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On a class of Discrete Maximum Value Problem

Abstract

In this paper, we discuss how to get f(m), the greatest number of grid points located on a circle in a grid net of m × m. For m≤15, a clear answer was given. For m≥16, we introduce a sequence similar to the *Farey Sequence*, and obtain a lower bound of f(m). Based on this estimate, we get the range of the radius of the circle. Then, we find out all possible circles and the corresponding numbers of grid points. Finally, the greatest number f(m) in a gird net of m × m is obtained.

Keywords

Grid point & Grid net, Euler Function, Farey Sequence, Relatively Prime Integers

1. Background and Results

There is a very interesting problem in the exam of junior high school graduation in Huzhou, Zhejiang province in 2010:

There is a 12 × 12 grid net as shown in Figure 1, which has 169 grid points. If we draw a circle arbitrary in the net, then the maximum value of the number of grid points located on a circle is ___________ . [1]

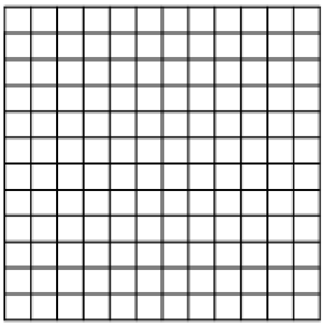

Figure I

The given answer is 12 in the exam. But this answer is wrong, the correct answer is 16. From this point, we are doing some research on this problem. What's more, we extend this question to a more interesting case of m≠12.

There are some people including some famous Mathematicians, such as Gauss, Hardy, Luogen Hua, Jingrun Chen and so on, focused on this problem. However, those studies were limited to the grid points located on a circle or an ellipse centered at the origin. When the grid net is less than 15×15, it is easy to solve if we use the result of Hardy's.[2]

Proposition Let $n = 2^{\alpha} \prod p^{r} \prod q^{s}$, p, q be the prime numbers of the form 4M+1 and 4M-1. Then the equation $x^2 + y^2 = n$ has $\delta(n)$ pairs of integer solutions, where $\delta(n) = \prod(r+1)\prod(\frac{1+(-1)^{s}}{2})$.

In fact, in his book *Introduction to Number Theory,* Luogen Hua discussed the following three questions:

1. the number of the grid points located on a circle centered at origin.
2. the number of the grid points located in an ellipse centered at origin.

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[1]LaigenLuo, "Discussion On a Math Question in a Junior High School Graduation Exam"

[2]G.H. Hardy and E.W. Wright, *An Introduction to the Theory of Numbers*, 5th ed. (Beijing: Posts and Telecom Press, 2008)

3. the number of the grid points located on a 3-dimensional or high dimensional sphere.[3]

By the above results, we have

Corollary Let a,n be positive integers, b_1,b_2 be the integers, $R=\frac{\sqrt{n}}{a}$. If the circle $(ax-b_1)^2+(ay-b_2)^2=n$ passes s grid points in the m×m grid net. If R>m, then $s\le\delta(n)$. If $\frac{\sqrt{2}}{2}m<R\le m$, then $s\le 2\delta(n)$. If $R\le\frac{\sqrt{2}}{2}m$, then $s\le 4\delta(n)$.

The proof will be given later in this paper.

It is known that, according to the properties of the circle, the center of the circle which passing the most grid points must be a rational point in a grid net. Without loss of generality, we assume the center of such a circle is a rational point.

When $m<16$, using the above Corollary and resolving the prime factor of the square of the radius , one can solve maximum value problem for the number of grid point located on a circle in a grid net $m\times m$. However, the method does not work for the case $m\ge 16$, the reason is that we cannot use *drawer principle* to solve this problem, since the number of the *Drawers* is greater than the number of the grid points. So, when the grid net is more than 16×16, we need to develop some new methods to solve it. There are some difficulties: the center of the circle is not known (but must be a rational point); the radius of the circle is not sure; in addition, we have to exclude more and more possibilities while the number of grid points grows, and so on.

One of the keys to solve the problem is the integer count method. Motivated by Farey sequences, we introduce the concept of the summation of two adjacent sides length(we use two grid points which constitutes the diagonal of the rectangle as the summation of two adjacent sides of this rectangle). We denote the summation by K_{EF}. The absolute value of the slope of diagonal is $|k|=\frac{a_1}{b_1}$, (a_1,b_1 are positive integers, they are prime each other), the relationship of K_{EF} and a_1+b_1 is set up. Thus, according to the counting methods that oral set up, we will solve the problem step by step. (We will get series of Lemma):

Definition. In a m×m grid net, f(m) is the greatest number of grid points located on a circle.

First, we give an estimate for the lower bound of f(m), i.e., f(m)≥s.

Next, we proof f(m)≤s by contradiction: Assume that f(m)>s, which is equivalent to f(m)≥s+1, we want to find the range of the radius of the circles that equal to or large than s+1. Then we write down all the equations of circles which radius in the

[3]LuogengHua, *Introduction to Number Theory* (Beijing: Science Press, China), 131-140

range we get in the former step. Finally, we will find that none of those circles we wrote down could actually pass through equal to or large than s+1 grid points no matter how we move these circles on the plane.

In this way, we can conclude that f(m) is equal to s.

By the method above, one can calculate f(m) for any positive integer m theoretically (we have calculated up to around f(100), but calculation became complex for large m).

In this paper, we take how to calculate f(23) as an example to illustrate our methods of finding f(m) when m is greater than 15.

Our main results are the following.

Results: f(1)=f(2)=4；f(3)=f(4)=f(5)=f(6)=8；f(7)=f(8)=f(9)=f(10)=12；f(11)=f(12)=f(13)=…=f(21)=f(22)=f(23)=16；f(24)=20；f(25)=24……

Theorem 1. f(15)=16.

Theorem 2. f(23)=16.

