The Relation between Difference Equation and Differential Equation

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Abstract:

In this paper, we will explain the solution to the difference equation by converting the difference equation into a corresponding differential equation. Then we can also derive the Euler-Maclaurin formula by using a similar method.

Keyword: difference equation, differential equation, qualitative analysis

Introduction:

We usually need to solve difference equations in theoretical research and practical situations. However, there are not so many articles on difference equations as those on differential equations. Throughout my investigation, I found that the results of difference equations are similar to those of their differential counterparts. As such we try to study difference equations by using the theory of differential equations.

Part1. On homogeneous linear difference equations with constant coefficients

Consider the general form of linear difference equation with constant coefficients:

$$\sum_{i=0}^{n} a_i x_{t+n-i} = 0 .$$

Being familiar with the theory of differential equations, we try to convert the difference equation into the related differential equation:

$$\sum_{i=0}^{n} b_i D^{n-i} x_t = 0,$$

where D is a differential operator. Then we can solve them. Though there are many forms of different differential equations, they all accord with the same difference equation. So we can just consider one of them.

In this paper we denote $x_{t+i} = V^i x_t$, where V is a recurrence operator.

Section 1: $Vx_t - kx_t = 0, k \in C$.

We regard x_t as a continuous function:

$$x_t = x(t),$$

where t is an independent variable.

Let $x_{t+\frac{1}{n}}, x_{t+\frac{2}{n}}, \ldots, x_{t+\frac{n-1}{n}}$ be points in the region between x_t and x_{t+1}, then we have:

$$x_{t+\frac{k+1}{n}} = V^{\frac{1}{n}} x_{t+\frac{k}{n}}, \left(k = 0,1,\cdots n-1\right),$$

and

$$\left(V^{\frac{1}{n}}\right)^n x_t = Vx_t .$$

We can suppose:

$$V^{\frac{1}{n}} x_t - \lambda x_t = 0, n \in N^+ .$$

Thus:

$$\begin{aligned} Vx_t &= x_{t+1} \\ &= x_{t+\frac{n-1+1}{n}} \\ &= V^{\frac{1}{n}} x_{t+\frac{n-1}{n}} \\ &= \lambda x_{t+\frac{n-1}{n}} \\ &= \cdots\cdots \\ &= \lambda^n x_t \\ &= kx_t . \end{aligned}$$

Comparing coefficients , we obtain the following equation:

$$\lambda = k^{\frac{1}{n}} .$$

Rearranging:

$$V^{\frac{1}{n}} x_t - k^{\frac{1}{n}} x_t = 0, n \in N^+ .$$

Because we have $x_{t+\frac{k+1}{n}} = V^{\frac{1}{n}} x_{t+\frac{k}{n}}, \left(k = 0,1,\cdots n-1\right)$ as well, we obtain:

$$V^{\frac{m}{n}} x_t = x_{t+\frac{m}{n}} .$$

By induction, first we suppose that:

$$V^{\frac{m}{n}}x_t - k^{\frac{m}{n}}x_t = 0\,, \forall m \in N^+ .$$

Then we obtain:

$$\begin{aligned} 0 &= V^{\frac{1}{n}}x_{t+\frac{m}{n}} - k^{\frac{1}{n}}x_{t+\frac{m}{n}} \\ &= V^{\frac{1}{n}} \cdot V^{\frac{m}{n}}x_t - k^{\frac{1}{n}} \cdot k^{\frac{m}{n}}x_t \\ &= V^{\frac{m+1}{n}}x_t - k^{\frac{m+1}{n}}x_t . \end{aligned}$$

Obviously the following equation is satisfied:

$$V^q x_t - k^q x_t = 0\,, q \in Q^+ .$$

This is:

$$V^q x_t = x_{t+q} = k^q x_t\,, q \in Q^+ .$$

We first find a series $\{q_i\}$ converging to $r \in R^+$, then we can obtain:

$$= k^{\frac{1}{n}} \cdot V^{\frac{1}{n}} x_{t+\frac{n-2}{n}} + \mu \cdot l^{t+\frac{n-1}{n}} - kx_t$$

$$= k^r x_t$$

Define $V^r x_t = \lim_{i \to +\infty} V^{q_i} x_t\,, x_{t+r} = \lim_{i \to +\infty} x_{t+q_i}$.

We can obtain:

$$V^r x_t - k^r x_t = 0\,, r \in R^+ .$$

This is:

$$\left(V^r - 1\right)x_t = \left(k^r - 1\right)x_t .$$

Taking the limit of the following equation as $r \to 0$, we have:

$$\begin{aligned} Dx_t &= \lim_{r \to 0} \frac{\left(x_{t+r} - x_t\right)}{r} \\ &= \lim_{r \to 0} \frac{V^r - 1}{r} x_t \\ &= \lim_{r \to 0} \frac{k^r - 1}{r} x_t \end{aligned}$$

$$= \ln k \cdot x_t .$$

The difference equation $Vx_t - kx_t = 0$ can be converted into the related differential equation $Dx_t - \ln k \cdot x_t = 0$.

We will give an example to explain the above method.

Example1: $x_{t+1} = -2x_t , x_0 = 1$ (This is a geometric progression).

By using above method, we obtain:

$$Dx_t = \ln(-2) \cdot x_t , x_0 = 1 .$$

The solution is:

$$x_t = e^{\ln(-2) \cdot t} = (-2)^t .$$

Section 2: $Vx_t - kx_t = \alpha \cdot l^t$, $k \in C$, $l \in C$, $k \neq l$, α is a constant coefficient.

