3, 7, k)$, i.e. the vertices y, z in Fig. 11 (b, c), can be treated in the same way as in Subcase 2.2.1.

Subcase 2.2.3: The heptagon is not adjacent to any k-gon for $32 \leq k \leq 41$. That is, all bad vertices are of the pattern $(4, 7, 9)$. Fix a vertex x on the heptagon of the pattern $(4, 7, 9)$, see Fig. 12. For the vertex y, we have the following possible situations:

Subcase 2.2.3(A): If the pattern of y is $(3, 7, 9)$ and z is not of the pattern $(3, 7, l)$ for $16 \leq l \leq 17$, then the curvature at y has not been distributed to other vertices before. And by the fact that $\Phi(y) = \frac{11}{126} > \frac{7}{132}$, y has enough curvature to be distributed to all bad vertices on the heptagon.

If the pattern of y is $(3, 7, 9)$ and z is of the pattern $(3, 7, l)$ for $16 \leq l \leq 17$, see Fig. 12(a), then the curvature at y might have been distributed to bad vertices of the l-gon in Step 1. However, since z and w, as depicted in Fig. 12(a), must be of the pattern $(3, 7, l)$ and $\min\{\Phi(z), \Phi(w)\} \geq \frac{25}{714}$, they have enough curvature to be distributed to all bad vertices of the heptagon.

Subcase 2.2.3(B): If the pattern of y is $(4, 7, 9)$, then we consider the vertex z, see Fig. 12(b). By the assumption, z cannot be of pattern $(4, 7, 7)$. If z is not of the pattern $(4, 7, 8)$ or $(4, 7, 9)$, then $\Phi(z) \geq \frac{5}{84} > \frac{7}{132}$, hence z has enough curvature to be distributed to bad vertices on the heptagon. If the pattern of z is $(4, 7, 8)$ $((4, 7, 9)$ resp.$)$ and the pattern of w is $(3, 7, 8)$ $((3, 7, 9)$ resp.$)$, we can get the conclusion by a similar argument as in the situation Subcase 2.2.3(A) above where y is of the pattern $(3, 7, 9)$. If
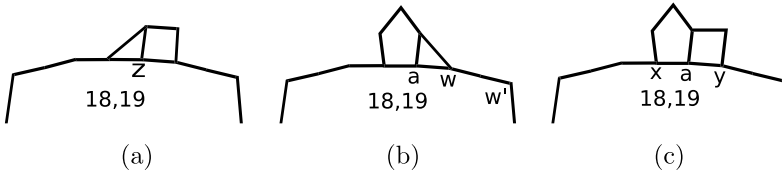
Fig. 13. Three cases for Step 3.

the pattern of w is $(4, 7, 8)$, which implies that the pattern of z must be $(4, 7, 8)$, then we have $\Phi(z) + \Phi(w) = \frac{1}{28}$. They have enough curvature to be distributed to bad vertices of the heptagon, since

$$\frac{1}{28} + \frac{5}{252} > \frac{7}{132}.$$

If the pattern of w is $(4, 7, 9)$, then the pattern of z must be $(4, 7, 9)$. Similarly, we consider s and t, if s or t is not $(4, 7, 9)$, the conclusion is obvious. If all of them are $(4, 7, 9)$, then u is of the pattern $(4, 4, 7)$ which has enough curvature to distribute.

Step 3. We consider the last case in which bad vertices are on a k-gon with $18 \leq k \leq 19$. That is, bad vertices are of the pattern $(4, 5, k)$ for $18 \leq k \leq 19$. We divide it into the following cases.

Case 3.1: There exists a vertex z on the boundary of k-gon with $18 \leq k \leq 19$ which is of the pattern $(3, 4, k)$, see Fig. 13(a). Note that $\Phi(z) = \frac{1}{12} + \frac{1}{k}$.
For $k = 18$, the curvature of a vertex with positive curvature on the 18-gon is at least $\frac{1}{180}$. Let l be the number of vertices with positive curvature on the 18-gon, $1 \leq l \leq 18$. Then

$$\Phi(z) + \frac{l-1}{180} > \frac{l}{132}.$$

Hence z has enough curvature to be distributed to all bad vertices on the 18-gon. For $k = 19$, the curvature on vertices of the 19-gon is at least $\frac{1}{380}$, which yields that

$$\Phi(z) + \frac{18}{380} > \frac{19}{132}.$$

Hence z also has enough curvature to be distributed to all bad vertices on the 19-gon.

Case 3.2: There exists a vertex z on the boundary of k-gon with $18 \leq k \leq 19$ which is of the pattern $(3, 5, k)$, see Fig. 13(b). Note that $\Phi(z) = \frac{1}{30} + \frac{1}{k}$. Hence, for the case $k = 18$ it is similar to Case 3.1 that
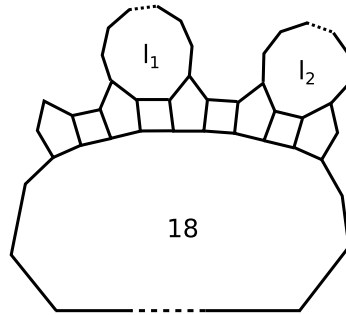
$$\Phi(z) + \frac{l-1}{180} > \frac{l}{132},$$

Fig. 14. Two squares, adjacent to the given 18-gon and a common pentagon, cannot be in the first class $\mathcal{M}_1$ simultaneously.

where l is the number of vertices with positive curvature on the 18-gon. For the case of $k = 19$, since $\Phi(w) + \Phi(w') \geq 2\left(\frac{1}{k} - \frac{1}{24}\right)$, we have

$$\frac{1}{30} + \frac{1}{k} + 2\left(\frac{1}{k} - \frac{1}{24}\right) + (k-3)\left(\frac{1}{k} - \frac{1}{20}\right) > \frac{k}{132}.$$

Hence, good vertices on the k-gon have enough curvature to be distributed to bad vertices on the k-gon.