2. Preliminaries

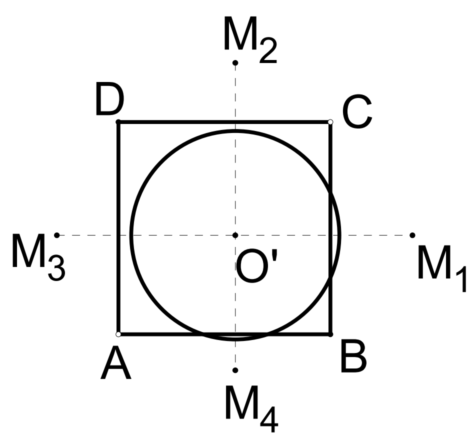
Definition 1. Let EF be a diagonal of a rectangle in a grid net, K_{EF} denote the summation of two adjacent sides of the rectangle. K_{EF} is a positive integer. Specially, when EF is on the grid net line, K_{EF} is the length of line segment EF. Moreover, if the absolute value of the slope of EF is $|k|=\frac{a_1}{b_1}$, where a_1,b_1 are positive integers and $(a_1,b_1)=1$, then $K_{EF}\geq a_1+b_1$.

Definition 2. Let s be a positive integer. Suppose that there is a positive integer j with $\sum_{i=1}^{j}\phi(i)\leq s<\sum_{i=1}^{j+1}\phi(i)$, where $\phi(i)$ is the Euler function, then $\eta(s)$ = $\sum_{i=1}^{j} i\phi(i)+[s-\sum_{i=1}^{j} i\phi(i)](j+1)$.

The following table shows some values of $\eta(s)$.

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\eta(s)$	1	3	6	9	13	17	22	27	32	37	43	49	56	63	70	77	84	91	99

Definition 3. Let O' be a circle in a grid net, M_1M_3 and M_1M_3 be lines which parallel to the grid and pass the center O' and separate the circle O' into four parts (as shown in the figure). Then the area which is above O'M_1 while is on the right side of O'M_2 is called the first area of the circle (containing the pointson line O'M1 while excluding the points on line O'M2); the area which is above O'M3 while is on the left side of O'M2 is called the second area of the circle (containing the points line O'M2 while excluding the points line O'M3);the area which is under O'M3 while is on the left side of O'M4 is called the third area of the circle (containing thepoints on line O'M3 while excluding the points on lineO'M4); the area which is under O'M1 while is on theright side of



O'M4 is called the forth area of the circle(containing the points on line O'M1 while excluding thepoints on line O'M2). We call one of the first, second, third or forth area of the circle one area.

Lemma 1. Let a, n, b_1, b_2 be positive integers. Suppose that, in one area, there are s grid points located on the circle $(ax-b_1)^2+(ay-b_2)^2=n$ in a grid net, and these s grid points can be covered by a $a_0 \times b_0$ grid net. Then $\eta(s)-1 \le a_0 + b_0$. If such a area is the union of the first area and the second area, then

$$\eta([\frac{s}{2}])+\eta([\frac{s+1}{2}])-1 \le a_0 + 2b_0.$$

If such a area is the all circle, then

$$\eta([\frac{s}{4}]) + \eta([\frac{s+1}{4}]) + \eta([\frac{s+2}{4}]) + \eta([\frac{s+3}{4}]) \le 2a_0 + 2b_0.$$

Lemma 2. Let a, n, b_1, b_2 be positive integers. Suppose that, in one area, there are s grid points located on the circle $(ax-b_1)^2+(ay-b_2)^2=n$ in a grid net, and these s grid points can be covered by a $a_0 \times b_0$ rectangle parallel to the grid net, then $s \le \min([a_0],[b_0])+1$.

Lemma 3. Suppose that there is a circle centered in a 23 × 23 grid net passing at least 17 grid points of 576 grid points, then the radius satisfies R≥8.

Lemma 4. Suppose that there is a circle centered in a 23 × 23 grid net passing at least 17 grid points of 576 grid points, then there is at most one grid point in the circle.

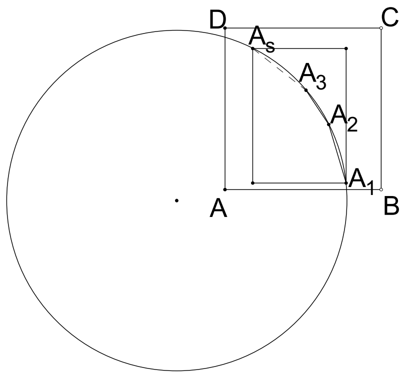
Lemma 5. Let M be a point belongs to a segment AB and |BM|=a (***a*** is a positive integer), and the circle center O is in ⊙M(R), then in the forth quadrant of ⊙O', there is no more than ***a*** grid points in it.

Lemma 6. For a grid net of 23 × 23, if the number of the grid points in grid net of 23 × 23 on the circle is no less than 17, then the radius of the circle R≤16.

Lemma 7. For a grid net of 23 × 23, if the number of the grid points in grid net of 23 × 23 on the circle is no less than 17, then the radius of the circle $R \le \frac{\sqrt{890}}{2}$.

3. Proof of Lemmas

Proof of lemma 1. We only prove (1) holds since the others are similar. Let A_1A_2, A_2A_3, ……, $A_{s-1}A_s$ be s grid point located on the circle $(ax-b_1)^2+(ay-b_2)^2=n$, AB=a_0, AD=b_0, and these grid points can be covered by a $a_0 \times b_0$ grid net (as shown in the figure). Denote by c_1, c_2,……c_{s-1} the absolute values of the sum of numerator and denominator of the slope of A_1A_2, A_2A_3, ……, $A_{s-1}A_s$ and sort c_1, c_2,……c_{s-1} in ascending order, then $c_1 \geq 2$, $c_3 \geq c_2 \geq 3$, $c_5 \geq c_4 \geq 4$, $c_9 \geq c_8 \geq c_7 \geq c_6 \geq 5$, …… (if the sum of numerator and denominator is i, the possible smallest value is at most $\phi(i)$). Then



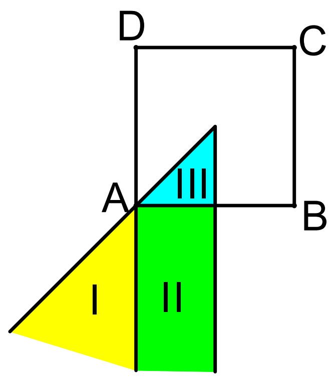
$$a_0+b_0 \geq K_{A_1A_2}+K_{A_2A_3}+\cdots+K_{A_{s-1}A_s}$$
$$\geq 2+3+3+4+4+5+5+\cdots=\eta(s)-1$$

(there are s-1 numbers in the lift hand of the last equality, starting from 2, the appear times of i is less than or equal to $\phi(i)$).

