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Title: New Study on Identities Involving *Euler* Numbers and *Bernoulli* Numbers

论文题目：含 *Euler* 数和 *Bernoulli* 数的恒等式新探

摘要：本文给出了求解$f_n(x_1,x_2,\cdots,x_m)$的简便运算方法, 通过对三角函数恒等式作适当变形, 利用生成函数法得到与*Euler*数和*Bernoulli*数有关的若干新颖恒等式, 由此可利用 *Euler* 数和 *Bernoulli* 数直接求解$f_n(1,2,\cdots,2^{m-1})$和$F_n(1,2,\cdots,2^{m-1})$, 最后给出$f_n(x_1,x_2,\cdots,x_m)$和$F_n(x_1,x_2,\cdots,x_m)$的恒等式关系.

关键词：*Euler* 数, *Bernoulli* 数, 恒等式, 生成函数

Title： New Study on Identities Involving *Euler* Numbers and *Bernoulli* Numbers

Abstract： The article studys on the simple methods to calculate $f_n(x_1, x_2, \cdots, x_m)$. Through some appropriate deformation of trigonometric identities, some new identities involving *Euler* numbers and *Bernoulli* numbers are discovered by generating function. Then $f_n(1, 2, \cdots, 2^{m-1})$ and $F_n(1, 2, \cdots, 2^{m-1})$ can be found directly by these identities involving *Euler* numbers and *Bernoulli* numbers. Lastly, the article reveals the relationship of $f_n(x_1, x_2, \cdots, x_m)$ and $F_n(x_1, x_2, \cdots, x_m)$ by an identity.

Key Words： *Euler* numbers, *Bernoulli* numbers, identities, generating function

Contents

1 Definition

Definition 1 Let $k_1 + \cdots + k_m = n$, $k_i \in \mathrm{N}$ $(i = 1,2,\cdots,n)$, define (cf. [1])

$$\binom{n}{k_1,\cdots,k_m} = \frac{n!}{k_1!\cdots k_m!}. \tag{1}$$

Especially, when $m = 2$ we get

$$\binom{n}{k_1,k_2} = \frac{n!}{k_1!k_2!} = \binom{n}{k_1} = \binom{n}{k_2}. \tag{2}$$

Definition 2 The *Bernoulli* numbers $B_n (n = 0,1,2,\cdots)$ are defined by the coefficients of $\frac{x^n}{n!}$ in the expansion of (cf. [1] and [2])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{3}$$

By (3), we have $B_n = 0$ for odd $n \geqslant 3$. For even $n \geqslant 2$, we have

$$B_n = -\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m} B_m. \tag{4}$$

Definition 3 The *Euler* numbers $E_n (n = 0,1,2,\cdots)$ are defined by the coefficients of $\frac{x^n}{n!}$ in the expansion of (cf. [1])

$$\frac{2e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}. \tag{5}$$

For $n \geqslant 1$, we have

$$E_0 = 1,\ E_{2n-1} = 0,\ \sum_{k=0}^{n}\binom{2n}{2k} E_{2k} = 0. \tag{6}$$

By Definition 2 and 3, we immediately obtain (cf. [1]):

n	0	1	2	4	6	8	10	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$
E_n	1	0	-1	5	-61	1385	-50521	2702765

Bernoulli numbers and *Euler* numbers appear frequently in the trigonometric generating functions as follows

$$x\cot x = \sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2x)^{2n}}{(2n)!}, \tag{7}$$

$$\sec x = \sum_{n=0}^{\infty}(-1)^n E_{2n}\frac{x^{2n}}{(2n)!}, \tag{8}$$

$$x\csc x = \sum_{n=0}^{\infty}(-1)^n (2 - 4^n) B_{2n}\frac{x^{2n}}{(2n)!}. \tag{9}$$

