A Study of Domino And Its Applications

多米诺骨牌模型的研究及其应用

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Abstract

The falling-down of dominoes is a process where gravitational energy converts to kinetic energy. Dominoes fall at an increasing speed as more gravitational energy converts. We derive the recurrence relations between angles, energy and time when an ideal domino queue (dominoes with negligible thickness and friction) collapse forwards, and analyze when energy loss and thickness of dominoes are inevitable.

Finally, approximation is introduced to allow fast computation.

It turns out that many positive feedback systems fall under this model. Specifically, we give application to:

1. Connection between CO_2 and global warming.

2. Domino effect when companies go bankrupt during financial crisis.

Keywords： Domino, velocity, approximation, global warming, financial crisis

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Chapter I Introduction

Domino has been a popular game worldwide ever since it was invented, for it strengthens the players' mind and helps to develop their creativity. Even though countless people have done some researches on domino, they only paid their attention on increasing the numbers of dominoes and innovating different ways to arrange the dominoes. However, necessary researches on the angular velocity and the energy relations of dominoes when they fall down are deficient.

The falling-down process of dominoes is a process where gravitational energy converts to kinetic energy. As the dominoes fall, gravitational energy releases and kinetic energy accumulates. As a result, the angular velocity of the dominoes increases rapidly.

In practice, the angular velocity of the falling dominoes has a limit due to loss of energy when one domino hits another. When the energy loss is equivalent to the energy released (so called equilibrium), the angular velocity becomes stable.

To give a mathematical model of domino we established: I. the recurrence relations between angles; II. time for one domino to hit the next one (t); III. time for N dominoes in a line to fall down (T_N); IV. the limit of angular velocity when loss of energy is taken into account by introducing a parameter Q to measure the loss of energy.

The extreme complexity of accurate function resulted in the computation disability of computers and the unpracticability of the model. To solve the problems, we introduced reasonable simplification and approximation to the model.

The accumulation of CO_2 shares similarity with the accumulation of t (time for one domino to hit the next) to T_N (time for N dominoes to fall down) in a domino line. The model contributes to the prediction of relationship between concentration of CO_2 in atmosphere and global mean temperature.

The model also has its application in economics. Domino effect takes place when an essential company bankrupts and a great many follow, resulting in a financial crisis. We studied the case of the great depression of 1929 in the US because the market at that time was acting freely without the interference from the government. The application is introduced to measure the graveness of a financial crisis and to provide suggestion to governments and financial institutions when enacting the fiscal and monetary policies.

Chapter II Modeling and Analysis

2.1 Preliminary & Notation

2.1.1 Lemmas

Lemma I

$$\text{When} \quad x \to 0,\ \sin x \sim x,\ \cos x \sim 1,\ \tan x \sim x$$

$$\text{When} \quad x \to 0,\ (1+x)^a \approx 1 + ax + \frac{a(a-1)}{2}x^2$$

Lemma II

The kinetic energy of a pole of weight m, length 2h, rotating at angular velocity ω is

$$E = \frac{2}{3} m\omega^2 h^2$$

Proof:

$$E = {}^1/_2\, m \left(\frac{ds}{dt}\right)^2$$

$$dE = {}^1/_2 \frac{dx}{2h} m(\omega h)^2$$

$$E = \int_0^{2h} {}^1/_2 \frac{m(\omega x)^2}{2h} dh = {}^1/_6\, m\omega^2 4h^2 = \frac{2}{3} m\omega^2 h^2$$

Lemma III

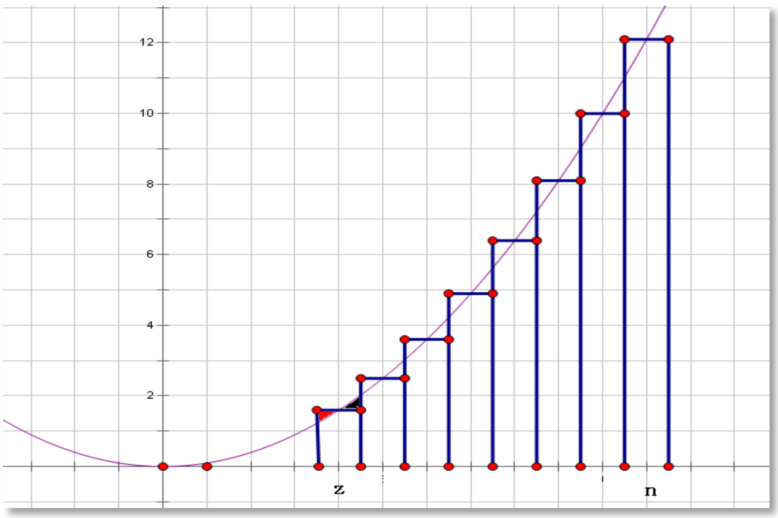

Figure 2.1.1

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According to the midpoint rule, it is easy to get

$$\sum_{i=z}^{n} f(i) \approx \int_{z-0.5}^{n+0.5} f(x)\,dx$$

Proof:

Take the graph of the function in Figure 2.1.1 as an example. We translate the y-value of ***z*** (an integer) half unit to the left and half unit to the right then draw a rectangle with the y-value of ***z*** as the length and 1 unit as the width.

As can be seen in figure 2.1.1, the area of the red (the redundant area) one and the black (the vacant area) are almost the same. Thus, the integration of the function can be expressed as the area of the sum of the rectangles:

$$\sum_{i=z}^{n} f(i) = f(z) + f(z+1) + f(z+2) \ldots \ldots + f(n)$$

$$= 1 * f(z) + 1 * f(z+1) + 1 * f(z+2) \ldots \ldots + 1 * f(n)$$

Obviously, the straighter the curve is, the more accurate the approximation is.

The equation above can be regarded as the area of the trapezium (circled by x=z-0.5, x=n+0.5, f(x)). Thus,

$$\sum_{i=z}^{n} f(i) \approx \int_{z-0.5}^{n+0.5} f(x)\,dx$$

2.1.2 Notations and Assumptions

As seen in Figure 2.1.2, dominoes are regarded as poles with negligible thickness and friction. Dominoes rotate around its intersecting point with the earth.

Energy loss during collision is ignored in Chapter 2.2 to 2.3 for simplicity and will be discussed in Chapter 2.4

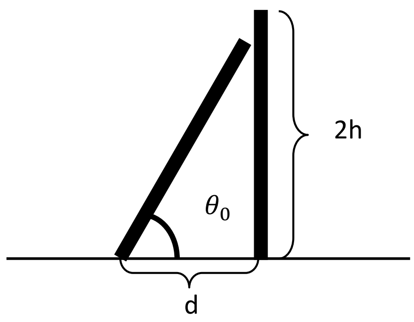

Figure 2.1.2

I. Each domino has length **2h,** thus its centroid is at the height of **h** when the domino is upright;

II. The gap between every two dominoes is ***d;***

III. When a domino falls down and hits the next piece, its angle against the ground is θ_0;

IV. Let **k**=$\frac{d}{2h}$=$\cos\theta_0$, thus $\theta_0 = \arccos(k)$

V. E_N be the total kinetic energy of N dominoes in line

VI. When taking the loss of energy in collision into account, each collision preserves *Q* of the original kinetic energy;

2.1.3 Setting up coordinate system and numbering dominoes

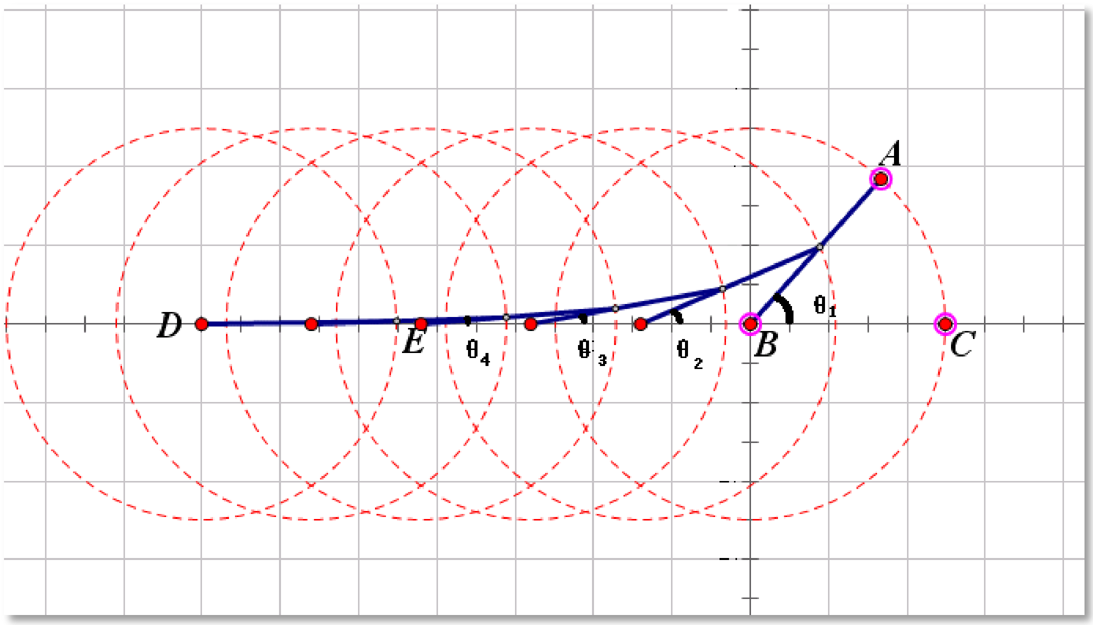

Figure 2.1.3

Figure 2.1.3 represents a moment of the falling-down process of dominoes.

It is obvious according to the figure that all the dominoes would not fall down to the ground, and each domino has kinetic energy. However, some posterior dominoes have very small angles against the ground that can be neglected.

At this very moment, shown in Figure 2.1.3, the rightmost domino is numbered the 1st domino (N=1), from right to left the dominoes are numbered the 2nd, the 3rd (N=2, 3…) and so on. If N dominoes have non-zero kinetic energy at this moment, then the one on the very left should be numbered as the N^{th}.

The motion of the top of a domino is a circle with a radius of its length (2h) centered at the bottom of the domino. Hence the position of the top the $\boldsymbol{N^{th}}$ domino is determined by that of the $\boldsymbol{(N\text{-}1)^{th}}$ domino: the circle of one domino ($\boldsymbol{n^{th}}$) and the prior domino ($\boldsymbol{n\text{-}1}$)$^{\boldsymbol{th}}$ intersect at one point, and the straight line between the intersection point and the rotating center of the nth domino is where the n^{th} domino is.

We set up the coordinate system as illustrated in Figure 2.1.3: rotating center of the 1^{st} domino as the origin, the direction for the dominoes to fall down as the positive direction of x-axis (Dominoes are supposed to fall down from left to right in the following deduction).

Denote that $\boldsymbol{\theta_n}$ be the angle between the ground and the $\boldsymbol{n^{th}}$ domino, and ***On***, be the circle describing the motion of the top of the domino.

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2.2 Solutions to the model

2.2.1 Relations between angles

For the ***n***th domino, the equation of O_n $(1 \le n \le N-1)$ is

$$[x+(n-1)d]^2+y^2=(2h)^2$$

In its parameter form

$$\begin{cases} x = \cos\theta_n \cdot (2h) - (n-1)d \\ y = \sin\theta_n \cdot (2h) \end{cases}$$

The ***(n-1)***th domino is in the position of the straight line:

$$\begin{cases} x = t - (n-2)d \\ y = t \cdot \tan\theta_{n-1} \end{cases}$$

Solve the simultaneous parameter equations, we get

$$\tan\theta_{n-1} = \frac{\sin\theta_n \cdot (2h)}{\cos\theta_n \cdot (2h) - d}$$

Let k=$\frac{d}{2h}$, the equation above can be simplified to

$$\tan\theta_{n-1} = \frac{\sin\theta_n}{\cos\theta_n - k} \qquad \dots\dots(1)$$

Thus,

$$(\tan^2\theta_{n-1}+1)\cos^2\theta_n - 2k\tan^2\theta_{n-1}\cdot\cos\theta_n + k^2\cdot\tan^2\theta_{n-1} - 1 = 0$$

Using extraction of root to solve, the solution is:

$$\cos\theta_n = \frac{k\tan^2\theta_{n-1} + \sqrt{\tan^2\theta_{n-1} - k^2\tan^2\theta_{n-1} + 1}}{\tan^2\theta_{n-1} + 1}$$

Also because

$$\tan\theta_n = \sqrt{\frac{1-\cos^2\theta_n}{\cos^2\theta_n}}$$

Recurrence relations for angles is

$$\tan\theta_{n+1} = \frac{\tan\theta_n \cdot \sqrt{(1-k^2)\tan^2\theta_n + 1 + k^2 - 2k\sqrt{(1-k^2)\tan^2\theta_n + 1}}}{k\cdot\tan^2\theta_n + \sqrt{(1-k^2)\tan^2\theta_n + 1}} \qquad \dots\dots(2)$$

The equation above is very complicated. When n is increasing, equation (1) can be simplified as

$$\theta_{n-1} = \frac{\theta_n}{1-k}$$

It is obvious that the equation above is a geometric progression, hence

$$\theta_{n+1} \approx \theta_1 (1-k)^n \qquad \ldots\ldots(3)$$

To examine the precision of the approximation, we used Matlab to calculate the values and used Excel to plot the function.

Series I&II are graphs drawn with k=0.6, $\theta = \arccos(0.6)$; series I is the graph of the accurate equation (2), series II is the graph of the simplified equation(3)

Series III&IV are graphs drawn with k=0.8, $\theta = \arccos(0.8)$; series III is the graph of the accurate equation (2), series IV is the graph of the simplified equation(3).