In the same way, when the grid points are in other two areas or four areas, we can also prove it.

Proof of lemma 2 If the grid points can be covered by a $a_0 \times b_0$ rectangle whose sides parallel to the grid net, then $s \leq \min([a_0],[b_0])+1$. The smallest one in these rectangles is $[a_0] \times [b_0]$. Then $s \leq [a_0]+1$, $s \leq [b_0]+1$ because, in one area of the circle, the grid net intersects the circle at most a point, and hence $s \leq \min([a_0],[b_0])+1$.

Proof of lemma 3 We assume that the radius R of the circle satisfies R＜8. By the drawer principle, the circle must be disjointed to grid edge AB or CD, and the same holds true for grid edge BC and AD. Then the grid points on the circle will be in a 15×15 grid net. However, the circle in the grid point can pass 16 grid points, which is a contradiction. Therefore, R≥8.

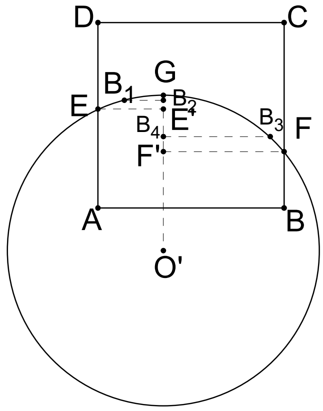


Next, we show that the center of the circle is in the grid net. Suppose to the contrary. By the symmetry of the square, we will discuss two situations, the center in area I and area II:

(a) When the center in area I(including boundary), because there are at least 17 grid points, they are all in area I then $\sum_{i=1}^{16} \kappa_{A_iA_{i+1}} \geq \eta(17)-1=83>2\times 23$. Obviously, according to lemma 1,

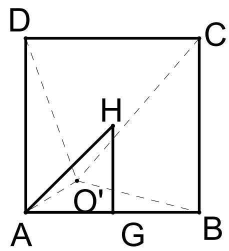
$\sum_{i=1}^{16} \kappa_{A_iA_{i+1}} \leq 2\times 23$. So the center is not in area I.

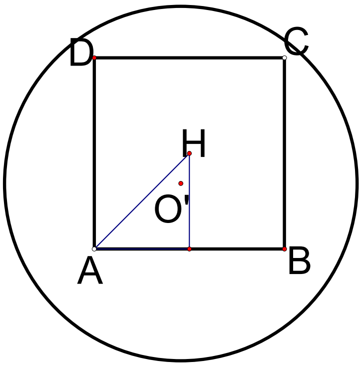
(b) When the center in area I(including boundary), they are all in area I and area II, $\sum_{i=1}^{16} \kappa_{A_iA_{i+1}} \geq \eta(9) + \eta(8) - 1 = 58 > 23+17\times 2+1$. By Lemma 1, a 23×17 grid net can cover 16 grid points which is on the circle at most.

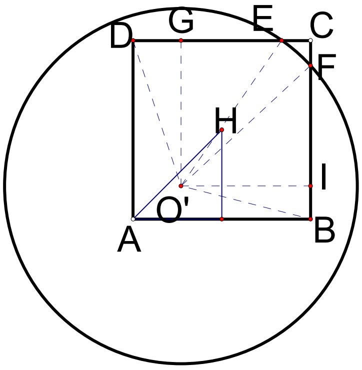


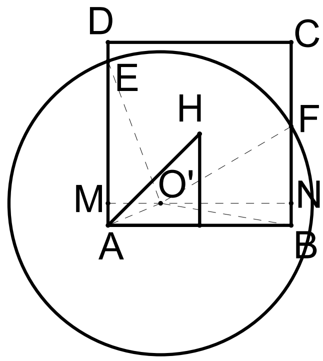
If $R \leq 17$, the center is out of the grid net. Choose a chord of the circle which pass A,B, the height of its bow is less than or equal to 17. Then all grid points on the circle can be covered by a 23×17 grid net. Thus the number of the grid points is not greater than 16.

If R>17, take radius O'G perpendicular to AB, $\eta(12)-1=48>23\times 2$, so there are at least 5 points on the left side of O'G, there are at least 9 points on one side. Take point B_2, B_4 on O'G, $GB_2 = 4$, $GB_4 = 8$. Do B_1B_2, B_3B_4 perpendicular left and right separately to O'G intersecting the circle at B_1, B_3 separately. Take E'E,F'F perpendicular to O'G intersecting O'G at E', F', respectively. Because GB2=4，GB4=8. By Lemma 1, the rectangle (GB_2 is its one side) has at most 5 grid points in the second area, the rectangle (GB_4 is its one side) has at most 5 grid points in the second area, as arc EG has5 grid points at least, arc FG has at least 5 grid points, B_2, B_4 is on segment G E'、G F' separately, $B_1B_2 \leq$ EE', $B_3B_4 \leq$ FF' 。 EE'$\geq B_1B_2 = \sqrt{4(2R-4)} > \sqrt{4\times(2\times 16-4)} = 4\times\sqrt{7} > 10$, FF'$\geq B_3B_4 = \sqrt{8(2R-8)} > \sqrt{8\times(2\times 16-8)} = 8\times\sqrt{3} > 13$. So AB=EE'+ FF'>10+13=23，which is contradict to “23=AB”. This completes the proof.









