The Diameter of (Directed) Graphs

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Abstract

The estimation for the diameter of connected simple graph has a wide application. Due to the lack of previous result of estimation for connected simple (directed) graphs, we focus our research on this topic. In the paper, we will give some bounds of diameter of connected simple (directed) graphs involving different parameters of graphs.

1 Introduction

We use Xu [1], and Bondy and Murty [2] for terminology and notation not defined here. Let $G(V, E)$ be a connected simple (directed) graph (without loops and parallel edges) with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge (arc) set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For $v \in V$, we use $N(v) = \{u \in V(G)|(u, v) \in E(G)\}$ ($N^-(v) = \{u \in V(G)|(u, v) \in E(G)\}$ or $N^+(v) = \{u \in V(G)|(v, u) \in E(G)\}$) to denote the neighbors (in-neighbors or out-neighbors) of v. Set $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ ($d^-(v) = |N^-(v)|$ or $d^+(v) = |N^+(v)|$) be the degree (in-degree or out-degree) of v. Specially, $\Delta = \Delta(G) = \max\{d(v)|v \in V\}$ and $\delta = \delta(G) = \min\{d(v)|v \in V\}$ denote the maximum and minimum degree of vertices of G, respectively. Let $\delta^- = \min\{d^-(v)|v \in V\}$ and $\delta^+ = \min\{d^+(v)|v \in V\}$ denote the minimum in-degree and minimum out-degree of G. Next we give some definitions which will be used in the following sections.

Definition 1. A sequence of vertices $v_1, v_2, \ldots, v_s$ in a (directed) graph G defines a **chain** of length $s - 1$ if $(v_i, v_{i+1}) \in E(G)$ for every $i, 1 \leq i \leq s - 1$. If all the vertices in the chain are distinct, we call such chain a **path**. The **distance** between v_i and v_j, denoted by $d(v_i, v_j)$, is the number of edges in a shortest path joining v_i and v_j.

Definition 2. Define $D(G) = \max\{d(u, v)|u, v \in V(G), u \neq v\}$ as the **diameter** of G.

Definition 3. Define $\sigma(G) = \sum_{u,v \in V(G), u \neq v} d(u,v)$, and $\mu(G) = \frac{1}{n(n-1)}\sigma(G)$ $= \frac{1}{n(n-1)} \sum_{u,v \in V(G), u \neq v} d(u,v)$. We call $\mu(G)$ as the **average diameter** of G.

Definition 4. Given a graph G, define its line graph $L(G)$ as a graph such that each vertex of $L(G)$ represents an edge of G; and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint (are incident) in G.

Definition 5. Let $e \in E(G)$, with x, y as its vertices. Define $\xi(e) = d(x) + d(y) - 2$ as the edge degree of e. Define $\xi(G)$ as the minimum of G's edge degrees, thus, $\xi(G) = \min\{\xi(e) | e \in E(G)\}$.

Definition 6. Let $\varnothing \neq S \subseteq V(G)$ and $\varnothing \neq B \subseteq E(G)$. Define $G - S$ as the graph by omitting all the edges or arcs relating to all the vertices of S, and $G - B$ as the graph by omitting all the edges or arcs of B.

Definition 7. Let $G(V,E)$ be a strongly connected directed graph. Define $\overrightarrow{d}(u,v)$ as the **distance** from u to v, and $\overrightarrow{D}(G) = \max\{\overrightarrow{d}(u,v) | u, v \in V(G), u \neq v\}$ as the **directed diameter** of G.

Definition 8. Define

$$\overrightarrow{\sigma}(G) = \sum_{u,v \in V(G), u \neq v} \overrightarrow{d}(u,v),$$

and

$$\overrightarrow{\mu}(G) = \frac{1}{n(n-1)}\overrightarrow{\sigma}(G) = \frac{1}{n(n-1)} \sum_{u,v \in V(G), u \neq v} \overrightarrow{d}(u,v).$$

We call $\overrightarrow{\mu}(G)$ as the **directed average diameter** of G.

Definition 9. For a one-vertex graph G, we define $D(G) = 0$.

Definition 10. Define $\kappa(G)$ as the minimum number of vertices that we need to delete in order to disconnect G. Define $\lambda(G)$ as the minimum number of edges that we need to delete in order to disconnect G. We call $\kappa(G)$ the vertex connectivity of G, $\lambda(G)$ the edge connectivity of G.

Notation. We denote the combination of n things taken k at a time without repetition by $\binom{n}{k}$. Here,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The diameter of a graph is a critical parameter in practical applications, for instance, when analyzing the performance and efficiency of a certain network, the maximum transmission delay, an essential factor, is actually the diameter in terms of graph theory. Therefore, estimations for the diameter of a graph has a practical significance. Also, the previous results, as is shown in [1] and [5], are based on graph theory characteristics only. We based our estimations on new perspectives like Hamilton cycles and line graphs. Moreover, we develop the previous results or give brand new estimations by certain characteristics.

2 Previous Results and Related Works

For the estimation of the diameter, reference [1] and other documents give the following results:

Result 1. (see [1]) Let $G(V, E)$ be a strongly connected directed simple graph of order n. If $n \geq 2$, then

$$\overrightarrow{D}(G) \geq \begin{cases} n-1 & ,\Delta = 1 \\ \lceil \log_{\Delta}(n(\Delta-1)+1) \rceil - 1 & ,\Delta \geq 2 \end{cases}$$

where $\Delta = \max\{d^+(v), d^-(v) | v \in V\}$.

Result 2. (see [1]) Let $G(V, E)$ be a connected undirected simple graph of order n. If $n \geq 3$, then

$$D(G) \geq \begin{cases} \left\lfloor \dfrac{n}{2} \right\rfloor & ,\Delta = 2 \\ \log_{\Delta-1} \dfrac{(\Delta-2)n+2}{\Delta} & ,\Delta \geq 3 \end{cases}.$$

Result 3. (see [1]) Let $G(V, E)$ be a connected undirected simple graph of order n. Then

$$D(G) \leq \frac{3n}{\delta+1}.$$

Result 4. (see [1]) Let $G(V, E)$ be a connected undirected simple graph of order n and size m. Then

$$m \leq D + \frac{1}{2}(n-D+4)(n-D+1).$$

Remark. Result 4 is equivalent to

$$D(G) \leq \frac{2n + 1 - \sqrt{8(m-n) + 17}}{2}.$$

Result 5. (see [1]) Let $G(V, E)$ be a connected undirected simple graph of order n and size m. Then

$$\sigma(G) \geq 2n(n-1) - 2m.$$

Result 6. (see [1]) Let $G(V, E)$ be a strongly connected directed simple graph of order n and size m. Then

$$\overrightarrow{\sigma}(G) \geq 2n(n-1) - m.$$

Result 7. (see [1]) Let $G(V, E)$ be a connected undirected simple graph of order n. Then

$$\mu(G) \leq \frac{n}{\delta + 1} + 2.$$

3 Lemmas

In this Section, we will give some lemmas which will be used in the proofs of our results.

Lemma 1. (See [1]) Let $G(V, E)$ be a connected undirected simple graph of order n. Then

$$D(G) \leq n - 1.$$

Lemma 2. (See [1]) Let $G(V, E)$ be a simple graph and $x, y \in V(G)$ with $x \neq y$. If there exists a chain of length t, then there exists a path whose length is not greater than t.

