A new method to measure and compute areas of ellipses & volumes of ellipsoids for industrial applications

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Abstract

A new method to rapidly and accurately measure and compute the areas of ellipses and the volumes of ellipsoids for industrial applications was introduced in this study.

We analyzed the lengths of certain lines on pictures taken by cameras flanking the measured objects from random angles and used analytic geometry and geometric optics to calculate the area of an ellipse and the volume of an ellipsoid.

In model 1, we introduced a method to calculate the area of an ellipse under specific situations using lines on pictures. In model 2, we provided a formula for more accurate measurement of the area of an ellipse under general conditions. Furthermore, for circumstances in which the center of the ellipse deviates from the center of the work surface, we used statistical analysis to prove that the error is negligible in real-world manufacturing.

In model 3, we explored a generic approach to compute the volume of an ellipsoid by analyzing the geometric characteristics of the measured objects and adopting the principles of geometric optics. A general expression for the calculation was deduced, and by statistical approaches, the error of the expression was demonstrated to be insignificant on industrial manufacturing scale.

In conclusion, this paper offers a fast, accurate and effective approach to measure and compute the area and volume of oval objects. Because of its efficiency, accuracy, low cost, practicability and innovativeness, this method has great potential for industrial manufacturing and quality control, especially for SMEs (small and medium enterprises), as it satisfies the enterprises' needs for low costs and maximum benefits.

Keywords: ellipse ellipsoid area & volume measurement camera techniques

Introduction

Ellipse and ellipsoid are common in daily life but most in industrial manufacturing. The area of ellipse and the volume of ellipsoid are often critical in quality control of the product, especially when it comes to the more accurate measuring. Current existing methods face the inevitable dilemma between accuracy and efficiency, such as the ray-scanning technique which has too high a cost, or buoyancy technique which is not efficient. Other methods either require complicated equipment and training or are limited to the physical properties of the objects.

Our method has overcome the limitation of precious methods by offering a novel approach to measure and compute area of ellipse and volume of ellipsoid that innovatively uses photographic techniques to film from the flank of the objects. The method is low in cost, requires less in equipment and grounds, and in the meantime guarantees accuracy and efficiency. Our method is especially suitable for SMEs (small and medium-sized enterprises) with its high utility and innovativeness maximizing companies' profits.

1. New Method to Measure and Compute Areas of Ellipses and Volumes of Ellipsoids

1.1 Measured objects

Generally speaking, the new method has a wide range of utility in industrial manufacturing. Measuring areas of ellipses: small and medium-sized objects with elliptical cross-sections, such as elliptical tubes, oval gear flowmeter, oval manhole cover, elliptical head, oval flange and elliptical cylindrical container. Measuring volumes of ellipsoids: small and medium-sized ellipsoidal objects, such as melted ellipsoidal glassware, ellipsoidal filler, ellipsoidal container, ellipsoidal capacitors etc.

1.2 Measurement principles

Measuring device: a circular work surface with its center marked, three cameras erected on the circumference, equidistantly and at the same height, pointing at the line perpendicular to the work surface and passing the center of the circular work surface, and all connected to a computer.

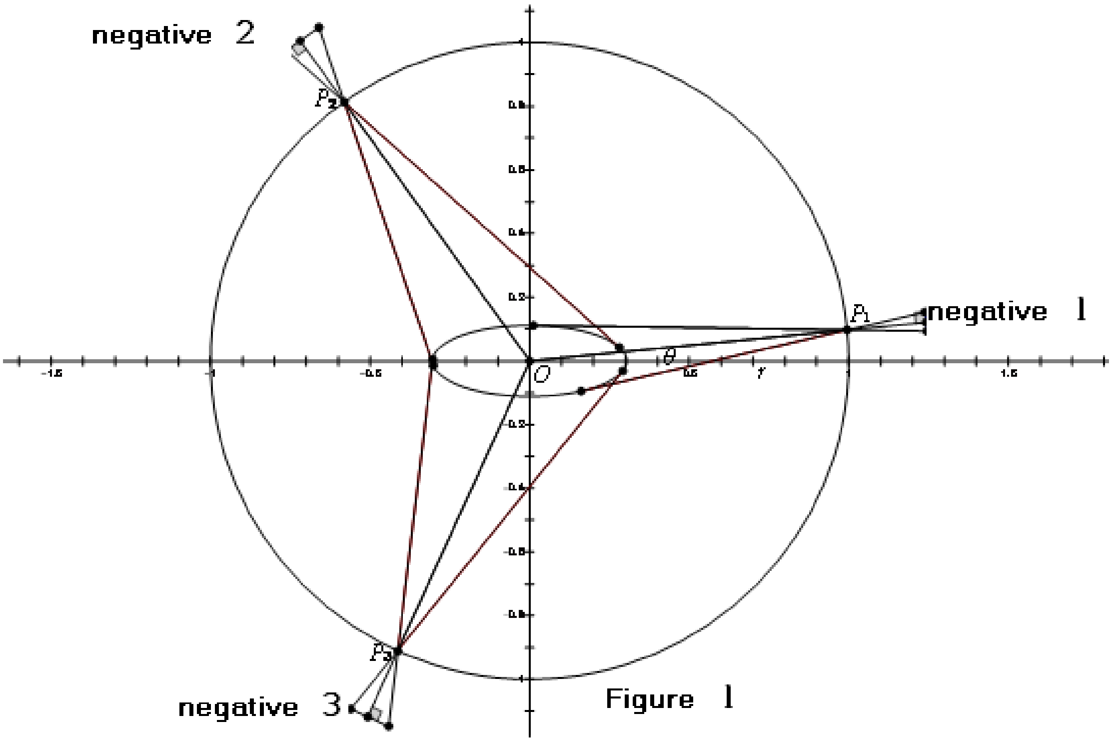

Figure 1

Measuring method: As it is shown in figure 1, when measuring the area of ellipse, we place the object at (or near) the center of the work surface so that the projection of the center of the ellipse on the work surface lies around the center of the work surface and the ellipse stays at the same height with the cameras. Three cameras P_1, P_2 and P_3 photograph the measured object simultaneously and meanwhile, images are transmitted to the computer. The computer takes the length of the segment lying on the axis of the symmetry on the vision on the computer screen and use these lengths X_1, X_2, X_3, a proportional coefficient of the scale of images displayed on the computer screen to that of images projected on negatives *K*, the radius of the circular work surface *r* and the focal length of cameras' lenses *f* to calculate the area of ellipse.

While measuring the volume of ellipsoid, we place the measured object at (or near) the center of the work surface so that the projection of the center of the ellipsoid on the work surface lies around the center of the work surface and the ellipsoid's equatorial plane stays at the same height with video cameras. Adjust

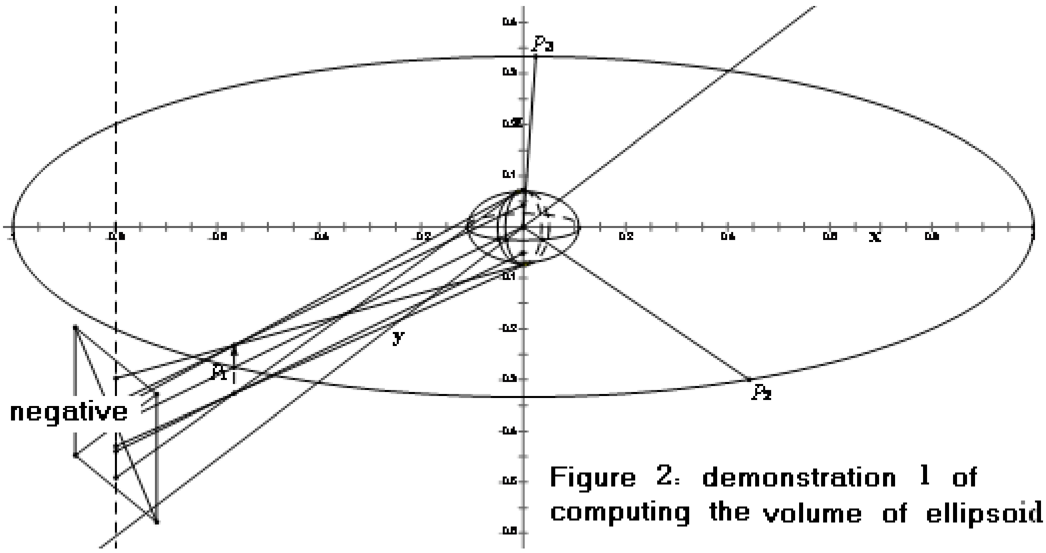

Figure 2: demonstration 1 of computing the volume of ellipsoid

the height of video cameras to set them below the ellipsoid's equatorial plane and then, let them ascends at a constant speed while filming. In the mean time, images are sent to the computer in real time. The computer automatically takes the lengths of segments lying on the vertical axis of symmetry of the vision shown on the screen. At the moment T_1, the length of the segment mentioned

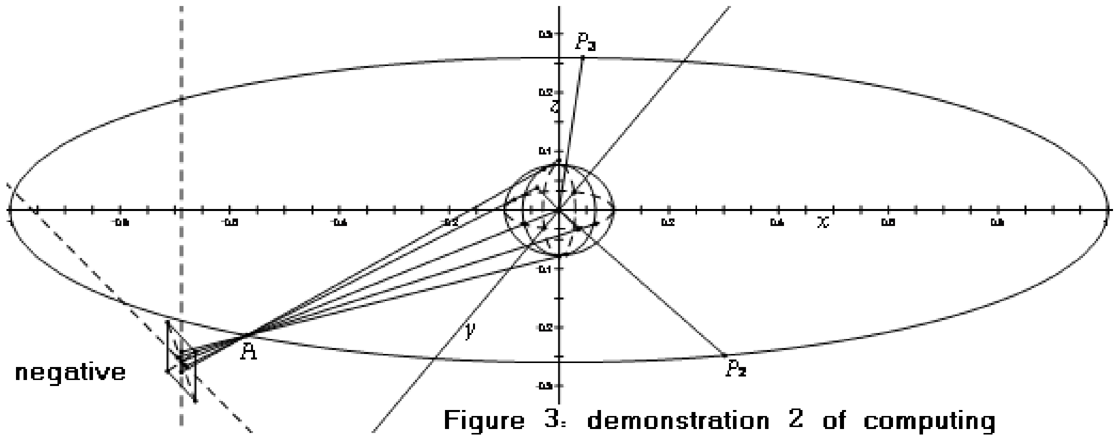

Figure 3: demonstration 2 of computing the volume of ellipsoids

above is the shortest. The computer then takes the lengths of segments lying on the horizontal axis of symmetry of the vision shown on the screen at the moment T_1. Thus, the volume of ellipsoid can be deduced with the lengths taken above, X_1, X_2, X_3, the moment T_1, the height of the video camera relative to the circular work surface before its uniform motion H_0, rising speed of video camera ΔH, a proportional coefficient of the scale of images displayed on the computer screen to that of images projected on negatives *K*, the radius of the circular work surface *r* and the focal length of cameras' lenses *f*.

1.3 Advantages of the measuring method

The new method's measurement device is simple, with mainly three video cameras and a computer. The device is easy to install and with a low cost, compared to other industrial methods. Moreover, the method is fast, efficient, accurate. When measuring the same type of objects in large scale, operators can set parameters beforehand so that the process can be run automatically, which saves labor force, time and cost. The method can be applied to quality control in industrial manufacturing and is able to promote enterprises' production efficiency.