Proof of lemma 4. By Lemma 3, the center of the circle is necessarily in the grid net. Assume that there are at least 2 grid points in the circle. We discuss there are 0, 1, 2 points out of the circle separately.

When the center in area III (not containing boundary), by Pythagorean Theorem, AO'≤BO'≤DO'≤CO'. There are three cases need to be considered.

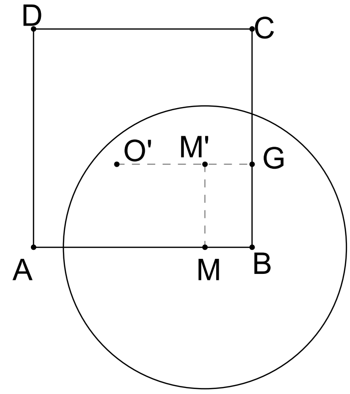
(i)There are 4 grid points in the interior of circle but no grid point located on the circle (as shown in the figure).

(ii) There are 3 grid net points in the interior of circle, AO'≤BO'≤DO'≤CO', C is out of the circle, A,B,D is in the interior of circle, BO'≤DO'＜R. Take O'I,O'G perpendicular to BC, CD intersecting BC, CD at I, G, respectively. Then GE＞DG、FI ＞BI, E, F is on the segment GC and IC. So arc EF is in the rectangle O'ICG, the grid points on EF is in the first area. Think about the distance between grid points on arc EF and AB. Assume the minimum of the distance value is d (d is a positive integer). Translate ⊙O' d unit downward. Then point O' is out of grid net ABCD, now the grid points on arc EF is still the grid points in grid net ABCD, we turn the problem to the center is out of grid net, so the circle can't pass 17 grid points.

(iii) There are 2 grid net points in the circle, AO'≤BO'≤DO'≤CO', C, D is out of the circle, A,B is in the circle, AO'≤BO'＜R. Take MN paralleling to AB intersecting AD, BC at M,N, AO'＜R＝EO', then EM＞AM, point E is on MD (not containing M). In the same way, point F is on NC (not containing N), arc EF is above MN. Assume the minimum of the distance between grid points on arc EF and AB is d (d is the positive integer). Then the distance between O' and AB is less than d. Translate ⊙O' d unit downward. Then point O' is out of grid net ABCD, now the grid points on arc EF are still grid points in grid net ABCD, we turn the problem to the center is out of grid net, so the circle can't pass17 grid points.

So, when the number of the grid net points in the circle is no less than 2, circle can't pass17 grid points.

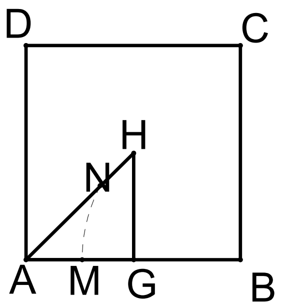
Proof of lemma 5. Take O'G perpendicular to BC intersecting BC at G, and M'M perpendicular to O'G intersecting O'G at M' (as shown in the figure). The intersected point E of ⊙O' and AB is on the segment BM(not containing endpoint). The grid points on the circle in the forth area are all in rectangle MBGM' (not containing M'M). So the grid points in the forth area can be covered by the rectangle whose width is a-1. By Lemma 2, there are at most a grid points

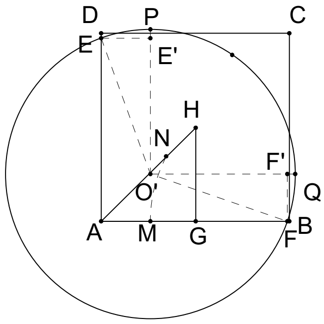


in the forth area of ⊙O'.

Proof of lemma 6 First, we show that R＜17. Assume R≥17, take a circle centered at B with radius 17. The circle intersects AH、AB at N and M, respectively (as shown in the figure).

When the center O' is in the region surrounded by the arc MN, segments MG, HG and HN (not containing arc MN), then AO'≤BO'＜17≤R, A, B is in the interior of circle. By Lemma 4, circle can't pass17 grid points in a 23×23 grid net.





When the center O' is in the region surrounded by the arcMN, segments AM and AN (not containing point N). Take O'P, O'Q paralleling to grid net line, and take E'E,F'F perpendicular to O'P,O'Q(as shown in the figure). Assume the coordinate of N is (a, a), then $(23-a)^2+a^2=17^2(a<11.5)$，a＝8. Thus N=(8,8). AO'＜2, $\sqrt{2}$ ＜R,E'E＜8, F'F＜8.

PE'= $R-\sqrt{R^2-E'E^2}<R-\sqrt{R^2-8^2}\le\frac{8^2}{17+\sqrt{17^2-8^2}}$ =2. In the same way, PE'＜2. By Lemma 2, there are at most 2 grid points on the arc PE and 2 grid points on the arc QF, so there are at most 2 grid points on the arc PE in the rectangle ABCD. Then there are 17-2-2＝13 grid points at on the arc PQ (not containing endpoints P,Q), they are all on the circle ⊙O' in the first area , $\eta(13)-1$＝55＞2×23. When the center O' is in the region surrounded by the arc MN, segments AM and AN (not containing point N), the circle can't pass17 grid points in a 23×23 grid nets.

When O' coincides with N, if R＞17，AO'≤BO'＝17＜R，A,B is in the circle, circle can't pass17 grid points in the grid net ABCD. If R=17, the equation of ⊙O' is $(x-8)^2+(y-8)^2=17^2$, it will pass 12 grid points, circle pass no more than12 grid points in grid nets ABCD, which is impossible. Therefore, R＜17.

