n-linear Coordinate System and Its Applications

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Abstract:

In this paper, the trilinear coordinate in plane is studied, and is generalized into the n-dimensional Euclidean space. n-linear coordinate system is established. The coplanar theorem of n-points and the concurrent theorem of n-hyperplanes are established in n-linear coordinate space. The author proposed the concepts of accompanying space, horizontal line in trilinear plane and horizontal plane in its accompanying space. The author established the parallel theorem and oriented line theorem in trilinear plane, etc.

Keywords: trilinear coordinate; n-linear coordinate; accompanying space; oriented line; hyperplane.

1 Introduction

In plane geometry, we often meet the problem on a line passing through three points. For example, the problem of Euler line: in $\triangle ABC$, H (orthocenter), G (centroid), O(circumcenter), are all collinear. I discovered that the value of third-order determinant is equal to zero consisting of distances d_{HAB} ,d_{HBC},d_{HCA}, d_{GAB} ,d_{GBC},d_{GCA}, d_{OAB} ,d_{OBC} and d_{OCA}, from H,G,O to three sidelines AB,BC,CA respectively. Let A, B and C be three angles of $\triangle ABC$, and let R be radius of circumcircle of $\triangle ABC$. Then

$$\begin{vmatrix} d_{HAB} & d_{HBC} & d_{HCA} \\ d_{GAB} & d_{GBC} & d_{GCA} \\ d_{OAB} & d_{OBC} & d_{OCA} \end{vmatrix} = \begin{vmatrix} 2R\cos A\cos B & 2R\cos B\cos C & 2R\cos C\cos A \\ \frac{2}{3}R\sin A\sin B & \frac{2}{3}R\sin B\sin C & \frac{2}{3}R\sin C\sin A \\ R\cos C & R\cos A & R\cos B \end{vmatrix} = 0.$$

I forward put question to: Any three points in plane are collinear if and only if that the value of third-order determinant is equal to zero consisting of distances from the three points to three sidelines of an any triangle in plane?

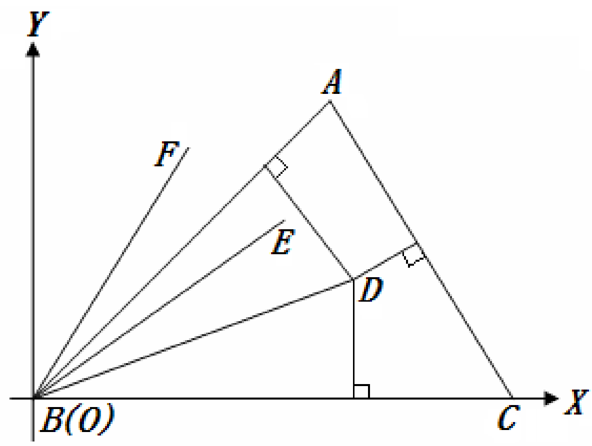

Fig 1

Theorem 1.1 (Collinear theorem on three points): Three points D,E,F in plane is collinear if and only if that the value of third-order determinant is equal to zero consisting of distances from the three points to three sidelines of an any triangle $\triangle ABC$. Namely, $\begin{vmatrix} d_{DBC} & d_{DCA} & d_{DAB} \\ d_{EBC} & d_{ECA} & d_{EAB} \\ d_{FBC} & d_{FCA} & d_{FAB} \end{vmatrix} = 0$.

Proof: As shown in Figure 1, Let XOY be the Descartes coordinate system in plane with B as the origin, with BC as the X-axis. Let $(x_D, y_D), (x_E, y_E), (x_F, y_F)$ be the Descartes coordinates of D,E,F. Let S be the area of $\triangle ABC$. $S = 2R^2 \sin A \sin B \sin C$. $\angle DBC=\alpha$, $\angle EBC=\beta$, $\angle FBC=\gamma$.

If any one of α, β, γ is not equal to zero, then $x_D = d_{DBC}\cot\alpha$, $y_D = d_{DBC}$. But

$\frac{d_{DBC}}{\sin\alpha}=\frac{d_{DAB}}{\sin(B-\alpha)}$, so $\cot\alpha=\frac{d_{DAB}}{d_{DBC}\sin B}+\cot B$, $x_D=d_{DBC}\cot\alpha=\frac{d_{DAB}}{\sin B}+d_{DBC}\cot B$.

Similarly $x_E=d_{EBC}\cot\beta=\frac{d_{EAB}}{\sin B}+d_{EBC}\cot B$, $y_E=d_{EBC}$.

$x_F=d_{FBC}\cot\gamma=\frac{d_{FAB}}{\sin B}+d_{FBC}\cot B$, $y_F=d_{FBC}$.

$d_{DCA}=\frac{2S-ad_{DBC}-cd_{DAB}}{b}$, $d_{ECA}=\frac{2S-ad_{EBC}-cd_{EAB}}{b}$, $d_{FCA}=\frac{2S-ad_{FBC}-cd_{FAB}}{b}$.

Three points *D, E, F* are collinear if and only if the area determined by the three points is equal to zero.

D, E, F is collinear $\Leftrightarrow \frac{1}{2}\begin{vmatrix} x_D & y_D & 1 \\ x_E & y_E & 1 \\ x_F & y_F & 1 \end{vmatrix}=0$

$$\Leftrightarrow \frac{1}{2}\begin{vmatrix} \frac{d_{DAB}}{\sin B}+d_{DBC}\cot B & d_{DBC} & 1 \\ \frac{d_{EAB}}{\sin B}+d_{EBC}\cot B & d_{EBC} & 1 \\ \frac{d_{FAB}}{\sin B}+d_{FBC}\cot B & d_{FBC} & 1 \end{vmatrix}=0$$

$$\Leftrightarrow \frac{1}{2\sin B}\begin{vmatrix} d_{DAB} & d_{DBC} & 1 \\ d_{EAB} & d_{EBC} & 1 \\ d_{FAB} & d_{FBC} & 1 \end{vmatrix}=0$$

$$\Leftrightarrow \frac{b}{4S\sin B}\begin{vmatrix} d_{DAB} & d_{DBC} & \frac{2S-ad_{DBC}-cd_{DAB}}{b} \\ d_{EAB} & d_{EBC} & \frac{2S-ad_{EBC}-cd_{EAB}}{b} \\ d_{FAB} & d_{FBC} & \frac{2S-ad_{FBC}-cd_{FAB}}{b} \end{vmatrix}=0$$

$$\Leftrightarrow \frac{1}{4R\sin A\sin B\sin C}\begin{vmatrix} d_{DBC} & d_{DCA} & d_{DAB} \\ d_{EBC} & d_{ECA} & d_{EAB} \\ d_{FBC} & d_{FCA} & d_{FAB} \end{vmatrix}=0.$$

If α, β, γ are all equal to zero, then α, β, γ are all equal to zero $\Leftrightarrow$ *D, E, F* are all in sideline *BC*.

If any two of α, β, γ is equal to zero, for example, α=0,β=0, then, the value of third-order determinant is equal to zero $\Leftrightarrow d_{FBC}=0 \Leftrightarrow$ γ=0 $\Leftrightarrow$ *D,E,F* are all in sideline *BC*.

If any one of α, β, γ is equal to zero, for example, α=0, then, the value of third-order determinant is equal to zero $\Leftrightarrow \frac{y_D - y_E}{x_D - x_E} = \frac{y_D - y_F}{x_D - x_F} \Leftrightarrow D,E,F$ are all collinear, and $y_D = 0$, D is in BC, and E, F are not in sideline BC. Q.E.D.

I put question to further: In n-dimensional Euclidean space, any n+1 points are all in same plane (hyperplane) if and only if that the value of (n+1)th-order determinant is equal to zero consisting of distances from each of the n+1 points to n+1 hyperplanes of an any n-dimensional simplex respectively?

This paper starts from the research to this question. In two-dimensional Euclidean space, I take three distances from one point to three sidelines of the triangle as the point's coordinate. I discovered this kind of coordinate by myself. However, I found later that in reference [1] it is called as trilinear coordinate. I studied deeply trilinear coordinate of two-dimensional Euclidean space, and generalized it into general Euclidean space. I established the system of n-linear coordinate in n-1-dimensional Euclidean space, and proposed n-linear coordinate of 0-dimensional, 1-dimensional,2-dimensional,…n-1-dimensional hyperplane, and so on.

0-dimensional hyperplane is a point. 1-dimensional hyperplane is a line. But for convenient description, they are all called as hyperplane or plane.

In this paper, E^n denotes n-dimensional Euclidean space. The concepts on the linear structure, linear dependence, linear independence, distance, angle, scalar product, vector product, mixed product, determinant, rank, etc, can be found in ordinary textbooks of advanced algebra. I directly cite these concepts without explanation. In this paper, n is a positive integer (1-dimensional Euclidean space is not discussed in this paper).

2 Establishment of n-linear Coordinate System

Definition 2.1 (The simplex in n-dimensional Euclidean space [2]): Let $P_0, P_1, \ldots, P_k\ (k \le n)$ be points of linear independence in n-dimensional Euclidean space, namely, the vectors $\vec{p}_i = P_i - P_0,\ i = 1,2,\ldots, k$ are linear independent. The Cartesian coordinate of P_i is given as $(p_{i,0}, p_{i,1}, \ldots, p_{i,n-1})$. The set of points

$\Omega_k = \left\{ X \mid X = \sum_{i=0}^{k} \lambda_i P_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0 \right\}$ is a K-dimensional simplex based on the vertices $P_0, P_1, ..., P_k$. It is expressed as notation $\sum p(k+1) = \{P_0, P_1, ..., P_k\}$. If $\lambda_i = 0$,then Ω_k is a (K-1)-dimensional simplex, and is called the plane (or hyperplane) corresponding to the vertex P_i. The K-dimensional simplex has K+1 vertices, K+1 planes, and C_{k+1}^2 edges.

Definition 2.2 (the n-linear coordinate system and the n-linear coordinate space): Based on any an n-1 dimensional simplex in E^{n-1}, and based on distances from the point to every hyperplane of the simplex, we can construct the coordinate system. We call the E^{n-1} as n-linear coordinate space, or n-space in short. We call the coordinate system as n-linear coordinate system, or n-coordinate in short.

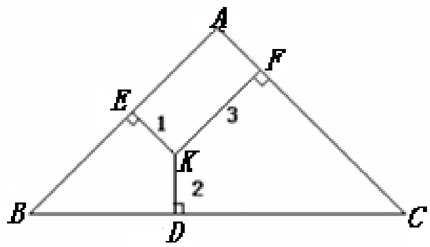

Fig 2

Definition 2.3 (The absolute coordinate of a point in n-space): The absolute coordinates are constructed by distances from one point K to n hyperplanes of n-1-dimensional simplex in n-space. $K = (d_{K,0}, d_{K,1}, ..., d_{K,n-1})$ denotes the absolute coordinates. If the point K and the corresponding vertex are in the same flank with the corresponding hyperplane, then $d_{K,i}$ is positive, otherwise $d_{K,i}$ is negative.