Case 3.3: No vertices on the k-gon is of the pattern $(3,4,k)$ or $(3,5,k)$, $18 \leq k \leq 19$. In this case, given a bad vertex a on the k-gon, we consider its neighbors x and y, see Fig. 13 (c). Since there are no vertices on the k-gon of the pattern $(3,4,k)$ or $(3,5,k)$, the pattern of x must be $(4,5,k)$ and the pattern of y must be $(4,4,k)$ or $(4,5,k)$. Applying similar arguments to other vertices on the k-gon, we conclude that all edges of k-gon are incident to squares or pentagons. Now we consider the cases $k = 18$ and $k = 19$ respectively.

Subcase 3.3.1: $k = 18$. We observe that if there is a vertex on the boundary of 18-gon which is of the pattern $(4,4,18)$, then this vertex has enough curvature to be distributed to bad vertices on the 18-gon, since $\frac{1}{18} + \frac{l-1}{180} > \frac{l}{132}$, where l is the number of the vertices on the 18-gon with positive curvature. Hence it suffices to consider that all patterns of vertices of 18-gon are $(4,5,18)$.

Note that all patterns which contain a square and a pentagon are $(3,4,5)$, $(4,4,5)$, $(4,5,l), 5 \leq l \leq 20$, $(3,3,4,5)$, $(3,4,4,5)$. We consider the squares which are adjacent to the given 18-gon. We divide these squares into two classes: The first class $\mathcal{M}_1$ consists of the squares which are adjacent to another l-gon with $7 \leq l \leq 20$; The second class $\mathcal{M}_2$ contains all the other squares. A useful observation is that if two squares, adjacent to the given 18-gon, are adjacent to a common pentagon, then these squares cannot be in the first class $\mathcal{M}_1$ simultaneously, see e.g. Fig. 14. Since there are 9 squares adjacent to the 18-gon and 9 is an odd number, there exist two squares in the second class $\mathcal{M}_2$ adjacent to a common pentagon, see Fig. 15(a). Hence possible patterns of a, b, c and d, depicted in Fig. 15(a), are $(3,4,5)$, $(4,4,5)$, $(4,5,5)$, $(4,5,6)$, $(3,3,4,5)$ and
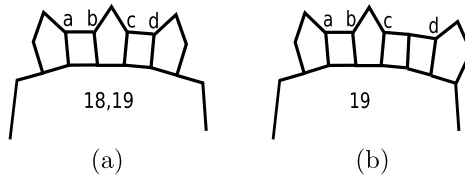
Fig. 15. There exist two squares in the second class $\mathcal{M}_2$ adjacent to a common pentagon.

$(3, 4, 4, 5)$. The curvature of a vertex of any pattern above is at least $\frac{1}{30}$, so that they have enough curvature to be distributed to bad vertices on the 18-gon, since

$$\frac{4}{30} + \frac{1}{180} \times 18 > \frac{18 + 4}{132}.$$

Subcase 3.3.2: $k = 19$. Note that if there are two vertices on the 19-gon which are of the pattern $(4, 4, 19)$, then these vertices have enough curvature to be distributed to bad vertices on the 19-gon, since $\frac{2}{19} + \frac{17}{380} > \frac{19}{132}$. So that it suffices to consider that all patterns of vertices of 19-gon are $(4, 5, 19)$ with an exceptional vertex of the pattern $(4, 4, 19)$. Adopting similar arguments as in Subcase 3.3.1, we have two situations, as depicted in Fig. 15(a) and Fig. 15(b). In both cases, the vertices a, b, c and d have enough curvature to be distributed to bad vertices of the 19-gon, since

$$\frac{4}{30} + \frac{1}{19} + \frac{1}{380} \times 18 > \frac{19 + 4}{132}.$$

Combining all cases above, we distribute the curvature at good vertices to bad vertices such that all vertices in T_G have final curvature $\widetilde{\Phi}$ uniformly bounded below by $\frac{1}{132}$. This proves the upper bound $\sharp T_G \leq 132$. For the equality case, suppose that G has $\sharp T_G = 132$, since all the inequalities in the discharge method are strict, then there are no bad vertices in T_G and all of them have curvature $\frac{1}{132}$. The vertex pattern of curvature $\frac{1}{132}$ is given by $(3, 3, 4, 11)$, $(4, 6, 11)$, or $(3, 11, 12)$. Hence, G has exactly 12 disjoint hendecagons. □

Now we study finite planar graphs with nonnegative curvature. First at all, we show that there are infinitely many finite planar graphs with a large face which are not prism-like graphs. This indicates that the upper bound of maximal facial degree 132 in Theorem 1.5 is somehow necessary.