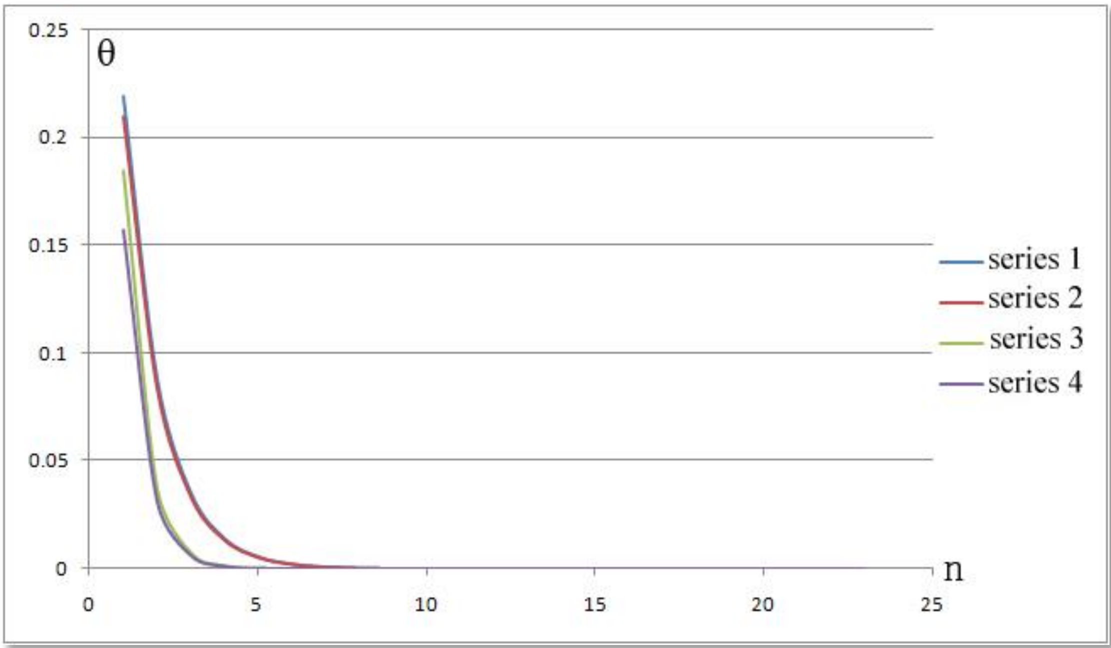

Figure 2.2.1

As can be seen in the figure, the approximation is highly precise. To simplify the following deduction and to enhance the practicability of the model, the formula (3) will be adopted to the recurrence relations between angles.

2.2.2 Analysis of Energy and Time

For a moment during the falling-down process of dominoes (assume **N** pieces are rotating at this moment), the sum of the kinetic energy of the domino line (**E_N**) is equivalent to the gravitational energy released:

$$E_N = nmgh - \sum_{i=1}^{N} mgh \cdot \sin\theta_i \qquad \ldots\ldots(4)$$

According to lemma II, the kinetic energy can be formulated with another formula:

$$E_N = \frac{2}{3}mh^2 \sum_{i=1}^{N} \omega_i{}^2$$

In 2.2.1 we have already known that the recurrence relations between angles (formula (3)) follow the geometric progression, thus the relation between pal stances also follows the geometric progression, and the common ratio is also (1-k)

$$\omega_{n+1} \approx \omega_1(1-k)^n$$

Thus,

$$E_N = \frac{2}{3}mh^2 \sum_{i=1}^{N} \omega_i{}^2 = \frac{2}{3}mh^2 \cdot \frac{\omega_1{}^2[1-(1-k)^{2N}]}{2k-k^2}$$

The two equations related to E_N yield,

$$\left(N - \sum_{i=1}^{N} \sin\theta_i\right) mgh = \frac{2}{3}mh^2 \cdot \frac{\omega_1{}^2[1-(1-k)^{2N}]}{2k-k^2}$$

Thus,

$$\omega_1 = \sqrt{\frac{3g}{2h}\left(N - \sum_{i=1}^{N} \sin\theta_i\right)\left(\frac{2k-k^2}{1-(1-k)^{2N}}\right)} \qquad \ldots\ldots(5)$$

It can be expressed in another expression

$$\frac{1}{\sqrt{\frac{3g}{2h}\left(N - \sum_{i=1}^{N} \sin\theta_i\right)\left(\frac{2k-k^2}{1-(1-k)^{2N}}\right)}} d\theta_1 = dt$$

We already know the rightmost domino has an angle $\theta_0 = \arccos\left(\frac{d}{2h}\right)$ with the ground when it hits the next domino, thus the integration to θ_1 is the time for each domino to hit the next one (t).

$$t = \sqrt{\frac{1-(1-k)^{2N}}{2k-k^2}} \sqrt{\frac{2h}{3g}} \int_{\theta_0}^{0.5\pi} \left(N - \sum_{i=1}^{N} \sin\theta_i\right)^{-0.5} d\theta_1 \qquad \ldots\ldots(6)$$

So the time for N dominoes to fall down (T_N) is

$$T_N = \sum t = \sqrt{\frac{2h}{3g}} \cdot \sum_{i=1}^{N} \sqrt{\frac{1-(1-k)^{2N}}{2k-k^2}} \int_{\theta_0}^{0.5\pi} \left(N - \sum_{i=1}^{N} \sin\theta_i \right)^{-0.5} d\theta_1 \qquad \ldots\ldots(7)$$

2.3 Simplification and Approximation to the model

From the previous deduction we can see that the falling-down process of dominoes is very complicated and the formulas we got were inconvenient to be used. If the formulas were not simplified, it would take a long time for personal computer to operate. Moreover, the summing function is unfittable. Thus, simplification and approximation to the formula is necessary. It was not after the simplifications were we able to put the model into application.

2.3.1 Approximation to Energy

From formula (5) we know that when n is very big, the first several dominoes have insignificant influence on the kinetic energy of the first domino.

Thus, we divide the domino line into the posterior ***p*** dominoes and prior ***N-p*** dominoes.

If the p posterior dominoes have very small angle with the ground ($\theta \to 0$), the completely released gravitational energy of the p dominoes, according to Lemma I, is:

$$E_{\text{posterior } \boldsymbol{p} \text{ dominoes}} \approx \text{pmgh} - mgh \sum_{i=\text{N}-\text{p}+1}^{N} \theta_i$$

According to formula (3)

$$\theta_{n+1} \approx \theta_1 (1-k)^n$$

Using summation formula of geometric progression, released gravitational energy of the posterior ***p*** dominoes is,

$$E_{\text{posterior p dominoes}} \approx \left(p - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} \right) \text{mgh}$$

Because the angles of the prior ***(n-p)*** cannot be neglected, the gravitational energy they released is calculated in accordance to the formula (4), that

$$E_{\text{prior N}-\text{p dominoes}} = \left[(N-p) - \sum_{i=1}^{N-p} \sin\theta_i \right] \text{mgh}$$

Thus, after the division of the domino line, the sum of the kinetic energy of each domino is

$$\mathrm{E}_N = \mathrm{E}_{\text{posterior } \boldsymbol{p} \text{ dominoes}} + \mathrm{E}_{\text{prior } \boldsymbol{N-p} \text{ dominoes}} \approx \left[\mathrm{N} - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} - \sum_{i=1}^{N-p} \sin\theta_i\right] mgh$$

In conclusion, the sum of the kinetic energy of all the dominoes is simplified as

$$\mathrm{E}_N = \left(\mathrm{N} - \sum_{i=1}^{N} \sin\theta_i\right) \mathrm{mgh} \approx \left[\mathrm{N} - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} - \sum_{i=1}^{N-p} \sin\theta_i\right] mgh \quad \ldots\ldots(\mathbf{8})$$

When ***p*** increases, the simplified equation above would be more accurate compared with the original formula. When p=0, simplified equation equals to the original equation.

2.3.2 Approximation to *t*

According to equation **(8)** and the deduction above, when **N** is big enough, the simultaneous equations about $\mathbf{E}_{\boldsymbol{N}}$ is

$$\left[\mathrm{N} - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} - \sum_{i=1}^{N-p} \sin\theta_i\right] mgh \approx \frac{2}{3} mh^2 \cdot \frac{\omega_1{}^2[1-(1-\mathrm{k})^{2\mathrm{N}}]}{2\mathrm{k}-\mathrm{k}^2}$$

In this occasion, the solution of ω_1 is

$$\Rightarrow \omega_1 = \sqrt{\frac{3\mathrm{g}}{2h}\left(\frac{2\mathrm{k}-\mathrm{k}^2}{1-(1-\mathrm{k})^{2\mathrm{N}}}\right)\left(\mathrm{N} - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} - \sum_{i=1}^{N-p} \sin\theta_i\right)}$$

Thus,

$$t \approx \sqrt{\frac{2\mathrm{h}(1+(1-\mathrm{k})^{\mathrm{N}})}{3\mathrm{g}[(1-\mathrm{k})^{\mathrm{N}}-1](\mathrm{k}^2-2\mathrm{k})}} \int_{\theta_0}^{0.5\pi} \left[\mathrm{N} - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} - \sum_{i=1}^{N-p} \sin\theta_i\right]^{-0.5} d\theta_1$$

When **N** is very big, prior several dominoes also have insignificant influence on the kinetic energy of the first domino, thus let ***p=N*** in the equation above.

We are able to get the integration of this equation:

$$\mathrm{t} = -2k\sqrt{\frac{2\mathrm{h}(1+(1-\mathrm{k})^{\mathrm{N}})}{3\mathrm{g}[(1-\mathrm{k})^{\mathrm{N}}-1](\mathrm{k}^2-2\mathrm{k})}}\left(\sqrt{\frac{(1-k)^N-1}{k}0.5\pi + \mathrm{N}} - \sqrt{\frac{(1-k)^N-1}{k}\theta_0 + \mathrm{N}}\right) \quad \ldots\ldots(9)$$

We used Excel to plot the function to prove the precision of the approximation.

Series I&II are drawn with k=0.6, $\theta = \text{acrcos}(0.6)$. Series I is the graph of the accurate function (6); series II is the graph of the simplified function (9)

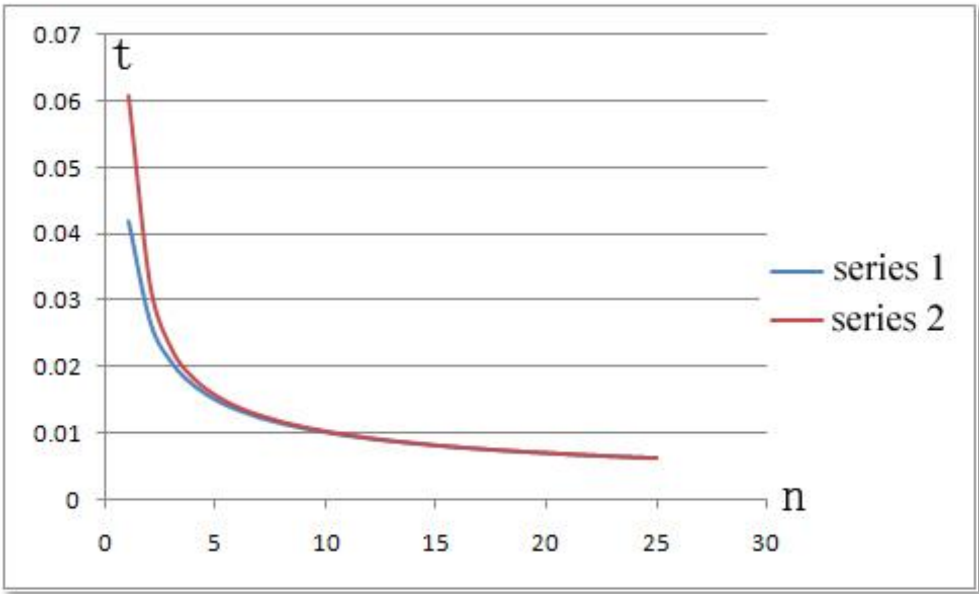

Figure 2.3.2

As can be seen in the figure 2.3.2, two curves coincide with each other when ***N***>5, thus the simplified function is practicable.

2.3.3 Approximation to T_N

The expression of T_N can be obtained according to the previous deduction that

$$T_N = \sum t = \sqrt{\frac{2h}{3g}} \cdot \sum_{i=1}^{N} \sqrt{\frac{1-(1-k)^{2N}}{2k-k^2}} \int_{\theta_0}^{0.5\pi} \left(N - \sum_{i=1}^{N} \sin\theta_i\right)^{-0.5} d\theta_1$$

$$\approx -2k\sqrt{\frac{2h}{3g}} \cdot \sum_{i=1}^{N} \sqrt{\frac{1+(1-k)^N}{[(1-k)^N-1](k^2-2k)}} \left(\sqrt{\frac{(1-k)^N-1}{k}0.5\pi + N} - \sqrt{\frac{(1-k)^N-1}{k}\theta_0 + N}\right)$$

According to Lemma III, the formula above can be simplified as

$$T_N \approx -2k\sqrt{\frac{2h}{3g}} \cdot \int_{0.5}^{N+0.5} \sqrt{\frac{1+(1-k)^N}{[(1-k)^N-1](k^2-2k)}} \left(\sqrt{\frac{(1-k)^N-1}{k}0.5\pi + N} - \sqrt{\frac{(1-k)^N-1}{k}\theta_0 + N}\right) dN$$

When **N** is very big, because k∈ (0, 1), hence $(1-k)^N \to 0$, the equation above can be further simplified as

$$T_N \approx -2k\sqrt{\frac{2h}{3g}} \cdot \sqrt{\frac{-1}{(k^2-2k)}} \int_{0.5}^{N+0.5} \left(\sqrt{\frac{-1}{k}0.5\pi + N} - \sqrt{\frac{-1}{k}\theta_0 + N}\right) dN \qquad \dots\dots(10)$$

$$= \frac{-4k}{3}\sqrt{\frac{-2h}{3g(k^2-2k)}}\left[\sqrt{\left(\frac{-\pi}{2k}+N+0.5\right)^3}-\sqrt{\left(\frac{-\theta_0}{k}+N+0.5\right)^3}-\sqrt{\left(\frac{-\pi}{2k}+0.5\right)^3}+\sqrt{\left(\frac{-\theta_0}{k}+0.5\right)^3}\right]\ldots\ldots(11)$$

2.3.4 Further simplification to T_N with Taylor Series Expansion Method

Considering the formula (11) is still quite inconvenient to be used, we further simplified the formula with Taylor series expansion.