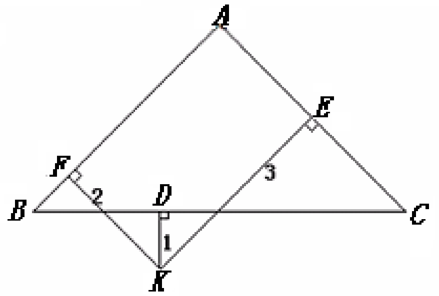

Fig 3

2-dimensional simplex in E^2 is shown in Fig 2 and Fig 3 as a triangle in plane. In Fig 2, d_{KD}=2, d_{KE}=3, d_{KF}=1, K=(2,3,1). In Fig 3, d_{KD}=-1, d_{KE}=3, d_{KF}=2, K=(-1,3,2). Such as d_{KD} in Fig 3, *K* and *A* are in the different flank of sideline *BC*, so d_{KD} is negative.

Definition 2.4 (The reduced coordinate in n-space): There is a point *K* in n-space. If $d_{K,0} : d_{K,1} ... : d_{K,n-1} = x_0 : x_1 ... : x_{n-1}$, then the reduced coordinate of *K* point is $(x_0, x_1, ..., x_{n-1})$.

If the ratio of $d_{K,0}, d_{K,1}, \ldots, d_{K,n-1}$ is determinate, then the location of point K is determinate.

Let 2-dimensional simplex in E^2 be an example shown in Fig 2, and Fig 3. Because $d_{KD} : d_{KE} : d_{KF} = x : y : z$, then $d_{KD} = xk$, $d_{KE} = yk$, $d_{KF} = zk$, $k \neq 0$. Suppose the areas of $\triangle ABC, \triangle BKC, \triangle CKA, \triangle AKB$ are S, S_1, S_2, S_3 respectively .Let *a, b, c* be the lengths of the three sidelines. The calculated area is oriented . If the height is negative, then area value is also negative.）

$$S_1 = \frac{1}{2} BC \cdot d_{KD} = \frac{1}{2} axk,\ S_2 = \frac{1}{2} CA \cdot d_{KE} = \frac{1}{2} byk,\ S_3 = \frac{1}{2} AB \cdot d_{KF} = \frac{1}{2} czk.$$

$$S = S_1 + S_2 + S_3,\ S_1 = \frac{SS_1}{S_1 + S_2 + S_3} = \frac{\frac{1}{2} axkS}{\frac{1}{2} axk + \frac{1}{2} byk + \frac{1}{2} czk} = \frac{axS}{ax + by + cz} = \frac{1}{2} ad_{KD}.$$

Therefore $d_{KD} = \dfrac{2xS}{ax + by + cz}, d_{KE} = \dfrac{2yS}{ax + by + cz}, d_{KF} = \dfrac{2zS}{ax + by + cz}$.

In the expression of d_{KD}, d_{KE} and d_{KF}, the numerator, denominator are homogeneous of *x, y, z*. Therefore they are not affected by proportional increase or decrease of *x, y, z*. To draw the parallel line of *BC* the distance of which to *BC* is d_{KD} (the distance is oriented, so it is only), and to draw the parallel line of *AB* the distance of which to *AB* is d_{KF}, the intersecting point of two parallel lines is *K*. Therefore, the unique point *K* can be located by the reduced coordinate (x, y, z). If $ax + by + cz = 0$, it is assumed that *K* represents the point infinitely far. The two-dimensional simplex in ordinary plane is the triangle. The three-dimensional simplex in ordinary space is the tetrahedron.

To n-dimensional simplex in E^n (n+1 vertices, n+1 hyperplanes):

$$d_{K,0} : d_{K,1} : \ldots : d_{K,n} = x_0 : x_1 : \ldots : x_n,\ d_{K,i} = x_i k, (i = 0,1,\ldots,n), k \neq 0.$$

The volume of a simplex is equal to the sum of n+1 volumes consisting of *K* to n+1 hyperplanes.

$$V=\sum_{i=0}^{n}V_i=\sum_{i=0}^{n}(\frac{1}{n}F_ix_ik).$$

$$V_i=\frac{VV_i}{\sum_{i=0}^{n}V_i}=\frac{\frac{1}{n}F_ix_ikV}{\sum_{i=0}^{n}(\frac{1}{n}F_ix_ik)}=\frac{1}{n}F_id_{K,i},\ d_{K,i}=\frac{nx_iV}{\sum_{i=0}^{n}F_ix_i},(i=0,1,...,n).$$

(This paragraph describes the computational method of V , F_i $(i=0,1,...,n)$ .This paragraph is a summary of related content on the reference 2.we list the results only)

V is volume of the simplex. F_i is area of n-1-dimensional hyperplane corresponding to vertex P_i. To the determinate simplex, V , F_i $(i=0,1,...,n)$ are all constant.

By Bartŏs formula, $V=\frac{1}{n}\left((n-1)!\left(\prod_{i=1}^{n}F_i\right)\sin\alpha_0\right)^{\frac{1}{n-1}}$.

α_k is n-dimensional angle of n vectors $P_kP_i(i=0,1,...k-1,k+1,...n)$ with P_k as the origin.

$$\sin\alpha_k=\begin{vmatrix}1 & \cos(0,1) & ... & \cos(0,k-1) & \cos(0,k+1) & ... & \cos(0,n)\\ \cos(1,0) & 1 & ... & \cos(1,k-1) & \cos(1,k+1) & ... & \cos(1,n)\\ ... & ... & ... & ... & ... & ... & ...\\ \cos(k-1,0) & \cos(k-1,1) & ... & 1 & \cos(k-1,k+1) & ... & \cos(k-1,n)\\ \cos(k+1,0) & \cos(k+1,1) & ... & \cos(k+1,k-1) & 1 & ... & \cos(k+1,n)\\ ... & ... & ... & ... & ... & ... & ...\\ \cos(n,0) & \cos(n,1) & ... & \cos(n,k-1) & \cos(n,k+1) & ... & 1\end{vmatrix}^{\frac{1}{2}},$$

$$\sin\alpha_0=\begin{vmatrix}1 & \cos(1,2) & ... & \cos(1.n)\\ \cos(2,1) & 1 & ... & \cos(2,n)\\ ... & ... & ... & ...\\ \cos(n,1) & \cos(n,2) & ... & 1\end{vmatrix}^{\frac{1}{2}}.$$

(i,j) represents the two-edges angle of P_kP_i,P_kP_j with P_k as the origin.

$\cos(i,j)=\frac{\vec{p}_i\cdot\vec{p}_j}{\left|(\vec{p}_i)\right|\cdot\left|(\vec{p}_j)\right|}$, $\vec{p}_i=P_i-P_k\,(i\neq k)$. We can prove $0\le\sin\alpha_k\le1$.

There is a circumscribed hypersphere to every n-dimensional simplex .Its radius R_n to n-dimensional simplex $\sum_{p(n+1)}=\{P_0,\ P_1,\ ...\ ,P_n\}$ in E^n satisfies

$R_n^2 = \frac{-D_0(P_0, P_1, \ldots, P_n)}{2D(P_0, P_1, \ldots, P_n)}$. ρ_{ij} represents the distance between vertices P_i, P_j.

$$D(P_0, P_1, \ldots, P_n) = \begin{vmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \rho_{01}^2 & \rho_{02}^2 & \ldots & \rho_{0n}^2 \\ 1 & \rho_{10}^2 & 0 & \rho_{12}^2 & \ldots & \rho_{1n}^2 \\ 1 & \rho_{20}^2 & \rho_{21}^2 & 0 & \ldots & \rho_{2n}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \rho_{n0}^2 & \rho_{n1}^2 & \rho_{n2}^2 & \ldots & 0 \end{vmatrix}, D_0(P_0, P_1, \ldots, P_n) = \begin{vmatrix} 0 & \rho_{01}^2 & \rho_{02}^2 & \ldots & \rho_{0n}^2 \\ \rho_{10}^2 & 0 & \rho_{12}^2 & \ldots & \rho_{1n}^2 \\ \rho_{20}^2 & \rho_{21}^2 & 0 & \ldots & \rho_{2n}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \rho_{n0}^2 & \rho_{n1}^2 & \rho_{n2}^2 & \ldots & 0 \end{vmatrix}.$$

The area of n-1-dimensional hyperplane corresponding to the vertex angle α_k is $F_k (0 \le k \le n)$, then

$\frac{F_k}{\sin\alpha_k} = \frac{(2R_n)^{n-1}}{(n-1)!}$, $F_k = \frac{(2R_n)^{n-1}\sin\alpha_k}{(n-1)!}$. $(0 \le k \le n)$) . (The sine theorem in E^n)

In the expression of $d_{K,i}$ $(i = 0,1,\ldots,n)$, the numerator, denominator are homogeneous of x_i $(i = 0,1,\ldots,n)$. Therefore $(x_0, x_1, \ldots, x_{n-1})$ can locate K point only. If $\sum_{i=0}^{n} F_i x_i = 0$, we assume that K represents the point infinitely far.

In this paper, if there is no special explanation we adopt the reduced coordinate as the n-linear coordinate.

In trilinear coordinate of E^2, for any k≠0:

$(x,0,0) = (1,0,0), (0,y,0) = (0,1,0), (0,0,z) = (0,0,1), (kx, ky, kz) = (x, y, z)$.

The n-linear coordinate can be reduced by a common factor. For example, *K=(2,4,8)* can be reduced as *K=(1,2,4)*.

Based on $\triangle ABC$, the followings are the trilinear coordinates of some special points.

Vertices: $A = (d_{ABC}, 0, 0) = (1,0,0), B = (0,1,0), C = (0,0,1)$.

Centroid: $G = \left(\frac{2S}{a}, \frac{2S}{b}, \frac{2S}{c}\right) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) = \left(\frac{1}{\sin A}, \frac{1}{\sin B}, \frac{1}{\sin C}\right)$.

Orthocenter: $H = (2R\cos B\cos C, 2R\cos A\cos C, 2R\cos A\cos B) = \left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$.

Circumcenter: $O = (R\cos A, R\cos B, R\cos C) = (\cos A, \cos B, \cos C)$.

Incenter: $I = (r, r, r) = (1,1,1)$.

midpoint of BC: $M=\left(0,d_{MAC},d_{MAB}\right)=\left(0,R\sin A\sin C,R\sin A\sin B\right)=\left(0,\frac{1}{\sin B},\frac{1}{\sin C}\right)$.

Definition 2.5 (Accompanying space): the accompanying space is a Cartesian coordinate space corresponding to n-space. The point $(x_0,x_1,...,x_{n-1})$ of n-linear coordinate space corresponds with the point $(x_0,x_1,...,x_{n-1})$ of Cartesian coordinate space. The former is n-linear coordinate, the latter is Cartesian coordinate.

The n-space is n-1-dimesional Euclidean space, but the accompanying space is n-dimensional Euclidean space. Through this correspondence, we can study problems of n-1-dimensional space in n-dimensional space. Similarly, we can study problems of n-dimensional space in n-1-dimensional space.

Definition 2.6 (The corresponding relation between n-space and accompanying space): In n-space, the points $(kx_0,kx_1,...,kx_{n-1})$ for all $k\neq 0$ are same point. In accompanying space, for all k, $(kx_0,kx_1,...,kx_{n-1})$ represents a line passing through the origin, namely, a oriented vector. We denote it with the notation $\vec{L_K}$ .Similarly, every oriented vector $\vec{L_K}$ of the accompanying space corresponds a point K of n-space. We can establish a one-one map between all points in n-space and all oriented vector in accompanying space, $f:K\leftrightarrow\vec{L_K}$.

