Shing-Tung Yau High School Mathematics Awards

The Structure and Topological Properties of Möbius Strip Dissection

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Abstract

The Möbius strip is a well-known surface with only one side and only one boundary. It has many curious properties, and has several technical applications in other disciplines such as physics and chemistry. In this paper, the structure and topological properties of Möbius strip dissection are studied.

A model of Möbius strip can be created by taking a paper strip and giving it a half-twist, and then joining the ends of the strip together to form a loop. Cutting a Möbius strip differently yields different strips including different length, width and half-twists. The structures of these strips are given respectively after $1/n$ cutting and n equal cutting of a Möbius strip. Moreover, the results are generalized to the dissection of Paradromic rings with extra twists.

Denote the result of bisecting a Paradromic ring with m half-twists by $P_{m,2}$. The structures and topological properties are investigated using knot theory. Firstly, it is proved that $P_{m,2}$ is equivalent to $(m,2)-$ torus link $T_{m,2}$. Secondly, the linking number, crossing number and unknotting number of $P_{m,2}$ are determined. Finally, the coloring of $P_{m,2}$ is discussed, the sufficient and necessary condition, the coloring scheme, and the minimum number of colors for p -coloring are given respectively.

Keywords: Möbius strip; Paradromic ring; dissection; structure; torus link; topological properties

(Note: The result in this English version is strengthened for that we find all p^2 p -colorings in Theorem 5, while in previous Chinese version we only gave one p -coloring)

莫比乌斯带分割的结构与拓扑性质

摘要：莫比乌斯带（Möbius Strip）是最具代表性的单侧曲面之一，它不但有许多神奇的结构和拓扑性质，而且在多个学科都有着十分广泛的应用。本文主要探讨对莫比乌斯带进行不同方式分割后得到的各种结构和拓扑性质。

莫比乌斯带是通过一个矩形纸带扭转半圈再把两端粘上之后得到的一个带环。设纸带的长度为l，宽度为w，扭转度数为π $(\pi=180^0)$。考虑两种分割方式，一种为$1/n$分割（即从距带边$1/n$宽度处沿长边切割，直至回到原处），另一种为n等分分割（即从宽边的n等分线沿长边切割）。得到了莫比乌斯带分割后得到的带环的结构。把一个矩形纸带扭转m（$m\geq 1$是任意正整数）个半圈再把两端粘上之后得到的一个带环称为Paradromic环。对Paradromic环分割也得到了相应的包括链接关系、长度、宽度、扭转度数和单双侧等性质的结论，推广了莫比乌斯带分割的结果。

利用纽结理论继续探讨Paradromic环$1/2$分割得到的带环（记为$P_{m,2}$）的结构和其它拓扑性质。首先证明了$P_{m,2}$等价于$(m,2)-$环面链环$T_{m,2}$。然后确定了$P_{m,2}$的环绕数、交叉数和解结数。最后研究了$P_{m,2}$的着色，得到了$P_{m,2}$是p（p是素数）可着色的充分必要条件，通过求解有限域F_p上的线性方程组给出了$P_{m,2}$的所有p^2个p-着色方案，还探讨了它的最小着色数。

关键词：莫比乌斯带；Paradromic 环；分割；结构；环面链环；拓扑性质

（备注：与以前中文版比较，内容较为精炼。另外，主要结论有所加强，即定理 5 中的着色方案，在现在英文版中通过求解有限域上的线性方程组给出了所有p^2个p-着色方案，而在原来中文版中只是给出了一个p-着色方案）

1. Introduction

The Möbius strip or Möbius band is a surface with only one side and only one boundary component. It was discovered independently by the German mathematicians August Ferdinand Möbius and Johann Benedict Listing in 1858. It can easily be created by taking a rectangular strip of paper and giving it a half-twist (180 degree twist), and then joining the ends of the strip together to form a loop, as shown in Figure 1.

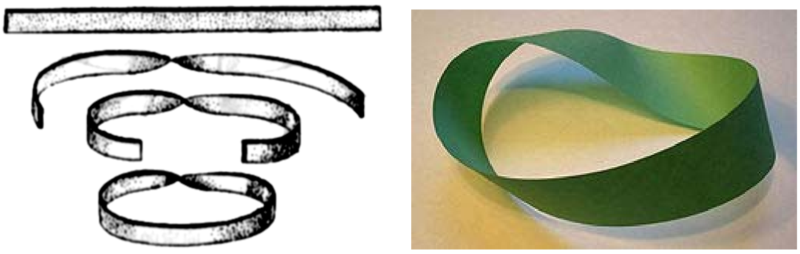


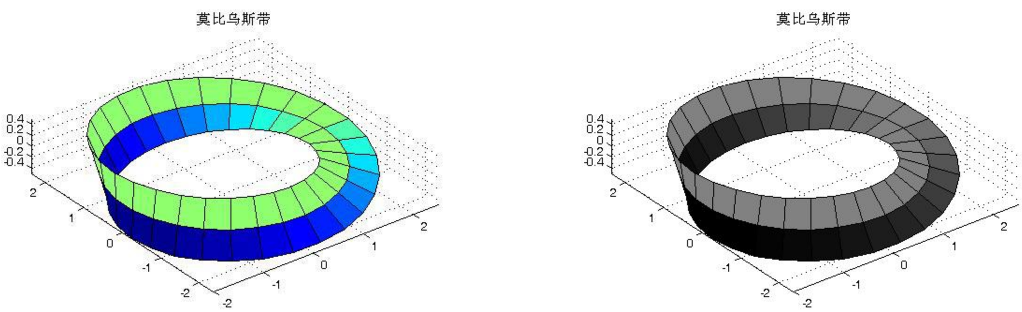
Figure 1 Möbius strip

The Möbius strip has fascinated both mathematicians and laypeople ever since Möbius discovered it in the nineteenth century and presented it as an object of mathematical interest. As the years passed, the popularity and application of the strip grew, and today it is an integral part of mathematics, magic, science, art, engineering, literature, and music (see [1-4] for details).