Example 3.3. For any even number $m \geq 8$, there is a finite planar graph $G_m \in \mathcal{PC}_{\geq 0}$, which is not a prism-like graph, such that it has a unique face of degree m and all the other faces are triangles and squares. Moreover,

$$\sharp T_G = m + 4$$

which yields that $K_{\mathbb{S}^2} = \infty$, see Problem 1.3 for the definition.
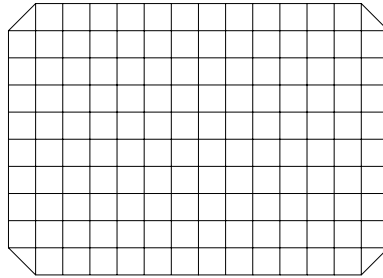
Fig. 16. A finite graph has many vertices with positive curvature, which is not prism-like.

Proof. Write $m = 2(a + b + 2)$ for some $a \geq 1, b \geq 1$. We construct a rectangle of side length a and b, consisting of $a \times b$ squares, and attach $2(a + b)$ squares along the sides of the rectangle and 4 triangles to the four corners. We obtain a convex domain as in Fig. 16 whose boundary consists of m edges. Now glue an m-gon along the boundary of the domain, we get a planar graph as desired. □

Proof of Theorem 1.5 (upper bound). Let G be a finite planar graph in $\mathcal{PC}_{\geq 0}$ with $D_G < 132$, which is not a prism-like graph. Note that our discharging method in the proof of Theorem 1.4 is local, one can distribute the curvature to bad vertices such that the modified curvature $\widetilde{\Phi} \geq \frac{1}{132}$ on T_G. Then by $\Phi(G) = 2$, one gets the upper bound 264. For the equality case, one can argue similarly as in the proof of Theorem 1.4. □

4. Constructions of large planar graphs with nonnegative curvature

In this section, we prove the lower bound estimates for the number of vertices in T_G for a planar graph G with nonnegative curvature in Theorem 1.4 and Theorem 1.5 by constructing examples.

In the first part of the section, we consider infinite planar graphs. As shown in the introduction, there is an infinite planar graph $G \in \mathcal{PC}_{\geq 0}$ which is not a prism-like graph with $\sharp T_G = 132$, see Fig. 1. By this example, we give the lower bound estimate of $K_{\mathbb{R}^2} \geq 132$ in Theorem 1.4. However, in a rigorous manner we need to show that the graph in Fig. 1 can be further extended up to infinity, so as to guarantee that it is a planar tessellation.

We cut off a central part of Fig. 1 and denote it by P_A, shown in Fig. 17. Then the annular part surrounding it in Fig. 1 is denoted by P_B, shown in Fig. 18, in which we have modified the sizes of faces to obtain a nice-looking annular domain.

We make infinitely many copies of P_B, denoted by $\{P_{B_i}\}_{i=1}^{\infty}$, and glue these pieces together with P_A as follows, see Fig. 19:

$$P_A \leftrightarrow P_{B_1} \leftrightarrow P_{B_2} \leftrightarrow P_{B_3} \leftrightarrow \cdots \leftrightarrow P_{B_i} \leftrightarrow \cdots \cdots ,$$
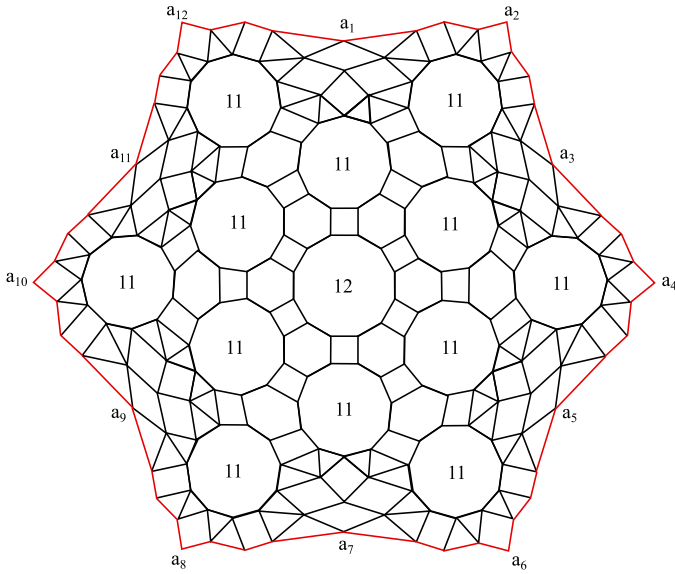
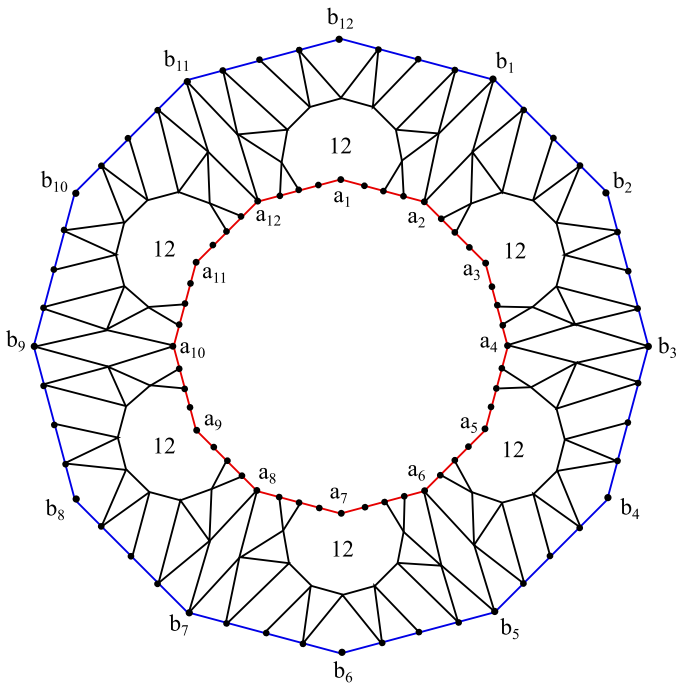**Fig. 17.** A central part in Fig. 1, denoted by P_A.

