Solar Clock

--- The method for Calculating Time

Xue Yuhao

Supervisor：Zhou Jianjun

The High School Attached to Tsinghua University

Dec. 2012

Contents

Abstract

This paper reports a study of calculating local time by the shadow length of an object.

In the sun, every object has a certain shadow length at a particular point of time. My study, titled *the solar clock*, is to calculate the local time by using four variables, namely the shadow length of an object on the horizontal surface, the height of the object, the local latitude and the date.

The function expression in my study is obtained through theoretical derivation, which is the relationship between the local time and the shadow length, the height of an object, the local latitude and the date. And then it is verified by actual measurement data. The errors between the theoretical calculation results and actual time value are carefully analyzed and reasonably interpreted.

Keywords: Shadow length, Local time, Modeling, Measurement data, Data analysis

1. Study Process

1) During the high enrollment of military training in August 2011, I got an idea about solar clock research when military training instructors didn't allow wearing the watch. So I want to estimate the time by the shadow length of an object on the level surface, but at that time not success. After returning home, a formula was successfully derived (needed data see 2.2):

$$t = t_0 \pm \frac{\arccos\left\{\dfrac{\sqrt{s^2+h^2}}{\cos\left[\alpha - \arcsin(\sin 23°26'21'' \cos\frac{2\pi x}{365})\right]\cdot\sqrt{s^2+h^2}}\right\}}{\omega}$$

At the beginning of the new semester (last year), I had some testing and found that the accuracy can be acceptable. But in the later (July 2012) by more data testing this formula has some problems.

2) From Nov. 22, 2011 I started to collect data (target object is 8.628cm height). A theory need to be verified by a large number of actual data, even this formula has some problems. So I decided to collect one year's data, in order to verify the correctness of each period of the year. But on different latitudes, it can be not verified as no actual data.

3) After March 2012, two items were modified. One is considering the effects of atmospheric refraction; the other is changing the measuring places to get more accurate data (see 3.1.3①). The new formula is:

$$t = t_0 \pm \frac{\arccos\left\{\dfrac{\sqrt{s^2+h^2-n^2s^2}}{\cos\left[\alpha - \arcsin(\sin 23°26'21'' \cos\frac{2\pi x}{365})\right]\cdot\sqrt{s^2+h^2}}\right\}}{\omega}$$

4) In July 2012, through more data analysis of more than six months, found the initial formula (see 4.4) defective, and re-derive the correct formula：

$$t = t_0 \pm \frac{\arccos\left\{1 - \dfrac{\sin(\alpha-2\beta)+\sin\alpha+2\cos^2\beta-2\cos\beta'\left[\cos\beta+\sin(\alpha-\beta)\right]}{2\cos^2\alpha\left[1+\sin(\alpha-2\beta)+\dfrac{\cos(\alpha-\beta)\sin\beta-\sin^2\beta}{1-\sin\alpha}\right]}\right\}}{\omega}$$

Here, $\beta = \alpha - \arcsin(\sin 23°26'21'' \cos\frac{2\pi x}{365})$ $\qquad \cos\beta' = \frac{\sqrt{h^2+s^2-n^2s^2}}{\sqrt{h^2+s^2}}$

Moreover, estimated the difference caused by not enough level surface (see 4.3 ①).

5) After Nov. 2012, considering the earth revolution elliptic orbit, revises the theoretical formula (see 2.1).

6) According to the opinions of the judges after Dec. 2012, measuring the flagpole and another two target objects on the level tablet made by myself (see 3.2 and 3.3), in order to eliminate the error of measurement. Measuring different height objects are more effective to verify the formula.

2. Theoretical Derivation

2.1 Theoretical formula

$$t = t_0 \pm \frac{\arccos\left\{1 - \frac{\sin(\alpha - 2\beta) + \sin\alpha + 2\cos^2\beta - 2\cos\beta'\left[\cos\beta + \sin(\alpha-\beta)\right]}{2\cos^2\alpha\left[1 + \sin(\alpha - 2\beta) + \frac{\cos(\alpha-\beta)\sin\beta - \sin^2\beta}{1-\sin\alpha}\right]}\right\}}{\omega}$$

Here, $\beta = \alpha - \arcsin(\sin 23°26'21'' \cos\gamma)$

γ is a solution of $\frac{1}{2}ab\arccos\left(\frac{a\cos\gamma - c}{a - c\cos\gamma}\right) + \frac{b^2 c\sin\gamma}{a - c\cos\gamma} = \frac{x}{365.25}ab\pi$, $\gamma \in [0, \pi]$

$$\cos\beta' = \frac{\sqrt{h^2 + s^2 - n^2 s^2}}{\sqrt{h^2 + s^2}}$$

2.2 Parameters

- t_0 : local midday time at that day (generally 12:00, not actually)
- ω : rotational angular velocity of the earth
- x : the number of days away from the summer solstice (x<365/2)
- α : the local latitude (Beijing: 39°58'N)
- h : object height
- s : shadow length
- n : air refractive index on the earth surface (generally, n≈1.00029)
- a : semi-major axis of the earth revolution orbit (149,600,000km)
- b : semi-minor axis of the earth revolution orbit (149,580,000km)
- c : semi-focal distance of the earth revolution orbit (2,500,000km)