A Möbius strip can be represented parametrically by[2, 5, 6]

$$\begin{cases} x = (r + s\cos(t/2))\cos t \\ y = (r + s\cos(t/2))\sin t \\ z = s\sin(t/2) \end{cases}$$

Where $s \in (-w/2, w/2)$, $t \in [0, 2\pi)$. This creates a Möbius strip of width w whose center circle has radius r, lies in the xy plane and is centered at (0, 0, 0). The parameter t runs around the strip while s moves from one edge to the other. Parametric surfaces of Möbius strip drawn with MATLAB are shown in Figure 2, and corresponding MATLAB program codes are provided in appendix 1 and appendix 2.



(a) color display (b) gray scale display

Figure 2 Möbius strip created with MATLAB

Cutting a Möbius strip differently yields different strips including different length, width and half-twists. In this paper, the structure and topological properties of Möbius strip dissection are studied. The structures of these strips are given respectively after $1/n$ cutting and n equal cutting of a Möbius strip. Moreover, the results are generalized to the dissection of Paradromic rings with extra twists.

Denote the result of bisecting a Paradromic ring with m half-twists by $P_{m,2}$. The structures and topological properties are investigated using knot theory. Firstly, it is proved that $P_{m,2}$ is equivalent to $(m,2)-$torus link $T_{m,2}$. Secondly, the linking number, crossing number and unknotting number of $P_{m,2}$ are determined. Finally, the coloring of $P_{m,2}$ is discussed, the sufficient and necessary condition, the coloring scheme, and the minimum number of colors for p-coloring are given respectively.

2. Structure of Möbius strip dissection

In this section, the structures of these strips are given respectively after $1/n$ cutting and n equal cutting of a Möbius strip. Moreover, the results are generalized to the dissection of Paradromic rings with extra twists.

Take a rectangular strip of paper and join the two ends of the strip together so that it has a 180 degree twist, we get a Möbius strip. Suppose that the length, width and twist degree of the paper strip be l, w, π $(\pi = 180^0)$。

Consider two cutting ways:

$1/n$ cutting --the strip is cut along a $1/n$ of the way from the edge;

n equal cutting --the strip is cut along n equally dividing lines.

2.1 $1/n$ cutting and n equal cutting of a Möbius strip

A result about $1/n$ cutting of a Möbius strip is stated without proof in [8] and [2]. Some part result about n equal cutting of a Möbius strip is found and proved in [5]. We summarize the results and give a general result in Theorem 1 below, and present a proof using MATLAB sketch.

Theorem 1 The results about $1/n$ cutting and n equal cutting of a Möbius strip are stated in Table 1.

Table 1 Structure of $1/n$ cutting and n equal cutting of a Möbius strip

cutting ways		number	linked	length	width	twist degree	sided
$1/n$ cutting	$n=2$	1		$2l$	$w/2$	4π	two-sided
	$n \neq 2$	1	linked	$2l$	w/n	4π	two-sided
		1		l	$w(n-2)/n$	π	one-sided
n equal cutting	$n=2k$	k	linked	$2l$	w/n	4π	two-sided
	$n=2k+1$	1	linked	l	w/n	π	one-sided
		k		$2l$	w/n	4π	two-sided

Proof (1) Consider $1/n$ cutting of a Möbius strip.

For $n = 2$, the cutting line is center circle lies in the xy plane. The cutting result is a strip with length $2l$, width $w/2$, twist degree 4π, and two-sided, as shown in Figure 3.

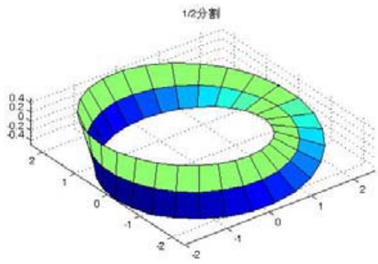
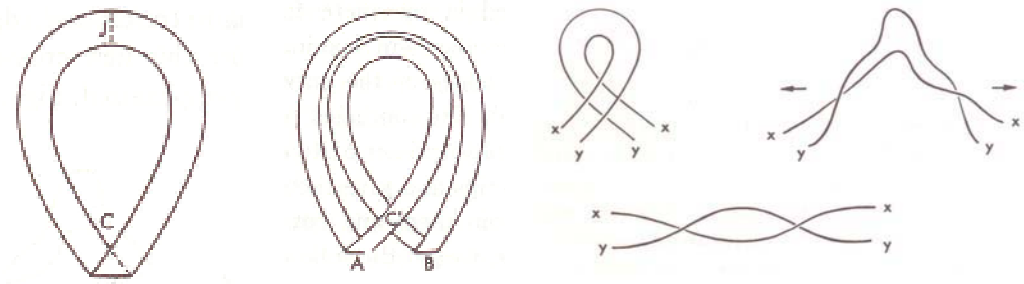


Figure 3 1/2 cutting of Möbius strip

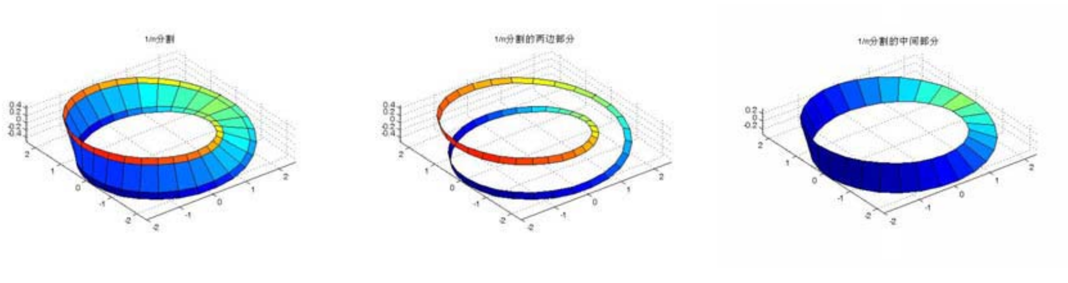
An explanation for twist degree 4π appeared in [7] and shown in Figure 4. Make a Möbius strip lie flat as Figure 4(a). There is one crossing of the edge with itself at C. When Fig.4(a) is cut along the center line and opened slightly, Fig.4(b), we see that the original twist shows in two places A and B, giving 2 twists to the new loop, but there is also a crossing of the loop with itself at C', corresponding to the crossed edge C in Fig.4(a). Consider this new crossed part C', with edges x and y emphasized in Fig.4(c). Pull it out as per the arrows to find that we get 2 more twists, which added to the first two give a total of 4.