Lemma 3. (See [1]) Let $G(V, E)$ be a connected simple (directed) graph of order at least 3. Assume x, y, z are three distinct vertices in $V(G)$. Then

$$d(x, y) \leq d(x, z) + d(z, y).$$

$$(\overrightarrow{d}(x, y) \leq \overrightarrow{d}(x, z) + \overrightarrow{d}(z, y).)$$

Lemma 4. Let $G(V, E)$ be a connected undirected simple graph of order at least 2. Then $L(G)$ is a connected graph, and

$$D(G) - 1 \leq D(L(G)) \leq D(G) + 1.$$

Proof. If $|V(G)| = 2$, then $D(G) = 1$, and $D(L(G)) = 0$. So we will assume $|V(G)| \geq 3$. Let $e_1, e_2 \in E(G)$ with $e_1 \neq e_2$. Since $e_1 \neq e_2$, there are two distinct vertices x, y of G such that x and y are end-vertex of e_1 and e_2, respectively.

By the definition of diameter, there is a path $P(w_1, w_2, \ldots, w_m)$ connecting x, y of length no larger than $D(G)$. Then $P(e_1, (w_1, w_2), (w_2, w_3), \ldots, (w_{m-1}, w_m), e_2)$ is a chain in $L(G)$ which connects e_1, e_2 of length no larger than $D(G)+1$. By Lemma 2, $d(e_1, e_2) \leq D(G) + 1$ which implies that $L(G)$ is a connected graph and $D(L(G)) \leq D(G) + 1$.

Suppose $D(L(G)) \leq D(G) - 2$. Since $G(V, E)$ is a connected graph, for any distinct vertices $x, y \in V(G)$, there are $e_1, e_2 \in E(G)$ with $e_1 \neq e_2$ such that x and y are endpoints of e_1 and e_2, respectively. By the definition of diameter, the length of the chain $P(e_1, (w_1, w_2), (w_2, w_3), \ldots, (w_{m-1}, w_m), e_2)$ is not greater than $D(G)-2$. It follows that $P(x, w_1, w_2, \ldots, w_m, y)$ is a chain connecting x, y whose length is not greater than $D(G)-1$. By Lemma 2, $D(G) \leq D(G)-1$, a contradiction. Hence $D(L(G)) \geq D(G)-1$. So,

$$D(G) - 1 \leq D(L(G)) \leq D(G) + 1.$$

□

Now, we are introducing a famous algorithm in graph theory——breadth first search algorithm. It is an algorithm that can mark the vertices of the graph and is normally used to calculate the distance between the vertices. (see [2]).

Breadth First Search Algorithm.

Algorithm. BREADTH-FIRST SEARCH (BFS) (see [2])

INPUT: a connected graph $G(r)$.

OUTPUT: an *r-tree* T in G with predecessor function p, a level function l such that $l(v) = d(r, v)$ for all $v \in V$, and a time function t

set $i := 0$ *and* $Q := \varnothing$
increment i *by* 1
colour r *black*
set $l(r) := 0$ *and* $t(r) := i$
append r *to* Q
while Q *is nonempty* ***do***
consider the head x *of* Q
if x *has an uncoloured neighbour* y ***then***
increment i *by* 1
colour y *black*
set $p(y) := x, l(y) := l(x) + 1$ *and* $t(y); = i$
append y *to* Q
else
remove y *from* Q
end if
end while
return (p, l, t)

The spanning tree T returned by BFS is called a *breadth-first search tree*, or *BFS-tree*, of G.

Lemma 5. (see [2]) Let T be a BFS-tree of a connected graph G, with root r. Then every edge of G joins vertices on the same or consecutive levels of T; that is, $|l(u) - l(v)| \leq 1$, for all $uv \in E$.

Lemma 6. (see [2]) Let G be a connected graph. Then the values of the level function l returned by BFS are the distances in G from the root r: $L(v) = d(r, v)$, for all $v \in V$.

Definition 10. Let $G(V, E)$ be a connected undirected simple graph. We pick a vertex $r \in V(G)$ as the root. We run the breadth-first search algorithm. When it ends, we define $F_i = \{u | l(u) = i\}$, $f_i = |F_i|$. Denote the set of all the edges from F_i to F_{i+1} by T_i.

Lemma 7. Let $G(V, E)$ be a connected undirected simple graph with $D = D(G) \geq 2$. Then for any $i \in \{0, 1, \ldots, D-2\}$, we have

$$f_i + f_{i+1} + f_{i+2} \geq \delta + 1.$$

Proof. Let $i \in \{0, 1, \ldots, D-2\}$. Denote the set of all edges of the vertices in F_{i+1} by E_{i+1}. Then

$$\begin{aligned}\binom{f_{i+1}}{2} &\geq |E_{i+1} - T_i - T_{i+1}| \\ &= |E_{i+1}| - |T_i| - |T_{i+1}| \\ &\geq \Big(\big(\sum_{x \in F_{i+1}} d(x)\big) - \binom{f_{i+1}}{2}\Big) - |T_i| - |T_{i+1}| \\ &\geq \Big(\delta f_{i+1} - \binom{f_{i+1}}{2}\Big) - |T_i| - |T_{i+1}|.\end{aligned}$$

Namely,

$$\begin{aligned}|T_i| + |T_{i+1}| &\geq \delta f_{i+1} - 2\binom{f_{i+1}}{2} \\ &= (\delta + 1) f_{i+1} - f_{i+1}^2.\end{aligned}$$

For any $x \in F_{i+1}$, denote $|N(x) \cap F_i|$ by $d'(x)$, and $|N(x) \cap F_{i+2}|$ by $d''(x)$. It follows that

$$\sum_{x \in F_{i+1}} d'(x) = |T_i|, \sum_{x \in F_{i+1}} d''(x) = |T_{i+1}|.$$

Therefore,

$$f_i \geq \max\{d'(x) | x \in F_{i+1}\} \geq \frac{\sum_{x \in F_{i+1}} d'(x)}{f_{i+1}} = \frac{|T_i|}{f_{i+1}}.$$

$$f_{i+2} \geq \max\{d''(x) | x \in F_{i+1}\} \geq \frac{\sum_{x \in F_{i+1}} d''(x)}{f_{i+1}} = \frac{|T_{i+1}|}{f_{i+1}}.$$

$$f_{i+1} = f_{i+1}.$$

By summing the above formulas,

$$\begin{aligned} f_i + f_{i+1} + f_{i+2} &\geq \frac{|T_i| + |T_{i+1}|}{f_{i+1}} + f_{i+1} \\ &\geq \frac{(\delta+1)f_{i+1} - f_{i+1}^2}{f_{i+1}} + f_{i+1} \\ &= \delta + 1. \end{aligned}$$

□

4 The Diameter of Graphs

In this chapter, we give some bounds of diameter in connected undirected simple graphs. For $v \in V(G)$, denote $N(v) = \{u|vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$.

Theorem 1. Let $G(V,E)$ be a connected graph of order n. Then

$$D(G) \leq n - \Delta + 1.$$

Proof. Let $v \in V(G)$ with $d(v) = \Delta$. If $|N[v]| = n$, then $D(G) \leq 2$. So we will assume that $|N[v]| < n$ in the following proof.