Definition 4 Let $b_i \in \mathrm{N}$, $x_i \in \mathrm{C}$ $(i = 1,2,\cdots,m)$, define

$$f_n(x_1,x_2,\cdots,x_m)=\sum_{\substack{b_1+\cdots+b_m=n\\ b_1,\cdots,b_m\geqslant 0}}\binom{2n}{2b_1,\cdots,2b_m}x_1^{2b_1}\cdots x_m^{2b_m}. \tag{10}$$

$$F_n(x_1,x_2,\cdots,x_m)=\sum_{\substack{b_1+\cdots+b_m=n\\ b_1,\cdots,b_m\geqslant 0}}\binom{2n}{2b_1,\cdots,2b_m}x_1^{2b_1}\cdots x_m^{2b_m}E_{2b_1}\cdots E_{2b_m}. \tag{11}$$

The main purpose of our thesis is to find out the relationship among *Bernoulli* numbers, *Euler* numbers, $f_n(1,2,\cdots,2^{m-1})$ and $F_n(1,2,\cdots,2^{m-1})$.

2 Identities involving $f_n(x_1,x_2,\cdots,x_m)$

It's not easy to calculate $f_n(x_1,x_2,\cdots,x_m)$ by (10). We find that $f_n(x_1,x_2,\cdots,x_m)$ is equivalent to a briefer form. Therefore, we can simplify the calculation and discover the relationship between $f_n(1,2,\cdots,2^{m-1})$ and *Bernoulli* numbers.

Theorem 1 If $i_1=0$, $i_2,\cdots,i_m\in\{0,1\}$, $W_m(\boldsymbol{x})=\sum_{k=1}^{m}(-1)^{i_k}x_k$, then

$$f_n(x_1,x_2,\cdots,x_m)=\frac{1}{2^{m-1}}\sum_{i_2,\cdots,i_m\in\{0,1\}}[W_m(\boldsymbol{x})]^{2n}. \tag{12}$$

Proof When $m=1$, clearly

$$f_n(x)=\sum_{\substack{b=n\\ b\geq 0}}\frac{(2n)!}{(2b)!}x^{2b}=x^{2n}. \tag{13}$$

When $m=2$, we get

$$f_n(x,y)=\sum_{\substack{a+b=n\\ a,b\geq 0}}\binom{2n}{2a,2b}x^{2a}y^{2b}=\sum_{a=0}^{n}\binom{2n}{2a}x^{2a}y^{2n-2a}=\frac{1}{2}[(x+y)^{2n}+(x-y)^{2n}]. \tag{14}$$

from binomial theorem, then

$$\sum_{j\geq 0}\binom{2n}{j}x^{j}y^{2n-j}=(x+y)^{2n}, \tag{15}$$

replace x by $-x$

$$\sum_{j\geq 0}\binom{2n}{j}(-x)^{j}y^{2n-j}=(-x+y)^{2n}, \tag{16}$$

combine (15)(16), then

$$\sum_{j\geq 0}\binom{2n}{2j}x^{2j}y^{2n-2j}=\frac{1}{2}[(x+y)^{2n}+(x-y)^{2n}]. \tag{17}$$

We use mathematical induction on m.

Suppose (12) holds when $m=k$. When $m=k+1$, we get

$$f_n(x_1,x_2,\cdots,x_k,x_{k+1})=\sum_{\substack{b_1+\cdots+b_{k+1}=n\\ b_1,\cdots,b_{k+1}\geqslant 0}}\frac{(2n)!}{(2b_1)!\cdots(2b_{k+1})!}x_1^{2b_1}\cdots x_{k+1}^{2b_{k+1}},$$

let $x_{k+1} = y$, $b_{k+1} = a$,then

$$f_n(x_1, x_2, \cdots, x_k, x_{k+1}) = \sum_{a=0}^{n} \sum_{\substack{b_1+\cdots+b_k=n-a \\ b_1,\cdots,b_k \geqslant 0}} \frac{(2n)!}{(2a)!(2n-2a)!} \cdot \frac{(2n-2a)!}{(2b_1)!\cdots(2b_k)!} x_1^{2b_1} \cdots x_k^{2b_k} y^{2a}$$

$$= \sum_{a=0}^{n} \binom{2n}{2a} y^{2a} \sum_{\substack{b_1+\cdots+b_k=n-a \\ b_1,\cdots,b_k \geqslant 0}} \frac{(2n-2a)!}{(2b_1)!\cdots(2b_k)!} x_1^{2b_1} \cdots x_k^{2b_k} = \sum_{a=0}^{n} \binom{2n}{2a} y^{2a} f_{n-a}(x_1, x_2, \cdots, x_k)$$