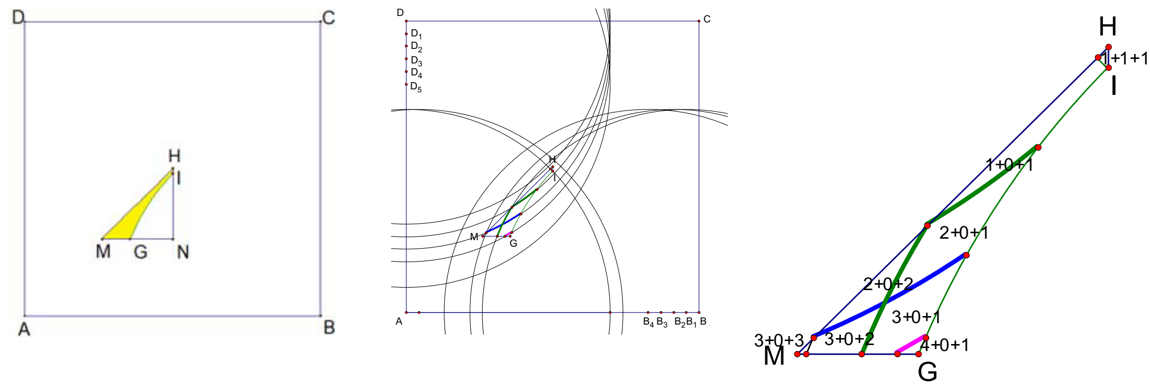
Next, let's prove R≤16.

By the above result, the center O' is in the circle and the radius is less than 17. Make suitable geometric transformation for the circle such that the number of the grid points on the circle is not reduced, and the center is in Rt△MNH (the distances between M and AB,CD, N and AB are both 6, H is the center of the square ABCD) (as shown in the figure).

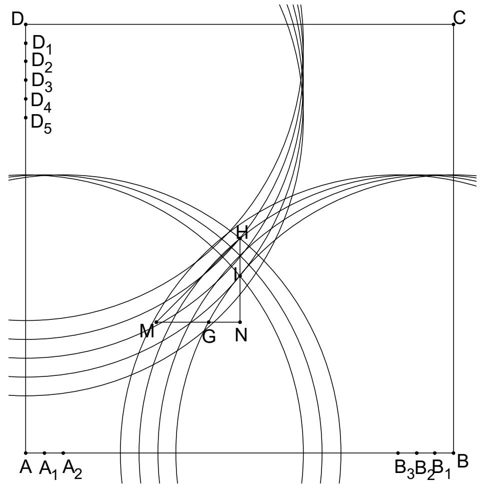
Suppose that there is a circle with radius R>16 such that the circle pass at least 17 grid points.

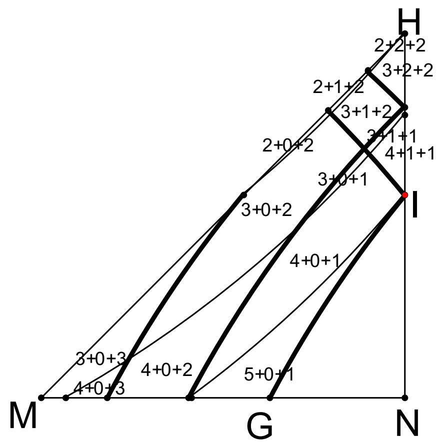
We may assume that the center is in Rt△MNH. If the center O’ is in the region surrounded by the arc GI, segments NI and GN, then points A, B are both in ⊙O'. So the center O’ is in the region surrounded by the arc GI, segments HI, MG and HM (see the yellow area).

Take B_1, B_2, B_3 on AB, $BB_1=1$, $BB_2=2$, $BB_3=3$, and let B_1, B_2, B_3 be the centers of circles with radius 16, respectively. By Lemma 5, if the center O' is in $\odot B_1$, $\odot B_2$, $\odot B_3$, then there are 1,2,3 grid points in the forth area of ⊙O'. Take circles centered at A and A_1 with radius 16. Similarly, take the integer points near D on the segment AD as centers and 16 as radius until there is a circle containing the point G (as shown in the figure). So the yellow area is divided into eight regions. Record every value for maximum of the grid points of these regions in the second, third, forth area (as shown in the figure , in the figure, the first, second, third addend correspond to the value for maximum of the grid points in the second, third, forth area). The sum of the grid in second, third forth is no more than 6, So the number of the grid points in the first area is no less than 11. $\eta(11)-1=52>2\times 23$. So the grid points in the first area are no less than 11. So R≤16.



Using Lemma 6, we know that the center is in Rt△MNH (the distances between M and AB,CD, N and AB are both 7, H is the center of square ABCD). By Lemma 2, the center O’ is in the region surrounded by the arc MH, segments HI and GI. In a similar way , take a sequence of circles and choose $R=\frac{\sqrt{890}}{2}$. Record every value for maximum of the grid points of these regions in the second, third, forth area (as shown in the figure , in the figure, the first, second, third addend correspond to the value for maximum of the grid points in the second, third, forth area). The summation of the grid in second, third forth is no more than 7, So the number of the grid points in the first area is no less than 10. $\eta(10)-1=36>2\times(23-7)$. The grid points in the first area of the circle can be covered by grid net (23-7) × (23-7). So we can get the following lemma.





One can prove Lemma 7 with small modifications. So we omit it here.

4.Proof of the Theorem

Proof of inference 1

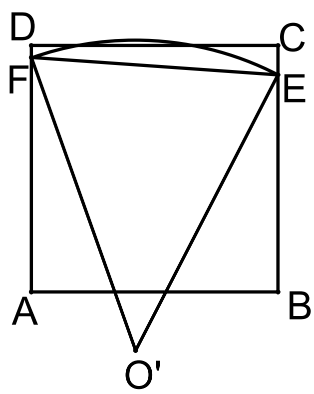
(1) If $R \le \frac{\sqrt{2}}{2}\mathrm{m}$, by the proposition, s≤4$\delta(\mathrm{n})$.

