# Study on the Properties of Circumscribed Ellipses of Convex Quadrilaterals

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# Abstract

In this paper, the author deals with the properties of circumscribed ellipses of convex quadrilaterals, using tools of parallel projective transformation and analytic geometry. And the procedures of research are always from the particular to the general. Moreover, for the sake of integrity, the author also studies the cases of the other two kinds of conics and concave quadrilaterals.

The main research conclusions are as follows:

1. Give a new geometric proof of the existence of circumscribed ellipses of convex quadrilaterals.
2. Figure out the coverage area of the circumscribed conics respectively, which equally means to divide the plane where a convex quadrilateral is located into three parts: con-elliptic, con-parabolic and con-hyperbolic with the four vertexes of the quadrilateral.
3. Figure out the locus of the center of circumscribed conics, both of convex quadrilaterals and concave quadrilaterals.
4. Figure out the minimal area of circumscribed ellipse of convex quadrilaterals.

Through the research, the author has insight into the innate connection of conic sections as well as a taste of the beauty and harmony of geometry.


**Key words:** Convex Quadrilateral, Circumscribed Ellipse, Parallel Projective Transformation, Conic Section

# Contents

# 1. Introduction

In plane geometry, work on four concyclic points is very sufficient, and there is a series of well-known nice conclusions, such as the judgment theorem, Ptolemy theorem. However, work on circumscribed ellipses of convex quadrilaterals is relatively deficient, so this paper hopes to make up for the shortage of this aspect and enrich knowledge about circumscribed ellipses of convex quadrilaterals.

In this paper, circles are defined as a special situation of ellipses, but lines are not.

# 2. The Existence of Circumscribed Ellipses of Convex Quadrilaterals

As we all know, if four points form a concave quadrilateral or any three points of them are on a line, there is no ellipse crossing these four points. However, does any convex quadrilateral have circumscribed ellipses? The answer should be yes. Next, the paper will prove the existence of circumscribed ellipses of convex quadrilaterals.

**Theorem 1** *There exists circumscribed ellipses of any convex quadrilateral.*

*Proof* To reduce the randomicity of convex quadrilaterals, we first fix the three points of them, and the forth point can move within the area which is enclosed by rays AB, AC and side BC, and we define this area as **Area M (Fig. 1)**. Since any triangle has circumscribed ellipses, we can also describe **Theorem 1** as follow:
*All circumscribed ellipses of $\Delta ABC$ can cover Area M.*

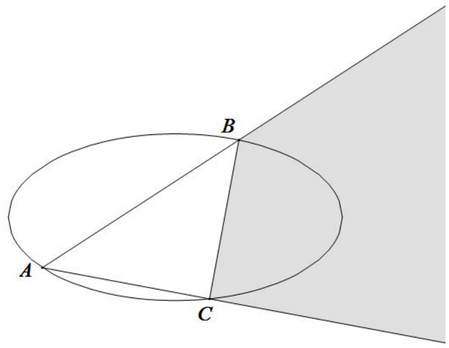


**Fig. 1** Area M of a scalene triangle

However, there is also much radomicity of $\Delta ABC$, so we need to simplify the above proposition further. We draw the following lemma to achieve this aim.

**Lemma 1** *There exists a parallel projective transformation to make any scalene triangle become an isosceles triangle.*

*Proof* Let $\Delta ABC$ be a scalene triangle and $\Delta A'B'C$ be its projection on **Plane Q**. BB′, CC′ are both perpendicular to Plane Q **(Fig. 2)**.

$\because AB' = AC'$

$\therefore \overline{AB} cos\angle BAB' = \overline{AC} cos\angle CAC'$

So it just needs to meet the following condition

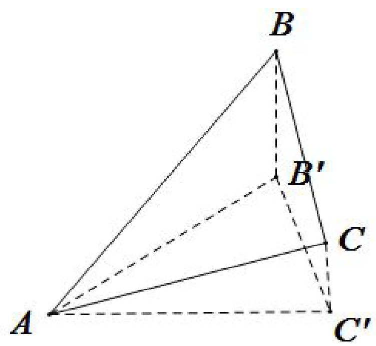


**Fig. 2** Proof of Lemma 1

$$\frac{cos\angle BAB'}{cos\angle CAC'} = \frac{\overline{AC}}{\overline{AB}}$$

At the same time, Area M of $\Delta ABC$ is projected to be Area M of $\Delta A'B'C$, and the circumscribed ellipses of $\Delta ABC$ are projected to be the circumscribed ellipses of $\Delta A'B'C$ [2].

Therefore, we just need to prove that *when* $\Delta ABC$ *is an isosceles triangle, all circumscribed ellipses of* $\Delta ABC$ *can cover Area M.*

We assume $AB = AC$. Set up a rectangular coordinate system in which A is the origin and *B*, *C* are symmetric about *x axis* (**Fig. 3**). First, we check the situation where one of the main diameters of the circumscribed ellipses of $\Delta ABC$ is coincide with *x axis*.

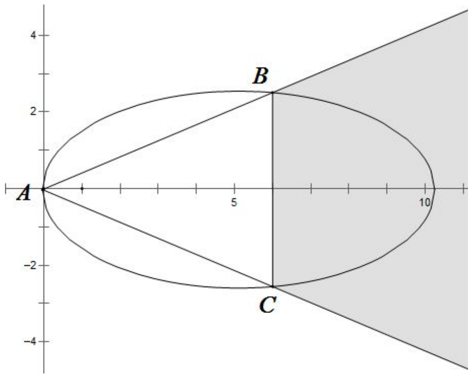


**Fig. 3** Set up a rectangular coordinate system for an isosceles triangle

In this case, the equation of the circumscribed ellipses is

$$Ax^2 + Cy^2 + 2Dx = 0 \tag{1}$$

Let $B(x_1, y_1)$, and $C(x_1, -y_1)$; $D(x_2, y_2)$ is a point in Area M and it is also above the *x axis*. So we have

$$x_2 > x_1 > 0,\ y_1 > 0,\ y_2 \geq 0,\ \frac{y_2}{x_2} < \frac{y_1}{x_1}.$$

Since the ellipses cross A, B, C, D, we can write down the equations

$$\begin{cases} x_1^2 A + y_1^2 C + 2x_1 D = 0 \\ x_2^2 A + y_2^2 C + 2x_2 D = 0. \end{cases}$$

Then we can get

$$\begin{cases} A = \dfrac{2(x_1 y_2^2 - x_2 y_1^2)}{x_2^2 y_1^2 - x_1^2 y_2^2} D \\ \\ C = \dfrac{2x_1 x_2 (x_1 - x_2)}{x_2^2 y_1^2 - x_1^2 y_2^2} D \end{cases}$$

We might as well assume $D > 0$, so $C < 0$. When $\frac{y_2^2}{y_1^2} < \frac{x_2}{x_1}$, $A < 0$.

For a general quadratic equation in two variable

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

we can use the following three discriminants to determine the locus of the equation:

$$I_1 = A + C\,, \qquad I_2 = AC - B^2,$$

$$I_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}. \tag{2}$$

If and only if $I_2 > 0$ and $I_3 \cdot I_1 < 0$, the locus is an ellipse.

For the equation (1), we can get the three discriminants:

$$I_1 = A + C < 0,\ I_2 = AC > 0,\ I_3 = -CD^2 > 0.$$

Therefore, the locus of the equation (1) is an ellipse.

In conclusion, when $\frac{y_2^2}{y_1^2} < \frac{x_2}{x_1}$, the circumscribed ellipses of $\Delta ABC$ whose one of the main diameters is coincide with *x axis* cross *D* (**Fig. 4**).

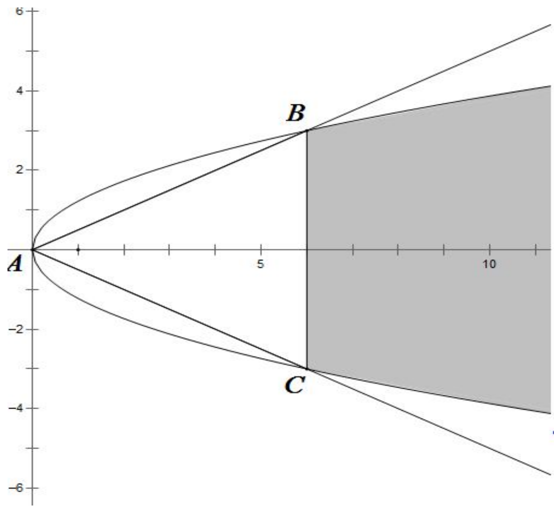


**Fig. 4** The circumscribed ellipses of $\Delta ABC$ whose one of the main diameters is coincide with *x axis* can cover the shaded part

Next we check the situation where $\frac{y_2}{y_1} < \frac{x_2}{x_1} < \frac{y_2^2}{y_1^2}$. We define the range of *D* that meet the inequation as **Area N**, which is in Area M. In this case, the equation of the circumscribed ellipses of $\Delta ABC$ is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey = 0. \tag{3}$$