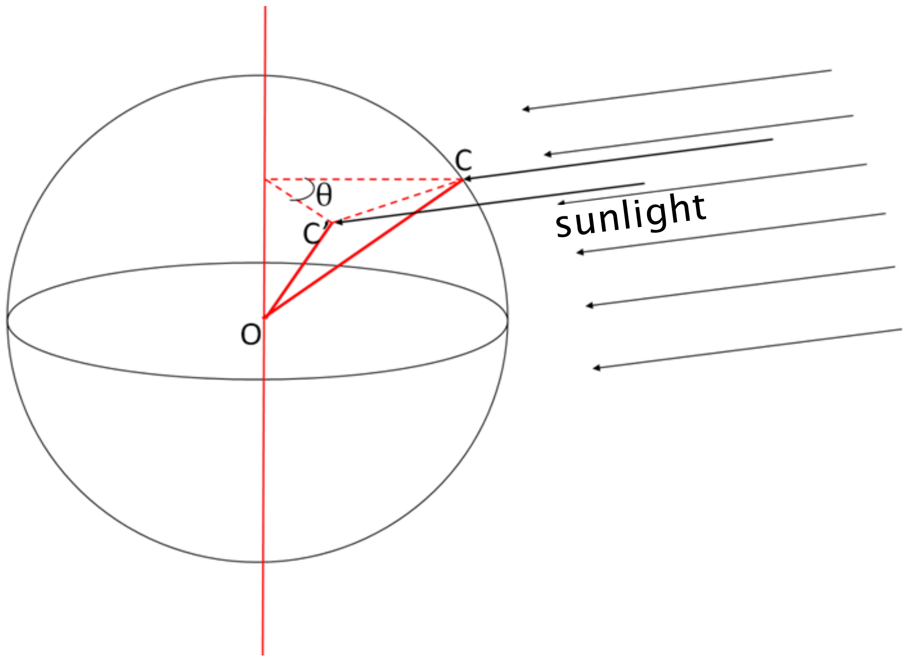
2.3 β and β'

β: complementary angle of Sun elevation angle at midday

β': complementary angle of Sun elevation angle at that time

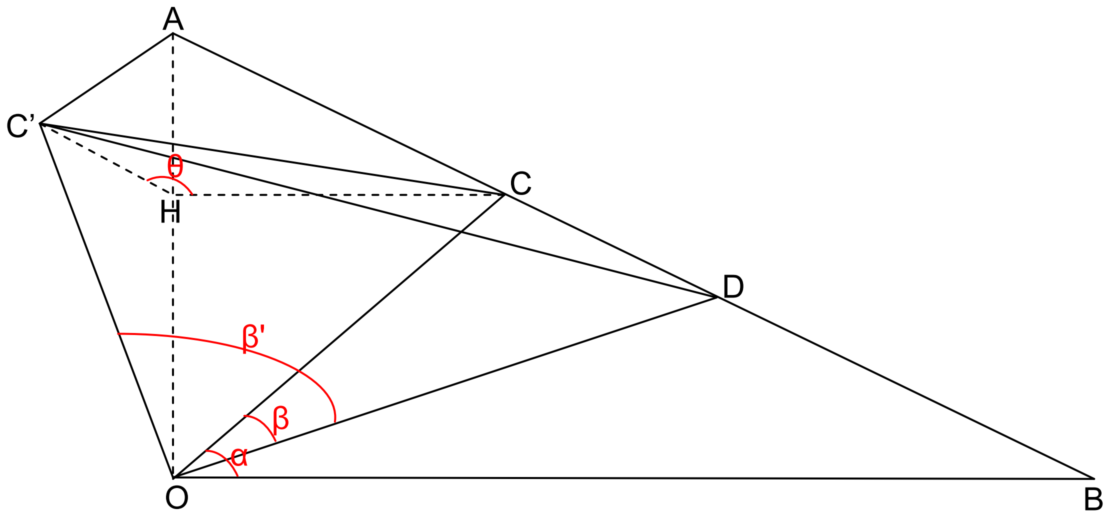
2.4 Deduction process

2.4.1 Model A



Sunlight comes from right. Place C is at local time midday (face to sun). After the earth rotation θ, the point C turns to point C'.

Simplified for the above model:



O: Earth center point

AO: Earth rotation axis

C: location C

C': location C'

OD: parallel to the sunlight

OB⊥AO, CH⊥AO, C'H⊥AO

α: latitude of C point

θ: angle of the Earth rotation

β: complementary angle of Sun elevation angle at midday

β': complementary angle of Sun elevation angle at that time

∠C'HC=θ, ∠C'OD=β', ∠COD=β, ∠COB=α

If OC'=OA=OC=R

$$\angle AOC = \angle AOC' = 90° - \alpha$$

$$\angle OAC = \angle OCA = 45° + \frac{1}{2}\alpha$$

$$\angle ODC = 45° - \beta + \frac{1}{2}\alpha$$

In △ AOC, △ AOC':

$$AC = AC' = 2R\ \cos(45° + \frac{1}{2}\alpha)$$

In △ OCD,

$$CD = \frac{\sin\beta}{\sin(45° - \beta + \frac{1}{2}\alpha)}R$$

$$OD = \frac{\sin(45° + \frac{1}{2}\alpha)}{\sin(45° - \beta + \frac{1}{2}\alpha)}R$$

In △ OHC, △ OHC':

$$CH = C'H = R\cos\alpha$$

In △CHC',

$$CC' = 2R\cos\alpha\sin\frac{1}{2}\theta$$

In △ACC',

$$\cos\angle ACC' = \frac{\cos\alpha\sin\frac{1}{2}\theta}{2\cos(45° + \frac{1}{2}\alpha)}$$

$$\therefore \cos\angle C'CD = -\cos\angle ACC' = -\frac{\cos\alpha\sin\frac{1}{2}\theta}{2\cos(45° + \frac{1}{2}\alpha)}$$

In $\triangle$ODC' and $\triangle$CC'D,

$$C'O^2 + OD^2 - 2\cos\angle C'OD \cdot OD \cdot C'O = C'D^2$$

$$C'C^2 + CD^2 - 2\cos\angle C'CD \cdot CD \cdot C'C = C'D^2$$

$$\therefore R^2 + \left[\frac{\sin\left(45° + \frac{1}{2}\alpha\right)}{\sin(45° - \beta + \frac{1}{2}\alpha)} R\right]^2 - 2\cos\beta' \frac{\sin\left(45° + \frac{1}{2}\alpha\right)}{\sin(45° - \beta + \frac{1}{2}\alpha)} R^2$$

$$= \left[\frac{\sin\beta}{\sin(45° - \beta + \frac{1}{2}\alpha)} R\right]^2 + \left(2R\cos\alpha\sin\frac{1}{2}\theta\right)^2$$

$$+ 2R^2 \frac{\cos\alpha\sin\frac{1}{2}\theta}{\cos(45° + \frac{1}{2}\alpha)} \frac{\sin\beta}{\sin(45° - \beta + \frac{1}{2}\alpha)} \cos\alpha\sin\frac{1}{2}\theta$$