$$= \sum_{a=0}^{n} \binom{2n}{2a} y^{2a} \cdot 2^{1-k} \sum_{i_2,\cdots,i_k \in \{0,1\}} [W_k(\boldsymbol{x})]^{2(n-a)} ,$$

according to (17)

$$f_n(x_1, x_2, \cdots, x_k, x_{k+1}) = 2^{1-k} \sum_{i_2,\cdots,i_k \in \{0,1\}} \frac{1}{2} [(W_k(\boldsymbol{x}) + y)^{2n} + (W_k(\boldsymbol{x}) - y)^{2n}]$$

$$= 2^{-k} \sum_{\substack{i_2,\cdots,i_k \in \{0,1\} \\ i_{k+1} \in \{0,1\}}} [W_k(\boldsymbol{x}) + (-1)^{i_{k+1}} x_{k+1}]^{2n} = 2^{-k} \sum_{i_2,\cdots,i_{k+1} \in \{0,1\}} [W_{k+1}(\boldsymbol{x})]^{2n} .$$

Therefore, (12) also holds when $m = k+1$. □

Especially, let $x_k = 2^{k-1} (k = 1, 2, \cdots, m)$ in Theorem 1 as follow

$$f_n(1, 2, \cdots, 2^{m-1}) = \frac{1}{2^{m-1}} \sum_{i_2,\cdots,i_m \in \{0,1\}} [1 + (-1)^{i_2} \cdot 2 + \cdots + (-1)^{i_m} \cdot 2^{m-1}]^{2n} . \tag{18}$$

In fact, there is a briefer form of (18).

Theorem 2 ($m, n \in \mathrm{N}^*$)

$$f_n(1, 2, \cdots, 2^{m-1}) = \frac{1}{2^{m-1}} \sum_{j=0}^{2^{m-1}-1} (2^m - 1 - 4j)^{2n} = \frac{1}{2^{m-1}} \sum_{j=0}^{2^{m-1}-1} (1 + 2j)^{2n} . \tag{19}$$

Proof (i) Define following sets:

$$P = \left\{1 + (-1)^{i_2} \cdot 2 + \cdots + (-1)^{i_m} \cdot 2^{m-1} \mid i_2, \cdots, i_m \in \{0,1\}\right\}, \tag{20}$$

$$Q = \left\{2^m - 1 - 4j \mid j = 0, 1, \cdots, 2^{m-1} - 1\right\}, \tag{21}$$

then prove $P = Q$.

When $m = 1$, $P = \{1\}$, $Q = \{1\}$, clearly $P = Q$.

When $m = 2$, $P = \{3, -1\}$, $Q = \{3, -1\}$, also have $P = Q$.

When $m = k$, suppose $P = Q$, then prove the situation of $m = k+1$.

We can easily infer that the number of elements in P depends on array $(i_2, \cdots, i_m)$, so the number of elements in P is 2^{m-1}, which is the same as that in Q.

When $m = k$, $P = Q = \{a_1, a_2, a_3, \cdots, a_{2^{k-1}}\}$, where $a_{i+1} - a_i = 4(i = 1, 2, \cdots, 2^{k-1} - 1)$.

When $m = k + 1$, $P = \{a_1 - 2^k, \cdots, a_{2^{k-1}} - 2^k, a_1 + 2^k, \cdots, a_{2^{k-1}} + 2^k\}$, obviously $\{a_n - 2^k\}_{n\geqslant 1}$ and $\{a_n + 2^k\}_{n\geqslant 1}$ are arithmetic progressions whose common difference is 4. And $Q = \{b_1, \cdots, b_{2^k}\}$, where $b_{i+1} - b_i = 4(i = 1, 2, \cdots, 2^k - 1)$. Apprently, we only need to prove $(a_1 + 2^k) - (a_{2^{k-1}} - 2^k) = 4$ and $a_{2^{k-1}} + 2^k = b_{2^k}$.

In fact, when $m = k$, $a_1 = -2^k + 3$, $a_{2^{k-1}} = 2^k - 1$, so $a_{2^{k-1}} - 2^k = -1$, $a_1 + 2^k = 3$.