Since Area N is symmetrical with *x axis*, we just need to discuss the situation where *D* is above *x axis*. So we have

$$x_2 > x_1 > 0,\ y_1 > 0,\ y_2 > 0.$$

Since the ellipses cross A, B, C, D, we can write down the equations

$$\begin{cases} x_1^2A + 2x_1y_1B + y_1^2C + 2x_1D + 2y_1E = 0 \\ x_1^2A - 2x_1y_1B + y_1^2C + 2x_1D - 2y_1E = 0. \\ x_2^2A + 2x_2y_2B + y_2^2C + 2x_2D + 2y_2E = 0 \end{cases}$$

Then we can get

$$\begin{cases} A = \dfrac{2(x_1y_2^2 - x_2y_1^2)D + 2y_1^2y_2\frac{x_2-x_1}{x_1}E}{x_2^2y_1^2 - x_1^2y_2^2} \\ B = -\dfrac{1}{x_1}E \\ C = -\dfrac{2x_1(x_2-x_1)(x_2D + y_2E)}{x_2^2y_1^2 - x_1^2y_2^2} \end{cases}.$$

We might as well assume $D > 0$. From $\frac{y_2}{y_1} < \frac{x_2}{x_1} < \frac{y_2^2}{y_1^2}$ we can get

$$-\frac{x_1y_2^2 - x_2y_1^2}{y_1^2y_2\frac{x_2-x_1}{x_1}} > -\frac{x_2}{y_2},$$

so

$$-\frac{x_1y_2^2 - x_2y_1^2}{y_1^2y_2\frac{x_2-x_1}{x_1}}D > E > -\frac{x_2}{y_2}D.$$

Here, $A < 0$, $C < 0$, so

$$I_1 = A + C < 0,\quad I_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = -\frac{2}{x_1}DE^2 - CD^2 - AE^2 > 0.$$

Therefore, the locus of equation (3) is an ellipse as long as variable *D*, *E* satisfy the condition that $I_2 = AC - B^2 > 0$.

Last we check the situation where $\frac{x_2}{x_1} = \frac{y_2^2}{y_1^2}$. In this case,

$$\begin{cases} A = \dfrac{2y_1^2y_2\frac{x_2-x_1}{x_1}E}{x_2^2y_1^2 - x_1^2y_2^2} \\ B = -\dfrac{1}{x_1}E \\ C = -\dfrac{2x_1(x_2-x_1)(x_2D + y_2E)}{x_2^2y_1^2 - x_1^2y_2^2} \end{cases}.$$

From $I_2 = AC - B^2 > 0$ we can get $AC > 0$. We might as well assume $D > 0$, so

$$-\frac{x_2}{y_2}D < E < 0$$

and

$$I_1 = A + C < 0\,,\quad I_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = -\frac{2}{x_1} DE^2 - CD^2 - AE^2 > 0\,.$$

Therefore, the locus of equation (3) is an ellipse as long as variable *D*, *E* satisfy the condition that $I_2 = AC - B^2 > 0$.

Now we can prove that all circumscribed ellipses of $\Delta ABC$ can cover Area M. In the other words, we can prove **Theorem 1**.

## 3. The Coverage Area of Circumscribed Ellipses

In the second chapter, this paper proves that the necessary and sufficient condition for four con-elliptic points is that they can form a convex quadrilateral. Then what about five points on the same ellipse? When four of them are fixed, this problem can also be equally described as the coverage area of the circumscribed ellipse of a convex quadrilateral.

### 3.1 Parallelogram

When studying this problem, we begin with a simple and special one, which is the coverage area of the circumscribed ellipse of a parallelogram. Due to the fact that any parallelogram can be an oblique section of a column with a square bottom (Since the ratio between line segments is certain in a parallel projection, the two diagonals bisecting each other can make it), we can go a step further and change it into a more simple and special case, which is known as a square, by using the powerful tool of parallel projective transformation.

Set up a rectangular coordinate system in which A is the origin and the axes are parallel to the sides of the square (**Fig. 5**). Set $B$ $(x_0, x_0)$, and the circumscribed ellipses are expressed by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

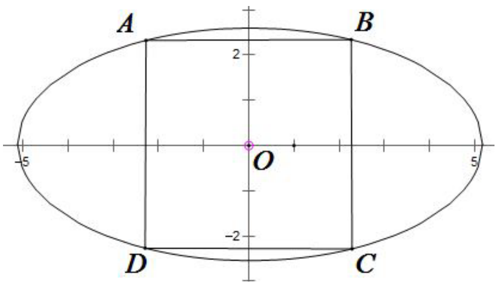

**Fig. 5** Set up a rectangular coordinate system for a square

Set the ellipse crosses the fifth point $E$ $(x_1, y_1)$, and we have

$$\begin{cases} \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1 \\ \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1. \end{cases}$$

So

$$\begin{cases} a^2 = \dfrac{x_0^2 y_1^2 - x_1^2 y_0^2}{y_1^2 - y_0^2} > 0 \\ b^2 = \dfrac{x_0^2 y_1^2 - x_1^2 y_0^2}{x_0^2 - x_1^2} > 0. \end{cases}$$

From the above inequations, we can get

$$(x_1^2 - x_0^2)(y_1^2 - y_0^2) < 0.$$

Therefore, the area of Point $E$, which also means the coverage area of the circumscribed ellipses of the square, is shown as the shaded part in **Fig. 6**.

Due to the fact that through parallel projective transformation, ellipses are still ellipses and squares change into any parallelograms[2], the coverage area of the circumscribed ellipses of a parallelogram is shown as the shaded part in **Fig. 7**.

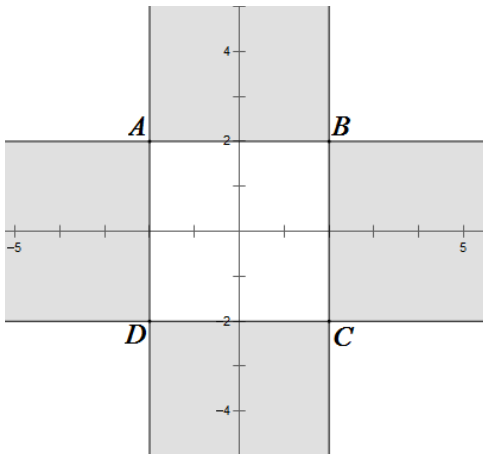


**Fig. 6** The coverage area of the circumscribed ellipses of a square is the shaded part

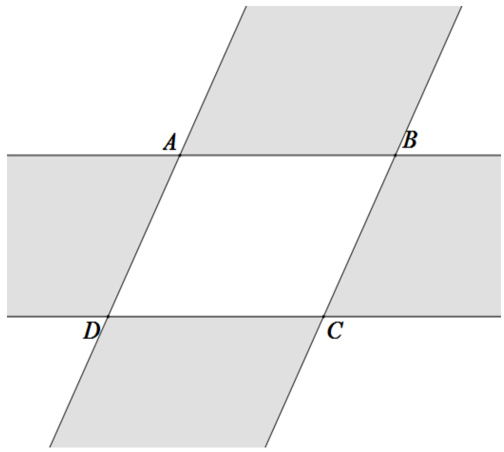


**Fig. 7** The coverage area of the circumscribed ellipses of a parallelogram is the shaded part

### 3.2 Trapezium

As the research on parallelograms, we begin with the simpler and more special one, which is a isosceles trapezium.

Let the coordinates of the four vertexes of a isosceles trapezium are $P_1(x_1, y_1)$, $P_2(x_2, -y_1)$, $P_3(-x_2, -y_1)$, $P_4(-x_1, y_1)$ (**Fig. 8**). We might as well assume$x_2 > x_1 > 0$, $y_1 > 0$. And the equations of the four sides are

$$P_1P_2: 2y_1x + (x_2 - x_1)y - (x_1 + x_2)y_1 = 0$$

$$P_3P_4: 2y_1x - (x_2 - x_1)y + (x_1 + x_2)y_1 = 0$$

$$P_1P_4: y - y_1 = 0$$

$$P_2P_3: y + y_1 = 0$$

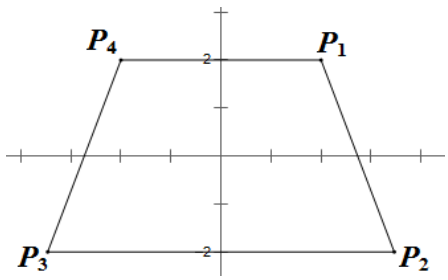


**Fig. 8** Set up a rectangular coordinate system for an isosceles trapezium

From the theorems of analytic geometry, if four points $P_1$, $P_2$, $P_3$, $P_4$ are given, the quadratic curve brunches crossing these four points can be expressed by

$$[2y_1x + (x_2 - x_1)y - (x_1 + x_2)y_1][2y_1x - (x_2 - x_1)y + (x_1 + x_2)y_1] + m(y - y_1)(y + y_1) = 0$$

After rearranging the above equation, we have the general quadratic equation

$$4{y_1}^2x^2 + [m - (x_1 - x_2)^2]y^2 + 2(x_2^2 - x_1^2)y_1y - [m + (x_1 + x_2)^2]{y_1}^2 = 0. \qquad (4)$$

