Study on the Relative periodicity and the Long-term behavior of Hypocycloid and Quasicycloid

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Abstract

The long term behaviors of hypocycloids and quasicycloids are studied in this paper. Quasicycloids, a new concept established in this essay (by changing two coefficients into two different irrational numbers), is a variation of hypocycloid and it is yet unknown to us. By introducing the new concept of relative period, a model was established to prove that certain hypocycloids are dense in a certain area and to seek out hidden connections between hypocycloids and quasicycloids. The model is also used to prove any real number has a sequence of good rational approximations and to find it. To view the long term behavior in a dynamical way, the alpha limit set of the hypocycloids equal the set of their limit points. The paper also includes some hypothesis waiting to be proved. The result of the study on hypocycloid can be used to perfect the hypocycloid internal-combustion Engine.

Key words: hypocycloids, quasicycloids, density, relative period.

Introduction

The raise of the problem

When doing my homework for the weekend calculus course in Tsinghua, I observed some interesting and attractive properties of the hypocycloidal image when the coefficients were changed into irrational numbers. Then I further wondered what patterns would appear if the two coefficients of parameter equation were changed into two different irrational numbers. After a lot of plotting observations in Geometer's Sketchpad, some basic properties of hypocycloids were discovered, and the general direction of research was determined.

Research methodology

The main idea of this study was to explore the general issues from the particular cases, and to transform complex situations into simple basic models.

This paper studies the parameter equations with two irrational coefficients. Firstly, I studied the parameter equations with two equal irrational coefficients. In order to simplify the problem, I first studied the singularity distribution, because they are the backbone of the entire image. For further simplification, a basic model of the relative period was established, which was, in vivid, to "cut" a excircle from the origin, transforming the circle into line. Then on the basis of the above process, the denseness properties of the hypocycloids was explored and proven. Finally, with the same idea, quasicycloid, the parameter equation with two different irrational coefficients was studied.

Background

Hypocycloid is a common type of curve with widely application in internal combustion engine systems. The majority of hypocycloid models used in the internal combustion engine are with two equal rational coefficients 2, and previous studies on hypocycloids have mostly stayed in the level of rational coefficients.

Internal combustion engine is the heart of the car, while the car industry is the pillar industry of the country, which means studies on the internal combustion engine have great significance. The new type of internal combustion engine with the hypocycloid gear structure can realize the complete linear motion of the link rod and the piston, which can effectively eliminate the cylinder-side pressure, reduce the wear of piston and cylinder, improve the economical efficiency and power of the internal combustion engine, and improve the service life of the engine.

Highlights of This Topic

Previous studies on hypocycloids have mostly stayed in the level of rational coefficients. In this study, long-term behavior of hypocycloids with irrational coefficients was explored, and basic model of the relative period was established. Furthermore, by the change of two coefficients of parameter equation, the concept quasicycloid was proposed. The results of this study can be applied in the new type of hypocycloid internal combustion engine, to improve the efficiency and the service life of the engine. The results of studies on hypocycloids could also be applied in the cryptology.

1. Basic properties of hypocycloid

Definition of hypocycloid

Hypocycloid is a special plane curve generated by the trace of a fixed point on a small circle that rolls within a larger circle, in this paper, the excircle. It is comparable to the cycloid but instead of the circle rolling along a line, it rolls within a circle.

Derivation of the parameter equation of hypocycloid

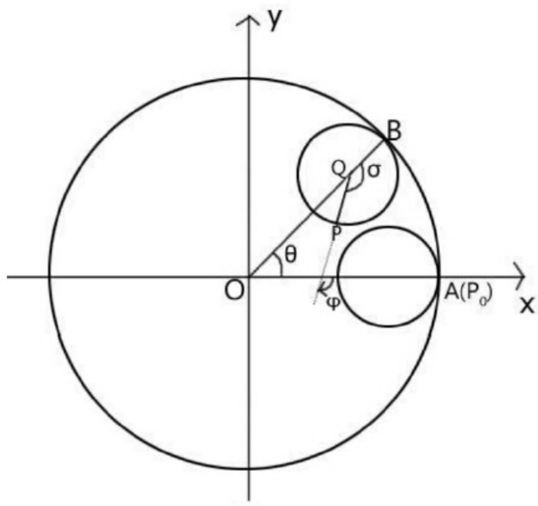
Coordinate system was established as shown in the figure. The excircle has radius R, and the moving circle has radius r and center C. The excircle intersected with X-axis at the point A. Initially the moving circle cut inside the excircle at point A, point P and A coincide at this time. θ is the turning angle of the center of moving circle relative to the center of excircle. σ is the turning angle of the point P relative to the center of moving circle.