Assume that $G - N[v] = G^*$. Also assume that G^* is the union of k connected components $G_1, G_2, \ldots, G_k$. Since G is connected, there is an edge between $N(v)$ and $G_i, i = 1, 2, \ldots, k$.

Let x, y be two distinct vertices of G. We will complete the proof by considering the following five cases.
(i) $x = v$ and $y \in N(v)$. Then $d(x,y) = 1$.
(ii) $x = u$ and $y \in V(G_j), j \in \{1, 2, \ldots, k\}$. In the case

$$d(x,y) \leq 2 + D(G_j) \leq 2 + |V(G_j)| - 1 \leq |V(G^*)| + 1 = n - d(v).$$

(iii) $x \in N(v)$ and $y \in V(G_j), j \in \{1, 2, \ldots, k\}$.

In this case, the path from x to y can reach v first, then reach one of the vertices in G_j that is a adjacent vertex to $N(v)$, and finally reach y. Therefore,

$$d(x,y) \leq 3 + D(G_j) \leq 3 + n - d(v) - 2 = n - d(v) + 1.$$

(iv) $x, y \in N(v)$. Then $d(x,y) \leq 2$.
(v) $x, y \in V(G^*)$. There are two possible cases:
(a) $x, y \in V(G_j)$. Then

$$d(x,y) \leq D(G_j) \leq |V(G_j)| - 1 \leq |V(G^*)| - 1 \leq n - d(v) - 2.$$

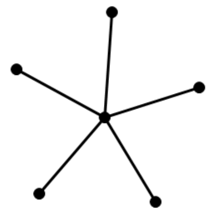

Fig.1

(b) $x \in V(G_i)$ and $y \in V(G_j), i \neq j$. Consider the following path:

$$x \to \cdots \to w_1 \to v \to w_2 \to \cdots \to y$$

where $w_1, w_2 \in N(v)$. Then

$$d(x,y) \leq D(G_i)+D(G_j)+4 \leq |V(G_i)|-1+|V(G_j)|-1+4 \leq |V(G^*)|+2 \leq n-d(v)+1.$$

Therefore,

$$D(G) \leq \max\{n-d(v)+1, 2\}.$$

Note that $d(v) \leq n-1$. Therefore, $n-d(v)+1 \geq 2$. Thus

$$D(G) \leq n-d(v)+1 = n-\Delta+1.$$

□

Remark. Theorem 1 simply use maximum degree to give the upper bound of the diameter. And the upper bounds in Theorem 1 is sharp. Let G be a star $K_{1,n}$ (see Fig.1),. Then $D=2, \Delta=n-1$. So $D(G)=n-\Delta+1$.

Theorem 2. Let $G(V,E)$ be a connected graph of order $n \geq 3$. Then

$$D(G) \leq \frac{3(n-1)}{\delta+1}-1.$$

Proof. Assume that $D(G)=D$, By the definition of diameter, there are vertices $u, v \in V(G)$ such that $d(u,v)=D$.

Assume that $P(v_1, v_2, \ldots, v_{D+1})$ is the shortest path of length D that connects u, v. Then $v_i v_j \notin E(G)$ if $|i-j| \geq 2$.

Assume $N'(v_i) = N(v_i) - \{v_1, v_2, \ldots, v_{D+1}\}, i=1,2,\ldots,D+1$. Then

$$|N'(v_1)| \geq \delta-1, |N'(v_i)| \geq \delta-2, i=2,\ldots,D+1.$$

Suppose there is $i \in \{1,2,\ldots,D+1\}$ and $k \geq 3$, such that $N'(v_i) \cap N'(v_{i+k}) \neq \emptyset$, say $w \in N'(v_i) \cap N'(v_{i+k})$. Then $P(v_1, v_2, \ldots, v_i, w, v_{i+k}, \ldots, v_{D+1})$ is a path shorter than P, a contradiction.

Therefore, for any $i \in \{1, 2, \ldots, D+1\}$ and $k \geq 3$, we have $N'(v_i) \cap N'(v_{i+k}) = \emptyset$. Denote $A = \{v_i | i \equiv 1(mod\ 3)\}$. Then for any $x, y \in A$, $N'(x) \bigcap N'(y) = \varnothing$. Therefore,

$$
\begin{aligned}
n = |V(G)| &\geq |\{v_1, v_2, \ldots, v_{D+1}\} \bigcup (\bigcup_{i=1}^{D+1} N'(v_i))| \geq |\{v_1, v_2, \ldots, v_{D+1}\} \bigcup (\bigcup_{x \in A} N'(x))| \\
&= |\{v_1, v_2, \ldots, v_{D+1}\}| + \sum_{x \in A} |N'(x)| \geq D + 1 + (\delta - 1) + (\delta - 2) \left\lfloor \frac{D}{3} \right\rfloor \\
&\geq D + 1 + (\delta - 1) + (\delta - 2) \frac{D-2}{3} = \frac{(D+1)(\delta+1)}{3} + 1.
\end{aligned}
$$

Hence

$$D(G) \leq \frac{3(n-1)}{\delta+1} - 1.$$

□

Remark. Theorem 2 uses minimum degree to give the upper bound of diameter. And the upper bound is sharp. Let G be a cycle of order 3. Then $D = 1, n = 3, \delta = 2$. So $D(G) = \dfrac{3(n-1)}{\delta+1} - 1$. By comparing it with Result 3, it is apparent that the upper in Theorem 2 is better than the upper bound in Result 3.

The proof of Theorem 2 is similar with the proof of Result 3 and is sophisticated to certain extent. The following proof of Result 3 is by the breadth first search algorithm.

Proof of Result 3. Assume that $D(G) = D$. By the definition of diameter, there are two vertices $u, v \in V(G), d(u, v) = D$. Take v as the root. We run the BFS algorithm. By Lemma 7, we have $f_0 = 1$, $f_1 \geq \delta$ and $f_{3i-1} + f_{3i} + f_{3i+1} \geq \delta + 1, i = 1, 2, \ldots, \lfloor \frac{D-1}{3} \rfloor$. Therefore,

$$n = \sum_{i=0}^{D} f_i \geq 1 + \delta + (\delta+1) \left\lfloor \frac{D-1}{3} \right\rfloor \geq \frac{(\delta+1)D}{3}$$

which implies $D(G) \leq \dfrac{3n}{\delta+1}$. □

Theorem 3. Let $G(V, E)$ be a connected graph of order n and $\xi(G) = \xi \geq 2$. Then

$$D(G) < \frac{8n - 12}{\xi + 6}.$$

Proof. Assume that $D(G) = D$, by the definition of diameter, there exist $u, v \in V(G)$ such that $d(u, v) = D$. Assume that $P(v_1, v_2, \ldots, v_{D+1})$ is the shortest path that connects u, v. Then $v_i v_j \notin E(G)$ if $|i - j| \geq 2$.

Assume that

$$N'(v_i) = N(v_i) - \{v_1, v_2, \ldots, v_{D+1}\}, i = 1, 2, \ldots, D+1,$$

$$A_j = N'(v_{4j-3}) \bigcup N'(v_{4j-2}), j = 1, 2, \ldots, \left\lfloor \frac{D+3}{4} \right\rfloor.$$

If there are $i, j \in \{1, 2, \ldots, \lfloor \frac{D+3}{4} \rfloor\}, i \neq j$, such that $A_i \cap A_j \neq \emptyset$, then we have a shorter path than P, a contradiction.