Let $x_{t+\frac{1}{n}}, x_{t+\frac{2}{n}}, \ldots, x_{t+\frac{n-1}{n}}$ be points in the region between x_t and x_{t+1}, then we have:

$$x_{t+\frac{k+1}{n}} = V^{\frac{1}{n}} x_{t+\frac{k}{n}} , (k = 0,1,\cdots n-1);$$

and

$$\left(V^{\frac{1}{n}} \right)^n x_t = Vx_t .$$

We can suppose:

$$V^{\frac{1}{n}} x_t - \lambda x_t = \mu \cdot l^t .$$

Thus:

$$Vx_t - kx_t = V^{\frac{1}{n}} x_{t+\frac{n-1}{n}} - kx_t$$

$$= \lambda \cdot x_{t+\frac{n-1}{n}} + \mu \cdot l^{t+\frac{n-1}{n}} - kx_t$$

$$= \lambda \cdot V^{\frac{1}{n}} x_{t+\frac{n-2}{n}} + \mu \cdot l^{t+\frac{n-1}{n}} - kx_t$$

$$= \cdots\cdots$$

$$= \lambda^n x_t + \mu \cdot \left(\lambda^{n-1} \cdot l^t + \cdots + \lambda \cdot l^{t+\frac{n-2}{n}} + l^{t+\frac{n-1}{n}} \right) - kx_t$$

$$= \lambda^n x_t - kx_t + \frac{k-l}{k^{\frac{1}{n}} - l^{\frac{1}{n}}} \cdot \mu \cdot l^t$$

$$= \alpha \cdot l^t .$$

Comparing coefficients , we obtain the following equation:

$$\lambda = k^{\frac{1}{n}}, \mu = \frac{k^{\frac{1}{n}} - l^{\frac{1}{n}}}{k-l} \cdot \alpha .$$

So the original equation can be converted into:

$$V^{\frac{1}{n}} x_t - k^{\frac{1}{n}} x_t = \frac{k^{\frac{1}{n}} - l^{\frac{1}{n}}}{k-l} \cdot \alpha \cdot l^t , n \in N^+ .$$

Because we have $x_{t+\frac{k+1}{n}} = V^{\frac{1}{n}} x_{t+\frac{k}{n}}, (k = 0,1,\cdots n-1)$ as well, we obtain:

$$V^{\frac{m}{n}} x_t = x_{t+\frac{m}{n}} .$$

By induction, we suppose that the following equation is satisfied:

$$V^{\frac{1}{n}} x_t - k^{\frac{1}{n}} x_t = \frac{k^{\frac{1}{n}} - l^{\frac{1}{n}}}{k-l} \cdot \alpha \cdot l^t , \forall m \in N^+ .$$

Then we can obtain:

$$\frac{k^{\frac{1}{n}} - l^{\frac{1}{n}}}{k-l} \cdot \alpha \cdot l^{t+\frac{m}{n}} = V^{\frac{1}{n}} x_{t+\frac{m}{n}} - k^{\frac{1}{n}} x_{t+\frac{m}{n}}$$

$$= V^{\frac{1}{n}} \cdot V^{\frac{m}{n}} x_t - k^{\frac{1}{n}} \cdot \left(k^{\frac{m}{n}} x_t + \frac{k^{\frac{m}{n}} - l^{\frac{m}{n}}}{k-l} \cdot \alpha \cdot l^t \right)$$

$$= V^{\frac{m+1}{n}} x_t - k^{\frac{m+1}{n}} x_t - \frac{k^{\frac{m+1}{n}} - l^{\frac{m}{n}} \cdot k^{\frac{1}{n}}}{k-l} \cdot \alpha \cdot l^t .$$

This is:

$$V^{\frac{m+1}{n}} x_t - k^{\frac{m+1}{n}} x_t = \frac{k^{\frac{m+1}{n}} - l^{\frac{m+1}{n}}}{k-l} \cdot \alpha \cdot l^t .$$

We can easily obtain that:

$$V^q x_t - k^q x_t = \frac{k^q - l^q}{k-l} \cdot \alpha \cdot l^t , q \in Q^+ .$$

This is:

$$V^q x_t = x_{t+q} = k^q x_t + \frac{k^q - l^q}{k-l} \cdot \alpha \cdot l^t , q \in Q^+ .$$

We first find a series $\{q_i\}$ converging to $r \in R^+$, then we can obtain:

$$\lim_{i \to +\infty} V^{q_i} x_t = \lim_{i \to +\infty} x_{t+q_i} = \lim_{i \to +\infty} \left(k^{q_i} x_t + \frac{k^{q_i} - l^{q_i}}{k-l} \cdot \alpha \cdot l^t \right)$$

$$= k^r x_t + \frac{k^r - l^r}{k-l} \cdot \alpha \cdot l^t .$$

Define $V^r x_t = \lim_{i \to +\infty} V^{q_i} x_t , x_{t+r} = \lim_{i \to +\infty} x_{t+q_i}$.

We can obtain:

$$V^r x_t - k^r x_t = \frac{k^r - l^r}{k-l} \cdot \alpha \cdot l^t , r \in R^+ .$$

This is:

$$\left(V^r - 1\right) x_t = \left(k^r - 1\right) x_t + \frac{k^r - l^r}{k-l} \cdot \alpha \cdot l^t .$$

Letting $r \to 0$, we can get:

$$Dx_t = \lim_{r \to 0} \frac{\left(x_{t+r} - x_t\right)}{r}$$

$$= \lim_{r \to 0} \frac{V^r - 1}{r} x_t$$

$$= \lim_{r \to 0} \frac{k^r - 1}{r} x_t + \lim_{r \to 0} \frac{k^r - l^r}{r \cdot (k-l)} \cdot \alpha \cdot l^t$$

$$= \ln k \cdot x_t + \beta \cdot l^t ,$$

where $\beta = \alpha \cdot \dfrac{\ln k - \ln l}{k - l}$.