φ is the directed angle of vector $\overrightarrow{OP}$ and X-axis,

$$\varphi = -(\sigma - \theta) = \frac{r - R}{r}\theta$$

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$$

$$\overrightarrow{OQ} = ((R - r)\cos\theta,\ (R - r)\sin\theta$$



So: $\overrightarrow{QP} = \left(r\cos\frac{R-r}{r}\theta, -r\sin\frac{R-r}{r}\theta\right)$

Therefore:

$$\overrightarrow{OP} = \left(\left[(R - r)\cos\theta + r\cos\frac{R-r}{r}\theta\right], \left[(R - r)\sin\theta - r\sin\frac{R-r}{r}\theta\right]\right)$$

Thus we could get the parameter equation of hypocycloid:

$$\begin{cases} x(\theta) = (R - r)\cos\theta + r\cos\dfrac{(R - r)}{r}\theta \\ y(\theta) = (R - r)\sin\theta - r\sin\dfrac{(R - r)}{r}\theta \end{cases}$$

For the convenience of following studies, θ is denoted as t, in order to simplify the parameter equation, Suppose $R - r = 1$, and α is an irrational number and $r = \frac{1}{\alpha}$, Consider the hypocycloid with the coefficients as follow:

$$\begin{cases} x_1(t) = \cos t + \dfrac{\cos \alpha t}{\alpha} \\ y_1(t) = \sin t - \dfrac{\sin \alpha t}{\alpha} \end{cases}$$

σ is defined as the turning angle of the moving circle.

$R = \frac{1}{\alpha} + 1$, $r = \frac{1}{\alpha}$, $\sigma = \frac{R}{r}\theta$

Thus $\sigma = (\alpha + 1)t$

Definition 1: Singularity

When $\sigma = 2k\pi$, the point turns back to the excircle.

This point in the excircle is defined as a Singularity.

$$\sigma = 2k\pi = (\alpha + 1)t_k \ k \in Z^{+}$$

$$t_k = \frac{2k\pi}{(\alpha + 1)}$$

All the cycloidal points can be expressed as

$$C_k : (x_1(t_k), y_1(t_k)),$$

$$t_k = \frac{2k\pi}{(\alpha + 1)},$$

of which,

$$\begin{cases} x_1(t_k) = \cos t_k + \dfrac{\cos \alpha t_k}{\alpha} \\ y_1(t_k) = \sin t_k - \dfrac{\sin \alpha t_k}{\alpha} \end{cases}$$

This is the basic of the following studies.

Noted that if to find the derivative of the hypocycloid, and make the derivative equal to zero,

$$\begin{cases} x'_1(t) = 0 \\ y'_1(t) = 0 \end{cases}$$

It is easy to find that

$$t = \frac{2k\pi}{(\alpha + 1)}$$

This is the reason why the point on the excircle is defined as singularity.

2. Establishment of the basic model of relative period

Although there is no strict periodicity in hypocycloids with irrational numbers, it presents a similar property of periodic regression on the impact point(singularity) of circle, which is defined as relative period. The especial property of relative period is that it comes back once in a while to a point very near to where it started in the first place.

Singularity motion is simplified to get a basic model of relative period:

Definition 2: Denote by Ω_t the mapping $t \to \Omega_t = \left\{\frac{t}{\alpha+1}\right\}$, where

$t \in N^+$ and α is an given irrational number. To view this in a dynamical way, the change in t causes the point "jump" in (1,0). Define $\Omega_t \to \Omega_{t+1}$ as a movement.

Ω_t is also the distance between the point's location after it moved "t" times and the point 0 where it started.

This original definition can be understood this way: firstly we minify the outer circumference to 1 in equal proportion, and then cut along the initial site of the singularity to form a straight line segment, while the point will still motion the same way as the singularity does on the hypocycloids. That is, a point moves forward at a time in a fixed interval. The core of the motion is that "go from the right side, and come out from the left side", just like the plumber Mario in the classic game.

When a unit length of motion $D_0 = \frac{1}{\alpha+1}$ reaches the position around the region right endpoint, it still needs to move a unit length. Thus the excess part is compensated by the left side, manifested as the point appearing again from the left and the distance from left endpoint is:

$$D_1 = \left[\frac{1}{D_0} + 1\right] D_0 - 1 \prec D_0 .$$

And so on, creating D_2, D_3, ……D_{N-1} and D_N. This kind of motion has an ergodicity property.