The many problems are convenient for treatment when points of n-space are transformed into oriented vectors of the accompanying space. Once the simplex of n-space is determinate, then the accompanying space corresponding with it is only.

Definition 2.7 (n-dimensional parallelotope[2]): Let $\vec{p}_1, \vec{p}_2,\dots, \vec{p}_n$ from the origin O be linear independent vectors in n-dimensional Euclidean space. The set $K=\left\{\vec{X}\mid\vec{X}=\sum_{i=1}^{k}t_i p_i, 0\le t_i\le 1, i=1,2,..., k\le n\right\}$ are called as n-dimensional parallelotope with the origin O as vertex and with $\vec{p}_1, \vec{p}_2,\dots, \vec{p}_n$ as edges.

If the Cartesian coordinate of $\vec{p_i}$ is $\left(x_{i,1}, x_{i,2}, \dots ,x_{i,n}\right)$ for $i=1,2,...,n$, then the volume of n-dimensional parallelotope can be expressed by determinant

$\begin{vmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{vmatrix}$. The mixed product of $\vec{p}_1, \vec{p}_2, ..., \vec{p}_n$ satisfies

$$(\vec{p}_1 \times \vec{p}_2 \times ... \times \vec{p}_{n-1}) \cdot \vec{p}_n = \begin{vmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{vmatrix}.$$

Therefore the volume can be represent by the mixed product of vectors.

3 The coplanar theorem and n-linear coordinate of hyperplane

Theorem 3.1 (The coplanar theorem of n+1 points): In n+1-space (n-dimensional Euclidean space), any n+1 points are all in same hyperplane if and only if that the value of (n+1)th-order determinant consisting of distances between n+1 points and every hyperplane of an any n-dimensional simplex is equal to zero, namely, that the value of (n+1)th-order determinant consisting of n+1 n+1-coordinates is equal to zero.

Proof: Let the n-dimensional simplex in n-dimensional Euclidean space be $\sum p(n+1) = \{P_0, P_1, ..., P_n\}$. Let the area of the hyperplane corresponding to P_i be F_i. Let the volume of the simplex be V. Let the n+1-coordinate of ith point in n+1 points relative to the simplex be $D_i = (d_{i,0}, d_{i,1}, ..., d_{i,n}), i = 0,1,2..., n$. Then

$$\frac{1}{n}\sum_{j=0}^{n} F_j d_{i,j} = V, i = 0,1,...n, F_i > 0, V > 0.$$

This is really linear equation $DS = V$.

$$D = \begin{pmatrix} d_{0,0} & d_{0,1} & \dots & d_{0,n} \\ d_{1,0} & d_{1,1} & \dots & d_{1,n} \\ \dots & \dots & \dots & \dots \\ d_{n,0} & d_{n,1} & \dots & d_{n,n} \end{pmatrix}, S = \begin{pmatrix} F_0 \\ F_1 \\ \dots \\ F_n \end{pmatrix}, V = \begin{pmatrix} nV \\ nV \\ \dots \\ nV \end{pmatrix}.$$

If all points $D_i = (d_{i,0}, d_{i,1}, ..., d_{i,n}), i = 0,1,2..., n$ are same hyperplane, then

$D_n = \sum_{i=0}^{n-1} \lambda_i D_i$, $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ are all not equal to zero.

$Det(D)=0 \Leftrightarrow rank(D)<(n+1)$,namely, n+1 row vectors in matrix D is linear dependent[3]$\Leftrightarrow$n+1 points are all same hyperplane [3]$\Leftrightarrow$ $D_n=\sum_{i=0}^{n-1}\lambda_i D_i$, $\lambda_0,\lambda_1,\ldots,\lambda_{n-1}$ are all not equal to zero. $Det(D)$ represents the determinant of matrix D, $rank(D)$ represents rank of matrix D. Q.E.D.

Theorem 3.2 (The coplanar theorem of n+1 points): In n+1-space (n-dimensional Euclidean space), any n+1 points $D_0,D_1,\ldots,D_n$ are all in same hyperplane if and only if that the corresponding oriented vectors $\vec{L}_{D_0},\vec{L}_{D_1},\ldots\vec{L}_{D_n}$ in accompanying space (n+1-dimensional Euclidean space) are all in same hyperplane.

Namely, the mixed product $\left(\vec{L}_{D_0}\times\vec{L}_{D_1}\times\ldots\vec{L}_{D_{n-1}}\right)\cdot\vec{L}_{D_n}$ =0.

Specially if n＝2,then this theorem is really collinear theorem of three points. Three points *D,E,F* in trilinear coordinate plane is collinear if and only if the oriented vectors $\vec{L}_D,\vec{L}_E,\vec{L}_F$ in accompanying space corresponding to *D,E,F* in trilinear coordinate plane are all same plane.

Proof: according to theorem 3.1

$D_0,D_1,\ldots D_n$ are all same hyperplane

$$\Leftrightarrow\begin{vmatrix} d_{0,0} & d_{0,1} & \ldots & d_{0,n} \\ d_{1,0} & d_{1,1} & \ldots & d_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ d_{n,0} & d_{n,1} & \ldots & d_{n,n} \end{vmatrix}=0\Leftrightarrow\begin{vmatrix} l_0 d_{0,0} & l_0 d_{0,1} & \ldots & l_0 d_{0,n} \\ l_1 d_{1,0} & l_1 d_{1,1} & \ldots & l_1 d_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ l_n d_{n,0} & l_n d_{n,1} & \ldots & l_n d_{n,n} \end{vmatrix}=0\cdot$$

$$l_i\neq 0,D_i=\left(l_i d_{i,0},\ \ l_i d_{i,1},\ldots\ \ ,l_i d_{i,n}\right)=\left(x_{i,0},\ \ x_{i,1},\ldots\ \ ,x_{i,n}\right),(i=0,1,\ldots,n).$$

In n+1-dimensional accompanying space, the value of mixed product on $\vec{L}_{D_0},\vec{L}_{D_1},\ldots\vec{L}_{D_n}$ is the value of determinant consisting of these vectors. It represents the volume of n+1-dimensional parallelotope. $\vec{L}_{D_0}$, $\vec{L}_{D_1}$, $\ldots\vec{L}_{D_n}$ are all same hyperplane if and only if the mixed product on $\vec{L}_{D_0},\vec{L}_{D_1},\ldots\vec{L}_{D_n}$ is equal to zero. Namely,

$$\left(\vec{L}_{D_0} \times \vec{L}_{D_1} \times \ldots \vec{L}_{D_{n-1}}\right) \cdot \vec{L}_{D_n} = \begin{vmatrix} l_0 d_{0,0} & l_0 d_{0,1} & \ldots & l_0 d_{0,n} \\ l_1 d_{1,0} & l_1 d_{1,1} & \ldots & l_1 d_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ l_n d_{n.0} & l_n d_{n,1} & \ldots & l_n d_{n,n} \end{vmatrix} = \begin{vmatrix} x_{0,0} & x_{0,1} & \ldots & x_{0,n} \\ x_{1,0} & x_{1,1} & \ldots & x_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n,0} & x_{n,1} & \ldots & x_{n,n} \end{vmatrix} = 0 \cdot$$

If n+1 points are all same hyperplane, then the mixed product on n+1 oriented vectors $\vec{L}_{D_0}, \vec{L}_{D_1}, \ldots \vec{L}_{D_n}$ in accompanying space is equal to zero. Vice versa, if the mixed product on n+1 oriented vectors $\vec{L}_{D_0}, \vec{L}_{D_1}, \ldots \vec{L}_{D_n}$ in accompanying space is equal to zero, then n+1 points are all same hyperplane.

$$D_0, D_1, \ldots D_n \text{ are coplane} \Leftrightarrow \begin{vmatrix} x_{0,0} & x_{0,1} & \ldots & x_{0,n} \\ x_{1,0} & x_{1,1} & \ldots & x_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n,0} & x_{n,1} & \ldots & x_{n,n} \end{vmatrix} = 0 \Leftrightarrow \left(\vec{L}_{D_0} \times \vec{L}_{D_1} \times \ldots \vec{L}_{D_{n-1}}\right) \cdot \vec{L}_{D_n} = 0$$

Q.E.D.

The theorem 3.2 is same as theorem 3.1 really. The theorem 3.1 deals with the problems from n-dimensional space. If n=2, theorem 3.1 deals with the problems from two-dimensional plane. The theorem 3.2 deals with the problems from n+1-dimensional accompanying space. If n=2, theorem 3.2 deals with the problems from three-dimensional space. The theorem 3.2 connects the points of n-dimensional space with the oriented vectors of n+1-dimensional accompanying space. The theorem 3.2 produces more associations between two spaces, provides us with wider eyeshot. The determinant consisting of oriented vectors represents the volume of n-dimensional parallelotope consisting of lines passing through origin. Because the n-linear coordinate is flexible, the volume can not be deduced by n-linear coordinate. Though we can not deduce the volume, the value of determinant is equal to zero represents that the volume is zero, namely, all points are same hyperplane. It represents the relation between point, line, plane when volume is equal to zero.

In Fig 1 (trilinear coordinate space), let trilinear coordinate $(x, \; y, \; z)$ be a moving point in line *DE*. If the trilinear coordinates of *D, E* are $D = (x_D, \; y_D, \; z_D), E = (x_E, \; y_E, \; z_E)$, then the line equation passing through *D, E* in trilinear coordinate form is

$$\begin{vmatrix} x & y & z \\ x_D & y_D & z_D \\ x_E & y_E & z_E \end{vmatrix} = 0 \cdot \quad \text{After reduction,} \quad \begin{vmatrix} y_D & z_D \\ y_E & z_E \end{vmatrix} x + \begin{vmatrix} z_D & x_D \\ z_E & x_E \end{vmatrix} y + \begin{vmatrix} x_D & y_D \\ x_E & y_E \end{vmatrix} z = 0 \cdot$$

In n-space, there are n-1 known points. In n-1 points, the n-linear coordinate of ith point is $X_i = (x_{i,0}, \quad x_{i,1}, \ldots \quad x_{i,n-1})$ $(i = 0,1,\ldots n-2)$. n-1 points are all same n-2-dimensional hyperplane. In this n-2-dimensional hyperplane there is a moving point $(x_0, \quad x_1, \ldots \quad x_{n-1})$. the hyperplane equation passing through n-1 known points in n-linear coordinate form is

$$\begin{vmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{0,0} & x_{0,1} & \cdots & x_{0,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,0} & x_{n-2,1} & \cdots & x_{n-2,n-1} \end{vmatrix} = 0$$

, After reduction, $\sum_{i=0}^{n-1} A_i x_i = 0$. A_i is algebra remain subdeterminant of x_i. Its expression with $X_0, X_1, \ldots X_{n-2}$ is

$$A_i = (-1)^i \begin{vmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,i-1} & x_{0,i+1} & \cdots & x_{0,n-1} \\ x_{0,1} & x_{1,1} & \cdots & x_{1,i-1} & x_{1,i+1} & \cdots & x_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-2,0} & .x_{n-2,1} & \cdots & x_{n-2,i-1} & x_{n-2,i+1} & \cdots & x_{n-2,n-1} \end{vmatrix}.$$

$$A_i = (-1)^{i(n-i)} \begin{vmatrix} x_{0,i\oplus 1} & x_{0,i\oplus 2} & \cdots & x_{0,i\oplus(n-1)} \\ x_{1,i\oplus 1} & x_{1,i\oplus 2} & \cdots & x_{1,i\oplus(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,i\oplus 1} & x_{n-2,i\oplus 2} & \cdots & .x_{n-2,i\oplus(n-1)} \end{vmatrix}.$$

The second subscript of elements in determinant includes addition by module n. If $(k+j) \le (n-1)$, then $k \oplus j = k + j$. If $(k+j) > (n-1)$, then $k \oplus j$ is equal to remainder of $(k+j)$ divided by n.

