A study on the properties of CG and XD figures

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Abstract: CG figures and XD figures are newly-developed mathematical conceptions. A CG figure consists of two polylines sharing the same end points with corresponding line segments perpendicular to each other. Let one polyline be the question, the other the solution of the question. If one polyline has only one solution, this figure is also considered an XD figure. There are already some conclusions about 2-dimensional XD figures. In this paper, these conclusions are extended using methods including vector analysis, contradiction and construction. In addition, the properties of 2-dimensional XD figures are compared to those of 3-dimensional XD figures. The properties proved are as follows:

(1) For a CG figure(C,C′) in a grid of m×n, if the solution set of C is A, the solution set of C′ is B, then the solution set of any element in A is B and vice versa.
(2) In a grid of n×n (with (n+1) × (n+1) grid points), there must exist an XD figure consisting of t line segments, where $n \geq 2, t = 4,6,8 \ldots 2n^2 + 4n$
(3) In a grid of m×n×p, if (C,C′)is an XD figure, (C′,C_1)is a CG figure, then (C_1,C′)is not necessarily an XD figure.
(4) For a CG figure (C,C′) in a grid of m×n×p, if the solution set of C is A, the solution set of C′ is B, then the solution set of any element in A is not necessarily B and vice versa.
(5) No 3-dimensional XD figure exist in infinite grids.

Key words: perpendicular, continuous, close, unique, uniform grid

I The introduction and definition about CG figures and XD figures

CG figures and XD figures are newly-developed mathematical conceptions. A CG figure consists of two polylines sharing the same end points with corresponding line segments perpendicular to each other. We call one polyline question, the other solution. If one polyline has only one solution, this figure is considered a special case more narrowly defined as an XD figure. For example, figure 1 is an XD figure.

The definitions of CG and XD figures are from problems involving perpendicular lines in uniform grids. Like Sudoku, XD figures can be used as mathematic games for mathematic entertainment. Apart from that, a 2-demensional XD figure looks like a beautiful picture while a 3-demensional one a unique architectural design, bringing not only intellectual challenge but also artistic enjoyment. Also, like many other mathematical conceptions, XD figures are likely to be useful in other fields.

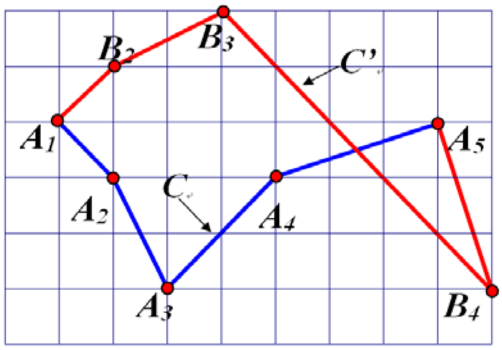


Figure 1

The definitions for CG and XD figures are as follows:

2-dimensional CG figure*: For a polyline C: A_1-A_2-…-A_n in a uniform grid of m×n (m, n∈ N^+), if there exists one polyline C′: A_1-B_2-B_3-…-B_{n-1}-A_n satisfying the conditions $A_1B_2 \perp A_1A_2$, $B_2B_3 \perp A_2A_3$, …, $B_{n-1}A_n \perp A_{n-1}A_n$, then polyline C is the question, polyline C′ is the solution, the closed figure consisting of the two polylines is a 2-dimensional CG figure(C,C′). (here, $A_1,A_2,A_3…A_n,B_2,B_3…B_{n-1}$are all lattice points andA_{i-1}, A_i, A_{i+1}(i=2,3…n-1) are not collinear)

2-dimensional XD figure**: For a polyline C: A_1-A_2-…-$A_n$in a uniform grid of m×n (m, n∈ N^+), if there exists only one polyline C′: A_1-B_2-B_3-…-B_{n-1}-A_n satisfying the conditions $A_1B_2 \perp A_1A_2$,$B_2B_3 \perp A_2A_3$,…,$B_{n-1}A_n \perp A_{n-1}A_n$, then polyline C is the question, polyline C′ is the solution, the closed figure consisting of the two polylines is a 2-dimensional XD figure (C,C′). (here, $A_1,A_2,A_3…A_n,B_2,B_3…B_{n-1}$ are all lattice points, A_{i-1}, A_i, A_{i+1}(i=2,3,…n-1) are not collinear)

3-dimensional CG figure***: For a polyline C: A_1-A_2-…-$A_n$in a uniform grid of m×n×p (m,n, p ∈ N^+), if there exists one polyline C′: A_1-B_2-B_3-…-B_{n-1}-A_n satisfying the conditions $A_1B_2 \perp A_1A_2$, $B_2B_3 \perp A_2A_3$, …, $B_{n-1}A_n \perp A_{n-1}A_n$, then polyline C is the question, polyline C′ is the solution, the closed figure consisting of the two polylines is a 3-dimensional CG figure (C,C′). (here, $A_1,A_2,A_3…A_n,B_2,B_3…B_{n-1}$ are lattice points on different planes, A_{i-1}, A_i, A_{i+1}(i=1,2,…n-1) are not collinear)

3-dimensional XD figure***: For a polyline C: A_1-A_2-…-$A_n$in a uniform grid of m×n×p (m,n, p ∈ N^+), if there exists only one polyline C′: A_1-B_2-B_3-…-B_{n-1}-A_n satisfying the conditions $A_1B_2 \perp A_1A_2$, $B_2B_3 \perp A_2A_3$, …, $B_{n-1}A_n \perp A_{n-1}A_n$, then polyline C is the question, polyline C′ is the solution, the closed figure consisting of the two polylines is a 3-dimensional XD figure(C,C′).(here, $A_1,A_2,A_3…A_n,B_2,B_3…B_{n-1}$ are lattice points on different planes, A_{i-1}, A_i, A_{i+1}(i=1,2,…n-1) are not collinear)

*The definition of CG figures is first introduced in the book 《点可点非常点——形独》

**The definition of XD figures is first publicly introduced in the science camp "Into the wonderful land of math" held by the Ministry of Education in 2011

***The definitions of 3-dimensional CG and XD figures are first introduced in the essay《点可点非常点之 CG 图在三维空间上的扩展》

corresponding line segments： If there are polyline C: A_1-A_2-A_3-…-A_n and polyline C′:A_1-B_2-B_3-…-B_{n-1}-A_n, then A_1B_2, A_1A_2; B_2B_3, A_2A_3; …; $B_{n-1}A_n$, $A_{n-1}A_n$ are corresponding line segments respectively.

(The m,n,p here indicates the numbers of the grids in three directions; limited grids can be extended to infinite grids. In this paper, all the CG and XD Figures are in uniform grids. Thus in the following text, "grids" implies "uniform grids"; a grid of m×n×p means there are a total of (m+1) × (n+1) ×(p+1)grid points)

II The known conclusions about 2-dimensional CG and XD figures

Below are conclusions from the recent studies about CG and XD figures:

Conclusion 1: In a grid of m×n, if (C,C′)is an XD figure and(C′,C_1) a CG figure, then (C_1,C′) is an XD figure.

Proof:

Since (C,C′)is an XD figure, (C′,C_1)is a CG figure,

the corresponding line segments of (C, C′) and (C′,C_1)are perpendicular to each other,

so the corresponding line segments of (C_1, C) are respectively collinear or parallel.

Suppose (C_1, C′) is not an XD figure, then there exists a polyline C_2 which makes (C_1, C_2) a CG figure.

Then the corresponding line segments of (C_1, C_2) are perpendicular to each other,

so the corresponding line segments of (C, C_2) are perpendicular to each other.

Thus, problem C has at least two solutions, C_2, C′, which shows (C,C′) is not an XD figure. This contradicts the condition that (C,C′)is an XD figure.

Therefore, the assumption '(C_1,C′)is not an XD figure' is not correct.

Thus, (C_1,C′)is an XD figure.

This completes the proof.

Conclusion 2: In the infinite grid, no 2-dimensional XD figure consisting of more than 4 line segments exist.

Proof*：

If there is a 2-dimensional CG figure(C,C′)

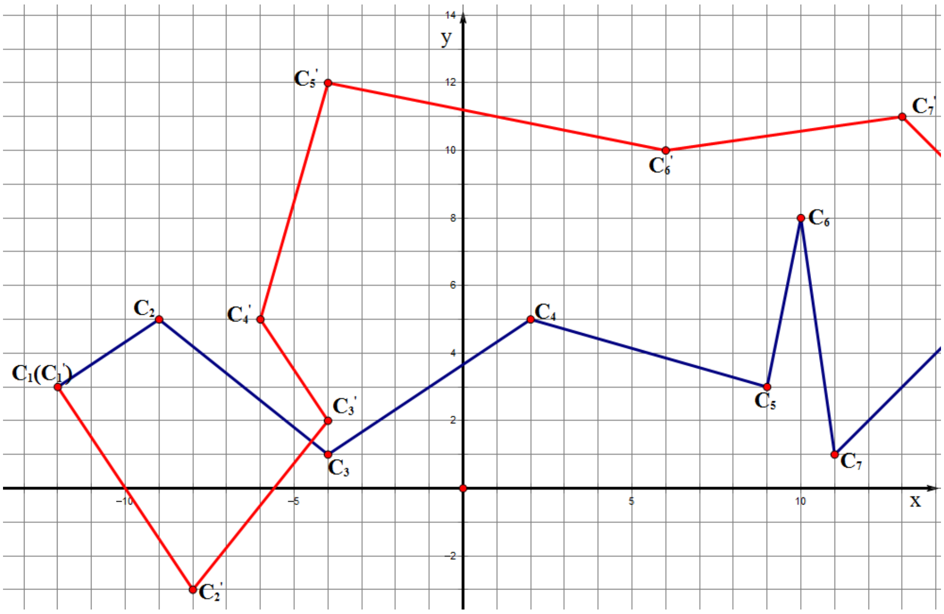
C:C_1-C_2-……-C_n; C_1'-C_2'-……-C_n'(C_1'=C_1,C_n'=C_n),

consider polyline $C_1C_2C_3C_4$and $C_1'C_2'C_3'C_4'$.