We have already got the conclusion about T_N that

$$T_N = \frac{-4k}{3}\sqrt{\frac{-2h}{3g(k^2-2k)}}\left[\sqrt{\left(\frac{-\pi}{2k}+N+0.5\right)^3}-\sqrt{\left(\frac{-\theta_0}{k}+N+0.5\right)^3}-\sqrt{\left(\frac{-\pi}{2k}+0.5\right)^3}+\sqrt{\left(\frac{-\theta_0}{k}+0.5\right)^3}\right]$$

With second order of Taylor series expansion we have

$$T_N = \sqrt{\frac{-2h}{3g(k^2-2k)}}\left[(2\theta_0-\pi)n^{0.5}-\frac{4{\theta_0}^2+\pi^2-2k(2\theta_0+\pi)}{8k}n^{-0,5}\right]\quad\ldots\ldots(12)$$

Since the formula above is still complex, we wanted to further simplify the formula to the one in the formation similar to

$$T_N = ax^{0.5}+bx^{-0.5}$$

We predicted a, b according to the equation **(12)** that

$$a = k\sqrt{\frac{t_1 h}{k^2-2k}}$$

$$b = (t_2k-t_3)\sqrt{\frac{h}{k^2-2k}}$$

(t_1, t_2, t_3 represent the parameters that need fitting)

By substituting series of k and h into formula (11), we obtained a series of graph which we fitted to obtain a series values of a and b. we set up a trilinear coordinate with k,h,a and k,h,b to be the x,y,z-axis of the coordinate and put the dots into the coordinate.

Fit the dots according to the expectation formula to obtain values of t_1, t_2, t_3

The results are as follows

1. Relations between a and k, h (interpolated)

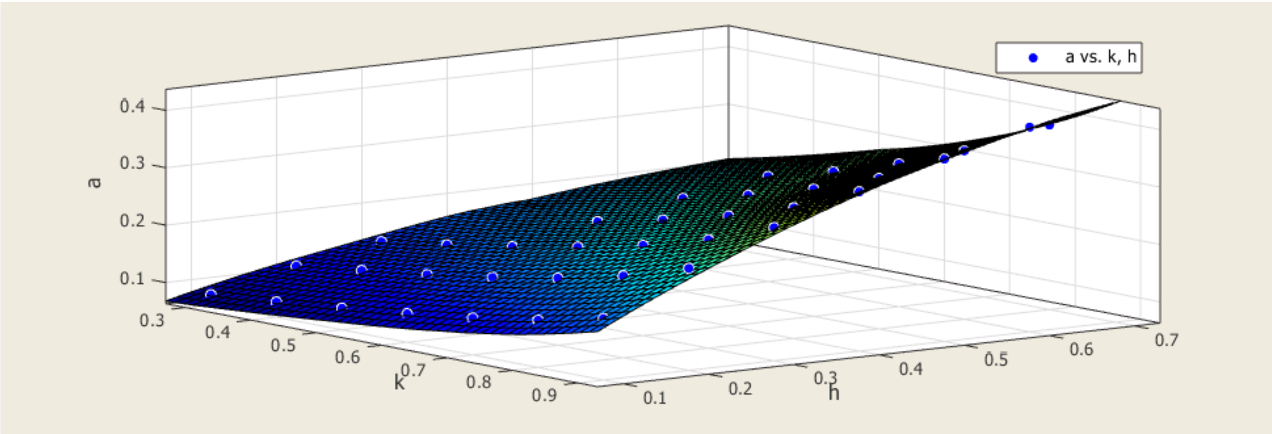

Figure 2.3.4(1)

Relations between a and k, h (fitted)

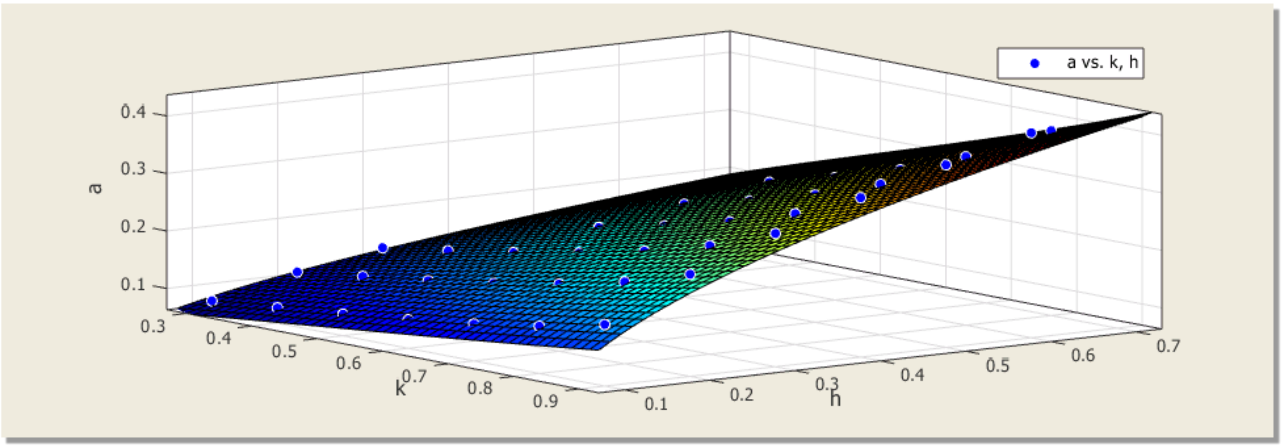

Figure 2.3.4(2)

2. Relations between b and k, h (interpolated)

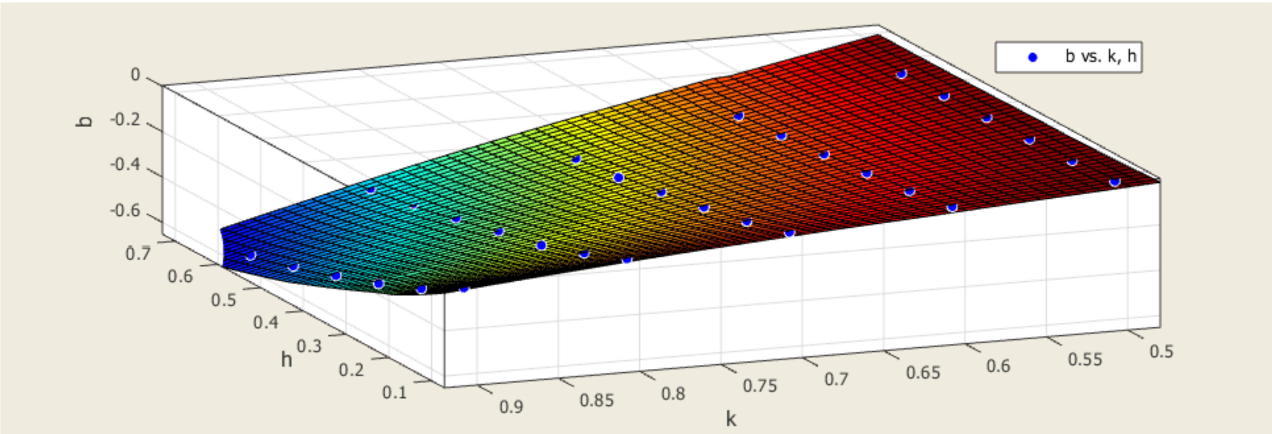

Figure 2.3.4(3)

Relations between a and k, h (fitted)

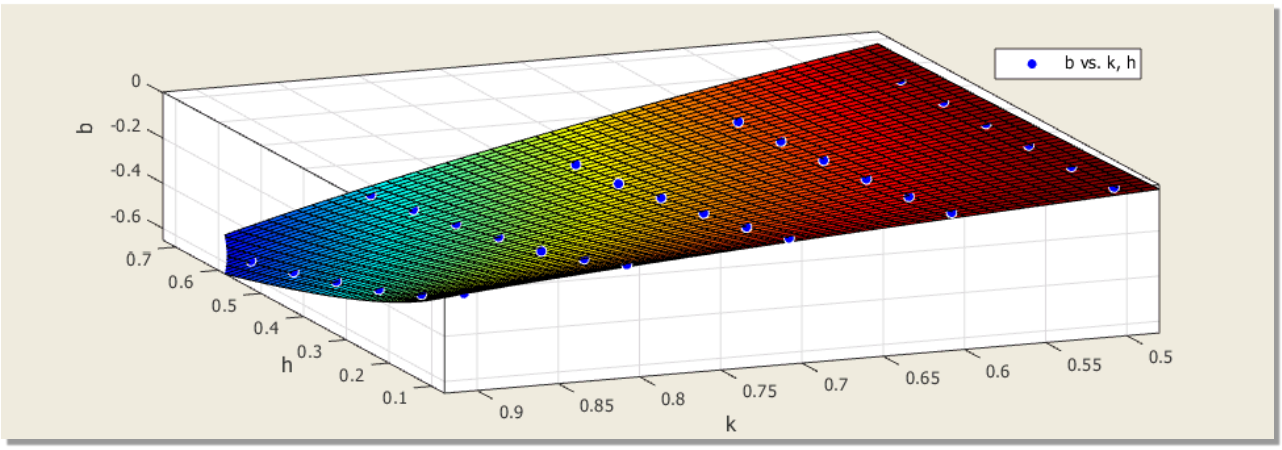

Figure 2.3.4(4)

Solution of fitting is

$$t_1 = 0.3099$$
$$t_2 = -1.796$$
$$t_3 = -0.8449$$

Because h and g have similar influence on the graph, so make g=9.7930,
Thus the **empirical formula** is

$$T_N = \sqrt{\frac{-h}{k^2 - 2k}}\left[0.5567k \cdot N^{\frac{1}{2}} + (0.845 - 1.8k)N^{-\frac{1}{2}}\right]$$

We draw graphs to examine its precision, the curves fit quite well. Thus, the empirical formula is practicable.

(origin 1,2,3 means the graph of the original formula, fitted 1,2,3, means the graph of the empirical formula)

Graphs of origin 1 and fitted 1 is drawn with k=0.4, h=0.1; origin 2 and fitted 2 is drawn with k=0.6, h=0.05; origin 3 and fitted 3 is drawn with k=0.6, h=0.1

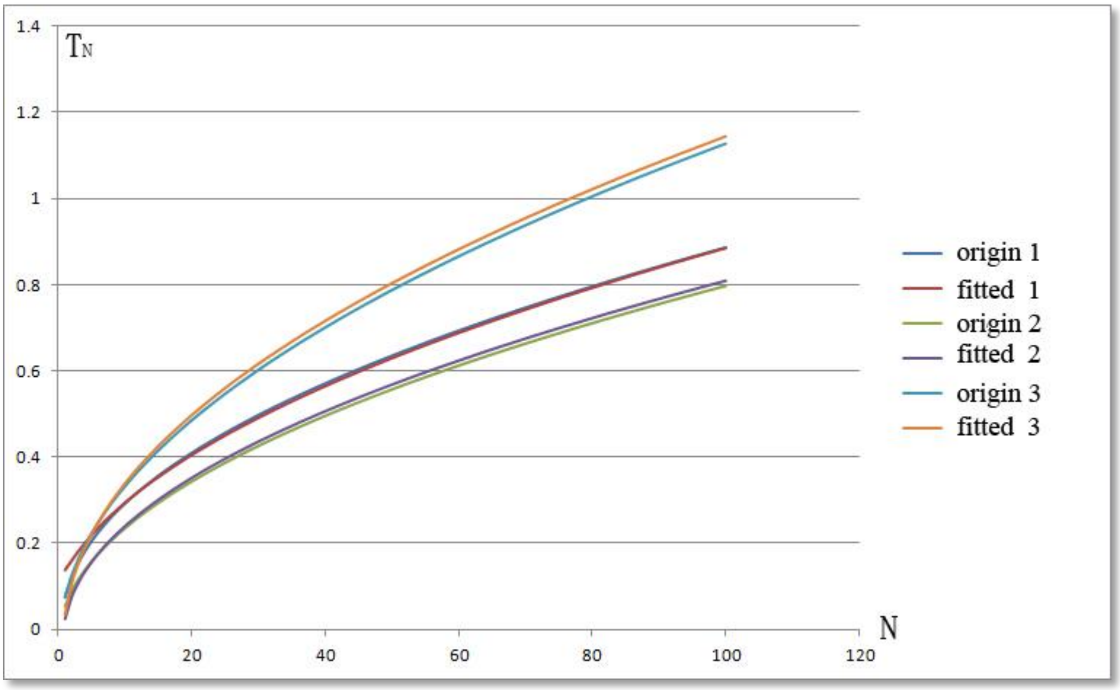

Figure 2.3.4(5)

2.4 Loss of Energy

The loss of energy would increase as the angular velocity of dominoes increase. In reality, the loss of energy cannot be neglected; otherwise the velocity of dominoes would not have its limit.