Both sides divided by R^2, then multiply by $2\sin^2(45° - \beta + \frac{1}{2}\alpha)$, simplified:

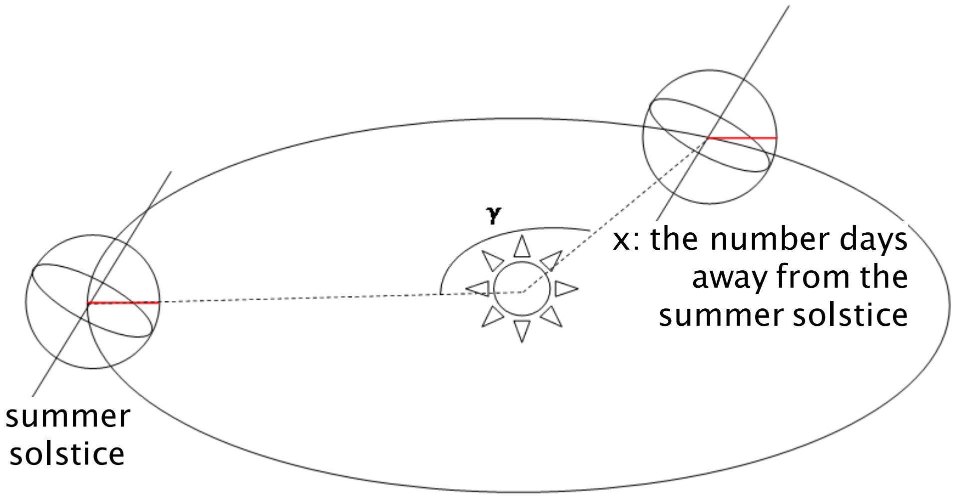
$$\cos\theta = 1 - \frac{\sin(\alpha - 2\beta) + \sin\alpha + 2\cos^2\beta - 2\cos\beta' \left[\cos\beta + \sin(\alpha - \beta)\right]}{2\cos^2\alpha\left[1 + \sin(\alpha - 2\beta) + \frac{\cos(\alpha - \beta)\sin\beta - \sin^2\beta}{1 - \sin\alpha}\right]}$$

∵ It takes time $\frac{\theta}{\omega}$ after the earth rotation θ

So,

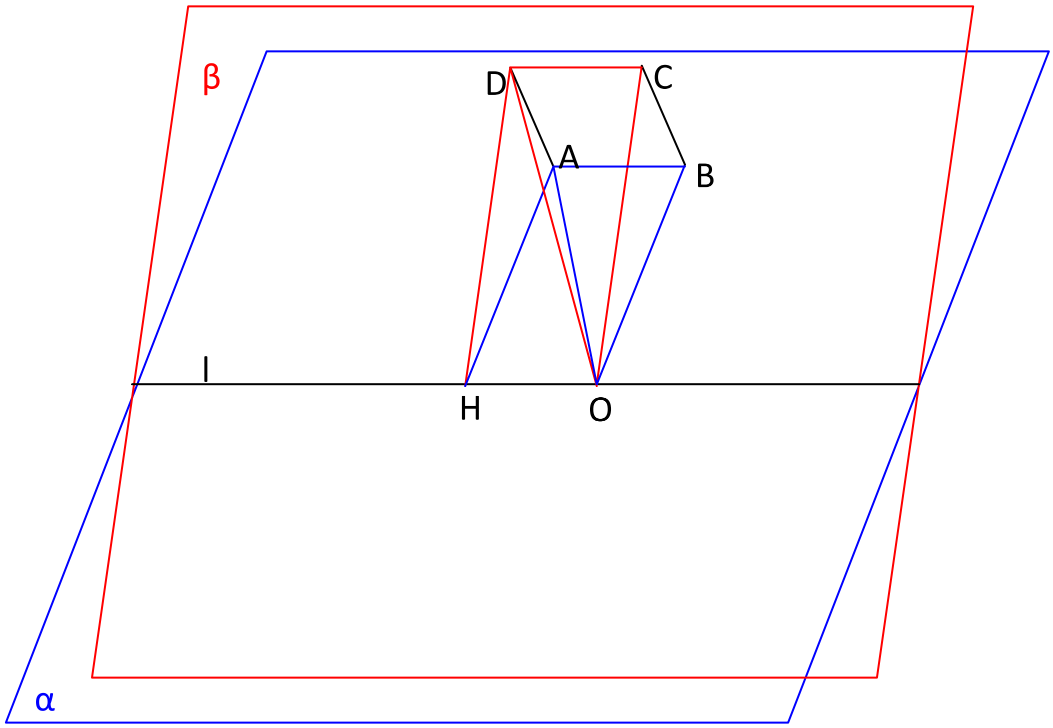
$$t = t_0 \pm \frac{\arccos\left\{1 - \frac{\sin(\alpha - 2\beta) + \sin\alpha + 2\cos^2\beta - 2\cos\beta' \left[\cos\beta + \sin(\alpha - \beta)\right]}{2\cos^2\alpha\left[1 + \sin(\alpha - 2\beta) + \frac{\cos(\alpha - \beta)\sin\beta - \sin^2\beta}{1 - \sin\alpha}\right]}\right\}}{\omega}$$

2.4.2 Model B



Model B is in order to calculate β in Model A (complementary angle of sun elevation angle at midday).