When C_1,C_2,C_3,C_4,C_1',$C_4'$are fixed points, if the coordinates of $C_2'$and C_3'have only one value, then (C,C′) may be an XD figure; if not,(C,C′)is not an XD figure.

As demonstrated in figure 2, build a 2-dimensional rectangular coordinate system in the grid where the origin is any lattice point. The x-axis and y-axis are the 2 grid lines passing through the origin. The unit length is same as the grid size.

* The proof here is not that of the original paper.



Let $\overrightarrow{C_1C_2}=(a_1,b_1)$, $\overrightarrow{C_2C_3}=(a_2,b_2)$, $\overrightarrow{C_3C_4}=(a_3,b_3)$, $\overrightarrow{C_1C_4'}=(a_0,b_0)$

$\overrightarrow{C_1'C_2'}=(x_1,y_1)$, $\overrightarrow{C_2'C_3'}=(x_2,y_2)$, $\overrightarrow{C_3'C_4'}=(x_3,y_3)$

Because (C,C′) is a 2-dimensional CG figure, we have:

$$\begin{cases} a_1x_1+b_1y_1=0 \\ a_2x_2+b_2y_2=0 \\ a_3x_3+b_3y_3=0 \\ x_1+x_2+x_3=a_0 \\ y_1+y_2+y_3=b_0 \end{cases} \qquad (1)$$

If (C,C′) is an XD figure, then this set of equations has integer solutions.

Let the solution be $x_1=m_1, x_2=m_2, x_3=m_3, y_1=n_1, y_2=n_2, y_3=n_3$,

then $\begin{cases} x_1=m_1+(a_2b_1b_3-a_3b_1b_2)k \\ x_2=m_2+(a_3b_1b_2-a_1b_2b_3)k \\ x_3=m_3+(a_1b_2b_3-a_2b_1b_3)k \\ y_1=n_1+(a_1a_3b_2-a_1a_2b_3)k \\ y_2=n_2+(a_1a_2b_3-a_2a_3b_1)k \\ y_3=n_3+(a_2a_3b_1-a_1a_3b_2)k \end{cases}$ must be the solution of Equations 1.

In addition, neither $C_1C_2C_3$ nor $C_2C_3C_4$ is collinear,

so none of $a_1b_2-a_2b_1, a_1b_3-a_3b_1, a_2b_3-a_3b_2$ equal to 0.

In addition, a_1, b_1 are not both 0.

which shows $(a_2b_1b_3-a_3b_1b_2)=0, (a_1a_3b_2-a_1a_2b_3)=0$ not to be both correct.

In addition, k may be any integer. Therefore, Equations 1 has innumerable solutions.

Thus, (C,C′) is not a 2-dimensional XD figure.

Therefore, in the infinite grid, no 2-dimensional XD figure consisting of more than 4 line segments exist.

This completes the proof.

III The properties of 2-dimensional CG and XD figures

By extending conclusion1 of XD figures, a property of CG figures is found.

Property 1: For a CG figure (C, C′) in a grid of m×n, if the solution set of C is A, the solution set of C′ is B, then the solution set of any element in A is B and vice versa.

Proof:

Since the solution set of C is A,
the corresponding line segments between any element from A and C are perpendicular to each other.
Thus, the corresponding line segments between any two elements of A are respectively collinear or parallel.
In addition, the solution set of C′ is B,
so the corresponding line segments between any element from B and C′ are perpendicular to each other.
Thus, the corresponding line segments between any element from B and any element from A are perpendicular to each other.
Suppose any element from B has another solution out of A, then this solution is also a solution of C.
Then the solution set of C is not A. This contradicts the condition that 'the solution set of C is A'.
Then the assumption 'an element from B has a solution addition to A' is incorrect.
Therefore, the solution set of any element from B is A.
In the same way, the solution set of any element from A is B.
This completes the proof.

Inspired by conclusion 2, the question of the existence of XD figures in limited grids was raised. In Reference 2, there also was an open problem about whether or not there is an upper limit of grid number for grids with XD figures. Here, we give the solution.
Property 2: In a grid of n×n (with (n+1) × (n+1) grid points, $n \geq 2$), there must exist an XD figure consisting of t line segments (t is the total number of line segments including question and solution, $t = 4,6,8 \ldots 2n^2 + 4n$*)
Proof:
In a grid of n×n, build a 2-dimensional rectangular coordinate system where the origin is defined to be the lower corner on the left, the x-direction going to the right, the y-direction going up, and the unit length the grid size.

Draw a polyline C: A_1-A_2-A_3-…$A_{\frac{t}{2}-1}$-$A_{\frac{t}{2}}$ in which the coordinate of A_1 is (a_0, b_0). Write the vector of each line segment as

$$\overrightarrow{A_iA_{i+1}}=(a_i, b_i) \quad (i=1,2\ldots\frac{t}{2}-1)$$

Draw another polyline C′: B_1-B_2-B_3-…$B_{\frac{t}{2}-1}$-$B_{\frac{t}{2}}$(B_1= A_1, $B_{\frac{t}{2}}$=$A_{\frac{t}{2}}$). Write the vector of each line segment as

$$\overrightarrow{B_iB_{i+1}}=(x_i, y_i) \quad (i=1,2\ldots\frac{t}{2}-1)$$

with coordinate of B_i being:

$$B_i\ (u_i, v_i) \quad (i=2,3\ldots\frac{t}{2})$$

If the following set of equations with x_i, y_i as unknowns

*When the points in the question of an XD figure are distinct, the number of the points in the question must range from 3 to $(n+1)^2$. Therefore, the range of t must be from 4 to $2n^2 + 4n$

$$\begin{cases} a_i x_i + b_i y_i = 0 \\ \sum a_i = \sum x_i \\ \sum b_i = \sum y_i \\ 0 \le a_0 + \sum_{i=1}^{j} x_i \le n \\ 0 \le b_0 + \sum_{i=1}^{j} y_i \le n \end{cases} \quad (i, j = 1,2 \ldots \tfrac{t}{2} - 1) \qquad (1)$$

has only one integer solution, then (C, C′) is an XD figure.

We also have $\begin{cases} u_i = a_0 + \sum_{j=1}^{i-1} x_j \\ v_i = b_0 + \sum_{j=1}^{i-1} y_j \end{cases} (i = 2,3 \ldots \frac{t}{2} - 1)$

Let's begin with Equations 1, if the above equations are satisfied, these are the following conclusions:

Lemma 1: When gcd $(a_i, b_i)=1$ while $|a_i|>n/2$ or $|b_i|>n/2$, then $x_i = \pm b_i$, $y_i = \mp a_i$ (gcd (a, b) means the greatest common divisor of a,b.)

Proof:

Since gcd $(a_i, b_i)=1$,

from $a_i x_i + b_i y_i = 0$ in Equations 1,

$x_i = s b_i$, $y_i = -s a_i$, $(s \in Z)$.

In addition, $0 \le a_0 + \sum_1^j x_i \le n$, $0 \le b_0 + \sum_1^j y_i \le n$,

so $-n \le x_i \le n$, $-n \le y_i \le n$.

Adding that $|a_i|>n/2$ or $|b_i|>n/2$,

so $s=\pm 1$.

Therefore, $x_i = \pm b_i, y_i = \mp a_i$

This completes the proof.

Lemma 2: If $\gcd(a_i,b_i)=1$ while $|a_i|=n/2$ or $|b_i|=n/2$, when(u_i, v_i)or$(u_{i+1}, v_{i+1})$is a fixed point with coordinates not equal to 0,n,n/2, then x_i,y_i have at most one integer solution.

Proof:

Since gcd $(a_i, b_i)=1$,

from $a_i x_i + b_i y_i = 0$ in Equations 1,

$x_i = s b_i$, $y_i = -s a_i$, $(s \in Z)$.