We assume that Q% of the energy would be preserved in collision. $Q \in (0, 1)$

In this occasion, the expression of E_N is

$$E_N = E_{N-1} \cdot Q + mg(h - h\sin\theta)$$

Thus, when the system comes to equilibrium, E_N follows the equation that

$$\lim_{N\to\infty} E_{N-1}(1-Q) = mgh$$

Thus, according to Lemma II,

$$\frac{mghQ}{1-Q} + mgh(1-\sin\theta) = {}^{1}\!/_{6}\, m\omega^2(2h)^2$$

$$\omega_{max} = \sqrt{\frac{3g}{2h}\left(\frac{1}{1-Q} + \sin\theta\right)} \qquad \dots\dots(13)$$

When the angular velocity reaches its limit, the minimum time for the first domino to hit the next is

$$t_{min} = \sqrt{\frac{2h}{3g}} \int_{\theta_0}^{0.5\pi} \left[\frac{1}{1-Q} + \sin\theta\right]^{-0.5} d\theta_0$$

$$t_{min} = \sqrt{\frac{2h}{3g}} \left(\frac{1}{1-Q}\right)^{-0.5} \int_{\theta_0}^{0.5\pi} [1 + (1-Q)\sin\theta]^{-0.5}\, d\theta_0$$

2.5 Conclusion

Following conclusions are drawn based on the above deduction.

1. The recurrence relations between angles

$$\theta_{n+1} \approx \theta_1 (1-k)^n$$

2. The sum of kinetic energy of **N** pieces in a domino line

$$E = \left(N - \sum_{i=1}^{N} \sin\theta_i\right) mgh \approx \left[N - \frac{1-(1-k)^p}{k} \cdot \theta_{N-p+1} - \sum_{i=1}^{N-p} \sin\theta_i\right] mgh$$

The angles between the domino and the ground are regarded as 0 from the ***p*** domino to the last one. Angles of the prior ***(N-p)*** pieces are unneglectable.

3. Time for the first domino to hit the next piece when there are totally N pieces in the line.

$$t = -2k \sqrt{\frac{2h(1+(1-k)^N)}{3g[(1-k)^N - 1](k^2 - 2k)}} \left(\sqrt{\frac{(1-k)^N - 1}{k} 0.5\pi + N} - \sqrt{\frac{(1-k)^N - 1}{k} \theta_0 + N}\right)$$

$(\arccos(k) = \theta_0)$

4. Time for an array of N dominoes to fall down (T_N) is

$$T_N = \frac{-4k}{3} \sqrt{\frac{-2h}{3g(k^2-2k)}} \left[\sqrt{\left(\frac{-\pi}{2k} + N + 0.5\right)^3} - \sqrt{\left(\frac{-\theta_0}{k} + N + 0.5\right)^3} - \sqrt{\left(\frac{-\pi}{2k} + u - 0.5\right)^3} + \sqrt{\left(\frac{-\theta_0}{k} + u - 0.5\right)^3}\right]$$

With Taylor series expansion, the equation above can be simplified and the empirical formula is

$$T_N = \sqrt{\frac{-h}{k^2 - 2k}} \left[0.5567k \cdot N^{\frac{1}{2}} + (0.845 - 1.8k) N^{-\frac{1}{2}}\right]$$

5. When the loss of energy during collision is taken into account and ***Q***% of the energy would be preserved, the minimum time for the first domino to hit the next is

$$t_{min} = \sqrt{\frac{2h}{3g}}\left(\frac{1}{1-Q}\right)^{-0.5}\int_{\theta_0}^{0.5\pi}[1+(1-Q)\sin\theta]^{-0.5}\,d\theta_0$$

Chapter III Verifying the Model

3.1 Measures to verify the Model

To prove the validity of our deduction to the mathematical model of domino, we arranged domino lines with 50, 100, 150, 200, 250,300, 400 dominoes. We recorded the time for each line to fall down and calculated the parameter ***Q*** in the experiment. Then we analyzed the video we shot during the experiment frame by frame to prove the validity of the formula (9) (related to ***t***).

3.2 Statistics from the Experiment

* Dominoes used in the experiment measures 4.3cm in height (2h=4.3 cm)and 0.7 in thickness (s=0.7cm) and are arranged with 1.0cm space in between (d=1.0cm).

Experiment 1: time for N dominoes to fall down

Total number of the dominoes	**Time to fall down (T_N)/s**
50	1.160
100	1.998
150	2.834
200	4.287
250	5.00
300	6.147
350	
400	7.928

Table 3.2(1)

The graph of the statistics in the table is shown in Figure 3.2(1)

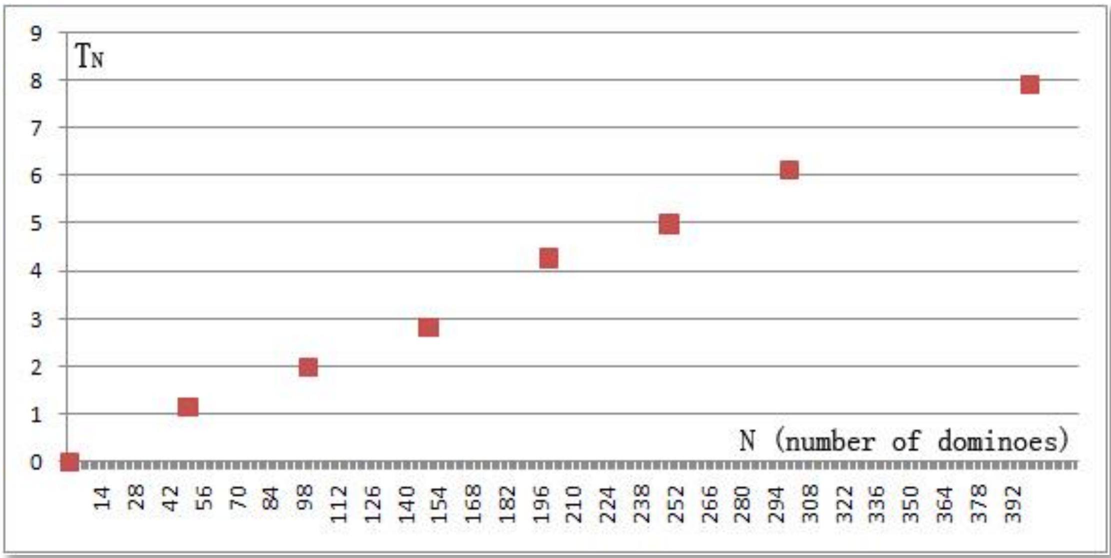

Figure 3.2(1)

Experiment 2: time for one domino to hit the next (analysis to the videos)

Number of the dominoes	Time to hit the next domino (t)/s
1	0.1678
2	0.0649
3	0.0351
4	
5	0.0332
6	
7	0.0335
8	
9	0.0165

10	
11	
12	
13	0.0147
14	
15	
16	
17	0.0149
18	
19	
20	0.0194
21	
22	
23	0.0186
24	
25	
26	0.0206

Table 3.2(2)

Because of the restriction of the video camera, some frames are not clear enough for us to obtain statistics. Consequently, statistics of some dominoes are missing.

The graph of the above statistics is shown in Figure 3.2(2)

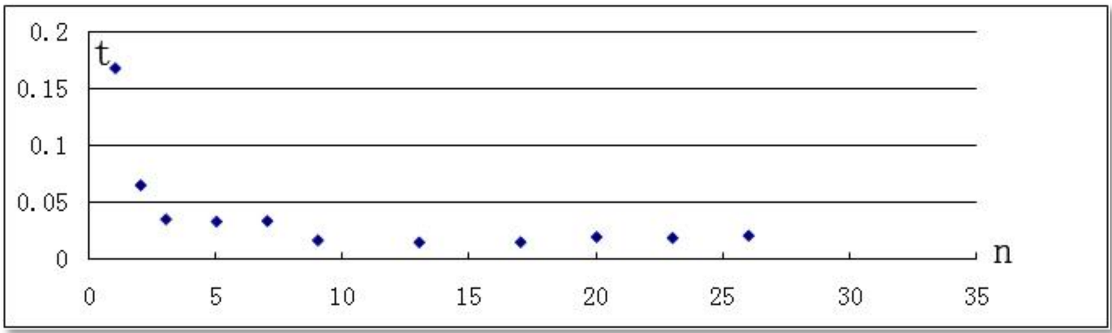

Figure 3.2(2)

3.3 The Experiment

3.3.1 Adjustment to the relations between angles

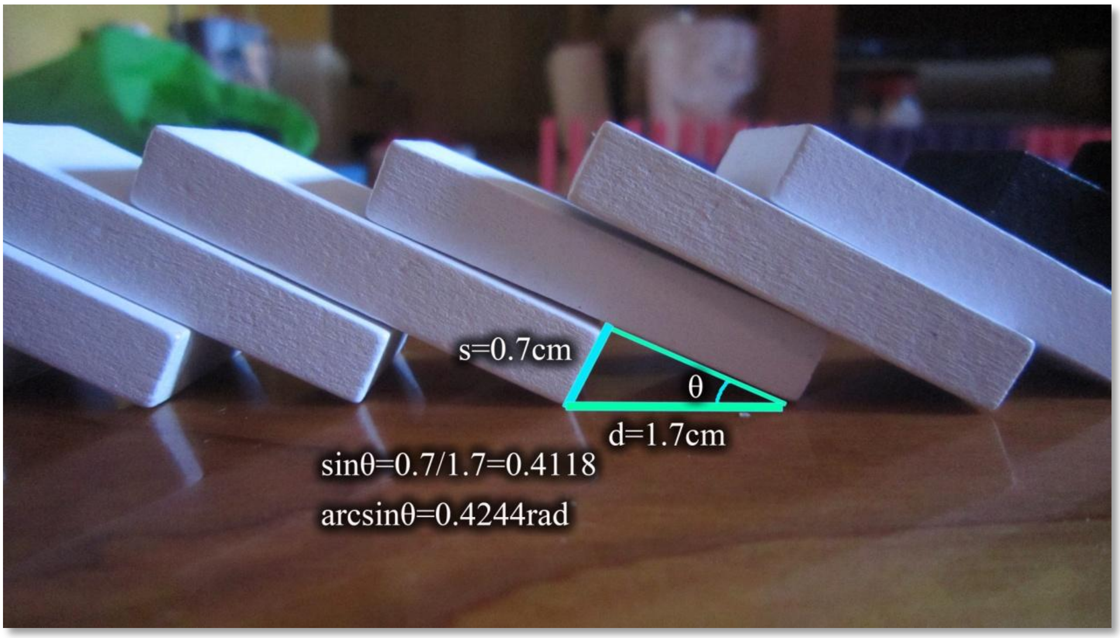

Figure3.3.1 (1)

As can be seen in figure 3.3.1(1), the dominoes would not fall down to the ground in practice, because of their thickness. Thus, adjustment to the recurrence relations between angles is necessary.

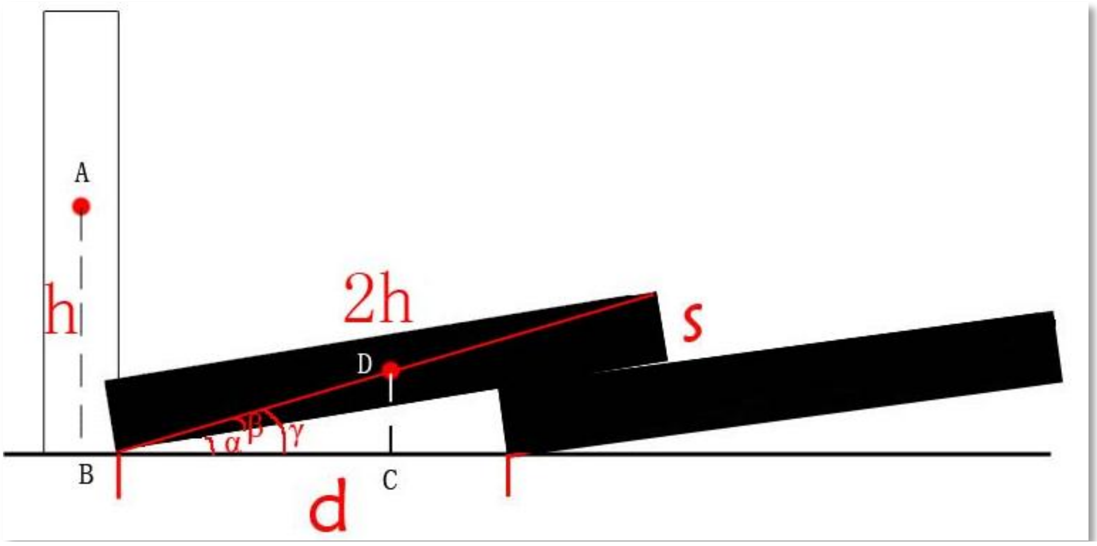

Figure 3.3.1(2)

We abstract the diagonal of the domino to another type of domino (without thickness)

$$2\tan\alpha_n \cdot \cos\alpha_{n+1} \cdot h = 2\sin\alpha_{n+1} \cdot h - \frac{s}{\cos^2\alpha_n} + d\tan\alpha_n \qquad \ldots\ldots(14)$$

When $\alpha \rightarrow 0.4244$, we used the Taylor series expansion

$$\sin\alpha = 0.911286\alpha + 0.025$$

$$\cos\alpha = -0.411774\alpha + 1.086$$

$$\tan\alpha = 1.204178\alpha - 0.059$$

Also, according to Figure 3.3.1(2)

$$\alpha = \gamma - \beta = \gamma - 0.1614$$

Substitute the above four equations to equation (14),

$$\gamma_{n+1} = \frac{0.147257293\gamma_n^2 - 0.4000134261\gamma_n - 0.03929680348}{0.087796599\gamma_n^2 - 0.07449846871\gamma_n - 0.414207213}$$

The approximate solution for the general formula is

$$\gamma_{n+1} = 0.43225230 + \frac{-0.000015656383}{1 - 0.99996289^{2^n - 1} \cdot \left(\frac{\gamma_1 - 0.42845666}{\gamma_1 - 0.42847283}\right)^{2^n}}$$