The difference equation $Vx_t - kx_t = \alpha \cdot l^t$ was converted into the differential equation $Dx_t - \ln k \cdot x_t = \beta \cdot l^t$.

From the following two equations:

$$Vx_t - kx_t = \alpha \cdot l^t,$$

$$Dx_t - \ln k \cdot x_t = \beta \cdot l^t,$$

we see that the functions' forms on the right are the same.

Section 3: $Vx_t - kx_t = \alpha \cdot k^t$, $k \in C$, α is a constant coefficient.

Let $x_{t+\frac{1}{n}}, x_{t+\frac{2}{n}}, \ldots, x_{t+\frac{n-1}{n}}$ be points in the region between x_t and x_{t+1}, then we have:

$$x_{t+\frac{k+1}{n}} = V^{\frac{1}{n}} x_{t+\frac{k}{n}}, \left(k = 0, 1, \cdots n-1\right);$$

$$\left(V^{\frac{1}{n}}\right)^n x_t = Vx_t.$$

We can suppose:

$$V^{\frac{1}{n}} x_t - \lambda x_t = \mu \cdot k^t.$$

Thus:

$$\begin{aligned} Vx_t - kx_t &= V^{\frac{1}{n}} x_{t+\frac{n-1}{n}} - kx_t \\ &= \lambda \cdot x_{t+\frac{n-1}{n}} + \mu \cdot k^{t+\frac{n-1}{n}} - kx_t \\ &= \lambda \cdot V^{\frac{1}{n}} x_{t+\frac{n-2}{n}} + \mu \cdot k^{t+\frac{n-1}{n}} - kx_t \\ &= \cdots\cdots \\ &= \lambda^n x_t + \mu \cdot \left(\lambda^{n-1} \cdot k^t + \cdots + \lambda \cdot k^{t+\frac{n-2}{n}} + k^{t+\frac{n-1}{n}}\right) - kx_t \end{aligned}$$

$$= \alpha \cdot l^t .$$

So we have:

$$\lambda = k^{\frac{1}{n}}, \mu = \frac{1}{n} \cdot k^{\frac{1}{n}-1} \cdot \alpha .$$

So the original equation can be converted into:

$$V^{\frac{1}{n}} x_t - k^{\frac{1}{n}} x_t = \frac{1}{n} \cdot k^{\frac{1}{n}-1} \cdot \alpha \cdot k^t , n \in N^+$$

Because we have $x_{t+\frac{k+1}{n}} = V^{\frac{1}{n}} x_{t+\frac{k}{n}}, (k = 0,1, \cdots n-1)$ as well, we have:

$$V^{\frac{m}{n}} x_t = x_{t+\frac{m}{n}} .$$

By induction, first we suppose that the following equation is satisfied:

$$V^{\frac{1}{n}} x_{t+\frac{m}{n}} - k^{\frac{1}{n}} x_{t+\frac{m}{n}} = \frac{1}{n} \cdot k^{\frac{1}{n}-1} \cdot \alpha \cdot k^{t+\frac{m}{n}} , \forall m \in N .$$

Then we can get:

$$\begin{aligned} \frac{1}{n} \cdot k^{\frac{1}{n}-1} \cdot \alpha \cdot k^{t+\frac{m}{n}} &= V^{\frac{1}{n}} x_{t+\frac{m}{n}} - k^{\frac{1}{n}} x_{t+\frac{m}{n}} \\ &= V^{\frac{1}{n}} \cdot V^{\frac{m}{n}} x_t - k^{\frac{1}{n}} \cdot \left(k^{\frac{m}{n}} x_t + \frac{m}{n} \cdot k^{\frac{m}{n}-1} \cdot \alpha \cdot k^t \right) \\ &= V^{\frac{m+1}{n}} x_t - k^{\frac{m+1}{n}} x_t - \frac{m}{n} \cdot k^{\frac{m+1}{n}-1} \cdot \alpha \cdot k^t . \end{aligned}$$

This is:

$$V^{\frac{m+1}{n}} x_t - k^{\frac{m+1}{n}} x_t = \frac{m}{n} \cdot k^{\frac{m+1}{n}-1} \cdot \alpha \cdot k^t .$$

Obviously the following equation is satisfied:

$$V^q x_t - k^q x_t = q \cdot k^{q-1} \cdot \alpha \cdot k^t , q \in Q^+ .$$

This is:

$$V^q x_t = x_{t+q} = k^q x_t + q \cdot k^{q-1} \cdot \alpha \cdot l^t , q \in Q^+ .$$

We first find a series $\{q_i\}$ converging to $r \in R^+$, then we can obtain:

$$\lim_{i\to+\infty} V^{q_i} x_t = \lim_{i\to+\infty} x_{t+q_i} = \lim_{i\to+\infty}\left(k^{q_i} x_t + q_i \cdot k^{q_i-1} \cdot \alpha \cdot l^t\right)$$

$$= k^r x_t + r \cdot k^{r-1} \cdot \alpha \cdot l^t .$$

Define $V^r x_t = \lim_{i\to+\infty} V^{q_i} x_t , x_{t+r} = \lim_{i\to+\infty} x_{t+q_i}$.