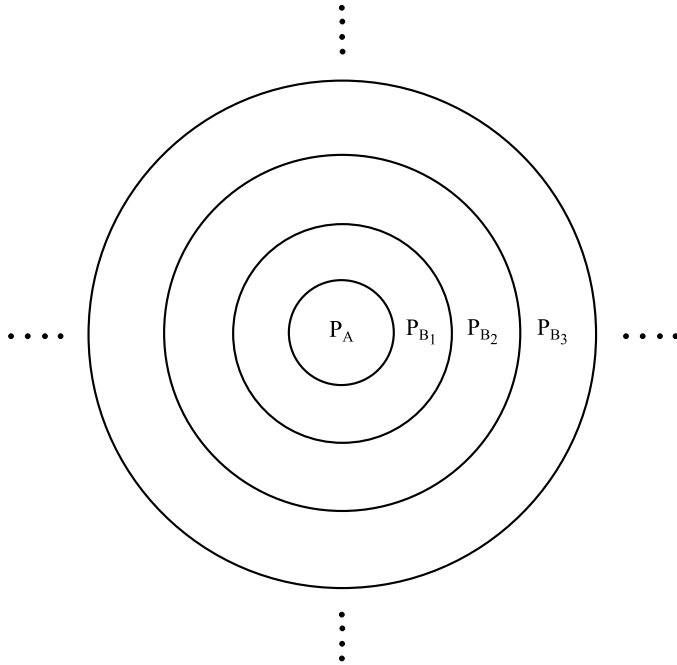**Fig. 18.** An annular part in Fig. 1, denoted by P_B.

Fig. 19. The gluing process for the construction of Fig. 1.

where we denote by $G_1 \leftrightarrow G_2$ the gluing of two graphs G_1 and G_2 along the boundaries following the rules:

(1) For $G_1 = P_A$ and $G_2 = P_{B_1}$, we glue the boundary in P_A with the innermost boundary of P_{B_1} by identifying the vertices $\{a_i\}_{i=1}^{12}$ in both graphs.

(2) For $G_1 = P_{B_i}$ and $G_2 = P_{B_{i+1}}$ with $i \geq 1$, we glue the outermost boundary in P_{B_i} with the innermost boundary of $P_{B_{i+1}}$ by identifying the vertices $\{b_i\}_{i=1}^{12}$ in P_{B_i} with the vertices $\{a_i\}_{i=1}^{12}$ in $P_{B_{i+1}}$.

This constructs the example in Fig. 1. One easily sees that this construction has a periodic structure, so that it extends to an infinite planar graph embedded into $\mathbb{R}^2$.

For the second part of the section, we consider finite planar graphs. We will construct a finite planar graph $G \in \mathcal{PC}_{\geq 0}$ with $\sharp T_G = 264$, which is not a prism-like graph and has $D_G = 12$. This will give the lower bound estimate of $\widetilde{K}_{\mathbb{S}^2} \geq 264$ in Theorem 1.5. We make two copies of P_A and P_B respectively, denote by P_{A_1}, P_{A_2}, P_{B_1} and P_{B_2}. We glue them together along the boundaries as follows

$$P_{A_1} \leftrightarrow P_{B_1} \leftrightarrow P_{B_2} \leftrightarrow P_{A_2},$$

where the boundary of P_{A_1} is glued with the innermost boundary of P_{B_1}, the outermost boundary of P_{B_1} is glued with the innermost boundary of P_{B_2} by identifying the vertices

$\{b_i\}_{i=1}^{12}$ in the former with the vertices $\{a_i\}_{i=1}^{12}$ in the latter, and the outermost boundary of P_{B_2} is glued with the boundary of P_{A_2} by identifying the vertices $\{b_i\}_{i=1}^{12}$ in the former with the vertices $\{a_i\}_{i=1}^{12}$ in the latter. This gives us an example of $\sharp T_G = 264$.

5. Automorphism groups of planar graphs with nonnegative curvature

In this section, we study automorphism groups of planar graphs with nonnegative curvature.

First, we introduce several definitions of isomorphisms on planar graphs.

Definition 5.1. Let $G_1 = (V_1, E_1, F_1)$ and $G_2 = (V_2, E_2, F_2)$ be two planar graphs.