3.3.2 Solving *Q*

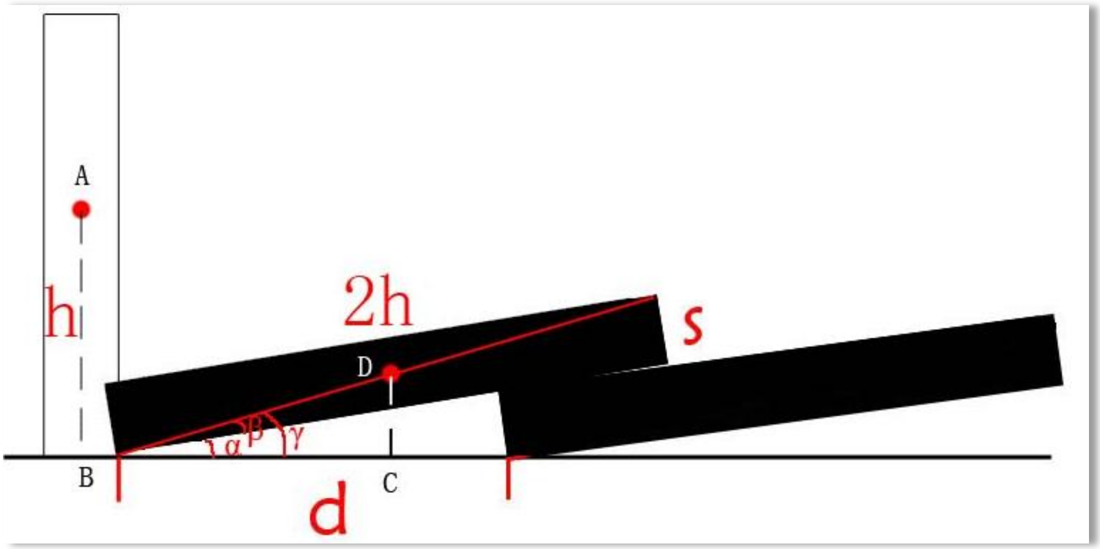

Figure 3.3.2(1)

The angle γ when a domino is fallen down is

$$\gamma = \alpha + \beta = \arcsin\frac{s}{d} + \arctan\frac{s}{2h} = 0.4244 + 0.1614 = 0.5858$$

Thus, the true gravitational energy released in reality is

$$E_{\text{each domino}} = mg(AB - CD) = mg\left(h - \frac{\sqrt{4h^2 + s^2}}{2}\sin\gamma\right) = 0.009457mg$$

When the system come to equilibrium,

$$\lim_{N\to\infty} E_{n-1}(1 - Q) = 0.009457mg$$

Thus,

$$\frac{0.009457\mathrm{mgQ}}{1-\mathrm{Q}}+\mathrm{mgh}(1-\sin(\beta+\theta))={}^{1}\!/_{6}\,\mathrm{m}\omega^{2}(\mathrm{s}^{2}+4\mathrm{h}^{2})$$

The solution for ω_1 is

$$\omega_1=\frac{1}{2}\sqrt{\frac{119.5827\mathrm{gQ}}{1-\mathrm{Q}}+271.8651\mathrm{g}[1-\sin(\beta+\theta)]}$$

Thus, the upper limit of integration should be$(0.5\pi+\beta)$, and the down limit should be$(\arccos\left(\frac{d-s}{2h}\right)+\beta)$. The solution for the minimum time for the first domino to hit the next is

$$\mathrm{t_{min}}=2\int_{\arccos\left(\frac{\mathrm{d-s}}{2\mathrm{h}}\right)+\beta}^{0.5\pi+\beta}\left[\frac{119.5827\mathrm{gQ}}{1-\mathrm{Q}}+271.8651\mathrm{g}[1-\sin(\beta+\theta)]\right]^{-0.5}\mathrm{d}\theta_0 \qquad \ldots\ldots(15)$$

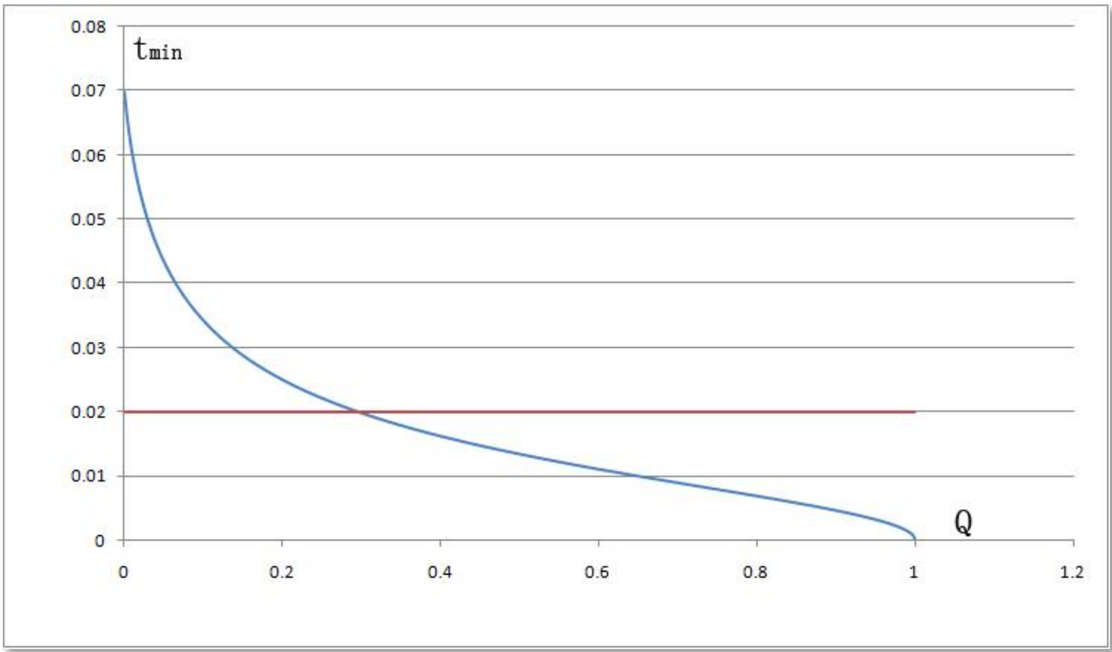

Figure 3.3.2(2)

In figure 3.3.2(2), the x-axis is Q$(0\le \mathrm{Q}\le 1)$, the y-axis is t_{min}.

According to statistics from the experiment, the minimum time is 0.02s.

The red straight line in the figure in y=0.02 and the blue curve is the equation (15).

From the intersection point, we get the percentage of the energy preserved is

$$Q = 0.2953 = 29.53\%$$

In other words, almost 70% of the energy would be lost in reality.

3.3.3 Calculation to the prior 50 dominoes

Q obtained, with the adjusted recurrence relations of angles, time for one domino to hit the next (t) can be calculated.

Dots are taken frame by frame from the video we shot, the green curve is calculated in accordance to the previous deduction, and they fit very well.

The red curve is the graph of the formula without the loss of energy and thickness of dominoes. The divergence between the green and the red is obvious.

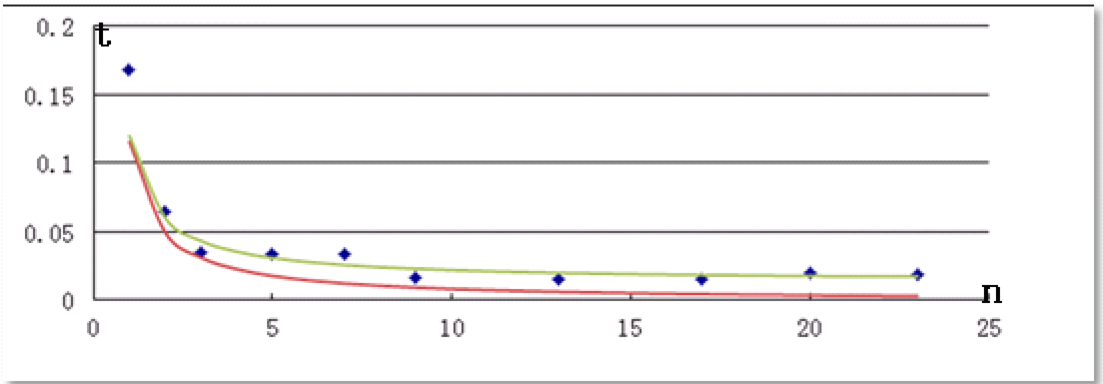

Figure 3.3.3

3.4 Conclusion

The above two experiment with t and T_N successfully proved the validity of our model as well as all the deduction. The success convinced us of the practicability of our model and the reasonability to put it into applications.

Chapter IV Application 1

Relationship between concentration of CO_2 in the atmosphere and global mean temperature

4.1 Background

Nowadays, the global warming is commanding more and more attention of the world. A great many studies showed that the carbon dioxide is the main composition of the greenhouse gases, and the global mean temperature is closely related to its concentration. From the time of Industrial Revolution in 1860s, the amount of the carbon dioxide emitted to the atmosphere by humans has been increasing year by year. With this, the global mean temperature has been on the trend to rise. This is what we call the greenhouse effect.

The principle is that the carbon dioxide reduces the radiation of the earth back to space. According to the law of thermodynamics, when the earth receives the same level of solar radiation, the temperature of the surface of the earth is supposed to rise, if the concentration of carbon dioxide increases.

We believe that a certain concentration of carbon dioxide in the atmosphere contributes to a certain rise in the global mean temperature. This conforms to the mathematical model of domino in the formula related to T_N.

We can draw an analogy between the model of domino and the rise in the global mean temperature caused by CO_2 that: a domino represents a unit volume of CO_2, the time for a domino to fall until it hit the next domino (t) is similar to the contribution of a unit volume of CO_2 makes to a certain rise in global mean temperature. As the dominoes continue to fall, the time for one to hit the next (t) will reduce ceaselessly. Analogously, with the CO_2 accumulation, the thermal contribution of an equivalent volume of CO_2 emitted into the atmosphere will subsequently reduce gradually. Then, the total falling time of N dominoes is equivalent to the accumulated rise of global mean temperature caused by N units' volume of CO_2.

4.2 Modeling and Calculation

4.2.1 Obtaining statistics

CO_2 concentration in the atmosphere from 1960 to 2010

*According to Hawaii Mauna Loa observatory

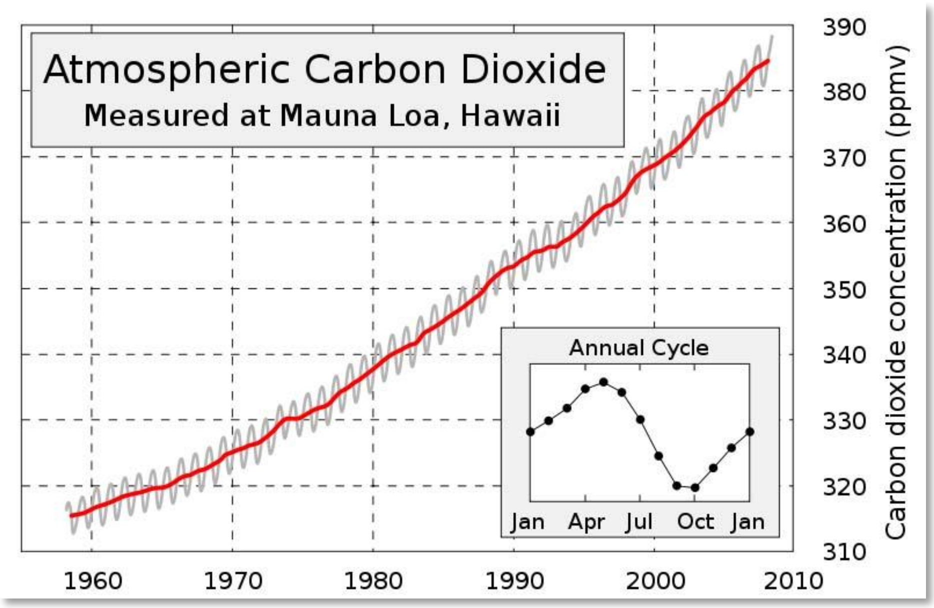

Figure 4.2.1(1)

Global mean temperature from 1860 to 2010

*According to IPCC 4th report WG1

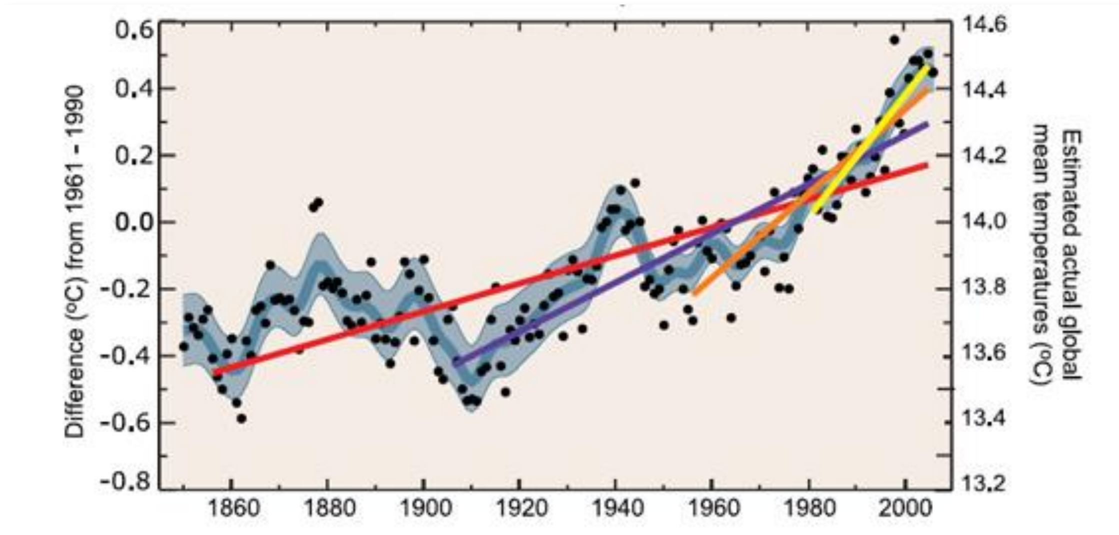

Figure 4.2.1(2)

Statistics from the above figures:

CO_2 concentration in the atmosphere (ppmv)	Global mean temperature（℃）
316	13.8

319	13.9
325	13.95
330	14
338	14.05
343	14.1
353	14.2
360	14.3
368	14.37
379	14.4
386	14.43

Table 4.2.1

We draw the graph of the statistics, concentration of CO_2 as the horizontal axis and global mean temperature as the vertical axis.