We can obtain:

$$V^r x_t - k^r x_t = r \cdot k^{r-1} \cdot \alpha \cdot k^t , r \in R^+ .$$

This is:

$$\left(V^r - 1\right)x_t = \left(k^r - 1\right)x_t + r \cdot k^{r-1} \cdot \alpha \cdot k^t .$$

When $r \to 0$, we can get:

$$Dx_t = \lim_{r\to 0} \frac{\left(x_{t+r} - x_t\right)}{r}$$

$$= \lim_{r\to 0} \frac{V^r - 1}{r} x_t$$

$$= \lim_{r\to 0} \frac{k^r - 1}{r} x_t + \lim_{r\to 0} \frac{1}{r} \cdot r \cdot k^{r-1} \cdot \alpha \cdot k^t$$

$$= \ln k \cdot x_t + \beta \cdot k^t ,$$

where $\beta = \alpha \cdot \frac{1}{k}$.

So the difference equation $Vx_t - kx_t = \alpha \cdot k^t$ has been converted into the corresponding differential equation $Dx_t - \ln k \cdot x_t = \beta \cdot k^t$.

From the two equations below:

$$Vx_t - kx_t = \alpha \cdot k^t ,$$

$$Dx_t - \ln k \cdot x_t = \beta \cdot k^t ,$$

we see that functions' forms on the right of the equations are the same.

Example2: $x_{t+1} + 2x_t = 2^t + \left(-2\right)^t , x_0 = 2$.

Considering these two equations:

$$y_{t+1} + 2y_t = 2^t,$$

$$z_{t+1} + 2z_t = (-2)^t.$$

Obviously we can get: $x_t = y_t + z_t$.

By using the way we have introduced above, we can obtain that the following two equations are satisfied:

$$Dy_t = \ln(-2)\cdot y_t + \frac{\ln 2 - \ln(-2)}{2-(-2)}\cdot 2^t$$
$$= \ln(-2)\cdot y_t + \frac{\ln(-1)}{4}\cdot 2^t,$$

where $y_0 = 1$,and

$$Dz_t = \ln 2\cdot z_t + \frac{1}{(-2)}\cdot(-2)^t$$
$$= \ln(-2)\cdot z_t - \frac{1}{2}\cdot(-2)^t,$$

where $z_0 = 1$.

By solving these two differential equations, we can get:

$$x_t = \frac{1}{4}\cdot 2^t + \frac{7}{4}\cdot(-2)^t - \frac{1}{2}\cdot t\cdot(-2)^t.$$

Section 4: $Vx_t - kx_t = P(t)\cdot l^t$ **,** $k \in C$ **,** $l \in C$ **, where** $P(t)$ **is a polynomial.**

Considering the conclusions of Section 1, 2 and 3, we conjecture the function's form on the right of the converted differential equation does not change. So the original equation can be converted into:

$$Dx_t - \ln k\cdot x_t = Q(t)\cdot l^t,$$

where $Q(t)$ is a function whose degree is the same as that of $P(t)$.

according to the theory of differential equations we know the solution of this equation is:

When $k \neq l$,

$$x_t = \alpha \cdot k^t + R_1(t) \cdot l^t .$$

where α is an indeterminate coefficient and $R_1(t)$ is a polynomial whose degree is the same as that of $Q(t)$.

When $k = l$,

$$x_t = R_2(t) \cdot k^t$$

where $R_2(t)$ is a function whose degree is one higher than that of $Q(t)$.

Substituting these two kinds of solution into $Vx_t - kx_t = P(t) \cdot l^t$, we can get:

If $k \neq l$,

$$\begin{aligned} & Vx_t - kx_t \\ &= \alpha \cdot k^{t+1} + R_1(t+1) \cdot l^{t+1} - k\left(\alpha \cdot k^t + R_1(t) \cdot l^t\right) \\ &= \left[l \cdot R_1(t+1) - k \cdot R_1(t)\right] \cdot l^t \\ &= P(t) \cdot l^t . \end{aligned}$$

If $k = l$,

$$\begin{aligned} & Vx_t - kx_t \\ &= R_2(t+1) \cdot k^{t+1} - k\left(R_2(t) \cdot k^t\right) \\ &= k \cdot \left[R_2(t+1) - R_2(t)\right] \cdot k^t \\ &= P(t) \cdot k^t . \end{aligned}$$

So the solution of $Dx_t - \ln k \cdot x_t = Q(t) \cdot l^t$ indeed satisfies the difference equation $Vx_t - kx_t = P(t) \cdot l^t$.

Then the difference equation $Vx_t - kx_t = P(t) \cdot l^t$ can be converted into the differential equation $Dx_t - \ln k \cdot x_t = Q(t) \cdot l^t$.

In fact, the form of $Q(t)$ can be determined by $P(t)$. But the actual expression of $Q(t)$, which will be given in Part 2, is complex.

Section 5: $\sum_{i=0}^{n} a_i V^{n-i} x_t = 0$ **,** $a_i \in R$ **.**

Because of the linear property of the recurrent operator, we can get:

$$\sum_{i=0}^{n} a_i V^{n-i} x_t = a_0 \cdot \prod_{i=1}^{n} (V - k_i) x_t .$$

The roots of $\sum_{i=0}^{n} a_i k^{n-i} = 0$, including repeated roots, are precisely the numbers in the sequence $k_1, k_2, k_3 \ldots$.

In fact, we can get this equation by factoring $\sum_{i=0}^{n} a_i V^{n-i}$.