2. Construction and Solution of Models And Analysis of Error

2.1 Establishment of Notations

Measuring the area of an ellipse:

a, b: ellipse's semi-major axis and semi-minor axis

r: radius of the circular work surface

$X_i (i=1,2,3)$: lengths of the segments lying on the axis of symmetry of the images displayed on the computer screen

K: proportionality coefficient of the scale of the images displayed on the computer screen to that of images projected on negatives

f: focal length of the video camera's lens

S: area of ellipse

Measuring the volume of an ellipsoid:

a, b, c: ellipsoid's semi-principal axes

$X_i (i=1,2,3)$: lengths of the segments lying on the axis of symmetry of the images displayed on the computer screen

T_1: the moment that the height of the optical center of the video camera's lens equals that of the center of the ellipsoid

H_0: original height of the optical center of the video camera's lens relative to the work surface

ΔH : ascension speed of the video camera

V: volume of the ellipsoid

(length unit: m, area unit: m^2 , volume unit: m^3 , angle unit: rad)

2.2 Construction and Solution of Models

In the plane of the measured elliptical cross-section (hereinafter referred to as the ellipse), we define a planar Cartesian coordinate system, with the center of the ellipse as the origin, the semi-major axis of the ellipse as the *x*-axis and the semi-minor axis of the ellipse as the *y*-axis. Let the orthographic projection of the center of the ellipse on the circular work surface (hereinafter referred to as the circle) coincides with the center of the circle (the error caused by the failure of the two centers to coincide exactly will be discussed in the "Analysis of Error" Section). Let us assume that the equation of the circle, when translated to the plane in which the ellipse lies, is $x^2+y^2=r^2$ and the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Because the optical centers of the three video cameras' lenses are all in the plane of the ellipse and the three cameras are placed equidistantly, the coordinates of the three video cameras are thus defined as:

$$P_1(r\cos\theta, r\sin\theta),\ P_2\left(r\cos\left(\theta+\frac{2\pi}{3}\right), r\sin\left(\theta+\frac{2\pi}{3}\right)\right)$$

$$\text{and } P_3\left(r\cos\left(\theta-\frac{2\pi}{3}\right), r\sin\left(\theta-\frac{2\pi}{3}\right)\right).$$

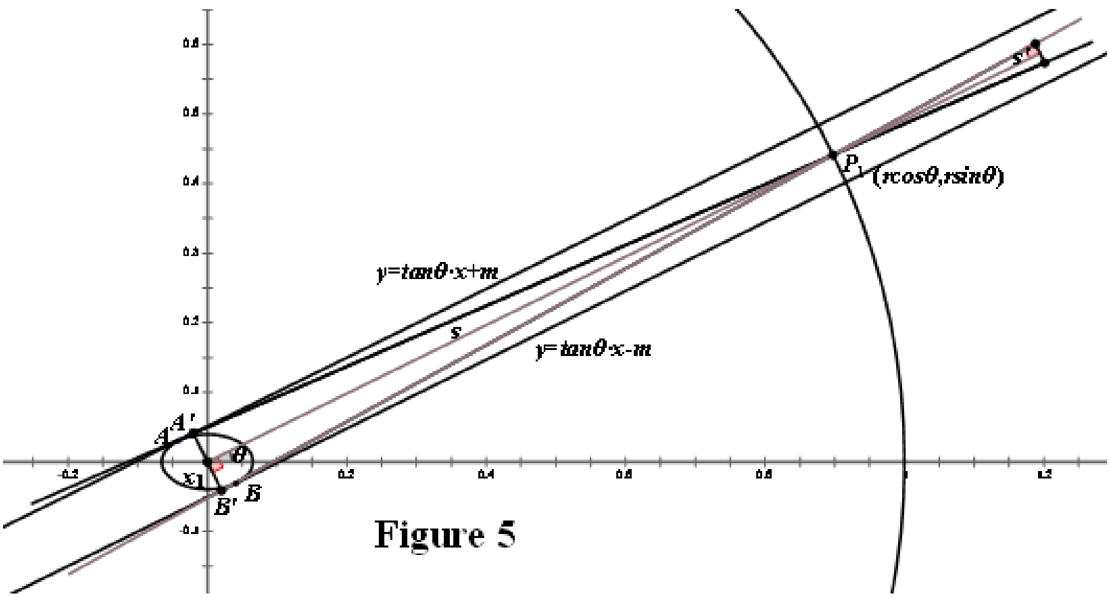

Figure 5

2.2.1 Model 1 We start with the following simplified situation: when the ellipse is sufficiently small compared with the circle (further explanation will be presented in the "Analysis of Error" Section), the model regards light rays pointing from every point on the ellipse to the lenses of the video cameras as approximately parallel. We will consider one of the video cameras $P_1\left(r\cos\theta, r\sin\theta\right)$; the cases of the other two are similar to the case of P_1. Because the video camera is always pointed at the center of the circle, the plane in which the negative is located is perpendicular to the line joining the video camera and the center of the circle $l_1: y = \tan\theta \bullet x$. Therefore, we can approximate the projection of the ellipse on the negative as that of the distance between two tangents of the ellipse that are parallel with l_1 on the negative, as shown in figure 5.

Let $x_1 = |A'B'|$ and two paralleling tangent lines are $y = \tan\theta \bullet x + m$ and $y = \tan\theta \bullet x - m$.

By combining the equation of the ellipse and that of tangents and eliminating y, we obtain:

$$\left(\frac{1}{a^2} + \frac{\tan^2\theta}{b^2}\right)x^2 + \frac{2m\tan\theta}{b^2}x + \frac{m^2 - b^2}{b^2} = 0$$

Because $\Delta = 0$, we have:

$$m^2 = b^2 + a^2\tan^2\theta \qquad (1)$$

Given equation 1, we substitute $b^2 + a^2\tan^2\theta$ for m^2 in the square form of the equation $x_1 = \frac{2|m|}{\sqrt{1+\tan^2\theta}}$, which yields:

$$\tan^2\theta = \frac{x_1^{\ 2} - 4b^2}{4a^2 - x_1^{\ 2}} \qquad (2)$$

Similarly, we have:

$$\tan^2\left(\theta+\frac{2\pi}{3}\right)=\frac{x_2^{\,2}-4b^2}{4a^2-x_2^{\,2}} \tag{3}$$

$$\tan^2\left(\theta-\frac{2\pi}{3}\right)=\frac{x_3^{\,2}-4b^2}{4a^2-x_3^{\,2}} \tag{4}$$

By transforming equation 3, we have

$$\frac{x_2^{\,2}-4b^2}{4a^2-x_2^{\,2}}=\left(\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}\right)^2=\frac{\tan^2\theta+3-2\sqrt{3}\tan\theta}{3\tan^2\theta+1+2\sqrt{3}\tan\theta}$$

Given equation 2, we substitute $\frac{x_1^{\,2}-4b^2}{4a^2-x_1^{\,2}}$ for $\tan^2\theta$ and similarly, $\sqrt{\frac{x_1^{\,2}-4b^2}{4a^2-x_1^{\,2}}}$ for $\tan\theta$ and we can obtain:

$$\frac{x_2^{\,2}-4b^2}{4a^2-x_2^{\,2}}=\frac{x_1^{\,2}-4b^2+3(4a^2-x_1^{\,2})-2\sqrt{3}\sqrt{(x_1^{\,2}-4b^2)(4a^2-x_1^{\,2})}}{4a^2-x_1^{\,2}+3(x_1^{\,2}-4b^2)+2\sqrt{3}\sqrt{(x_1^{\,2}-4b^2)(4a^2-x_1^{\,2})}} \tag{5}$$

Assume $A_i=\sqrt{4a^2-x_i^{\,2}}$ and $B_i=\sqrt{x_i^{\,2}-4b^2}\,(i=1,2,3)$, i.e.,

$$4a^2=A_1^{\,2}+x_1^{\,2}=A_2^{\,2}+x_2^{\,2}=A_3^{\,2}+x_3^{\,2}$$

$$4b^2=x_1^{\,2}-B_1^{\,2}=x_2^{\,2}-B_2^{\,2}=x_3^{\,2}-B_3^{\,2}$$

or $B_1^{\,2}+(x_2^{\,2}-x_1^{\,2})=B_2^{\,2}$, $A_1^{\,2}-(x_2^{\,2}-x_1^{\,2})=A_2^{\,2}$,

Substituting $\sqrt{4a^2-x_i^{\,2}}$ for A_i and $\sqrt{x_i^{\,2}-4b^2}$ for B_i, for $(i=1,2,3)$ in equation 5 yields:

$$\frac{B_2^{\,2}}{A_2^{\,2}}=\frac{B_1^{\,2}+3A_1^{\,2}-2\sqrt{3}A_1B_1}{3B_1^{\,2}+A_1^{\,2}+2\sqrt{3}A_1B_1}$$

Given $B_1^{\,2}+(x_2^{\,2}-x_1^{\,2})=B_2^{\,2}$ and $A_1^{\,2}-(x_2^{\,2}-x_1^{\,2})=A_2^{\,2}$, we have:

$$\frac{B_1^{\,2}+(x_2^{\,2}-x_1^{\,2})}{A_1^{\,2}-(x_2^{\,2}-x_1^{\,2})}=\frac{B_1^{\,2}+3A_1^{\,2}-2\sqrt{3}A_1B_1}{3B_1^{\,2}+A_1^{\,2}+2\sqrt{3}A_1B_1}$$

Simplify the expression above to obtain:

$$3(B_1^{\,2}-A_1^{\,2})+4(x_2^{\,2}-x_1^{\,2})+2\sqrt{3}A_1B_1=0 \tag{6}$$

Similarly, from equation 4, we can obtain:

$$3(B_1^{\ 2}-A_1^{\ 2})+4(x_3^{\ 2}-x_1^{\ 2})-2\sqrt{3}A_1B_1=0 \qquad (7)$$

Because the square of the area of the ellipse is $S^2=\pi^2a^2b^2$, we substitute $\frac{A_1^{\ 2}+x_1^{\ 2}}{4}$ for a^2 and $\frac{x_1^{\ 2}-B_1^{\ 2}}{4}$ for b^2 and obtain:

$$S^2=\frac{\pi^2}{16}(A_1^{\ 2}+x_1^{\ 2})(x_1^{\ 2}-B_1^{\ 2})=\frac{\pi^2}{16}\left[x_1^{\ 2}(A_1^{\ 2}-B_1^{\ 2})+x_1^{\ 4}-(A_1B_1)^2)\right]$$

Assume that $u=A_1^{\ 2}-B_1^{\ 2}$ and $v=A_1B_1$ and substitute u for $A_1^{\ 2}-B_1^{\ 2}$ and v for A_1B_1 in equations 6 and 7; then, we obtain:

$$-3u+4(x_2^{\ 2}-x_1^{\ 2})+2\sqrt{3}v=0$$

$$-3u+4(x_3^{\ 2}-x_1^{\ 2})-2\sqrt{3}v=0$$

By equating the two equations above and we can obtain:

$$u=\frac{2}{3}(x_2^{\ 2}+x_3^{\ 2}-2x_1^{\ 2})$$

$$v=\frac{\sqrt{3}}{3}(x_3^{\ 2}-x_2^{\ 2})$$

Substituting $\frac{2}{3}(x_2^{\ 2}+x_3^{\ 2}-2x_1^{\ 2})$ for $A_1^{\ 2}-B_1^{\ 2}$ and $\frac{\sqrt{3}}{3}(x_3^{\ 2}-x_2^{\ 2})$ for A_1B_1 in the expression for the square of the area of the ellipse yields:

$$S^2=\frac{\pi^2}{16}\left[x_1^{\ 2}\frac{2}{3}(x_2^{\ 2}+x_3^{\ 2}-2x_1^{\ 2})+x_1^{\ 4}-\frac{1}{3}(x_3^{\ 2}-x_2^{\ 2})^2\right]$$

$$=\frac{\pi^2}{48}(2x_1^{\ 2}x_2^{\ 2}+2x_1^{\ 2}x_3^{\ 2}+2x_2^{\ 2}x_3^{\ 2}-x_1^{\ 4}-x_2^{\ 4}-x_3^{\ 4})$$

Thus:

$$S=\frac{\sqrt{3}\pi}{12}\sqrt{2x_1^{\ 2}x_2^{\ 2}+2x_1^{\ 2}x_3^{\ 2}+2x_2^{\ 2}x_3^{\ 2}-x_1^{\ 4}-x_2^{\ 4}-x_3^{\ 4}} \qquad (8)$$

In the special case, when the ellipse degenerates to circle and thus $x_1=x_2=x_3$, we have $S_{circle}=\pi\frac{x_1^{\ 2}}{4}$.

2.2.2 Model 2 We do not consider the simplified situation described above for model 2: this model does not regard light rays pointing from every point on the ellipse to the lenses of video cameras as parallel but rather describes the actual circumstances. We still just consider only one of the cameras $P_1(r\cos\theta, r\sin\theta)$; the situations concerning the other two cameras are similar to that of P_1. Also, we still start from a simpler situation, when the ellipse degenerates into the circle. Because the video camera is always pointed at the center of the circle, the plane in which the negative is located is perpendicular to the line joining the video camera and the center of the circle $l_1: y = \tan\theta \bullet x$. Therefore, we can regard the projection of the ellipse on the negative as that of the segment *A'B'* produced by the two tangents of the circle from point P_1 intercepting the line crossing the center and perpendicular to l_1, as shown in figure 6.

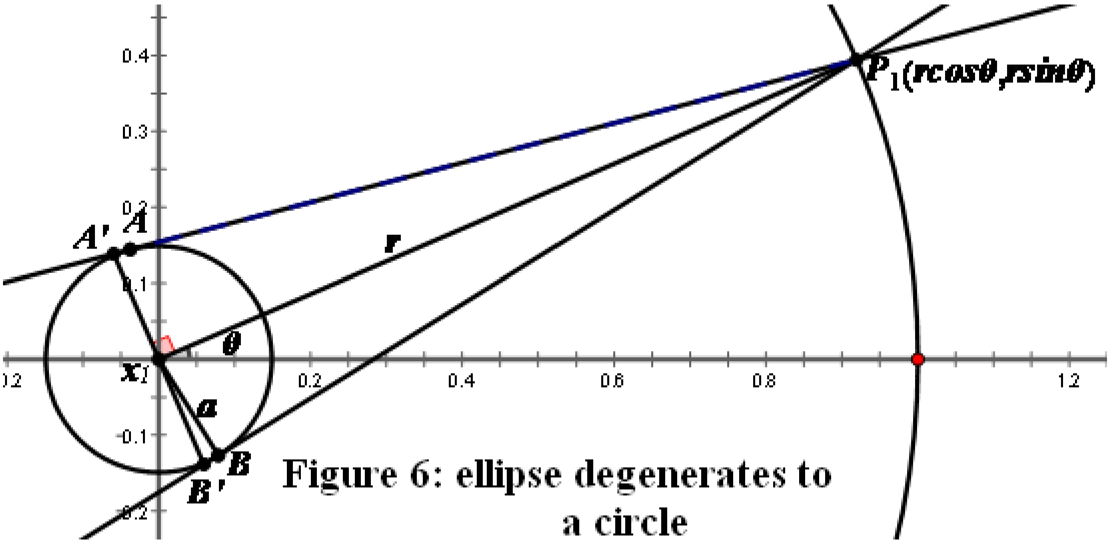

Figure 6: ellipse degenerates to a circle

Assuming $x_1 = |A'B'|$, because $\frac{r}{\sqrt{r^2 - a^2}} = \frac{\frac{x_1}{2}}{a}$, we have $a^2 = \frac{r^2 x_1^2}{4r^2 + x_1^2}$,

Therefore,

$$S_{circle} = \pi a^2 = \pi \frac{r^2 {x_1}^2}{4r^2 + {x_1}^2} \qquad (9)$$

Now, we consider the general situation in which the ellipse does not degenerate. In the same manner, we regard the projection of the ellipse on the negative as that of the segment *A'B'* produced by the two tangents of the ellipse from point P_1 intercepting the line crossing the center and perpendicular to l_1, shown in figure 7.