Simplified for Model B:



Plane α: equatorial plane

Plane β: ecliptic plane

l: intersection line of plane α and β

O: earth center point

γ: angle of two lines, earth and sun into lines while the summer solstice and that day

Red line: ecliptic plane and lines on it

Blue line: equatorial plane and lines on it

OC: sunlight at midday on summer solstice

OD: sunlight at midday on x days away from the summer solstice

CD∥l, CB⊥α, DA⊥α, DH⊥l, BO⊥l, AH⊥l, △OBC≌△HAD

$\angle\mathrm{COB} = \angle\mathrm{DHA} = 23°26'21''$ (Obliquity of the ecliptic)

$\angle\mathrm{DOA} = \alpha - \beta$ (in Model A)

$\angle\mathrm{DOC} = \angle\mathrm{HDO} = \gamma$

If $\mathrm{OC} = \mathrm{DH} = 1$

In △HAD,

$$\mathrm{AH} = \cos 23°26'21''$$

$$\mathrm{AD} = \sin 23°26'21''$$

In △ HOD,

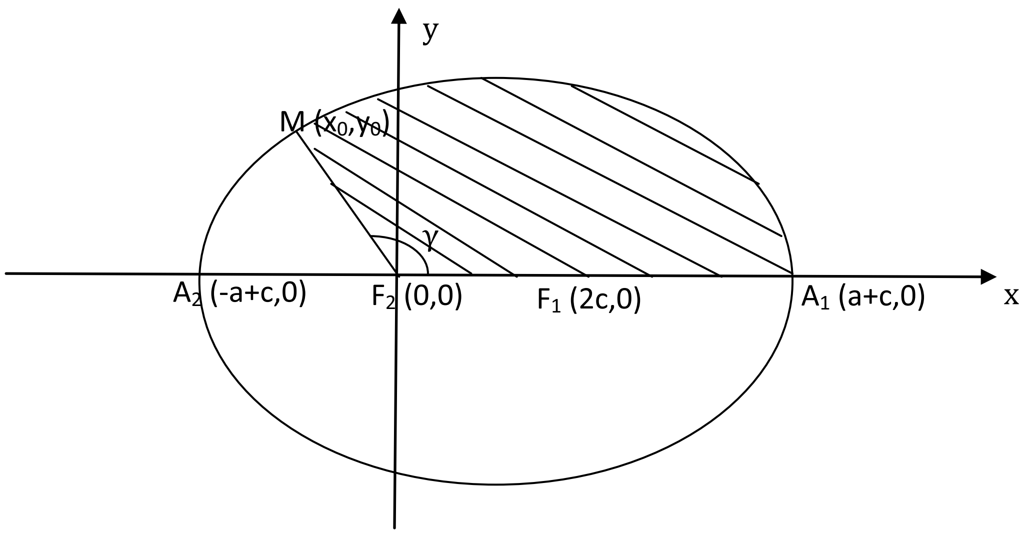
$$\mathrm{DO} = \frac{1}{\cos\gamma}$$

In △ OAD,

$$\sin(\alpha - \beta) = \frac{\mathrm{AD}}{\mathrm{DO}} = \sin 23°26'21'' \cdot \cos\gamma$$

$$\therefore \beta = \alpha - \arcsin(\sin 23°26'21'' \cos\gamma)$$

2.4.3 Model C



Model C is in order to calculate γ in Model B (the angle of two lines, earth and sun into lines while the summer solstice and that day).

F_2 (0, 0)： left focal point (sun location)

F_1 (2c, 0)： right focal point

A_1 (a+c, 0)： aphelion

A_2 (-a+c, 0)： perihelion

M (x_0, y_0)： earth location on x-day to summer solstice

$\angle MF_2A_1=\gamma$

Sγ ： focal radius passing area (shaded area)

S： total area of the ellipse

elliptic equation: $\frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}=1$

top half elliptic equation: $y=\frac{b}{a}\sqrt{a^2-(x-c)^2}$

KeplerNo, 2 law: $\frac{S_\gamma}{S}=\frac{x}{365.25}$

Sγ : $S_\gamma=\frac{1}{2}x_0y_0+\int_{x_0}^{a+c}\frac{b}{a}\sqrt{a^2-(x-c)^2}\,dx$

order: $x=a\cos\theta+c\quad \theta\in[0,\pi]$

$$\therefore S_\gamma=\frac{1}{2}x_0y_0+\int_{\arccos\left(\frac{x_0-c}{a}\right)}^{0}\frac{b}{a}\sqrt{a^2-(a\cos\theta)^2}\,d(a\cos\theta+c)$$

$$=\frac{1}{2}x_0y_0+ab\int_{\arccos\left(\frac{x_0-c}{a}\right)}^{0}-\sin\theta\sqrt{1-\cos^2\theta}\,d\theta$$

$$=\frac{1}{2}x_0y_0+ab\int_{0}^{\arccos\left(\frac{x_0-c}{a}\right)}\sin^2\theta\,d\theta$$

$$=\frac{1}{2}x_0y_0+ab\int_{0}^{\arccos\left(\frac{x_0-c}{a}\right)}\frac{1-\cos 2\theta}{2}d\theta$$

$$=\frac{1}{2}x_0\frac{b}{a}\sqrt{a^2-(x_0-c)^2}+\frac{1}{2}ab\arccos\left(\frac{x_0-c}{a}\right)-\frac{1}{4}ab\sin\arccos\left(\frac{x_0-c}{a}\right)$$

$$=\frac{1}{2}ab\arccos\left(\frac{x_0-c}{a}\right)-\frac{bc}{2a}\sqrt{a^2-(x_0-c)^2}$$

$\because\ \tan\gamma=\frac{y_0}{x_0}=\frac{b}{a}\frac{\sqrt{a^2-(x_0-c)^2}}{x_0}$ $\quad\therefore\ x_0=\frac{b^2}{\frac{a}{\cos\gamma}-c}$

$$\therefore S_\gamma = \frac{1}{2}ab\arccos\left(\frac{\frac{b^2}{\frac{a}{\cos\gamma}-c}-c}{a}\right) + \frac{bc}{2a}\sqrt{a^2-\left(\frac{b^2}{\frac{a}{\cos\gamma}-c}-c\right)^2}$$