This kind of motion presents a property similar to periodicity. After a limited time, the point can only infinite approach the value of initial point but cannot take it. We can also observe that through t times of motion, the point can move back to the position D_{N+1} distant from the initial point; through t more times of motion, it will move back to the position $2D_{N+1}$ distant from the initial point. The process is considered as a whole that the point moves from t to 2t times holds the equivalent of all the points at time t moving forward for D_{N+1}, thus the details can be omitted and considered as an equivalent motion.

Definition 3: Distance of equivalent motion

Take any natural numbers m, n, N, and m<n, then $\|\Omega_m - \Omega_n\| = D_N$, $(\text{defining } \Omega_0 = 0)$

$$\text{If } \forall t \in (m, n),\ |\Omega_t - \Omega_m| > D_N$$

(Subscript N is for the ordinal number under such circumstances), then define D_N as N-stage distance of equivalent motion for point column, and $D_0 = \dfrac{1}{\alpha + 1}$

That is, beginning from the m time motion, the distance between two points is minimum at the n time motion. After n-m times of motion, the actual distance of the point motion is D_N, presenting the “equivalency”.

Definition of Relative period is an extended definition of the period. The definition of relative period is that after a period time of Δt, the point moves back to the position Ω_n with a minimum distance $\|\Omega_m - \Omega_n\| = D_{N+1}$ away from the initial point Ω_m.

Definition 4: Equivalent breach

$\forall k \in Z^{+}$ meets with $D_N - kD_{N+1} \geq 0$,

$Q_N = \min\{D_N - kD_{N+1}\}$, Q_N is defined as N-stage equivalent breach.

Definition 5: Equivalent interval

Take any natural numbers m, n, N, and m<n, then $\|\Omega_m - \Omega_n\| = D_N$, (defining $\Omega_0 = 0$)

If $\forall t \in (m, n)$, $|\Omega_t - \Omega_m| > D_N$

Then define D_N as $(N+1)$-stage equivalent interval. Define $D_0 = \dfrac{1}{\alpha + 1}$

Here D_N is not only the N-stage motion distance, but also the $(N+1)$-stage equivalent interval because the N-stage equivalent motion itself will create a higher stage $(N+1)$ equivalent interval every time. And the length of equivalent interval is less than the N-stage equivalent interval.

After the $(N+1)$-stage equivalent motion moved several times in the N-stage equivalent interval, it "jumped out" of the N-stage equivalent interval and created a shorter distance between the two points, named $(N+1)$-stage equivalent interval. At the same time a N-stage breach was generated. Thus we get the first obvious lemma:

Lemma 1:

When $N \to \infty$, $D_{N+1} \to 0$

Proof by contradiction:

It is easy to figure out that D_N is divided into Q_N and D_{N+1} by the moving point, thus $D_{N+1} < D_N$, for $D_{N+1} + Q_N = D_N$. D_N is monotone decreasing and $D_N > 0$.

Thus, depending on the monotone bounded theorem, When $N \to \infty$,

$D_{N+1} \to a$, suppose $a \neq 0$, then $\exists D_{N+2} = D_{N+1} - Q_{N+1} < D_{N+1}$, contradiction.

Thus

$$D_{N+1} \to 0$$

It is worth noting that the starting point of equivalent motion is not unique. Several N-stage equivalent motions divide N-stage interval into several $(N+1)$-stage equal equivalent intervals and an N-stage breach. $(N+1)$-stage equal equivalent intervals generated in the former stage will be divided by D_{N+1}, which is obtained by $D_N - Q_N$ as per unit length of the equivalent motion.

The process presents a property similar to self-similarity to a certain extent. with The change over time, as the stage of the equivalent motions go up, the point's motion repeat itself in a smaller and smaller area. Meanwhile the motion patterns of different area with the same stage are completely the same. Therefore, we can focus on only one interval of the numerous N stage equivalent intervals when researching the (N+1)-stage equivalent motion.

Definition 6: N-stage relative period

Take any natural numbers m, n, N, and m<n, then $\left\|\Omega_m - \Omega_n\right\| = D_N, \left(\text{defining } \Omega_0 = 0, D_0 = \frac{1}{\alpha + 1}\right)$

$$\text{If } \forall t \in (m, n),\ \left|\Omega_t - \Omega_m\right| > D_N$$

$$S_N = \Delta t = n - m$$

Define S_N as N-stage relative period.