(a) lie flat (b) 1/2 cutting (c) 2 more twists

Figure 4 twist degree is 4π

For $n \neq 2$, the cutting sketch with MATLAB is shown in Figure 5, and corresponding MATLAB codes are provided in appendix 3. We obtain two linked strips. One is the fringe part, which is a strip with length $2l$, width w/n, twist degree 4π, and two-sided, as shown in Fig.5(b). The other is the middle part, which is a strip with length l, width $w(n-2)/n$, twist degree π, and one-sided, as shown in Fig.5(c).



(a) $1/n$ cutting (b) the fringe part (c) the middle part

Figure 5 $1/n$ cutting of Möbius strip

(2) Consider n equal cutting of a Möbius strip.

There are two cases: n is even and n is odd. We only consider the case for n is even.

We prove by induction on n.

When $n=2$, we know that 2 equal cutting is 1/2 cutting. So from (1) by 2 equal cutting we obtain a strip with length $2l$,width $w/2$, twist degree 4π ,and two-sided.

Suppose the result holds when $n=2k$, that is, $2k$ equal cutting yields k strips with length $2l$,width $w/(2k)$, twist degree 4π ,and two-sided.

Consider $n=2k+2$. The $2k+2$ equal cutting can be completed through two steps: firstly $1/(2k+2)$ cutting, by (1) this produces two linked strips: the fringe part is a strip with length $2l$,width $w/(2k+2)$, twist degree 4π ,and two-sided; the middle part is a strip with length l ,width

$$w-2w/(2k+2)=w(2k)/(2k+2)$$

twist degree π, and one-sided. Secondly, $2k$ equal cutting of the middle part, by the induction hypothesis we get k strips with length $2l$,width

$$(w(2k)/(2k+2))/(2k)=w/(2k+2)$$

twist degree 4π ,and two-sided. Thus, we obtain $k+1$ strips with length $2l$,width $w/(2k+2)$, twist degree 4π ,and two-sided.

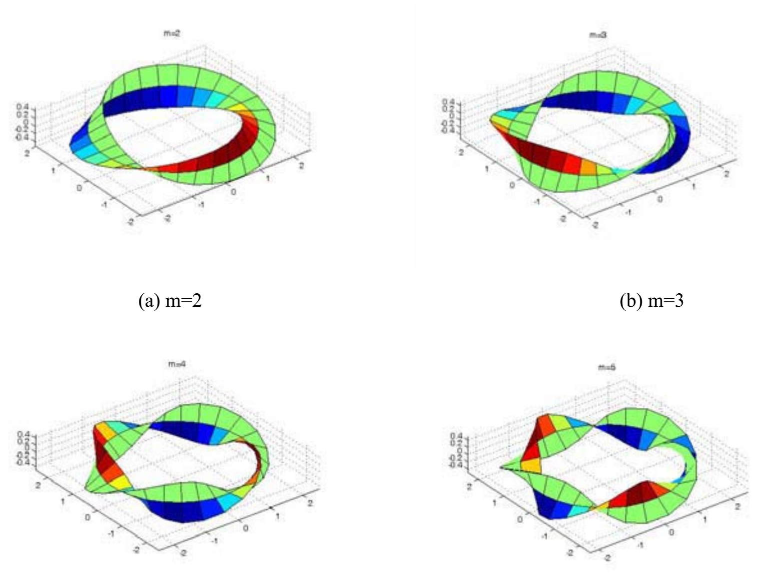
2.2 $1/n$ cutting and n equal cutting of a Paradromic ring

Taking a rectangular strip of paper, giving it m half-twists and reconnecting the ends of the strip produces figure called a Paradromic ring[10,11,2]. Suppose that the length, width and twist degree of the paper strip be l, w, $m\pi$, $m\geq 1$. A Paradromic ring is a generation of a Möbius strip. Paradromic ring is one-sided if m is odd, two-sided if m is even. Note that the new strips obtained in Theorem 1 are Paradromic rings, or linked Paradromic rings.

The parametric equation of Paradromic ring is

$$\begin{cases} x=(r+s\cos(mt/2))\cos t \\ y=(r+s\cos(mt/2))\sin t \\ z=s\sin(mt/2) \end{cases}$$

where $s\in(-w/2,w/2)$, $t\in[0,2\pi)$。The sketches by MATLAB are given in Figure 6 when $m=2,3,4,5$.



(a) m=2 (b) m=3

(c) m=4 (d) m=5

Figure 6 Paradromic ring with half-twists 2,3,4,5

It is known that a Paradromic ring with m half-twists, when bisected, becomes a strip with $m+1$ full twists [2]. Some examples was given about cutting of a Paradromic ring in [10-13]. Our second result gives the structure of cutting of a Paradromic ring.

Theorem 2 The structure of cutting of a Paradromic ring is stated in Table 2.

Table 2 Structure of $1/n$ cutting and n equal cutting of a Paradromic ring

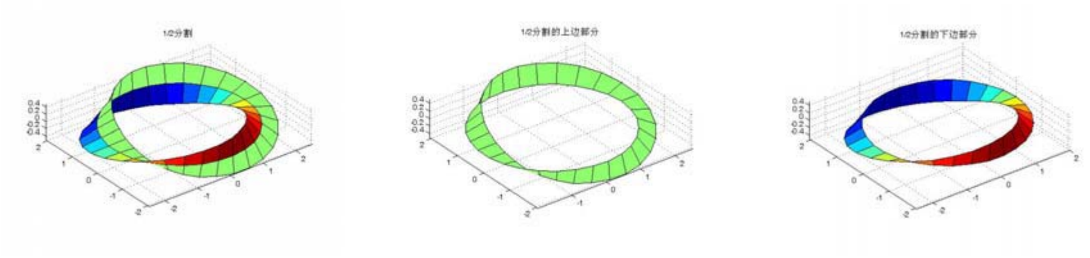
cutting ways		m	number	linked	length	width	twist degree	sided
$1/n$ cutting	$n=2$	odd	1		$2l$	$w/2$	$(2m+2)\pi$	2-sided
		even	2	linked	l	$w/2$	$m\pi$	2-sided
	$n\neq 2$	odd	1		$2l$	w/n	$(2m+2)\pi$	2-sided
			1	linked	l	$w(n-2)/n$	$m\pi$	1-sided
		odd	2	linked	l	w/n, $w(n-1)/n$	$m\pi$	2-sided
n equal cutting	$n=2k$	odd	k	linked $(k>1)$	$2l$	w/n	$(2m+2)\pi$	2-sided
		even	$2k$	linked	l	w/n	$m\pi$	2-sided
	$n=2k+1$	odd	1		l	w/n	$m\pi$	1-sided
			k	linked	$2l$	w/n	$(2m+2)\pi$	2-sided
		odd	$2k+1$	linked	l	w/n	$m\pi$	2-sided