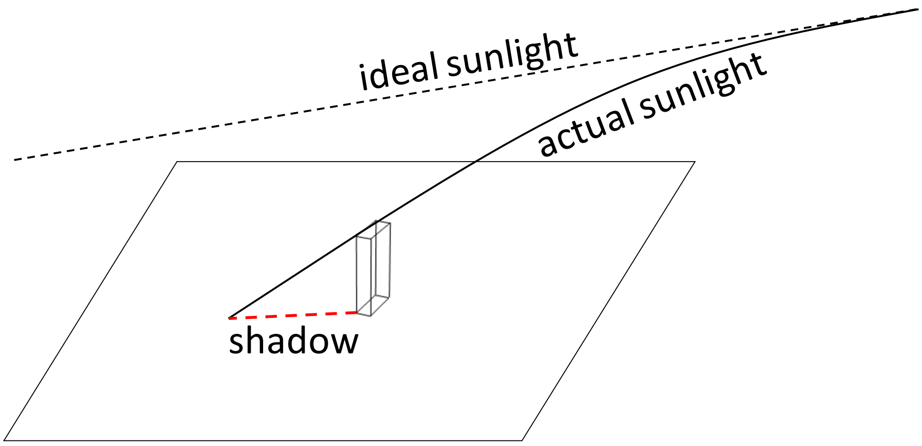
$$= \frac{1}{2}ab\arccos\left(\frac{a\cos\gamma - c}{a - c\cos\gamma}\right) + \frac{b^2 c\sin\gamma}{a - c\cos\gamma}$$

$$\because \frac{S_\gamma}{S} = \frac{x}{365.25}$$

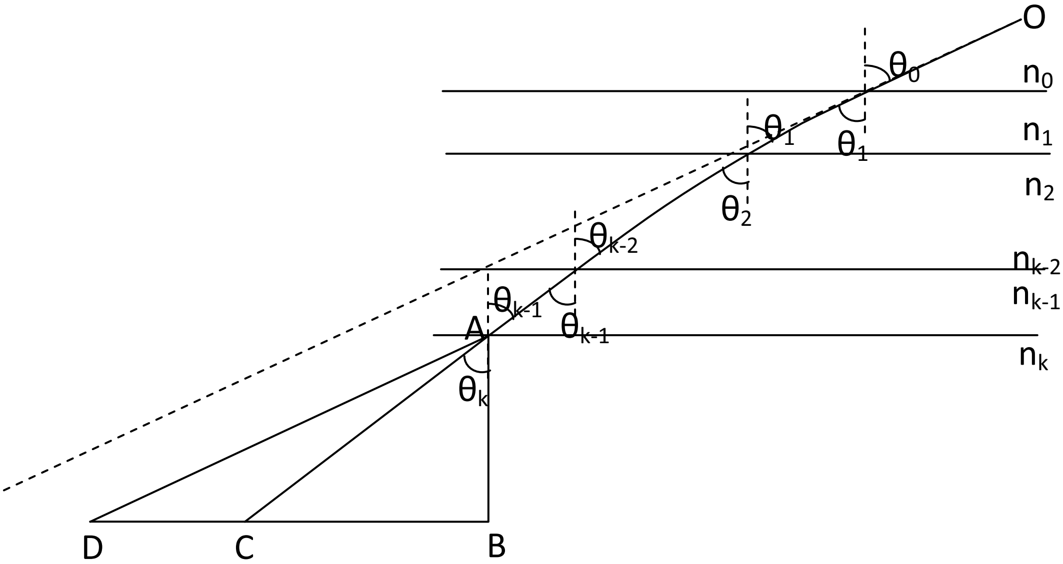
γ is a solution of $\frac{1}{2}ab\arccos\left(\frac{a\cos\gamma - c}{a - c\cos\gamma}\right) + \frac{b^2 c\sin\gamma}{a - c\cos\gamma} = \frac{x}{365.25}ab\pi$, $\gamma \in [0, \pi]$

specially, when the earth revolution circle orbit approximately, $\gamma = \frac{2\pi x}{365.25}$

2.4.4 Model D



Model D is in order to re-calculate β' in Model A, simplified for it:



AB： object

BC： shadow

AC： actual sunlight

AD, dotted line: ideal sunlight

OA： concave curve of actual sunlight

The area between O and A: atmosphere

n_0 , n_1 ,……n_k ： air refractive index

θ_0 , θ_1 ,……θ_k ： angle of incidence/refraction

AB⊥BD

$\theta_0 = \beta'$

$n_0 = 1$ ， $n_k = n \approx 1.00029$

refraction law: $n_k \sin\theta_k = n_{k-1}\sin\theta_{k-1} = n_{k-2}\sin\theta_{k-2} = \cdots\cdots = n_0 \sin\theta_0$

$$\therefore \sin\beta' = n \cdot \sin\theta_k = n\frac{BC}{AC} = n\frac{s}{\sqrt{h^2+s^2}}$$

$$\therefore \cos\beta' = \frac{\sqrt{h^2+s^2-n^2s^2}}{\sqrt{h^2+s^2}}$$

From the above four models, the theoretic formula is:

$$t = t_0 \pm \frac{\arccos\left\{1 - \dfrac{\sin(\alpha-2\beta)+\sin\alpha+2\cos^2\beta-2\cos\beta'\,[\cos\beta+\sin(\alpha-\beta)]}{2\cos^2\alpha\left[1+\sin(\alpha-2\beta)+\dfrac{\cos(\alpha-\beta)\sin\beta-\sin^2\beta}{1-\sin\alpha}\right]}\right\}}{\omega}$$

Here, $\beta = \alpha - \arcsin(\sin 23°26'21'' \cos\gamma)$

γ is a solution of $\frac{1}{2}ab\arccos\left(\frac{a\cos\gamma - c}{a - c\cos\gamma}\right) + \frac{b^2 c\sin\gamma}{a - c\cos\gamma} = \frac{x}{365.25}ab\pi, \gamma \in [0，\pi]$

$$\cos\beta' = \frac{\sqrt{h^2+s^2-n^2s^2}}{\sqrt{h^2+s^2}}$$

2.5 Notes

① ω: rad/h

② "-" morning; "+" afternoon

③ Obliquity of the ecliptic: 23°26′21″

④ Air refractive index: n≈1.00029

⑤ Local time to be calculated (Beijing time earlier 16min than local time in Beijing)