There is another description, compared to the definition of the traditional period: $\left|\Omega_t - \Omega_{t+S_N}\right| \prec D_{N-1}$

Lemma 2:

D_{N+1}, D_N and S_N defined above meet the iteration relation as follows:

$$D_{N+1} = S_N D_N - D_{N-1}$$
$$\text{and } S_N = \left[\frac{D_{N-1}}{D_N} + 1\right],$$

Proof: mathematical induction

Due to the selection of m, the starting point m, is with arbitrary equivalence, we could define m = 0.

When $N = 1$, $\Omega_1, \Omega_2 \ldots\ldots$ divide the interval into several one stage equivalent intervals with the same distance of D_1, thus

$$\exists \Omega_n, (1 - \Omega_n) < D_1$$

And $n = \left[\frac{1}{D_1}\right]$, the next point Ω_{n+1} will return the starting point of the interval to create the next relative period.

Thus $S_1 = 1 + \left[\frac{1}{D_1}\right] = \left[\frac{1}{D_1} + 1\right]$, match condition, and

$$D_2 = \left[\frac{1}{D_1} + 1\right] D_1 - 1 = S_1 D_1 - 1$$

Thus, when $N = 1$, it matches the condition.

Suppose that when $N = k$, it matches the condition, $D_{k+1} = S_k D_k - D_{k-1}$, in which $n - 0 = S_k = \left[\frac{1}{D_k} + 1\right]$ and $|\Omega_0 - \Omega_n| = D_{k+1}$, meets $\forall t \in (0, n), |\Omega_t - 0| > D_{k+1}$.

Thus we need find the next $\Omega_{n'}$, $|\Omega_0 - \Omega_{n'}| = D_{k+2}$, meet $\forall t \in (0, n'), |\Omega_t - 0| > D_{k+2}$.

Considering the motion is equivalent to that of equivalent interval D_k with distance of D_{k+1}. The motion makes the point far away from left endpoint of the interval until $\left[\frac{D_k}{D_{k+1}}\right]$ times,

$$\left(D_k - \left[\frac{D_k}{D_{k+1}}\right]D_{k+1}\right) < D_{k+1}$$

Thus, after $S_{k+1} = \left[\frac{D_k}{D_{k+1}}\right] + 1$ times, the point enters into next adjacent equivalent interval. The distance between the point and the left endpoint of next interval:

$$D_{k+2} = \left(\left[\frac{D_k}{D_{k+1}}\right] + 1\right)D_{k+1} - D_k < D_{k+1} < \left|\Omega_t - 0\right|,\ \forall t \in \left(0, n^{'}\right)$$

Similarly, $\Omega_{n^{'}}$, the point of the prior equivalent interval, enters into the equivalent interval with left endpoint being zero. And

$$\left|\Omega_0 - \Omega_{n^{'}}\right| = D_{k+2},$$

Thus $n^{'} = nS_N$.

Thus there must be $n^{'} = nS_N$ such that $\left|\Omega_0 - \Omega_{n^{'}}\right| = D_{k+2}$, meet $\forall t \in \left(0, n^{'}\right), \left|\Omega_t - 0\right| > D_{k+2}$, so $N = k + 1$ matches the condition.

Thus lemma 2 is proved.

At this point, the basic model of relative period is set up completely. Deeper studies will be performed as follows.

Firstly, an analogous Diophantine approximation conclusion depending on the relative period will be studied.

Construct the following sequence:

Definition 7: The sequence of "Ω" mapping

Sequence A : $A = \left\{a_n \middle| a_n = \frac{n}{\alpha + 1} - t, t = \sum_{i=1}^{n}\left[\frac{n}{S_i}\right], \mathrm{n} \in \mathrm{N}^{+}\right\}$. Define A as the sequence of "Ω" mapping.

A is also a one-to-one mapping of singularities on the hypocycloids.

Definition 8: N-stage column map

Sequence B_N : $B_N = \left\{b_{N_k} \middle| b_{N_k} = a_n, n = kS_N, \mathrm{n, k} \in \mathrm{N}^{+}\right\}$, Sequence B_N is a subset of A, describing every N-stage equivalent motion on the axis. Thus define

B_N as a N-stage sequence of "Ω" mapping.

So $B_N \subseteq B_{N-1} \subseteq B_{N-2} \cdots\cdots\cdots\cdots \subseteq B_2 \subseteq B_1 \subseteq A$

Definition 9: Approximating sequence

Take the subset of A by a rule of taking the first item in the N-stage sequence of "Ω" mapping corresponding to every N value, to constitute a approximating sequence A_c.

$$A_C = \{c_n | c_n = b_{n_1}, \mathrm{n} = 1, 2 \ldots\ldots \mathrm{N}, \}$$

This is also the first point of every N-stage relative period. These points can be arranged to form a point set.