Clearly, $(a_1 + 2^k) - (a_{2^{k-1}} - 2^k) = 4$.

And $b_{2^k} = a_{2^{k-1}} + 2^k = 2^{k+1} - 1$, $c_{2^k} = 2^{k+1} - 1$, so $b_{2^k} = c_{2^k}$. According to the property of arithmetic progression, we can prove that $P = Q$.

（ii）By $n^2 = (-n)^2$, we can simplify $\sum_{j=0}^{2^{m-1}-1} (2^m - 1 - 4j)^{2n}$ as

$$
\begin{aligned}
\sum_{j=0}^{2^{m-1}-1} (2^m - 1 - 4j)^{2n} &= \sum_{j=1}^{2^{m-2}} (-1 + 4j)^{2n} + \sum_{j=0}^{2^{m-2}-1} (-1 - 4j)^{2n} \\
&= \sum_{j=1}^{2^{m-2}} (-1 + 4j)^{2n} + \sum_{j=0}^{2^{m-2}-1} (1 + 4j)^{2n} \\
&= \sum_{j=0}^{2^{m-1}-1} (1 + 2j)^{2n}. \qquad \square
\end{aligned}
$$

From Theorem 2, we find that $f_n(1, 2, \cdots, 2^{m-1})$ is equivalent to the sum of （$2n$）th power of the first 2^{m-1} odd numbers. It is well-known that the sum of n^{th} powers of the first m natural numbers can be expressed in terms of *Bernoulli* numbers:

$$
\sum_{k=1}^{m} k^n = \frac{m+1}{n+1} \sum_{i=0}^{n} (m+1)^i \binom{n+1}{i+1} B_{n-i}. \tag{22}
$$

For example, Chen and Li (cf. [3]) used *Bernoulli* numbers to calculate the sum of n^{th} powers of the first m natural numbers. Liu and Luo (cf. [4, Equation 4]) have found similar results concerning the first m odd positive integers.

Therefore, we will study the relationship between $f_n(1, 2, \cdots, 2^{m-1})$ and *Bernoulli* numbers, and get some new combinatorial identities.

3 Identities involving *Bernoulli* numbers and $f_n(1, 2, \cdots, 2^{m-1})$

There have been lots of research about combinatorial identities involving *Bernoulli* numbers and *Euler* numbers, and the speed of innovation is accelerated day by day. According to the latest achievement, Professor Chu (see [5] and [6]) obtained many new identities involving *Bernoulli* and *Euler* numbers by generating function. Professor Liu (see [4, Equation 4]) have found the identity to calculate the sum of the first *m* odd positive integers. Motivated by them, we choose some trigonometric identities which differ from theirs to find out some new identities involving $f_n(1,2,\cdots,2^{m-1})$ and *Bernoulli* numbers.

Here is an ordinary trigonometric identity.

$$\cos x\cos 2x\cdots\cos 2^{m-1}x=\frac{\sin(2^m x)}{2^m\sin x}. \tag{23}$$

It's obvious that (23) is equivalent to the equation

$$2^m\prod_{k=1}^{m}\cos(2^{k-1}x)=\sin(2^m x)\cdot\csc x\,, \tag{24}$$

Applying (9), we get the following power series expansion

$$2^m\prod_{k=1}^{m}\left(\sum_{i=0}^{\infty}(-1)^i(2^{k-1})^{2i}\frac{x^{2i}}{(2i)!}\right)=\left(\sum_{j=0}^{\infty}(-1)^j 2^{m(2j+1)}\frac{x^{2j}}{(2j+1)!}\right)\left(\sum_{i=0}^{\infty}(-1)^i(2-4^i)B_{2i}\frac{x^{2i}}{(2i)!}\right), \tag{25}$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (25), then get

Theorem 3 $(m,n\in\mathrm{N}^*)$

$$\sum_{k=0}^{n}\binom{2n+1}{2k}4^{m(n-k)}(2-4^k)B_{2k}=(2n+1)f_n(1,2,\cdots,2^{m-1}). \tag{26}$$

In fact, Theorem 3 can be deduced by [4, Equation 4] and [5, Theorem 1], but we prove it in a new way.