Therefore, for any $i, j \in \{1, 2, \ldots, \lfloor \frac{D+3}{4} \rfloor\}, i \neq j$, we have $A_i \cap A_j = \emptyset$. Also

$$|A_1| = |N'(v_1) \bigcup N'(v_2)| \geq \max\{|N'(v_1)|, |N'(v_2)|\} \geq \frac{|N'(v_1)| + |N'(v_2)|}{2} \geq \frac{\xi - 1}{2}.$$

$$\begin{aligned} |A_j| &= |N'(v_{4j-3}) \bigcup N'(v_{4j-2})| \\ &\geq \max\{|N'(v_{4j-3})|, |N'(v_{4j-2})|\} \\ &\geq \frac{|N'(v_{4j-3})| + |N'(v_{4j-2})|}{2} \\ &\geq \frac{\xi - 2}{2}, j = 2, \ldots, \left\lfloor \frac{D+3}{4} \right\rfloor. \end{aligned}$$

Therefore,

$$\begin{aligned} n = |V(G)| &\geq |\{v_1, v_2, \ldots, v_{D+1}\} \bigcup (\bigcup_{j=1}^{\lfloor \frac{D+3}{4} \rfloor} A_j)| && (1) \\ &= |\{v_1, v_2, \ldots, v_{D+1}\}| + \sum_{j=1}^{\lfloor \frac{D+3}{4} \rfloor} |A_j| && (2) \\ &\geq D + 1 + \frac{\xi - 1}{2} + \frac{\xi - 2}{2} \left\lfloor \frac{D-1}{4} \right\rfloor && (3) \\ &\geq \frac{(\xi + 6)D + 12}{8} && (4) \end{aligned}$$

which implies

$$D \leq \frac{8n - 12}{\xi + 6}.$$

The equality in (1) holds if and only if $V(G) = \{v_1, v_2, \ldots, v_{D+1}\} \bigcup (\bigcup_{j=1}^{\lfloor \frac{D+3}{4} \rfloor} A_j)$. However the equality in (4) holds if and only if $4|D$. The two conditions cannot be satisfied at the same time.

Thus,

$$D < \frac{8n - 12}{\xi + 6}.$$

□

Remark. Theorem 3 can be regarded as the analogy of Theorem 2. By replacing minimum degree with minimum edge degree, we give the upper bound of diameter from a different perspective. The advantage of this estimation is that none of the accurate vertex degrees are necessary, we simply need to know the sum of two adjacent vertices' degrees, then we can have an estimation.

Theorem 4. Let $G(V, E)$ be a connected undirected simple graph of order n and size e. Then

$$D(G) \leq \frac{(\Delta + 3)(n-1) - 2e}{\Delta + 1}.$$

Proof. Assume that $D(G) = D$. By the definition of diameter, there are two vertices $u, v \in V(G)$ such that $d(u, v) = D$. Let $P(v_1, v_2, \ldots, v_{D+1})$ is the shortest path that connects u, v. Assume that $G - \{v_1, v_2, \ldots, v_{D+1}\} = G^*$, we will consider two cases.

Case 1 $|V(G^*)| \geq 1$.

Denote $A_i = \{v_j | v_j \in \{v_1, v_2, \ldots, v_{D+1}\}, j \equiv i (mod\ 3)\}, i = 1, 2, 3$.

Denote the set of edges of all the vertices $\{v_1, v_2, \ldots, v_{D+1}\}$ by B_1, denote the set of edges of all the vertices in $V(G^*)$ by B_2, denote the set of edges that has one vertex in $\{v_1, v_2, \ldots, v_{D+1}\}$ and the other vertex in $V(G^*)$ by B_3, and denote the set of edges that has a vertex in A_i, and the other vertex in $V(G^*)$ by $R_i, i = 1, 2, 3$.

Assume that $N'(v_i) = N(v_i) - \{v_1, v_2, \ldots, v_{D+1}\}, i = 1, 2, \ldots, D + 1$. Then,

$$N'(v_i) \subseteq G^*, i = 1, 2, \ldots, D + 1.$$

By the same argument as the proof of Theorem 1, we have $\forall i \in \{1, 2, \ldots, D+1\}$ and $k \geq 3$ $N[v_i] \cap N[v_{i+k}] \neq \emptyset$. Thus for any $x, y \in A_i, x \neq y$, $N'(x) \bigcap N'(y) = \emptyset, i = 1, 2, 3$. It follows that

$$|B_3| = |R_1 \cup R_2 \cup R_3| = |R_1| + |R_2| + |R_3| \leq |V(G^*)| + |V(G^*)| + |V(G^*)| = 3(n - D - 1).$$

For any $v \in V(G^*)$, assume that $d'(v) = |N(v) - \{v_1, v_2, \ldots, v_{D+1}\}|$. Then

$$|B_2| = \frac{1}{2} \sum_{v \in V(G^*)} d'(v) \leq \frac{1}{2}(|V(G^*)|\Delta - |B_3|) = \frac{1}{2}((n - D - 1)\Delta - |B_3|).$$

Also,

$$|B_1| = D.$$

It follows that

$$\begin{aligned} |E(G)| &= |B_1| + |B_2| + |B_3| \\ &\leq D + \frac{1}{2}((n - D - 1)\Delta - |B_3|) + |B_3| \\ &= (1 - \frac{\Delta}{2})D + \frac{\Delta(n-1)}{2} + \frac{1}{2}|B_3| \\ &\leq (1 - \frac{\Delta}{2})D + \frac{\Delta(n-1)}{2} + \frac{3}{2}(n - D - 1) \\ &= -\frac{\Delta + 1}{2}D + \frac{(\Delta + 3)(n-1)}{2}. \end{aligned}$$

$$\Rightarrow D(G) \leq \frac{(\Delta + 3)(n-1) - 2e}{\Delta + 1}.$$

Case 2 $|V(G^*)| = 0$.

In this case, $D(G) = n-1, e = n-1, \Delta = 2, \frac{(\Delta+3)(n-1)-2e}{\Delta+1} = n-1$, It follows that

$$D(G) \leq \frac{(\Delta+3)(n-1)-2e}{\Delta+1}.$$

So,

$$D(G) \leq \frac{(\Delta+3)(n-1)-2e}{\Delta+1}.$$

□

Remark. In Theorem 4, we use the maximum degree and the number of edges to give the upper bound of the diameter. The two characteristics are essential, but when we obtain both of them, we can get a very precise estimation. (Comparing to Result 4 and Theorem 1). Note that the bound is sharp. Let G be a path of order n. Then $D = n-1, \Delta = 2, e = n-1$. So $D(G) = \frac{(\Delta+3)(n-1)-2e}{\Delta+1}$.