⑥ To be apply to the north temperate zone (23°26′21″N~66°33′39″N)

2.6 Ideal Conditions

① Regarding the universe as vacuum space, ignoring the effecting of atmospheric scattering

② Ignoring the effecting of the earth revolution in 12 hours

③ Regarding sunlight as parallel light

④ Regarding the earth as sphere and regarding the ground surface as plane

⑤ Taking aphelion for the summer solstice and perihelion for the winter (Actual: aphelion on July 5th, and perihelion on Jan. 5th)

⑥ Each atmospheric layer equally

⑦ 365.25 days one year, 23.93 hours one day

⑧ Ignoring the interaction between other planets and the earth

3. Actual Measurement

3.1 Measuring 8.628cm height object (from Nov. 2011 till now)

3.1.1 Measuring tools

15cm ruler, 30cm ruler, 15cm vernier calipers (scale division 0.02mm), 2m steel tape, 8.628cm height box

3.1.2 Some measurement problems

① Ground surface: not enough level

② Shadow tail end: fuzzy, especially in the early morning and afternoon

③ Midday time in Beijing: in theory Beijing time 12:16, but actually Beijing time 12:00~12:30

④ Air refractive index, n≈1.00029, different with air cleanliness index

⑤ Measuring error

3.1.3 The way to deal with the measurement problems

① The way to "Ground surface not enough level"

a) Watering on the ground: according to water flow direction and velocity, to judge whether the surface is enough level.

b) Changed the measurement place after March 2012, as the old place is not enough level.

Selected places for measuring (Beijing, northern latitude 39°58’)**:**

a) On the balcony in my room (measured at home)

b) The center of the fountain in our school (measured 3 times along two sides tangential direction in the morning, at noon, and in the afternoon)

c) In front of the teaching building (measured in the morning)

d) At the fire fighting access of teaching building unit B (measured after 3 p.m.)

e) At the corridor of the teaching building (measured in the afternoon)

f) On the table tennis table in front of the teaching building (measured in the morning and at noon)

② **Air refractive index, n≈1.00029**

Actually different with air cleanliness index each day

③ **The way to "fuzzy shadow tail end, especially in the morning and afternoon"**

a) Maybe result from atmospheric scattering, unparallel sunlight or other lights, etc.

b) Take the solid part of the shadow tail end for measuring.

④ **The way to "midday time in Beijing is Beijing time 12:16 in theory, but actually is Beijing time between 12:00 to 12:30"**

a) The duration of one day is the interval time when the same meridian face to the sun twice, rather than the earth rotates a circle time.

b) As the earth's revolution, one day actually should be slight longer than the time required for the earth to make one rotation.

c) The length of time each day is different. This is because the earth's revolution velocity is different, for its elliptical revolution orbit.

d) It is defined 24 hours per day, so the above time error accumulation will result that it is not a constant midday time every day.

e) The midday time is fluctuated twice every year in equilibrium position (i.e. local midday time, not Beijing time). Fluctuation range is different, and the biggest can reach more than ten minutes.

See "*NEWTON Science World*", 2011.6, Time is more accurate

⑤ **The way to "Measuring error"**

To take average value by multi-measuring

3.1.4 Discontinuous measurement

When it is raining or has no good place for measuring shadow out of town, the data will be discontinuous.

3.1.5 Estimate measuring accuracy

Advantage: convenient to collect and statistic data.

Disadvantage: no enough level surface, reading error of shadow tail end.

3.2 Measuring flagpole (Dec. 2012 till now)

3.2.1 Measuring tools

20m leather band tape, 2m steel tape

3.2.2 Some measurement problems

① No exact flagpole height

② Shadow length too long to confirm the solid tail end of shadow

③ Ground surface not enough level

④ Measuring reading error

3.2.3 The way to deal with the measurement problems

① No exact flagpole height

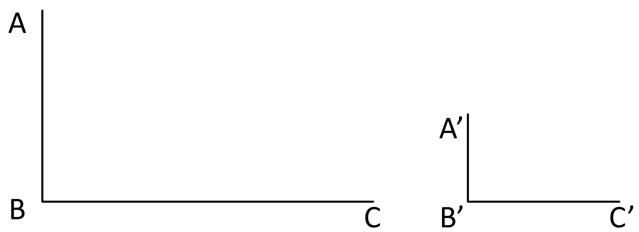
a) Method A: On third floor of building near the flagpole, the approximate altitude with flagpole height, putting the other end of leather tape down onto the ground to measure the height of the position.

Measured the flagpole height is 10.8m.

b) Method B: With the help of flag-raising device, taking the tape up to the top flagpole, to measure the height of that point, and then estimate the distance to the highest end by shadow length. Adding the above two height to get the height of flagpole.

Measured the flagpole height is 11.2m.

c) Method C: Standing a shorter object easy to measure near the flagpole



AB: flagpole, BC: shadow A’B’: object easy to measure, B’C’: shadow

$$\therefore AB = \frac{BC}{B'C'} A'B'$$

measured: BC=27.8 B’C’=313cm A’B’=122.5cm

Calculated the flagpole height is 10.9m.

Based on the above three results, take 11.0m for the height of flagpole.

② **The way to "fuzzy shadow tail end"**

In winter at noon, about 20cm shadow tail end id fuzzy and selected the solid tail end to measure. When the sun is not too strong, put my eyes, the top end flagpole and the sun into line, according myself shadow position to confirm the flagpole shadow tail end.

③ **Ground not enough level**

This is no solution at present, because the flagpole stands on the fixed position, which is unable to move to level place, no solution.

④ **Measuring reading error**

This is no solution at present.

3.2.4 Estimate measuring accuracy

Advantage: larger object, measuring accuracy requirement lower than small object (10cm accuracy correspond to 1mm that of last object).