Proof We complete the proof by the following steps.

(1) First consider three basic cuttings such as 1/2 cutting for $m=2$, 1/2 cutting for $m=3$, and $1/n$ ($n\neq 2$) cutting for $m=3$.

The 1/2 cutting of a Paradromic ring for $m=2$ creates a link, consisting of 2 two-side strips with length l, width $w/2$, and twist degree 2π. This result is shown in Fig.7, where the upper part and lower part are depicted respectively in Fig.7(b), $s\in(0,w/2)$ and (c),

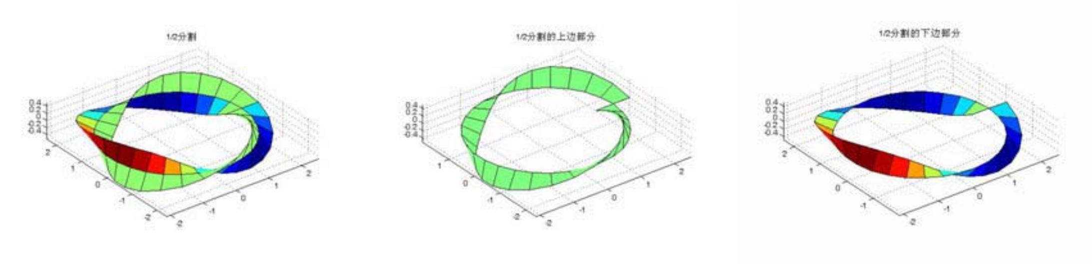
$s \in (-w/2, 0)$.



(a) 1/2 cutting (b) the upper part of 1/2 cutting (c) the lower part of 1/2 cutting

Figure 7 1/2 cutting of Paradromic ring with $m = 2$

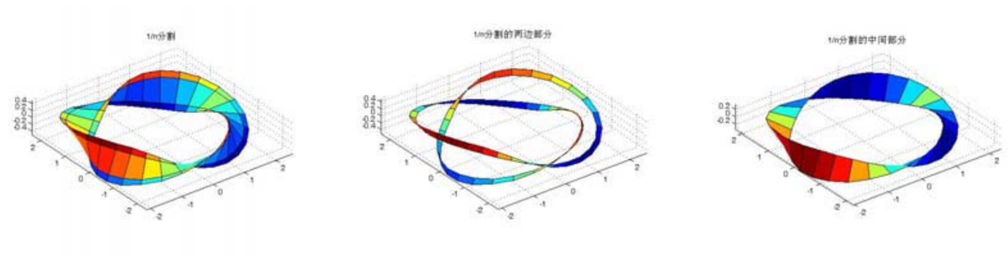
The 1/2 cutting of a Paradromic ring for $m = 3$ creates a loop, which is a two-side strip with length $2l$, width $w/2$, and twist degree 8π. This result is shown in Fig.8, where the upper part and lower part are depicted respectively in Fig.7(b) and (c). Note that these two parts connect together at the plane $z = 0$.



(a) 1/2 cutting (b) the upper part of 1/2 cutting (c) the lower part of 1/2 cutting

Figure 8 1/2 cutting of Paradromic ring with $m = 3$

The $1/n$ ($n \neq 2$) cutting of a Paradromic ring for $m = 3$ creates two linked strips. One is the fringe part, which is a strip with length $2l$, width w/n, twist degree 8π, and two-sided, as shown in Fig.9(b). The other is the middle part, which is a strip with length l, width $w(n-2)/n$, twist degree 3π, and one-sided, as shown in Fig.9(c).



(a) $1/n$ cutting of Paradromic ring (b) the fringe part (c) the middle part

Figure 9 $1/n$ cutting of Paradromic ring with $m = 3$

(2) Consider n equal cutting of a Paradromic ring.

We give the structure using recursive method. We only consider the case when m is odd.

$2k$ equal cutting can be completed through $1/n$ cutting in turn.

Firstly, $1/2k$ cutting of a Paradromic ring results in two linked strips. Where the fringe part is a two-sided strip with length $2l$, width $w/(2k)$, twist degree $(2m+2)\pi$; the middle part is a one-sided strip with length l,width

$$w-2w/(2k)=w(2k-2)/(2k)$$

twist degree $m\pi$.

Secondly, $1/(2k-2)$ cutting of the middle part results in two linked strips. Where the fringe part is a two-sided strip with length $2l$,width

$$(w(2k-2)/(2k))/(2k-2)=w/(2k)$$

twist degree $(2m+2)\pi$; the middle part is a one-sided strip with length l,width

$$w(2k-2)/(2k)-2w/(2k)=w(2k-4)/(2k)$$

twist degree $m\pi$.

Go on $1/(2k-4)$ cutting for the new middle part, after $(k-1)$-th cutting, we obtain two linked strips: the fringe part is a two-sided strip with length $2l$,width $w/(2k)$,twist degree $(2m+2)\pi$; the middle part is a one-sided strip with length l,width $w(2)/(2k)$,twist degree $m\pi$.

Finally, $1/2$ cutting this middle part creates a two-sided strip with length $2l$,width $w/(2k)$,twist degree $(2m+2)\pi$.