Theorem 5. Let $G(V, E)$ be a connected graph. Denote $V(G) = \{v_1, v_2, \ldots, v_n\}, \Delta(G) = \Delta$. Then

$$D(G) \leq \frac{(2\Delta+3)\sum_{i=1}^{n} d(v_i) - 2\sum_{i=1}^{n} d(v_i)^2 - 4}{4\Delta-2}.$$

Proof. We observe the line graph $L(G)$ of G. Then

$$|V(L(G))| = |E(G)| = \frac{\sum_{i=1}^{n} d(v_i)}{2}.$$

On the other hand,

$$|E(L(G))| = \frac{1}{2}\sum_{e\in E(G)} \xi(e) = \frac{\sum_{i=1}^{n} d(v_i)^2 - \sum_{i=1}^{n} d(v_i)}{2}.$$

By Lemma 4 and Theorem 4,

$$\begin{aligned} D(G) \leq D(L(G)) + 1 &\leq \frac{(\Delta(L(G))+3))(|V(L(G))|-1) - 2|E(L(G))|}{\Delta(L(G))+1} + 1 \\ &\leq \frac{(2\Delta+3)\sum_{i=1}^{n} d(v_i) - 2\sum_{i=1}^{n} d(v_i)^2 - 4}{4\Delta-2}. \end{aligned}$$

□

Remark. The bound in Theorem 5 is sharp. Let G be a path of order n. Then $D = n-1, \sum_{i=1}^{n} d(v_i) = 2(n-1)$, $\sum_{i=1}^{n} d(v_i)^2 = 2 + 4(n-2)$ and $\Delta = 2$. So

$$D = \frac{(2\Delta+3)\sum_{i=1}^{n} d(v_i) - 2\sum_{i=1}^{n} d(v_i)^2 - 4}{4\Delta - 2}.$$

Theorem 6. Let $G(V,E)$ be a Hamiltonian graph of order n and $\delta(G) = \delta$. Then

$$D(G) \leq \max\{2, \left\lfloor \frac{n}{2} \right\rfloor - \delta + 2\}.$$

Proof. Let $v_1 v_2 \ldots v_n v_1$ be the Hamiltonian cycle of G. For $v_i, v_j \in V(G)$, we define its circumferential distance $f(v_i, v_j)$ as

$$f(v_i, v_j) = \min\{|i-j|, n - |i-j|\}.$$

Let $v_i, v_j \in V(G)$ and $v_i \neq v_j$. We consider the following two cases.

Case 1 $N[v_i] \bigcap N[v_j] \neq \emptyset$. Then $d(v_i, v_j) \leq 2$.

Case 2 $N[v_i] \bigcap N[v_j] = \emptyset$. Assume that $l = \min\{f(x,y) | x \in N[v_i], y \in N[v_j]\}$.

For a vertex p_1 in $N[v_i]$, we start from p_1, and write the vertices in $N[v_i] \bigcup N[v_j]$ clockwisely on the cycle. In the process, we denote the kth point in $N[v_i]$ by p_k, $k = 2, 3, \ldots, m$, and we denote the kth point in $N[v_j]$ by q_k, $k = 1, 2, \ldots t$. In this idea, we go through all the vertices in $N[v_i] \bigcup N[v_j]$.

It follows that we obtain a vertex sequence $\{h_k\}_{k=1}^{m+t}$. From the first term in $\{h_k\}_{k=1}^{m+t}$, the process of p's changing into q or q's changing into p is called a jump. It follows that in the process $h_1 \to h_2 \to \cdots \to h_{m+t} \to h_1$, there are at least two jumps.

Assume that the number of p type vertex between two adjacent jumps are $P_1, P_2, \ldots, P_M$ from the first term; the number of q type vertex between two adjacent jumps are $Q_1, Q_2, \ldots, Q_T$ from the first term. Since the consecutive p type vertices and q type vertices are interlaced, then $M = T$.

We observe the location of two vertices x, y on the cycle after a jump. Since the number of edges between x and y is no less than l, and

$$\sum_{k=1}^{M} P_k = |N[v_i]| \geq \delta + 1, \sum_{k=1}^{T} Q_k = |N[v_j]| \geq \delta + 1,$$

Then,

$$
\begin{aligned}
n &\geq \sum_{k=1}^{M}(P_k-1)+\sum_{k=1}^{M}(Q_k-1)+2Ml \\
&= \sum_{k=1}^{M}P_k+\sum_{k=1}^{M}Q_k+2M(l-1) \\
&\geq 2\delta+2+2(l-1).
\end{aligned}
$$

$$\Rightarrow l \leq \left\lfloor \frac{n}{2} \right\rfloor - \delta.$$

Therefore

$$D(G) \leq l+2 \leq \left\lfloor \frac{n}{2} \right\rfloor - \delta + 2.$$

So

$$D(G) \leq \max\{2, \left\lfloor \frac{n}{2} \right\rfloor - \delta + 2\}.$$

□

Remark. Theorem 6 uses minimum degree to give the the upper bound of diameter when the graph has a Hamilton cycle. With the given characteristics of graph theory, the diameters of these graphs are much shorter than those graphs without the Hamilton cycle. However, there has not been an ordinary graph theory characteristic to illustrate whether a graph contains a Hamilton cycle or not. Therefore, we give an estimation for Hamiltonian graph. Note that the bound is sharp. Let G be a cycle of order n. Then $D = \lfloor \frac{n}{2} \rfloor$, $\delta = 2$. So $D(G) = \max\{2, \lfloor \frac{n}{2} \rfloor - \delta + 2\}$.

Theorem 7. Let $G(V, E)$ be a connected undirected simple graph of order n. If there exist L vertex-disjoint cycles in G, then

$$D(G) \leq n - L - 1.$$

Proof. Assume that the L vertex-disjoint cycles are $C_1, C_2, \ldots, C_L$, denote the length of C_i by c_i. Then $c_i \geq 3, i = 1, 2, \ldots, L$. Assume that M vertices remain in G after we have omitted all the vertices of $C_1, C_2, \ldots, C_L$.

Observe the graph after contracting $C_1, C_2, \ldots, C_L$. Namely, the graph by replacing the cycle C_i with a single vertex v_i, preserving the edges except those from $C_1, C_2, \ldots, C_L$ and substituting the edges of G from C_i by edges from v_i. It follows that we obtain a graph G^*. Assume that $V(G^*) = \{u_1, u_2, \ldots, u_{L+M}\}$, it follows that u_i corresponds to a vertex or a cycle in G. Assume that the set of all the vertices in G corresponds to u_i is U_i.

Since G is connected, then G^* is connected. By Lemma 1,

$$D(G^*) \leq |V(G^*)| - 1 = L + M - 1.$$

Thus, for $x, y \in V(G)$, assume that $x \in U_i, y \in U_j$,

$$d(u_i, u_j) \leq L + M - 1.$$

Therefore,

$$\begin{aligned} d(x,y) &\leq L + M - 1 + \sum_{i=1}^{L} D(C_i) \\ &= L + n - \sum_{i=1}^{L} c_i - 1 + \sum_{i=1}^{L} D(C_i) \\ &\leq L + n - \sum_{i=1}^{L} c_i - 1 + \sum_{i=1}^{L} \left\lfloor \frac{c_i}{2} \right\rfloor \\ &\leq L + n - 1 + \sum_{i=1}^{L} (\left\lfloor \frac{c_i}{2} \right\rfloor - c_i) \\ &\leq L + n - 1 - 2L = n - L - 1. \end{aligned}$$

Therefore,

$$D(G) \leq n - L - 1.$$

□

Remark. Theorem 7 only uses the number of disjoint cycles in G to give the upper bound of the diameter. And the bound is sharp. Let G be a graph of order 6 obtained by connecting two cycles of length 3 with an edge. Then $D = 3$ and $L = 2$. So $D(G) = n-L-1$. Generally speaking, a tree-like graph with fewer branches has a longer diameter, while the graph with more cycles has a shorter diameter. Theorem 7 actually gives an quantitative estimation.