So the equation $\sum_{i=0}^{n} a_i V^{n-i} x_t = 0$ can be converted into a system of equations:

$$\begin{cases} (V - k_1) y_1 = 0 \\ (V - k_2) y_2 = y_1 \\ \vdots \\ (V - k_n) y_n = y_{n-1} \end{cases} .$$

where $y_n = x_t$, $y_i = \prod_{j=i+1}^{n} (V - k_j) x_t$, $i = 1, 2, \cdots, n-1$.

By using the result in Section 4, we can get:

The expression of y_i is $\sum R_j(t) k_j^t$, where $R_i(t)$ is a polynomial and k_i's are different from each other. The degree of $R_i(t)$ is the same as the repetitions of k_i .

So we can convert the equation into the following system of differential equations:

$$\begin{cases} (D - \ln k_1) y_1' = 0 \\ (D - \ln k_2) y_2' = y_1' \\ \vdots \\ (D - \ln k_n) y_n' = y_{n-1}' \end{cases}$$

where the form of $y_i^{'}$ is the same as that of y_i.

The above system of differential equations is equivalent to:

$$b_0 \sum_{i=1}^{n} b_i (D - \ln k_i) x_t = 0 .$$

So the difference equation $\sum_{i=0}^{n} a_i V^{n-i} x_t = a_0 \cdot \prod_{i=1}^{n} (V - k_i) x_t = 0$ has been converted into the differential equation $\sum_{i=0}^{n} b_i D^{n-i} x_t = b_0 \cdot \prod_{i=1}^{n} (D - \ln k_i) x_t = 0$.

Through the discussion of the above 5 sections, we have shown how to convert the homogeneous linear difference equations with constant coefficients into the homogeneous linear differential equations with constant coefficients. From the properties of the differential equations, we know that the solution to the differential equation is:

$$\sum R_i(t) e^{\ln k_i \cdot t} = \sum R_i(t) k_i^t .$$

where $R_i(t)$ is a polynomial, and the degree of $R_i(t)$ is the same as the repetitions of k_i. It is the result that we can know from the Eigenvalue method.

Example3: $x_{t+2} = x_{t+1} + x_t, x_1 = 1, x_2 = 1$ (This is the Fibonacci Sequence)

By the method we introduced above, the equations can be converted into:

$$\begin{cases} (V - k_1) y_1 = 0 \\ (V - k_2) y_2 = y_1 \end{cases} .$$

where $y_2 = x_t$ and k_1, k_2 are the solutions to $k^2 = k + 1$.

Then we can get:

$$\begin{cases} \left(V - \dfrac{1+\sqrt{5}}{2}\right) y_1 = 0 \\ \left(V - \dfrac{1-\sqrt{5}}{2}\right) y_2 = y_1 \end{cases} ,$$

which can be converted into the following system of differential equaitons:

$$\begin{cases}\left[D-\ln\left(\frac{1+\sqrt{5}}{2}\right)\right]y_1^{'}=0\\ \left[D-\ln\left(\frac{1-\sqrt{5}}{2}\right)\right]y_2^{'}=y_1^{'}\end{cases},$$

where $y_2^{'}=x_t$.

By comparison with the result for differential equations, the solution is:

$$x_t=c_1\left(\frac{1+\sqrt{5}}{2}\right)^t+c_2\left(\frac{1-\sqrt{5}}{2}\right)^t,$$

Using the method of undetermined coefficient, we can get:

$$x_t=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^t-\left(\frac{1-\sqrt{5}}{2}\right)^t\right].$$

Section 6: Equations that can be converted into the above forms.

1) $x_{t+m}=c\cdot\prod_{i=0}^{m-1}x_{t+i}^{k_i}, k_i\in C, c\in C$.

We simply take the natural logarithm, from which we can obtain:

$$\ln x_{t+m}=\sum_{i=0}^{m-1}k_i\cdot\ln x_{t+i}+\ln c.$$

Letting $y_{t+i}=\ln x_{t+i}$, this becomes:

$$y_{t+m}-\sum_{i=0}^{m-1}k_i\cdot y_{t+i}=\ln c.$$

This can be solved by using the method introduced in Section 5.

2) $x_{t+1}=\frac{ax_t+c}{bx_t+d}$, where the constants are all real number and $b\neq 0$.

This may be rewritten as:

$$x_{t+1}-m=\frac{ax_t+c}{bx_t+d}-m=\frac{(a-mb)x_t+c-md}{bx_t+d}.$$

We would like the form on the left of the equation to the same as that on the right, so we consider the following condition:

$$-m(a-mb)=c-md\ .$$

This is:

$$bm^2+(d-a)m-c=0\ .$$

If $m_1 \neq m_2$:

$$\frac{x_{t+1}-m_1}{x_{t+1}-m_2}=\frac{a-m_1 b}{a-m_2 b}\cdot\frac{x_t-m_1}{x_t-m_2}\ .$$

If $m_1 = m_2$:

$$\frac{1}{x_{t+1}-m}=\frac{d}{a-mb}+\frac{b}{a-mb}\cdot\frac{1}{x_t-m}\ .$$

It can be solved by using the way we have introduced in Section 1 and 2.

Example4: 1) $x_{t+1}=x_t^2, x_0=2$;

2) $y_{t+1}=\dfrac{2y_t+1}{y_t+2}, y_1=2$.

1) Taking the natural logarithm, we can obtain:

$$\ln x_{t+1}=2\cdot\ln x_t\ .$$

This is $a_{t+1}=2a_t$.

So the solution is $x_t=2^{2^t}$.

2) It can be rewritten as:

$$\frac{y_{t+1}+1}{y_{t+1}-1}=3\cdot\frac{y_t+1}{y_t-1}\ .$$

This is $a_{t+1}=3a_t$.