Disadvantage: complex operation, rough ground surface, measuring reading error within 10cm.

Anyway, measuring flagpole is not better than the former small object.

3.3 Measuring by making water level (Dec. 4, 2012 till now)

3.3.1 Measuring tools and operating method

Tools: 6.080cm and 10.564cm height objects, 15cm vernier callipers (scale division 0.02mm), 30cm rulers, a piece of typing paper (A4), plastic tablet (size as A4 paper), sink made by a drawer and plastic, camera

The height of object was measured by using the vernier callipers. The plastic tablet, posted a piece of typing paper (shadow imaging on it clearly), was sealed by plastic wrap against water wetting within a short time and 3 rulers was paralleling put on it. Put it into the special sink made by a drawer posted plastic sheeting, and free floating. Put the target object onto the tablet zero point position, and put a same object onto the symmetrical position in order to be balance and level. Adjust the tablet let the shadow parallel to the direction of the scale, which is convenient to reading. Take pictures and enlarge them to analyze the shadow tail end and reading accuracy. Time reading accuracy is within 10 second.

3.3.2 The problems and solutions

The typing paper is easy to wet, need to replace new paper with a plastic seal keeping level surface.

3.3.3 Estimate measuring accuracy

Advantage: accuracy, perfect level surface, shadow length reading with a precision 0.5mm

Disadvantage: complex operation, limited size tablet

Anyway, this measuring method is more accuracy than the formers.

3.4 Actual obtained data and calculated results

Most difference of actual and theoretical time is within 10 minutes.

(See *"Solar clock measuring data and results table.xlsx"* in Annex)

4. Data Analysis

Totally about 800 pieces of data were obtained by measuring (Nov. 2011-Dec. 2012), covering the different seasons of one year and different time of one day.

4.1 Parameters effecting

① Air refractive index

If change the air refraction index from 1.00029 to 1.001 or 1, the difference will change less than 10%, which indicates that it is a little effective factor.

② The number days away from the summer solstice

Increase or decrease 1 day, the difference changes range from 10% to 100%, from less than 1min. to 10min., which indicates that it is an effective factor.

③ Local latitude

Increase or decrease 0.5°, more than half the differences change >100%, more than 8 min., which indicates that it is a big effective factor.

④ Regard earth revolution elliptic orbit as circular orbit

Around winter/summer solstice, the difference change little, most are within 5 min.. Around spring/autumn equinox, the difference change a lot, most are up to 5 min., which indicates that it is an effective factor.

4.2 Analyze the difference of actual and theoretical time

(Measuring 8.628cm height object)

① After March 2012, by changing the measurement place as level surface, the difference becomes smaller.

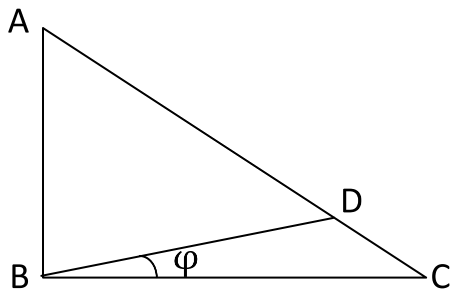
Difference	Obtained data (Nov. 2011 to Dec. 6, 2012)		
	Before March 2012	After March 2012	All
Within 5 minutes	68.8%	89.7%	83.2%
Within 10 minutes	88.8%	96.6%	94.1%

② Some data around the midday result in bigger difference, and even no results (*error "arccos"*). This maybe because the limited shadow length range at noon, 1mm deviation may cause the difference of more than 5 minutes, while in the morning and afternoon, the shadow can lengthen or shorten 1cm or more per minutes. If changing the measure data 1-2mm, the difference at noon will be within 10 minutes, even no *error "arccos"*.

4.3 The difference analysis

(Here only analyze the difference of measuring 8.628cm height object. As for the other two measuring methods, there are no enough measuring data for analyzing.)

① **Not enough level ground surface**



AB: object, BC: ideal shadow, BD: actual shadow

AB⊥BC, AB=h, BC=s, BD=s', ∠DBC=φ

When φ>0,

$$\sin\angle BDC = \sin\angle BDA = \sin\angle BCA\cos\varphi + \sin\varphi\cos\angle BCA$$

△BDC, Sine theorem:

$$\frac{s}{\sin\angle BDC} = \frac{s'}{\sin\angle BCA}$$

$$\therefore \frac{s}{\sin\angle BCA\cos\varphi + \sin\varphi\cos\angle BCA} = \frac{s'}{\sin\angle BCA}$$

$$\therefore s' = \frac{s}{\cos\varphi + \sin\varphi\cot\angle BCA} = \frac{hs}{h\cdot\cos\varphi + s\cdot\sin\varphi}$$

$$\therefore s = \frac{s'h\cos\varphi}{h - s'\sin\varphi}$$

When φ<0, the formula is same.

By watering on the ground, meet the needs of -1°< φ <1°.

Put “s” into theoretical formula and get the difference. Most is in 5 minutes, but in the morning and afternoon it is bigger, sometimes even 10 minutes.

Generally, -1°< φ <1°, so the difference is smaller.

② **Fuzzy shadow tail end**

Generally in summer the fuzzy part of shadow tail end at noon is less than 0.5mm, which can cause 3-6 minutes difference, maximum up to 8 minutes. And in the morning and afternoon it may be 1cm, which causes only 1minute difference.

In winter the fuzzy part of shadow tail end is relatively longer, but which cause only smaller difference.

③ **Different air refractive index each day**

Unable to calculate “n” exactly. (n≈1.00029).

By changing the value of “n”, the difference doesn’t change much.

④ **Reading error**

The shadow length reading error can be controlled within 0.5mm, and that is negligible in the morning and afternoon.

The timing reading error can be controlled within 20ms, which is negligible.

⑤ **Ideal conditions result in some difference** (see 2.6)

From 4.1, three factors, the number days away from that summer solstice, local latitude and the earth revolution orbit model, are sensitive to the difference.