In a different aspect, Theorem 7 can be used to give the upper bound of the number of vertex-disjoint cycles in G.

Corollary. Let $G(V, E)$ be a connected graph of order n. Then the number of vertex-disjoint cycles in G is not greater than $n - D(G) - 1$.

Theorem 8. Let $G(V, E)$ be a connected graph of order n with $\kappa(G) = \kappa, \lambda(G) = \lambda$. Then

(1) $D(G) \leq \dfrac{n-2}{\kappa} + 1$.

(2) $D(G) < \dfrac{n}{\sqrt{\lambda}} + 2 - \sqrt{\lambda} - \dfrac{1}{\sqrt{\lambda}}$.

Proof. (1) Assume that $D(G) = D$, by the definition of diameter, then there are two vertices $u, v \in V(G)$ such that $d(u, v) = D$. By $Menger's$ theorem, there are κ vertex-disjoint pathes that connects u, v. Denote them by $P_1, P_2, \ldots, P_\kappa$. Since $d(u, v) = D$, then $|P_i| \geq D, i = 1, 2, \ldots, \kappa$. Therefore,

$$n = |V(G)| \geq \sum_{i=1}^{\kappa} (|P_i| - 1) + 2 \geq \kappa(D - 1) + 2,$$

which implies

$$D(G) \leq \frac{n-2}{\kappa} + 1.$$

(2) Assume that $D(G) = D$. By the definition of diameter, there are $u, v \in V(G)$ such that $d(u, v) = D$. By $Menger's$ theorem, there are λ edge-disjoint pathes that connects u, v. Denote them by $P_1, P_2, \ldots, P_\lambda$. Pick u as the root, we run the breadth first search algorithm. Since $d(u, v) = D$ and $D(G) = D$, the largest marked number is D, and $l(v) = D$.

By Lemma 5, for any $j \in \{1, 2, \ldots, \lambda\}$ and $i \in \{0, 1, \ldots, D-1\}$, there is an edge $e \in T_i$, such that $e \in P_j$. Since all the edges in $P_1, P_2, \ldots, P_\lambda$ are disjoint, then $|T_i| \geq \lambda$. For any $i \in \{0, 1, \ldots, D-1\}$, assume that $F_i = \{v_1, v_2, \ldots, v_t\}$, denote the edges in T_i from v_j by $N'(v_j)$, and $d'(v_j) = |N'(v_j)|$. Therefore,

$$\sum_{j=1}^{t} d'(v_j) = |T_i|.$$

It follows that

$$f_{i+1} \geq \max\{d'(v_j) | j \in \{1, 2, \ldots, t\}\} \geq \frac{\sum_{j=1}^{t} d'(v_j)}{t} = \frac{|T_i|}{f_i} \geq \frac{\lambda}{f_i},$$

$$f_i + f_{i+1} \geq f_i + \frac{\lambda}{f_i} \geq 2\sqrt{f_i \frac{\lambda}{f_i}} = 2\sqrt{\lambda}.$$

Therefore,

$$n = \sum_{i=1}^{D} f_i \geq 1 + \lambda + \left\lfloor \frac{D-1}{2} \right\rfloor 2\sqrt{\lambda} \geq 1 + \lambda + \frac{D-2}{2} 2\sqrt{\lambda} = \sqrt{\lambda} D - 2\sqrt{\lambda} + 1 + \lambda, \quad (5)$$

$$\Rightarrow D(G) \leq \frac{n + 2\sqrt{\lambda} - \lambda - 1}{\sqrt{\lambda}} = \frac{n}{\sqrt{\lambda}} + 2 - \sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}.$$

The first equality in (5) holds if and only if D is odd, however, the second equality in (5) holds if and only if D is even. The two conditions cannot be satisfied at the same time. Thus,

$$D(G) < \frac{n}{\sqrt{\lambda}} + 2 - \sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}$$

□

Remark. Theorem 8 uses connectivity to give the upper bound of diameter. As for connected graphs, their connectivities are not always the same. In general, a graph with better connectivity has a shorter diameter. Theorem 8 gives the quantitative estimation. Note that the bound in (1) is sharp. Let G be a path of order n. Then $D = n - 1$, $\kappa = 1$. So $D(G) = \dfrac{n-2}{\kappa} + 1$.

Theorem 9. Let $G(V,E)$ be a connected graph of order n and $D(G)=D$. Then

$$\mu(G) \geq \frac{2\binom{D+2}{3} + (n-D-1)\binom{D+1}{2} + 2\binom{n-D-1}{2}}{n(n-1)}.$$

Proof. By the definition of diameter, there are two vertices $u, v \in V(G)$ such that $d(u,v) = D$. Assume that $P(v_1, v_2, \ldots, v_{D+1})$ is the shortest path that connects u, v. Then for any $i, j \in \{1, 2, \ldots, D+1\}, i < j$, $d(v_i, v_j) = j - i$. It follows that

$$\sum_{\substack{x,y \in \{1,2,\ldots,D+1\},\\ x \neq y}} d(x,y) = 2 \sum_{1 \leq i < j \leq D+1} (j-i) = 2\binom{D+2}{3}.$$

Assume that $G - \{v_1, v_2, \ldots, v_{D+1}\} = G^*$. We consider two cases.

Case 1 $V(G^*) \neq \emptyset$.

For any $v \in G^*$, by Lemma 3, $d(v_i, x) + d(x, v_j) \geq j - i$. Therefore,

$$\begin{aligned}
\sum_{\substack{x \in V(G^*),\\ y \notin V(G^*)}} d(x,y) + \sum_{\substack{x \notin V(G^*),\\ y \in V(G^*)}} d(x,y) &= 2 \sum_{x \in V(G^*)} \sum_{i=1}^{\lfloor \frac{D+1}{2} \rfloor} (d(v_i, x) + d(x, v_{D+2-i})) \\
&\geq 2 \sum_{x \in V(G^*)} \sum_{i=1}^{\lfloor \frac{D+1}{2} \rfloor} (D + 2 - 2i) \\
&= 2 \left\lfloor \frac{D+1}{2} \right\rfloor (D - \left\lfloor \frac{D+1}{2} \right\rfloor + 1)(n - D - 1) \\
&\geq \frac{D(D+1)(n-D-1)}{2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sigma(G) = & \sum_{\substack{x,y \notin V(G^*),\\ x \neq y}} d(x,y) + \sum_{\substack{x \in V(G^*),\\ y \notin V(G^*)}} d(x,y) + \sum_{\substack{x \notin V(G^*),\\ y \in V(G^*)}} d(x,y) + \sum_{\substack{x,y \in V(G^*),\\ x \neq y}} d(x,y) \\
& \geq 2\binom{D+2}{3} + \frac{D(D+1)(n-D-1)}{2} + 2\binom{n-D-1}{2} \\
& = 2\binom{D+2}{3} + (n-D-1)\binom{D+1}{2} + 2\binom{n-D-1}{2}.
\end{aligned}$$

So,

$$\mu(G) \geq \frac{2\binom{D+2}{3} + (n-D-1)\binom{D+1}{2} + 2\binom{n-D-1}{2}}{n(n-1)}.$$

Case 2 $V(G^*) = \emptyset$.