So the solution is $y_t=\dfrac{3^t+1}{3^t-1}$.

Part2. Non-homogeneous linear differential equation with

constant coefficients and a derivation of the Euler-Maclaurin formula

In this part, we research how to convert a difference equation such as $Vx_t - kx_t = f(t)$ into a differential equation. Meanwhile, we solve the problem left in Section 4 of Part 1.

Section 1: $Vx_t - kx_t = f(t)$.

At first, consider the equation $Dx_t - \ln k \cdot x_t = g(t)$, by the theory of differential equation we can change the differential equation to:

$$\frac{1}{e^{\ln k \cdot t}} \cdot Dx_t - \frac{1}{e^{\ln k \cdot t}} \cdot \ln k \cdot x_t = \frac{1}{e^{\ln k \cdot t}} \cdot g(t).$$

This is:

$$D\left(\frac{1}{e^{\ln k \cdot t}} \cdot x_t\right) = \frac{1}{e^{\ln k \cdot t}} \cdot g(t).$$

Comparing with the above method, we can hope to use a similar method to deal with the equation $Vy_t - ky_t = f(t)$.

Obviously, we have:

$$\frac{1}{k^{t+1}} \cdot Vy_t - \frac{1}{k^{t+1}} \cdot ky_t = \frac{1}{k^{t+1}} \cdot f(t).$$

This is:

$$V\frac{y_t}{k^t} - \frac{y_t}{k^t} = \frac{1}{k^{t+1}} \cdot f(t).$$

Note $h(t) = \frac{1}{k^{t+1}} \cdot f(t), x_t = \frac{y_t}{k^t}$,then we obtain $Vx_t - x_t = h(t)$.

Section 2: The situation of $h(t) = t^k$.

1) At first, we will consider the situation where n is an integer.

Assume:

$$V^{\frac{1}{n}}x_t - x_t = H_{\frac{1}{n}}(t).$$

Our aim is to calculate:

$$Dx_t = \lim_{n\to+\infty} n\left(V^{\frac{1}{n}} - 1\right)x_t = \lim_{n\to+\infty} n\cdot H_{\frac{1}{n}}(t)\cdots(1).$$

From the assumption we can get:

$$\begin{cases} V^{\frac{1}{n}}x_t - x_t = H_{\frac{1}{n}}(t) \\ V^{\frac{1}{n}}x_{t+\frac{1}{r}} - x_{t+\frac{1}{n}} = H_{\frac{1}{n}}\left(t+\frac{1}{n}\right) \\ \vdots \\ V^{\frac{1}{n}}x_{t+\frac{n-1}{n}} - x_{t+\frac{n-1}{n}} = H_{\frac{1}{n}}\left(t+\frac{n-1}{n}\right) \end{cases}.$$

Adding them up, we can obtain:

$$Vx_t - x_t = \sum_{i=0}^{n-1} H_{\frac{1}{n}}\left(t+\frac{i}{n}\right) = t^k\cdots(2).$$

Obviously we can suppose $H_{\frac{1}{n}}(t)$ is a polynomial with degree k:

$$H_{\frac{1}{n}}(t) = \sum_{j=0}^{k} a_j\cdot t^{k-j}\cdots(3).$$

where a_j depends on j.

Substituting (3) into (2), we have:

$$\sum_{i=0}^{n}\sum_{j=0}^{k} a_j\cdot\left(t+\frac{i}{n}\right)^{k-j}$$

$$=\sum_{i=0}^{n}\sum_{j=0}^{k}\left[a_j\cdot\sum_{l=0}^{k-j}C_{k-j}^{l}\cdot t^{k-j-l}\cdot\left(\frac{i}{n}\right)^{l}\right]$$

$$=t^k\cdots(4).$$

At the same time, substituting (3) into (1), we have:

$$\lim_{n\to+\infty} n\left(V^{\frac{1}{n}}-1\right)x_t = \lim_{n\to+\infty} n\cdot H_{\frac{1}{n}}(t)$$

$$=\lim_{n\to+\infty} n\cdot\sum_{j=0}^{k} a_j\cdot t^{k-j}$$

$$=\sum_{j=0}^{k} b_j\cdot t^{k-j}\ .$$

where $b_j=\lim_{n\to+\infty} n\cdot a_j, n\in N^+$.

We use equation (4) to calculate b_j.

Rearranging equation (4) in descending order, we can get:

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\left[a_j\cdot\sum_{l=0}^{k-j} C_{k-j}^{l}\cdot t^{k-j-l}\cdot\left(\frac{i}{n}\right)^{l}\right]$$

$$=\sum_{i=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j} a_l C_{k-l}^{j-l}\cdot\left(\frac{i}{n}\right)^{j-l}\cdot t^{k-j}$$

$$=\sum_{j=0}^{k} t^{k-j}\cdot\left\{\sum_{l=0}^{j} a_l C_{k-l}^{j-l}\cdot\left[\sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^{j-l}\right]\right\}$$

$$=\sum_{j=0}^{k} t^{k-j}\cdot\left\{\sum_{l=0}^{j} n\cdot a_l C_{k-l}^{j-l}\cdot\left[\sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^{j-l}\cdot\frac{1}{n}\right]\right\}$$

$$=t^k\ .$$

As $n\to+\infty$, taking the limit of the above equation, we can get:

$$\lim_{n\to+\infty}\sum_{j=0}^{k} t^{k-j}\cdot\left\{\sum_{l=0}^{j} n\cdot a_l C_{k-l}^{j-l}\cdot\left[\sum_{i=0}^{r-1}\left(\frac{i}{n}\right)^{j-l}\cdot\frac{1}{n}\right]\right\}$$