It's obvious that (24) is equivalent to the equation

$$2^m\prod_{j=1}^{m}\cos(2^{j-1}x)\cdot\cos x=\sin(2^m x)\cdot\cot x\,, \tag{27}$$

Applying (7), we get the following power series expansion

$$2^m\left(\prod_{j=1}^{m}\sum_{i=0}^{\infty}(-1)^i\frac{(2^{j-1}x)^{2i}}{(2i)!}\right)\left(\sum_{i=0}^{\infty}(-1)^i\frac{x^{2i}}{(2i)!}\right)=\left(\sum_{j=0}^{\infty}(-1)^j 2^{m(2j+1)}\frac{x^{2j}}{(2j+1)!}\right)\left(\sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2x)^{2n}}{(2n)!}\right) \tag{28}$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (28), then get

Theorem 4 $(m,n\in\mathrm{N}^*)$

$$(2n+1)\sum_{k=0}^{n}\binom{2n}{2k}f_k(1,2,\cdots,2^{m-1})=\sum_{k=0}^{n}\binom{2n+1}{2k}4^{m(n-k)+k}B_{2k}. \tag{29}$$

Set $m=1$, we immediately obtain [6, Corollary 13].

Set $m=2,3$, we obtain following new corollaries other than [5][6].

Corollary 4. 1 $$(2n+1)\sum_{k=0}^{n}\binom{2n}{2k}f_k(1,2)=\sum_{k=0}^{n}\binom{2n+1}{2k}4^{2n-k}B_{2k}\,. \quad (30)$$

Corollary 4. 2 $$(2n+1)\sum_{k=0}^{n}\binom{2n}{2k}f_k(1,2,4)=\sum_{k=0}^{n}\binom{2n+1}{2k}4^{3n-2k}B_{2k}\,. \quad (31)$$

It's obvious that (24) is equivalent to the equation

$$2^m\csc(2^m x)\prod_{j=1}^{m}\cos(2^{j-1}x)=\csc x\,. \quad (32)$$

Applying (9), we get the following power series expansion

$$\left(\sum_{k=0}^{\infty}(-1)^k(2-4^k)B_{2k}\frac{(2^m x)^{2k}}{(2k)!}\right)\left(\prod_{j=1}^{m}\sum_{i=0}^{\infty}(-1)^i\frac{(2^{j-1}x)^{2i}}{(2i)!}\right)=\sum_{n=0}^{\infty}(-1)^n(2-4^n)B_{2n}\frac{x^{2n}}{(2n)!}. \quad (33)$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (33), then get

Theorem 5 ($m,n\in\mathrm{N}^*$)

$$\sum_{k=0}^{n}\binom{2n}{2k}4^{mk}(2-4^k)B_{2k}f_{n-k}(1,2,\cdots,2^{m-1})=(2-4^n)B_{2n}\,. \quad (34)$$

In fact, Theorem 5 can be deduced by [4, Equation 11], but we prove it in a new way.
It's obvious that (32) is equivalent to the equation

$$2^m\cot(2^m x)\prod_{j=1}^{m}\cos(2^{j-1}x)=\cos(2^m x)\cdot\csc x\,. \quad (35)$$

Applying (7)(9), we get the following power series expansion

$$\left(\sum_{j=0}^{\infty}(-1)^j B_{2j}\frac{(2^{m+1}x)^{2j}}{(2j)!}\right)\left(\prod_{j=1}^{m}\sum_{i=0}^{\infty}(-1)^i\frac{(2^{j-1}x)^{2i}}{(2i)!}\right)=\left(\sum_{i=0}^{\infty}(-1)^i\frac{(2^m x)^{2i}}{(2i)!}\right)\left(\sum_{j=0}^{\infty}(-1)^j(2-4^j)B_{2j}\frac{x^{2j}}{(2j)!}\right). \quad (36)$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (36), then get

Theorem 6 ($m,n\in\mathrm{N}^*$)

$$\sum_{k=0}^{n}\binom{2n}{2k}2^{(m+1)(2n-2k)}B_{2n-2k}f_k(1,2,\cdots,2^{m-1})=\sum_{k=0}^{n}2^{m(2n-2k)}\binom{2n}{2k}(2-4^k)B_{2k}\,. \quad (37)$$

Set $m=1$, we immediately obtain [6, Corollary 15].
Set $m=2,3$, we obtain following new corollaries other than [5][6].