(2) If $\frac{\sqrt{2}}{2}\mathrm{m} < \mathrm{R} \le \mathrm{m}$, i.e., $\sqrt{2}\mathrm{m} < 2\mathrm{R}$. The diameter of the grid net is less than the diameter of the circle, one of two endpoints of the same diameter must be out of the grid net. The grid points on the circle of $(\mathrm{ax}-\mathrm{b}_1)^2+(\mathrm{ay}-\mathrm{b}_2)^2=\mathrm{n}$ is no more than on the circle of $\mathrm{x}_0{}^2+\mathrm{y}_0{}^2=\mathrm{n}$. Then, s≤$\frac{4\delta(\mathrm{n})}{2}=2\delta(\mathrm{n})$.

(3) If R>m, according to the Symmetry of the circle, we can discuss the center in two situations: (a) it is out of the grid net; (b) it is in the grid net. No matter where the center is, we can find a Sector to cover the grid points on the circle, the radius of Sector is R, the Central angle is the acute angle.

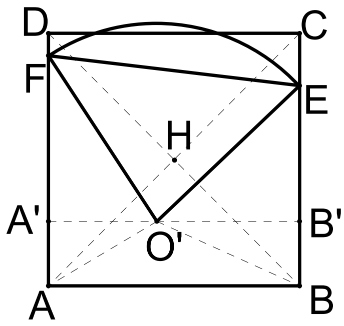
(a) When the center is out of the grid net, according to the symmetry of the circle, we can assume the center O' is under AB(as shown in the figure). The Sector O'-EF can be covered by semicircle whose diameter is paralleling to AB, then the Sector O'-EF is minor arc. EF≤AC$=\sqrt{2}\mathrm{AB}=\sqrt{2}\mathrm{m}<\sqrt{2}$ R, according to cosine theorem,

$$\cos\angle \mathrm{EO'F}=\frac{\mathrm{O'E}^2+\mathrm{O'F}^2-\mathrm{EF}^2}{2\mathrm{O'E}\cdot\mathrm{O'F}}>\frac{\mathrm{R}^2+\mathrm{R}^2-(\sqrt{2}\mathrm{R})^2}{2\mathrm{R}\cdot\mathrm{R}}=0,$$ then $\angle\mathrm{EO'F}$ is the acute angle.



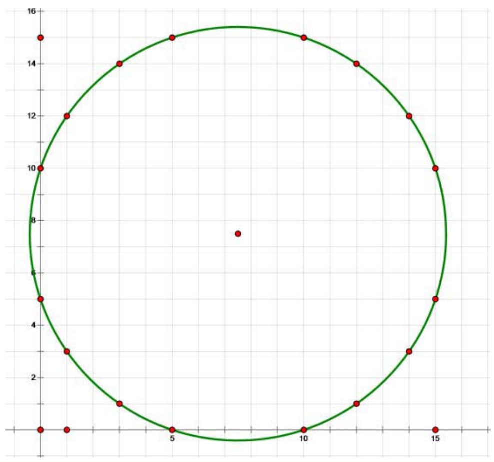
(b) When the center is out of the grid net, according to the symmetry of the circle, we can assume that the center O' is in △ABH (H is the center of ABCD), as shown in the figure . Take segments O'A' and O'B' paralleling to AB intersecting AD and BC at A' and B'. Because O' is in △ABH, ∠AHB=90°, so∠A O'B is a obtuse angles, according to the larger angle makes larger edge, A O'≤AB<R, B O'≤AB<R, so the points in the rectangle ABB'A' must be in ⊙O', the grid points on the circle must be above A'B'. The Sector O'-EF can be covered by semicircle whose diameter is paralleling to AB, then the Sector O'-EF is minor arc. EF≤AC$=\sqrt{2}\mathrm{AB}=\sqrt{2}\mathrm{m}<\sqrt{2}$ R, according to cosine theorem,

$\cos\angle EO'F = \dfrac{O'E^2 + O'F^2 - EF^2}{2O'E \cdot O'F} > \dfrac{R^2 + R^2 - (\sqrt{2}R)^2}{2R \cdot R} = 0$, then $\angle EO'F$ is the acute angle.



If the circle $(ax - b_1)^2 + (ay - b_2)^2 = n$ passes s grid points in the grid net m×m, all grid points are on sector O'-EF, the Central angle is the acute angle. When A is the center of the grid net, let coordinate system magnification of a, the center O' is O'' (b_1, b_2) (it is the grid point in new coordinate system), E is E', F is F', there is s grid point at least on the arc O''- E' F. the equation of the circle is $(x - b_1)^2 + (y - b_2)^2 = n$ will pass 4s grid points at least, then4s≤4 $\delta(n)$. So s≤ $\delta(n)$

Proof of Theorem 1:



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As shown in the figure above, we give the estimation of f(15): f(15)≥16; then let's prove f(15)= 16, We proof by contradiction: assuming f(15)≥17

Establish a rectangular coordinate system as the grid point at lower left quarter is the origin of coordinate, the square length of a side is per unit length, the right level is X axis, the vertical direction is y axis. If there are 17 grid points at least on the circle, according to drawer principle, there are two grid points at the same lever, assume them E,F. As they are grid points on the circle, so the abscissa of the center of the circle must be the integer or half of the integer. In the same way, the ordinate of the center of the circle must be the integer or half of the integer.

Assume the coordinate of the center is $(\frac{a}{2}, \frac{b}{2})$ (a, b are positive integer), the radius is R, the equation of the circle is $(2x-a)^2+(2y-b)^2=4R^2$, so the number of the integer solve is the number the grid points on the circle. Then when left of the equation is the integer, so $4R^2$ is the integer.

As there are17grid points on the circle at least, so the equation has 17 integer solutions at least.