Then plug the fifth point $P_5(x_5, y_5)$ into equation (4), and we can get

$$m = \frac{4{y_1}^2{x_5}^2 - (x_1 - x_2)^2{y_5}^2 + 2({x_2}^2 - {x_1}^2)y_1y_5 - (x_1 + x_2)^2{y_1}^2}{{y_1}^2 - {y_5}^2} \qquad (5)$$

Plug equation (5) into equation (4) and rearrange it, and then we have the general quadratic equation

$$4y_1x^2 + \frac{4y_1{x_5}^2 + 2({x_2}^2 - {x_1}^2)y_5 - 2({x_2}^2 + {x_1}^2)y_1}{{y_1}^2 - {y_5}^2}y^2 + 2({x_2}^2 - {x_1}^2)y$$
$$- \frac{4{y_1}^2{x_5}^2 + 2({x_2}^2 - {x_1}^2)y_1y_5 - 2({x_2}^2 + {x_1}^2){y_5}^2}{{y_1}^2 - {y_5}^2}y_1 = 0$$

Since the locus of the above equation is an ellipse, we have an inequation for it by using the discriminant (2)

$$I_2 = AC = 4y_1 \cdot \frac{4y_1{x_5}^2 + 2({x_2}^2 - {x_1}^2)y_5 - 2({x_2}^2 + {x_1}^2)y_1}{{y_1}^2 - {y_5}^2} > 0.$$

Then, we get the result that when $-y_1 < y_5 < y_1$,

$$y_5 > \frac{2y_1}{{x_2}^2 - {x_1}^2}{x_5}^2 + \frac{{x_2}^2 + {x_1}^2}{{x_2}^2 - {x_1}^2}y_1\,;$$

when $y_5 < -y_1$ or $y_5 > y_1$,

$$y_5 < \frac{2y_1}{{x_2}^2 - {x_1}^2}{x_5}^2 + \frac{{x_2}^2 + {x_1}^2}{{x_2}^2 - {x_1}^2}y_1\,.$$

And

$$y = \frac{2y_1}{{x_2}^2 - {x_1}^2}x^2 + \frac{{x_2}^2 + {x_1}^2}{{x_2}^2 - {x_1}^2}y_1$$

is the parabola crossing the four vertexes $P_1(x_1, y_1)$, $P_2(x_2, -y_1)$, $P_3(-x_2, -y_1)$, $P_4(-x_1, y_1)$. Therefore, the coverage area is shown as the **light grey part** in **Fig. 9**. Besides, it is worth noting that the area enclosed by the parabola and Side $P_1P_2$ or Side $P_3P_4$ cannot be covered.

Next, we check $I_1$ and $I_3$ of the equation.
As

$$A = 4y_1 > 0,\ I_2 = AC - B^2 = AC > 0,$$

we have

$$C > 0, \text{ so } I_1 = A + C > 0.$$

To make $I_1 I_3 < 0$, which is namely

$$I_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = A(CF - E^2) < 0.$$

We need to prove that

$$CF - E^2 < 0.$$

When $-y_1 < y_5 < y_1$, from

$$y_5 > \frac{2y_1}{{x_2}^2 - {x_1}^2}{x_5}^2 + \frac{{x_2}^2 + {x_1}^2}{{x_2}^2 - {x_1}^2} y_1 ,$$

$$x_2 > x_1 > 0,\ y_1 > 0$$

we have

$$F = -2y_1 \frac{2{y_1}^2{x_5}^2 + ({x_2}^2 - {x_1}^2)y_1 y_5 - ({x_2}^2 + {x_1}^2){y_5}^2}{{y_1}^2 - {y_5}^2}$$
$$< -2y_1 \frac{4{y_1}^2{x_5}^2 + ({x_2}^2 + {x_1}^2)({y_1}^2 - {y_5}^2)}{{y_1}^2 - {y_5}^2} < 0.$$

When $y_5 < -y_1$ or $y_5 > y_1$, from

$$y_5 < \frac{2y_1}{{x_2}^2 - {x_1}^2}{x_5}^2 + \frac{{x_2}^2 + {x_1}^2}{{x_2}^2 - {x_1}^2} y_1 ,$$

$$x_2 > x_1 > 0,\ y_1 > 0$$

we have

$$F = -2y_1 \frac{2{y_1}^2{x_5}^2 + ({x_2}^2 - {x_1}^2)y_1 y_5 - ({x_2}^2 + {x_1}^2){y_5}^2}{{y_1}^2 - {y_5}^2}$$
$$< -2y_1 \frac{4{y_1}^2{x_5}^2 + ({x_2}^2 + {x_1}^2)({y_1}^2 - {y_5}^2)}{{y_1}^2 - {y_5}^2} < 0.$$

Hence, when $P_5$ is in the light grey part shown in **Fig. 9**, we always have $F < 0$. Then we get $CF - E^2 < 0$, so $I_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = A(CF - E^2) < 0.$

Now we can prove that the coverage area of the circumscribed ellipses of the isosceles trapezium is the **light grey part** in **Fig. 9**.

Due to the fact that parallelism does not change through parallel projective transformation[2], the coverage area of circumscribed ellipses of trapezium is shown as the **light grey part** in **Fig. 10**.

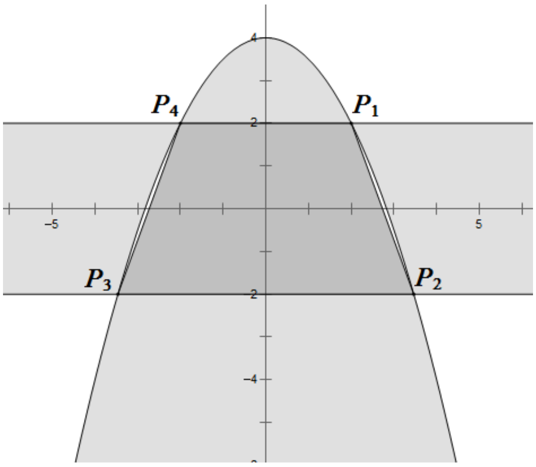


**Fig. 9** The coverage area of the circumscribed ellipses of an isosceles trapezium is the light grey part

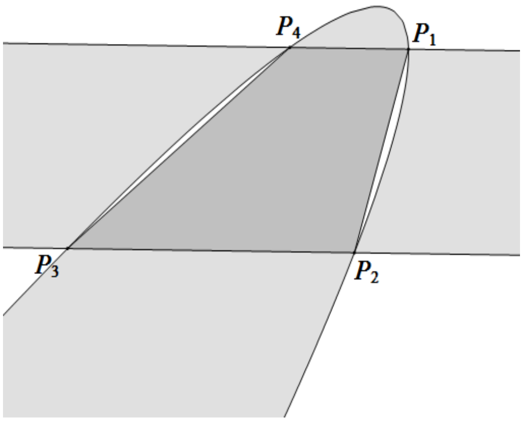


**Fig. 10** The coverage area of the circumscribed ellipses of a trapezium is the light grey part

### 3.3 General Convex Quadrilateral

From the discussion about the coverage area of the circumscribed ellipses of the two kinds of special convex quadrilaterals above, we found that when a pair of opposite sides are parallel, the boundary of the coverage area takes place on the parallel lines where the two sides are located; on the other hand, when the two opposite sides are not parallel, the boundary is the parabola crossing its four vertexes. In this case, the parallel lines can be regarded as a degenerated form of a parabola. Therefore, we guess that for a general convex quadrilateral whose opposite side are both not parallel, the boundaries of its coverage area are two parabolas crossing its four vertexes. Next, we will prove the guess.

Let the coordinates of the four vertexes of a general convex quadrilateral are $P_1(x_1, y_1)$, $P_2(x_2, 0)$, $P_3(0,0)$, $P_4(x_4, y_4)$ (**Fig. 11**). ( $y_1 \neq y_4$, $\frac{y_4}{x_4} \neq \frac{y_1}{x_1 - x_2}$ )

Using the same procedure as in Chapter 3.2, we first list the equations of the four sides

$$P_1P_2: y_1x + (x_2 - x_1)y - x_2y_1 = 0,$$

$$P_3P_4: y_4x - x_4y = 0,$$

$$P_1P_4: (y_1 - y_4)x - (x_1 - x_4)y - x_4y_1 + y_4x_1 = 0,$$

$$P_2P_3: y = 0.$$

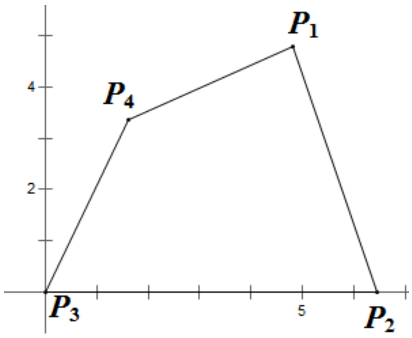


**Fig. 11** Set up a rectangular coordinate system for a general convex quadrilateral

Then, the quadratic curve brunches crossing these four points can be expressed by

$$[y_1x+(x_2-x_1)y-x_2y_1](y_4x-x_4y)+my[(y_1-y_4)x-(x_1-x_4)y-x_4y_1+y_4x_1]=0$$