$$=\sum_{j=0}^{k} t^{k-j}\cdot\left\{\sum_{l=0}^{j}\lim_{n\to+\infty} n\cdot a_l C_{k-l}^{j-l}\cdot\left[\lim_{n\to+\infty}\sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^{j-l}\cdot\frac{1}{n}\right]\right\}$$

$$=\sum_{j=0}^{k} t^{k-j}\cdot\left(\sum_{l=0}^{j} b_l C_{k-l}^{j-l}\cdot\int_0^1\xi^{j-l}d\xi\right)$$

$$=\sum_{j=0}^{k} t^{k-j}\cdot\left(\sum_{l=0}^{j} b_l\cdot\frac{C_{k-l}^{j-l}}{j-l+1}\right)$$

$$=\lim_{r\to+\infty} t^k$$

$$= t^k .$$

In the process of calculating the above equation, we used the definition of Riemann integral.

So the corresponding coefficient is equal:

$$\sum_{l=0}^{j} \frac{C_{k-l}^{j-l}}{j-l+1} \cdot b_l = \begin{cases} 1 & (j=0) \\ 0 & (j \geq 1) \end{cases}.$$

So we can get $b_l = B_l \cdot C_k^l$, in this equation, B_l is Bernoulli number.

2) The situation where r is a real number.

Because the conclusions above are independent of t, so we have:

$$\lim_{n \to +\infty} n \cdot H(t) = \sum_{i=0}^{k} b_i \cdot t^{k-i} \quad , \quad \forall t \in R .$$

Define $H_{\frac{m}{n}}(t) = \left(V^{\frac{m}{n}} - 1 \right) x_t = \sum_{i=0}^{m-1} H_{\frac{1}{n}} \left(t + \frac{i}{n} \right)$.

At first, let r is a rational number, we write it in the form of $\frac{m}{n}$,and we will get:

$$\lim_{\frac{n}{m} \to +\infty} \frac{n}{m} \left(V^{\frac{m}{n}} - 1 \right) x_t$$

$$= \lim_{\frac{n}{m} \to +\infty} \frac{n}{m} \cdot \sum_{i=0}^{m-1} H \left(t + \frac{i}{n} \right)$$

$$= \lim_{\frac{n}{m} \to +\infty} \frac{\sum_{i=0}^{m-1} n \cdot H \left(t + \frac{i}{n} \right)}{m}$$

where $m, n \in N^+$. If $\frac{n}{m} \to +\infty$, then $n \to +\infty$.

So we have:

$$\lim_{\frac{n}{m} \to +\infty} \frac{\sum_{i=0}^{m-1} n \cdot H \left(t + \frac{i}{n} \right)}{m}$$

$$=\lim_{\frac{n}{m}\to+\infty}\frac{\sum_{i=0}^{k}\sum_{j=0}^{m-1}b_i\cdot\left(t+\frac{j}{n}\right)^{k-i}}{m}$$

$$=\lim_{\frac{n}{m}\to+\infty}\frac{\sum_{i=0}^{k}\sum_{j=0}^{m-1}b_i\cdot t^{k-i}}{m}$$

$$=\frac{m\cdot\sum_{i=0}^{k}b_i\cdot t^{k-j}}{m}$$

$$=\sum_{i=0}^{k}b_i\cdot t^{k-i}\ .$$

This is:

$$\lim_{q\to0}\frac{V^q-1}{q}x_t=\sum_{i=0}^{k}b_i\cdot t^{k-i}\ ,q\in Q^+\ .$$

Then by using rational approximations for real numbers, we get:

$$\lim_{r\to0}\frac{V^r-1}{r}x_t=\sum_{i=0}^{k}b_i\cdot t^{k-i}\ ,r\in R^+\ .$$

So we can obtain the below equations:

$$Dx_t=\lim_{r\to0}\frac{V^r-1}{r}x_t$$

$$=\sum_{i=0}^{k}b_i\cdot t^{k-i}$$

$$=\sum_{i=0}^{k}B_i\cdot C_k^i\cdot t^{k-i}\ .$$

The difference equation $Vx_t-x_t=t^k$ has been converted into the differential equation $Dx_t=\sum_{i=0}^{k}B_i\cdot C_k^i\cdot t^{k-i}$.

Example 5: $x_{t+1} = x_t + t^3, x_1 = 0$.

Using the method stated above, the equation can be converted into:

$$Dx_t = \sum_{i=0}^{k} B_i \cdot C_k^i \cdot t^{k-i} = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

The solution to it is：

$$x_t = \frac{1}{4}t^4 - \frac{1}{2}t^3 + \frac{1}{4}t^2 + c = \left[\frac{t(t-1)}{2}\right]^2 + c.$$

By the method of undetermined coefficient, we can get:

$$x_t = \left[\frac{t(t-1)}{2}\right]^2.$$

Section 3:Euler- Maclaurin formula.