(1) G_1 and G_2 are said to be *graph-isomorphic* if there is a graph isomorphism between (V_1, E_1) and (V_2, E_2), i.e. $R : V_1 \to V_2$ such that for any $v, w \in V$, $v \sim w$ if and only if $R(v) \sim R(w)$.
(2) G_1 and G_2 are said to be *cell-isomorphic* if there is a cellular isomorphism $H = (H_V, H_E, H_F)$ between (V_1, E_1, F_1) and (V_2, E_2, F_2) in the sense of cell complexes, i.e. three bijections $H_V : V_1 \to V_2$, $H_E : E_1 \to E_2$ and $H_F : F_1 \to F_2$ preserving the incidence relations, that is, for any $v \in V, e \in E, \sigma \in F$, $v \prec e$ if and only if $H_V(v) \prec H_E(e)$ and $e \prec \sigma$ if and only if $H_E(e) \prec H_F(\sigma)$;
(3) G_1 and G_2 are said to be *metric-isomorphic* if there is an isometric map in the sense of metric spaces $L : S(G_1) \to S(G_2)$, such that the restriction map L is cell-isomorphic between (V_1, E_1, F_1) and (V_2, E_2, F_2).

One is ready to see that metric-isomorphic planar graphs are cell-isomorphic, and hence graph-isomorphic. For a planar graph G, a graph (cellular, metric resp.) isomorphism between G and G, i.e. setting $G_1 = G_2 = G$ in the above definitions, is called a graph (cellular, metric resp.) automorphism of G. We denote by $\mathrm{Aut}(G)$, $(\widetilde{\mathrm{Aut}}(G), \mathcal{L}(G)$ resp.) the group of graph (cellular, metric resp.) automorphisms of a planar graph G. By the standard identification,

$$\mathcal{L}(G) \leq \widetilde{\mathrm{Aut}}(G) \leq \mathrm{Aut}(G),$$

where $\leq$ indicates that the former can be embedded as a subgroup of the latter. By our definition of polyhedral surfaces, it is easy to see that

$$\mathcal{L}(G) \cong \widetilde{\mathrm{Aut}}(G).$$

Moreover, by the results in [33,46] for a 3-connected planar graph G, any graph automorphism R of G can be uniquely realized as a cellular automorphism H such that $H_V = R$, which is called the associated cellular automorphism of R. This implies that

$$\mathcal{L}(G) \cong \widetilde{\mathrm{Aut}}(G) \cong \mathrm{Aut}(G).$$

For any $G \in \mathcal{PC}_{\geq 0}$, let $H = (H_V, H_E, H_F)$ be a cellular automorphism of G. Since for any vertex $v \in V$, $\Phi(v) = \Phi(H_V(v))$, $H_V : T_G \to T_G$. This yields a group homomorphism,

$$\rho : \widetilde{\mathrm{Aut}}(G) \to S_{T_G}, \;\; H \mapsto H_V|_{T_G} \tag{6}$$

where S_{T_G} is the permutation group on T_G and $H_V|_{T_G}$ is the restriction of H_V to T_G. The kernel of the homomorphism ρ, denoted by $\ker \rho$, consists of $H \in \widetilde{\mathrm{Aut}}(G)$ such that $H_V|_{T_G}$ is the identity map on T_G. By the group isomorphism theorem,

$$\widetilde{\mathrm{Aut}}(G)/\ker \rho \cong \mathrm{im}(\rho) \leq S_{T_G},$$

where $\mathrm{im}(\rho)$ denotes the image of the map ρ in S_{T_G}. To show that $\widetilde{\mathrm{Aut}}(G)$ is a finite group, it suffices to prove that $\ker \rho$ is finite.

We need some basic properties of cellular automorphisms of planar graphs. Recall that for $\sigma_1, \sigma_2 \in F$, we denote by $\sigma_1 \sim \sigma_2$ if there is an edge e such that $e \prec \sigma_1$ and $e \prec \sigma_2$. By the definition of planar tessellation in the introduction, the edge e satisfying the above property is unique. For any face $\sigma \in F$, we write the *vertex boundary* of σ as

$$\partial \sigma = \{v_1, v_2, \cdots, v_{\deg(\sigma)}\}$$

such that $v_i \sim v_{i+1}$ for any $1 \leq i \leq \deg(\sigma)$ by setting $v_{\deg(\sigma)+1} = v_1$. For a cellular automorphism H of a planar graph G, we say that H *fixes a face* σ if $H_V(v) = v$, $H_E(e) = e$ and $H_F(\sigma) = \sigma$ for any $v \in \partial \sigma, e \prec \sigma, e \in E$. It is easy to check that H fixes σ if and only if $H_V(v) = v$ for any $v \in \partial \sigma$. The following lemma is useful.

Lemma 5.2. *Let $G = (V, E, F)$ be a planar graph and $H \in \widetilde{\mathrm{Aut}}(G)$. Suppose that there are a face σ and $\{v_1, v_2\} \subset \partial \sigma$ with $v_1 \sim v_2$ satisfying*

$$H_F(\sigma) = \sigma, H_V(v_1) = v_1, H_V(v_2) = v_2,$$

then H fixes the face σ, i.e. $H_V(v) = v$ for any $v \in \partial \sigma$. Moreover, H is the identity map on G.