Definition 3.1 (The n-linear coordinate of hyperplane): In n-space (n-1-dimensional Euclidean space), there are n-1 known points, $D_0, D_1, \ldots, D_{n-2}$ .Their n-linear coordinates is $D_i = (x_{i,0}, \quad x_{i,1}, \quad \ldots \quad , x_{i,n-1})$, for $i = 0,1,\ldots,n-2$.

We call $D_0, D_1, \ldots, D_{n-2}$ as n-linear coordinate of 0-dimensional hyperplane passing through one point, namely, the n-linear coordinate of point. All are n-1. The vectors $\overrightarrow{L_{D_0}}, \overrightarrow{L_{D_1}}, ... \overrightarrow{L_{D_{n-2}}}$ are oriented vectors of corresponding accompanying space.

We call $(A_0, \ A_1, \ ... \ A_{n-1})$ as n-linear coordinate of 1-dimensional hyperplane passing through two points D_i, D_j, namely, the n-linear coordinate of line.

$$A_k = (-1)^{k(n-k)} \begin{vmatrix} x_{i,(n-2)\oplus k} & x_{i,(n-1)\oplus k} \\ x_{j,(n-2)\oplus k} & x_{j,(n-1)\oplus k} \end{vmatrix}, (k = 0,1,...,n-1).$$

All are C_{n-1}^2.The oriented vector is $(A_0, \ A_1, \ ... \ , A_{n-1})$ corresponding to the hyperplane passing through the origin in accompanying space.

We call $(A_0, A_1, ..., A_{n-1})$ as n-linear coordinate of 2-dimensianl hyperplane passing through three points D_i, D_j, D_m, namely, n-linear coordinate of plane.

$$A_k = (-1)^{k(n-k)} \begin{vmatrix} x_{i,(n-3)\oplus k} & x_{i,(n-2)\oplus k} & x_{i,(n-1)\oplus k} \\ x_{j,(n-3)\oplus k} & x_{j,(n-2)\oplus k} & x_{j,(n-1)\oplus k} \\ x_{m,(n-3)\oplus k} & x_{m,(n-2)\oplus k} & x_{m,(n-1)\oplus k} \end{vmatrix}, \ (k = 0,1,...,n-1).$$

All are C_{n-1}^3. The oriented vector is $(A_0, A_1, ..., A_{n-1})$ corresponding to the hyperplane passing through the origin in accompanying space.

…

We call the vector product of all n-1 points $D_0 \times D_1 \times ... \times D_{n-2} = (A_0, A_1, ..., A_{n-1})$ as n-linear coordinate of n-2 dimensional hyperplane passing through n-1 points. All is one. the vector product $\overrightarrow{L_{D_0}} \times \overrightarrow{L_{D_1}} ... \times \overrightarrow{L_{D_{n-2}}} = (A_0, A_1, ..., A_{n-1})$ is oriented vector corresponding to the hyperplane through the origin in accompanying space.

$$A_k = (-1)^{k(n-k)} \begin{vmatrix} x_{0,1\oplus k} & x_{0,2\oplus k} & \cdots & x_{0,(n-1)\oplus k} \\ x_{1,1\oplus k} & x_{1,2\oplus k} & \cdots & x_{1,(n-1)\oplus k} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,1\oplus k} & x_{n-2,2\oplus k} & \cdots & .x_{n-2,(n-1)\oplus k)} \end{vmatrix}, \ (k = 0,1,...,n-1).$$

If a point D is located in the hyperplane passing through the n-1 points, then $(D_0 \times D_1 \times ... D_{n-2}) \cdot D = 0$.

If there is not an explanation, in this paper the n-linear coordinate of hyperplane is n-linear coordinate of n-2 dimensional hyperplane passing through n-1 points in n-space.

In accompanying space, n-coordinate of hyperplane passing through the origin is normal vector of this hyperplane. Let m be a hyperplane of n-space. Let α_m be hyperplane passing through the origin of accompanying space corresponding to m. Then $m=\alpha_m=[A_0,A_2,...,A_{n-1}]$. We adopt the round brackets for n-coordinate of point, the square brackets for n-coordinate of hyperplane.

In the 2-dimensional plane (the trilinear coordinate plane), let X, Y be two points of a line m, $X=(x_1,y_1,z_1)$, $Y=(x_2,y_2,z_2)$. $X\times Y$ is the trilinear coordinate of this line connecting X, Y. In 3-dimensional accompanying space, the vector product is

$$\vec{L}_X\times\vec{L}_Y=\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2\end{vmatrix},\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2\end{vmatrix},\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}\right).$$

It presents normal vector of plane determined by two oriented vectors.

Therefore $m=\alpha_m=\left[\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2\end{vmatrix},\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2\end{vmatrix},\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}\right]$, $X\times Y=m$.

In n-space (n-1 dimensional Euclidean space), let n-coordinate of n-1 hyperplanes be $l_i=[x_{i,0},x_{i,1},...,x_{i,n-1}], i=0,1,...,n-2$ .Then the intersecting point of n-1 hyperplanes is $(x_0,x_1,...,x_{n-1})$, such that

$$\begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,n-1} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,0} & x_{n-2,1} & \cdots & x_{n-2,n-1}\end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ \cdots \\ x_{n-1}\end{pmatrix}=0, \text{ namely, } \frac{x_0}{A_0}=\frac{x_1}{A_1}=...=\frac{x_{n-1}}{A_{n-1}}.$$

$$A_i=(-1)^{i(n-i)}\begin{vmatrix} x_{0,i\oplus 1} & x_{0,i\oplus 2} & \cdots & x_{0,i\oplus n-1} \\ x_{1,i\oplus 1} & x_{1,i\oplus 2} & \cdots & x_{1,i\oplus n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,i\oplus 1} & x_{n-2,i\oplus 2} & \cdots & .x_{n-2,i\oplus n-1}\end{vmatrix}, \quad (i=0,1,...n-1).$$

$$(x_0,x_1,...x_{n-1})=(A_0,A_1,...A_{n-1}).$$

In two-dimensional plane, let trilinear coordinates of two lines be $l_1=[a_1,\ b_1,\ c_1], l_2=[a_2,\ b_2,\ c_2]$ respectively.

Then the intersecting point of two lines is the solution of $\begin{cases} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \end{cases}$,

namely,

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ (x,y,z)=\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix},\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right).$$

Definition 3.2 (The n-coordinate of intersecting point of n-1 hyperplanes): In n-space(n-1-dimensiona), let n-coordinates of n-1 hyperplanes (n-2-dimension) be $l_i=[x_{i,0},x_{i,1},...,x_{i,n-1}], i=0,1,...,n-2$ .Then n-coordinate of intersecting point of hyperplanes is $(A_0,A_1,...,A_{n-1})$.

$$A_i=(-1)^{i(n-i)}\begin{vmatrix} x_{0,i\oplus 1} & x_{0,i\oplus 2} & \cdots & x_{0,i\oplus n-1} \\ x_{1,i\oplus 1} & x_{1,i\oplus 2} & \cdots & x_{1,i\oplus n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,i\oplus 1} & x_{n-2,i\oplus 2} & \cdots & .x_{n-2,i\oplus n-1} \end{vmatrix},\ (i=0,1,...,n-1).$$

In trilinear coordinate plane (two-dimensional plane), the trilinear coordinate of intersecting point of lines l,m is $X=l\times m$.

If $l=[x_1,y_1,z_1], m=[x_2,y_2,z_2]$, then $X=\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix},\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix},\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$.

In accompanying space, it is really trilinear coordinate of intersecting line $\vec{L_X}$ by α_m, α_l.

Because $\vec{L_X}\in\alpha_m,\alpha_l$, so $\vec{L_X}\perp\vec{L_m}$, $\vec{L_X}\perp\vec{L_l}$.

So $\vec{L_m}\times\vec{L_l}=\vec{L_X}$, $\alpha_m\times\alpha_l=\vec{L_X}$, $l\times m=X$.

In two-dimensional plane, the vector product by two trilinear coordinates on two points is trilinear coordinate of line connecting two points. The vector product by two trilinear coordinates on two lines is trilinear coordinate of intersecting point by two lines.

4 The notations and relationship in n-linear coordinate system

For convenient description, the upper letters such as M represent a point. The lower letters such as m represent a line. The greek letters without subscript such as α represent the hyperplane. Two upper letters such as AD represent a line connecting two points. Several upper letters such as ABD represent the hyperplane connecting several points. The upper letters with arrow such as $\vec{L}_M$ represent oriented vector in accompanying space. The greek letters with subscript such as α_m represent the hyperplane passing through the origin in accompanying space. The n-coordinate of point and oriented vector in accompanying space are denoted by round brackets. The n-coordinate of hyperplane is denoted by square brackets.

Definition 4.1 (The n-coordinate of hyperplane passing through the origin in accompanying space): The oriented vector of a hyperplane in accompanying space is its normal vector. The map one by one between line passing through the origin and hyperplane passing through the origin in accompany space can be established $g:\alpha_m \leftrightarrow \vec{L}_M$, $\vec{L}_M$ is perpendicular to α_m . If $\vec{L}_M = (d_0, d_1, ..., d_{n-1})$,then $\alpha_m = [d_0, d_1, ..., d_{n-1}]$. $[d_0, d_1, ..., d_{n-1}]$ represent n-coordinate of α_m. The equation $\sum_{i=0}^{n-1} d_i x_i = 0$ expresses the hyperplane α_m in accompanying space.

Let X be a point in n-space. Let α be a hyperplane in n-space. It is evident that $X \in \alpha$ is true if and only if the scalar product $\vec{L}_X \cdot \vec{L}_\alpha = 0$. And $\vec{L}_X \perp \vec{L}_\alpha$.

Definition 4.2 (The accompanying relation, the accompanying point, the accompanying hyperplane): Let the hyperplane α in n-space be corresponding to α_m passing through the origin in accompanying space. Let the normal vector of α_m be $\vec{L}_M$. Let $\vec{L}_M$ in accompanying space be corresponding to point M in n-space. $M = \vec{L}_M = (d_0, d_1, ..., d_n)$, and $\alpha = \alpha_m = [d_0, d_1, ..., d_n]$. M is called as accompanying point of α . α is called as accompanying hyperplane of M. M and α become accompanying relation.