To sum up, $1/2k$ cutting of a Paradromic ring results in k linked strips, all with length $2l$,width $w/(2k)$,twist degree $(2m+2)\pi$.

The case for $2k+1$ cutting of a Paradromic ring is similar.

3 The structure and topological properties of a Paradromic ring cutting

For a Paradromic ring cutting, we have known some properties including length, width, twist degree, one-sided or two-sided in Section 2. In this section, we continue to investigate the structure and other topological properties such as linking number, crossing number, unknotting number and coloring number.

We only consider $1/2$ cutting of a Paradromic ring. Let $P_{m,2}$ denote the new strip obtained by $1/2$ cutting of a Paradromic ring.

3.1 The structure of a Paradromic ring cutting

It follows from Theorem 2 we know that if m is even, then $P_{m,2}$ is a link consisting of two two-sided strips with length l, width $w/2$ and m half-twists; if m is odd, then $P_{m,2}$ is a two-sided strip with length $2l$, width $w/2$ and $2m+2$ half-twists, and it is a knot in deed. See Fig.10 for details.

(a) $P_{m,2}$ (m=2) (b)Hopf link

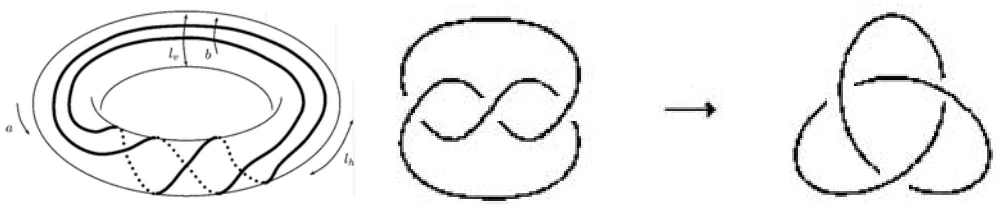
(b) $P_{m,2}$ (m=3) (d) trefoil knot

Figure 10 $P_{m,2}$ and knot (link)

Please refer to [14,15] for knot theory. It was guessed that there is some relationship between a Paradromic ring and a torus links in [16,17]. We will prove that $P_{m,2}$ is torus link in Theorem 3.

A knot (or link) is a torus knot (or link) if it is equivalent to a knot (or link) that can be drawn without any points of intersection on the trivial torus. A $(q,r)-$torus knot (link) $T_{q,r}$ is a torus knot (or link) characterized by the number of times q that it circles around the meridian of the torus and the number of times r that it circles around the longitude of the torus. $T_{q,r}$ is a knot if and only if $(q,r)=1$, that is, q and r are relatively prime. $T_{q,r}$ is a link with $d=(q,r)$ component if $(q,r)>1$, and each component are knot equivalent to $T_{q/d,r/d}$.

A torus knot $T_{3,2}$ and its diagram are depicted in Fig.11. Two torus knots $T_{5,2}$ and $T_{9,2}$ are depicted in Fig.12. Where the MATLAB program sketching $T_{5,2}$ is given in appendix 4.



(a) torus knot $T_{3,2}$ (b) diagram of $T_{3,2}$

Figure 11 torus knot $T_{3,2}$ and diagram

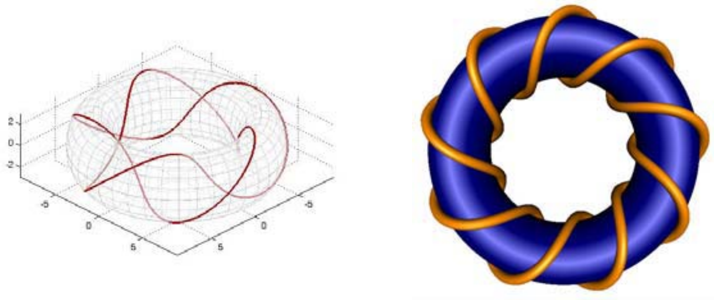


Figure 12 torus knot $T_{5,2}$ and $T_{9,2}$

It follows from Fig.10, Fig.11 and the property of torus link we know $P_{m,2} = T_{m,2}$ when $m = 2,3$. The statement is true for all m .

Theorem 3 $P_{m,2} = T_{m,2}$.

Proof First , note that the torus link $T_{m,2}$ is the boundary of a Paradromic ring with m half-twists for all m .

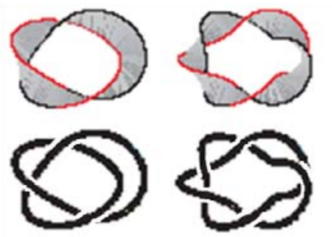
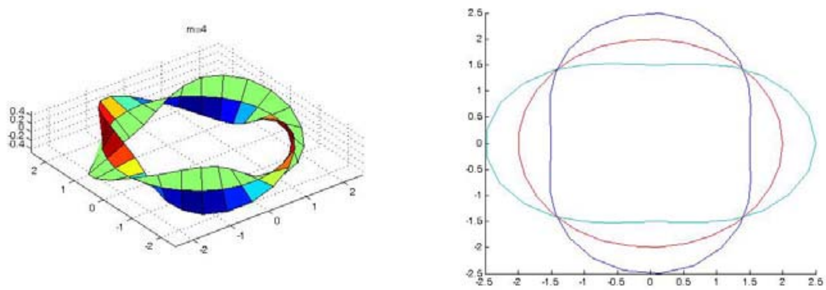
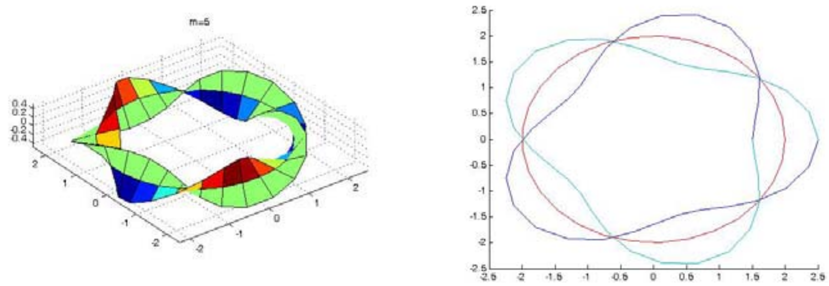


Figure13 torus link $T_{m,2}$ is the boundary of a Paradromic ring

Secondly, the diagram of $P_{m,2}$ is equivalent to that of the boundary of a Paradromic ring. Fig.14 shows the projections of the boundaries and center circles of Paradromic rings $P_{m,2}$ for $m = 4,5$ to plane $Z = 0$, where the MATLAB program is given in appendix 5.