Lemma 3:

$$N \to \infty, \quad c_n \to 0$$

that is, the limit of sequence A_c is 0.

Proof:

Every number in A_c corresponds to an individual point in the equivalent motion, starting from the origin and is the first point in the N-stage equivalent motion corresponding to every N value. Therefore, the distance between c_n and the origin is D_N, and when $N \to \infty$, $D_{N+1} \to 0$

so

$$\lim_{\mathrm{N}\to\infty} D_N = 0$$

Accordingly

$$N \to \infty, \quad c_n \to 0$$

That is, the limit of this sequence is 0.

Suppose $p = \prod_{i=1}^{N} S_i$, $G_m = \dfrac{p}{\alpha + 1}$, for any fractional part of G_m, call it $g_m = \left\{\dfrac{p}{\alpha + 1}\right\}$, it is obvious g_m can be described as $g_m = c_n = b_{n_1}$.

Now let: $q = \left[\dfrac{p}{\alpha + 1} \right]$

Theorem 1: any given rational numbers of irrational numbers infinitely approximation theorem:

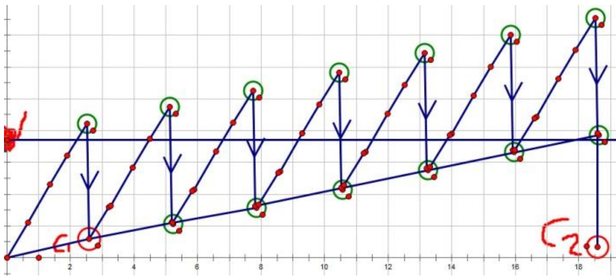
When $N \to \infty$, we can get:

$$\lim_{N \to \infty} \frac{\prod_{i=1}^{N} S_i}{\left[\dfrac{\prod_{i=1}^{N} S_i}{\alpha + 1} \right]} = \alpha + 1$$

That is:

$$\left| (\alpha + 1) - \frac{p}{q} \right| \to 0$$

This is a basic conclusion of Diophantine approximation; this paper gives a proof from a new Angle. The value of this theorem is that a new iterative algorithm it has given, and as long as calculated according my algorithm in this paper, we can get a definite recursion approximation sequence, which can approach any given α .



Inference 1: The ratio of rotation and revolution

For hypocycloid, there is a strict ratio only when taking the rational numbers. When it comes to hypocycloids with two same irrational number as coefficients, we can only get the limit of the ratio.

The rotation turns of the moving circle divides the revolution turns of the moving circle, when moving circle get close to the starting point the limit is:

$$\lim_{N \to \infty} \frac{\prod_{i=1}^{N} S_i}{\left[\dfrac{\prod_{i=1}^{N} S_i}{\alpha + 1} \right]} = \alpha + 1 = \frac{R}{r}$$

The limitation is the radius R of excircle divides radius r of moving circle.

3. The proof of denseness

All of the above discussed is the foundation of the denseness proof.

Lemma 4: $\{\Omega_t\}$ is dense on the interval $(0.\ 1)$

Proof:

As $\exists \Omega_N \in \{\Omega\}, \mathit{subset}\{\Omega_N\} = \left\{\dfrac{N}{\alpha+1}\right\}, (\mathrm{N} \in \mathrm{N}^{+})$ it has been proved already,

D_N ,the distance between Ω_N and axis origin D_N , meets:

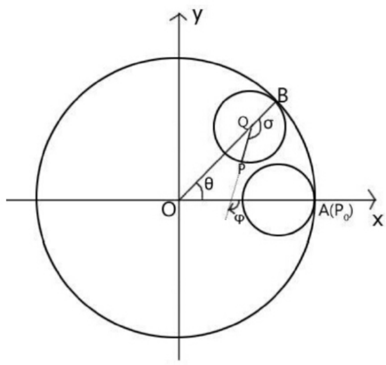
$$\lim_{N\to\infty} D_N = 0$$

Thus a subset from $\{\Omega_t\}$ can be found to approximate 0. Now the only thing needed to be proved is that starting from 0 the point can jump or get close to any point.