After rearranging the above equation, we have the general quadratic equation

$$y_1y_4x^2+[-x_4y_1-y_4x_1+y_4x_2+m(y_1-y_4)]xy+[x_4(x_1-x_2)-m(x_1-x_4)]y^2$$
$$-x_2y_1y_4x+[x_2x_4y_1+m(-x_4y_1+y_4x_1)]y=0. \qquad (6)$$

Then plug the fifth point $P_5(x_5,y_5)$ into equation (6), and we can get

$$m=-\frac{[y_1x_5+(x_2-x_1)y_5-x_2y_1][y_4x_5-x_4y_5]}{y_5[(y_1-y_4)x_5-(x_1-x_4)y_5-x_4y_1+y_4x_1]} \qquad (7)$$

Since the locus of the equation (6) is an ellipse, we have an inequation for it by using the discriminant (2)

$$I_2=y_1y_4[x_4(x_1-x_2)-m(x_1-x_4)]-\frac{1}{4}[-x_4y_1-y_4x_1+y_4x_2+m(y_1-y_4)]^2>0$$

Namely, we have

$$(y_1-y_4)^2m^2+[2(-x_4y_1-y_4x_1+y_4x_2)(y_1-y_4)+4y_1y_4(x_1-x_4)]$$
$$+(-x_4y_1-y_4x_1+y_4x_2)^2-4y_1y_4x_4(x_1-x_2)<0 \qquad (8)$$

Since $y_1\neq y_4$, inequation (8) is a quadratic one of $m$. If the inequation has no solution, there exists no circumscribed ellipse of the convex quadrilateral. However, in Chapter 2 we have proved that there exists circumscribed ellipse of any convex quadrilateral, from which we can release the controversial result. Therefore, inequation (8) must have solution.

As the coefficient of the quadratic term $(y_1-y_4)^2>0$, we can assume the solution of inequation (8) is $k_1<m<k_2$.

When $m=k_1,\ k_2$, the left side of inequation (8) equals zero, which namely means $I_2=0$.

Since

$$\frac{y_4}{x_4}\neq\frac{y_1}{x_1-x_2},$$

the constant term

$$(-x_4y_1-y_4x_1+y_4x_2)^2-4y_1y_4x_4(x_1-x_2)\neq 0.$$

Hence, $k_1,k_2\neq 0$.

Plug equation (7) into $k_1<m<k_2$, we have

$$k_1 < -\frac{[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5]}{y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1]} < k_2.$$

When

$$y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] > 0,$$

which means $P_5$ is above $P_1P_4$ or below $P_2P_3$,
we have

$$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] < 0 \quad (9)$$

$$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_2y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] > 0 \quad (10)$$

Since
$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1](y_4x_5 - x_4y_5) + my_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1]=0$
is the quadratic curve crossing $P_1$, $P_2$, $P_3$, $P_4$. And when $m = k_1$, $k_2$, $I_2 = 0$. Hence, the left side of inequation (9), (10) represent the two parabolas crossing $P_1$, $P_2$, $P_3$, $P_4$. In the other words, $P_5$ is in the parabola whose equation is
$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1](y_4x_5 - x_4y_5) + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1]=0$;
and $P_5$ is also out of the parabola whose equation is
$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1](y_4x_5 - x_4y_5) + k_2y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1]=0$.

It is worth mentioning that since $I_2 = 0$ is a quadratic equation of *m*, it has only two roots, which means there are only two parabolas crossing the four vertexes of the convex quadrilateral.

Using the same argument, when

$$y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] < 0,$$

which means $P_5$ is between $P_1P_4$ and $P_2P_3$,
we have

$$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] > 0$$

$$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] < 0$$

So $P_5$ is out of the parabola whose equation is
$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1](y_4x_5 - x_4y_5) + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1]=0$;
and $P_5$ is also in the parabola whose equation is
$[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1](y_4x_5 - x_4y_5) + k_2y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1]=0$.

In conclusion, the coverage area of circumscribed ellipses of general convex quadrilateral is shown as the **light grey part** in **Fig. 12**. Like the case of trapezium, the area enclosed by two parabolas and four sides cannot be covered.

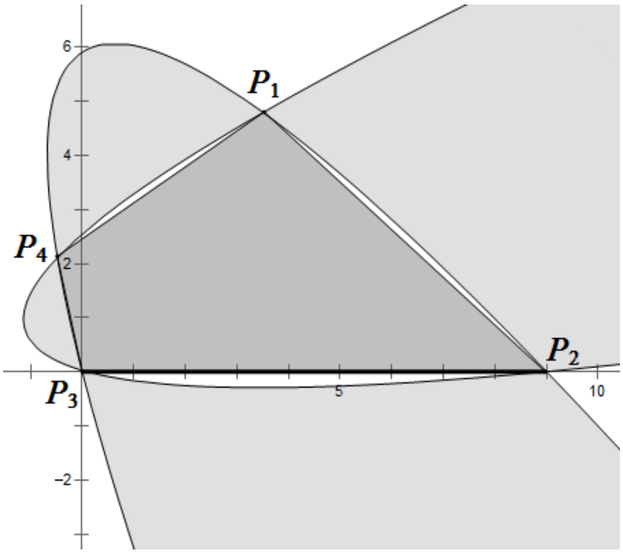


**Fig. 12** The coverage area of the circumscribed ellipses of a general convex quadrilateral is the light grey part

### 3.4 Circumscribed Conics of Convex Quadrilaterals

Through the studies in the three chapters above, we conclude that the boundaries of the coverage area of circumscribed ellipses are another conic, which is parabola. So what about the third conic, which is hyperbola? Next, we will generalize the conclusion about *the coverage area of circumscribed ellipses* to the *coverage area of circumscribed conics*.

As mentioned above, for a general quadratic equation in two variables, we can use discriminants (2) to determine its locus. When $I_2 > 0$, the graph of the quadratic curve is elliptic; and it needs to satisfy $I_1 I_3 < 0$ to become a real ellipse. When $I_2 = 0$, the graph is parabolic; and moreover, when $I_3 \neq 0$, it is a parabola; when $\mathrm{I}_3 = 0$, it degenerates into two parallel lines. When $\mathrm{I}_2 < 0$, the graph is hyperbolic; and moreover, when $\mathrm{I}_3 \neq 0$, it is a hyperbola; when $I_3 = 0$, it degenerates into two intersecting lines.

In Chapter 3.3, we conclude that the light grey part in **Fig. 12** is the coverage area of the circumscribed ellipses, where $I_2 > 0$. Therefore, the area which cannot be covered by the circumscribed ellipse (except the boundary region) can be covered by the circumscribed hyperbolas (including the degenerated condition, namely two intersecting lines), where $I_2 < 0$ (as shown the shaded part in **Fig. 13**). Besibes, the boundary of this two coverage areas is the parabola crossing the four vertexes, where $I_2 = 0$.

So far, the plane where a convex quadrilateral is located can be divided into three parts: one part con-elliptic with its four vertexes (**Fig. 12**); another part con-parabolic with the vertexes (namely two parabolas crossing the four vertexes of the convex quadrilateral); and the third part, which is con-hyperbolic with the vertexes (**Fig. 13**).

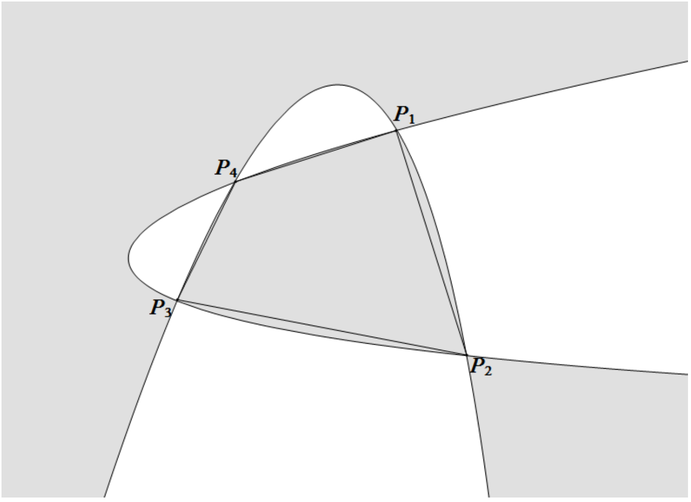


**Fig. 13** The coverage area of the circumscribed hyperbolas of a general convex quadrilateral is the shaded part

Normally, the circumscribed hyperbolas should be classified into three categories:

- **"2+2" hyperbola** *Two vertexes are on one branch of the hyperbola, while the other two are on the other branch* (**Fig. 14**).
- **"3+1" hyperbola** *Three vertexes are on one branch of the hyperbola, while the forth is on the other branch*
- **"4+0" hyperbola** *All of four vertexes are on one branch of the hyperbola* (**Fig. 15**).