Corollary 6. 1 $$\sum_{k=0}^{n}\binom{2n}{2k}8^{2n-2k}B_{2n-2k}f_k(1,2)=\sum_{k=0}^{n}\binom{2n}{2k}16^{n-k}(2-4^k)B_{2k}\,. \quad (38)$$

Corollary 6. 2 $$\sum_{k=0}^{n}\binom{2n}{2k}16^{2n-2k}B_{2n-2k}f_k(1,2,4)=\sum_{k=0}^{n}\binom{2n}{2k}64^{n-k}(2-4^k)B_{2k}\,. \quad (39)$$

4 Identities involving *Euler* numbers, *Bernoulli* numbers and $F_n(1,2,\cdots,2^{m-1})$

By analogy, we can acquire some identities involving *Euler* numbers, *Bernoulli* numbers and $F_n(1,2,\cdots,2^{m-1})$ by generating function.

It's obvious that (23) is equivalent to the equation

$$\sec x\sec 2x\cdots\sec(2^{m-1}x)=2^m\sin x\cdot\csc(2^m x)\,. \tag{40}$$

Applying (8), we get the following power series expansion

$$\prod_{k=1}^{m}\left(\sum_{i=0}^{\infty}(-1)^i E_{2i}\frac{(2^{k-1}x)^{2i}}{(2i)!}\right)=\left(\sum_{j=0}^{\infty}(-1)^j\frac{x^{2j}}{(2j+1)!}\right)\left(\sum_{k=0}^{\infty}(-1)^k(2-4^k)B_{2k}\frac{(2^m x)^{2k}}{(2k)!}\right). \tag{41}$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (41), then get

Theorem 7 $(m,n\in\mathrm{N}^*)$

$$(2n+1)F_n(1,2,\cdots,2^{m-1})=\sum_{k=0}^{n}\binom{2n+1}{2k}4^{mk}(2-4^k)B_{2k}\,. \tag{42}$$

Set $m=1$, we immediately obtain [5, Equation 19a].

Set $m=2,3$, we obtain following new corollaries other than [5][6].

Corollary 7. 1 $$\sum_{k=0}^{n}\binom{2n+1}{2k}4^{2k}(2-4^k)B_{2k}=(2n+1)F_n(1,2)\,. \tag{43}$$

Corollary 7. 2 $$\sum_{k=0}^{n}\binom{2n+1}{2k}4^{3k}(2-4^k)B_{2k}=(2n+1)F_n(1,2,4)\,. \tag{44}$$

It's obvious that (40) is equivalent to the equation

$$\csc x\prod_{k=1}^{m}\sec(2^{k-1}x)=2^m\csc(2^m x)\,. \tag{45}$$

Applying (8)(9), we get the following power series expansion

$$\left(\sum_{i=0}^{\infty}(-1)^i(2-4^i)B_{2i}\frac{x^{2i}}{(2i)!}\right)\left[\prod_{k=1}^{m}\left(\sum_{j=0}^{\infty}(-1)^j E_{2j}\frac{(2^{k-1}x)^{2j}}{(2j)!}\right)\right]=\sum_{n=0}^{\infty}(-1)^n(2-4^n)B_{2n}\frac{(2^m x)^{2n}}{(2n)!}. \tag{46}$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (46), then get

Theorem 8 $(m,n\in\mathrm{N}^*)$

$$\sum_{k=0}^{n}\binom{2n}{2k}(2-4^{n-k})B_{2n-2k}F_k(1,2,\cdots,2^{m-1})=(2-4^n)4^{mn}B_{2n}\,. \tag{47}$$

Set $m=1,2,3$, we immediately obtain following new corollaries.