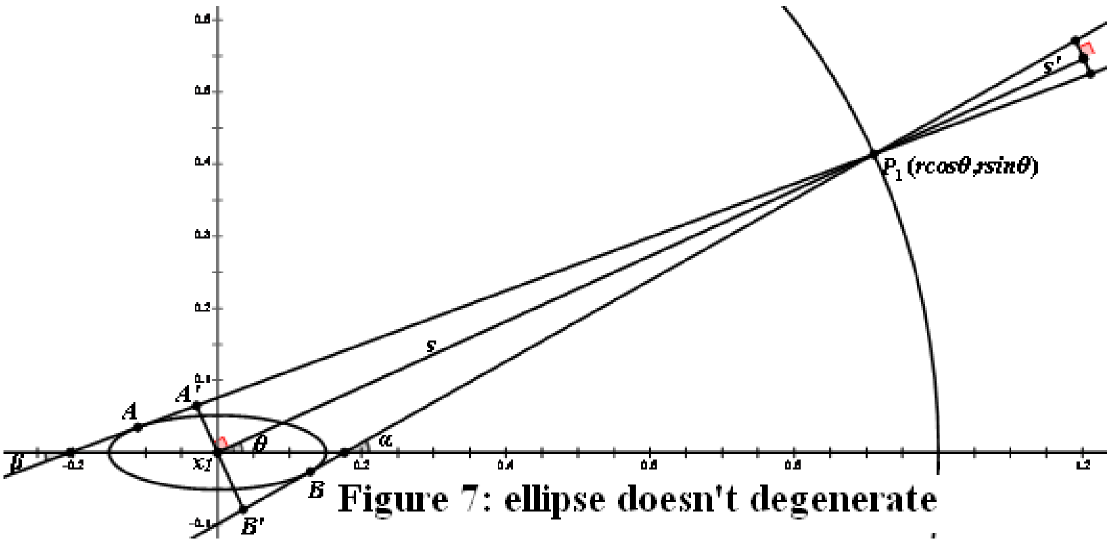

Figure 7: ellipse doesn't degenerate

We refer to the two tangent points of two of the ellipse's tangent lines, both of which originate from P_1, as point *A* and point *B*. The coordinates of point *A* are (x_a, y_a), and those of point *B* are (x_b, y_b). Thus, the equation of line *AB* is $\frac{r\cos\theta}{a^2}x + \frac{r\sin\theta}{b^2}y = 1$.

Substitute $\frac{r\sin\theta - y_a}{r\cos\theta - x_a}$ for $\tan\alpha$ and $\frac{r\sin\theta - y_b}{r\cos\theta - x_b}$ for $\tan\beta$ in

$$x_1 = |A'B'| = r[\tan(\alpha-\theta) + \tan(\theta-\beta)] = r(\frac{\tan\alpha - \tan\theta}{1+\tan\theta\tan\alpha} + \frac{\tan\theta - \tan\beta}{1+\tan\theta\tan\beta})$$

We obtain:

$$x_1 = r\left[\frac{(r\sin\theta - y_a) - \tan\theta(r\cos\theta - x_a)}{(r\cos\theta - x_a) + \tan\theta(r\sin\theta - y_a)} + \frac{\tan\theta(r\cos\theta - x_b) - (r\sin\theta - y_b)}{(r\cos\theta - x_b) + \tan\theta(r\sin\theta - y_b)}\right]$$

$$= r\left(\frac{x_a\sin\theta - y_a\cos\theta}{r - x_a\cos\theta - y_a\sin\theta} + \frac{y_b\cos\theta - x_b\sin\theta}{r - x_b\cos\theta - y_b\sin\theta}\right)$$

$$= r\frac{r\sin\theta(x_a - x_b) - r\cos\theta(y_a - y_b) - x_a y_b + x_b y_a}{r^2 - r\cos\theta(x_a + x_b) - r\sin\theta(y_a + y_b) + \cos^2\theta x_a x_b + \sin^2\theta y_a y_b + \cos\theta\sin\theta(x_a y_b + x_b y_a)} \quad (10)$$

According to the equation for line *AB*, we have

$$y_a = -\frac{b^2\cos\theta}{a^2\sin\theta}x_a + \frac{b^2}{r\sin\theta},\quad y_b = -\frac{b^2\cos\theta}{a^2\sin\theta}x_b + \frac{b^2}{r\sin\theta},$$

Therefore,

$$y_a - y_b = -\frac{b^2\cos\theta}{a^2\sin\theta}(x_a - x_b),\quad y_a + y_b = -\frac{b^2\cos\theta}{a^2\sin\theta}(x_a + x_b) + \frac{2b^2}{r\sin\theta},$$

$$y_a y_b = \frac{b^4\cos^2\theta}{a^4\sin^2\theta}x_a x_b - \frac{b^4\cos\theta}{ra^2\sin\theta}(x_a + x_b) + \frac{b^4}{r^2\sin^2\theta},$$

$$x_a y_b = -\frac{b^2\cos\theta}{a^2\sin\theta}x_a x_b + \frac{b^2}{r\sin\theta}x_a,\quad x_b y_a = -\frac{b^2\cos\theta}{a^2\sin\theta}x_a x_b + \frac{b^2}{r\sin\theta}x_b,$$

Plug these above into equation 10, we have:

$$x_1 = r\frac{(x_a - x_b)\dfrac{r^2b^2\cos^2\theta + r^2a^2\sin^2\theta - a^2b^2}{ra^2\sin\theta}}{-(x_a + x_b)\dfrac{\cos\theta(a^2 - b^2)(r^2 - b^2)}{ra^2} + x_a x_b\dfrac{\cos^2\theta(a^2 - b^2)^2}{a^4} + \dfrac{(r^2 - b^2)^2}{r^2}} \quad (11)$$

Joining the equation for line *AB* and that of the ellipse, we obtain:

$$\left(\frac{b^2\cos\theta + a^2\sin\theta}{a^4}\right)x^2 - \frac{2b^2\cos\theta}{a^2 r}x + \frac{b^2 - r^2\sin^2\theta}{r^2} = 0$$

Therefore,

$$x_a + x_b = \frac{2a^2b^2\cos\theta}{r(b^2\cos^2\theta + a^2\sin^2\theta)},\quad x_a x_b = \frac{a^4(b^2 - r^2\sin^2\theta)}{r^2(b^2\cos^2\theta + a^2\sin^2\theta)},$$

$$x_a - x_b = \frac{2a^2\sin\theta\sqrt{r^2b^2\cos^2\theta + r^2a^2\sin^2\theta - a^2b^2}}{r(b^2\cos^2\theta + a^2\sin^2\theta)},$$

Plug these above into equation 11, we can have:

$$x_1 = r\frac{2(r^2b^2\cos^2\theta + r^2a^2\sin^2\theta - a^2b^2)\sqrt{r^2b^2\cos^2\theta + r^2a^2\sin^2\theta - a^2b^2}}{-2b^2\cos^2\theta(a^2-b^2)(r^2-b^2) + (b^2 - r^2\sin^2\theta)\cos^2\theta(a^2-b^2)^2 + (b^2\cos^2\theta + a^2\sin^2\theta)(r^2-b^2)^2}$$

$$= r\frac{2(r^2b^2\cos^2\theta + r^2a^2\sin^2\theta - a^2b^2)\sqrt{r^2b^2\cos^2\theta + r^2a^2\sin^2\theta - a^2b^2}}{b^2\cos^2\theta(r^2-a^2)^2 + a^2\sin^2\theta(r^2-b^2)^2 - r^2\sin^2\theta\cos^2\theta\left[(r^2-a^2)^2 + (r^2-b^2)^2 - 2(r^2-a^2)(r^2-b^2)\right]}$$

$$= r\frac{2\left[b^2\cos^2\theta(r^2-a^2) + a^2\sin^2\theta(r^2-b^2)\right]\sqrt{b^2\cos^2\theta(r^2-a^2) + a^2\sin^2\theta(r^2-b^2)}}{\cos^2\theta(r^2-a^2)^2\left[r^2\cos^2\theta - (r^2-b^2)\right] + \sin^2\theta(r^2-b^2)^2\left[r^2\sin^2\theta - (r^2-a^2)\right] + 2r^2\sin^2\theta\cos^2\theta(r^2-a^2)(r^2-b^2)}$$

(12)

Assume $u = r^2 - a^2$ and $v = r^2 - b^2$, i.e., $a^2 = r^2 - u$ and $b^2 = r^2 - v$.

Substitute $r^2 - u$ for a^2 and $r^2 - v$ for b^2 in equation 12:

$$x_1 = r\frac{2\left[(r^2-v)u\cos^2\theta + (r^2-u)v\sin^2\theta\right]\sqrt{(r^2-v)u\cos^2\theta + (r^2-u)v\sin^2\theta}}{u^2\cos^2\theta(r^2\cos^2\theta - v) + v^2\sin^2\theta(r^2\sin^2\theta - u) + 2r^2uv\sin^2\theta\cos^2\theta}$$

$$= r\frac{2\left[r^2(u\cos^2\theta + v\sin^2\theta) - uv\right]\sqrt{r^2(u\cos^2\theta + v\sin^2\theta) - uv}}{r^2(u\cos^2\theta + v\sin^2\theta)^2 - uv(u\cos^2\theta + v\sin^2\theta)}$$

$$= r\frac{2\sqrt{r^2(u\cos^2\theta + v\sin^2\theta) - uv}}{u\cos^2\theta + v\sin^2\theta}$$

Simplify the expression above:

$$x_1^2(u\cos^2\theta + v\sin^2\theta)^2 = 4r^4(u\cos^2\theta + v\sin^2\theta) - 4r^2uv$$

Thus, we have:

$$u\cos^2\theta + v\sin^2\theta = 2\frac{r^4 \pm \sqrt{r^6 - x_1^2uv}}{x_1^2} \qquad (13)$$

Because of the approximation made in this model, the radius of the circle must be at least 5.53 times longer than the semi-major axis of the ellipse for this model's margin of error to be less than 0.5%. A specific explanation will be presented in the "Analysis of Error" Section.

Because $r > 5.53a > x_1$, we have $2\frac{r^4 + \sqrt{r^6 - x_1^2uv}}{x_1^2} > 2\frac{r^4}{x_1^2} > 2\frac{r^4}{r^2} > r^2$, Again, because $u\cos^2\theta + v\sin^2\theta = r^2 - a^2\cos^2\theta - b^2\sin^2\theta < r^2$, we have $2\frac{r^4 + \sqrt{r^6 - x_1^{\ 2}uv}}{x_1^{\ 2}} > u\cos^2\theta + v\sin^2\theta$, which contradicts equation 13. Therefore, we can rule out the possibility of the plus sign and obtain:

$$u\cos^2\theta + v\sin^2\theta = 2\frac{r^4 - \sqrt{r^6 - x_1^{\ 2}uv}}{x_1^{\ 2}} \quad (14)$$

Similarly, we have

$$u\cos^2(\theta + \frac{2\pi}{3}) + v\sin^2(\theta + \frac{2\pi}{3}) = 2\frac{r^4 - \sqrt{r^6 - x_2^{\ 2}uv}}{x_2^{\ 2}}$$

$$u\cos^2(\theta - \frac{2\pi}{3}) + v\sin^2(\theta - \frac{2\pi}{3}) = 2\frac{r^4 - \sqrt{r^6 - x_3^{\ 2}uv}}{x_3^{\ 2}}$$

Assume

$$2p_1 = u\cos^2\theta + v\sin^2\theta \quad (15)$$

$$2p_2 = u\cos^2(\theta + \frac{2\pi}{3}) + v\sin^2(\theta + \frac{2\pi}{3})$$

$$= u(\frac{1}{4}\cos^2\theta + \frac{3}{4}\sin^2\theta + \frac{\sqrt{3}}{2}\sin\theta\cos\theta) + v(\frac{3}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta - \frac{\sqrt{3}}{2}\sin\theta\cos\theta) \quad (16)$$

$$2p_3 = u\cos^2(\theta - \frac{2\pi}{3}) + v\sin^2(\theta - \frac{2\pi}{3})$$

$$= u(\frac{1}{4}\cos^2\theta + \frac{3}{4}\sin^2\theta - \frac{\sqrt{3}}{2}\sin\theta\cos\theta) + v(\frac{3}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta + \frac{\sqrt{3}}{2}\sin\theta\cos\theta) \quad (17)$$

From equation 15, 16 and 17, we can obtain:

$$u + v = \frac{4}{3}(p_1 + p_2 + p_3) \quad (18)$$

From $((16)-(17))^2$, we can obtain:

$$3\cos^2\theta\sin^2\theta(u-v)^2 = 4(p_2 - p_3)^2 \quad (19)$$

From equation 15, we can obtain: $\cos^2\theta = \frac{2p_1 - v}{u - v}$ and $\sin^2\theta = \frac{2p_1 - u}{v - u}$.