For example, in two-dimensional plane, the trilinear coordinate of vertex A is A=(1,0,0), and the trilinear coordinate of sideline BC is a=[1,0,0]. We call A as the accompanying point of a, call a as the accompanying line of A, A and a become accompanying relation. In accompanying space, $\vec{L}_A$ is OX axis, α_a is OYZ plane. AB=[0,0,1], AC=[0,1,0] are given. Because c, b intersect in A, so $A=b\times c$=[0,0,1]×[0,1,0]=(1,0,0).

Theorem 4.1 (The concurrent theorem of n hyperplanes): In n-space (n-1-dimensional Euclidean space), n hyperplanes are concurrent if and only if the mixed product of n-coordinates of n hyperplanes is equal to zero. In other words, the value of determinant consisting of n n-coordinates is equal to zero. (In two-dimensional plane: The three lines are concurrent, if and only if the value of determinant consisting of trilnear coordinates of three lines is equal to zero).

Proof: In n-space, that n points $X_0,X_1,\ldots,X_{n-1}$ are coplanar is equivalent with that n oriented vectors in accompanying space $\vec{L}_{X_0},\vec{L}_{X_1},\ldots,\vec{L}_{X_{n-1}}$ are coplanar. Let the hyperplane determined by the n oriented vectors in accompanying space be α_k .Its normal vector is $\vec{L}_K$. The hyperplane α in n-space corresponding to $\vec{L}_K$ is hyperplane determined by $X_0,X_1,\ldots,X_{n-1}$ in n-space .Its accompanying point is K.

$$\because\ \vec{L}_{X_0},\vec{L}_{X_1},\ldots,\vec{L}_{X_{n-1}}\in\alpha_k \qquad \therefore\ \vec{L}_{X_0},\vec{L}_{X_1},\ldots,\vec{L}_{X_{n-1}}\perp\vec{L}_K .$$

$$\because\ \vec{L}_{X_0}\perp\alpha_{x_0} \quad \therefore \vec{L}_K\in\alpha_{x_0} \qquad \therefore\ \vec{L}_K\in\alpha_{x_0},\alpha_{x_1},\ldots,\alpha_{x_{n-1}} .$$

In n-space ,n points X_0、X_1、…X_{n-1} are coplanar in $\alpha\Leftrightarrow$ the accompanying hyperplanes of n points X_0、X_1、…X_{n-1} are concurrent in K.

In n hyperplanes of n-space, the n-coordinate of ith hyperplane is

$$l_i=\left(l_{i,0},l_{i,1},\ldots,l_{i,n-1}\right),\ \left(i=0,1,\ldots,n-1\right).$$

The intersecting point is $\left(x_0,x_1,\ldots,x_{n-1}\right)$, then

$$\begin{pmatrix} l_{0,0} & l_{0,1} & \cdots & l_{0,n-1}\\ l_{1,0} & l_{1,1} & \cdots & l_{1,n-1}\\ \cdots & \cdots & \cdots & \cdots\\ l_{n-1,0} & l_{n-1,1} & \cdots & l_{n-1,n-1}\end{pmatrix}\begin{pmatrix} x_0\\ x_1\\ \cdots\\ x_{n-1}\end{pmatrix}=0 \text{. Only if } \begin{vmatrix} l_{0,0} & l_{0,1} & \cdots & l_{0,n-1}\\ l_{1,0} & l_{1,1} & \cdots & l_{1,n-1}\\ \cdots & \cdots & \cdots & \cdots\\ l_{n-1,0} & l_{n-1,1} & \cdots & l_{n-1,n-1}\end{vmatrix}=0,$$

the solution $\left(x_0,x_1,\ldots,x_{n-1}\right)$ is non-zero. Q.E.D.

5 The applications of trilinear coordinate in plane geometry

In this section,the triangle denotes with $\triangle ABC$. and three angles denote with A,B,C. The corresponding sidelines denote with a,b,c. R is radius of circumcircle of $\triangle ABC$. r is radius of inscribed circle. G,O,H,I are centroid,circumcenter,orthocenter, incenter respectively.

Example 1. In $\triangle ABC$, O is circumcenter, G is centroid, H is orthocenter. Find the trilinear coordinate of line connecting O,H. Prove that O ,G, H are collinear (Euler line theorem).

Proof: Constructs trilinear coordinate system based on $\triangle ABC$. The trilinear coordinates of points O, G, H are respectively

$$O=(\cos A,\quad \cos B,\quad \cos C),\ G=\left(\frac{1}{\sin A},\frac{1}{\sin B},\frac{1}{\sin C}\right),H=\left(\frac{1}{\cos A},\frac{1}{\cos B},\frac{1}{\cos C}\right).$$

Let m be trilinear coordinate of line connecting O,H, then

$$m=O\times H=\left[\begin{vmatrix}\cos B & \cos C\\ \frac{1}{\cos B} & \frac{1}{\cos C}\end{vmatrix},\begin{vmatrix}\cos C & \cos A\\ \frac{1}{\cos C} & \frac{1}{\cos A}\end{vmatrix},\begin{vmatrix}\cos A & \cos B\\ \frac{1}{\cos A} & \frac{1}{\cos B}\end{vmatrix}\right]$$

$$=\left[\frac{\cos^2 B-\cos^2 C}{\cos B\cos C},\frac{\cos^2 C-\cos^2 A}{\cos C\cos A},\frac{\cos^2 A-\cos^2 B}{\cos A\cos B}\right]$$

$$=\left[\frac{\sin A\cos A\sin(C-B)}{\cos A\cos B\cos C},\frac{\sin B\cos B\sin(A-C)}{\cos A\cos B\cos C},\frac{\sin C\cos C\sin(B-A)}{\cos A\cos B\cos C}\right]$$

$$=[\sin 2A\sin(C-B),\quad \sin 2B\sin(A-C),\quad \sin 2C\sin(B-A)].$$

$$\left(\vec{L}_O\times\vec{L}_G\right)\cdot\vec{L}_H=\begin{vmatrix}\cos A & \cos B & \cos C\\ \frac{1}{\sin A} & \frac{1}{\sin B} & \frac{1}{\sin C}\\ \frac{1}{\cos A} & \frac{1}{\cos B} & \frac{1}{\cos C}\end{vmatrix}=\sum\cos A\left(\frac{1}{\sin B\cos C}-\frac{1}{\cos B\sin C}\right)$$

$$=\sum\cos A\left(\frac{\sin C\cos B-\cos C\sin B}{\sin B\sin C\cos B\cos C}\right)=\sum\cos A\left(\frac{\sin(C-B)}{\sin B\sin C\cos B\cos C}\right)$$

$$=\sum\frac{1}{2}\left(\frac{\sin 2B-\sin 2C}{\sin B\sin C\cos B\cos C}\right)=\sum\left(\frac{1}{\sin C\cos C}-\frac{1}{\sin B\cos B}\right)=0.$$

Therefore G, O, H are collinear. Q.E.D.

Example 2. Prove Menelaus' theorem. As shown In Fig 4, D, E ,F are collinear if and only if $\frac{BF\cdot AE\cdot CD}{FA\cdot EC\cdot DB}=1$.

Proof: Constructs trilinear coordinate system based on $\triangle ABC$. The trilinear coordinates of points *D,E,F* are respectively $F=(BF\sin B,\quad FA\sin A,\quad 0)$, $E=(EC\sin C,\quad 0,\quad AE\sin A)$, $D=(0,\quad -CD\sin C,\quad DB\sin B)$.

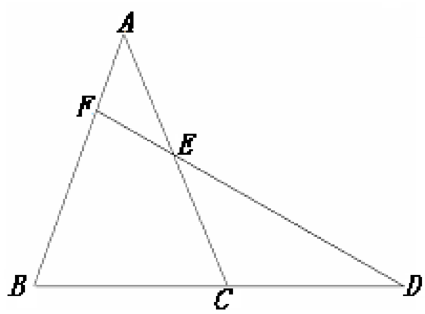

Fig 4

$$\left(\vec{L}_D\times\vec{L}_E\right)\cdot\vec{L}_F=\begin{vmatrix} BF\sin B & FA\sin A & 0 \\ EC\sin C & 0 & AE\sin A \\ 0 & -CD\sin C & DB\sin B \end{vmatrix}$$

$=BF\cdot AE\cdot CD\sin A\sin B\sin C-DB\cdot EC\cdot FA\sin A\sin B\sin C$.

D, E, F are collinear

$\Leftrightarrow \sin A\sin B\sin C(BF\cdot AE\cdot CD-DB\cdot EC\cdot FA)=0\Leftrightarrow\dfrac{BF\cdot AE\cdot CD}{FA\cdot EC\cdot DB}=1$.   Q.E.D.

Example 3. Prove Ceva's theorem. As shown In Fig 5, in $\triangle ABC$, *K,P,Q* are respectively points on sidelines *BC,CA* and *AB* or on their extend line. *AK,BP, CQ* are concurrent if and only if $\dfrac{BK}{KC}\cdot\dfrac{CP}{PA}\cdot\dfrac{AQ}{QB}=1$.(angle form: $\dfrac{Sin\alpha Sin\gamma Sin\lambda}{Sin\beta Sin\theta Sin\omega}=1$).

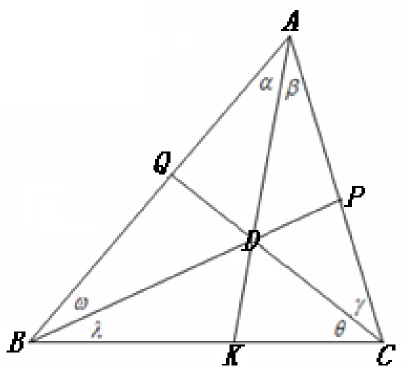

Fig 5

Proof: Constructs trilinear coordinate system based on $\triangle ABC$.

1 angle form

$K=(0,\quad \text{Sin}\beta,\quad \text{Sin}\alpha)$, $A=$ (1,0,0) .

$AK=A\times K=(1,0,0)\times(0,\quad \sin\beta,\quad \sin\alpha)=[0,\quad \sin\alpha,\quad -\sin\beta]$.

similarly $BP=[\sin\omega,\quad 0,\quad -\sin\lambda]$, $CQ=[\sin\gamma,\quad -\sin\theta,\quad 0]$.

The mixed product of *AK, BP, CQ is*

$$(AK\times BP)\cdot CQ=\begin{vmatrix} 0 & \sin\alpha & -\sin\beta \\ \sin\omega & 0 & -\sin\lambda \\ \sin\gamma & -\sin\theta & 0 \end{vmatrix}=\sin\omega\sin\theta\sin\beta-\sin\alpha\sin\gamma\sin\lambda .$$

Then *AK, BP, CQ* are concurrent $\Leftrightarrow (AK\times BP)\cdot CQ=0\Leftrightarrow\dfrac{\sin\alpha\sin\gamma\sin\lambda}{\sin\beta\sin\theta\sin\omega}=1$

2 side form

$K=(0,\quad KC\sin C,\quad BK\sin B)$

$$AK = A\times K = (1,0,0)\times\left(0,\quad KC\sin C,\quad BK\sin B\right)=\left[0,\quad -BK\sin B,\quad KC\sin C\right].$$

Similarly $BP=\left[PA\sin A,\quad 0,\quad -CP\sin C\right]$, $CQ=\left[AQ\sin A,\quad -QB\sin B,\quad 0\right]$.