Proof. For the first assertion, by $H_F(\sigma) = \sigma$ we have $H_V(\partial \sigma) = \partial \sigma$. We write $\partial \sigma = \{v_1, v_2, \cdots, v_{\deg(\sigma)}\}$ such that $v_i \sim v_{i+1}$ for any $1 \leq i \leq \deg(\sigma)$ by setting $v_{\deg(\sigma)+1} = v_1$. Noting that $H_V(v_1) = v_1$ and $H_V(v_2) = v_2$, we get that $H_V(v_3) = v_3$ since the cellular automorphism H preserves the incidence structure. The result follows from the induction argument on v_i for $i \geq 3$.

For the second assertion, it suffices to show that H fixes any face $\tilde{\sigma} \in F$. Since the dual graph of the planar graph G is connected, there is a sequence of faces $\{\sigma_j\}_{j=0}^M \subset F$, $M \geq 1$, such that

$$\sigma = \sigma_0 \sim \sigma_1 \sim \cdots \sim \sigma_M = \tilde{\sigma}.$$

By the above result, H fixes the face $\sigma_0 = \sigma$. Let $\{v, w\} = \partial\sigma_0 \cap \partial\sigma_1$. Noting that $H_F(\sigma_0) = \sigma_0, H_V(v) = v$ and $H_V(w) = w$, we have $H_F(\sigma_1) = \sigma_1$ by the incidence preserving property of H. Then one applies the first assertion to σ_1, and yields that H fixes σ_1. Similarly, the induction argument implies that H fixes σ_j for any $0 \le j \le M$. Hence H fixes $\tilde{\sigma}$ which proves the second assertion. $\quad \square$

For any $v \in T_G$, we denote by $N(v)$ the set of neighbors of v and by $S_{N(v)}$ the permutation group on $N(v)$. For any $H \in \ker\rho$, it is easy to see that $H_V : N(v) \to N(v)$ since $H_V(v) = v$. This induces a group homomorphism

$$\pi : \ker\rho \to S_{N(v)}, \ H \mapsto H_V|_{N(v)}. \tag{7}$$

Lemma 5.3. *Let $G = (V, E, F)$ be a planar graph with nonnegative combinatorial curvature and positive total curvature. For any $v \in T_G$, the map π, defined as in* (7)*, is a group monomorphism.*

Proof. It suffices to show that the map π is injective. For any $H \in \ker\pi$, $H_V(w) = w$, for any $w \in N(v) \cup \{v\}$. Let $w_1, w_2 \in N(v)$ such that there is a face σ such that $\{w_1, w_2, v\} \subset \partial\sigma$. By the incidence preserving property of H, $H_F(\sigma) = \sigma$. Then Lemma 5.2 yields that H is the identity on G. This implies that $\ker\pi$ is trivial and π is injective. $\quad \square$

Now we are ready to estimate the order of the automorphism group of a planar graph with nonnegative curvature and positive total curvature.

Theorem 5.4. *Let $G = (V, E, F)$ be a planar graph with nonnegative combinatorial curvature and positive total curvature. Then the automorphism group of G is finite. Set $Q(G) := \sharp\widetilde{\mathrm{Aut}}(G)$, $a := \sharp T_G$ and $b := \max_{v \in T_G} \deg(v)$. We have the following:*

(1) *If $D_G \le 42$,*

$$Q(G) \mid a!b!.$$

(2) *If $D_G > 42$, then*

$$Q(G) \text{ divides } \begin{cases} 2D_G, & G \text{ is infinite}, \\ 4D_G, & G \text{ is finite}. \end{cases}$$

Proof. Suppose that $D_G \le 42$, then by Lemma 5.2,

$$\sharp(\widetilde{\mathrm{Aut}}(G)/\ker\rho) \mid a!,$$

where ρ is defined in (6). By Lemma 5.3,

$$\sharp \ker \rho \mid b!.$$

This yields the result.

Suppose that $D_G > 42$, then the planar graph G has some special structure. For simplicity, we write $m := D_G$. We divide it into two cases:

Case 1.: G is infinite. In this case, G is a prism-like graph. By Theorem 2.3, there is only one face σ with $\deg(\sigma) = m > 42$, and $T_G = \partial\sigma$. For any $H \in \ker\rho$, $H_F(\sigma) = \sigma$ and $H_V(v) = v$ for any $v \in T_G$, so that by Lemma 5.2, H is the identity map on G. This yields that

$$\widetilde{\mathrm{Aut}}(G) \cong \mathrm{im}(\rho) \leq S_{T_G}.$$

For any $H \in \widetilde{\mathrm{Aut}}(G)$, $\rho(H) \in \mathrm{im}(\rho)$ induces a graph isomorphism on the cycle graph C_m, which is given by the dihedral group D_m. That yields that

$$\mathrm{im}(\rho) \leq D_m \text{ and } Q(G)|2m.$$

Case 2.: G is finite. We have two subcases.