In addition, $0 \le a_0 + \sum_1^j x_i \le n$, $0 \le b_0 + \sum_1^j y_i \le n$, $\begin{cases} u_i = a_0 + \sum_{j=1}^{i-1} x_j \\ v_i = b_0 + \sum_{j=1}^{i-1} y_j \end{cases} (i = 2,3 \ldots \frac{t}{2} - 1)$,

so $0 \le u_i + x_i \le n$, $0 \le v_i + y_i \le n$, $0 \le u_{i+1} - x_i \le n$, $0 \le v_{i+1} - y_i \le n$

Adding that $|a_i|=n/2$ or $|b_i|=n/2$, (u_i, v_i)or$(u_{i+1}, v_{i+1})$is a fixed point with coordinates not equal to 0,n,n/2, we have:

If $a_i=n/2$,

Then, if $v_i < \frac{n}{2}, s = 1$; if $v_i > \frac{n}{2}, s = -1$; if $v_{i+1} < \frac{n}{2}, s = -1$; if $v_{i+1} > \frac{n}{2}, s = 1$;

If $a_i=-n/2$,

Then, if $v_i < \frac{n}{2}, s = -1$; if $v_i > \frac{n}{2}, s = 1$; if $v_{i+1} < \frac{n}{2}, s = 1$; if $v_{i+1} > \frac{n}{2}, s = -1$;

If $b_i=n/2$,

Then, if $u_i < \frac{n}{2}, s = 1$; if $u_i > \frac{n}{2}, s = -1$; if $u_{i+1} < \frac{n}{2}, s = -1$; if $u_{i+1} > \frac{n}{2}, s = 1$;

If b_i=-n/2,

Then, if $u_i < \frac{n}{2}, s = -1$; if $u_i > \frac{n}{2}, s = 1$; if $u_{i+1} < \frac{n}{2}, s = 1$; if $u_{i+1} > \frac{n}{2}, s = -1$;

Thus, when(u_i, v_i)or(u_{i+1}, v_{i+1})is a fixed point with coordinates not equal to 0,n,n/2, x_i,y_i have at most one integer solution.
This completes the proof.

Lemma 3: If gcd(a_i, b_i)=1 while $|a_i|$>n/2 or $|b_i|$>n/2, when(u_i, v_i)or(u_{i+1}, v_{i+1})is a fixed point, then x_i,y_i have at most one integer solution.
Proof:
Since gcd (a_i, b_i)=1,
from $a_i x_i + b_i y_i = 0$ in Equations 1,
$x_i = s b_i,\ y_i = -s a_i,\ (s \in Z)$

In addition, $0 \le a_0 + \sum_1^j x_i \le n,\ 0 \le b_0 + \sum_1^j y_i \le n, \begin{cases} u_i = a_0 + \sum_{j=1}^{i-1} x_j \\ v_i = b_0 + \sum_{j=1}^{i-1} y_j \end{cases} (i = 2,3 \ldots \frac{t}{2} - 1)$

so $0 \le u_i + x_i \le n,\ 0 \le v_i + y_i \le n,\ 0 \le u_{i+1} - x_i \le n,\ 0 \le v_{i+1} - y_i \le n$
Adding that $|a_i|$>n/2 or $|b_i|$>n/2, (u_i, v_i)or(u_{i+1}, v_{i+1})is a fixed point, we have
If a_i>n/2,

Then, if $v_i < \frac{n}{2}, s = 1$; if $v_i > \frac{n}{2}, s = -1$; if $v_{i+1} < \frac{n}{2}, s = -1$; if $v_{i+1} > \frac{n}{2}, s = 1$ (if $v_i = \frac{n}{2}$, there's no solution to the set);
If a_i<-n/2,

Then, if $v_i < \frac{n}{2}, s = -1$; if $v_i > \frac{n}{2}, s = 1$; if $v_{i+1} < \frac{n}{2}, s = 1$; if $v_{i+1} > \frac{n}{2}, s = -1$ (if $v_i = \frac{n}{2}$, there's no solution to the set);
If b_i>n/2,

Then, if $u_i < \frac{n}{2}, s = 1$; if $u_i > \frac{n}{2}, s = -1$; if $u_{i+1} < \frac{n}{2}, s = -1$; if $u_{i+1} > \frac{n}{2}, s = 1$ (if $u_i = \frac{n}{2}$, there's no solution to the set);
If b_i<-n/2,

Then, if $u_i < \frac{n}{2}, s = -1$; if $u_i > \frac{n}{2}, s = 1$; if $u_{i+1} < \frac{n}{2}, s = 1$; if $u_{i+1} > \frac{n}{2}, s = -1$ (if $u_i = \frac{n}{2}$, there's no solution to the set).
Thus, when (u_i, v_i)or(u_{i+1}, v_{i+1})is a fixed point, x_i,y_i have at most one integer solution.
This completes the proof.

Lemma 4: When gcd(a_i,b_i)=1, gcd (a_{i+1}, b_{i+1})=1, if $|a_i|$>n/2,$|a_{i+1}|$>n/2, $y_i=\pm a_i$,then $y_{i+1}=\mp a_{i+1}$; if $|b_i|$>n/2, $|b_{i+1}|$>n/2, $x_i=\pm b_i$,then $x_{i+1}=\mp b_{i+1}$.
As the conditions $|a_i|$>n/2,$|a_{i+1}|$>n/2, $y_i=\pm a_i$ and $|b_i|$>n/2, $|b_{i+1}|$>n/2, $x_i=\pm b_i$ are not essentially different, here we assume the first one to be correct.
Since gcd (a_i, b_i)=1, $|a_i|$>n/2
from Lemma 1,$x_i = \pm b_i,\quad y_i = \mp a_i$
In addition, gcd (a_{i+1}, b_{i+1})=1, $|a_{i+1}|$>n/2

From Lemma 1, $x_{i+1} = \pm b_{i+1}$, $y_{i+1} = \mp a_i$

Adding that $|a_i|>n/2, |a_{i+1}|>n/2$

$|y_i|>n/2, |y_{i+1}|>n/2$

In addition, $0 \le a_0 + \sum_1^j x_i \le n$, $0 \le b_0 + \sum_1^j y_i \le n$

so $-n \le y_i + y_i + 1 \le n$,

Thus, if $y_i=\pm a_i$, then $y_{i+1}=\mp a_{i+1}$

This completes the proof.

Here we discuss the case by constructing specific line segments as problems and proving their solutions to be unique under the following situations respectively:

A. $\mathbf{t} \in [\mathbf{8}, \mathbf{2n^2} + \mathbf{4n}]$

1） When n≡1 (mod 4), let n=4p+1(p≥1) .

i)If t=$2n^2 + 4n$, construct one polyline as follows (see Figure 3 as an example)

let $a_0=0, b_0=0$

$$a_i = a_i{}' = \begin{cases} 2p+1 & i=1 \\ 2p+1 & i-1 \equiv (-1)^e e, i \le 16p^2+12p+1 \\ -2p-1 & i-1 \equiv -(-1)^e e, i \le 16p^2+12p+1 \\ 1 & i-1 \equiv 0 \text{ or } 4p+2, i \in [2,8p^2+8p+2] \\ 2 & i = 8p^2+8p+3 \\ 1 & i-1 \equiv 0 \text{ or } 4p+2, i \in [8p^2+8p+2,16p^2+12p+1] \\ 1 & i = 16p^2+12p+2 \\ 2p+1 & i \equiv -(-1)^e e, i \in [16p^2+12p+3,16p^2+16p+2] \\ -2p-1 & i \equiv (-1)^e e, i \in [16p^2+12p+3,16p^2+16p+2] \\ 2p & i = 16p^2+16p+3 \end{cases} (\text{mod } 8p+4)$$

$$b_i = b_i{}' = \begin{cases} 0 & i=1 \\ 1 & i-1 \not\equiv 0, i-1 \not\equiv 4p+2, i \le 16p^2+12p+1 \\ -4p-1 & i-1 \equiv 0 \text{ or } 4p+2, i \in [2,16p^2+12p+1] \\ -4p+1 & i = 16p^2+12p+2 \\ 1 & i \in [16p^2+12p+3,16p^2+16p+2] \\ 0 & i = 16p^2+16p+3 \end{cases} (\text{mod } 8p$$

$+ 4)$

$(e \in [1,4p+1])$

Obviously, for $i \ne 1, i \ne 16p^2+16p+3$, we have $\gcd(a_i{}', b_i{}')$=1.

Substitute a_i, b_i into Equations 1, we have $x_1=0, u_2=0$, so $x_2>0$.

In addition, $a_i(i-1 \not\equiv 0, i-1 \not\equiv 4p+2 (\text{mod } 8p+4), i \le 8p^2+4p+1)>n/2$.

According to lemma 1,y_2 has a definite value.

According to lemma 4,$y_3, y_4, \ldots y_{4p+2}$ each has a definite value, so $x_3, x_4, \ldots x_{4p+2}$ each has a definite value as well.

From these conditions we can figure out $u_{4p+3}=4p+1$.

In addition, $b_i(i-1 \equiv 0 \text{ or } 4p+2 (\text{mod } 8p+4), i \le 8p^2+4p+1)=-4p-1$.

Therefore, $x_{4p+3}=-4p-1$, $u_{4p+4}=0$.

In the same way, we can deduce that $x_i, y_i (i \in [2,8p^2+4p+1])$ each has a fixed value.

Therefore, we can figure out $v_3 - v_{8p^2+4p+2} = 4p+1$

which leads to $v_3 = 4p+1, v_{8p^2+4p+2} = 0$.

Therefore, according to lemma 3,$x_i, y_i (i \in [2,16p^2+16p+2])$ each has a fixed value .

Therefore, $\begin{cases} x_i = b_i \\ y_i = -a_i \\ x_1 = 0 \\ y_1 = 2p \\ x_{16p^2+16p+3} = 0 \\ y_{16p^2+16p+3} = 2p+1 \end{cases}$ $(i \in [2, 16p^2+16p+2])$ is the only solution to Equations 1.

Therefore, this polyline C here is the question of an XD figure.