The mixed product of AK、BP、CQ *is*

$$\left(AK\times BP\right)\cdot CQ=\begin{vmatrix} 0 & -BK\sin B & KC\sin C \\ PA\sin A & 0 & -CP\sin C \\ AQ\sin A & -QB\sin B & 0 \end{vmatrix}$$

$$=\sin A\sin B\sin C\left(AQ\cdot BK\cdot CP - PA\cdot QB\cdot KC\right).$$

AK, BP, CQ are concurrent $\Leftrightarrow \dfrac{AQ\cdot BK\cdot CP}{AP\cdot BQ\cdot CK}=1$. Q.E.D.

Example 4. Prove Desargues' theorem. As shown In Fig 6,there are three projecting lines from *K, A* and *D, B* and *E, C* and *F.* Lines *BC* and *EF* are intersecting on *O.* Lines *AC* and *DF* are intersecting on *N.* Lines *AB* and *DE* are intersecting on *M.* Prove that points *M,N,O* are collinear.

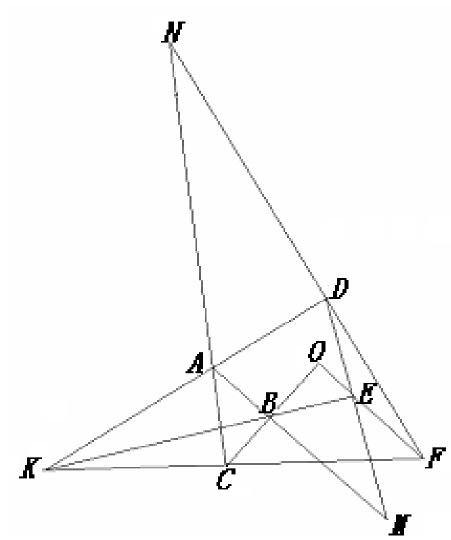

Fig 6

Proof: Constructs trilinear coordinate system based on $\triangle ABC$. Suppose the trilinear coordinate of K is $\left(x_k,\quad y_k,\quad z_k\right)$.

Because D is on AK, then $D=\left(x,\quad y_k,\quad z_k\right)$.

Similarly $E=\left(x_k,\quad y,\quad z_k\right), F=\left(x_k,\quad y_k,\quad z\right)$. $AB=\left[0,\quad 0,\quad 1\right]$.

$$DE=D\times E=\left(x,\quad y_k,\quad z_k\right)\times\left(x_k,\quad y,\quad z_k\right)=\left[y_k z_k - y z_k,\quad x_k z_k - x z_k,\quad xy - x_k y_k\right].$$

AB and *DE* are intersecting on *M.*

$$M = AB\times DE=\left[0,\quad 0,\quad 1\right]\times\left[y_k z_k - y z_k,\quad x_k z_k - x z_k,\quad xy - x_k y_k\right]$$

$$=\left(xz_k - x_k z_k,\quad y_k z_k - y z_k,\quad 0\right)=\left((x_k - x),\quad -(y_k - y),\quad 0\right).$$

Similarly $N=\left((x_k - x),\quad 0,\quad -(z_k - z)\right)$, $O=\left(0,\quad (y_k - y),\quad -(z_k - z)\right)$.

$$\left(\vec{L}_M\times\vec{L}_N\right)\cdot\vec{L}_O=$$

$$\begin{vmatrix} (x_k - x) & -(y_k - y) & 0 \\ (x_k - x) & 0 & -(z_k - z) \\ 0 & (y_k - y) & -(z_k - z) \end{vmatrix} = (x_k - x)(y_k - y)(z_k - z) - (x_k - x)(y_k - y)(z_k - z) = 0\cdot$$

Therefore *M, N, O* are collinear. Q.E.D.

Example 5. Construct trilinear coordinate system based on $\triangle ABC$. *K* is a point in trilinear coordinate plane. Suppose the trilinear coordinate of *K* is (x_k, y_k, z_k), then

K is on circumcircle of $\triangle ABC$ if and only if $\frac{a}{x_k}+\frac{b}{y_k}+\frac{c}{z_k}=0$.

Here x_k, y_k, z_k are not equal to zero (If *K* is one of *A,B,C*, x_k, y_k, z_k are equal to zero).

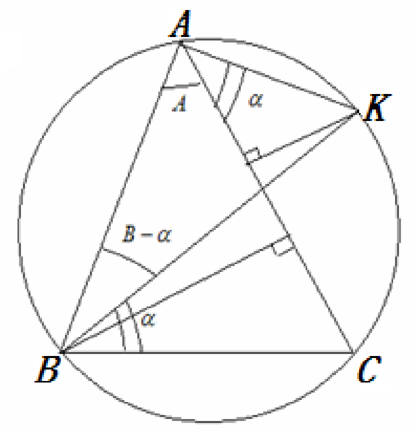

Fig 7

Proof: necessity

As shown in Fig 7, ∠KBC=α. $BK = 2R\sin(A+\alpha)$.

$x_k = BK\sin\alpha = 2R\sin(A+\alpha)\sin\alpha$.

$y_k = -AK\sin\alpha = -2R\sin(B-\alpha)\sin\alpha$.

$z_k = BK\sin(B-\alpha) = 2R\sin(A+\alpha)\sin(B-\alpha)$.

$K=\left(2R\sin(A+\alpha)\sin\alpha,\ \ -2R\sin\alpha\sin(B-\alpha),\ \ 2\mathrm{R}\sin(B-\alpha)\sin(A+\alpha)\right)$

$=\left(\frac{1}{\sin(B-\alpha)},\ \ -\frac{1}{\sin(A+\alpha)},\ \ \frac{1}{\sin\alpha}\right)$.

Then $\frac{a}{x_k}+\frac{b}{y_k}+\frac{c}{z_k}=\sin A\sin(B-\alpha)-\sin B\sin(A+\alpha)+\sin C\sin\alpha$

$=\frac{1}{2}\left(\cos(A-B+\alpha)-\cos(A+B-\alpha)-\cos(A-B+\alpha)+\cos(A+B+\alpha)+\cos(C-\alpha)-\cos(C+\alpha)\right)=0$.

Sufficiency

The distances between K and three sidelines:

∠KBC=α, $d_{KBC}=KB\sin\alpha$, $d_{KAB}=KB\sin(B-\alpha)$.

$$d_{KCA}=\frac{2S-ad_{KBC}-cd_{KAB}}{b}=\frac{4R^2\sin A\sin B\sin C-2R\cdot KB(\sin A\sin\alpha+\sin C\sin(B-\alpha))}{2R\sin B}.$$

The area of triangle is $S=2R^2\sin A\sin B\sin C$.

Because $\frac{a}{x_k}+\frac{b}{y_k}+\frac{c}{z_k}$=0, then

$$\frac{2R\sin A}{KB\sin\alpha}+\frac{2R\sin C}{KB\sin(B-\alpha)}+\frac{4R^2\sin^2 B}{4R^2\sin A\sin B\sin C-2R\cdot KB(\sin A\sin\alpha+\sin C\sin(B-\alpha))}=0.$$

After reduction, $KB = 2R\sin(A+\alpha)$. Therefore K is on circumcircle of $\triangle ABC$.

Q.E.D.

Example 6. Prove Pascal's theorem. The three intersecting points from three-pair(opposite) sidelines of hexagon inscribed in a circle are collinear.

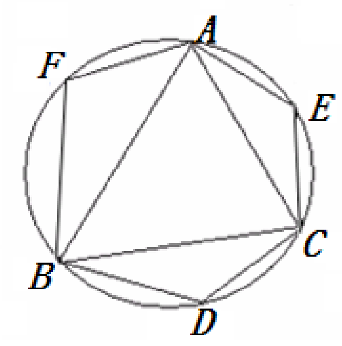

Fig 8

Proof: As shown in Fig 8, Constructs trilinear coordinate system based on $\triangle ABC$.

$$F=(x_F,\ \ y_F,\ \ z_F),B=(0,\ \ 1,\ \ 0).$$

$$FB = F\times B = [-z_F,\ \ 0,\ \ x_F].$$

Similarly $EC = E\times C = [y_E,\ \ -x_E,\ \ 0]$.

Suppose P is intersecting point of FB, EC, then

$$P = FB\times EC = [-z_F,\ \ 0,\ \ x_F]\times[y_E,\ \ -x_E,\ \ 0]$$

$$=(x_E x_F,\ \ x_F y_E,\ \ z_E x_E)=\left(1,\ \ \frac{y_E}{x_E},\ \ \frac{z_F}{x_F}\right)\ .$$

Similarly, Suppose Q is intersecting point of CD, AF, $Q=\left(\frac{x_D}{y_D},\ \ 1,\ \ \frac{z_F}{y_F}\right)$. Suppose

R is intersecting point of AE,BD, $R=\left(\frac{x_D}{z_D},\ \ \frac{y_E}{z_E},\ \ 1\right)$.

$$\left(\vec{L}_P\times\vec{L}_Q\right)\cdot\vec{L}_R=\begin{vmatrix}1 & \frac{y_E}{x_E} & \frac{z_F}{x_F}\\ \frac{x_D}{y_D} & 1 & \frac{z_F}{y_F}\\ \frac{x_D}{z_D} & \frac{y_E}{z_E} & 1\end{vmatrix}=\frac{x_D y_E z_F}{abc}\begin{vmatrix}\frac{a}{x_D} & \frac{b}{y_D} & \frac{c}{z_D}\\ \frac{a}{x_E} & \frac{b}{y_E} & \frac{c}{z_E}\\ \frac{a}{x_F} & \frac{b}{y_F} & \frac{c}{z_F}\end{vmatrix}=\frac{x_D y_E z_F}{abc}\begin{vmatrix}\frac{a}{x_D}+\frac{b}{y_D}+\frac{c}{z_D} & \frac{b}{y_D} & \frac{c}{z_D}\\ \frac{a}{x_E}+\frac{b}{y_E}+\frac{c}{z_E} & \frac{b}{y_E} & \frac{c}{z_E}\\ \frac{a}{x_F}+\frac{b}{y_F}+\frac{c}{z_F} & \frac{b}{y_F} & \frac{c}{z_F}\end{vmatrix}.$$

According to example 5, because D, E, F are all on circumcircle of $\triangle ABC$, so $\frac{a}{x_D}+\frac{b}{y_D}+\frac{c}{z_D}=0$, $\frac{a}{x_E}+\frac{b}{y_E}+\frac{c}{z_E}=0$, $\frac{a}{x_F}+\frac{b}{y_F}+\frac{c}{z_F}=0$ .Then $\left(\frac{1}{x_E},\ \ \frac{1}{y_E},\ \ \frac{1}{z_E}\right)$, $\left(\frac{1}{x_F},\ \ \frac{1}{y_F},\ \ \frac{1}{z_F}\right)$,$\left(\frac{1}{x_D},\ \ \frac{1}{y_D},\ \ \frac{1}{z_D}\right)$ are all on plane $ax+by+cz=0$ of accompanying space.