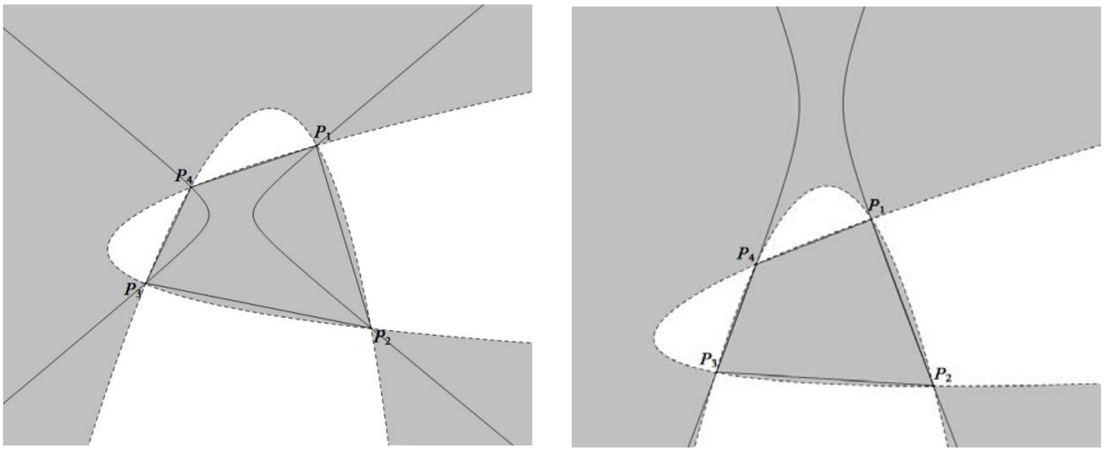


**Fig. 14** "2+2" hyperbola, which means two vertexes are on one branch while the other two are on the other branch

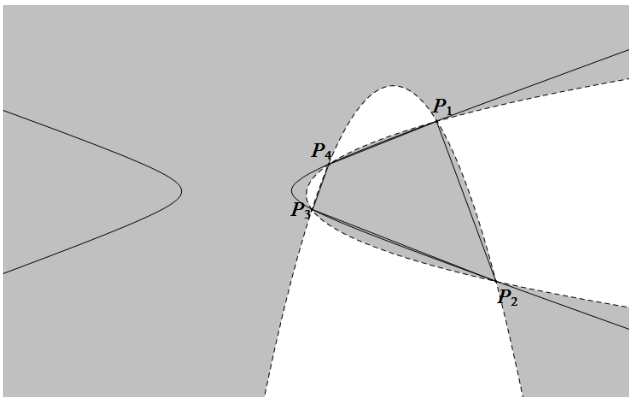


**Fig. 15** "4+0" hyperbola, which means all of four vertexes are on one branch

However, not all of these three kinds can be circumscribed hyperbolas of a convex quadrilateral. Through some studies, we put forward the following proposition.

**Proposition 1** *There exists no "3+1" hyperbola which is circumscribed about a convex quadrilateral.*

*Proof* Suppose the vertex of $P_1$ is on the left branch of the hyperbola, and the other three vertexes $P_2$, $P_3$, $P_4$ are on the right branch. According to the definition of convex quadrilaterals, we can infer that $P_4$ must be located in the area enclosed by rays formed by any two of the sides of $\Delta P_1P_2P_3$ and its third side (the **light grey part** in **Fig. 16**).

Due to the fact that each line has and only has two intersections with the hyperbola, and the slope of $P_2P_3$ is greater than the asymptote while that of $P_1P_3$ is smaller than it, we can infer that the right branch must be out of the light grey area which is to the right of $P_2P_3$ (as shown in **Fig. 16**). And the rest can be proved in the same way.

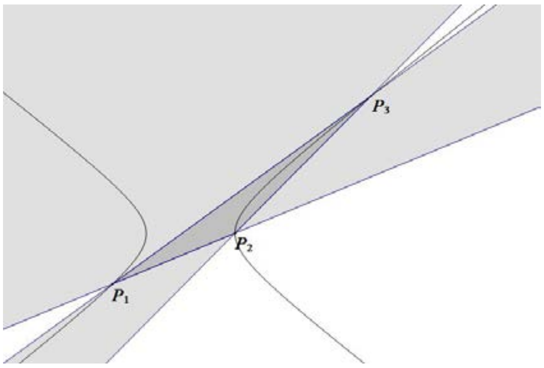


**Fig. 16** There exists no "3+1" hyperbola which is circumscribed about a convex quadrilateral

Therefore, the right branch must be outside of the light grey area, and therefore the proposition holds true.

In conclusion, the circumscribed hyperbolas only can be classified into two categories: **"2+2" hyperbolas** (**Fig. 14**) & **"4+0" hyperbolas** (**Fig. 15**).

Furthermore, we have discussed the coverage area of these two kinds of hyperbolas by using limit thought, and the results are shown in **Fig. 17**. The **"2+2" hyperbolas** cover the **light grey area**, while the **"4+0" hyperbolas** cover the **dark grey area**.

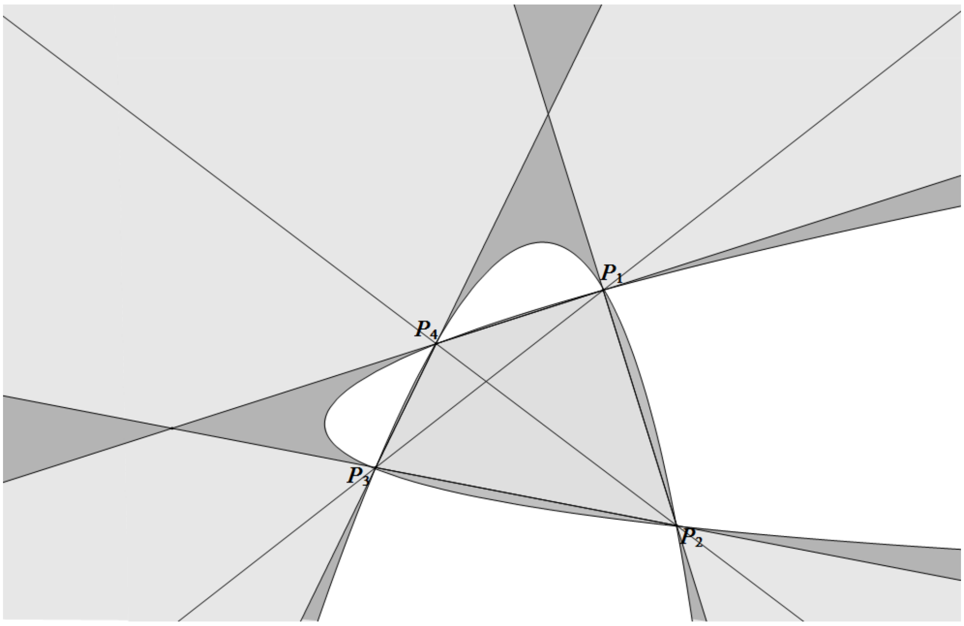


**Fig. 17** The “2+2” hyperbolas cover the light grey area, while the “4+0” hyperbolas cover the dark grey area

# 4. The Locus of the Center of Circumscribed Conics

## 4.1 Convex Quadrilateral

*For a parallelogram, the center of its circumscribed conics is necessarily the center of the parallelogram.* For an isosceles trapezium, since conics and isosceles trapeziums are both mirror-symmetrical graphs, the center of its circumscribed conics must locate at the line which crosses the midpoints of its bases. Due to the fact that through parallel projective projection, the ratio of segments and parallelism do not change, *the center of circumscribed conics of a trapezium must locate at the line which crosses the midpoints of its bases.* Then, what is the locus of the center of circumscribed conics of a general convex quadrilateral? To answer this question, we put forward the following theorem through a series of research.

**Theorem 2** *The locus of the center of circumscribed conics of a convex quadrilateral is its nine-point curve, which is a hyperbola. Moreover, the locus of the center of circumscribed ellipses is a branch of it, and the locus of the center of circumscribed hyperbolas is the other branch* (**Fig. 18**).

Below, we will simply introduce the nine-point curve.

For a quadrilateral, the intersection of the diagonals, two intersections of the opposite sides, four midpoints of the sides and two midpoints of diagonals are necessarily on a conic, which is called the nine-point curve of it. The nine-point curve is always a centered conic. Whether it is a hyperbola or an ellipse depends on whether the quadrilateral is convex or concave. Moreover, the center of the nine-point curve is the barycenter of the quadrilateral.