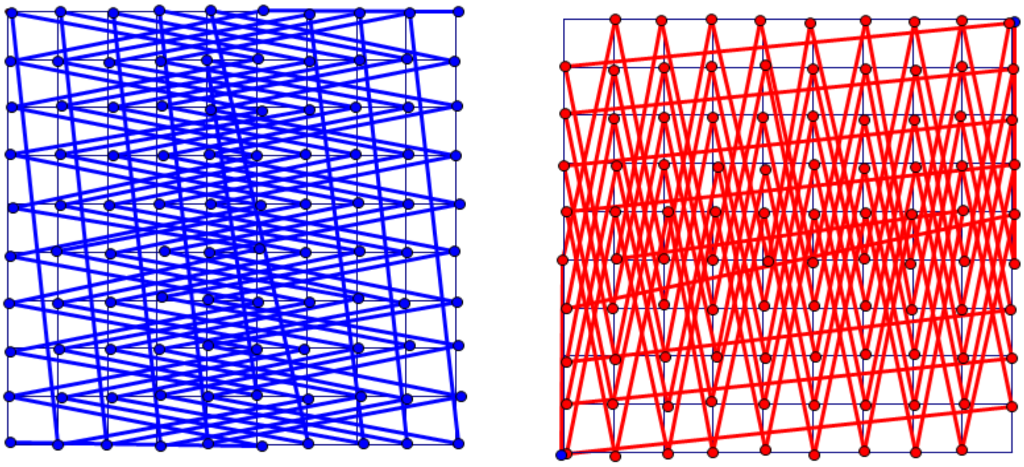


Figure 3

An XD figure consisting of 198 line segments in the grid of 9*9

(the blue line is the question, the red one the solution)

ii) If $\frac{t}{2} \in [8p^2+4p+3, 16p^2+16p+2]$

let $a_0'' = 0, b_0'' = 0$

$$\begin{cases} a_i'' = a_i' \\ b_i'' = b_i' \\ a''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+16p+2} a' \\ b''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+16p+2} b' \\ a''_{\frac{t}{2}} = 2p \\ b''_{\frac{t}{2}} = 0 \end{cases} \quad \left(i \in \left[1, \frac{t}{2}-2\right]\right)$$

If $b_{\frac{t}{2}-1}'' \neq 0$, let $a_i = a_i'', b_i = b_i'' \left(i \in \left[0, \frac{t}{2}\right]\right)$

If $b''_{\frac{t}{2}-1} = 0$, let $a_0 = 0, b_0 = 0$

$$\begin{cases} a_i = a_i{}' \\ b_i = b_i' \\ a_{4p} = -1 \\ b_{4p} = -4p + 1 \\ a_j = a_{j+2}' \\ b_j = b_{j+2}' \\ a_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+16p+2} a' \\ b_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+16p+2} b' \\ a_{\frac{t}{2}} = 2p \\ b_{\frac{t}{2}} = 0 \end{cases} \quad (i \in [1,4p-1], j \in \left[4p+1, \frac{t}{2}-2\right])$$

In the same way as with t=$2n^2 + 4n$, we can deduce that x_i,y_i($i \in [2,8p^2 + 4p + 1]$)each has a fixed value. From that, we figure out x_i,y_i($i \in \left[1, \frac{t}{2} - 2\right]$)each has a fixed value. Obviously, $x_{\frac{t}{2}} = 0$. Therefore, in Equations 1 there remain 3 equations with 3 unknowns. Their coefficients are not equal to 0, which means Equations 1 has at most one integer solution.

In addition, $\begin{cases} x_i = b_i \\ y_i = -a_i \\ x_1 = 0 \\ y_1 = 2p \\ x_{\frac{t}{2}} = 0 \\ y_{\frac{t}{2}} = 2p + 1 \end{cases} \quad (i = 2,3, \dots \frac{t}{2} - 1)$is a solution of Equations 1.

Therefore, Equations 1 has only one integer solution.
Therefore, this polyline C here is the question of an XD figure.

iii)If $\frac{t}{2} \in [6, 8p^2 + 4p + 2]$, let a_0''=0,b_0''=0

$$\begin{cases} a_3'' = 4p + 1 \\ b_3'' = 1 \\ a_4'' = -4p - 1 \\ b_4'' = 1 \\ a_i'' = a_i{}' \\ b_i'' = b_i{}' \\ a''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+16p+2} a' \\ b''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+16p+2} b' \\ a''_{\frac{t}{2}} = 2p \\ b''_{\frac{t}{2}} = 0 \end{cases} \quad (i \in \{1,2\} \cup \left[5, \frac{t}{2} - 2\right])$$

If $b''_{\frac{t}{2}-1} \neq 0$,

let $a_i = a_i'', b_i = b_i''(i \in \left[0, \frac{t}{2}\right])$

If $b''_{\frac{t}{2}-1} = 0$,

let a_0=0,b_0=0

$$\begin{cases} a_3 = 4p+1 \\ b_3 = 1 \\ a_4 = -4p-1 \\ b_4 = 1 \\ a_i = a_i' \\ b_i = b_i' \\ a_{4p} = -1 \\ b_{4p} = -4p+1 \\ a_j = a_{j+2}' \\ b_j = b_{j+2}' \\ a_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+16p+2} a' \\ b_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+16p+2} b' \\ a_{\frac{t}{2}} = 2p \\ b_{\frac{t}{2}} = 0 \end{cases} \quad (i \in \{1,2\} \cup [5, 4p-1], j \in \left[4p+1, \frac{t}{2}-2\right])$$

In the same way as mentioned above, we can prove that Equations 1 has at most one integer solution.

In additon, $\begin{cases} x_i = b_i \\ y_i = -a_i \\ x_1 = 0 \\ y_1 = 2p \\ x_{\frac{t}{2}} = 0 \\ y_{\frac{t}{2}} = 2p+1 \end{cases} \left(i = 2,3, \dots \frac{t}{2}-1\right)$ is a solution of Equations 1.

Therefore, this polyline C here is the question of an XD figure

iv)If t=10,

let a_0=0,b_0=0

$$\begin{cases} a_1 = 2p+1 \\ b_1 = 0 \\ a_2 = -2p-1 \\ b_2 = 1 \\ a_3 = 4p+1 \\ b_3 = 1 \\ a_4 = -2p \\ b_4 = 4p-1 \\ a_5 = 2p \\ b_5 = 0 \end{cases}$$

Substitute these into Equations 1, then the set has only one solution.Therefore, this polyline C here is the question of an XD figure.

v)If t=8

let a_0=0,b_0=0

$$\begin{cases} a_1 = 1 \\ b_1 = 0 \\ a_2 = -1 \\ b_2 = 4p + 1 \\ a_3 = 4p + 1 \\ b_3 = -1 \\ a_4 = -4p \\ b_4 = 0 \end{cases}$$

Substitute these into Equations 1, then the set has only one solution.

Therefore, this polyline C here is the question of an XD figure.

Therefore, for t=8,10…$2n^2 + 4n$,p≥1,in the grid of (4p+1)×(4p+1), XD figures consisting of t line segments exist.

2） If n≡3 (mod 4),n>3

Let n=4p+3(p≥1) .

i)If t=$2n^2 + 4n$, construct line segments as follows (see Figure 4 as an example)

let a_0=0,b_0=0

$a_i = a_i{}', \ b_i = b_i{}'$

$$a_i{}' = \begin{cases} 2p+2 & i = 1 \\ 2p+2 & i-1 \equiv (-1)^e e, i \le 8p^2+16p+7 \\ -2p-2 & i-1 \equiv -(-1)^e e, i \le 8p^2+16p+7 \\ 1 & i-1 \equiv 0 \text{ or } 4p+4, i \in [2, 8p^2+16p+7] \\ -1 & i = 8p^2+16p+8 \\ 2p+2 & i \equiv -(-1)^e e, i \in [8p^2+16p+9, 8p^2+24p+15] \\ -2p-2 & i \equiv (-1)^e e, i \in [8p^2+16p+9, 8p^2+24p+15] \\ 1 & i = 8p^2+20p+12 \text{ or } 8p^2+24p+16 \\ 2p+2 & i+1 \equiv -(-1)^e e, i \in [8p^2+24p+17, 16p^2+32p+14] \\ -2p-2 & i+1 \equiv (-1)^e e, i \in [8p^2+24p+17, 16p^2+32p+14] \\ 1 & i+1 \equiv 0 \text{ or } 4p+4, i \in [8p^2+24p+17, 16p^2+32p+14] \\ 2p+4 & i = 16p^2+32p+15 \end{cases} \pmod{8p+8}$$

$$b_i{}' = \begin{cases} 0 & i = 1 \\ 1 & i-1 \not\equiv 0, i-1 \not\equiv 4p+4, i \le 8p^2+16p+7 \\ -4p-3 & i-1 \equiv 0 \text{ or } 4p+4, i \in [2, 8p^2+16p+7] \\ -4p-2 & i = 8p^2+16p+8 \\ 1 & i \in [8p^2+16p+9, 8p^2+20p+11] \\ -4p-3 & i = 8p^2+20p+12 \\ 1 & i \in [8p^2+20p+13, 8p^2+24p+15] \\ -4p-2 & i = 8p^2+24p+16 \\ 1 & i+1 \not\equiv 0, i+1 \not\equiv 4p+4, i \in [8p^2+24p+17, 16p^2+32p+14] \\ -4p-3 & i+1 \equiv 0 \text{ or } 4p+4, i \in [8p^2+24p+17, 16p^2+32p+14] \\ 0 & i = 16p^2+32p+15 \end{cases} \pmod{8p+8}$$