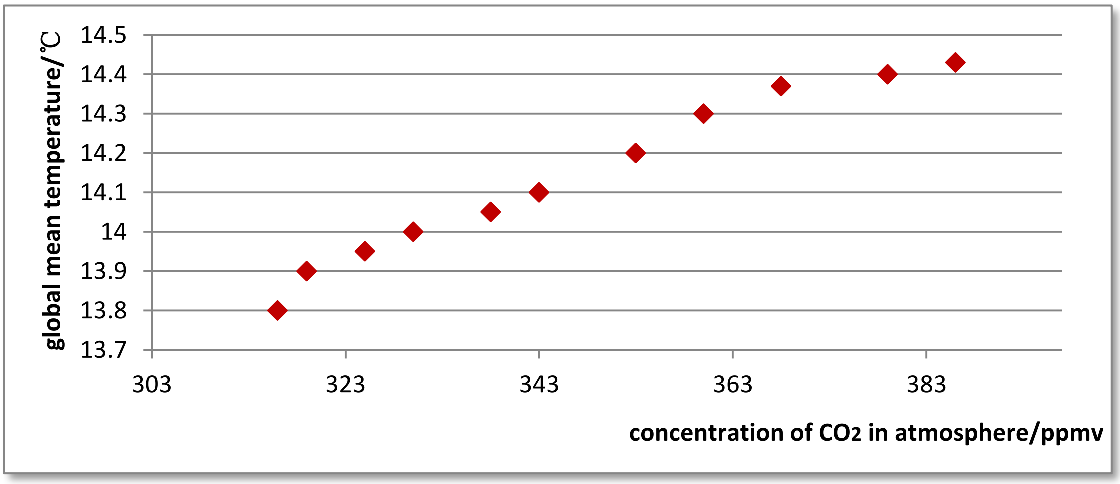

Figure 4.2.1(3)

4.2.2 Analysis and Calculation

Because the study on the global mean temperature requires high precision, we used formula (11) to fit the statistics. The intercept of the graph has its meaning as the global mean temperature without any CO_2 in the atmosphere.

The formula (11) is

$$T_N = \frac{-4k}{3}\sqrt{\frac{-2h}{3g(k^2-2k)}}\left[\sqrt{\left(\frac{-\pi}{2k}+n+0.5\right)^3}-\sqrt{\left(\frac{-\theta_0}{k}+n+0.5\right)^3}-\sqrt{\left(\frac{-\pi}{2k}+0.5\right)^3}+\sqrt{\left(\frac{-\theta_0}{k}+0.5\right)^3}\right]+C$$

Combining the constant terms, we get

$$T_N = \frac{-4k}{3}\sqrt{\frac{2h}{3g}}\sqrt{\frac{-1}{(k^2-2k)}}\left[\sqrt{\left(\frac{-\pi}{2k}+n+0.5\right)^3}-\sqrt{\left(\frac{-\theta_0}{k}+n+0.5\right)^3}\right]+C'$$

*make $\frac{-4}{3}\sqrt{\frac{2h}{3g}}$=p as a new parameter while fitting. Thus,

$$T_N = kp\sqrt{\frac{-1}{(k^2-2k)}}\left[\sqrt{\left(\frac{-\pi}{2k}+n+0.5\right)^3}-\sqrt{\left(\frac{-\theta_0}{k}+n+0.5\right)^3}\right]+C'$$

The solution of fitting is

K=0.9629 (0<k<1)

P=-0.171 (p<0)

C'=7.939

The parameters are all in its domain of definition.

Substitute the parameters into the formula (7) (accurate), the graph is shown in figure 4.2.2(1)

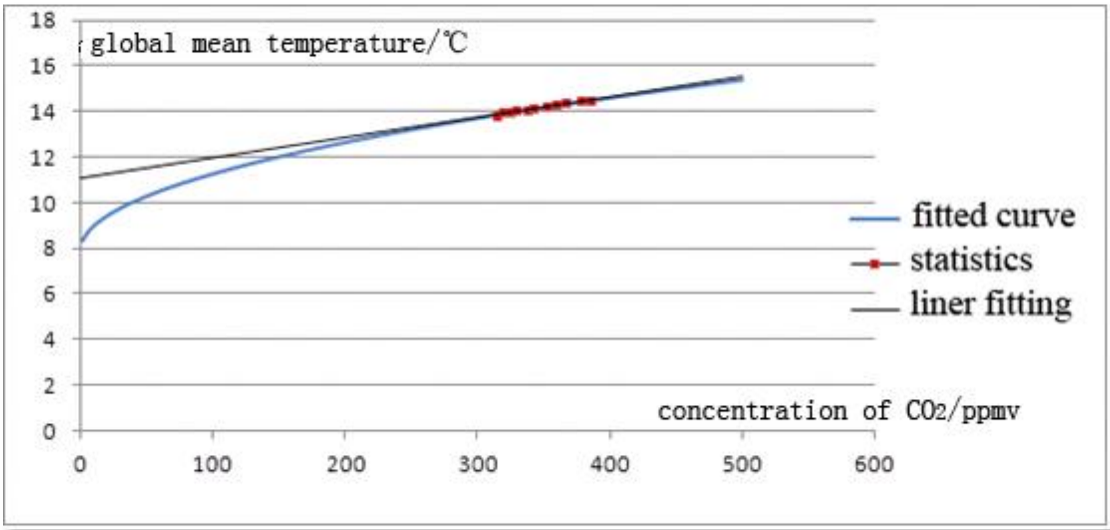

Figure 4.2.2(1)

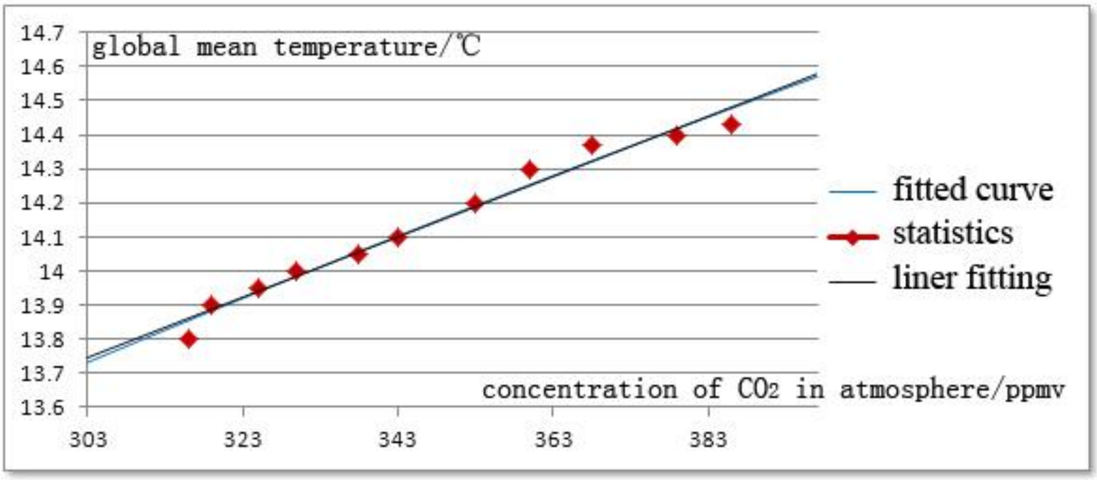

Figure 4.2.2(2)

The high precision of the fitting implies the potential relationship between concentration of CO_2 in the atmosphere and global mean temperature.

Despite the fitting curve has similarity with the linear function in a narrow range, the tendencies are different in the long run, as can be seen in Figure 4.2.2(1).

4.3 Conclusion

The figure mentioned above shows that the calculated results tally with the real facts very well and the usage of the simplified formula won't cause too much error. Obviously, the model has its universality to a considerable degree.

Though the concentration of CO2 is not the only factor (other factors are the effect of surface of the earth, the emission of other kinds of greenhouse gases, the activity of the sun, etc.), we hope to try to establish a model of relations between the concentration of CO2 and the rise of global mean temperature with reasonable assumptions, and offer our suggestion on the situation and tendency of the change of global temperature. According to our model, each 100 ppmv rise of concentration of CO2 will cause 1.1℃ of rise in global mean temperature.

It will bring very severe influence on the atmospheric circulation and the entire ecosystem. With the present growth rate of concentration of CO2, the foreseeable tremendous changes will come soon in the near future. Therefore, it is becoming more and more important to reduce the emission of greenhouse gases and to save energy.

Chapter V Application 2
Domino effect in Financial Crisis

5.1 Background

In present world economy, the financial crisis occurs periodically. Under the background of globalization of the world economy, it is necessary to establish related mathematical models so that we can monitor the process of economic crisis better.

Every financial crisis is usually sparked by a minor event. At the stage of the beginning, the influence is very limited. As the crisis develop, more and more enterprises and fields will be involved and the deterioration of economic situation will accelerate. For example, a company went bankrupt because of a financial crisis and affected the related companies involved in its business or debts. Some ones among them broke consequently. Subsequently, the same process happened and more companies followed. Hence, the crisis swept through.

In the field of economics, this phenomenon is often referred to as domino effects. The inner link of these two indicates the possibility to describe the outbreak of financial crisis with the model of domino.

Because of this, we obtained the related data of the financial crisis started in USA in 1929 in the MBER database. Because Hoover government was pursuing a policy of non-intervention at the stage of the beginning of crisis, the performance of the market most directly presented the Domino effects in economics.

In our model, a bankrupted company is similar to a fallen domino, N of the total number of bankrupted companies is analogized to N fallen dominoes. The time T for the N companies to bankrupt is equivalent to the total time it takes for the N dominoes to fall down (T_N).

The following figure is the graphs of the statistics of the bankrupted companies from Jan 1919 to Aug 1938 (specific statistics is attached in the appendix).

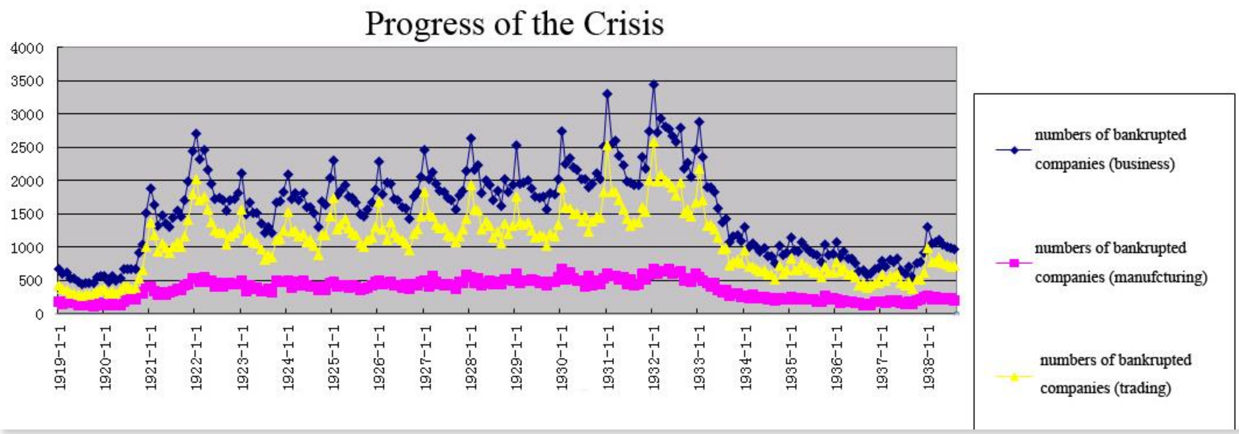

Figure 5.1

5.2 Calculation and Analysis

Studies suggest that the first climax of bankruptcy appears before the great depression. There were dramatic changes in the American economy. Because of the excessive optimism among Americans, the economic bubbles were not taken seriously. Massive newly-established corporations went bankrupt and lost the ability to pay their debts. This resulted in a break in the capital chain and sparked the great depression.

We summed up the total number of bankrupt companies from January of 1919, got the graph with the total number of bankrupt companies as the x-axis and the time (months) as the y-axis.

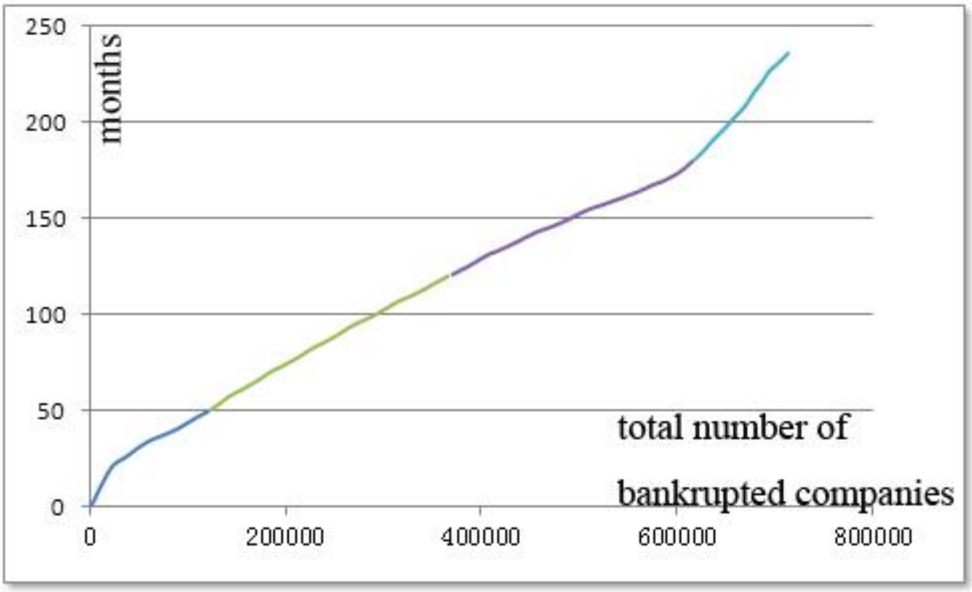

Figure 5.2 (1)

It is obvious that the bigger the slope is, the slower the corporations go bankrupt.