The following situations are two special cases of nine-point curve:

- *When the quadrilateral has a circumscribed circle, its nine-point curve is a rectangular hyperbola.*
- *When the four vertexes of the quadrilateral form an orthocentric system, its nine-point curve is a circle, which is known as the nine-point circle.*

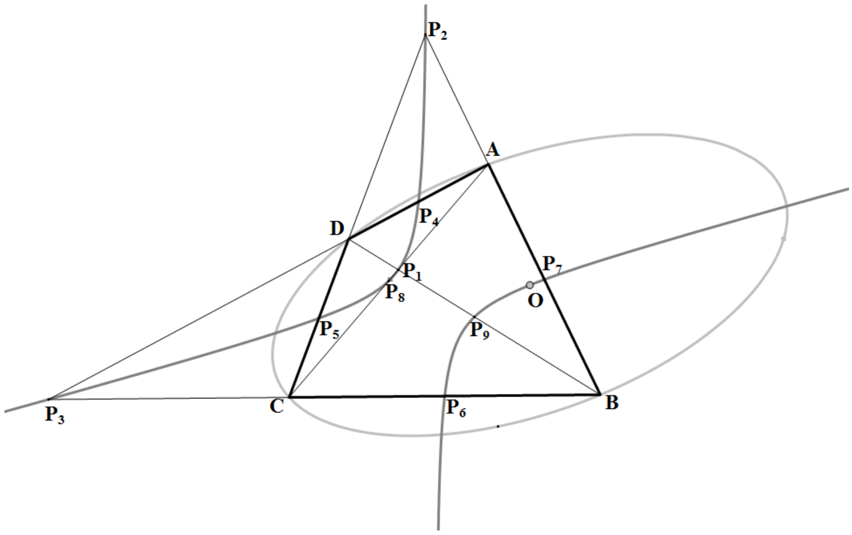


**Fig. 18** The locus of the center of circumscribed conics of a convex quadrilateral is its nine-point curve, which is a hyperbola

To prove the theorem, we can use equation (6) to express the circumscribed conics of the quadrilateral, and figure out the equation of the center of the conics. The equation is quadratic, and the nine special points which the nine-point curve crosses satisfy the equation. Since the nine point can and only can determine one quadratic curve, the locus of the center of circumscribed conics is the nine-point curve.

Besides, we can draw the following corollary from **Theorem 2**.

**Corollary 1** *The axes of symmetry of a general convex quadrilateral's two circumscribed parabolas of are parallel to the asymptotic lines of its nine-point curve respectively* **(Fig. 19)**.

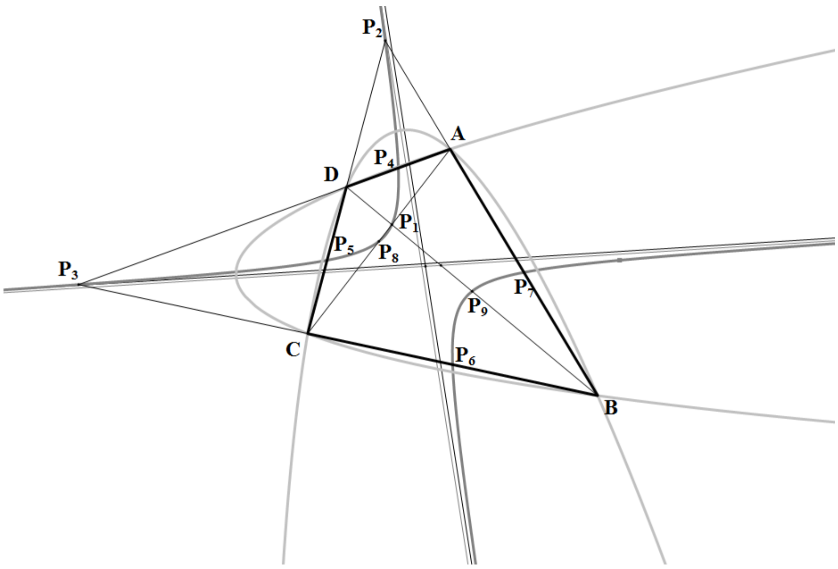


**Fig. 19** Corollary 1

*Proof* A parabola is a non-centered conic, and it can also be regarded as the critical situation of an ellipse or a hyperbola. Hence, if we also regard it as a centered conic, we can have the following definition:

**Definition 1** *The center of a parabola is the point at infinity which is determined by its axis of symmetry.*

From **Theorem 2**, we know that the locus of the center of a general convex quadrilateral is its nine-point curve. Therefore, the centers of the two circumscribed parabolas are necessarily on the nine-point curve. On the other hand, since the nine-point curve of a convex quadrilateral is a hyperbola, it is infinitely near its asymptotic lines at infinity. Therefore, the asymptotic lines cross the centers of the two circumscribed parabolas.

In conclusion, the axes of symmetry of the two circumscribed parabolas of a general convex quadrilateral are parallel to the asymptotic lines of its nine-point curve respectively.

### 4.2 Concave Quadrilateral

As mentioned in Chapter 4.2, a concave quadrilateral also has a nine-point curve, which is an ellipse. Taking a step further, we generalized **Theorem 2** to concave quadrilaterals.

**Theorem 3** *The locus of the center of circumscribed conic of a concave quadrilateral is its nine-point curve, which is an ellipse* **(Fig. 20)**.

It is worth mentioning that the circumscribed conics of a concave quadrilateral can only be **"3+1" hyperbolas**, while the circumscribed conics of a convex quadrilateral only cannot be **"3+1" hyperbolas**, which is proved in Chapter 3.4.

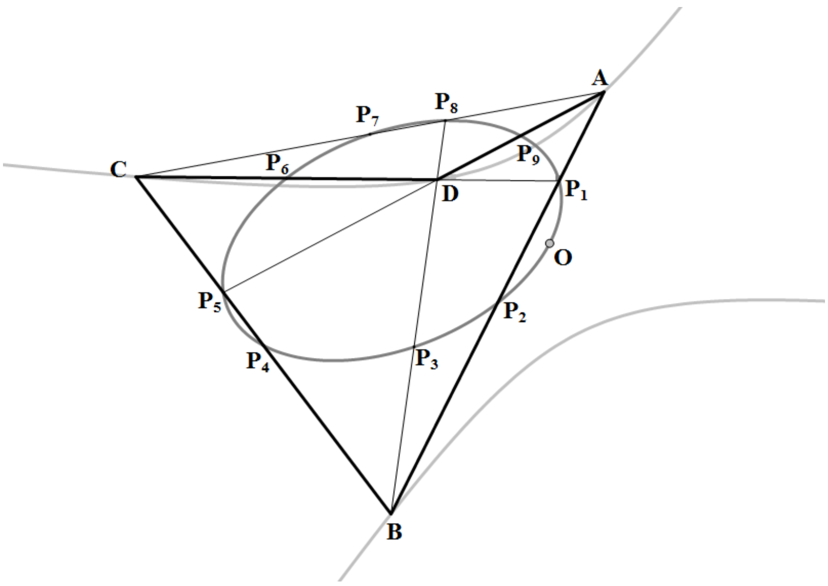


**Fig. 20** The center O of the circumscribed conics of a concave quadrilateral is on its nine-point curve, which is an ellipse

## 5. The Minimal Area of Circumscribed Ellipses

Since ellipses are widely used in architecture, engineering and medicine, the conclusions in this section may have potential applications in real life.

### 5.1 Parallelogram

It is well known that among the inscribed parallelograms of an ellipse, the area of the parallelogram is maximal when its diagonals are the conjugate diameters of the ellipse. In that way, among the circumscribed ellipses of a parallelogram, is the area of the ellipse minimal when its conjugate diameters are the diagonals of the parallelogram? The answer should be yes. Next, the paper will prove the theorem.

**Theorem 4** *When the conjugate diameters of the circumscribed ellipse are the diagonals of the parallelogram, the area of the circumscribed ellipse is minimal.*

*Proof* As mentioned in Chapter 3.1, any parallelogram can be an oblique section of a column with a square bottom. Let $\theta$ is the dihedral angle between the oblique section and the bottom and $\theta$ is unique. Suppose $E$ is a circumscribed ellipse of the parallelogram, so its projection on the oblique section *E'* is a circumscribed ellipse of the square, which is the bottom. According to the fundamental theorem in projective geometry, we know that $S_{E'} = S_E \cdot \cos\theta$. Therefore, when and only when $S_{E'}$ is minimal, $S_E$ is minimal.

Thus we drew the following lemma.

**Lemma 2** *Among the circumscribed ellipses of a square, the area of the circumscribed circle is the minimum.*

*Proof* Since squares and ellipses are both central symmetry, the center of its circumscribed conics is necessarily the center of the parallelogram.

Suppose a square *ABCD*, and its center is *O*. Set up a rectangular coordinate system in which the center *O* is the origin. We first consider the situation where the axes of the circumscribed ellipses are not parallel to the square's sides (**Fig. 21**). Let $A\ (x_1, y_1)$ , $B\ (x_2, y_2)$ , and the circumscribed ellipses are expressed by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since *A*, *B* are on the ellipse, we have

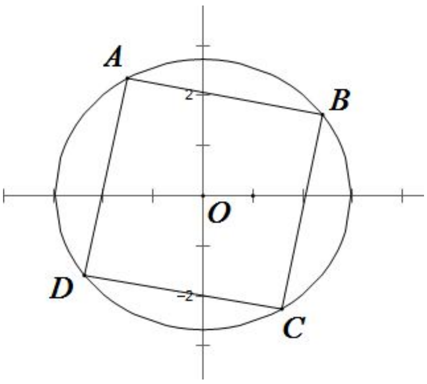


**Fig. 21** The axes of the circumscribed ellipses are not parallel to the square's sides

$$\begin{cases} \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \\ \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1 \end{cases}$$

Then we can get

$$\begin{cases} a^2 = \frac{x_1^2 y_2^2 - x_2^2 y_1^2}{y_2^2 - y_1^2} \\ b^2 = \frac{x_1^2 y_2^2 - x_2^2 y_1^2}{x_1^2 - x_2^2} \end{cases}$$

Plug $x_1^2 + y_1^2 = x_2^2 + y_2^2$ in to the above equations, and we have

$$a^2 = b^2.$$

Hence, there exists one and only one circumscribed ellipse of the square, which is the circumscribed circle.