    Subcase 2.1.: Suppose that $\sharp\{\sigma \in F : \deg(\sigma) > 42\} \geq 2$, then G is a finite prism-like graph. By Theorem 2.4, there are exactly two faces σ_1 and σ_2, which are disjoint, of same degree $m > 42$, and $T_G = \partial\sigma_1 \cup \partial\sigma_2$. For any $H \in \widetilde{\mathrm{Aut}}(G)$, $H_F : \{\sigma_1, \sigma_2\} \to \{\sigma_1, \sigma_2\}$, which yields a group homomorphism

$$\widetilde{\rho} : \widetilde{\mathrm{Aut}}(G) \to S_{\{\sigma_1, \sigma_2\}}, \ H \mapsto H_F|_{\{\sigma_1, \sigma_2\}},$$

where $S_{\{\sigma_1, \sigma_2\}}$ is the permutation group on $\{\sigma_1, \sigma_2\}$. This implies that

$$\sharp(\widetilde{\mathrm{Aut}}(G)/\ker\widetilde{\rho}) = 1 \text{ or } 2. \tag{8}$$

For any $H \in \ker\widetilde{\rho}$, $H_F(\sigma_i) = \sigma_i$ for $i = 1, 2$. Hence $H_V : \partial\sigma_1 \to \partial\sigma_1$. This yields a group homomorphism

$$\eta : \ker\widetilde{\rho} \to S_{\partial\sigma_1}, H \mapsto H_V|_{\partial\sigma_1},$$

where $S_{\partial\sigma_1}$ is the permutation group on $\partial\sigma_1$. The same argument as in Case 1 implies that $\ker\eta$ is trivial and $\mathrm{im}(\eta) \leq D_m$. Hence

$$\sharp\ker\widetilde{\rho} \mid 2m.$$

By combining it with (8) we have

$$Q(G)|4m.$$

Subcase 2.2.: Suppose that $\sharp\{\sigma \in F : \deg(\sigma) > 42\} = 1$, then there is only one face $\sigma \in F$ such that $\deg(\sigma) = m$. The same argument as in Case 1 yields that

$$Q(G)|2m.$$

Combining all cases above, we prove the theorem. □

By combining the estimates of the size of T_G in Theorem 1.4 and Theorem 1.5 with the above result, we obtain the estimates for the orders of cellular automorphism groups.

Proof of Theorem 1.6. For any $G \in \mathcal{PC}_{\geq 0}$ with $D_G \leq 42$, we obtain that $\sharp T_G \leq 132$ if G is infinite by Theorem 1.4, and $\sharp T_G \leq 264$ if G is finite by Theorem 1.4. Then the theorem follows from Theorem 5.4. □

Acknowledgments

References

[1] K. Appel, W. Haken, Every planar map is four colorable. I. Discharging, Illinois J. Math. 21 (3) (1977) 429–490.
[2] A.D. Alexandrov, Convex Polyhedra, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005, translated from the 1950 Russian edition by N.S. Dairbekov, S.S. Kutateladze and A.B. Sossinsky, with comments and bibliography by V.A. Zalgaller and appendices by L.A. Shor and Yu.A. Volkov.
[3] L. Babai, Automorphism groups of planar graphs. II, Colloq. Math. Soc. János Bolyai 10 (1975) 29–84.
[4] D. Burago, Yu. Burago, S. Ivanov, A Course in Metric Geometry, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.
[5] G. Brinkmann, A.W.M. Dress, A constructive enumeration of fullerenes, J. Algorithms 23 (2) (1997) 345–358.
[6] V. Buchstaber, N. Erokhovets, Finite sets of operations sufficient to construct any fullerene from C_{20}, Struct. Chem. 28 (1) (2017) 225–234.
[7] V.M. Buchstaber, N.Yu. Erokhovets, Constructions of families of three-dimensional polytopes, characteristic patches of fullerenes, and Pogorelov polytopes, Izv. Ross. Akad. Nauk Ser. Mat. 81 (5) (2017) 15–91.
[8] G. Brinkmann, J. Goedgebeur, B.D. McKay, The generation of fullerenes, J. Chem. Inf. Model. 52 (11) (2012) 2910–2918.
[9] O. Baues, N. Peyerimhoff, Curvature and geometry of tessellating plane graphs, Discrete Comput. Geom. 25 (1) (2001) 141–159.
[10] O. Baues, N. Peyerimhoff, Geodesics in non-positively curved plane tessellations, Adv. Geom. 6 (2) (2006) 243–263.
[11] J.H. Conway, H. Burgiel, C. Goodman-Strauss, The Symmetries of Things, A K Peters, Ltd., Wellesley, MA, 2008.

[12] B. Chen, G. Chen, Gauss–Bonnet formula, finiteness condition, and characterizations of graphs embedded in surfaces, Graphs Combin. 24 (3) (2008) 159–183.

[13] B. Chen, The Gauss–Bonnet formula of polytopal manifolds and the characterization of embedded graphs with nonnegative curvature, Proc. Amer. Math. Soc. 137 (5) (2009) 1601–1611.

[14] M. DeVos, B. Mohar, An analogue of the Descartes–Euler formula for infinite graphs and Higuchi's conjecture, Trans. Amer. Math. Soc. 359 (7) (2007) 3287–3300.

[15] B. Galebach, n-Uniform tilings, available online at http://probabilitysports.com/tilings.html, 2009.

[16] L. Ghidelli, On the largest planar graphs with everywhere positive combinatorial curvature, arXiv: 1708.08502, 2017.