Proof by contradiction: assuming the point traveling from 0 cannot approximate, say, Ω_m from the left, we assume that the nearest point to Ω_m on the left is Ω_x with the distance is X. So now we view Ω_x as a starting point, by the previous proof, we can prove that Ω_x has a approximating subset, and thus we can find out a subsequence getting close to Ω_x from the right. There must be a point Ω_y in these subsequences, of which the distance between Ω_y and Ω_m is less than X. Contradiction found. Proved.

Consider the hypocycloid with the coefficients as follow:

$$\begin{cases} x_1(t) = \cos t + \dfrac{\cos \alpha t}{\alpha} \\ y_1(t) = \sin t - \dfrac{\sin \alpha t}{\alpha} \end{cases}$$



Where $t \in R$, α must be irrational numbers, we can establish a mapping $\varphi(\mathrm{t}) \to (\mathrm{x}(t), y(t))$.

Definition 10: The image of $\varphi(\mathrm{t})$:

The image of $\varphi(\mathrm{t})$ is $H = (\mathrm{x}_1(t), y_1(t))$. from the definition of hypocycloids, H is at a Closed annulus E

$$E = \left\{ (x, y) \middle| \frac{1}{\alpha} = r \le \sqrt{x^2 + y^2} \le R = \frac{1}{\alpha} + 1 . x, y \in R \right\}$$

Lemma 5:

The singularity of hypocycloids is dense at the excircle.

Proof:

as mentioned above,: $C_k : (x_1(t_k), y_1(t_k))$ stands for the singularity

The distance between two singularities C_k and C_{k+1} at the excircle is the perimeter of the moving circle:

$$c = 2n\pi r = \frac{2n\pi}{\alpha}, \quad n \in Z^+$$

And the perimeter of the excirlce is:

$$C = \frac{2(\alpha + 1)\pi}{\alpha}, \quad n \in Z^+$$

The ratio of two lengths is:

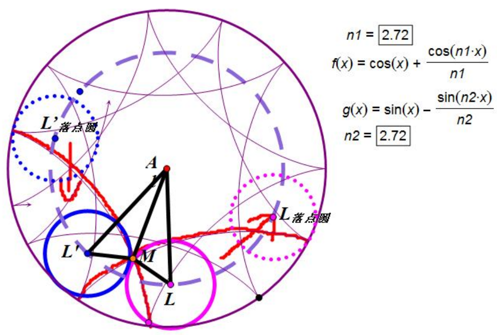
$$\frac{c}{C} = \frac{n}{\alpha + 1}, \quad n \in Z^+$$

As it turns a whole loop of excircle and $\frac{n}{\alpha + 1} > 1$, what we actually consider is $\left\{ \frac{n}{\alpha + 1} \right\}$, this problem has been completely transformed to lemma 4 problem. Proof completed.

Lemma 6: there is no period of hypocycloids which coefficient is irrational numbers.

Proof:

Take any arbitrary point on hypocycloid M, there are only two axisymmetric moving circle L' and L corresponds to two respective curves only and the two curves only have 4 singularities. As is proved in Lemma 6 each singularity is unique, so each curve cannot be repeated. By the definition of periodic



function, any point on the image will appear infinite times periodically. But for hypocycloids of irrational Numbers, any point on the image will only be passed twice. So clearly there is no strict period for hypocycloids of irrational numbers.

Lemma 7: Do the hypocycloid image and a circle within a radius of a circle r, the center is the hypocycloid, and the intersection point of circle C, arc length between any two points of intersection for less than two ipsilateral recent cycloidal point between the arc length.

Theorem 2:

The image of $\varphi(\mathrm{t})$, which is the mapping of real number collection R, is at a close circle: $E : \left((x, y) \middle| r \le \sqrt{x_0^{\ 2} + y_0^{\ 2}} \le R \right).$ And It is dense.

$$H = \overline{E}$$

Another expression is:

As $\forall A_0(x_0, y_0)$,

$if\ \frac{1}{\alpha} = r \le \sqrt{x_0^{\ 2} + y_0^{\ 2}} \le R = \frac{1}{\alpha} + 1. x_0, y_0 \in R$, there is a Subset in H with limit A_0.

Lemma 5 has proved that singularities in **excircle** are dense. To apply the conclusion of singularity to the whole hypocycloid, it is necessary to prove that for every single point on hypocycloid, a approximating subsequence can be found.