Therefore $\left(\vec{L_P}\times\vec{L_Q}\right)\cdot\vec{L_R}=\begin{vmatrix}0 & \frac{b}{y_D} & \frac{c}{z_D}\\ 0 & \frac{b}{y_E} & \frac{c}{z_E}\\ 0 & \frac{b}{y_F} & \frac{c}{z_F}\end{vmatrix}=0$.

P, Q, R are collinear. Q.E.D.

Example 7. As shown in Fig 9, the triangle consisting of three tangent lines on three points *A,B,C* of circumcircle of $\triangle ABC$ is $\triangle DEF$. Prove that *AD,BE,CF* are concurrent. We call this point as tripod center W. The trilinear coordinate of W is

$$W=\vec{L_W}=(\sin A,\ \sin B,\ \sin C)=(a,\ b,\ c).$$

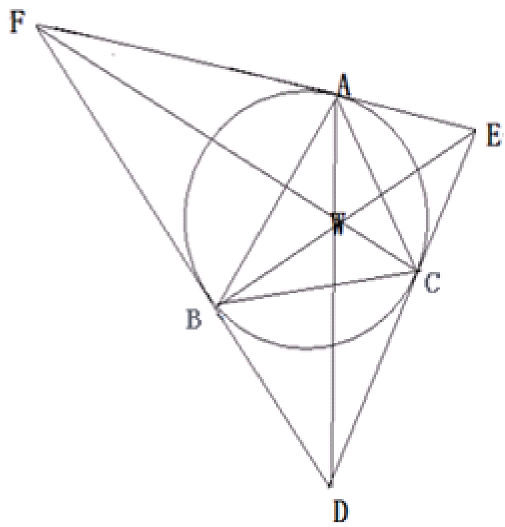

Fig 9

Proof: Because *AF=BF*, so the ratio of distances from *F* to *AC,CB,* $\sin\angle CAF:\sin\angle CBF=\sin B:\sin A$. All points on line *CF* can be expressed as $F_1=(\sin A,\sin B,z)$. Similarly, all points on line *AD* can be expressed as

$D_1=(x,\sin B,\sin C)$. All points on line *BE* can be expressed as $E_1=(\sin A,y,\sin C)$.

$A=(1,\ 0,\ 0), B=(0,\ 1,\ 0), C=(0,\ 0,\ 1)$.

$$AD_1=A\times D_1=\left[\begin{vmatrix}0 & 0\\ \sin B & \sin C\end{vmatrix},\ \begin{vmatrix}0 & 1\\ \sin C & x\end{vmatrix},\ \begin{vmatrix}1 & 0\\ x & \sin B\end{vmatrix}\right]=[0,\ -\sin C,\ \sin B].$$

$$CF_1=C\times F_1=\left[\begin{vmatrix}0 & 1\\ \sin B & z\end{vmatrix},\ \begin{vmatrix}1 & 0\\ z & \sin A\end{vmatrix},\ \begin{vmatrix}0 & 0\\ \sin A & \sin B\end{vmatrix}\right]=[-\sin B,\ \sin A,\ 0].$$

$$W=AD_1\times CF_1=[0,-\sin C,\sin B]\times[-\sin B,\ \sin A,\ 0]=(\sin A,\ \sin B,\ \sin C).$$

Similarly, $AD_1\times BE_1=(\sin A,\ \sin B,\ \sin C)=W$. So *AD, CF, BE* are concurrent in *W*. Q.E.D.

Definition 5.1 (The tripod center): We call $W=\vec{L_W}=(\sin A,\sin B,\sin C)=(a,b,c)$ as the tripod center in the trilinear coordinate plane.

Definition 5.2 (The horizontal line and horizontal plane): All points of accompanying line *w* of the tripod center W are on infinity far. Because α_w is perpendicular to $\vec{L_W}$ in accompanying space, the plane expressed by α_w is $ax+by+cz=0$ .All points (x,y,z) of α_w are satisfied by $ax+by+cz=0$ .The distance from K to BC is $d_{KD}=\dfrac{2xs}{ax+by+cz}$ .When $ax+by+cz=0$, d_{KD} is not significant. So all oriented vectors on α_w have not corresponding points in trilinear plane. We assume every point of w is on infinity far in trilinear plane. So the line expressed by w is a line in infinity far, we call visually w as horizontal line, call α_w as horizontal plane.

Constructs trilinear coordinate system based on regular $\triangle ABC$. $a=b=c$, $-2a+b+c=0$ .Then point K=(−2,1,1) is on infinity far.

Let $m=[x_M,\ y_M,\ z_M]$, $n=[x_N,\ y_N,\ z_N]$ be lines on trilinear plane. Let α_m, α_n be corresponding planes passing through the origin in accompanying space. The intersecting line $\vec{L_K}$ of α_m and α_n passing through the origin is exist certainly. The point K is the intersecting point of *m, n* in trilinear plane correspondingly. If m, n are parallel, they have not the intersecting point in trilinear plane. But in accompanying space, α_m and α_n all are passing through the origin, the intersecting line passing through the origin is exist certainly .We assume that the intersecting point is on infinity far when *m ,n* are parallel, namely, the intersecting point is on *w*. So, if the intersecting line $\vec{L_K}$ of α_m, α_n is on α_w, then *m, n* are parallel.

Theorem 5.1 (the theorem of parallel lines): *m, n* are lines in trilinear plane. *m, n* are parallel if and only if the mixed product of *m, n, w* is zero.

Proof: Suppose $m\times n$=*K*. If *m, n* are parallel, then the intersecting point is on infinity far, The mixed product $(m\times n)\cdot w$=0, and $\vec{L_K}\cdot\vec{L_w}=0$ .Then $\vec{L_K}\in\alpha_w, K\in w$. If the intersecting point of m, n is on infinity far, then m,n are parallel. If *m, n* are parallel, namely, the accompanying points *M,N,W* are collinear, then the mixed product of *m,n,w* is zero, namely, $(m\times n)\cdot w$=0. Q.E.D.

Definition 5.3 (The oriented line): If m, n are parallel, the angles between *m, n* and sideline BC respectively are same in trilinear plane. The intersecting lines between α_m, α_n and horizontal plane α_w are same in accompanying space. If the line m in trilinear plane is corresponding with the plane α_m passing through the origin in accompanying space, the intersecting line of α_m and α_w is line w_m, we call w_m as the oriented line of m. $w_m = m \times w$.

w_m is a line passing through the origin in accompanying space, and it is on α_w. Therefore w_m have not corresponding point in trilinear plane (on infinity far), the line direction of m in trilinear plane is dependent with w_m.

Theorem 5.2 (oriented line theorem): Given $w_m = [x_0, y_0, z_0]$, then the slope that m respect to sideline BC of $\triangle ABC$ is $\dfrac{x_0 \sin B}{z_0 + x_0 \cos B}$ (tangent of angle).

Proof: As shown in Fig 10, m is a line in trilinear plane. Draws BK parallel to m. If $K = (x, y, z)$, $\angle KBD = \alpha$,then the slope of m respect to sideline BC is the slope of line BK respect to sideline BC. The trilinear coordinate of BK is

$$BK = B \times K = (0,1,0) \times (x, y, z) = [z, 0, -x].$$

According to theorem 5.1, $(m \times w) \cdot BK = 0$,and $m \times w = w_m$.

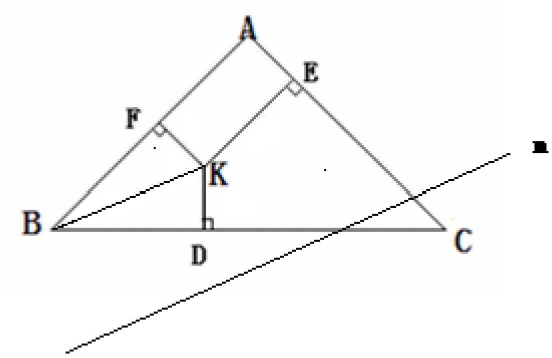

Fig 10

Namely, $[x_0, y_0, z_0] \cdot [z, 0, -x] = 0$, $x_0 z - x z_0 = 0$.

$$\frac{z_0}{x_0} = \frac{z}{x} = \frac{\sin(B-\alpha)}{\sin \alpha} = \sin B \cot \alpha - \cos B.$$

$$\tan \alpha = \frac{x_0 \sin B}{z_0 + x_0 \cos B}. \qquad \text{Q.E.D.}$$

According to symmetry, the slopes of m respect to sideline CA, sideline AB of $\triangle ABC$ respectively are

$$\tan\beta = \frac{y_0 \sin C}{x_0 + y_0 \cos C}, \quad \tan\gamma = \frac{z_0 \sin A}{y_0 + z_0 \cos A}.$$

Theorem 5.3 (Theorem of the angle between line and line): If $w_m = [x_1, y_1, z_1]$, $w_n = [x_2, y_2, z_2]$, the tangent of angle between *m* and *n* is

$$\frac{(x_1 z_2 - x_2 z_1)\sin B}{x_1 x_2 + z_1 z_2 + (x_1 z_2 + x_2 z_1)\cos B}.$$

Specially *m, n* are parallel $\Leftrightarrow x_1 z_2 - x_2 z_1 = 0$,

m, n are perpendicular $\Leftrightarrow x_1 x_2 + z_1 z_2 + (x_1 z_2 + x_2 z_1)\cos B = 0$.

Proof: Let α be angle between m and sideline *BC*. Let β be angle between *n* and sideline *BC*. The angle between m and n is $\alpha - \beta$.

So $\tan\alpha = \dfrac{x_1 \sin B}{z_1 + x_1 \cos B}$, $\tan\beta = \dfrac{x_2 \sin B}{z_2 + x_2 \cos B}$.

Then $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\beta \cdot \tan\alpha} = \dfrac{(x_1 z_2 - x_2 z_1)\sin B}{x_1 x_2 + z_1 z_2 + (x_1 z_2 + x_2 z_1)\cos B}$.

Therefore *m, n* are parallel $\Leftrightarrow x_1 z_2 - x_2 z_1 = 0$.

m, n are perpendicular $\Leftrightarrow x_1 x_2 + z_1 z_2 + (x_1 z_2 + x_2 z_1)\cos B = 0$.

Q.E.D.

According to symmetry , we can derive easily,

$$\tan(\alpha - \beta) = \frac{(y_1 x_2 - y_2 x_1)\sin C}{y_1 y_2 + x_1 x_2 + (y_1 x_2 + y_2 x_1)\cos C} = \frac{(z_1 y_2 - z_2 y_1)\sin A}{z_1 z_2 + y_1 y_2 + (z_1 y_2 + z_2 y_1)\cos A}.$$

m, n are parallel $\Leftrightarrow y_1 x_2 - y_2 x_1 = 0$.

m,n are perpendicular $\Leftrightarrow y_1 y_2 + x_1 x_2 + (y_1 x_2 + y_2 x_1)\cos C = 0$.

m, n are parallel $\Leftrightarrow z_1 y_2 - z_2 y_1 = 0$.

m, n are perpendicular $\Leftrightarrow z_1 z_2 + y_1 y_2 + (z_1 y_2 + z_2 y_1)\cos A = 0$.