$(e \in [1, 4p+3])$

In the same way as in 1), substitute a_i,b_i into Equations 1. We can prove that it has only one integer solution, which is

$$\begin{cases} x_i = -b_i \\ y_i = a_i \\ x_1 = 0 \\ y_1 = 2p-1 (i = 2,3, \dots \frac{t}{2}-1) \\ x_{\frac{t}{2}} = 0 \\ y_{\frac{t}{2}} = 2p+2 \end{cases}$$

Therefore, this polyline C here is the question of an XD figure.

ii) if $\frac{t}{2} \in [8p^2+12p+8, 16p^2+32p+15]$

let $a_0'' = 0, b_0'' = 0$

$$\begin{cases} a_i'' = a_i' \\ b_i'' = b_i' \\ a''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+32p+14} a' \\ b''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+32p+14} b' \quad (i \in \left[1, \frac{t}{2}-2\right]) \\ a''_{\frac{t}{2}} = 2p+4 \\ b''_{\frac{t}{2}} = 0 \end{cases}$$

If $b_{\frac{t}{2}-1}'' \neq 0$,

let $a_i = a_i'', b_i = b_i'' (i \in \left[0, \frac{t}{2}\right])$

If $b''_{\frac{t}{2}-1} = 0$,

let $a_0=0, b_0=0$

$$\begin{cases} a_i = a_i' \\ b_i = b_i' \\ a_{4p+2} = -1 \\ b_{4p+2} = -4p-1 \\ a_j = a_{j+2}' \\ b_j = b_{j+2}' \\ a_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+32p+14} a' \quad (i \in [1, 4p+1], j \in \left[4p+3, \frac{t}{2}-2\right]) \\ b_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+32p+14} b' \\ a_{\frac{t}{2}} = 2p+4 \\ b_{\frac{t}{2}} = 0 \end{cases}$$

In the same way as in 1), we can prove that Equations 1 has only one integer solution.
Therefore, this polyline C here is the question of an XD figure.

iii) If $\frac{t}{2} \in [6, 8p^2+12p+7]$

let $a''_0=0, b''_0=0$

$$\begin{cases} a''_3 = 4p+3 \\ b''_3 = 1 \\ a''_4 = -4p-3 \\ b''_4 = 1 \\ a''_i = a_i' \\ b''_i = b_i' \\ a''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+32p+14} a' \\ b''_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{16p^2+32p+14} b' \\ a''_{\frac{t}{2}} = 2p+4 \\ b''_{\frac{t}{2}} = 0 \end{cases} \quad (i \in \{1,2\} \cup \left[5, \frac{t}{2}-2\right])$$

If $b''_{\frac{t}{2}-1} \neq 0$,

let $a_i = a''_i, b_i = b''_i (i \in \left[0, \frac{t}{2}\right])$

If $b''_{\frac{t}{2}-1} = 0$,

let $a_0=0, b_0=0$

$$\begin{cases} a_3 = 4p+3 \\ b_3 = 1 \\ a_4 = -4p-3 \\ b_4 = 1 \\ a_i = a_i' \\ b_i = b_i' \\ a_{4p+2} = -1 \\ b_{4p+2} = -4p-1 \\ a_j = a'_{j+2} \\ b_j = b'_{j+2} \\ a_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+32p+14} a' \\ b_{\frac{t}{2}-1} = \sum_{\frac{t}{2}+1}^{16p^2+32p+14} b' \\ a_{\frac{t}{2}} = 2p+4 \\ b_{\frac{t}{2}} = 0 \end{cases} \quad (i \in \{1,2\} \cup [5, 4p+1], j \in \left[4p+1, \frac{t}{2}-2\right])$$

In the same way as in 1), we can prove that Equations 1 has only one integer solution. Therefore, this polyline C here is the question of an XD figure.

iv)If t=10,
let $a_0=0, b_0=0$,

$$\begin{cases} a_1 = 2p + 2 \\ b_1 = 0 \\ a_2 = -2p - 2 \\ b_2 = 1 \\ a_3 = 4p + 3 \\ b_3 = 1 \\ a_4 = -4p - 3 \\ b_4 = 1 \\ a_5 = 2p + 2 \\ b_5 = 0 \end{cases}$$

Substitute these into Equations 1. The set has only one solution.
Therefore, this polyline C here is the question of an XD figure.

v) If t=8,let a_0=0,b_0=0,

$$\begin{cases} a_1 = 1 \\ b_1 = 0 \\ a_2 = -1 \\ b_2 = 4p + 3 \\ a_3 = 4p + 3 \\ b_3 = -1 \\ a_4 = 0 \\ b_4 = -4p - 2 \end{cases}$$

Substitute these into Equations 1. Then the set has only one integer solution, which indicates that this polyline C here is the question of an XD figure.
Therefore, for t=8,10…$2n^2 + 4n$,p≥1,in the grid of (4p+3)×(4p+3), XD figures consisting of t line segments exist.

3) When n≡2 (mod 4) n>6,
Let n=4p+2(p≥2).
i)If t=$2n^2 + 4n$, construct line segments as follows (see Figure 5 as an example)

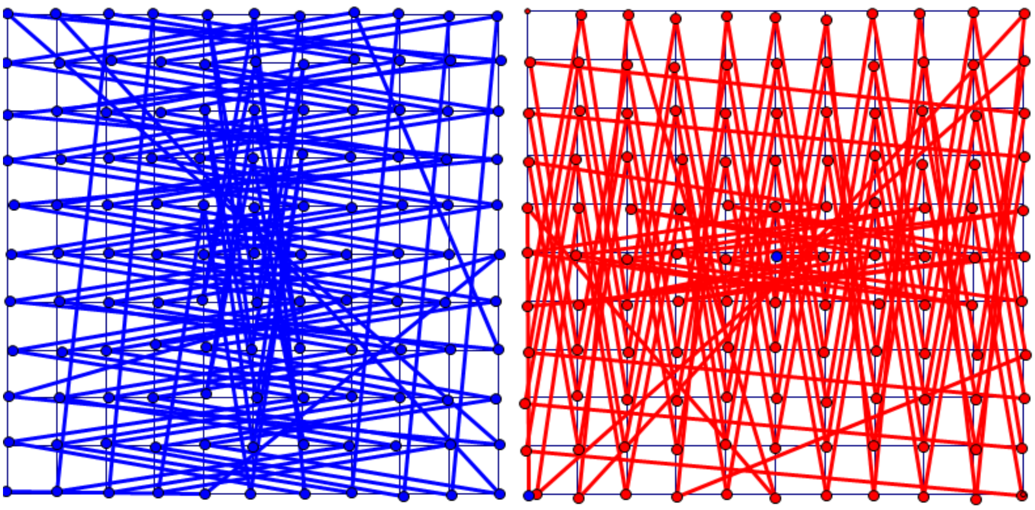


Figure 5
An XD figures consisting of 240 line segments in the grid of 10x10
(the blue line is the question, the red one the solution)

let $a_0=2p+1, b_0=2p+1$,

$a_i = a_i', b_i = b_i'$,

$$a_i' = \begin{cases} 0 & i = 1 \\ 1 & i \equiv 1 \text{ or } 2 (\text{mod } 4), i \in [2,4p+2] \\ -1 & i \equiv 0 \text{ or } 3 (\text{mod } 4), i \in [2,4p+2] \\ -2p-2 & i \equiv 1 (\text{mod } 2), i \in [4p+3,8p+4] \\ 2p+2 & i \equiv 0 (\text{mod } 2), i \in [4p+3,8p+4] \\ -1 & i \equiv 1 \text{ or } 2 (\text{mod } 4), i \in [8p+5,12p+6] \\ 1 & i \equiv 0 \text{ or } 3 (\text{mod } 4), i \in [8p+5,12p+6] \\ -1 & i = 12p+7 \\ 2p+2 & i+2 \equiv -(-1)^e e (\text{mod } 4p+3), i \in [12p+8, 8p^2+14p+3] \\ -2p-2 & i+2 \equiv (-1)^e e (\text{mod } 4p+3), i \in [12p+8, 8p^2+14p+3] \\ -1 & i+2 \equiv 0 (\text{mod } 4p+3), i \in [12p+8, 8p^2+14p+3] \\ 4p+1 & i = 8p^2+14p+4 \\ -2p-2 & i+2 \equiv -(-1)^e e (\text{mod } 4p+3), i \in [8p^2+14p+5, 16p^2+20p+3] \\ 2p+2 & i+2 \equiv (-1)^e e (\text{mod } 4p+3), i \in [8p^2+14p+5, 16p^2+20p+3] \\ -1 & i+2 \equiv 0 (\text{mod } 4p+3), i \in [8p^2+14p+5, 16p^2+20p+3] \\ 2p-1 & i = 16p^2+20p+4 \\ -4p-3 & i \equiv 1 (\text{mod } 2), i \in [16p^2+20p+5, 16p^2+22p+1] \\ 4p+3 & i \equiv 0 (\text{mod } 2), i \in [16p^2+20p+5, 16p^2+22p+1] \\ 2p+2 & i = 16p^2+22p+2 \\ -2p-2 & i = 16p^2+22p+3 \\ 4p+3 & i \equiv 0 (\text{mod } 2), i \in [16p^2+22p+4, 16p^2+24p+3] \\ -4p-3 & i \equiv 1 (\text{mod } 2), i \in [16p^2+22p+4, 16p^2+24p+3] \\ 4p+2 & i = 16p^2+24p+4 \\ -4p-2 & i = 16p^2+24p+5 \\ 4p+2 & i = 16p^2+24p+6 \\ -2p-2 & i = 16p^2+24p+7 \\ -2p & i = 16p^2+24p+8 \end{cases}$$

$$b_i' = \begin{cases} 1 & i = 1 \\ -2p-1 & i \equiv 0 (\text{mod } 2), i \le 4p \\ 2p+2 & i \equiv 1 (\text{mod } 2), i \in [3,4p+1] \\ -4p-2 & i = 4p+2 \\ 1 & i \in [4p+3,8p+4] \\ -2p-2 & i = 8p+5 \\ 2p+2 & i \equiv 0 (\text{mod } 2), i \in [8p+6,12p+6] \\ -2p-3 & i \equiv 1 (\text{mod } 2), i \in [8p+6,12p+6] \\ -2p-2 & i = 12p+7 \\ 1 & i+2 \not\equiv 0 (\text{mod } 4p+3), i \in [12p+8, 16p^2+20p+3] \\ -4p-2 & i+2 \equiv 0 (\text{mod } 4p+3), i \in [12p+8, 16p^2+20p+3] \\ 1-4p & i = 16p^2+20p+4 \\ 1 & i \in [16p^2+20p+5, 16p^2+24p+3] \\ -4p-1 & i = 16p^2+24p+4 \\ 1 & i = 16p^2+24p+5 \\ 2p-1 & i = 16p^2+24p+6 \\ -2p-1 & i = 16p^2+24p+7 \\ 0 & i = 16p^2+24p+8 \end{cases}$$