Substitute $\dfrac{2p_1 - v}{u - v}$ for $\cos^2\theta$ and $\dfrac{2p_1 - u}{v - u}$ for $\sin^2\theta$ in equation 19 and simplify the equation. Again, substitute $\dfrac{4}{3}(p_1 + p_2 + p_3)$ for $u + v$ and simplify the equation to obtain:

$$2p_1p_2 + 2p_1p_3 + 2p_2p_3 - \frac{3}{4}uv = p_1^2 + p_2^2 + p_3^2 \qquad (20)$$

in which, $p_i = \dfrac{r^4 - \sqrt{r^6 - x_i^2 uv}}{x_i^2} (i = 1,2,3)$

Assuming $uv = x$, substitute x for uv in equation 20; then we have:

$$2\left(\frac{r^4 - \sqrt{r^6 - x_1^2 x}}{x_1^2}\frac{r^4 - \sqrt{r^6 - x_2^2 x}}{x_2^2} + \frac{r^4 - \sqrt{r^6 - x_1^2 x}}{x_1^2}\frac{r^4 - \sqrt{r^6 - x_3^2 x}}{x_3^2} + \frac{r^4 - \sqrt{r^6 - x_2^2 x}}{x_2^2}\frac{r^4 - \sqrt{r^6 - x_3^2 x}}{x_3^2}\right)$$

$$= \left(\frac{r^4 - \sqrt{r^6 - x_1^2 x}}{x_1^2}\right)^2 + \left(\frac{r^4 - \sqrt{r^6 - x_2^2 x}}{x_2^2}\right)^2 + \left(\frac{r^4 - \sqrt{r^6 - x_3^2 x}}{x_3^2}\right)^2 + \frac{3}{4}x$$

in which, r, x_1, x_2 and x_3 are character constants and x is unknown.

Because the equation provided above does not have an analytical expression, certain approximation is required.

For convenience of expression, we assume that $r = 1$. When the value of r is not 1, the corresponding deduction process is similar.

For $\sqrt{1 - x_i^2 x}$, when $1 = r > 5.53a > x_i > x_i^2$, the first through fifth derivatives at $x = 1$ all exist. We may use the expression's Taylor series expansion at $x = 1$ for an approximation:

$$\sqrt{1 - x_i^2 x} = \sqrt{1 - x_i^2} - \frac{x_i^2(x-1)}{2\sqrt{1 - x_i^2}} - \frac{x_i^4(x-1)^2}{8\sqrt{(1 - x_i^2)^3}} - \frac{x_i^6(x-1)^3}{16\sqrt{(1 - x_i^2)^5}} - \frac{5x_i^8(x-1)^4}{128\sqrt{(1 - x_i^2)^7}} + o[x-1]^5$$

Because $1 = r > 5.53a$, we have

$$0.9362 < (1 - 0.18^2)^2 < (1 - a^2)^2 < x = (1 - a^2)(1 - b^2) < 1,$$

$$0 < x_i^2 < \left(\frac{1}{\sqrt{1 - a^2}} a \times 2\right)^2 < \left(\frac{1}{\sqrt{1 - 0.18^2}} \times 0.18 \times 2\right)^2 < 0.1339,$$

Substitute the second-degree Taylor series of $\sqrt{1 - x_i^2 x}$, i.e.,

$\sqrt{1-x_i^2}-\frac{x_i^2(x-1)}{2\sqrt{1-x_i^2}}$, for the original expression and transform the original equation into a quadratic equation. The relative error produced by this process is

$$\varepsilon(x,x_i^2)=\frac{\left[\sqrt{1-x_i^2}-\frac{x_i^2(x-1)}{2\sqrt{1-x_i^2}}\right]-\sqrt{1-x_i^2x}}{\sqrt{1-x_i^2x}}$$

We use Mathematica to plot the graph of $\varepsilon(x,x_i^2)$ ($x\in(0.9362,1)$, $x_i^2\in(0,0.1339)$):

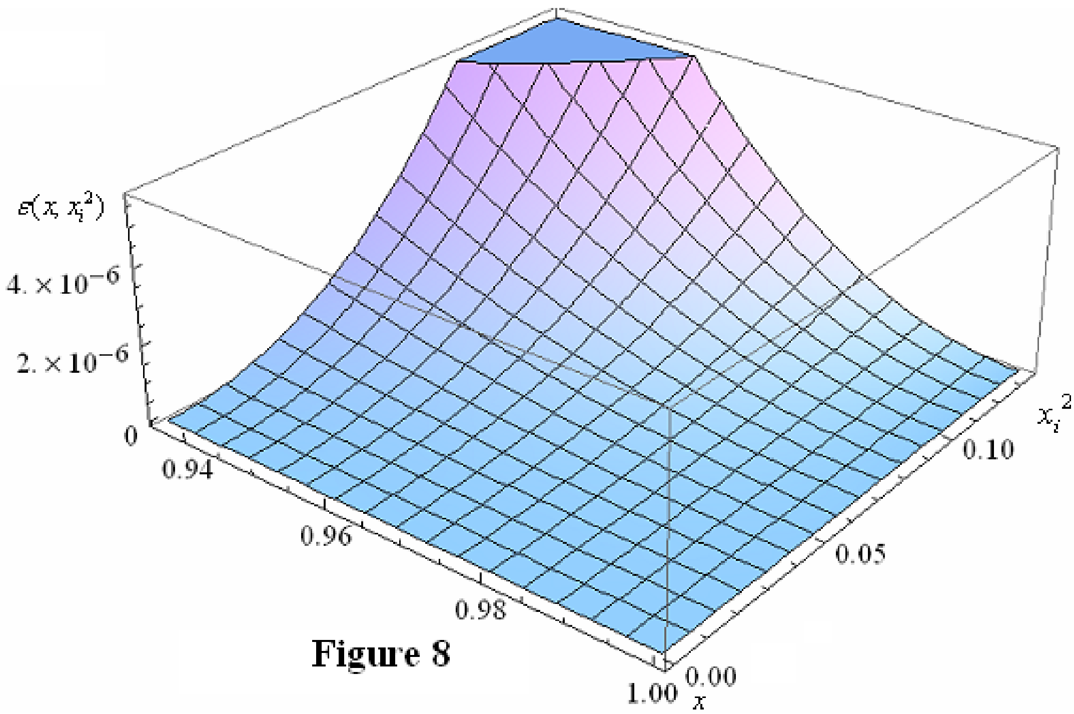

Figure 8

The graph demonstrates that the relative error is significantly smaller than 0.5%. Therefore, we can use the second-degree Taylor series of $\sqrt{1-x_i^2x}$, $\sqrt{1-x_i^2}-\frac{x_i^2(x-1)}{2\sqrt{1-x_i^2}}$, to replace the original expression in the approximation. Whereas if we used the third-degree Taylor series for the substitution, the solution would be more accurate ($\varepsilon<1\times10^{-8}$), the original equation would become a quartic equation, whose solution is not suitable for expression. Thus, we only use the first two terms and transform the equation into:

$$2\frac{1-\left[\sqrt{1-x_1^2}-\frac{x_1^2(x-1)}{2\sqrt{1-x_1^2}}\right]}{x_1^2}\frac{1-\left[\sqrt{1-x_2^2}-\frac{x_2^2(x-1)}{2\sqrt{1-x_2^2}}\right]}{x_2^2}+2\frac{1-\left[\sqrt{1-x_1^2}-\frac{x_1^2(x-1)}{2\sqrt{1-x_1^2}}\right]}{x_1^2}\frac{1-\left[\sqrt{1-x_3^2}-\frac{x_3^2(x-1)}{2\sqrt{1-x_3^2}}\right]}{x_3^2}$$

$$+2\frac{1-\left[\sqrt{1-x_2^2}-\frac{x_2^2(x-1)}{2\sqrt{1-x_2^2}}\right]}{x_2^2}\frac{1-\left[\sqrt{1-x_3^2}-\frac{x_3^2(x-1)}{2\sqrt{1-x_3^2}}\right]}{x_3^2}=\frac{\left\{1-\left[\sqrt{1-x_1^2}-\frac{x_1^2(x-1)}{2\sqrt{1-x_1^2}}\right]\right\}^2}{x_1^4}+\frac{\left\{1-\left[\sqrt{1-x_2^2}-\frac{x_2^2(x-1)}{2\sqrt{1-x_2^2}}\right]\right\}^2}{x_2^4}$$

$$+\frac{\left\{1-\left[\sqrt{1-x_3^2}-\frac{x_3^2(x-1)}{2\sqrt{1-x_3^2}}\right]\right\}^2}{x_3^4}+\frac{3}{4}x$$

Then, we can transform the equation into the general form of $Ax^2+Bx+C=0$ and use Mathematica to solve the equation. Because the expression of x is particularly long, we does not present it in full here.

Given the expression of x, or uv, we use it to replace the uv in equation 18: $u+v=\frac{4}{3}(p_1+p_2+p_3)$, in which $p_i\approx\frac{1-\left[\sqrt{1-x_i^2}-\frac{x_i^2(uv-1)}{2\sqrt{1-x_i^2}}\right]}{x_i^2}(i=1,2,3)$ to obtain the expression of $u+v$.

Because

$$uv=(1-a^2)(1-b^2)=1-(a^2+b^2)+a^2b^2,$$

$$u+v=(1-a^2)+(1-b^2)=2-(a^2+b^2),$$

we have:

$$ab=\sqrt{uv-[1+(u+v)-2]}=\sqrt{uv-(u+v)+1}$$

Thus, $S=\pi ab=\pi\sqrt{uv-(u+v)+1}$. The expression of S then can be determined after the substitution of uv and $u+v$. For the convenience of expression, we assume that $t_i=x_i^2(i=1,2,3)$ and simplify the expression using Mathematica to obtain:

$S = (1/\sqrt{6})$ [Sqrt] $((-4 (-1+\sqrt{1-t_1}) t_1 (-1+t_2) t_2^2 (-1+t_3) t_3^2 (-5-\sqrt{1-t_2} -\sqrt{1-t_3} + t_3$

$+ \sqrt{1-t_2} t_3 + t_2(1+ \sqrt{1-t_3} +3t_3))-2t_1^3 (-1+t_2) t_2(-1+t_3) t_3(-2 (-1+ \sqrt{1-t_2})$

$t_3(-6-\sqrt{1-t_1} -2\sqrt{1-t_3} + (-2+\sqrt{1-t_1}) t_3)+ t_2^2 (-2 (-2+\sqrt{1-t_1}) (-1+\sqrt{1-t_3})+(-7+3$

$\sqrt{1-t_1}) t_3+3 (-3+\sqrt{1-t_1}) t_3^2)+t_2 (2 (6+\sqrt{1-t_1} +2 \sqrt{1-t_2}) (-1+\sqrt{1-t_3})-(-55+9$

$\sqrt{1-t_1} +12 \sqrt{1-t_2} +12 \sqrt{1-t_3})t_3 +(-7+3 \sqrt{1-t_1}) t_3^2))- t_1^4(-1+t_2) t_2(-1+t_3)$

$t_3(-4(-1+\sqrt{1-t_2}) t_3(1+\sqrt{1-t_3} +3 t_3)+ 3t_2^2(4-4\sqrt{1-t_3} +(-3+2\sqrt{1-t_3})t_3+3 t_3^2)+ t_2(-4$

$(1+\sqrt{1-t_2}) (-1+\sqrt{1-t_3})+(-23+6 \sqrt{1-t_2} +6 \sqrt{1-t_3}) t_3 +(-9+6 \sqrt{1-t_2}) t_3^2))+t_1^2$

$(-1+t_2) t_2 (-1+t_3) t_3 (-4 (-1+\sqrt{1-t_2}) t_3 (-5-\sqrt{1-t_1} - \sqrt{1-t_3} +t_3+ \sqrt{1-t_1} t_3)+ t_2^2 (-4$

$(1+\sqrt{1-t_1}) (-1+\sqrt{1-t_3})+(-27+10 \sqrt{1-t_1} +2 \sqrt{1-t_3} +4 \sqrt{1-t_1} \sqrt{1-t_3}) t_3+3 (-7+6$

$\sqrt{1-t_1}) t_3^2)+t_2 (4 (5+\sqrt{1-t_1}+\sqrt{1-t_2}) (-1+\sqrt{1-t_3})-(-107+38 \sqrt{1-t_1} +14 \sqrt{1-t_2} +4$

$\sqrt{1-t_1} \sqrt{1-t_2} +14 \sqrt{1-t_3} +4 \sqrt{1-t_1} \sqrt{1-t_3})t_3 +(-27+10 \sqrt{1-t_1} +2 \sqrt{1-t_2} +4$