According to inference 1, when $4R^2 \le 4\times(15\times\frac{\sqrt{2}}{2})^2=450$ ($R\le\frac{\sqrt{2}}{2}m$), $4\delta(4R^2)\ge17$, ∴ $\delta(4R^2)\ge5$ ($\delta(4R^2)$ is the integer); when $450< 4R^2\le4\times15^2=900$ ($\frac{\sqrt{2}}{2}m<R\le m$), $2\delta(4R^2)\ge17$, ∴ $\delta(4R^2)\ge9$; when $4R^2>900$ 时 (R>m), $\delta(4R^2)\ge17$.

Let's estimate the range of $4R^2$, as the center is in the grid net, then $R\le15\sqrt{2}$, $4R^2\le1800$. Think about the prime number in the form of $4M+1$: because $5\times13\times17\times29=32045$, so there are three prime numbers in the form of $4M+1$ at most in $4R^2$. Think about $4R^2\le450$, $450<4R^2\le900$ and $4R^2>900$.

When $4R^2\le450$, $\delta(4R^2)\ge5$. As $5\times13\times17=1105>450$, , so there are two prime number in the form of $4M+1$ at most in $4R^2$. Then we will discuss $4R^2$ has one prime number in the form of $4M+1$ and two prime number in the form of $4M+1$.

$4R^2$ has one prime number in the form of $4M+1$, as $\delta(4R^2)\ge5$, so the value for minimum of $4R^2$ has $5^4=625$, which is contradict to $4R^2\le450$. So this is impossible.

$4R^2$ has two prime numbers in the form of $4M+1$, $\delta(4R^2)\ge5$, then one exponent of prime number in the form of $4M+1$ is no less than 2. There are two possible situations: $4R^2=5^2\times13=325$ or $4R^2=5^2\times17=425$. when $4R^2\le450$, $\delta(4R^2)\ge5$, $4R^2$ may be 325 425.

When $4R^2=325, 425$, $4R^2$ is the odd number, $\delta(4R^2)=6$, (x_0, y_0) is the integer solution of $x_0{}^2+y_0{}^2=4R^2$, then (y_0, x_0) is the integer solution of $x_0{}^2+y_0{}^2=4R^2$, so one of x_0、y_0 is even number, the other is odd number. so one of a,b is even number, the other is odd number.

As the $\begin{cases}2x-a=x_0\\2y-b=y_0\end{cases}$ and $\begin{cases}2x-a=x_0\\2y-b=y_0\end{cases}$ there is only one of them has integer solution. According to proposition, $x_0{}^2+y_0{}^2=4R^2$ has 24 integer solutions. So

$(2x-a)^2+(2y-b)^2=4R^2$ has 12 integer solutions, which is contradict to s≥17. So this is impossible.

（1）When 450＜$4R^2$≤900, $\delta(4R^2)$≥9, there are three prime numbers in the form of $4M+1$ at most in $4R^2$. So one exponent of prime number in the form of $4M+1$ is no less than 2. When $4R^2$ has one prime number in the form of $4M+1$, the value for minimum of $4R^2$ is 5^8 =390625＞900; When $4R^2$ has two prime number in the form of $4M+1$, the value for minimum of $4R^2$ is $5^2\times13^2$ =4225>900; When $4R^2$ has three prime number in the form of $4M+1$, the value for minimum of $4R^2$ is $5^2\times13\times17$ =5525＞900. So this is impossible.

In sum, f(15)≥17 is wrong, so f(15)＝16.

Proof of Theorem 2:

If the result is wrong, so f(23)≥17, then it must exist a circle that there are 17 grid points in 23×23 grid net on the circle. The range of the radius R: 8≤R≤$\frac{\sqrt{890}}{2}$.

According to the inference of definition 1 , there exist grid points E、F、G on ⊙O', F is on arc EG, satisfied $\kappa_{EF}+\kappa_{FG}\le[\frac{8m}{s}]=[\frac{8\times23}{17}]=10$. Apparently, $\kappa_{EF}+\kappa_{FG}\ge\kappa_{EG}$, so $\kappa_{EG}\le10$.

Let the circle do some translation transformation up or down and symmetry transform as the coordinate axis is symmetry axis. Point E,G are E’（0，0）, G’（a，b）(a, b are integer points, a≥b, a+b≤10, their greatest common divisor is expressed asgcd（a，b）), think about the grid point not on E'G', the distance between it and E'G' for minimum is no less than $\frac{\gcd(a,b)}{\sqrt{a^2+b^2}}$. So the height of the bow of arc E'F' is no less than $\frac{\gcd(a,b)}{\sqrt{a^2+b^2}}\ge\frac{1}{\sqrt{a^2+b^2}}$

As R≥$\frac{17}{2}$, the height of the bow of arc E'G' is $R-\sqrt{R^2-\frac{a^2+b^2}{4}}=$

$$\frac{\frac{a^2+b^2}{4}}{R^2+\sqrt{R^2-\frac{a^2+b^2}{4}}}\le\frac{\frac{a^2+b^2}{4}}{\left(\frac{17}{2}\right)^2+\sqrt{\left(\frac{17}{2}\right)^2-\frac{a^2+b^2}{4}}}=\frac{17}{2}-\sqrt{(\frac{17}{2})^2-\frac{a^2+b^2}{4}}。$$

$$\therefore\frac{\gcd(a,b)}{\sqrt{a^2+b^2}}\le\frac{17}{2}-\sqrt{(\frac{17}{2})^2-\frac{a^2+b^2}{4}}$$

$$\therefore \gcd(a, b) \le [\frac{17}{2} - \sqrt{(\frac{17}{2})^2 - \frac{a^2 + b^2}{4}}]\sqrt{a^2 + b^2} \text{。(1)}$$

According to （1）: a is a positive integer, b is nonnegative integer, a≥b，a+b≤10. Let's test (a, b) , we have 20 possibility.