In this case, $D = n-1$. Since $(n-D-1)\binom{D+1}{2} = 2\binom{n-D-1}{2} = 0$, it is easy to compute that

$$\mu(G) = \frac{2\binom{D+2}{3}}{n(n-1)} = \frac{2\binom{D+2}{3} + (n-D-1)\binom{D+1}{2} + 2\binom{n-D-1}{2}}{n(n-1)}.$$

.

To conclude,

$$\mu(G) \geq \frac{2\binom{D+2}{3} + (n-D-1)\binom{D+1}{2} + 2\binom{n-D-1}{2}}{n(n-1)}.$$

□

Remark. Theorem 9 uses the diameter of G to give the lower bound of average diameter. Different from the present result, the estimation of average diameter doesn't involve with other characteristics of the graph. In Result 5, we find that its estimation for the average diameter is constantly smaller than 2. However, in Theorem 9, the estimation is not just restricted in that limited range. It shows that Theorem 9 has a better applicability. Note that the bound is sharp. Let G be a path of order 3. Then $\mu(G) = \frac{4}{3}$ and $D = 2$. So $\mu(G) = \dfrac{2\binom{D+2}{3} + (n-D-1)\binom{D+1}{2} + 2\binom{n-D-1}{2}}{n(n-1)}$.

Theorem 10. Let $G(V,E)$ be a connected graph of order n and $D(G) = D$. Then

$$\mu(G) < \frac{D((7-D)(\delta+1)+6n)}{6(n-1)}.$$

Proof. Assume $V(G) = \{v_1, v_2, \ldots, v_n\}$. For every $i \in \{1, 2, \ldots, n\}$, pick v_i as the root, we run the BFS algorithm. Assume the largest value of the level function is m_i. Then by Lemma 6 and the definition of diameter, $m_i \leq D, i = 1, 2, \ldots, n$. Let $s_i = \sum_{j=1}^{n} d(v_i, v_j)$. Then by Lemma 6, $s_i = \sum_{j=0}^{m_i} jf_j$. By Lemma 7 and the rearrangement inequality, we have

$$\begin{aligned}
s_i &< \sum_{j=1}^{\lfloor \frac{m_i}{3} \rfloor} ((3j-2)\cdot 0 + (3j-1)\cdot 0 + 3j(\delta+1)) + m_i(n - \left\lfloor \frac{m_i}{3} \right\rfloor (\delta+1)) \\
&= \frac{3(\delta+1)}{2} \left\lfloor \frac{m_i}{3} \right\rfloor (\left\lfloor \frac{m_i}{3} \right\rfloor + 1) + m_i n - \left\lfloor \frac{m_i}{3} \right\rfloor m_i(\delta+1) \\
&\leq \frac{\delta+1}{2} m_i \frac{m_i+3}{3} + m_i n - \frac{m_i-2}{3} m_i(\delta+1) \\
&= m_i(\frac{\delta+1}{6}(m_i+3) + n - \frac{\delta+1}{3}(m_i-2)) \\
&= m_i(-\frac{\delta+1}{6} m_i + \frac{7}{6}(\delta+1) + n).
\end{aligned}$$

By the property of quadratic function and Result 3,

$$m_i(-\frac{\delta+1}{6} m_i + \frac{7}{6}(\delta+1) + n) \leq D(-\frac{\delta+1}{6} D + \frac{7}{6}(\delta+1) + n).$$

Therefore,

$$\sigma(G) = \sum_{i=1}^{n} s_i < n(D(-\frac{\delta+1}{6}D + \frac{7}{6}(\delta+1) + n)).$$

Thus,

$$\mu(G) < \frac{D((7-D)(\delta+1)+6n)}{6(n-1)}.$$

□

Remark. Theorem 10 uses minimum degree and the diameter of the graph to give the upper bound of average diameter.

5 The Diameter of Directed Graphs

In this section, we will assume that $G(V,E)$ is a strongly connected directed graph and give bounds of the diameter.

Theorem 11. Let $G(V,E)$ be a strongly connected graph of order n. Let $\delta^+(G) = \delta^+$ and $\delta^-(G) = \delta^-$. Then

$$\overrightarrow{D}(G) \leq n - (\delta^+ + \delta^-) + 1.$$

Proof. Since $\delta^+ + \delta^- \leq n-1$, $n - (\delta^+ + \delta^-) + 1 \geq 2$. So we assume $\overrightarrow{D}(G) \geq 3$. Let $\overrightarrow{D}(G) = \overrightarrow{D}$. By the definition of directed diameter, there are two vertices $x, y \in V(G)$ such that $\overrightarrow{d}(u,v) = \overrightarrow{D}$. Assume that $P(v_1, v_2, \ldots, v_{D+1})$ is the shortest path that connects u, v. Denote

$$(N^+)'(v_i) = (N^+)(v_i) - \{v_1, v_2, \ldots, v_{\overrightarrow{D}+1}\}, i = 1, 2, \ldots, \overrightarrow{D}+1,$$

$$(N^-)'(v_i) = (N^-)(v_i) - \{v_1, v_2, \ldots, v_{\overrightarrow{D}+1}\}, i = 1, 2, \ldots, \overrightarrow{D}+1.$$

It follows that the number of arcs from v_1 to another vertex in $\{v_1, v_2, \ldots, v_{\overrightarrow{D}+1}\}$ is at most one, and the number of arcs from $\{v_1, v_2, \ldots, v_{\overrightarrow{D}+1}\}$ to $v_{\overrightarrow{D}+1}$ is at most one. Therefore,

$$|(N^+)'(v_1)| \geq \delta^+ - 1, |(N^-)'(v_{\overrightarrow{D}+1})| \geq \delta^- - 1.$$

If $(N^+)'(v_1) \cap (N^-)'(v_{\overrightarrow{D}+1}) \neq \emptyset$, say $w \in (N^+)'(v_1) \bigcap (N^-)'(v_{\overrightarrow{D}+1})$, then $P(v_1, w, v_{\overrightarrow{D}+1})$ is a shorter path to connect u, v, a contradiction. Therefore $(N^+)'(v_1) \cap (N^-)'(v_{\overrightarrow{D}+1}) = \emptyset$. It follows that

$$n \geq \overrightarrow{D} + 1 + |(N^+)'(v_1)| + |(N^-)'(v_{\overrightarrow{D}+1})| \geq \overrightarrow{D} + 1 + \delta^+ - 1 + \delta^- - 1 = \overrightarrow{D} + \delta^+ + \delta^- - 1.$$

So,

$$\overrightarrow{D}(G) \leq n - (\delta^+ + \delta^-) + 1.$$

□

Remark. Theorem 11 uses minimum in-degree and out-degree to give the upper bound of directed diameter. And the bound is sharp. Let G be a directed cycle of order n. Then $\delta^+ = 1, \delta^- = 1$, and $\overrightarrow{D}(G) = n = n - (\delta^+ + \delta^-) + 1$.