Corollary 8. 1 $$\sum_{k=0}^{n}\binom{2n}{2k}(2-4^{n-k})B_{2n-2k}E_{2k}=(2-4^n)4^n B_{2n}\,. \tag{48}$$

Corollary 8. 2 $$\sum_{k=0}^{n}\binom{2n}{2k}(2-4^{n-k})B_{2n-2k}F_k(1,2)=(2-4^n)4^{2n}B_{2n}. \tag{49}$$

Corollary 8. 3 $$\sum_{k=0}^{n}\binom{2n}{2k}(2-4^{n-k})B_{2n-2k}F_k(1,2,4)=(2-4^n)4^{3n}B_{2n}. \tag{50}$$

It's obvious that (45) is equivalent to the equation

$$\cot x\prod_{k=1}^{m}\sec(2^{k-1}x)=2^m\csc(2^m x)\cdot\cos x. \tag{51}$$

Applying (7)(8)(9), we get the following power series expansion

$$\left(\sum_{i=0}^{\infty}(-1)^i B_{2i}4^i\frac{x^{2j}}{(2j)!}\right)\left[\prod_{k=1}^{m}\left(\sum_{j=0}^{\infty}(-1)^j E_{2j}(2^{k-1})^{2j}\frac{x^{2j}}{(2j)!}\right)\right]$$
$$=\left(\sum_{i=0}^{\infty}(-1)^i(2-4^i)B_{2i}4^{mi}\frac{x^{2i}}{(2i)!}\right)\left(\sum_{j=0}^{\infty}(-1)^j\frac{x^{2j}}{(2j)!}\right). \tag{52}$$

Comparing the coefficient of $\frac{x^{2n}}{(2n)!}$ on both sides of (52), then get

Theorem 9 ($m,n\in\mathrm{N}^*$)

$$\sum_{k=0}^{n}\binom{2n}{2k}4^{n-k}B_{2n-2k}F_k(1,2,\cdots,2^{m-1})=\sum_{k=0}^{n}\binom{2n}{2k}4^{mk}(2-4^k)B_{2k}. \tag{53}$$

Set $m=1$, we immediately obtain [5, Equation 11a].
Set $m=2,3$, we obtain following new corollaries other than [5][6].

Corollary 9. 1 $$\sum_{k=0}^{n}\binom{2n}{2k}4^{n-k}B_{2n-2k}F_k(1,2)=\sum_{k=0}^{n}\binom{2n}{2k}4^{2k}(2-4^k)B_{2k}. \tag{54}$$

Corollary 9. 2 $$\sum_{k=0}^{n}\binom{2n}{2k}4^{n-k}B_{2n-2k}F_k(1,2,4)=\sum_{k=0}^{n}\binom{2n}{2k}4^{3k}(2-4^k)B_{2k}. \tag{55}$$

It's obvious that (40) is equivalent to the equation

$$\sin(2^m x)\cdot\prod_{k=1}^{m}\sec(2^{k-1}x)=2^m\sin x. \tag{56}$$

Applying (8), we get the following power series expansion

$$\left(\sum_{i=0}^{\infty}(-1)^i 2^{m(2i+1)}\frac{x^{2i+1}}{(2i+1)!}\right)\left[\prod_{k=1}^{m}\left(\sum_{j=0}^{\infty}(-1)^j(2^{k-1})^{2j}E_{2j}\frac{x^{2j}}{(2j)!}\right)\right]=2^m\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}. \tag{57}$$

Comparing the coefficient of $\frac{x^{2n+1}}{(2n+1)!}$ on both sides of (57), then get

Theorem 10 ($m,n\in\mathrm{N}^*$)

$$\sum_{k=0}^{n}\binom{2n+1}{2k}4^{m(n-k)}F_k(1,2,\cdots,2^{m-1})=1. \tag{58}$$

It's amazing that the right side of (58) not depends on m.
Set $m=1$, we immediately obtain [6, Corollary 33].

Set $m=2,3$, we obtain following new corollaries other than [5][6].

Corollary 10. 1 $$\sum_{k=0}^{n}\binom{2n+1}{2k}4^{2(n-k)}F_k(1,2)=1. \quad (59)$$

Corollary 10. 2 $$\sum_{k=0}^{n}\binom{2n+1}{2k}4^{3(n-k)}F_k(1,2,4)=1. \quad (60)$$

5 Further Research and Expectation

5. 1 Further Research

During the research, we have found that the identities involving $f_n(1,2,\cdots,2^{m-1})$ are similar to those involving $F_n(1,2,\cdots,2^{m-1})$. We will find out the relationship between $f_n(x_1,x_2,\cdots,x_m)$ and $F_n(x_1,x_2,\cdots,x_m)$.