$(e \in [1,2p+1])$

Substitute $a_{t/2}, b_{t/2}, a_{t/2-1}, b_{t/2-1}, a_{t/2-2}, b_{t/2-2}$ into Equations 1. $x_{t/2}, y_{t/2}, x_{t/2-1}, y_{t/2-1}, x_{t/2-2}, y_{t/2-2}$ each has at most one value.

In addition, when i=4p+1,4p+2…t/2-1,gcd(a_i,b_i)=1 while a_i>n/2 or b_i>n/2,

so according to lemma 3, B_{4p+1},B_{4p+2}……$B_{t/2}$ are all fixed points.

Also, $x_i=b_i,y_i=-a_i$(i=4p+1,4p+2…t/2-1) is a solution of Equations 1.

Therefore, $x_i=b_i,y_i=-a_i$(i=4p+1,4p+2…t-1) is the only solution of Equations 1.

Moreover, $B_{4p+1}$and (a_{4p},b_{4p})meet the precondition of lemma 2,

so it can be figured out that $x_{4p}=b_{4p},y_{4p}=-a_{4p}$.

In addition, (a_i,b_i) (i=3,5…4p-1) all meet the precondition of lemma 3.

In the same way, we can prove that for(a_i,b_i) (i=2,4.6…4p), the coordinates of $B_{i+1}$are

$$\begin{cases}\left(2p+2,\frac{i}{2}\right)(i\equiv 2)\\ \left(2p,\frac{i}{2}\right)(i\equiv 0)\end{cases}$$ (mod 4), which all meet the precondition of lemma 2.

Therefore, Equations 1 has at most one integer solution.

In addition, $$\begin{cases}x_i=b_i{}'\\ y_i=-a_i{}'\\ x_{\frac{t}{2}}=0\\ y_{\frac{t}{2}}=2p+2\end{cases}\quad (i=1,2,\ldots\frac{t}{2}-1)$$ is a solution to Equations 1.

Therefore, this polyline C here is the question of an XD figure

ii) If 6<t<$2n^2+4n$,

let a_0=n/2,b_0=n/2,

$$\begin{cases}a_1=\sum_1^{n^2+2n+1-\frac{t}{2}}a'\\ a_i=a'_{i+n^2+2n-\frac{t}{2}}\\ b_1=\sum_1^{n^2+2n+1-\frac{t}{2}}b'\\ b_i=b'_{i+n^2+2n-\frac{t}{2}}\end{cases}\quad (i=2,3\ldots.t/2)$$

In the same way as with t=$2n^2+4n$, this polyline C here is the question of an XD figure.

Therefore, for t=8,10…$2n^2+4n$,p≥2,in the grid of (4p+2)×(4p+2), XD figures consisting of t line segments exist.

4) When n≡0 (mod 4) n>4, let n=4p (p≥2).

i) If t=$2n^2+4n$, construct line segments as follows (see Figure 6 as an example)

let a_0=2p,b_0=2p; $a_i=a_i{}',b_i=b_i{}'$,

$$a_i' = \begin{cases} -2p & i = 1 \\ 2p-1 & i = 2 \\ 1 & i \equiv 0 \text{ or } 3 (\text{mod } 4), i \le 4p+1 \\ -1 & i \equiv 1 \text{ or } 2 (\text{mod } 4), i \in [3, 4p+1] \\ 1 & i = 4p+2 \\ -2p-1 & i \equiv 1 (\text{mod } 2), i \in [4p+3, 8p+2] \\ 2p+1 & i \equiv 0 (\text{mod } 2), i \in [4p+3, 8p+2] \\ 1 & i \equiv 1 \text{ or } 2 (\text{mod } 4), i \in [8p+3, 12p+2] \\ -1 & i \equiv 0 \text{ or } 3 (\text{mod } 4), i \in [8p+3, 12p+2] \\ 1 & i = 12p+3 \\ 2p+1 & i \equiv (-1)^e e (\text{mod } 4p+1), i \in [12p+4, 8p^2+10p+1] \\ -2p-1 & i \equiv -(-1)^e e (\text{mod } 4p+1), i \in [12p+4, 8p^2+10p+1] \\ 1 & i \equiv 0 (\text{mod } 4p+1), i \in [12p+4, 8p^2+10p+1] \\ -2p-1 & i = 8p^2+10p+2 \\ 2p & i = 8p^2+10p+3 \\ -2p-1 & i+1 \equiv -(-1)^e e (\text{mod } 4p+1), i \in [8p^2+10p+4, 16p^2+4p-2] \\ 2p+1 & i+1 \equiv (-1)^e e (\text{mod } 4p+1), i \in [8p^2+10p+4, 16p^2+4p-2] \\ -1 & i+1 \equiv 0 (\text{mod } 4p+1), i \in [8p^2+10p+4, 16p^2+4p-2] \\ 4p-1 & i = 16p^2+4p-1 \\ 4p & i \equiv 1 (\text{mod } 2), i \in [16p^2+4p, 16p^2+8p-2] \\ -4p & i \equiv 0 (\text{mod } 2), i \in [16p^2+4p, 16p^2+8p-2] \\ -2p-2 & i = 16p^2+8p-1 \\ -2p+2 & i = 16p^2+8p \end{cases}$$

$$b_i' = \begin{cases} 2p & i = 1 \\ -4p & i = 2 \\ 2p+1 & i \equiv 1 (\text{mod } 2), i \in [3, 4p+1] \\ -2p & i \equiv 0 (\text{mod } 2), i \in [3, 4p+1] \\ -4p & i = 4p+2 \\ 1 & i \in [4p+3, 8p+2] \\ -2p-1 & i = 8p+3 \\ 2p+1 & i \equiv 0 (\text{mod } 2), i \in [8p+4, 12p+2] \\ -2p-2 & i \equiv 1 (\text{mod } 2), i \in [8p+4, 12p+2] \\ -2p-1 & i = 12p+3 \\ 1 & i \not\equiv 0 (\text{mod } 4p+1), i \in [12p+4, 8p^2+10p+1] \\ -4p & i \equiv 0 (\text{mod } 4p+1), i \in [12p+4, 8p^2+10p+1] \\ -2p & i = 8p^2+10p+2 \\ -2p+1 & i = 8p^2+10p+3 \\ 1 & i+1 \not\equiv 0 (\text{mod } 4p+1), i \in [8p^2+10p+4, 16p^2+4p-2] \\ -4p & i+1 \equiv 0 (\text{mod } 4p+1), i \in [8p^2+10p+4, 16p^2+4p-2] \\ -1 & i \in [16p^2+4p-1, 16p^2+8p-1] \\ 0 & i = 16p^2+8p \end{cases}$$

$(e \in [1, 2p])$

In the same way as in 3), we can prove that

$$\begin{cases} x_i = b_i' \\ y_i = -a_i' \\ x_{\frac{t}{2}} = 0 \\ y_{\frac{t}{2}} = -2p-2 \end{cases} \quad (i = 1, 2 \ldots \frac{t}{2} - 1)$$

is the only integer solution for Equations 1.

Therefore, this polyline C here is the question of an XD figure.