$\sqrt{1-t_1} \sqrt{1-t_2}) t_3^2))+(2 \sqrt{1-t_1} \sqrt{1-t_2} +2 \sqrt{1-t_1} \sqrt{1-t_3} +2 \sqrt{1-t_2} \sqrt{1-t_3} -3$

$\sqrt{1-t_1} \sqrt{1-t_2} \sqrt{1-t_3})$ [Sqrt] $(-(-1+t_1)^2 t_1^2 (-1+t_2)^2 t_2^2 (-1+t_3)^2 t_3^2 (128 (-1+\sqrt{1-t_1})$

$\sqrt{1-t_2} t_2^2 \sqrt{1-t_3} t_3^2+3 t_1^3 t_2 t_3 (8 (-1+\sqrt{1-t_2}) (1+\sqrt{1-t_3} -t_3) t_3 + t_2 (8 (1+\sqrt{1-t_2})$

$(-1+ \sqrt{1-t_3})+(11+8 \sqrt{1-t_2} +8 \sqrt{1-t_3} -8 \sqrt{1-t_2} \sqrt{1-t_3}) t_3-7 t_3^2)+t_2^2 (8-8$

$\sqrt{1-t_3} -7 t_3+3 t_3^2))-8 t_1 t_2 t_3 (8 (-1+\sqrt{1-t_1}) (-1+\sqrt{1-t_2}) (1+2 \sqrt{1-t_3})t_3 -3 (-1+\sqrt{1-t_1})$

$t_2^2 (1+\sqrt{1-t_3} -t_3)t_3 +t_2 (8 (-1+\sqrt{1-t_1}) (1+2 \sqrt{1-t_2}) (-1+\sqrt{1-t_3})+(-19+19 \sqrt{1-t_1} -11$

$\sqrt{1-t_2} +11 \sqrt{1-t_1} \sqrt{1-t_2} -11 \sqrt{1-t_3} +11 \sqrt{1-t_1} \sqrt{1-t_3} -8 \sqrt{1-t_2} \sqrt{1-t_3})$

$t_3-3 (-1+\sqrt{1-t_1}) (1+\sqrt{1-t_2}) t_3^2))+t_1^2 (128 \sqrt{1-t_1} (-1+\sqrt{1-t_2}) \sqrt{1-t_3} t_3^2-3 t_2^3 t_3$

$(-8 (1+\sqrt{1-t_1}) (-1+\sqrt{1-t_3})+(-11-8 \sqrt{1-t_1} -8 \sqrt{1-t_3} +8 \sqrt{1-t_1} \sqrt{1-t_3})t_3 +7 t_3^2)+$

$8t_2t_3(-8 (1+2 \sqrt{1-t_1}) (-1+\sqrt{1-t_2}) (-1+\sqrt{1-t_3})+(19+11 \sqrt{1-t_1} -19 \sqrt{1-t_2} -11$

$\sqrt{1-t_1} \sqrt{1-t_2} +11 \sqrt{1-t_3} +8 \sqrt{1-t_1} \sqrt{1-t_3} -11 \sqrt{1-t_2} \sqrt{1-t_3})t_3 +3 (1+\sqrt{1-t_1})$

$(-1+ \sqrt{1-t_2}) t_3^2)+ t_2^2 (128 \sqrt{1-t_1} \sqrt{1-t_2} (-1+ \sqrt{1-t_3})+8 (19+11 \sqrt{1-t_1} +11$

$\sqrt{1-t_2} +8 \sqrt{1-t_1} \sqrt{1-t_2} -19 \sqrt{1-t_3} -11 \sqrt{1-t_1} \sqrt{1-t_3} -11 \sqrt{1-t_2} \sqrt{1-t_3})t_3$

$+3(-79-16 \sqrt{1-t_1} -16 \sqrt{1-t_2} +8 \sqrt{1-t_1} \sqrt{1-t_2} -16 \sqrt{1-t_3} +8 \sqrt{1-t_1} \sqrt{1-t_3} +8$

$\sqrt{1-t_2} \sqrt{1-t_3})t_3^2 +3(11+8 \sqrt{1-t_1} +8 \sqrt{1-t_2} -8 \sqrt{1-t_1} \sqrt{1-t_2}) t_3^3))))) /$

$(\sqrt{1-t_1} t_1^2 \sqrt{1-t_2} t_2^2 \sqrt{1-t_3} t_3^2(-2 \sqrt{1-t_1} -2 \sqrt{1-t_2} -2 \sqrt{1-t_3} +3 \sqrt{1-t_1}$

$\sqrt{1-t_2} \sqrt{1-t_3} + 2 \sqrt{1-t_1} t_3 + 2 \sqrt{1-t_2} t_3 + 2 \sqrt{1-t_3} t_3 - 2 \sqrt{1-t_1} \sqrt{1-t_2}$

$\sqrt{1-t_3} t_3+ t_2 (2 (\sqrt{1-t_1} + \sqrt{1-t_2}+ \sqrt{1-t_3} - \sqrt{1-t_1} \sqrt{1-t_2} \sqrt{1-t_3}) + (-2$

$\sqrt{1-t_1} - 2 \sqrt{1-t_2} - 2 \sqrt{1-t_3} + \sqrt{1-t_1} \sqrt{1-t_2} \sqrt{1-t_3}) t_3) + t_1 (2$

$(\sqrt{1-t_1} + \sqrt{1-t_2} + \sqrt{1-t_3} - \sqrt{1-t_1} \sqrt{1-t_2} \sqrt{1-t_3}) + (-2 \sqrt{1-t_1} -2 \sqrt{1-t_2} -2$

$\sqrt{1-t_3} \sqrt{1-t_3} + \sqrt{1-t_1} \sqrt{1-t_2} \sqrt{1-t_3})t_3 + t_2 (-2 \sqrt{1-t_1} -2 \sqrt{1-t_2} -2$

$$\sqrt{1-t_3}+\sqrt{1-t_1}\quad\sqrt{1-t_2}\quad\sqrt{1-t_3}+2\,(\sqrt{1-t_1}+\sqrt{1-t_2}+\sqrt{1-t_3}\,)\,t_3\,))))) \qquad (21)$$

In the special case in which the ellipse degenerates to circle and thus $x_1 = x_2 = x_3$, we have:

$$S_{circle-estimated}=\frac{1}{\sqrt{2}}\pi\frac{1}{(1-t_1)^2t_1^3}\sqrt{\begin{array}{l}(4-4\sqrt{1-t_1})t_1^5-(15-14\sqrt{1-t_1})t_1^6+(19-16\sqrt{1-t_1})t_1^7\\+(-6+4\sqrt{1-t_1})t_1^8+(-6+4\sqrt{1-t_1})t_1^9+(5-2\sqrt{1-t_1})t_1^{10}-t_1^{11}\\-(2-\sqrt{1-t_1})t_1^5(t_1-1)^4\sqrt{t_1(8-8\sqrt{1-t_1}-3t_1-t_1^2)}\end{array}}$$

According to equation 9, we have:

$$S_{circle-real}=\pi a^2=\pi\frac{x_1^2}{4+x_1^2}=\pi\frac{t_1}{4+t_1}$$

and therefore,

$$\varepsilon_{circle}(t_1)=\frac{\left|S_{circle-real}-S_{circle-estimated}\right|}{S_{circle-real}}$$

We use Mathematica to plot the graph of $\varepsilon_{circle}(t_1)$ ($t_1\in(0,0.1339)$):

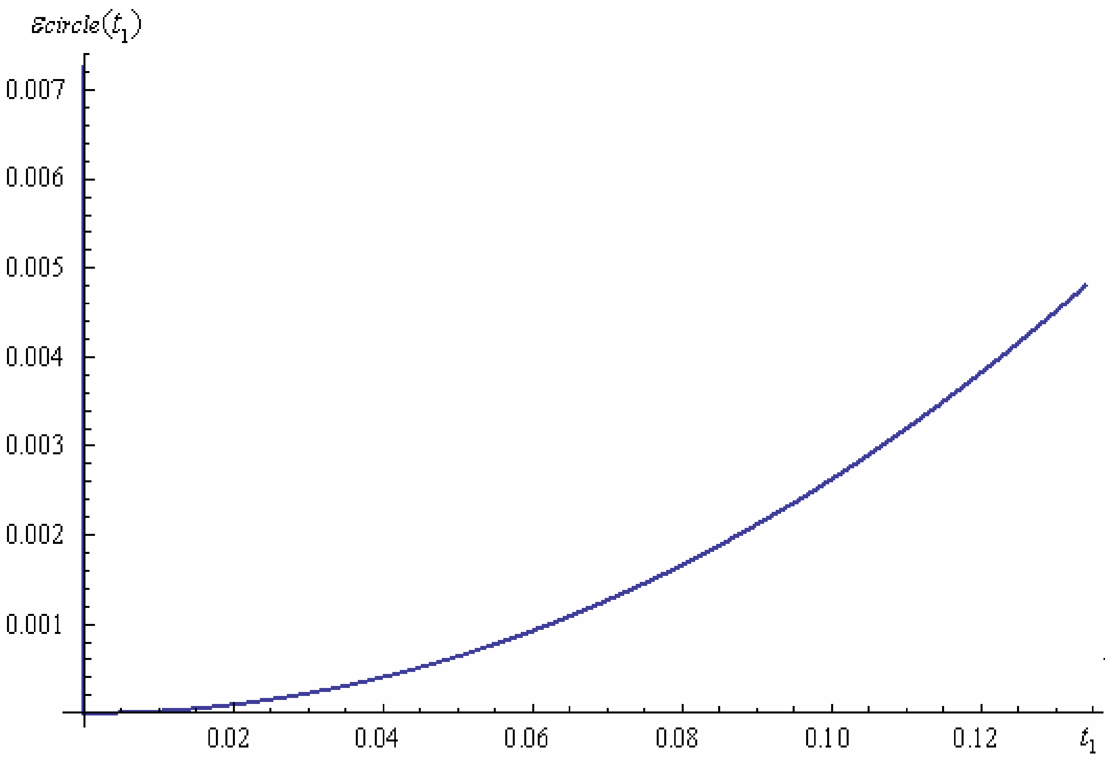

Figure 9

The graph indicates that the expression's relative error is less than 0.5% when the ellipse degenerates to circle. The margin of error in the general situation in which the ellipse does not degenerate will be discussed in detail in the "Analysis of Error" Section.

The solution of Model 1 (Equation 8) and that of Model 2 (Equation 21) are both the expression for S in terms of $x_i (i=1,2,3)$. Although the specific meaning of x_i is different in the two models, it can always be regarded as the length of the line segment whose projection on the negative can be considered an equivalent to that of the ellipse on the negative. Thus, x_i can be understood as the length of the object and the length of the projection of the ellipse on negative can be understood as the length of the image.

Assume that $x_i'(i=1,2,3)$ is the length of the image; the distances from the object to the lens and from the lens to the image are s and s', respectively, and the focal length is f.

Because in Model 1 and Model 2, $s=r$, according to the thin lens formula: $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$, we have $s'=\frac{rf}{r-f}$. Therefore, $\frac{s}{s'}=\frac{r-f}{f}$. Because $\frac{x_i}{x_i'}=\frac{s}{s'}=\frac{r-f}{f}$, we obtain $x_i=\frac{r-f}{f}x_i'$.

Assume that the length of the projection of $A'B'$ on the negative, when displayed on the computer screen, is $X_i(i=1,2,3)$ and $\frac{X_i}{x_i'}=K$. We have $x_i=\frac{r-f}{f}\frac{X_i}{K}$. By substituting $\frac{r-f}{f}\frac{X_i}{K}$ for x_i in the equation 8 and equation 21, and we obtain an the expression for S in terms of X_1, X_2, X_3, r, f and K. Because equation 21 is excessively long, we do not present the expression in full here.

2.2.3 Model 3 The models presented above have treated the measurement and computation of the area of an elliptical cross-section. Now we will consider measuring the volume of an ellipsoidal object (hereinafter referred to as the ellipsoid). First, we define the three-dimensional coordinate system 1 with the center of the ellipsoid as the origin, the semi-minor axis of the equatorial ellipse as the x-axis, the semi-major axis of the equatorial ellipse as the y-axis, and the polar radius as the z-axis. We referring to the ellipsoid's semi-principal axes as a, b and c, the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Because the measured ellipsoid is placed at the center of the circle and the orthographic projection of the center of the ellipsoid on the circular work surface (hereinafter referred to as the circle) coincides with the center of the circle, we assume that the equation of the circle, when translated to the ellipsoid's equatorial plane, is $x^2+y^2=r^2(z=0)$ and the coordinates of the three video cameras are, $P_1\left(r\cos\theta, r\sin\theta, h\right)$, $P_2\left(r\cos(\theta+\frac{2\pi}{3}), r\sin(\theta+\frac{2\pi}{3}), h\right)$, $P_3\left(r\cos(\theta-\frac{2\pi}{3}), r\sin(\theta-\frac{2\pi}{3}), h\right)$.