Theorem 11 $(m,n\in \mathrm{N}^*)$

$$\sum_{k=0}^{n}\binom{2n}{2k}f_k(x_1,\cdots,x_m)F_{n-k}(x_1,\cdots,x_m)=0. \quad (61)$$

Proof Define a formal power series as $\sum_{n=0}^{\infty}F_n(x_1,\cdots,x_m)\frac{t^{2n}}{(2n)!}$, then

$$\sum_{n=0}^{\infty}F_n(x_1,\cdots,x_m)\frac{t^{2n}}{(2n)!}=\prod_{j=1}^{m}\left(\sum_{n=0}^{\infty}E_{2n}\frac{(x_jt)^{2n}}{(2n)!}\right)=\prod_{j=1}^{m}\frac{2}{e^{x_jt}+e^{-x_jt}}=\prod_{j=1}^{m}\frac{1}{\cosh(x_jt)}. \quad (62)$$

Define a formal power series as $\sum_{n=0}^{\infty}f_n(x_1,\cdots,x_m)\frac{t^{2n}}{(2n)!}$, then

$$\sum_{n=0}^{\infty}f_n(x_1,\cdots,x_m)\frac{t^{2n}}{(2n)!}=\prod_{j=1}^{m}\left(\sum_{n=0}^{\infty}\frac{(x_jt)^{2n}}{(2n)!}\right)=\prod_{j=1}^{m}\frac{e^{x_jt}+e^{-x_jt}}{2}=\prod_{j=1}^{m}\cosh(x_jt). \quad (63)$$

It's obvious that

$$\left(\sum_{n=0}^{\infty}\frac{F_n(x_1,\cdots,x_m)}{(2n)!}t^{2n}\right)\left(\sum_{n=0}^{\infty}\frac{f_n(x_1,\cdots,x_m)}{(2n)!}t^{2n}\right)=1. \quad (64)$$

Comparing the coefficient of $\frac{t^{2n}}{(2n)!}$ on both sides of (64), we obtain (61). □

By Theorem 11, we can structure n-element equations to calculate $F_n(x_1,x_2,\cdots,x_m)$. This is how to use $f_n(x_1,x_2,\cdots,x_m)$ to calculate $F_n(x_1,x_2,\cdots,x_m)$ and the result shows their quite close relationship.

We will give more numberical value for $f_n(x_1,x_2,\cdots,x_m)$ and $F_n(x_1,x_2,\cdots,x_m)$ to find more combinatorial identities. For example, we may try $(x_1,x_2,\cdots,x_m)=(1,3,\cdots,3^{m-1})$.

Because of the shortage of knowledge, we haven't use Bernoulli inversion on the identities above. Maybe we can use the theory of congruence in number theory for the identities in the future.

5. 2 Innovation and Deficiency

In Theorem 1, we gain an easy way to calculate $f_n(x_1, x_2, \cdots, x_m)$. In Theorem 11, we obtain a way to calculate $F_n(x_1, x_2, \cdots, x_m)$ by using $f_n(x_1, x_2, \cdots, x_m)$ and the relationship between $f_n(x_1, x_2, \cdots, x_m)$ and $F_n(x_1, x_2, \cdots, x_m)$. Especially, we can calculate $f_n(1, 2, \cdots, 2^{m-1})$ and $F_n(1, 2, \cdots, 2^{m-1})$ by *Euler* numbers and *Bernoulli* numbers, then we acquire some new identities involving *Euler* and *Bernoulli* numbers.

But we haven't gained a way to calculate $f_n(x_1, x_2, \cdots, x_m)$ and $F_n(x_1, x_2, \cdots, x_m)$ directly by *Euler* and *Bernoulli* numbers or other briefer identities. That will be the direction of our study in the future.

6 References

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国际友谊杯数学竞赛　二等奖
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2012年8月 国际天文学和天体物理学奥赛(IOAA)金牌
2012年5月 全国天文奥赛金牌
2011年3月 全国初中应用物理竞赛一等奖
2011年3月 "天原杯"全国初中化学竞赛一等奖
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