[17] M. Gromov, Hyperbolic groups, in: Essays in Group Theory, in: Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[18] B. Grünbaum, G.C. Shephard, Tilings and Patterns, A Series of Books in the Mathematical Sciences, W. H. Freeman and Company, New York, 1989.

[19] Y. Higuchi, Combinatorial curvature for planar graphs, J. Graph Theory 38 (4) (2001) 220–229.

[20] O. Häggström, J. Jonasson, R. Lyons, Explicit isoperimetric constants and phase transitions in the random-cluster model, Ann. Probab. 30 (1) (2002) 443–473.

[21] B. Hua, J. Jost, S. Liu, Geometric analysis aspects of infinite semiplanar graphs with nonnegative curvature, J. Reine Angew. Math. 700 (2015) 1–36.

[22] B. Hua, Y. Lin, Curvature notions on graphs, Front. Math. China 11 (5) (2016) 1275–1290.

[23] Y. Higuchi, T. Shirai, Isoperimetric constants of (d, f)-regular planar graphs, Interdiscip. Inform. Sci. 9 (2) (2003) 221–228.

[24] B. Hua, Y. Su, Total curvature of planar graphs with nonnegative curvature, arXiv:1703.04119, 2017.

[25] W. Imrich, On Whitney's theorem on the unique embeddability of 3-connected planar graphs, in: Recent Advances in Graph Theory, 1975, pp. 303–306 (loose errata).

[26] M. Ishida, Pseudo-curvature of a graph, in: Lecture at Workshop on Topological Graph Theory, Yokohama National University, 1990.

[27] M. Keller, The essential spectrum of the Laplacian on rapidly branching tessellations, Math. Ann. 346 (1) (2010) 51–66.

[28] M. Keller, Curvature, geometry and spectral properties of planar graphs, Discrete Comput. Geom. 46 (3) (2011) 500–525.

[29] H.W. Kroto, J.R. Heath, S.C. O'Brien, R.F. Curl, R.E. Smalley, C_{60}: buckminsterfullerene Nature 318 (1985) 162–163.

[30] M. Keller, N. Peyerimhoff, Cheeger constants, growth and spectrum of locally tessellating planar graphs, Math. Z. 268 (3–4) (2011) 871–886.

[31] S. Lawrencenko, M.D. Plummer, X. Zha, Isoperimetric constants of infinite plane graphs, Discrete Comput. Geom. 28 (3) (2002) 313–330.

[32] P. Mani, Automorphismen von polyedrischen Graphen, Math. Ann. 192 (1971) 279–303.

[33] B. Mohar, Embeddings of infinite graphs, J. Combin. Theory Ser. B 44 (1) (1988) 29–43.

[34] B. Mohar, Light structures in infinite planar graphs without the strong isoperimetric property, Trans. Amer. Math. Soc. 354 (8) (2002) 3059–3074.

[35] R. Nevanlinna, Analytic Functions, translated from the second German edition by Phillip Emig, Die Grundlehren der mathematischen Wissenschaften, vol. 162, Springer-Verlag, New York–Berlin, 1970.

[36] R. Nicholson, J. Sneddon, New graphs with thinly spread positive combinatorial curvature, New Zealand J. Math. 41 (2011) 39–43.

[37] B.-G. Oh, On the number of vertices of positively curved planar graphs, Discrete Math. 340 (6) (2017) 1300–1310.

[38] P.R. Oldridge, Characterizing the polyhedral graphs with positive combinatorial curvature, thesis, available at https://dspace.library.uvic.ca/handle/1828/8030, 2017.

[39] T. Réti, E. Bitay, Z. Kosztolányi, On the polyhedral graphs with positive combinatorial curvature, Acta Polytech. Hung. 2 (2) (2005) 19–37.

[40] N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-colour theorem, J. Combin. Theory Ser. B 70 (1) (1997) 2–44.

[41] B. Servatius, H. Servatius, Symmetry, automorphisms, and self-duality of infinite planar graphs and tilings, in: International Scientific Conference on Mathematics. Proceedings, Žilina, 1998, Univ. Žilina, Žilina, 1998, pp. 83–116.

[42] D.A. Stone, A combinatorial analogue of a theorem of Myers, Illinois J. Math. 20 (1) (1976) 12–21.

[43] L. Sun, X. Yu, Positively curved cubic plane graphs are finite, J. Graph Theory 47 (4) (2004) 241–274.
[44] C. Thomassen, Duality of infinite graphs, J. Combin. Theory Ser. B 33 (2) (1982) 137–160.
[45] W.P. Thurston, Shapes of polyhedra and triangulations of the sphere, in: The Epstein Birthday Schrift, in: Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 511–549.
[46] H. Whitney, 2-isomorphic graphs, Amer. J. Math. 55 (1–4) (1933) 245–254.
[47] W. Woess, A note on tilings and strong isoperimetric inequality, Math. Proc. Cambridge Philos. Soc. 124 (1998) 385–393.
[48] A. Żuk, On the norms of the random walks on planar graphs, Ann. Inst. Fourier (Grenoble) 47 (5) (1997) 1463–1490.
[49] L. Zhang, A result on combinatorial curvature for embedded graphs on a surface, Discrete Math. 308 (24) (2008) 6588–6595.