Proof:

Denote U_k as the intersection points of circle U (with radius u and the same center of hypocycloid) and hypocycloid. $u \in [r, R]$

$$\begin{cases} x_1(t_U) = \cos t + \dfrac{\cos \alpha t}{\alpha} \\ y_1(t_U) = \sin t - \dfrac{\sin \alpha t}{\alpha} \\ x_1^{\ 2}(t_U) + y_1^{\ 2}(t) = d^2 \end{cases}$$

Then

$$\cos(\alpha + 1)t = \frac{\alpha}{2}\left(d^2 - 1\right) - \frac{1}{\alpha},$$

So

$$t_U = \frac{\arccos\left[\frac{\alpha}{2}\left(d^2 - 1\right) - \frac{1}{\alpha}\right] + 2k\pi}{(\alpha + 1)}, \quad k \in Z^+$$

The t for singularity is $t_k = \dfrac{2k\pi}{(\alpha + 1)}$

For $\forall k_1, k_2 \in Z^+, j_1$: the length of arc for two intersection points Z and V, is:

$$j_1 = \Delta\theta \bullet \mathrm{d} = (t_{U1} - t_{U2})d = \left[\frac{2(k_1 - k_2)\pi}{(\alpha + 1)}\right] \bullet d$$

For the same k_1, k_2 , j_2 :the length of arc between two singularities C_{k_1}, C_{k_2}, is:

$$j_2 = \Delta t_k \bullet \mathrm{R} = (t_{k_1} - t_{k_2})R = \left[\frac{2(k_1 - k_2)\pi}{(\alpha + 1)}\right] \bullet R$$

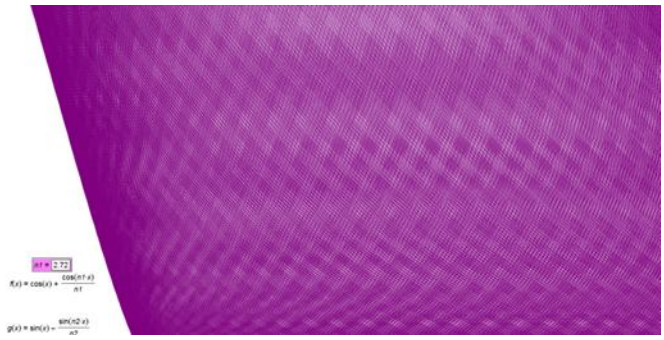
$d \in [r, R]$, so

$$j_1 \le j_2 .$$

For every two singularities C_{k_1}, C_{k_2}, it is proven that there are approximating subsequences for both singularities. For every point C_k on the approximating subsequences between C_{k_1}, C_{k2_1}, denote by μ the mapping: $C_k \to U_k$. Thus a approximating subsequence $\{U_k\}$ is found. Proof complete.

Furthermore, due to $j_1 \le j_2$, it is even “denser” on the circle *U*. This in one of the basis for a later hypothesis

The graph is a small part of the computer simulated hypocycloid with a large enough t. it is reasonable to deduce that certain area can be completely covered when t is infinitely large.



4. The study of long-term behavior

Lemma 7: the image of hypocycloid and the original image are measurable.
Proof:
All the singularity can be expressed as:

$$C_k \ : \ \left(x_1\left(t_k\right), y_1\left(t_k\right)\right), \ t_k = \frac{2k\pi}{\left(\alpha + 1\right)}$$

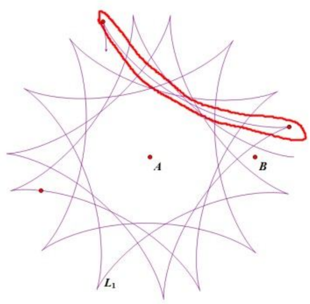
When $2\left(k - 1\right) < \sigma \leq 2k\pi$, that is

$$\frac{2\left(k - 1\right)\pi}{\left(\alpha + 1\right)} < t \leq \frac{2k\pi}{\left(\alpha + 1\right)}$$

In this way, the hypocycloid can be divided into curves.
For $\forall A \ \left(x_1\left(t\right), y_1\left(t\right)\right)$, consider the following one-to-one correspondence:

$$A \rightarrow C_k$$

Thus each section of the hypocycloid can be corresponded to a singularity. $k \rightarrow C_k$, thus, $\left\{C_k\right\}$ is denumerable, that is, A is denumerable. Every curve is a bounded open set, which means it is measurable.



So take an open set of H K, it contains all arcs can be corresponding to the singularity, so the number of arcs are measurable. And each section curve of the original image is measurable, and predecessors have proved that the collection of the measurable set and measurable set is measurable set, so K can be measured.

And $F = \overline{K}$, so F is measurable.