Next, we consider the situation where the axes of the circumscribed ellipses are parallel to the square's sides (**Fig. 22**). Let $B(x_0, x_0)$, and the circumscribed ellipses are expressed by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since $B$ is on the ellipse, we have

$$\frac{x_0^2}{a^2} + \frac{x_0^2}{b^2} = 1.$$

By using the fundamental inequations, we have

$$\frac{1}{x_0^2} = \frac{1}{a^2} + \frac{1}{b^2} \geq 2\sqrt{\frac{1}{a^2 b^2}} = \frac{2}{ab}.$$

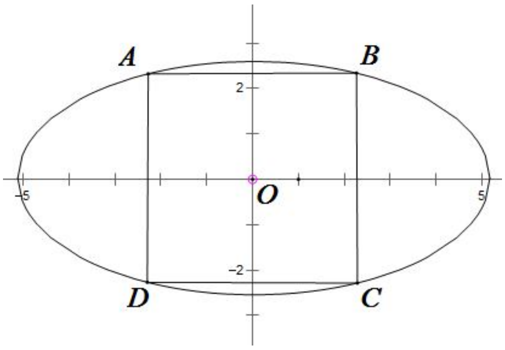


**Fig. 22** The axes of the circumscribed ellipses are parallel to the square's sides

Namely, we have

$$ab \geq 2x_0^2,$$

so

$$S = \pi ab \geq 2\pi x_0^2.$$

When and only when $a = b = x_0$, the equality holds, which namely means the area is the minimum. At this time, the circumscribed ellipse of the square is the circumscribed circle.

In conclusion, among the circumscribed ellipses of a square, the area of the circumscribed circle is the minimum.

It is obvious that the diagonals of the square are the perpendicular diameters of its circumscribed circle. So what about their projections on the section? To answer the question, we need to use the following lemma.

**Lemma 3** *If the projections of an ellipse's two diameters on its bottom are perpendicular, these two diameters must be conjugate* (**Fig. 23**).

*Proof* Suppose there is a cylinder. Let *A'B'* and *C'D'* are the perpendicular diameters of the bottom. Draw *M'N'*// *A'B'*. Let *K'* is the midpoint of chord *M'N'*, so its projection *K* is the midpoint of *MN*[2]. Since parallelism does not change through projection, we also have *MN*// *AB*. Thus *AB* and *CD* are conjugate.

Now we can prove **Lemma 3**.

From **Lemma 3**, we can conclude that the projections of the circle's two diameters on its section are conjugate. Hence, we can prove **Theorem 4**.

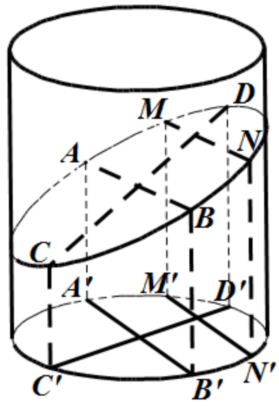

**Fig. 23** Lemma 3

Moreover, to figure out the value of the minimal area, we drew the following theorem.

**Theorem 5** *The minimal area is* $\frac{\pi}{2}$ *times of the area of the parallelogram.*

*Proof* From **Lemma 2**, when *E* is the minimal circumscribed ellipse of the parallelogram, which is a section, its projection *E'* is the circumscribed circle of the square, which is the bottom. Let the area of the parallelogram is *S*, while the area of the square is *S'*. According to the fundamental theorem in projective geometry, we know that $S' = S \cdot \cos\theta$ and $S_{E'} = S_E \cdot \cos\theta$. Thus we can get the following equations.

$$\frac{S_E}{S} = \frac{S_{E'}}{S'} = \frac{\pi r^2}{2r^2} = \frac{\pi}{2}$$

Hence, we can prove **Theorem 5**.

**Drawing Method of the Minimal Circumscribed Ellipse of a Parallelogram**

Since we have found out the minimal circumscribed ellipse of parallelograms, we need to find the drawing method. According to **Theorem 4**, this proposition is equivalent to drawing an ellipse of which the conjugate diameters are the diagonals of the parallelogram. So next we will point out the drawing method of the ellipse when we know a pair of conjugate diameters of it (**Fig. 24**).

If we set up a corresponding relationship between *AD*, *BC* and a pair of conjugate diameters of a circle, the corresponding ellipse to the circle is namely what we want.

**Drawing Steps**:

1. Let *BC* be coincide with *B'C'* and draw another diameter $A'D' \perp B'C'$. Then we can determine the corresponding relationship between the circle and the ellipse by the axis *BC* (*B'C'*) and a pair of corresponding points *A*-*A'*.

2. Through any point *N'* on the circle, draw $\Delta N'NM \backsim \Delta A'AO$. *N* is on the ellipse and *M* is on the axis *BC*.

3. Same as above, draw a series of points similar to point N and join them together with a smooth curve. This is namely the ellipse we want.

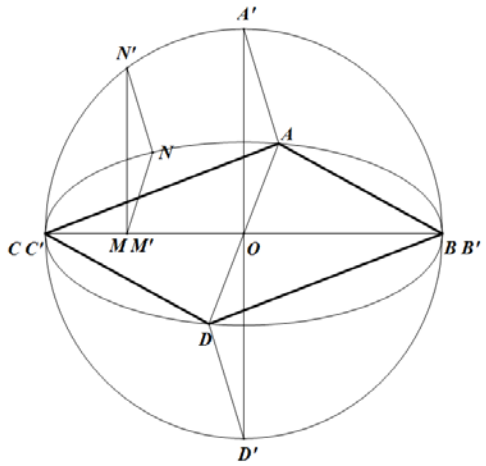


**Fig. 24** The drawing method of the minimal circumscribed ellipse of a parallelogram

From the drawing method, we can easily draw the following theorem and corollary:

**Theorem 5** *There exists one and only one ellipse of which the conjugate diameters are the diagonals of a parallelogram.*

**Corollary 2** *There exists one and only one minimal circumscribed ellipse of a parallelogram.*

**5.2 Trapezium**

**Theorem 6** *Let M, N are the midpoints of the bases of a trapezium. The center O of the minimal circumscribed ellipse of a trapezium is on MN, and it satisfies the ratio*

$$\frac{MO}{ON} = \left|\frac{(1-2m^2)+\sqrt{m^4-m^2+1}}{(2-m^2)-\sqrt{m^4-m^2+1}}\right|,$$

*where m is the ratio between two bases* (**Fig. 25**).

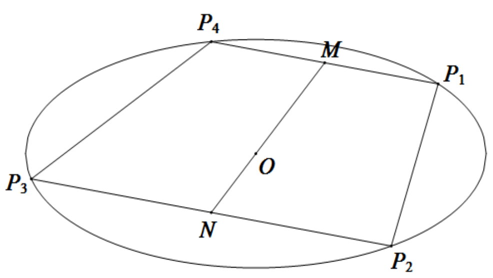


**Fig. 25** The minimal circumscribed ellipse of a trapezium

*Proof* As the procedures above, we begin with the simpler and more special one, which is a isosceles trapezium.

If an isosceles trapezium and the shape of its circumscribed ellipse (namely the eccentricity) are given, the ellipse is determined. In order to facilitate the research, the paper uses the ratio between major and minor axes $k=\frac{b}{a}\left(k\in R^{+}\right)$ to describe the shape of a circumscribed ellipse.

Suppose the equation of the circumscribed ellipses is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and $k=\frac{b}{a}\left(k\in R^{+}\right)$. Due to the fact that ellipses and isosceles trapeziums are both graphs of axial symmetry, the symmetry axis of the isosceles trapezium must be the major or minor axis of the circumscribed ellipses. We might as well assume *y axis* is its symmetry axis (**Fig. 26**). Let $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(-x_2,y_2)$, $P_4(-x_1,y_1)$, and $x_2>x_1>0$, $y_1>0$, $y_2<0$. In this case, $x_1$ and $x_2$, which are half of the two bases of the isosceles trapezium , are fixed values; $y_1$ and $y_2$ are non-fixed values, while they must satisfy $y_1-y_2=y_0$. Here $y_0$ is the height of the isosceles trapezium, which is also a fixed value.