(a) Paradromic ring with m=4 (b) the projections of the boundaries and center circle



(c) Paradromic ring with m=5 (d) the projections of the boundaries and center circle

Figure 14 Paradromic ring and the projections of the boundaries

Finally, we get $P_{m,2} = T_{m,2}$.

3.2 Topological properties of of a Paradromic ring cutting

Please refer to [14,15] for the definitions and the properties of linking number, crossing number, and unknotting number of knot (link). The determination for these topological invariants is an important and extremely difficult problem. Fortunately, the linking number $l(T_{q,r})$, crossing number $c(T_{q,r})$, and unknotting number $u(T_{q,r})$ for torus link $T_{q,r}$ is known. So, we get the corresponding numbers for $T_{q,2}$: $l(T_{q,2}) = q/2$ if q is even, $c(T_{q,2}) = q$, $u(T_{q,2}) = (q-1)/2$ if q is odd. Thus by Theorem 3 we get

Corollary 4 The linking number, crossing number, and unknotting number of $P_{m,2}$ are

$l(P_{m,2}) = m/2$ if m is even,

$c(P_{m,2}) = m$,

$u(P_{m,2}) = (m-1)/2$ if m is odd.

Let L be a knot (link), and D be a diagram of L, p an integer greater than 1. Let x, y, z denote integers which label the over arc and two under arcs, respectively, at a crossing of D. The crossing satisfies the condition of p-colorability if

$$2x = y + z \pmod{p}$$

We say L is p- colorable if there is a diagram D of L such that the arcs of D can be labeled, or colored, with the numbers $0, 1, \cdots, p-1$ so that at least two numbers are used and every crossing satisfies the condition of p-colorability. The numbers $0, 1, \cdots, p-1$ are called colors. The specific colors assigned to the arcs make up a p-coloring of D. A p-coloring where every arc is assigned the same color is called a trivial coloring.

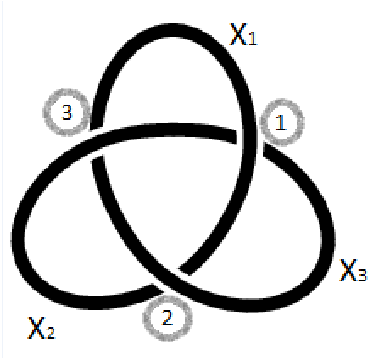


Figure 17 a 3- coloring of $T_{3,2}$

The minimum number of p-coloring of L, is defined to be the minimum number of colors for all p-coloring of all diagram D of L, denoted by $\min \mathrm{col}_p L$. This is an invariant which is, in general, very difficult to evaluate.

There are some researches on p-colorability, minimum number of p-coloring of torus link $T_{q,r}$. We list two results here [19,20,21].

(1) Suppose $T_{q,r}$ is a torus knot and p is prime.

(1.1) If q and r are both odd, then $T_{q,r}$ is not p-colorable.

(1.2) If q is odd and r is even, then $T_{q,r}$ is p-colorable if and only if $p \mid q$

$$(2)\min \mathrm{col}_p L \begin{cases} =<m,p>, & \text{if } <m,p>\in\{2,3\} \\ =4, & \text{if } <m,p>=5 \\ \in(3,k+2], & \text{if } <m,p>=2k+1, k>2 \end{cases}$$

Where $<m,p>$ stand for the least common prim divisor of m,p.

So, p-colorability, minimum number of p-coloring of $P_{m,2}$ has been known. In the following, we focus on p-coloring scheme of $P_{m,2}$.

The diagram of $P_{m,2}$ $(m=2k+1)$ is shown in Fig.18. By Corollary 4, there are m crossings, denoted by $c_1,c_2,\cdots,c_m$. The over arcs corresponding to crossing $c_1,c_2,\cdots,c_m$ denoted by $a_1,a_2,\cdots,a_m$. Let the color of a_i be x_i, $i=1,2,\cdots,m$. Then p-coloring (scheme) of $P_{m,2}$ can be denoted by a vector $x=(x_1,\cdots,x_i,\cdots,x_m)$.

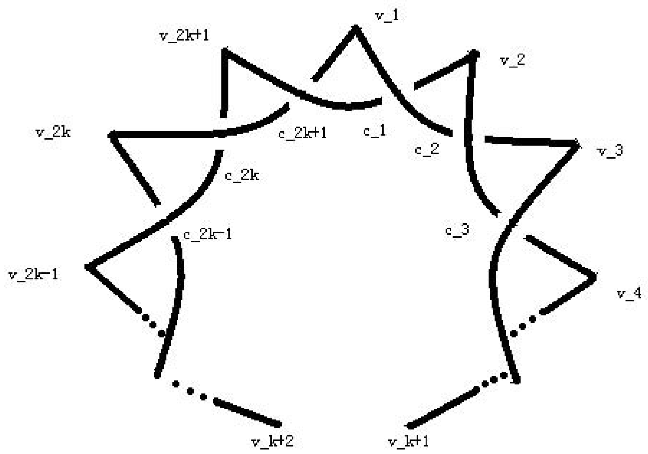


Figure 18 diagram of $P_{m,2}$

The p-coloring of $P_{m,2}$ is given in the Theorem 5.

Theorem 5 (1) Suppose p is prime, then $P_{m,2}$ is p- colorable if and only if $p \mid m$.

(2) $\min\operatorname{col}_p P_{m,2} \begin{cases} =p, & \text{if } p=2,3 \\ =4, & \text{if } p=5 \\ \in (3,(p+3)/2], & \text{if } p>5 \end{cases}$

Where p is prime divisor of m.

(3) If m is even, then $P_{m,2}$ is 2- colorable.