Theorem 12. Let $G(V, E)$ be a strongly connected graph of order n and $\overrightarrow{D}(G) = \overrightarrow{D}$. Then

$$\overrightarrow{\mu}(G) \geq \frac{4\binom{\overrightarrow{D}+2}{3} + 4\binom{\overrightarrow{D}+1}{2} + 4\overrightarrow{D} + \overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1) + 12\binom{n-\overrightarrow{D}-1}{2}}{4n(n-1)}.$$

Proof. By the definition of directed diameter, there are two vertices $x, y \in V(G)$ such that $\overrightarrow{d}(u, v) = \overrightarrow{D}$. Assume that $P(v_1, v_2, \ldots, v_{\overrightarrow{D}+1})$ is the shortest path that connects u, v. Then for any $i, j \in \{1, 2, \ldots, \overrightarrow{D}+1\}, i < j$, $\overrightarrow{d}(v_i, v_j) = j - i$. Assume that $G - \{v_1, v_2, \ldots, v_{\overrightarrow{D}+1}\} = G^*$. To complete our proof, we will consider two cases.

Case 1 $V(G^*) \neq \emptyset$.

By Lemma 3, for any $i, j \in \{1, 2, \ldots, \overrightarrow{D}+1\}, i < j, x \in V(G^*)$, we have

$$\overrightarrow{d}(v_i, x) + \overrightarrow{d}(x, v_j) \geq \overrightarrow{d}(v_i, v_j) = j - i.$$

Since $\forall x, y \in V(G), x \neq y$, if $\overrightarrow{d}(x, y) = 1$, $\overrightarrow{d}(y, x) \geq 2$. Therefore for any $x, y \in V(G)$ with $x \neq y$, $\overrightarrow{d}(x, y) + \overrightarrow{d}(y, x) \geq 3$. It follows that

$$\begin{aligned}
&\sum_{\substack{x\in V(G^*),\\ y\notin V(G^*)}} \overrightarrow{d}(x, y) + \sum_{\substack{x\notin V(G^*),\\ y\in V(G^*)}} \overrightarrow{d}(x, y) \\
&\geq \sum_{x\in V(G^*)} \Big(\sum_{i=1}^{\lfloor \frac{\overrightarrow{D}+1}{2} \rfloor} (\overrightarrow{d}(v_i, x) + \overrightarrow{d}(x, v_{\overrightarrow{D}+2-i}) + \overrightarrow{d}(v_{\overrightarrow{D}+2-i}, x) + \overrightarrow{d}(x, v_i)) \Big) \\
&\geq \sum_{x\in V(G^*)} \Big(\sum_{i=1}^{\lfloor \frac{\overrightarrow{D}+1}{2} \rfloor} (\overrightarrow{D} + 2 - 2i + 2) \Big) \\
&\geq \sum_{x\in V(G^*)} \Big(\sum_{i=1}^{\lfloor \frac{\overrightarrow{D}+1}{2} \rfloor} (\overrightarrow{D} - 2i + 4) \Big) \\
&= \left\lfloor \frac{\overrightarrow{D}+1}{2} \right\rfloor (\overrightarrow{D} - \left\lfloor \frac{\overrightarrow{D}+1}{2} \right\rfloor + 3)(n - \overrightarrow{D} - 1) \\
&\geq \frac{\overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1)}{4}.
\end{aligned}$$

Therefore,

$$\overrightarrow{\sigma}(G) = \sum_{\substack{x,y\notin V(G^*),\\ x\neq y}} \overrightarrow{d}(x,y) + \sum_{\substack{x\in V(G^*),\\ y\notin V(G^*)}} \overrightarrow{d}(x,y) + \sum_{\substack{x\notin V(G^*),\\ y\in V(G^*)}} \overrightarrow{d}(x,y) + \sum_{\substack{x,y\in V(G^*),\\ x\neq y}} \overrightarrow{d}(x,y)$$

$$\geq \binom{\overrightarrow{D}+2}{3} + \left(\binom{\overrightarrow{D}+1}{2} + \overrightarrow{D}\right) + \frac{\overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1)}{4} + 3\binom{n-\overrightarrow{D}-1}{2}.$$

It follows that

$$\overrightarrow{\mu}(G) \geq \frac{4\binom{\overrightarrow{D}+2}{3} + 4\binom{\overrightarrow{D}+1}{2} + 4\overrightarrow{D} + \overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1) + 12\binom{n-\overrightarrow{D}-1}{2}}{4n(n-1)}.$$

Case 2 $V(G^*) = \emptyset$.

In this case

$$\begin{aligned}\overrightarrow{\sigma}(G) &= \sum_{1\leq i<j\leq D+1} \overrightarrow{d}(v_i,v_j) + \sum_{1\leq i<j\leq D+1} \overrightarrow{d}(v_j,v_i)\\ &\geq \sum_{1\leq i<j\leq D+1} (j-i) + \Big(\sum_{1\leq i<j\leq D+1} 1\Big) + \overrightarrow{D}\cdot 1\\ &= \binom{D+2}{3} + \binom{D+1}{2} + \overrightarrow{D}.\end{aligned}$$

Therefore,

$$\begin{aligned}\overrightarrow{\mu}(G) &\geq \frac{\binom{D+2}{3} + \binom{D+1}{2} + \overrightarrow{D}}{n(n-1)}\\ &= \frac{4\binom{\overrightarrow{D}+2}{3} + 4\binom{\overrightarrow{D}+1}{2} + 4\overrightarrow{D} + \overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1) + 12\binom{n-\overrightarrow{D}-1}{2}}{4n(n-1)}.\end{aligned}$$

To conclude,

$$\overrightarrow{\mu}(G) \geq \frac{4\binom{\overrightarrow{D}+2}{3} + 4\binom{\overrightarrow{D}+1}{2} + 4\overrightarrow{D} + \overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1) + 12\binom{n-\overrightarrow{D}-1}{2}}{4n(n-1)}.$$

□

Remark. Theorem 12 uses the directed diameter to give the lower bound of average directed diameter. Theorem 12 is actually an analogy of Theorem 9 in directed graphs. And the bound is sharp. Let G be a directed cycle of order 3. Then $\overrightarrow{D}(G) = 2$ and $\overrightarrow{\mu}(G) = \frac{3}{2}$. So

$$\overrightarrow{\mu}(G) = \frac{4\binom{\overrightarrow{D}+2}{3} + 4\binom{\overrightarrow{D}+1}{2} + 4\overrightarrow{D} + \overrightarrow{D}(\overrightarrow{D}+5)(n-\overrightarrow{D}-1) + 12\binom{n-\overrightarrow{D}-1}{2}}{4n(n-1)}.$$

6 Conclusion

Through this paper, we give several estimations for the diameter and the average diameter of connected simple graph, and we also do the comparison between our results and the present results. Theorem 2 is the development of Result 3, while Theorem 1, 4, 5, 11 are the estimations by using other ordinary characteristics of graph. Other estimations are based on new perspectives, including minimum edge degree (Theorem 3), number of vertex-disjoint cycles (Theorem 7), connectivity (Theorem 8) and Hamilton cycle (Theorem 6). We also use the diameter of the connected simple graph to estimate its average diameter (Theorem 9,10,12).

Apart from our present result, we plan to further study the diameter. Firstly, we plan to estimate characteristics of the graph by the diameter. Secondly, since the estimations for directed graph are simply the analogy of those for undirected graph, we plan to give more estimations for directed graph from distinct perspectives. Thirdly, we will do research on estimations for the diameter when several characteristics are given at the same time. Finally, we are going to study the diameter of weighted graph.

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