Empirical formula is used in this application, because it is more straightforward to find out the relationship between k, h and the stability of a market.

The empirical formula is

$$\mathrm{T}_N = \sqrt{\frac{-h}{k^2 - 2k}}\left[0.5567k \cdot N^{\frac{1}{2}} + (0.845 - 1.8k)N^{-\frac{1}{2}}\right]$$

Blue dots are from the database

Parameters of the red fitting curve is

h =0.1229

k =0.7873

They are all in domain of definition.

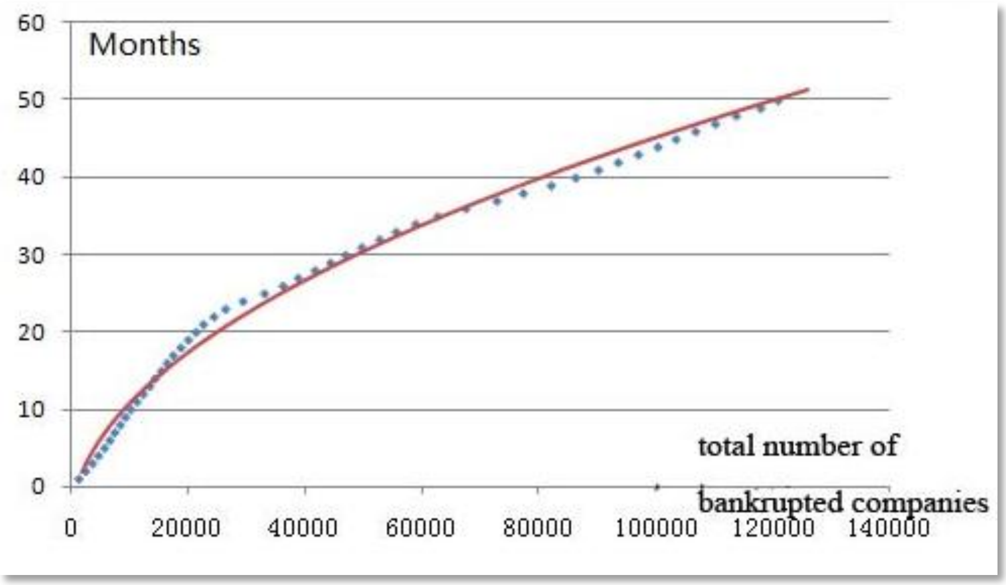

Figure 5.2(2)

The Figure 5.2(1) showed that the bankruptcy rate in the beginning 50 months conforms to the model of domino. Historical materials suggest that the Hoover's government didn't take any actions to stop the bankruptcy of corporations, thus the performance of the corporation in this period of time is the most direct reflect of domino effect. This set the stage for a period of wide trade imbalances and for the depression that followed.

From the 50^{th} month to the 120^{th} month, economy of America is in prosperity (the green curve in the graph). Thus, the curve separated from the curve of the function. The bankruptcy rate slowed down, and is slower than our prediction.

From the 120^{th} month to the 180^{th} month, America was in the 'great depression'. Though the austerity financial program adopted by the Hoover's government aroused fiery criticism towards it, in fact, the program did not have much negative influence on the economy, according to the graph.

It was 181 months after the outburst of the crisis that Roosevelt come into his power and revived the badly-weakened US economy. As can be seen in Figure 5.2(3),the curve goes steeper (the blue part of the curve).

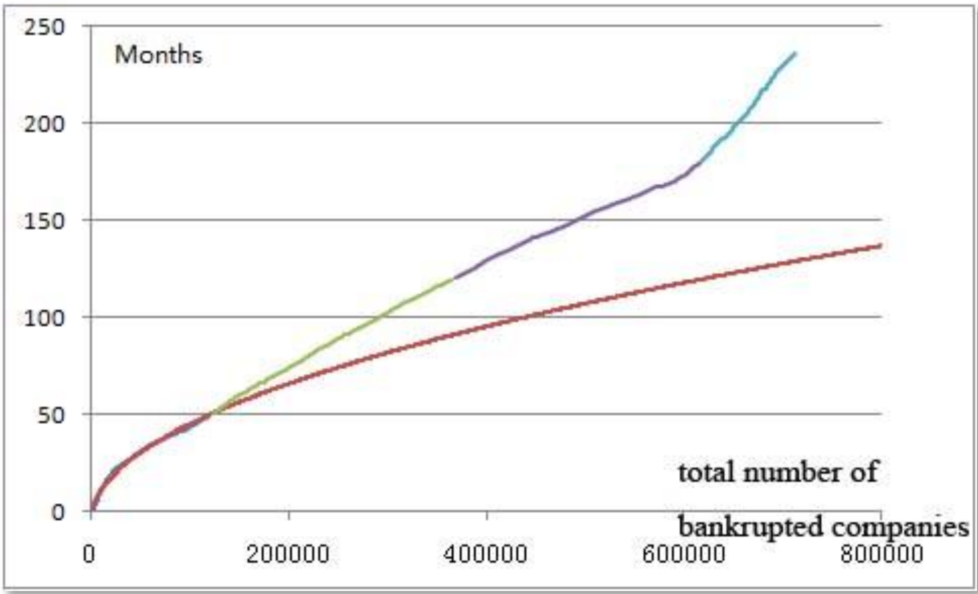

Figure 5.2(3)

5.2 Further studies on the empirical formula

Apart from the fitting above, we analyzed bankruptcy of business, manufacturing and trading companies in that financial crisis. By doing this, we aimed to prove the comprehensive practicability of our model (especially the empirical formula) and to conjecture the meaning of k and h in reality.

The empirical formula is

$$\mathrm{T}_N = \sqrt{\frac{-h}{k^2-2k}}\left[0.5567k \cdot N^{\frac{1}{2}} + (0.845 - 1.8k)N^{-\frac{1}{2}}\right]$$

The results of fitting are shown in the below figures.

Business companies

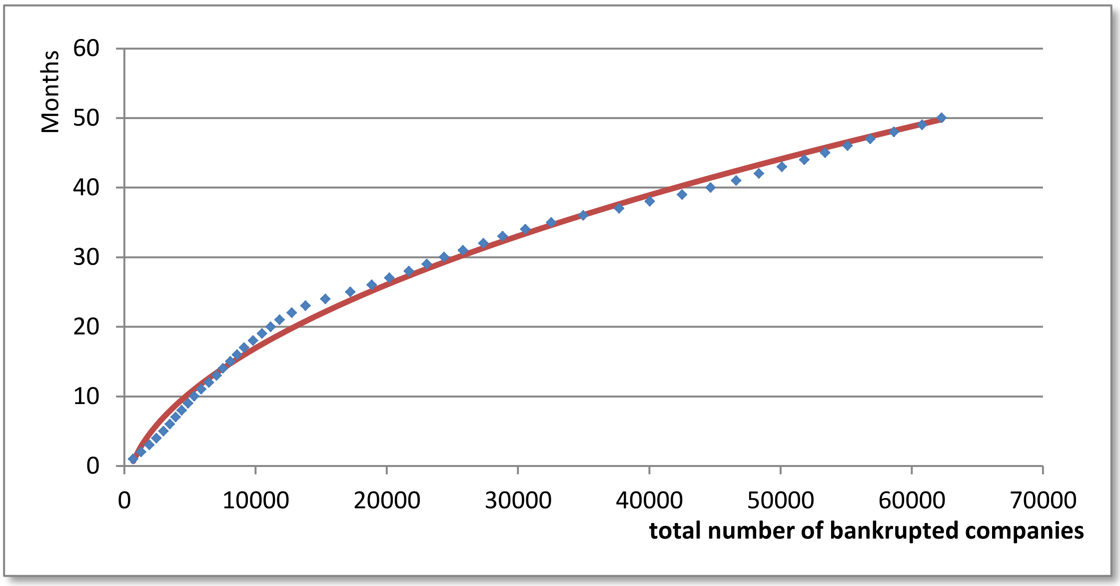

Figure 5.2(3)

Parameters:

k=0.9744 h=0.1638

(r^2=0.9935)

Manufacturing companies

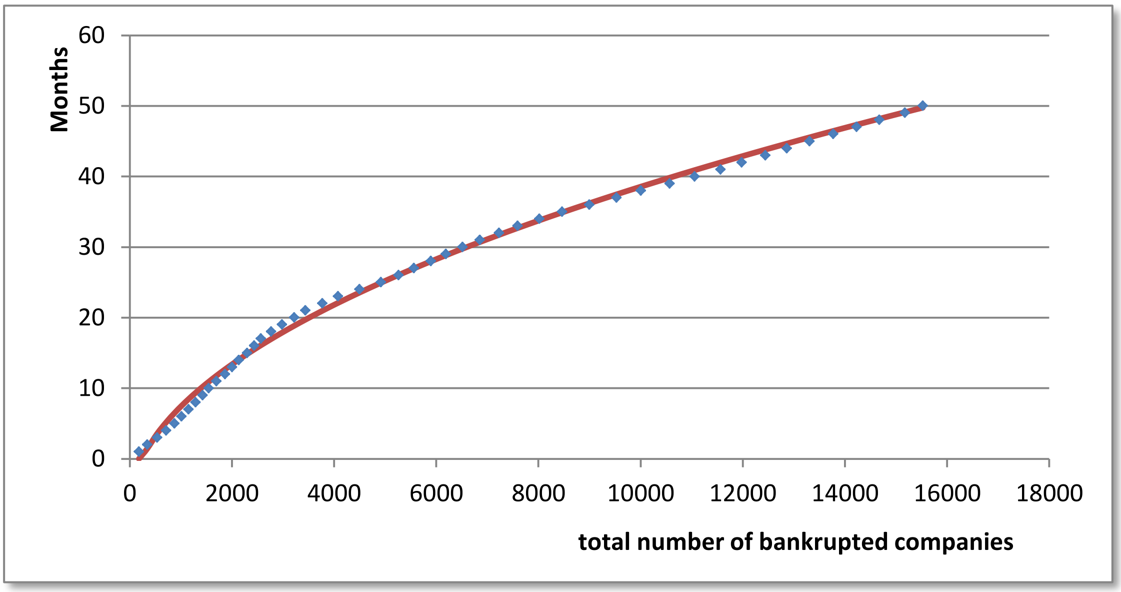

Figure 5.2(4)

Parameters:

k=0.4032 h=2.639

(r^2=0.9967)

Trading companies

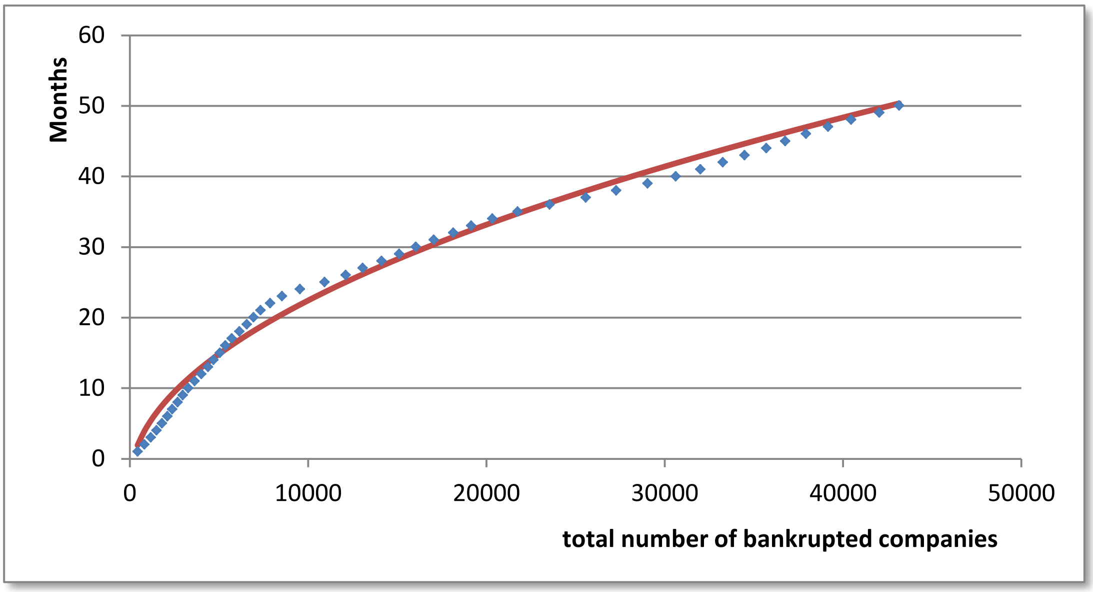

Figure 5.2(5)

Parameters:

k=0.5873 h=0.5217

(r^2=0.9903)

***Analysis to the empirical formula and the fitting result**

$$T_N = \sqrt{\frac{-h}{k^2 - 2k}}\left[0.5567k \cdot N^{\frac{1}{2}} + (0.845 - 1.8k)N^{-\frac{1}{2}}\right]$$

	business	manufacturing	trade
property involved	45058628.01	11124012.01	31360019.01
total number	44326537	10945663	30848020
average property	1.016515863	1.016294034	1.016597467

We summed up the number of bankrupted companies and their assets in each fields. The even scales of the corporations are almost the same in the three fields, thus the implied meaning of k and h of the three territories can be compared with each other.