So, the three ideal conditions, ignoring the effecting of the earth revolution in 12 hours; regarding the earth as sphere and regarding the ground surface as plane; and taking aphelion for the summer solstice and perihelion for the winter solstice, are more effective to the difference.

From the above analysis, I think the difference from measuring at noon come from reading error for shadow length, and that of in the morning and afternoon mainly caused by the ground level degree.

The theoretical difference may be result from the above three ideal conditions (⑤).

4.4 The initial error formula analysis

Initial formula is:

$$t = t_0 \pm \frac{\arccos\left\{\frac{\sqrt{s^2+h^2-n^2s^2}}{\cos\left[\alpha - \arcsin(\sin 23°26'21'' \cos\frac{2\pi x}{365.25})\right]\cdot\sqrt{s^2+h^2}}\right\}}{\omega}$$

t_0 ： local midday time at that day (generally 12:00, not actually)

ω ： rotational angular velocity of the earth

x ： the number of days away from the summer solstice (x<365/2)

α ： the local latitude (Beijing: 39°58’N)

h ： object height

s ： shadow length

n ： air refractive index on the earth surface (generally, n≈1.00029)

(the same notes and ideal conditions with right formula)

This formula was deducted in August last year and modified in March this year. But in July by analyzing data of half the year, it was found some mistake.

From the data table, the differences are seasonal variation. That is when around the spring equinox, the difference is smaller at a.m. & p.m. and bigger at noon, and when around the summer & winter solstice, it is otherwise.

In winter, difference<0;

In summer, difference>0;

In spring equinox, difference: ±25min;

In summer & winter solstice, difference: ± 50min

So, found that the formula may be inaccurate, or without considering seasonal factors. And then re-deducted the right formula, which shows the difference is not related with seasons.

5. Application

5.1 To calculate time

5.2 To estimate the space between buildings

The space between buildings: the distance from one building to another.

According to "Civil Design Principles", in No. 3.1.3 rules the sunlight standard shows that one bedroom or half of rooms should have at least one hour sunshine time of full window on the winter solstice, for the residential in each floor.

Considering the two buildings located on the same level place:

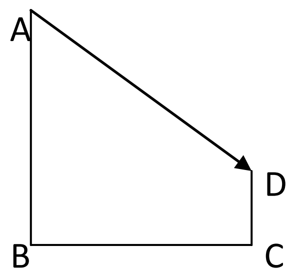
AB: the front building

D: the windowsill of the first floor behind building

AD: sunlight

BC: level ground

$\mathrm{AB} = \mathrm{H} \quad \mathrm{CD} = \mathrm{h} \quad \mathrm{BC} = \mathrm{s}$ (shadow length)



$\mathrm{T} \in [-0.5\mathrm{h}, 0.5\mathrm{h}]$ (half an hour to midday, sunshine time one hour)

The theoretical formula for calculating time：

$$t = t_0 \pm \frac{\arccos\left\{1 - \frac{\sin(\alpha - 2\beta) + \sin\alpha + 2\cos^2\beta - 2\cos\beta'\,[\cos\beta + \sin(\alpha - \beta)]}{2\cos^2\alpha\left[1 + \sin(\alpha - 2\beta) + \frac{\cos(\alpha - \beta)\sin\beta - \sin^2\beta}{1 - \sin\alpha}\right]}\right\}}{\omega}$$

Here, $\beta = \alpha - \arcsin(\sin 23°26'21'' \cos\gamma)$

γ is a solution of $\frac{1}{2}ab\,arccos\left(\frac{a\cos\gamma - c}{a - c\cos\gamma}\right) + \frac{b^2 c\sin\gamma}{a - c\cos\gamma} = \frac{x}{365.25}ab\pi, \gamma \in [0，\pi]$

$$\cos\beta' = \frac{\sqrt{h^2 + s^2 - n^2 s^2}}{\sqrt{h^2 + s^2}}$$

While to the winter solstice, $\gamma = \pi$

$\therefore \beta = \alpha + 23°26'21''$

$$\because \frac{\arccos\left\{1 - \dfrac{\sin(\alpha - 2\beta) + \sin\alpha + 2\cos^2\beta - 2\cos\beta'\,[\cos\beta + \sin(\alpha - \beta)]}{2\cos^2\alpha\left[1 + \sin(\alpha - 2\beta) + \dfrac{\cos(\alpha - \beta)\sin\beta - \sin^2\beta}{1 - \sin\alpha}\right]}\right\}}{\omega} = T$$

$\therefore s = (H - h)\tan\beta'$

Here,

$$\cos\beta' = \frac{\sin(\alpha - 2\beta) + \sin\alpha + 2\cos^2\beta - 2\cos^2\alpha\,(1 - \cos\omega T)\left[1 + \sin(\alpha - 2\beta) + \dfrac{\cos(\alpha - \beta)\sin\beta - \sin^2\beta}{1 - \sin\alpha}\right]}{2[\cos\beta + \sin(\alpha - \beta)]}$$

$\beta = \alpha + 23°26'21''$ $\quad T \in [-0.5h, 0.5h]$

To calculating the shadow length "s" by using the above formula, but in fact it is not the actual space between buildings. The angle between the building and shadow is defined as θ. Minimum space between buildings is the maximum of $s * \sin\theta$ ($T \in [-0.5h, 0.5h]$). θ is related with the building's facing direction, T and local latitude.

References

1. NEWTON Science World, 2011.6
2. Day shuttle perpetual calendar V2007 online

http://www.time.ac.cn/calendar/calendar.htm

3. Wikipedia term "rotation axis inclination"

http://zh.wikipedia.org/wiki/%E8%BD%AC%E8%BD%B4%E5%80%BE%E8%A7%92

4. Wikipedia term "earth revolution"

http://zh.wikipedia.org/wiki/%E5%9C%B0%E7%90%83%E5%85%AC%E8%BD%AC

5. Baidu baike "space between buildings"

http://baike.baidu.com/view/4066944.htm

Annex

Solar clock measuring data and results table.xlsx