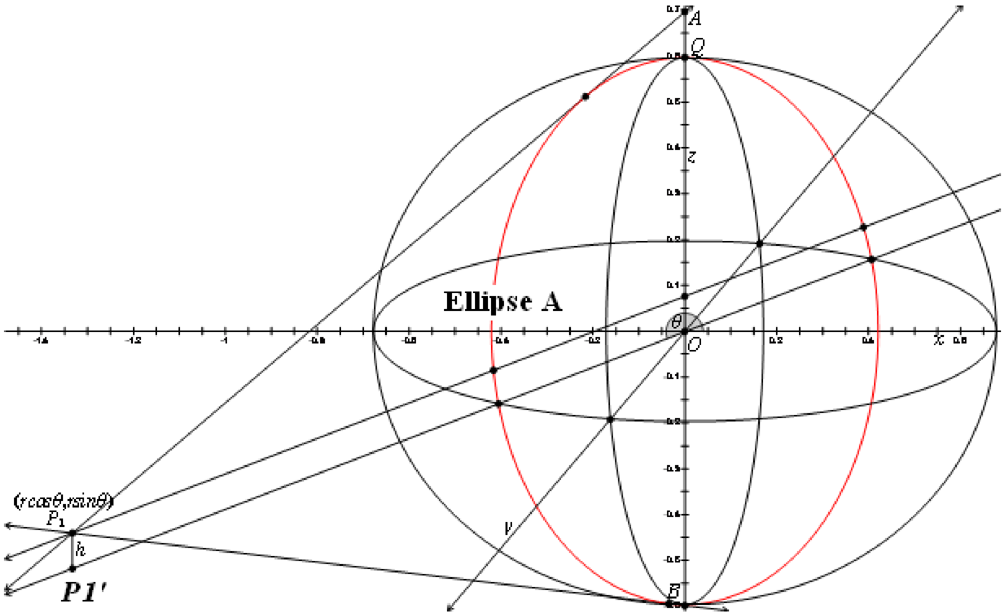

Figure 10 coordinate system 1

As shown in figure 10, while filming, the lenses of the video cameras were kept vertical, at the same height and aimed at the *z*-axis, and were raised uniformly from their original positions, which were below the ellipse's equatorial plane, to positions above it. We will consider one of the video cameras, $P_1(r\cos\theta, r\sin\theta, h)$; the situations regarding the other two cameras are similar. Because the video camera always points at the *z*-axis, the negative is always parallel to the *z*-axis. Therefore, the projection of segment *AB*, the segment produced by the two tangents of the ellipse *A* (the ellipse produced by the plane defined by the *z*-axis and camera P_1 intercepting the ellipsoid, or the red ellipse shown in figure 6) from point P_1 intercepting the *z*-axis, can be regarded as equivalent to the projection of the ellipse *A*. Thus, the computer uses the length of the line segment lying on the vertical axis of symmetry of the image displayed on the computer screen, i.e., the length of the projection of line *AB*, when displayed on the computer. The following section will discuss the correlation between *AB* and *h* (shown in figure 10).

We define the planar Cartesian coordinate system 2 with point *O* as the origin, the line $P_1'O$ (shown in figure 10) as the x-axis, and the line AB as the y-axis (as shown in figure 11).

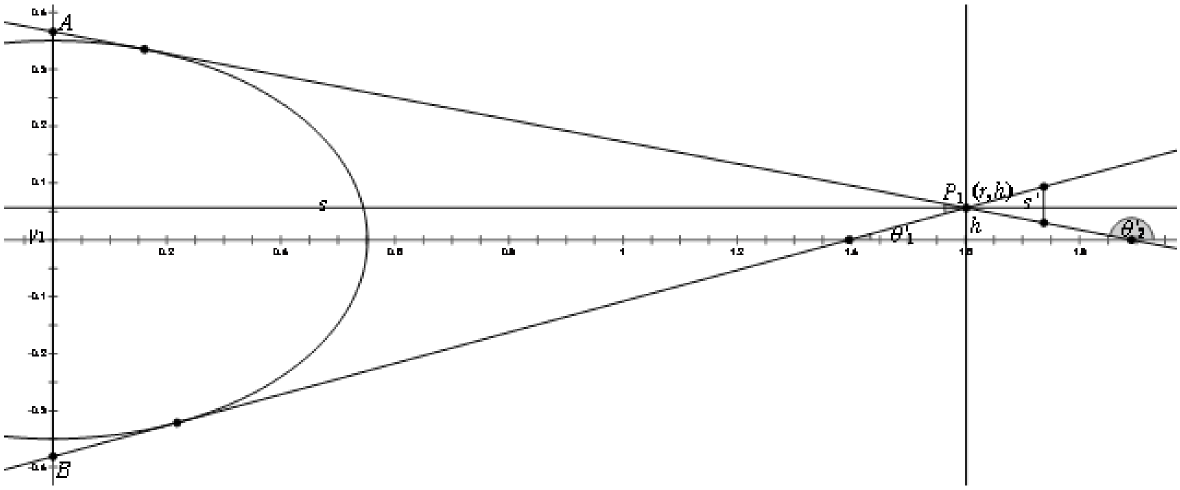

Figure 11 coordinate system 2

Assume that $y_1 = |AB|$ and that the equation of the ellipse in Cartesian coordinate system 2 is $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$. The coordinates of P_1 in Cartesian coordinate system 2 are (r,h) $(h < b')$, and the equation of line l_{AP_1} is $y = \tan\theta'(x-r) + h$. Join two equations above to obtain:

$$(b'^2 + a'^2\tan^2\theta')x^2 - 2(r\tan\theta' - h)\tan\theta' a'^2 x + a'^2(r\tan\theta' - h)^2 - a'^2 b'^2 = 0$$

Given $\Delta = 0$, we have:

$$b'^2 + a'^2\tan^2\theta' = h^2 + r^2\tan^2\theta' - 2hr\tan\theta'$$

Thus, $(r^2 - a'^2)\tan^2\theta' - 2hr\tan\theta' + (h^2 - b'^2) = 0$.

Because $h < b'$, we have

$$\tan\theta'_1 + \tan(\pi - \theta'_2) = \frac{2\sqrt{a'^2 h^2 + b'^2 r^2 - a'^2 b'^2}}{r^2 - a'^2}$$

and therefore,

$$y_1 = r(\tan\theta'_1 + \tan(\pi - \theta'_2)) = \frac{2r\sqrt{a'^2 h^2 + b'^2 r^2 - a'^2 b'^2}}{r^2 - a'^2}$$

Then, we use quantities in Cartesian coordinate system 1 to express quantities *a'* and *b'* in Cartesian coordinate system 2.

By combining $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = \tan\theta \bullet x$, we have:

$$x^2 + y^2 = \frac{a^2 b^2 (1 + \tan^2\theta)}{b^2 + a^2 \tan^2\theta}$$

Thus, $a'^2 = \frac{a^2 b^2 (1 + \tan^2\theta)}{b^2 + a^2 \tan^2\theta} = \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}$.

Because $b' = c$, we have

$$y_1 = \frac{2r\sqrt{\frac{a^2 b^2}{a^2 \sin^2\theta + b^2\cos^2\theta} h^2 + c^2 r^2 - \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta} c^2}}{r^2 - \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}}$$

$$= \frac{2r\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\sqrt{c^2 r^2 (a^2 \sin^2\theta + b^2\cos^2\theta) - a^2 b^2 (c^2 - h^2)}}{r^2 (a^2 \sin^2\theta + b^2 \cos^2\theta) - a^2 b^2}$$

(22)

According to equation 22, when $h = 0$, i.e., when the video camera is at the same height as the center of the ellipsoid, y_1, the length of *AB*, is minimal. Refer to the moment when the length of *AB* is minimal as T_1, the height of the video camera relative to the circular work surface before its uniform motion begin as H_0 and the ascension speed of the video camera as ΔH. Hence, the height of the video camera relative to the work surface at T_1 is $H = H_0 + T_1 \Delta H$. Additionally, because at T_1, $h = 0$, we have $c = H$.

Furthermore, at T_1, the line segment lying on the horizontal axis of symmetry of the image displayed on the computer screen is actually the projection of equatorial ellipse, when displayed on the computer. By combining the expression for πab in terms of $t_i (i=1,2,3)$, i.e., equation 21, where $t_i = x_i^2 (i=1,2,3)$ and x_i denotes the length of the projection of the line that is equivalent alternative to that of the ellipse, with the equation $V=\frac{4}{3}\pi abc=\frac{4}{3}\pi abH=\frac{4}{3}\pi ab(H_0+T_1\Delta H)$, we can then write an expression for V in terms of $x_i (i=1,2,3)$, T_1, H_0 and ΔH: Equation 23 (Because equation 21 is too long, we do not present equation 23 in full here).

Refer to the length of the line segment lying on the horizontal axis of symmetry of the image on the computer as $X_i (i=1,2,3)$ and the coefficient of proportionality as *K*. Substituting $\frac{r-f}{f}\frac{X_i}{K}$ for x_i in equation 23 yields an expression for *V* in terms of X_1, X_2, X_3, *r*, *f* and *K*. Because equation 21 is very long, we do not present it in full here.

2.3 Analysis of error

We first analyze the error of model 1, which is the approximation error between the hypothetical parallel model and the actual situation. In this model, we assume that the projection of the ellipse on the negative is *AB*, the distance between the two tangents of the ellipse, from which equation 8 is derived.

In the actual situation, however, light rays pointing from every point on the ellipse to the lenses of the video cameras are not parallel. Instead, we should regard the projection of the ellipse on the negative as that of the segment *CD* produced by the two tangents of the ellipse from the camera intercepting the line crossing the center and perpendicular to the line joining the video camera and the center, displayed in figure 12. Therefore, the model is not completely accurate.

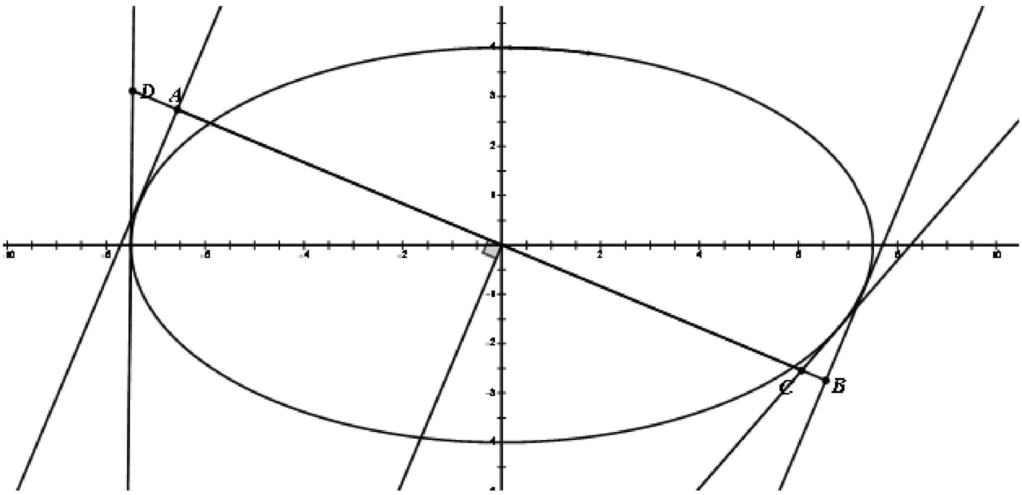

Figure 12

Using statistical measures (Matlab and The Geometer's Sketchpad), we analyze the error of the model. Considering a regular ellipse with a semi-major axis of 7.5 and a semi-minor axis of 4, we alter the radius of the circular work surface, which is *r*, and calculate the area accordingly to determine the error of *AB* relative to *CD*, $\varepsilon_1 = \frac{(l_{AB} - l_{CD})}{l_{CD}}$.

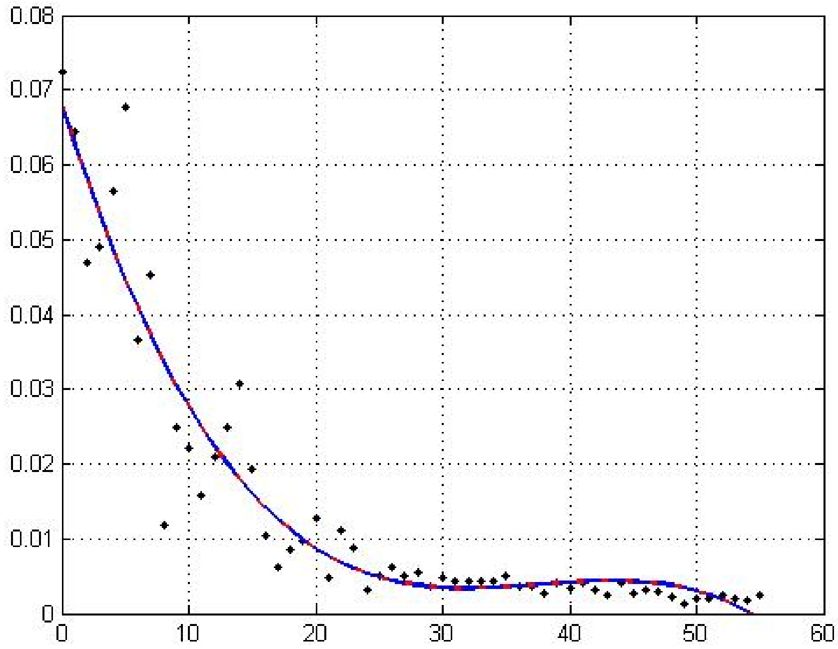

Figure 13

In the graph above, the *y*-axis represents error ε_1 and the *x*-axis represents the serial number of the data points obtained. For data points 1 through 9, $r = 20$; for data points 10 through 18, $r = 30$; for data points 19 through 27, $r = 40$; for data points 28 through 36, $r = 50$; for data points 37 through 45, $r = 60$; for data points 46 through 54, $r = 75$; for data points 55 through 63, $r = 90$. By fitting the data above, we can see that approximately, when $r \geq 50$, i.e., $\frac{r}{a} \geq 6.67$, the error of *AB* relative to *CD*, ε_1 is less than 0.5%.