If m is odd, and if p is the least prime divisor of m, then $P_{m,2}$ is p- colorable. Further, there are a total of p^2 p-colorings. In particular, if $m=p$, the total of p^2 p-colorings are

$$\begin{aligned} x &= (x_1,\cdots,x_i,\cdots,x_m)' \\ &= c_1(p-1,p-2,\cdots,2,1,0)' + c_2(2,3,\cdots,p-2,p-1,0,1)' \end{aligned}$$

Where c_1, c_2 are arbitrary elements in $F_p=\{0,1,\cdots,p-1\}$, and the operations are in the finite field F_p.

Proof We only prove (3) for odd m.

Suppose that p is the least prime divisor. Let a p -coloring of $P_{m,2}$ be $x=(x_1,\cdots,x_i,\cdots,x_m)$. Then $x=(x_1,\cdots,x_i,\cdots,x_m)$ satisfies the following system (1) of linear equations module p.

$$\begin{cases} 2x_1 - x_2 - x_m = 0(\bmod p) \\ -x_1 + 2x_2 - x_3 = 0(\bmod p) \\ \vdots \\ -x_{m-2} + 2x_{m-1} - x_m = 0(\bmod p) \\ -x_1 - x_{m-1} + 2x_m = 0(\bmod p) \end{cases} \qquad (1)$$

Its coefficient matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

Constant matrix

$$b = (0, 0, \cdots, 0)'$$

Where b' is the transpose of b, A is a matrix of order m.

Note that

$$0 \le x_i \le p-1,\ i = 1, 2, \cdots, m$$

The system (1) of linear equations module p is equivalent to the system (2) of linear equations over the finite field F_p.

$$\begin{cases} 2x_1 - x_2 - x_m = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ \vdots \\ -x_{m-2} + 2x_{m-1} - x_m = 0 \\ -x_1 - x_{m-1} + 2x_m = 0 \end{cases} \qquad (2)$$

The coefficient matrix over F_p is denoted still by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

A is a matrix of order m.

If the rank of A is r, then the number of the fundamental system of solutions of (2) is m - r, the number of total solutions of (2) is p^{m-r}.

After elementary row transformation, A becomes

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 3/2 & -1 & \cdots & 0 & 0 & -1/2 \\ 0 & 0 & 4/3 & \cdots & 0 & 0 & -1/3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (m-1)/(m-2) & -1 & -1/(m-2) \\ 0 & 0 & 0 & \cdots & 0 & m/(m-1) & -m/(m-1) \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

The rank of A is $m-2$, the number of the fundamental system of solutions of (2) is 2, the number of total solutions of (2) is p^2.

In particular, if $m=p$, after calculation, we get a fundamental system of solutions

$$(p-1, p-2, \cdots, 2, 1, 0)', (2, 3, \cdots, p-2, p-1, 0, 1)'.$$

So, the general solution is given by:

$$\begin{aligned} x &= (x_1, \cdots, x_i, \cdots, x_m)' \\ &= c_1(p-1, p-2, \cdots, 2, 1, 0)' + c_2(2, 3, \cdots, p-2, p-1, 0, 1)' \end{aligned}$$

Where c_1, c_2 are arbitrary elements in $F_p = \{0, 1, \cdots, p-1\}$. Thus we obtained the total of p^2 p-colorings of $P_{m,2}$.

Remark It follows from Theorem 5(3) that if $m=p$, the total of p^2 p-colorings are

$$\begin{aligned} x &= (x_1, \cdots, x_i, \cdots, x_m)' \\ &= c_1(p-1, p-2, \cdots, 2, 1, 0)' + c_2(2, 3, \cdots, p-2, p-1, 0, 1)' \end{aligned}$$

Where c_1, c_2 are arbitrary elements in $F_p = \{0, 1, \cdots, p-1\}$.

So, write $c_1 = c_2 + j$, where j is arbitrary element in F_p. We can state the p-colorings in following way:

$$\begin{aligned} x &= c_1(p-1, p-2, \cdots, 2, 1, 0)' + c_2(2, 3, \cdots, p-2, p-1, 0, 1)' \\ &= (c_2 + j)(p-1, p-2, \cdots, 2, 1, 0)' + c_2(2, 3, \cdots, p-2, p-1, 0, 1)' \\ &= c_2(1, 1, \cdots, 1, 1)' + j(p-1, p-2, \cdots, 2, 1, 0)' \qquad (*) \end{aligned}$$

Where c_2, j are arbitrary elements in $F_p = \{0, 1, \cdots, p-1\}$.

Two p-colorings of $P_{m,2}$ are equivalent if they are the same under the rotation of $P_{m,2}$. So, for fixed j, p-colorings in (*) for different c_2 are equivalent. Thus, In all p^2 p-colorings of $P_{m,2}$, we have p different p-colorings of $P_{m,2}$, which corresponding to different $j = 0, 1, \cdots, p-2, p-1$ (let $c_2 = 0$):

$$(0,0,\cdots,0,0)',$$
$$(p-1,p-2,\cdots,1,0)',$$
$$(p-2,p-4,\cdots,2,0)',$$
$$\cdots,$$
$$(2,4,\cdots,p-2,0)'$$
$$(1,2,\cdots,p-1,0)'$$

If $m=kp$, the total of p^2 p-colorings can be obtained from the case of $m=p$ by repeating k times.

By the above theorems, we have the structure and topological properties of $1/2$ cutting of a Paradromic ring $P_{m,2}$, as shown in Table 3.

Table 3 structure and topological properties of cutting of a Paradromic ring

	$1/2$ cutting	
m	odd	even
number	1	2
linked	knot	link
length	$2l$	l
width	$w/2$	$w/2$
twist degree	$(2m+2)\pi$	$m\pi$
one-sided or two-sided	two-sided	two-sided
linking number		$m/2$
crossing number	m	m
unknotting number	$(m-1)/2$	
p -colorable	the least prime divisor of m	2
minimum number of colors	2,3,4，or $(3,(p+3)/2]$	2

For example, when m =5, 1/2 cutting of Paradromic ring with length l ,width w ,twist degree 5π produces a two-sided strip with length $2l$,width $w/2$,twist degree 12π ;it is a knot with crossing number 5, unknotting number 2, 5- colorable， the minimum number of colors is 4. When m =6, 1/2 cutting of Paradromic ring with length l ,width w ,twist degree 6π produces two linked two-sided strip with length l ,width $w/2$,twist degree 6π ;it is a knot with linking number 3, crossing number 6, 2- colorable， the minimum number of colors is 2.