Serialnumber	a	b	Serialnumber	a	b
1	4	1	11	6	4
2	4	3	12	7	1
3	5	1	13	7	2
4	5	2	14	7	3
5	5	3	15	8	0
6	5	4	16	8	1
7	5	5	17	8	2
8	6	1	18	9	0
9	6	2	19	9	1
10	6	3	20	10	0

On base of the figure above, we use Geometer's Sketchpad program to test: according to the automatic adsorption properties of the program .take E（0，0）、G（a，b）,make two circles, E,G is the center, radius is 8, the intersection point above EG is O_1, .take E(0，0)、G(a，b),make two circles, E,G is the center, radius is $\frac{\sqrt{890}}{2}$, the intersection point above EG is O_1, , the intersection point above EG is O_2, collect O_1O_2. make two arc, O_1,O_2 is the center, E, G are the end points ofminor arcENG、EMG. Take the integer point F between the two arc, collect EF, do the perpendicular bisector of KL, think about the intersection point of O1O2 and KL, if there is no intersection point, it can not satisfy $8 \le R \le \frac{\sqrt{890}}{2}$; else, there exist intersection point which satisfies $8 \le R \le \frac{\sqrt{890}}{2}$, we can know when $\kappa_{EF} + \kappa_{FG} \le 10$, record the coordinate of F, G, then figure out the equation of the circle. Then move F,G, we can have 29 possibilities:

Serial number	a	b	c	d	Equation of the circle	4δ(n)
1	4	3	3	2	$(2x + 11)^2 + (2y - 23)^2 = 650$	24
2	5	1	1	0	$(2x - 1)^2 + (2y - 21)^2 = 442$	16
3	5	3	4	2	$(x + 2)^2 + (y - 9)^2 = 85$	16
4	5	4	2	1	$(2x + 7)^2 + (2y - 19)^2 = 410$	16
5	5	5	3	2	$(2x + 7)^2 + (2y - 17)^2 = 338$	12

6	6	1	2	0	$(2x-2)^2+(2y-25)^2=629$	16
7	6	1	3	0	$(2x-3)^2+(2y-19)^2=370$	16
8	6	2	1	0	$(2x-1)^2+(2y-17)^2=290$	16
9	6	2	4	1	$(2x+3)^2+(2y-29)^2=850$	24
10	6	3	3	1	$(2x+5)^2+(2y-25)^2=650$	24
11	6	3	5	2	$(2x+1)^2+(2y-17)^2=290$	16
12	6	4	4	2	$(x+3)^2+(y-11)^2=130$	16
13	6	4	5	3	$(x+5)^2+(y-14)^2=221$	16
14	7	1	3	0	$(2x-3)^2+(2y-29)^2=850$	24
15	7	1	4	0	$(x-2)^2+(y-11)^2=125$	16
16	7	2	1	0	$(2x-1)^2+(2y-23)^2=530$	16
17	7	2	2	0	$(4x-4)^2+(4y-39)^2=1537$	16
18	7	2	5	1	$(6x+1)^2+(6y-83)^2=6890$	32
19	7	3	1	0	$(2x-1)^2+(2y-17)^2=290$	16
20	7	3	4	1	$(10x+7)^2+(10y-113)^2=12818$	32
21	7	3	6	2	$(2x-1)^2+(2y-17)^2=290$	16
22	8	0	4	-1	$(2x-8)^2+(2y-15)^2=289$	12
23	8	1	5	0	$(2x-5)^2+(2y-25)^2=650$	24
24	8	1	6	0	$(2x-6)^2+(2y-17)^2=325$	24
25	8	2	2	0	$(x-1)^2+(y-13)^2=170$	16
26	8	2	3	0	$(2x-3)^2+(2y-22)^2=493$	16
27	8	2	4	0	$(x-2)^2+(y-9)^2=85$	16

28	8	2	6	1	$(4x-3)^2+(4y-56)^2=3145$	32
29	9	1	6	0	$(x-3)^2+(y-14)^2=205$	16
30	9	1	7	0	$(2x-7)^2+(2y-19)^2=410$	16

When $4\delta(n)\leq 16$, there are 16 grid points on the circle at most in the figure above. So we just discuss the situation $4\delta(n)>16$, the following figure is the situation when $4\delta(n)>16$ is in the figure above, we can work out the number of grid points on the circle:

Serial number	a	b	c	d	Equation of the circle	The number of grid points
1	4	3	3	2	$(2x+11)^2+(2y-23)^2=650$	24
2	6	2	4	1	$(2x+3)^2+(2y-29)^2=850$	24
3	6	3	3	1	$(2x+5)^2+(2y-25)^2=650$	24
4	7	1	3	0	$(2x-3)^2+(2y-29)^2=850$	24
5	7	2	5	1	$(6x+1)^2+(6y-83)^2=6890$	8
6	7	3	4	1	$(10x+7)^2+(10y-113)^2=12818$	8
7	8	1	5	0	$(2x-5)^2+(2y-25)^2=650$	24
8	8	1	6	0	$(2x-6)^2+(2y-17)^2=325$	12
9	8	2	6	1	$(4x-3)^2+(4y-56)^2=3145$	8

The number of the circle which there are no less than 24 grid points on is 5. Do the translation of the circle, the circle can coincide $(2x-1)^2+(2y-1)^2=650$ or $(2x-1)^2+(2y-1)^2=850$ .Depicture the 24 points, we can move the 23×23 grid net, then the grid net can cover 16 grid points for maximum. Thus it is contradicted to the assumption. Circles of $(6x+1)^2+(6y-83)^2=6890$, $(10x+7)^2+(10y-113)^2=12818$, $(4x-3)^2+(4y-56)^2=3145$ can pass 8 grid points, the circle of

$(2x-6)^2+(2y-17)^2=325$ can pass 12 grid points. All is contradicted to the assumption.
In sum, there isn't 17 grid points on the circle in 23×23 grid net.

5.Reflection and Prospection

We know if m is larger and larger, we need to exclude more possibilities, the calculation is more and more complicated. In other words, our method lacks universality to m as all the integers.

Appriciation We want to thank Professor Yang Xiaoming from Zhejiang University for good advice and Math Teacher Zhang Chuanpeng from Hangzhou Foreign Languages School for comprehensive instruction.

Appendix: References

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