Example 8. (Rumania Olympic test question in 2005) Let *R* be radius of circumcircle of $\triangle ABC$, O is center. Let r be radius of inscribed circle of $\triangle ABC$, I is center, and $I\neq O$. G is centriod. Prove: $IG\perp BC$ if and only if b=c or b+c=3a.

Proof: Constructs trilinear coordinate system based on $\triangle ABC$, centriod, $G=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$, incenter, I=(1,1,1）,tripod center, $W=(a,\ b,\ c)$, sideline BC, BC=[1,0,0].

$$IG = I\times G = \left[\begin{vmatrix}1 & 1\\ \frac{1}{b} & \frac{1}{c}\end{vmatrix},\begin{vmatrix}1 & 1\\ \frac{1}{c} & \frac{1}{a}\end{vmatrix},\begin{vmatrix}1 & 1\\ \frac{1}{a} & \frac{1}{b}\end{vmatrix}\right] = \left[\frac{1}{c}-\frac{1}{b},\ \ \frac{1}{a}-\frac{1}{c},\ \ \frac{1}{b}-\frac{1}{a}\right].$$

Oriented lines:

$$w_{BC} = \left[\begin{vmatrix}0 & 0\\ b & c\end{vmatrix},\ \begin{vmatrix}0 & 1\\ c & a\end{vmatrix},\ \begin{vmatrix}1 & 0\\ a & b\end{vmatrix}\right] = [0,\ \ -c,\ \ b].$$

$$w_{IG} = \left[\begin{vmatrix}\frac{1}{a}-\frac{1}{c} & \frac{1}{b}-\frac{1}{a}\\ b & c\end{vmatrix},\ \begin{vmatrix}\frac{1}{b}-\frac{1}{a} & \frac{1}{c}-\frac{1}{b}\\ c & a\end{vmatrix},\ \begin{vmatrix}\frac{1}{c}-\frac{1}{b} & \frac{1}{a}-\frac{1}{c}\\ a & b\end{vmatrix}\right] = \left[2-\frac{b+c}{a},\ \ 2-\frac{c+a}{b},\ \ 2-\frac{a+b}{c}\right]$$

$$IG\perp BC \Leftrightarrow x_1x_2+z_1z_2+(x_1z_2+x_2z_1)\cos B=0 \quad \text{(reduced after substitution)}$$

$$\Leftrightarrow z_2+x_2\cos B=0 \Leftrightarrow (2-\frac{a+b}{c})+(2-\frac{b+c}{a})\cos B=0$$

$$\Leftrightarrow \frac{a(2c-a-b)}{b+c-2a}=\frac{a^2+c^2-b^2}{2a} \qquad \text{(after substitution } \cos B=\frac{a^2+c^2-b^2}{2ac}\text{)}$$

$$\Leftrightarrow b^3-c^3+2ac^2-2ab^2+3a^2c-3a^2b+cb^2-bc^2=0$$

$$\Leftrightarrow (b-c)(b+c-3a)(a+b+c)=0 \Leftrightarrow b=c \ \text{ or } \ b+c=3a.\ \text{Q.E.D.}$$

Remark: The author does many exercises of geometrical question in process of studied trilinear coordinate system. Because the limitation of space, there are no its lists. Here are only a few famous questions in mathematical history and Olympic question. The common method (geometrical) for Euler line question is: Connects AH. Draws OD perpendicular to BC. It is proved by proving $\triangle AGH$ resemble with $\triangle DGO$. The common method for Menelaus theorem is: Draws parallel line with DEF passing through A. It is proved by resembling ratio. The common method for Desargues theorem is that Menelaus' theorem and Ceva theorem are applied in many times. It is very complicated, and it lacks of aesthetic feeling. These methods break the balance and symmetry of *A,B,C,a,b,c*. The advantage of triline coordinate system in plane is that we not need auxiliary lines nearly to solve geometrical problem. In form, the expressions of *A, B, C, a, b, c* are balanced, symmetrical, beautiful.

The trilinear coordinate system provides us new viewpoint to deal with geometrical problems from trilinear coordinate ,accompanying space. Puts planar problems into space. Solves planar problems in space. The trilinear coordinate system provides us new viewpoint to view the relationship between point. Line, plane. If we are experienced in applied this kind of method, many problems can be solved for very brief and convenience.

6 The applications of n-linear coordinate system in multi-variables regression analysis

In statistics, the multi-regression analysis is often made with the help of computer. For example, The synthetic makings of students in middle school are related with mark of moral education ,physical education , and all courses. Its regression expression is

$x_{n-1} = a_0x_0 + a_1x_1 + ... + a_{n-2}x_{n-2}$, or $x_{n-1} = \sum_{p=0}^{n-2} a_p x_p$. ($x_{n-1}$ is mark of synthetic makings, $x_0, x_1, ..., x_{n-2}$ are mark of moral education, physical education, and all courses.)

In multi-regression analysis, the least square method is common method.

Here takes method of n-linear coordinate system.

There are m sampled students in the class. There are n examining indexes to every student. The value of full mark is 100, m>n. In multi-regression analysis, m students are considered as m points in n-space, every mark is considered as one of n-linear coordinates.

At first, students are divided into some groups. There are n students in every group. According to combinatorics，all are C_m^n groups. Every student is in C_{m-1}^{n-1} groups. Let $k = C_m^n$.

To jth group, $A_{j,0}x_0 + A_{j,1}x_1 + ... + A_{j,n-2}x_{n-2} + A_{j,n-1}x_{n-1} = 0$, or $\sum_{p=0}^{n-1} A_{j,p}x_p = 0$.

$$A_{j,i} = (-1)^{i(n-i)} \begin{vmatrix} x_{0,1\oplus i} & x_{0,2\oplus i} & \cdots & x_{0,(n-1)\oplus i} \\ x_{1,1\oplus i} & x_{1,2\oplus i} & \cdots & x_{1,(n-1)\oplus i} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-2,1\oplus i} & x_{n-2,2\oplus i} & \cdots & x_{n-2,(n-1)\oplus i)} \end{vmatrix}, (j = 0,1,...,k-1; i = 0,1,...,n-1).$$

$A_{j,0}, A_{j,1}, ..., A_{j,n-2}, A_{j,n-1}$ are n-linear coordinates of n-1-dimensional hyperplane.

The mean values $b_i = \dfrac{\sum_{j=0}^{k-1} A_{j,i}}{k}$, $i = 0,1,...,n-1$.

$b_0x_0 + b_1x_1 + ... + b_{n-2}x_{n-2} + b_{n-1}x_{n-1} = 0$, or $\sum_{p=0}^{n-1} b_p x_p = 0$.

Let $a_i = -\dfrac{b_i}{b_{n-1}}$, $i = 0,1,...,n-2$.

then $x_{n-1} = a_0x_0 + a_1x_1 + ... + a_{n-2}x_{n-2}$, or $x_{n-1} = \sum_{p=0}^{n-2} a_p x_p$.

Here $k = C_m^n$. If k is very great, the computational load of $A_{j,i}$ is very great for $j = 0,1,...,k-1; i = 0,1,...,n-1$.So the computational method must be modified.

If m is divided exactly by n, then let $k = \dfrac{m}{n}$. If m is not divided exactly by n, then let $k = \left[\dfrac{m}{n}\right] + 1$. The last group have not n students (suppose q students in last group, q is equal to remainder of m divided by n, q<n). All are k groups. Every student is only in one group. To frontal k-1 groups, n-linear coordinate of n-1 dimensional hyperplane is computed. To last group, n-linear coordinate of q-1-dimensional hyperplane is computed. In grouping, k groups can be arranged at random. can be also arranged according to level principle.

7 The further problems to study and conjecture

To the system of n-linear coordinate, the study starts justly .There are much work to do.

The followings are some problems in the system of n-linear coordinate that is considered as further studies by the author.

One. Applies n-linear coordinate system into computational geometry. There are many relative problems on point, line and plane, such as, how to ascertain two line segments are intersecting? how to ascertain a point is in a line? how to ascertain a line is in polygon? and so on.

Conjecture: After applying n-linear coordinate system, the arithmetic method in computational geometry can be reduced.

Two. Applies n-linear coordinate system into linear programming. The restrict domain of some linear programming problems is a simplex. After applying n-linear coordinate system, can the arithmetic efficiency of linear programming be improved?

Conjecture: After applying n-linear coordinate system, the arithmetic efficiency of linear programming can be improved.

Three In figure 1, in absolute trilinear coordinate. the length of DE satisfies

$$DE^2 = \frac{\cos A}{\sin B \sin C}(x_1 - x_2)^2 + \frac{\cos B}{\sin A \sin C}(y_1 - y_2)^2 + \frac{\cos C}{\sin B \sin A}(z_1 - z_2)^2$$

$$= \sum \frac{\cos A}{\sin B \sin C}(x_1 - x_2)^2 \cdot$$

In reduced trilinear coordinate, the length of DE satisfies

$$DE^2 = k \sin 2A \left(\begin{vmatrix} b & \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \\ c & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \end{vmatrix} \right)^2 + k \sin 2B \left(\begin{vmatrix} c & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ a & \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \end{vmatrix} \right)^2 + k \sin 2C \left(\begin{vmatrix} a & \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \\ b & \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \end{vmatrix} \right)^2$$

$$= k \sum \sin 2A \left(\begin{vmatrix} b & \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \\ c & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \end{vmatrix} \right)^2 \cdot \quad \text{And} \quad k = \frac{2S^2}{(ax_1 + by_1 + cz_1)^2 (ax_2 + by_2 + cz_2)^2 \sin A \sin B \sin C} \cdot$$

Whether applying absolute or reduced trilinear coordinate, the expressions of length are all complicated. And here is in trilinear case. In general n-space, the expressions is very more complicated.

Definition 7.1 metric, metric space[4]: (X, d) is a metric space. X is a set, d is a metric on X. d is a function to definite in X×X. And satisfy following four axioms to all x, y, z∈X.

① d is a real number, finite and non-negative.

② d(x,y)=0. If and only if x=y.

③ d(x,y)=d(y,x)

④ d(x,y)≦d(x,z)+d(z,y)

According to this definition, studies a kind of metric, under which the expressions of length in n-space is simple, symmetrical,easily to compute.

Four. In n-space, how to compute the angle of two hyperplanes the dimensions of which are different?

Five. The oriented line theorem in trilinear plane can be generalized into n-space?

Definition 7.2 Inner product, inner product space[4]: The real vector space X in which inner product is defined is called as inner product space. The inner product is a map from X×X to scalar domain K of X. To all pair x、y in X, a scalar is corresponding with it, denoted by $< x, y >$. Satisfies

①$< x + y, z >=< x + z > + < y + z >$

②$< \alpha x, y >= \alpha < x, y >$

③$< x, y >=< y, x >$

④$< x, x >\geq 0, < x, x >= 0 \Leftrightarrow x = 0$

According to this definition, studies a kind of inner product, under which the expressions of slope in n-space is simple, symmetrical, easily to compute.

The length and angle are two main foundation stones in the Euclidean space. The simple, symmetrical expressions and computational methods on length and angle can make n-linear coordinate system to play a powerful role in geometry.

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