Because of the error ε_1, the area of the ellipse derived by using equation 8 slightly differs from the actual area of the ellipse. In the same manner, we analyze ε_2, the error between the two areas $\varepsilon_2 = \frac{S_{(8)estimated} - S_{real}}{S_{real}}$, by again altering *r*. (The ellipse is regular with the same size, a semi-major axis of 7.5 and a semi-minor axis of 4)

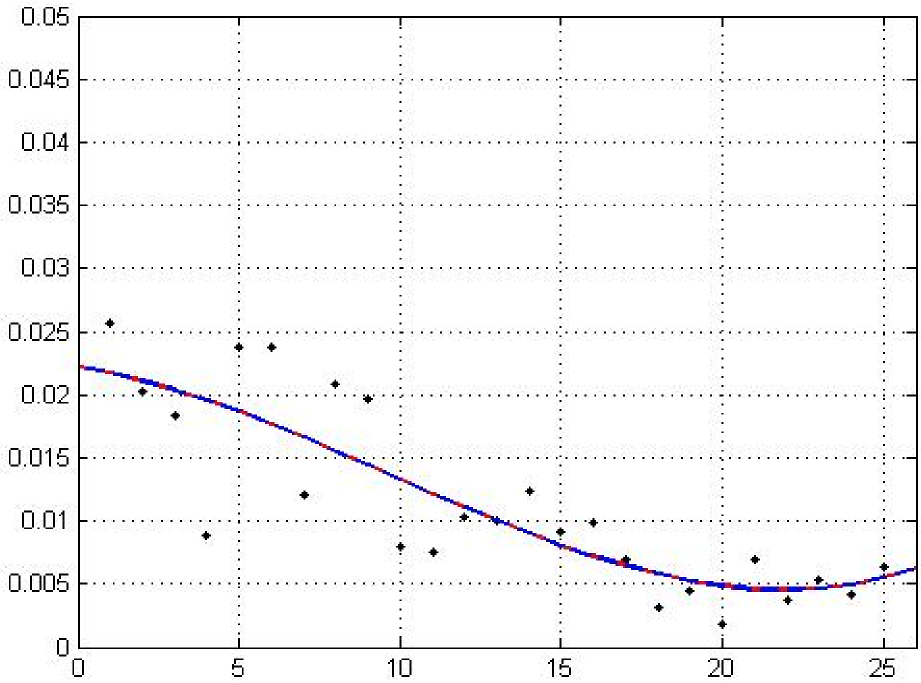

Figure 14

In the graph above, the *y*-axis represents the error ε_2 and the *x*-axis represents the serial number of the data points obtained. For data points 1 through 9, $r = 20$; for data points 10 through 18, $r = 35$; for data points 19 through 27, $r = 50$. By fitting the data above, we can see that approximately, when $r \geq 50$, i.e., $\frac{r}{a} \geq 6.67$, error ε_2 is less than 0.5%.

In conclusion, when the orthographic projection of the center of the ellipse on the circular work surface coincides with the center of the circle and the radius of the circle *r* is at least 6.67 times longer than the semi-major axis of the ellipse *a*, the error between the area of the ellipse derived by using equation 8 and the actual area of the ellipse is less than 0.5%.

We then analyze the error of Model 2, which is a more precise calculation of the ellipse's area. Considering the deduction, we can see that the primary error originates from the use of Taylor series to simplify certain steps in the calculation.

To prove that the error introduced by the use of Taylor series is marginal, we again use statistical measures and tools to analyze the error ε_3, the error between the area derived using equation 21 and the actual area $\varepsilon_3 = \dfrac{S_{(21)estimated} - S_{real}}{S_{real}}$, by altering *r* while maintaining the ellipse with a semi-major axis of 7.5 and a semi-minor axis of 4 unchanged.

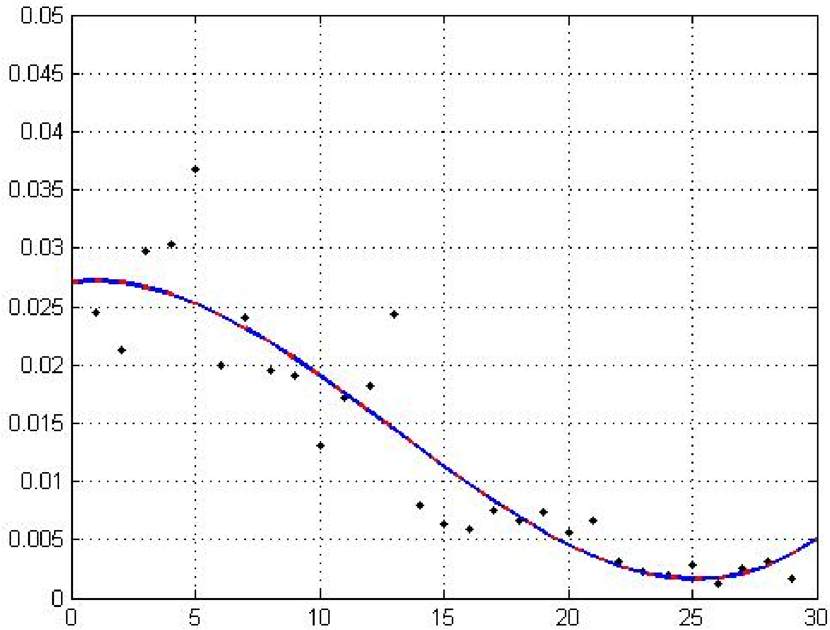

Figure 15

In the graph above, *y*-axis represents the error ε_3 and *x*-axis represents the serial number of the data points obtained. For data points 1 to 8, $r = 20$; for data points 9 to 16, $r = 30$; for data points 17 to 24, $r = 35$; and for data points 25 to 30, $r = 45$. By fitting the data above, we can see that approximately when $r \geq 40$, i.e., $\dfrac{r}{a} \geq 5.53$, the error ε_3 is less than 0.5%.

In conclusion, when the orthographic projection of the center of the ellipse on the circular work surface coincides with the center of the circle and the radius of the circle *r* is at least 5.53 times longer than the semi-major axis of the ellipse *a*, the error between the area of the ellipse derived by using equation 8 and the actual area of the ellipse is less than 0.5%.

However, in real-world applications, the projection of the oval object's center on the circular work surface does not always coincide with the center of the circular work surface, regardless of whether the object is placed manually or mechanically. We refer to such error as the deviation error. In the passage below, we will further discuss the relation among the deviation *d*, the radius of the circular work surface *r*, the semi-major axis of the ellipse *a*, and the deviation error.

Under such circumstances, the projection of the ellipse on the negative can be regarded as that of the segment *A'B'* produced by the two tangents of the ellipse from the camera intercepting the line crossing the center of the ellipse and perpendicular to l_1 (displayed in the figure 16).

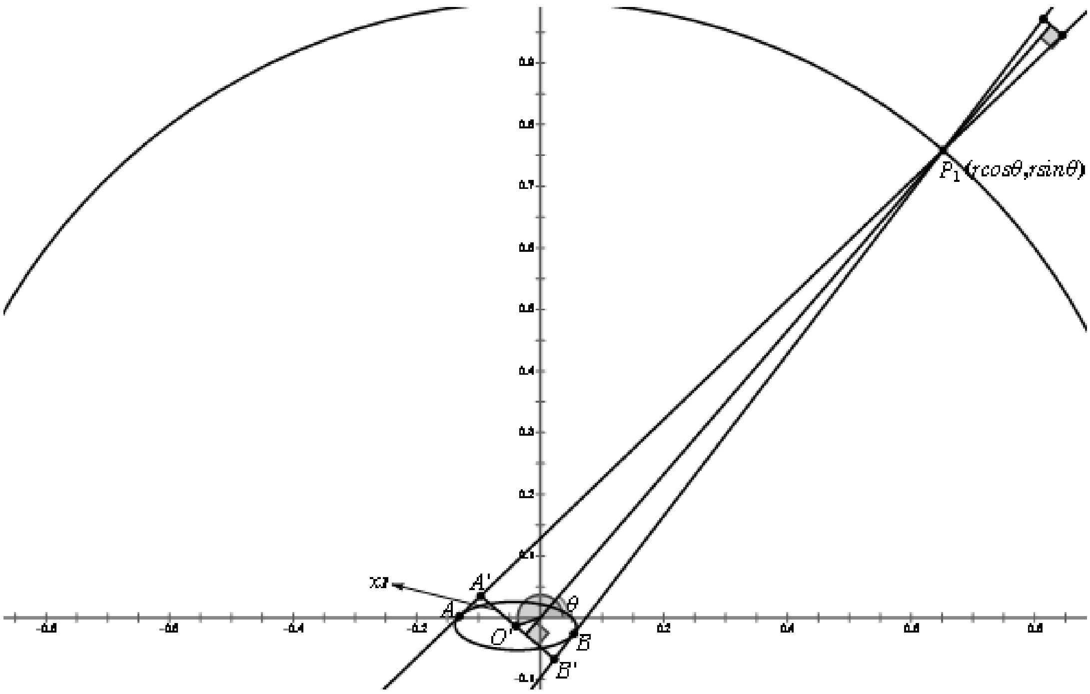

Figure 16

Based on the result of model 1's analysis, that the error of the area deduced by using equation 8 is marginal when approximately $\frac{r}{a} \geq 6.67$, we alter d under the extreme condition of $\frac{r}{a} = 6.67$ (the ellipse's semi-major axis is 7.5 and semi-minor axis is 4), to discuss the deviation error of equation 8, $\varepsilon_4 = \frac{S_{(8,d)estimated} - S_{real}}{S_{real}}$.

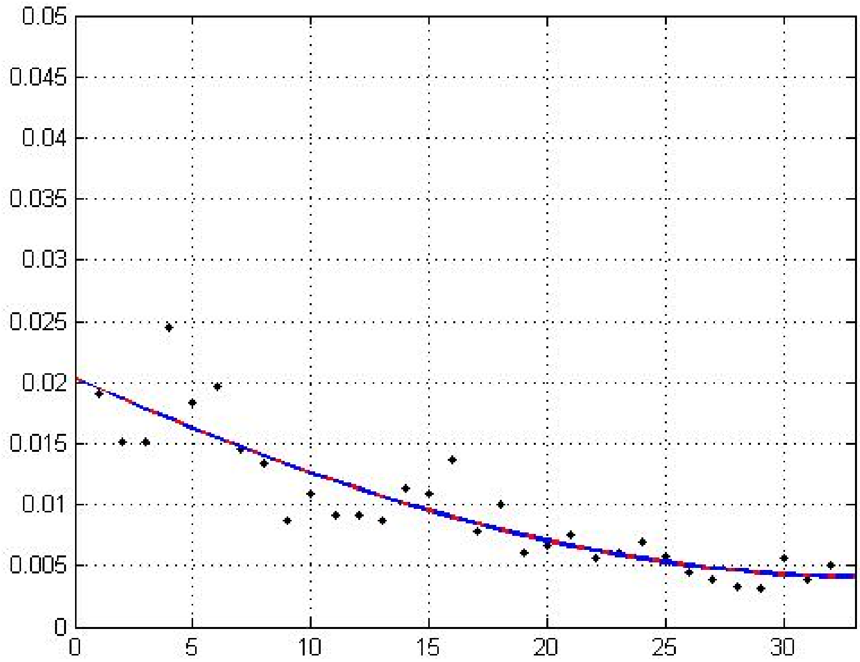

Figure 17

In the graph above, the y-axis represents error ε_4 and the x-axis represents the serial number of the data points obtained. For data points 1 to 8, $d = 7.5$; for data points 9 to 16, d=5.625; for data points 17 to 24, d=3.75 and for data points 25 to 32, d=1.875. By fitting the data above, we can see that approximately, when $d \geq 1.875$, i.e., $\frac{d}{a} \geq 0.25$, the deviation error ε_4 of model 1 under the extreme condition of $\frac{r}{a} = 6.67$, is greater than 0.5%. Thus, the error of equation 8 is relatively large when the projection of the oval object's center on the circular work surface does not coincide with the center of the circular work surface.

Then we analyze the deviation error of model 2: based on model 2's analysis, that the error of the area deduced by using equation 21 is marginal when approximately $\frac{r}{a} \geq 5.53$, we alter d under the extreme condition of $\frac{r}{a} = 5.53$ (the ellipse's semi-major axis is 7.5 and semi-minor axis is 4), to discuss the deviation error of equation 21, $\varepsilon_5 = \frac{S_{(21,d)estimated} - S_{real}}{S_{real}}$.