4 Summary

We consider $1/n$ cutting and n equal cutting of a Möbius strip, obtain the structures of the new strips. Moreover, the results are generalized to the dissection of Paradromic rings with extra twists. The results are complete and proofs are given.

We use MATLAB software to sketch the Möbius strip, Paradromic ring and their dissection. This method is effective and is proved to be feasible.

We prove that the result of bisecting a Paradromic ring with m half-twists is equivalent to torus knot, and then give all p -colorings of 1/2 cutting of a Paradromic ring through solving the system of linear equations over finite field F_p.

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Appendices

Appendix 1. The MATLAB program to plot Mobius strip

```
function z=mobius(r,w)
r=2;w=1; %中心圆的半径为2,带的宽度为1
s=linspace(-w/2,w/2,3);
t=linspace(0,2*pi,30);
[s,t]=meshgrid(s,t);
x=(r+w.*s.*cos(t/2)).*cos(t);
y=(r+w.*s.*cos(t/2)).*sin(t);
z=w.*s.*sin(t/2);
surf(x,y,z)
axis equal
title(‘莫比乌斯带’)
```

Appendix 2. Animation for definition of Mobius strip

```
clear all;clc;close all;
u=linspace(-1/2,1/2,3);
t=linspace(0,2*pi,40);
[X,Y]=meshgrid(u,t);
x =(2+X.*cos(0.5*Y)/2).*cos(Y);
y =(2+X.*cos(0.5*Y)/2).*sin(Y);
z=X.*sin(0.5*Y)/2;
SH=surf(x,y,z);
shading interp
axis([-2 2 -2 2 -2 2]);
for k=1:40;
    t=linspace(0,pi*k/20,k*20);
    [X,Y]=meshgrid(u,t);
    x =(1+X.*cos(0.5*Y)/2).*cos(Y);
    y =(1+X.*cos(0.5*Y)/2).*sin(Y);
    z=X.*sin(0.5*Y)/2;
    set(SH,'XData',x,'YData',y,'ZData',z,'CData',z);
    saveas(gcf,['P',num2str(k),'.jpg']);
    pause(0.1);
end
clc;clear;close all;
A=imread('P40.jpg');
B=A(300:570,375:830,1:end);
```

```
imshow(B,[]);
SS=sum(A,3);
[x,y]=find(SS<255*3);
x1=min(x(1:end));
x2=max(x(1:end));
y1=min(y(1:end));
y2=max(y(1:end));
for k=1:40;
    A=imread(['P',num2str(k),'.jpg']);
    B=A(x1:x2,y1:y2,1:end);
    imwrite(B,['Q',num2str(k),'.jpg']);
end
```

Appendix 3. 1/n cutting of Mobius strip

```
clear
close all
clc
r=2;w=1;
m=1;
p=1/3;
s=[-w/2,-p,p,w/2];
t=linspace(0,2*pi,30);
[s,t]=meshgrid(s,t);
x=(r+w.*s.*cos(t.*m/2)).*cos(t);
y=(r+w.*s.*cos(t.*m/2)).*sin(t);
z=w.*s.*sin(t.*m/2);
surf(x,y,z)
axis equal
title('1/n分割')
figure
s=[-p,p];
t=linspace(0,2*pi,30);
[s,t]=meshgrid(s,t);
x=(r+w.*s.*cos(t.*m/2)).*cos(t);
y=(r+w.*s.*cos(t.*m/2)).*sin(t);
z=w.*s.*sin(t.*m/2);
surf(x,y,z)
axis equal
title('1/n分割的中间部分')
figure
s=[-w/2,-p];
t=linspace(0,2*pi,30);
[s,t]=meshgrid(s,t);
x=(r+w.*s.*cos(t.*m/2)).*cos(t);
y=(r+w.*s.*cos(t.*m/2)).*sin(t);
z=w.*s.*sin(t.*m/2);
surf(x,y,z)
hold on
s=[p,w/2];
```

```
t=linspace(0,2*pi,30);
[s,t]=meshgrid(s,t);
x=(r+w.*s.*cos(t.*m/2)).*cos(t);
y=(r+w.*s.*cos(t.*m/2)).*sin(t);
z=w.*s.*sin(t.*m/2);
surf(x,y,z)
hold off
axis equal
title('1/n分割的两边部分')
end
```

Appendix 4. Torus knot

```
a=7;b=2;c=3;
u=linspace(0,2*pi,20);
v=linspace(0,2*pi,40);
[u,v]=meshgrid(u,v);
x=(a+b*cos(u)).*cos(v);
y=(a+b*cos(u)).*sin(v);
z=c*sin(u);
mhndl=mesh(x,y,z);
set(mhndl,'EdgeColor',[.6,.6,.6],'FaceAlpha',0.5,'EdgeAlpha',0.5);
axis equal
t=linspace(0,2*pi,200);
x=(a+b*cos(5*t)).*cos(2*t);
y=(a+b*cos(5*t)).*sin(2*t);
z=c*sin(5*t);
lhndl=line(x,y,z);
set(lhndl,'Color',[.625,0,0],'LineWidth',2)
view(135,30)
```

（3.5 Space Curves in
Matlab http://msenux.redwoods.edu/Math4Textbook/Plotting/SpaceCurves.pdf）

Appendix 5. Projection of Paradromic ring

```
clear
close all
clc
r=2;w=1; m=5;
p=0;
s=[-w/2,-p,p,w/2];
t=linspace(0,2*pi,30);
[s,t]=meshgrid(s,t);
x=(r+w.*s.*cos(t.*m/2)).*cos(t);
y=(r+w.*s.*cos(t.*m/2)).*sin(t);
z=w.*s.*sin(t.*m/2);
surf(x,y,z)
axis equal
figure
plot3(x,y,z)
view(0,90)
```