Since the four vertexes are on the ellipse, we have

$$\begin{cases}\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1\end{cases}$$

So

$$\begin{cases}\frac{y_1}{b}=\sqrt{1-\frac{x_1^2}{a^2}}\\ \frac{y_2}{b}=-\sqrt{1-\frac{x_2^2}{a^2}}\end{cases}$$

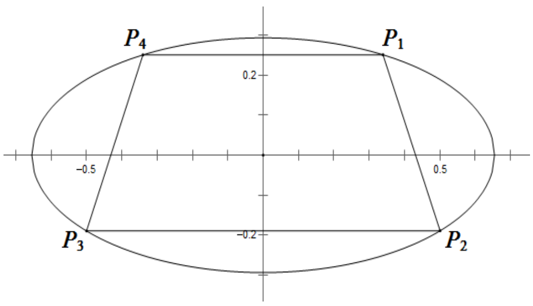


**Fig. 26** Set up a rectangular coordinate system for an isosceles trapezium

Subtract the above equations

$$\frac{y_1}{b}-\frac{y_2}{b}=\frac{y_0}{b}=\sqrt{1-\frac{x_1^2}{a^2}}+\sqrt{1-\frac{x_2^2}{a^2}}.$$

Plug $b=ak$ into it, then we get

$$a^2=\frac{y_0^4+k^4(x_2^2-x_1^2)^2+2k^2y_0^2(x_2^2+x_1^2)}{4k^2y_0^2}.$$

Thus the area of the circumscribed ellipses can be expressed by

$$S=\pi ab=\pi a^2k=\pi\frac{y_0^4+k^4(x_2^2-x_1^2)^2+2k^2y_0^2(x_2^2+x_1^2)}{4ky_0^2}.$$

Set

$$f(k) = \frac{4y_0^2}{\pi} S = \left(x_2^2 - x_1^2\right)^2 k^3 + 2y_0^2 (x_2^2 + x_1^2) k + \frac{y_0^4}{k}.$$

To get the minimum of $S$, $f(k)$ should achieve its minimum, where its derivative is equal to zero.

$$f'(k) = 3(x_2^2 - x_1^2)^2 k^2 + 2y_0^2 (x_2^2 + x_1^2) - \frac{y_0^4}{k^2} = 0$$

So we can get

$$k^2 = \frac{-y_0^2 (x_2^2 + x_1^2) + 2y_0^2 \sqrt{x_1^4 + x_2^4 - x_1^2 x_2^2}}{3(x_2^2 - x_1^2)^2}$$

On the other hand, from

$$\begin{cases} \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1 \\ \dfrac{x_2^2}{a^2} + \dfrac{y_2^2}{b^2} = 1, \end{cases}$$

we have

$$\frac{x_2^2 - x_1^2}{a^2} = \frac{y_1^2 - y_2^2}{b^2} = \frac{(y_1 - y_2)(y_1 + y_2)}{a^2 k^2} = \frac{y_0 (y_1 + y_2)}{a^2 k^2}.$$

So

$$y_1 + y_2 = \frac{k^2 (x_2^2 - x_1^2)}{y_0}.$$

Since $y_1 - y_2 = y_0$, we can get

$$\left|\frac{y_1}{y_2}\right| = \left|\frac{y_0^2 + k^2 (x_2^2 - x_1^2)}{y_0^2 - k^2 (x_2^2 - x_1^2)}\right|.$$

Plug

$$k^2 = \frac{-y_0^2 (x_2^2 + x_1^2) + 2y_0^2 \sqrt{x_1^4 + x_2^4 - x_1^2 x_2^2}}{3(x_2^2 - x_1^2)^2}$$

into the above equation, we get

$$\left|\frac{y_1}{y_2}\right| = \left|\frac{(x_2^2 - 2x_1^2) + \sqrt{x_1^4 + x_2^4 - x_1^2 x_2^2}}{(2x_2^2 - x_1^2) - \sqrt{x_1^4 + x_2^4 - x_1^2 x_2^2}}\right|.$$

Let $m = \frac{x_1}{x_2}$, which namely means the ration between the bases. Plug $m = \frac{x_1}{x_2}$ into the above equation, we finally get

$$\left|\frac{y_1}{y_2}\right| = \left|\frac{(1 - 2m^2) + \sqrt{m^4 - m^2 + 1}}{(2 - m^2) - \sqrt{m^4 - m^2 + 1}}\right|.$$

As mentioned is Chapter 4.1, for an isosceles trapezium, the center of its circumscribed conics must locate at the line which crosses the midpoints of its bases. Therefore, the value of $\left|\frac{y_1}{y_2}\right|$ indicates the location of the center of the minimal circumscribed ellipse. Now we can prove **Theorem 6** holds in this special case of isosceles trapeziums (Fig. 27).

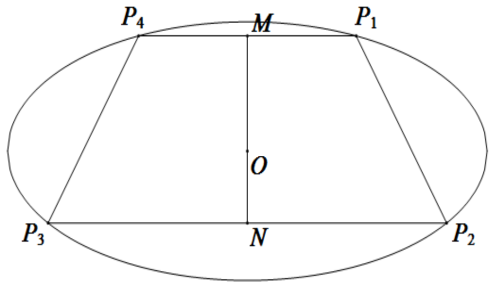


**Fig. 27** The minimal circumscribed ellipse of an isosceles trapezium

Due to the fact that parallelism and ratio between line segments do not change through parallel projective transformation[2], we can prove **Theorem 6** holds in all cases of trapeziums.

## 6. Circumscribed Ellipses of Cyclic Quadrilaterals

Since any convex quadrilateral can be transformed to a cyclic quadrilateral through parallel projection, it is helpful for research on the general convex quadrilaterals to study the properties of cyclic quadrilaterals. Therefore, this paper has researched the connection between cyclic quadrilaterals and their circumscribed ellipses.

**Theorem 7**: *Let A, B, C, D are four points on the given ellipse of which the major diameters are parallel to the coordinate axis. If A, B, C, D are cyclic, the opposite sides of the quadrilateral formed by these four points will locate on two lines of which the slope angles are complementary* (**Fig. 28**).

*Proof* For a given ellipse, set up a rectangular coordinate system of which the coordinate axis is parallel to its major axis (**Fig.28**), then the equation of the ellipse can be expressed by

$$A_1x^2 + C_1y^2 + D_1x + E_1y + F_1 = 0.$$

Suppose a cyclic quadrilateral *ABCD*, whose vertexes are on the ellipse above. Let the four vertexes are on the circle

$$x^2 + y^2 + D_0x + E_0y + F_0 = 0.$$

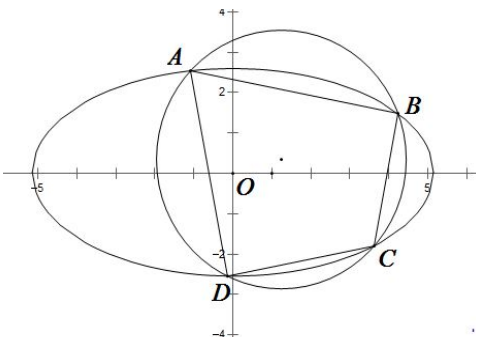


**Fig. 28** Set up a rectangular coordinate system for a cyclic quadrilateral

Then the quadratic curve crossing these four points can surely be expressed by

$$A_1x^2 + C_1y^2 + D_1x + E_1y + F_1 + \lambda(x^2 + y^2 + D_0x + E_0y + F_0) = 0. \qquad (11)$$

On the other hand, the equation of a quadratic curve crossing the four points can also be expressed by

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0. \tag{12}$$

Obviously, there are no terms containing $xy$ in equation(11). So by comparing equation(11) and (12), we can get

$$a_1b_2 + a_2b_1 = 0.$$

Since the curve expressed in equation (11) crosses all the four given points *A*, *B*, *C*, *D* and there are no collinear points, the two lines expressed by equation (12) each respectively crosses two of the points. In brief, equation (12) can express any quadratic curve made up of the two lines where any pair of opposite sides of the quadrilateral *A*, *B*, *C*, *D* is located.

Therefore, we can know that any pair of opposite sides of the complete quadrilateral *A*, *B*, *C*, *D* locates on two lines of which the slope angles are complementary. And the slopes of the two lines are opposite numbers if they exist, namely

$$\frac{a_1}{b_1} = -\frac{a_2}{b_2}.$$

Now, we can prove **Theorem 7**.

From the theorem above, we can infer two corollaries as following:

**Corollary 3** *For a circumscribed ellipse of a cyclic quadrilateral, the included angles between its axis and the opposite sides of the quadrilateral are equal.*

**Corollary 4** *For the circumscribed ellipses of a cyclic quadrilateral, their major axes are parallel to each other.*

## 7. Postscript and Perspective

During the course of researching, the author had some insight into the innate connection of conic sections as well as a taste of the beauty and harmony of geometry. As a few beautiful properties and theorems were found, due to the limited time and knowledge, some questions yet remained to be solved:

- In the discussion of hyperbola in Chapter 3.4, we found out the coverage area of 2 kinds of hyperbola by using intuitive figures and the concept of limit, while rigorous proof remains to be given.
- In the discussion of circumscribed conic sections in Chapter 4, rigorous proof of our conclusion is also uncompleted.

Meanwhile, further study has been scheduled as following:

- We will research other properties of the circumscribed conic curves of quadrilaterals (such as the locus of the foci, the range of the eccentricity) so as to reveal the deeper connection between convex quadrilaterals and their circumscribed ellipses.
- If possible, we hope to extend the above results to three-dimensional situations,

which is namely the circumscribed quadric surface of a solid figure. Since quadrics are widely applied to construction engineering, I believe such study has bright perspective.

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