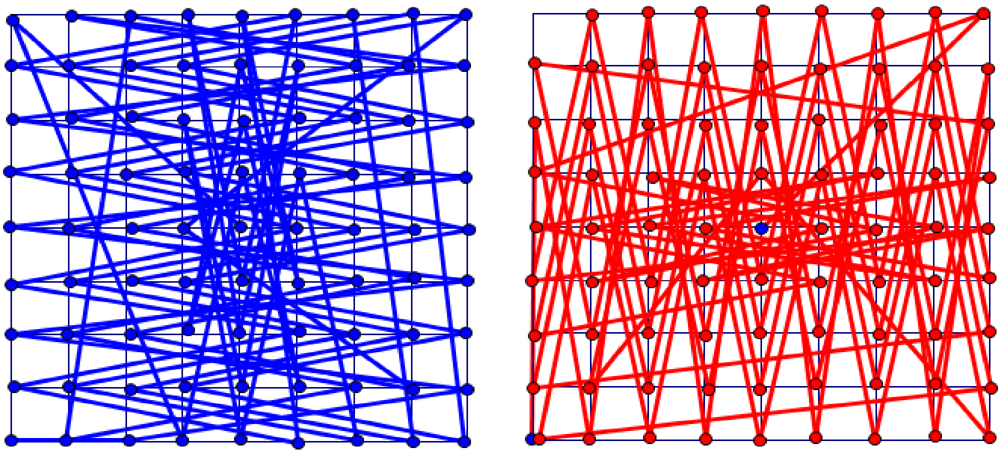


Figure 6

An XD figure consisting of 160 line segments in the grid of 8*8

(the blue line is the question, the red one the solution)

ii)If $6<t<2n^2+4n$,

let $a_0=n/2, b_0=n/2$

$$\begin{cases} a_1 = \sum_1^{n^2+2n+1-\frac{t}{2}} a' \\ a_i = a'_{i+n^2+2n-\frac{t}{2}} \\ b_1 = \sum_1^{n^2+2n+1-\frac{t}{2}} b' \\ b_i = b'_{i+n^2+2n-\frac{t}{2}} \end{cases} \quad (i=2,3\ldots.t/2)$$

In the same way as with $t=2n^2+4n$, this polyline C here is the question of an XD figure. Therefore, for $t=8,10\ldots 2n^2+4n, p\geq 2$, in the grid of $4p\times 4p$, XD figures consisting of t line segments exist.

5)When n=2

i)If t=16, construct line segments as follows (see Figure 7as an example)

let $a_0=1, b_0=1$, the line segments of polyline C be (1,-1) (-1,2) (1,-1) (-2,1) (1,-2) (1,2) (-2,-1) (0,-1).$(a_i = a_i', b_i = b_i')$

Substitute these into Equations 1, We can then figure out that C is the question of an XD figure.

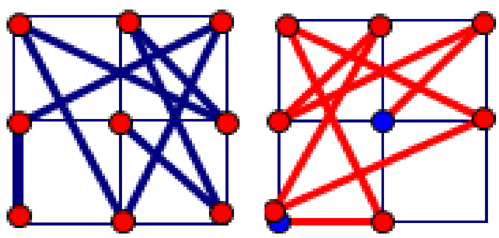


Figure 7

An XD figure consisting of 16 line segments in the grid of 2*2

(the blue line is the question, the red one the solution) ii)If $t \in [8,14]$,in the same way as in 3),let a_0=1,b_0=1

$$\begin{cases} a_1 = \sum_{1}^{9-\frac{t}{2}} a' \\ b_1 = \sum_{1}^{9-\frac{t}{2}} b' \, (i = 2,3 \ldots \frac{t}{2}) \\ a_i = a'_{i+8-\frac{t}{2}} \\ b_i = b'_{i+8-\frac{t}{2}} \end{cases}$$

Similar to t=16, this polyline C here is the question of an XD figure.

Therefore, for t=8,10…16, in the grid of 2×2, XD figures consisting of t line segments exist.

6)When n=3

i)If t=30, construct line segments as follows (see Figure 8 as an example)

let a_0=0, b_0=0 and the line segments of polyline C be represented by vectors (1,0) (2,1) (-2,1) (1,-2) (-2,1) (2,1) (-2,1) (1,-2) (2,1) (-2,1) (1,-2) (-2,1) (3,-2) (-1,3) (1,0).($a_i = a_i', b_i = b_i'$)

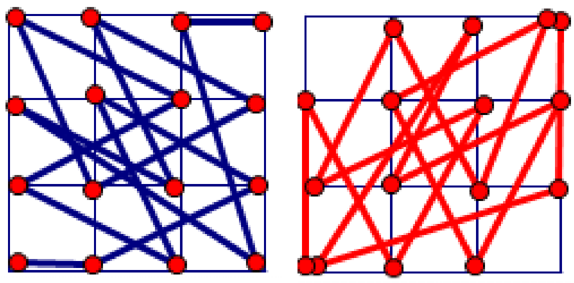


Figure 8

An XD figure consisting of 30 line segments in the grid of 3*3

(the blue line is the question, the red one the solution)

ii)If $t = 14,16,20,22,26,28$,in the same way as in 2),

let a_0=0,b_0=0,

$$\begin{cases} a_i = a_i' \\ b_i = b_i' \\ a_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{29} a' \, (i = 1,2 \ldots \frac{t}{2} - 2 \text{ or } \frac{t}{2}) \\ b_{\frac{t}{2}-1} = \sum_{\frac{t}{2}-1}^{29} b' \end{cases}$$

iii)If $t = 12$, let a_0=0, b_0=0 and the line segments of polyline C be represented by vectors (1,0) (2,1) (-3,1) (3,-2)(-1,3)(1,0).

iv)If $t = 10$, let a_0=0, b_0=0 and the line segments of polyline C be represented by vectors (1,0)(2,1) (-3,1) (2,1)(1,0).

v)If $t = 8$, let a_0=0, b_0=0 and the line segments of polyline C be represented by vectors (1,0) (-1,3) (3,-1)(0,-2).

vi)If t=18, let a_0=0, b_0=0 and the line segments of polyline C be represented by vectors (1,0) (2,1) (-2,1) (1,-2) (-2,1) (2,1) (1,-2) (-1,3) (1,0)

vii)If t=24, let a_0=0, b_0=0 and the line segments of polyline C be represented by vectors (1,0) (2,1)

(-2,1) (1,-2) (-2,1) (2,1) (-2,1) (3,-1) (-2,1) (1,-2) (0,2) (1,0)
Substitute these into Equations 1. We can figure out the C here are all questions of XD figures. Therefore, for t=8,10…30, in the grid of 3×3, XD figures consisting of t line segments exist.

7) When n=4
i)If t=48, construct line segments as follows (see Figure 9 as an example)
let a_0=2,b_0=2,the line segments of polyline C represented by vector be respectively

$$a_i=\begin{cases}1 & i=1,2,3,5,12,13,14\\ -1 & i=4,10,11\\ -3 & i=6,8,15,17,19\\ 3 & i=7,9,16,18,20\\ 4 & i=22\\ -4 & i=21,23\\ -2 & i=24\end{cases}\qquad b_i=\begin{cases}0 & i=1\\ 3 & i=2,4,11,13\\ -2 & i=3,19,24\\ -4 & i=5,12\\ 1 & i=6,7,8,9,15,16,17,18,21,22,23\\ -3 & i=10,14\\ -1 & i=20\end{cases}$$

Substitute these into Equations 1. We can then figure out that C is the question of an XD figure.

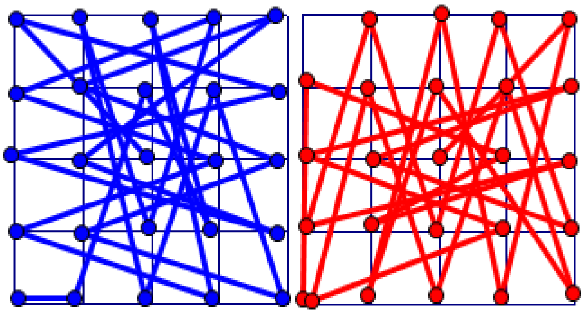


Figure 9
An XD figure consisting of 48 line segments in the grid of 4*4
(the blue line is the question, the red one the solution)

ii)If t ∈ [8,48], in the same way as in 4),
let a_0=2,b_0=2,

$$\begin{cases}a_1=\sum_1^{25-\frac{t}{2}}a'\\ b_1=\sum_1^{25-\frac{t}{2}}b' & (i=2,3\ldots\frac{t}{2})\\ a_i=a'_{i+24-\frac{t}{2}}\\ b_i=b'_{i+24-\frac{t}{2}}\end{cases}$$

In the same way as with t=48, we can prove that this polyline C here is the question of an XD figure.
Therefore, for t=8,10…48, in the grid of 3×3, XD figures consisting of t line segments exist.

8) When n=6
i) If t=96, construct line segments as follows (see Figure 10 as an example)
let a_0=3,b_0=3,the line segments of polyline C represented by vector be respectively

$$a_i=\begin{cases}2 & i=1\\ 1 & i=2,3,6,7,16,17,41\\ -1 & i=4,5,14,15,18,19,20,34,48\\ -4 & i=8,10,12,22,24,26,28,30,32,35,37,39\\ 4 & i=9,11,13,21,23,25,29,31,33,36,38,40,47\\ 5 & i=27\\ -6 & i=42,44,46\\ 6 & i=43,45\end{cases}$$

$$b_i=\begin{cases}0 & i=1,48\\ 4 & i=2,4,6,15,17,19\\ -3 & i=3,5,47\\ -6 & i=7\\ -4 & i=14,20\\ -5 & i=16,18,41\\ 1 & i\in[8,13]\,or\,[21,26]\,or[28,33]\,or[35,40]\,or[42,46]\\ -6 & i=27,34\end{cases}$$

Substitute these into Equations 1. We can figure out that C is the question of an XD figure.