In this section we will study the situation of $Vx_t - x_t = h(t)$. From the Weierstrass approximation theorem, we know $h(t)$ can be approximated by a series of polynomials. So the original equation can be changed to:

$$Vx_t - x_t = \sum_{i=0}^{+\infty} a_i t^i.$$

This difference equation can be changed to a differential equation:

$$\begin{aligned} Dx_t &= \sum_{i=0}^{+\infty} a_i \left(\sum_{j=0}^{i} B_j \cdot C_i^j \cdot t^{i-j} \right) \\ &= \sum_{j=0}^{+\infty} B_j \left(\sum_{i=j}^{+\infty} a_i \cdot C_i^j \cdot t^{i-j} \right) \\ &= \sum_{j=0}^{+\infty} \frac{B_j}{j!} \left(\sum_{i=j}^{+\infty} a_i \cdot \frac{i!}{(i-j)!} \cdot t^{i-j} \right). \end{aligned}$$

Because $\frac{i!}{(i-j)!} \cdot t^{i-j}$ is j-th order derivative of t^i, we can get:

$$\sum_{i=j}^{+\infty} a_i \cdot \frac{i!}{(i-j)!} \cdot t^{i-j} = h^{(j)}(t).$$

So we have:

$$Dx_t = \sum_{j=0}^{+\infty} \frac{B_j}{j!} \left(\sum_{i=j}^{+\infty} a_i \cdot \frac{i!}{(i-j)!} \cdot t^{i-j} \right)$$

$$= \sum_{j=0}^{+\infty} \frac{B_j}{j!} \cdot h^{(j)}(t).$$

This is one form of the Euler- Maclaurin formula.

This requires the series to be absolutely convergent. In general, we can express $h(t)$ in partial form with the remainder. It's easy to prove that in this situation, the final result will has a remainder.

In fact, the Euler- Maclaurin formula can be calculated directly. Suppose the solution to the equation $Vx_t - x_t = h(t)$ is in the form $x_t = H(t)$.

Using Taylor's theorem, we can get:

$$H(t+1) - H(t) = \sum_{i=1}^{+\infty} \frac{H^{(i)}(t)}{i!} = h(t),$$

and the following system of equations:

$$\begin{cases} \int h(t+1)d(t+1) - \int h(t)dt = \sum_{i=1}^{+\infty} \frac{h^{(i-1)}(t)}{i!} \\ h(t+1) - h(t) = \sum_{i=1}^{+\infty} \frac{h^{(i)}(t)}{i!} \\ \vdots \end{cases}.$$

So we can guess that the original equation was changed to $Dx_t = H'(t) = \sum_{i=0}^{+\infty} a_i \cdot h^{(i)}(t)$. Then we can use the method used in Section 2 to get the Euler- Maclaurin formula.

At this point, we have solved how to solve a non-homogeneous linear differential equation with constant coefficients, and obtained Euler- Maclaurin formula.

Example 6: $x_{t+1} = x_t + \frac{1}{t}$, $x_1 = 0$.

We can convert this equation into:

$$Dx_t = \sum_{j=0}^{+\infty} \frac{B_j}{j!} \cdot f^{(j)}(t) = \sum_{j=0}^{+\infty} (-1)^j B_j \cdot \frac{1}{t^{j+1}}.$$

The solution to it is:

$$x_t = \ln t - \sum_{j=1}^{+\infty} \frac{(-1)^j B_j}{j} \cdot \frac{1}{t^j} + c.$$

Using the undetermined coefficients method, we can get:

$$c = \gamma = \sum_{j=1}^{+\infty} \frac{(-1)^j B_j}{j} \cdot \frac{1}{t^j} \approx 0.577\cdots,$$

which is Euler constant.

So we have:

$$x_t = \ln t - \sum_{j=1}^{+\infty} \frac{(-1)^j B_j}{j} \cdot \frac{1}{t^j} + \gamma.$$

Obviously, if $t \to +\infty$, the difference between x_t and $\ln t$ will approach γ, which is the same as the known result.

Section 4: $X_{t+m} + \sum_{i=1}^{m} A_i \cdot X_{t+n-i} = F(t)$, the capital letters represent matrices.

In this section we study how to solve a system of linear difference equations with constant coefficients.

By using recurrence operator, we can rewrite the system of equations as:

$$\left(V^m \cdot I + \sum_{i=1}^{m} V^{m-i} \cdot A_i\right) X_t = F(t).$$

where I represents the identity matrix.

By using Cramer theorem, we can get:

$$\det\left(V^m \cdot I + \sum_{i=1}^{m} V^{m-i} \cdot A_i\right) x_{ti} = D_i = g_i(t).$$

where D_i is an determinant, which is a function of t.

Then we will get:

$$\det\left(V^m \cdot I + \sum_{i=1}^{m} V^{m-i} \cdot A_i\right) X_t = G(t).$$

where $G(t)$ can be calculated by Cramer theorem.

We can solve these m equations using the method introduced in the section before.

In fact, this result can also be provided more conventionally. This is the relationship between the system of difference equations and the system of differential equations.

Example 7: $\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

We can change this system of equations to:

$$\det\left(V^m \cdot I + \sum_{i=1}^{m} V^{m-i} \cdot A_i\right) \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$= \left[(V-2)^2 - (-1)^2\right] \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is:

$$\left(V^2 - 4V + 3\right) \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution to it is:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cdot 1^t + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \cdot 3^t.$$

Using the method of undetermined coefficients, we can get:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

So the solution is:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot 3^t .$$

Conclusion:

In this article, we solve some problems of difference equations by converting them to differential equations and explain the traditional way of solving linear difference equations with constant coefficients. We also derive the Euler-Maclaurin formula.

Reference documentation:

张筑生，《数学分析新讲》（第三册），北京大学出版社.

王高雄、周之铭等，《常微分方程》（第三版），高等教育出版社.

朝伦巴根、贾德彬，《数值计算方法》，中国水利水电出版社.

章纪民、萧树铁，《大学数学·多元微积分及其应用》,高等教育出版社.

张禾瑞、郝鈵新，《高等代数》（第五版），高等教育出版社.