Considering the meaning of k and h in economics, the bigger the k is, the less connected the corporations are (the bigger T_N is); the bigger the h is, the more stable the corporations are (the bigger the T_N is). They exert impact on the bankruptcy speed of companies at the same time.

Theoretically, business is engaged in the exchanging of products. The scarcity of raw materials and capital will firstly strike business companies, and secondly, manufacturing companies. As can be seen in the fitting parameters, the k for business is 0.9744 and the k for manufacturing is 0.9921. The two values are close and are both very big. This showed the potential relations between the companies in the two fields. Because of the less-related inner structure and the rigid demand from the society, manufacturing has the highest stability (h=0.7815), while business has the weakest stability (h=0.1638)

Trading companies involve in the international market, therefore, the connection between them is the tightest among the three economics fields (k=0.5873). Its stability (h=0.5217) is higher than the business but weaker than the manufacturing

5.4 Conclusion

By analyzing the most far-reaching economic crisis in American history, we can conclude that in a market without intervention or with stable policies, the relations between the number of bankrupted companies and the corresponding time tallies with our model.

The model of domino can precisely predict the number of bankrupted companies and provide timely market forecast.

We hope, through this attempt, to provide reference suggestions to the government and financial institutions in economic forecasting and early-warning of financial crisis and other important issues so that they can adjust the policies and take proper measures according to the changes of the situation, finally to ensure and reduce the financial risks.

Chapter VI Retrospect

The considerable application of domino effect aroused our interests in setting up the mathematical model of domino. People in the past described a certain situation as domino effect only to indicate the speeding-up impact, but didn't quantify the phenomenon. Through the study, we expressed the domino effect with simple mathematical formulas.

The study led us to the usage of mathematical software which is impossible for us to experience in daily study. Physical knowledge such as kinetic energy, gravitational energy, angular velocity, etc. is introduced in our mathematical model of domino.

Though we encountered a great many difficulties during the deduction, we tackled them one by one with the limit knowledge of mathematics we possessed. To enhance the practicability of our model, we reasonably simplified the model five times in our deduction and made adjustment to reduce the error caused by the simplification. The pursuit of preciseness encouraged us to conduct an experiment to prove the validity of our mathematical model.

However, in the process of encountering and tackling problems, the study has greatly improved our capacity of mathematical thinking, and has helped us to realize the beauty of mathematics.

From modeling to analyzing till application, one thing intrigued us is the close connection between mathematics and the real world. Throughout this project, we obtained the ability to analyze the world as well as to predict the future with mathematical models.

Despite all our effort to tackle with difficulties we encountered during the study, there are still some flaws in the paper that are far beyond our capability to deal with:

1. We did not manage to use the formula with thickness and loss of energy because of the extreme complexity and our unmanagement with elliptic function.

2. We simplified the multiple compositions of the atmosphere and complex market environment in our application. Besides, we are lack of the ability to obtain more statistics to further convince people about the reasonability of our application.

3. Apart from the loss of energy and thickness, there are many other factors to influence the falling-down process of dominoes, which are ignored by us.

At the very end, we want to express our sincere thankfulness to those who helped us out from troubles and provided us with instructions. This is surely not the destination of our project, we hope the model will continue to be perfected in the future as our mathematical knowledge grows.

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Appendix I: Programme of MATLAB & 1STOPT

Relations between angles

```
clear all;
clc;
n=1;
i=input('input theta 0=')
k=input('input k=')
i=tan(i)
while n<=150;
j=i*sqrt((1-k^2)*(i)^2+1+k^2-2*k*sqrt((1-k^2)*(i)^2+1))/(k*(i)^2+sqrt((1-k^2)*(i)^2+1))
p=atan(j)
mat(n)=p;
i=j;
n=n+1;
end
```

t-N

```
clear all;
clc;
z=2;
g=9.7930
syms x;
k=input('input k=');
i=input('input n=');
h=input('input h=');
y=input('input theta 0=');
while z<=i;
    if ceil(z)==i
       return;
    else
         r=z;
         n=1;
         m=sin(x);
    while n<=(r-1);
         b=(1-k)^(n)*x;
         w=m+sin(b);
         m=w;
         n=n+1;
     end
       t=(r-m)^(-0.5)
       fun=inline(t)
```

```
        j=quad(fun,y,0.5*pi)
        mat(z)=real(sqrt(2*h/(3*g))*j);
        z=z+1
    end
end

clear all;
clc;
n=2;
g=9.7930
k=input('input k=');
i=input('input n=');
h=input('input h=');
y=input('input theta 0=');
while n<=i;

j=2*k/((1-k)^(n)-1)*(sqrt(((1-k)^(n)-1)*0.5*pi/k+n)-sqrt(((1-k)^(
n)-1)*y/k+n));
        mat(n)=real(sqrt(2*h/(3*g))*j);
        n=n+1;
end
```

Q

```
clear all;
clc;
n=1;
g=9.7930
syms x
while n<=1000;
      Q=n/1000

j=(119.5827.*9.7930.*Q./(1-Q)+271.8651.*9.793.*(1-sin(0.1614+x)))
.^-0.5
      p=inline(j)
      r=quadl(p,1.49749,1.7322)
      mat(n)=2*r;
 n=n+1;
end
```

T_N-N

```
clear all;
clc;
z=2;
```

```
g=9.7930
syms x
k=0.6
h=0.05
i=input('input n=')
y=pi/3
u=0
for v=1:(i-1)
while z<=(v+1);
        r=z
        n=1;
        a=x
        m=sin(a)
   while n<=(r-1);
        b=(1-k)^(n)*a
        w=m+sin(b)
        m=w
        n=n+1;
   end
      t=(r-m)^(-0.5)
      fun=inline(t)
      j=quad(fun,y,0.5*pi)
      u=real(j)+u;
      z=z+1
      mat(v)=sqrt(2*h/(3*g))*u
end
end

clear all;
clc;
z=1;
g=9.7930
syms x
k=0.23256
h=0.043
i=50
y=1.3361
u=0;
for v=1:i
        r=z;
        s=r+1;
        n=1;
        a=x;
        m=sin(a);
   while n<=r;
          b=(1-k)^(n)*a;
```

```
          w=m+sin(b);
          m=w;
          n=n+1;
      end
         t=(s-m)^(-0.5)
         fun=inline(t)
            j=quad(fun,y,0.5*pi)
j1=real(2*k/((1-k)^(v+1)-1)*(sqrt(((1-k)^(v+1)-1)*0.5*pi/k+v+1)-s
qrt(((1-k)^(v+1)-1)*y/k+v+1)))
      if abs(((j1-j)/j))<=0.01
         j=j1
         break
       end
         u=j+u;
         z=z+1
       mat(v)=sqrt(2*h/(3*g))*u
   end
   for v=v:i
      w=v+1;
j=real(2*k/((1-k)^(w+1)-1)*(sqrt(((1-k)^(w+1)-1)*0.5*pi/k+w+1)-sq
rt(((1-k)^(w+1)-1)*y/k+w+1)))
      u=j+u;
      mat(v)=sqrt(2*h/(3*g))*u
   end
```

solving Q

```
fzero(@(Q)quadl(@(x)2*(119.5827.*9.7930.*Q./(1-Q)+271.8651.*9.
793.*(1-sin(0.1614+x))).^-0.5, 1.49749,1.7322)-0.02 ,0.4)
```

fitting a (as an example)

```
function [fitresult, gof] = createFit(k, h, b)
%      X Input : k
%      Y Input : h
%      Z Output: b

%% Fit: 'untitled fit 4'.
[xData, yData, zData] = prepareSurfaceData( k, h, b );

% Set up fittype and options.
ft = fittype( 'loess' );
opts = fitoptions( ft );
opts.Robust = 'LAR';
opts.Span = 0.5;
opts.Normalize = 'on';
```

```
% Fit model to data.
[fitresult, gof] = fit( [xData, yData], zData, ft, opts );

% Plot fit with data.
figure( 'Name', 'untitled fit 4' );
h1 = plot( fitresult, [xData, yData], zData );
legend( h1, 'untitled fit 4', 'b vs. k, h', 'Location',
'NorthEast' );
% Label axes
xlabel( 'k' );
ylabel( 'h' );
zlabel( 'b' );
grid on
view( 12.5, 66 );
```

fitting h and k in the application of CO2

```
function [fitresult, gof] = createFit(b, n)
%  Data for 'untitled fit 1' fit:
%      X Input : b
%      Y Output: n

%% Fit: 'untitled fit 1'.
[xData, yData] = prepareCurveData( b, n );

% Set up fittype and options.
ft =
fittype( 'sqrt(-h/(k^2-2*k))*(0.5567*k*x^0.5+(-1.8*k+0.845)*x^(-0
.5))+c', 'independent', 'x', 'dependent', 'y' );
opts = fitoptions( ft );
opts.Display = 'Off';
opts.Lower = [-Inf 0 0];
opts.MaxFunEvals = 60000;
opts.Robust = 'Bisquare';
opts.StartPoint = [0.350727103576883 0.939001561999887
0.875942811492984];
opts.Upper = [Inf Inf 1];

% Fit model to data.
[fitresult, gof] = fit( xData, yData, ft, opts );

% Plot fit with data.
figure( 'Name', 'untitled fit 1' );
h = plot( fitresult, xData, yData );
legend( h, 'n vs. b', 'untitled fit 1', 'Location', 'NorthEast' );
```

```
% Label axes
xlabel( 'b' );
ylabel( 'n' );
grid on
```

fitting the graph with 1stopt

```
Title "hello good morning";
Parameters k=[0.5,1],d,c,z;
Variable n,y;
Function
y=k*d*sqrt((2*k-k^2)^-1)*(sqrt((-pi/(2*k)+n+0.5)^3)-sqrt((-z/k+n+
0.5)^3)-sqrt((-pi/(2*k)+300.5)^3)+sqrt((-z/k+300.5)^3))+c;
Data;
n y
316 13.8
319 13.9
325 13.95
330 14
338 14.05
343 14.1
353 14.2
360 14.3
368 14.37
379 14.4
386 14.43
```

Appendix II: Statistics used to generate the figures

Statistics of figure 2.2.1

series 1	series 2	series 3	series 4
0.20303	0.20944	0.15005	0.15708
0.080815	0.083776	0.029955	0.031416
0.032301	0.03351	0.0059906	0.0062832
0.012919	0.013404	0.0011981	0.0012566
0.0051674	0.0053617	0.00023963	0.00025133
0.0020669	0.0021447	4.79E-05	5.03E-05
0.00082677	0.00085786	9.59E-06	1.01E-05
0.00033071	0.00034315	1.92E-06	2.01E-06
0.00013228	0.00013726	3.83E-07	4.02E-07
5.29E-05	5.49E-05	7.67E-08	8.04E-08
2.12E-05	2.20E-05	1.53E-08	1.61E-08
8.47E-06	8.78E-06	3.07E-09	3.22E-09
3.39E-06	3.51E-06	6.13E-10	6.43E-10
1.35E-06	1.41E-06	1.23E-10	1.29E-10
5.42E-07	5.62E-07	2.45E-11	2.57E-11
2.17E-07	2.25E-07	4.91E-12	5.15E-12
8.67E-08	9.00E-08	9.82E-13	1.03E-12
3.47E-08	3.60E-08	1.96E-13	2.06E-13
1.39E-08	1.44E-08	3.93E-14	4.12E-14
5.55E-09	5.76E-09	7.85E-15	8.24E-15
2.22E-09	2.30E-09	1.57E-15	1.65E-15
8.88E-10	9.21E-10	3.14E-16	3.29E-16
3.55E-10	3.68E-10	6.28E-17	6.59E-17
1.42E-10	1.47E-10	1.26E-17	1.32E-17
5.68E-11	5.90E-11	2.51E-18	2.64E-18

Statistics of Figure 2.3.2

series 1	series 2	percentage of deviation
0.76854	0.82356	0.071590288
0.59381	0.62268	0.048618245
0.50045	0.51815	0.035368169
0.44044	0.45246	0.027290891
0.39781	0.40657	0.022020563
0.36555	0.37228	0.018410614
0.34006	0.34541	0.015732518
0.31925	0.32364	0.013750979
0.30185	0.30554	0.012224615
0.28701	0.29017	0.011010069
0.27417	0.27691	0.009993799
0.26292	0.26532	0.009128252
0.25294	0.25508	0.008460504
0.24402	0.24593	0.007827227
0.23599	0.23771	0.007288444

0.22204	0.22347	0.006440281
0.21594	0.21725	0.0060665
0.21031	0.21152	0.005753412
0.2051	0.20622	0.005460751
0.20026	0.2013	0.005193249
0.19574	0.19672	0.005006641
0.19152	0.19243	0.004751462
0.18756	0.18842	0.004585199
0.18384	0.18464	0.00435161

Statistics for application of Financial Crisis in 5.2

Months	total number of bakrupted companies
1	1291
2	2438
3	3631
4	4667
5	4667
6	6590
7	7461
8	8361
9	9266
10	10155
11	11210
12	12329
13	13419
14	14356
15	15432
16	16385
17	17430
18	18722
19	20030
20	21315
21	22613
22	24397
23	26424
24	29377
25	33075
26	36251
27	38836
28	41723
29	44361
30	46919
31	49726
32	52746
33	55591
34	58905
35	62753
36	67523

25	33075
26	36251
27	38836
28	41723
29	44361
30	46919
31	49726
32	52746
33	55591
34	58905
35	62753
36	67523
37	72812