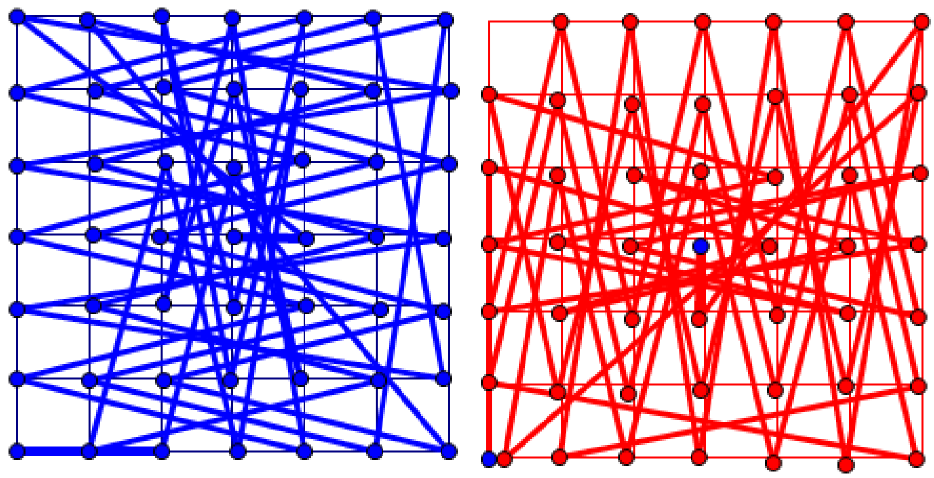


Figure 10

An XD figure consisting of 96 line segments in the grid of 6*6

(the blue line is the question, the red one the solution)

In the same way as in 4),

let a_0=3,b_0=3, $\begin{cases}a_1=\sum_1^{49-\frac{t}{2}}a' \\ b_1=\sum_1^{49-\frac{t}{2}}b' \\ a_i=a'_{i+48-\frac{t}{2}} \\ b_i=b'_{i+48-\frac{t}{2}}\end{cases}$ $(i=2,3\ldots\frac{t}{2})$

Similar to t=48, we can prove that this polyline C here is the question of an XD figure.
Therefore, for t=8,10…48, in the grid of 3×3, XD figures consisting of t line segments exist.
In summary, in the grid of n×n, XD figures consisting of t line segments exist.
$(n\geq 2, t=8,10,12\ldots 2n^2+4n)$

B. t=6,

$$\text{let}\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = n - 1 \\ a_3 = 0 \\ b_0 = 0 \\ b_1 = 0 \\ b_2 = 1 \\ b_3 = -1 \end{cases}$$

Substitute these intoEquations 1,it can be figured out that Equations 1 has only one solution. Therefore, this polyline C here is the question of an XD figure.

C.t=4

Any two adjacent sides of a rectangle is the question of an XD figure. Therefore, such XD figures exist in any grid.

Therefore, in summary, an XD figure consisting of t line segments exists in a grid of n×n.

$(n \geq 2, t = 4,6,8 \ldots 2n^2 + 4n)$

This completes the proof.

IV The properties of 3-dimensional CG and XD figures*

Compared with 2-dimensional XD figures, 3-dimensional XD figures do not process conclusion 1 and property 1

Property 3: In a grid of m×n×p, if (C,C′)is an XD figure, (C′,C_1)is a CG figure, then (C_1,C′)is not necessarily an XD figure.

Proof:

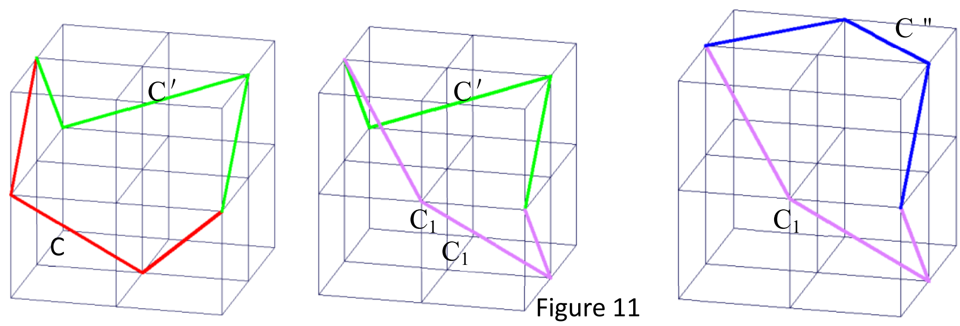


Figure 11

In figure 11, let the question be the red line C, the solution the green line C′. In the grid of 2×2×2, (C, C′) is an XD figure. Let the solution of the green line C′ be the purple line C_1 in the figure, then (C′, C_1) is a CG figure. However, except for C′, C_1 also has C " (the blue line in the figure)as a solution. Therefore, (C_1, C′) is not an XD figure.

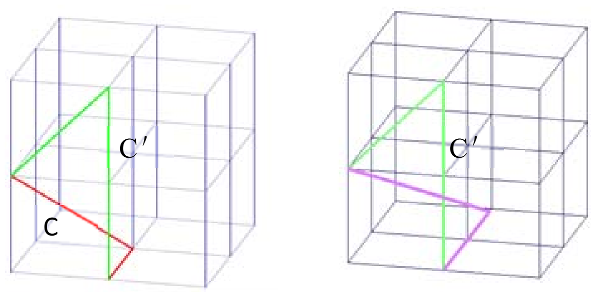


Figure 12

*The property 3 and 5 here and the proof of property 3 are from the third part of the first chapter of the book 《形独》which was written by the author of this paper.

Meanwhile, in figure 12, let the question be the red line C, the solution the green line C′. In the grid of 2×2×2, (C,C′) is an XD figure. Let the solution of the green line C′ be the purple line C_1, then, (C′,C_1) is a CG figure. In addition, C_1 has only one solution which is C′. Therefore, (C_1,C′) is an XD figure.
In summary, in a grid of m×n×p, if (C,C′)is an XD figure, (C′,C_1)is a CG figure, then (C_1,C′)is not necessarily an XD figure.
This completes the proof.

For 2-dimensional XD figures, conclusion 1 is a special case of property 1. Therefore, since 3-dimensional CG figures do not follow conclusion 1, they do not follow property 1 either. Thus, we have the following conclusion:
Property 4： In a grid of m×n×p, there is a CG figure (C,C′), if the solution set of C is A, the solution set of C′ is B, then the solution set of any element from A is not necessarily B and vice versa.

As for conclusion 2 of 2-dimensional XD figures, in infinite grids, no 3-dimensional XD figure exist no matter how many line segments they consist of.
Property 5： No 3-dimensional XD figure exist in an infinite grid.
Proof：
If a 3-dimensional CG figure exists in an infinite grid, let it be CG figure(C,C′)

C: C_1-C_2-……-C_n

C′: C_1'-C_2'-……-C_n'(C_1'=C_1,C_n'=C_n).

Consider polyline $C_1C_2C_3$and $C_1'C_2'C_3'$,

When C_1, C_2, C_3, C_1', $C_3'$are fixed points, if the coordinates of C_2' have only one value, then

(C,C′) might be an XD figure; if not (C,C′) is not an XD figure.

Build a 3-dimensional rectangular coordinate system in the grid where the origin is any lattice point. The x-axis, y-axis, z-axis are respectively the three grid lines passing the origin. The unit length is same as the grid.
Let$\overrightarrow{C_1C_2}$=(a_1,b_1,c_1),$\overrightarrow{C_2C_3}$=(a_2,b_2,c_2),$\overrightarrow{C_1C_3'}$=(a_0,b_0,c_0)
$\overrightarrow{C_1'C_2'}$=$(x_1,y_1,z_1)$,$\overrightarrow{C_2'C_3'}$=$(x_2,y_2,z_2)$
Because(C,C′)is a 3-dimensional CG figure, we get

$$\begin{cases} a_1x_1+b_1y_1+c_1z_1=0 \\ a_2x_2+b_2y_2+c_2z_2=0 \\ x_1+x_2=a_0 \\ y_1+y_2=b_0 \\ z_1+z_2=c_0 \end{cases} \qquad (1)$$

If(C,C′)is an XD figure, then Equations 1 has one integer solution.
Let that solution be $x_1=m_1, x_2=m_2, y_1=n_1, y_2=n_2, z_1=p_1, z_2=p_2$.

If k is an integer, then, $\begin{cases} x_1=m_1-(b_1c_2-b_2c_1)k \\ x_2=m_2+(b_1c_2-b_2c_1)k \\ y_1=n_1+(a_1c_2-a_2c_1)k \\ y_2=n_2-(a_1c_2-a_2c_1)k \\ z_1=p_1-(a_1b_2-a_2b_1)k \\ z_2=p_2+(a_1b_2-a_2b_1)k \end{cases}$ must be the solution of Equations 1.

If $a_1c_2 - a_2c_1 = 0, a_1b_2 - a_2b_1 = 0, b_1c_2 - b_2c_1 = 0$

Then $\cos\langle\overrightarrow{C_1C_2},\overrightarrow{C_2C_3}\rangle = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$= \pm\sqrt{\frac{a_1^2a_2^2+b_1^2b_2^2+c_1^2c_2^2+2a_1^2b_2^2+2b_1^2c_2^2+2a_1^2c_2^2}{a_1^2a_2^2+b_1^2b_2^2+c_1^2c_2^2+2a_1^2b_2^2+2b_1^2c_2^2+2a_1^2c_2^2}}$

$=\pm 1$

which shows $C_1C_2C_3$ to be collinear. This does not match the definition of XD figures.

Thus, $a_1c_2 - a_2c_1 = 0, a_1b_2 - a_2b_1 = 0, b_1c_2 - b_2c_1 = 0$ are not all correct.

In addition, k can be any integer, so the set of equations has innumerable solutions,

so (C,C′)is not an XD figure

Therefore, no 3-dimensional XD figure exist in an infinite grid.

This completes the proof.

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