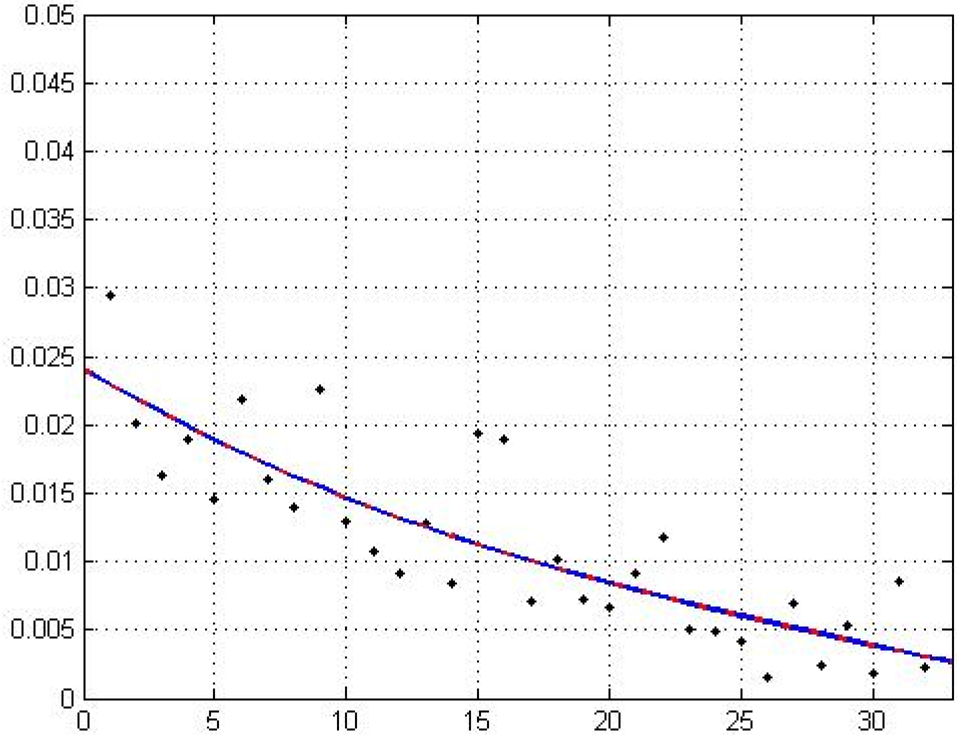

Figure 18

In the graph above, the y-axis represents error ε_5 and the x-axis represents the serial number of the data points obtained. For data points 1 to 8, d=15; for data points 9 to 16, d=11.25; for data points 17 to 24, d=7.5 and for data points 25 to 32, d=3.75. By fitting the data above, we can see that approximately, when $d \leq 3.75$, i.e., $\frac{d}{a} \leq 0.50$, the deviation error ε_4 of model 2 under the extreme condition of $\frac{r}{a} = 5.53$, is less than 0.5%.

In conclusion, Equation 21 is more accurate than Equation 8. When $\frac{r}{a} \geq 5.53$ and $\frac{d}{a} \leq 0.50$, using equation 21 to calculate the are of the ellipse can ensure that the margin of error is controlled within 0.5%.

In model 3, the error originates from (1) the deviation of the projection of the ellipsoid's center from the center of the circular work surface; (2) the error introduced by equation 21. The analysis of the former one is similar to that of the deviation error of model 1 and model 2 discussed above. The analysis of the latter one has been covered in that of model 2. Therefore, detailed presentation is omitted here.

2.4 Conclusions

Through constructing, solving, analyzing and testing models, we developed a new method to calculate the cross-sectional area of ellipse and the volume of ellipsoids. Two area formulas, which are given by the Equation 8 in Model 1: $S = \frac{\sqrt{3}\pi}{12}\sqrt{2{x_1}^2{x_2}^2 + 2{x_1}^2{x_3}^2 + 2{x_2}^2{x_3}^2 - {x_1}^4 - {x_2}^4 - {x_3}^4}$ and Equation 21 in Model 2 (too long to be listed), were derived. Furthermore, the error of each model was analyzed with the aid of statistical methods and mathematical software. Equation 8 provides a sufficiently accurate solution when $\frac{r}{a} \geq 6.67$ and $\frac{d}{a} \leq 0.25$. Equation 21 provides a sufficiently accurate solution when $\frac{r}{a} \geq 5.53$ and $\frac{d}{a} \leq 0.50$. Moreover, the formula for the volume of ellipsoids, Equation 23, was derived and its error of was discussed.

3. The advantages and disadvantages of the models and possible improvements

3.1 The advantages and disadvantages

Model 1: this model is used to calculate the cross-sectional area of an ellipse. The formula is relatively simple, and the use of professional calculation software is not necessary. However, the error is relatively large because the light rays pointing from every point on the ellipse to the lenses of video cameras are regarded as approximately parallel.

Model 2: this model is used to calculate the cross-sectional area of an ellipse. The error is relatively small because the parallel assumption above is not made. Compared with Model 1, Model 2 is less sensitive to the ratio of r to a, the deviation d, and the mechanical error. Thus, compared to Model 1, this model poses fewer requirements on the grounds and the accuracy of placing measured objects and saves the general costs. However, the formula is relatively complex, and it is necessary to use mathematical software.

Model 3: this model is used to calculate the volume of ellipsoids. The accuracy of this model is relatively high and the costs are relatively low. However, camera techniques and filming are needed. Therefore, the time required for this method is relatively long.

3.2 Improvements to the models

To improve the models, we could further investigate other aspects of the errors to reduce the errors and increase the scale for which the model functions properly. Furthermore, we could thoroughly discuss the deviation errors by introducing new variables, such as the coordinates of the deviated projection center of the ellipsoids. Additionally, a larger sample size could be introduced such that the analysis of errors could be more accurate. Finally, the actual measurement process could by tailored to the individual characteristics of different objects to be measured.

4. The utility and innovation of the new method

4.1 Utility

The high utility of this method comes from the simple requirements for space, equipment and structures; errors are contained within 0.5%, which is marginal for industrial applications. The measurement and calculation time is relatively short, and automation of the method could be performed with ease. Such characteristics of the method match the interests of certain manufacturers.

4.2 Innovation

The new method innovatively calculates the cross-sectional area and volume using cameras filming from the sides of the object. The method solves the ray-scanning limitation by offering an efficient and low-cost calculation method. The method also overcomes the limitations of traditional physics techniques, such as the buoyancy method, that require information about the physical properties of the object.

Summary

In this project, we developed a new method for calculating the cross-sectional area of ellipses and the volume of ellipsoids with low cost and high efficiency.

In the process of the study, we successfully deduced area formulas for two different sets of assumptions and a formula for the volume of ellipsoids. Furthermore, we discussed the errors and determined under what conditions the error is less than 0.5%, which is insignificant for industrial applications.

We learned how to use mathematical software such as Matlab, Mathematica and the Geometer's Sketchpad to produce more accurate results.

Working on this project taught us an application of math outside classrooms. The power of mathematics for optimizing and predicting inspires us to study applied math and to use mathematical perspectives to solve problems.

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Appendix

Statistics：

Table 1：

Serial number	r	l_{AB}	l_{CD}	Error ε_1
1		14.220	13.190	7.243%
2		15.820	14.800	6.448%
3		18.950	18.060	4.697%
4	r=20	16.540	15.730	4.897%
5		13.780	13.000	5.660%
6		12.560	11.710	6.768%
7		13.880	13.370	3.674%
8		17.900	17.090	4.525%
9		18.590	18.370	1.183%
10		16.490	16.080	2.486%
11		15.380	15.040	2.211%
12	r=30	15.110	14.870	1.588%
13		12.940	12.670	2.087%
14		14.130	13.780	2.477%
15		17.890	17.340	3.074%
16		13.980	13.710	1.931%
17		15.190	15.030	1.053%
18		19.020	18.900	0.631%
19		14.030	13.910	0.855%

20	r=40	16.590	16.430	0.964%
21		14.980	14.790	1.268%
22		18.210	18.120	0.494%
23		14.450	14.290	1.107%
24		17.190	17.040	0.873%
25		18.970	18.910	0.316%
26		13.990	13.920	0.500%
27		12.830	12.750	0.624%
28	r=50	17.890	17.800	0.503%
29		14.670	14.590	0.545%
30		13.910	13.860	0.359%
31		16.450	16.370	0.486%
32		15.930	15.860	0.439%
33		15.940	15.870	0.439%
34		13.910	13.850	0.431%
35		13.510	13.450	0.444%
36	r=60	17.890	17.800	0.503%
37		16.310	16.250	0.368%
38		16.880	16.820	0.355%
39		14.660	14.620	0.273%
40		17.210	17.140	0.407%
41		22.730	22.650	0.352%
42		14.580	14.520	0.412%
43		15.760	15.710	0.317%
44	r=75	16.210	16.170	0.247%
45		16.890	16.820	0.414%
46		17.560	17.510	0.285%
47		18.790	18.730	0.319%
48		20.940	20.880	0.287%
49		22.960	22.930	0.131%
50		21.830	21.780	0.229%
51		21.010	20.970	0.190%
52	r=90	21.320	21.280	0.188%
53		19.870	19.820	0.252%
54		18.820	18.780	0.213%
55		17.740	17.710	0.169%
56		16.740	16.700	0.239%

Table 2：

Serial	r	$S_{estimated}$	S_{real}	Error ε_3

number				
1		44.88	43.78	2.451%
2		44.73	43.78	2.124%
3		45.12	43.78	2.970%
4	r=20	45.15	43.78	3.034%
5		45.45	43.78	3.674%
6		44.67	43.78	1.992%
7		44.86	43.78	2.407%
8		44.65	43.78	1.948%
9		145.99	143.81	1.493%
10		146.45	143.81	1.803%
11		147.42	143.81	2.449%
12	r=30	146.61	143.81	1.910%
13		145.72	143.81	1.311%
14		146.32	143.81	1.715%
15		146.48	143.81	1.823%
16		147.39	143.81	2.429%
17		202.98	201.38	0.788%
18		202.67	201.38	0.637%
19		202.58	201.38	0.592%
20	r=35	202.91	201.38	0.754%
21		202.71	201.38	0.656%
22		202.88	201.38	0.739%
23		202.51	201.38	0.558%
24		202.72	201.38	0.661%
25		88.210	87.930	0.317%
26		88.130	87.930	0.227%
27		88.110	87.930	0.204%
28	r=45	88.190	87.930	0.295%
29		88.040	87.930	0.125%
30		88.150	87.930	0.250%
31		88.200	87.930	0.306%
32		88.080	87.930	0.170%

Table 3:

Serial number	d	$S_{estimated}$	S_{real}	Error ε_4
1		91.280	89.54	1.906%
2		90.920	89.54	1.518%
3		90.910	89.54	1.507%
4	d=7.5	91.790	89.54	2.451%

5		91.210	89.54	1.831%
6		91.340	89.54	1.971%
7		90.870	89.54	1.464%
8		90.760	89.54	1.344%
9		17.780	17.626	0.866%
10		17.820	17.626	1.089%
11		17.790	17.626	0.922%
12	d=5.625	17.790	17.626	0.922%
13		17.780	17.626	0.866%
14		17.830	17.626	1.144%
15		17.820	17.626	1.089%
16		17.870	17.626	1.365%
17		54.990	54.560	0.782%
18		55.110	54.560	0.998%
19		54.890	54.560	0.601%
20	d=3.75	54.920	54.560	0.655%
21		54.970	54.560	0.746%
22		54.870	54.560	0.565%
23		54.890	54.560	0.601%
24		54.940	54.560	0.692%
25		35.980	35.770	0.584%
26		35.930	35.770	0.445%
27		35.910	35.770	0.390%
28	d=1.875	35.890	35.770	0.334%
29		35.880	35.770	0.307%
30		35.970	35.770	0.556%
31		35.910	35.770	0.390%
32		35.950	35.770	0.501%

Table 4：

Serial number	d	$S_{estimated}$	S_{real}	Error ε_5
1		149.980	145.560	2.947%
2		148.540	145.560	2.006%
3		147.970	145.560	1.629%
4	d=15	148.370	145.560	1.894%
5		147.710	145.560	1.456%
6		148.810	145.560	2.184%
7		147.920	145.560	1.595%
8		148.920	145.560	2.256%

9		80.010	78.890	1.400%
10		79.930	78.890	1.301%
11		79.740	78.890	1.066%
12	d=11.25	79.620	78.890	0.917%
13		79.910	78.890	1.276%
14		79.560	78.890	0.842%
15		80.440	78.890	1.927%
16		80.410	78.890	1.890%
17		233.29	231.64	0.707%
18		234.02	231.64	1.017%
19		233.31	231.64	0.716%
20	d=7.5	233.21	231.64	0.673%
21		233.76	231.64	0.907%
22		234.39	231.64	1.173%
23		232.81	231.64	0.503%
24		232.77	231.64	0.485%
25		123.210	122.690	0.422%
26		122.870	122.690	0.146%
27		123.540	122.690	0.688%
28	d=3.75	122.980	122.690	0.236%
29		123.340	122.690	0.527%
30		122.910	122.690	0.179%
31		123.760	122.690	0.865%
32		122.970	122.690	0.228%