Definition 11: if $P(x, y)$, meets the condition $\forall \varepsilon > 0, t_n \to \infty$

$$\sqrt{|x(t_n) - x|^2 + |y(t_n) - y|^2} < \varepsilon$$

Among them t_n is the time subsequence, $namely \{t_n\} \subset \{t : t \in |R\}$.

Define alpha limit set:

$$L = \{(x, y)\}$$

Inference 2: Hypocycloid's limit set is equal to alpha limit set. Namely:

$$L = \overline{H}$$

Because the hypocycloid is dense on the annulus E, this is an obvious conclusion.

Inference 3: The measure of the boundary of the Hypocycloid is 0. Namely:

$$m(\partial \overline{H}) = 0$$

M stands for Lebesgue measure.

Proof: the Boundary of the images consists of the singularities $C_k : (x_k(t), y_k(t))$, and each point corresponds to an integer k, so cardinal number of the set $\{C_k\}$ a, so:

$$m(\partial \overline{H}) = 0 .$$

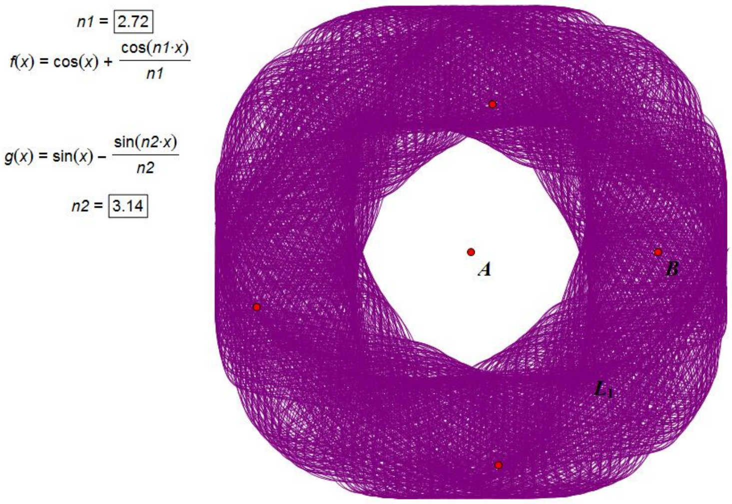
Definition 12:

$$\begin{cases} x_2(\theta) = \cos\theta + \dfrac{\cos\alpha\theta}{\alpha} \\ y_2(\theta) = \sin\theta - \dfrac{\sin\beta\theta}{\beta} \end{cases}$$

When the two coefficient α and β is irrational Numbers and $\alpha \neq \beta$, define this as a quasicycloid. $t \in R$

Similarly establish the mapping $\varphi_2(t) \to (x_2(t), y_2(t))$, , the image is $H_2 = \{x_2(t), y_2(t)\}$.

Computer simulated quasicycloid



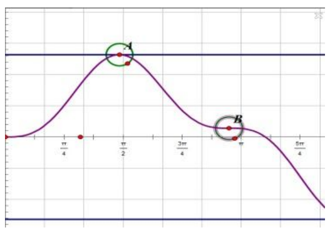
The essential difference between hypocycloid and quasicycloid is that the singularities in the hypocycloid degenerated to stationary points in quasicycloid.

To get the distribution of the stationary points, Let: $x'_2(t) = 0, t = \frac{2k\pi}{(\alpha + 1)}$.

Let $y'_2(t) = 0, t = \frac{2k\pi}{(\beta + 1)}, k \in Z^+$.

The computer simulated image of $\left\{t \middle| t = \frac{2k\pi}{(\alpha + 1)}\right\}$ is

The computer simulated image of $\left\{t \middle| t = \frac{2k\pi}{(\beta + 1)}\right\}$ is



With The geometric sketchpad simulation, it is consistent with the conclusion.

Obviously at this time there is no t makes the transverse and longitudinal derivatives equal to zero at the same time, that is, no singularity.

Theorem 3: Quasicycloid is measurable.

Proof: Use the same methodology as that in hypocycloid.

$$C_\alpha : \{x_2(t_\alpha)\}, \quad t_\alpha = \frac{2k\pi}{(\alpha + 1)} k \in Z^+$$

$$C_\beta : \{y_2(t_\beta)\}, \quad t_\beta = \frac{2k\pi}{(\beta + 1)} k \in Z^+$$

Get the collection of the two set:

$$C = C_\alpha \bigcup C_\beta$$

For arbitrary point $P(x_2(t), y_2(t))$ on the quasicycloid, if it meets

$$\frac{2(k_\alpha - 1)\pi}{(\alpha + 1)} < t \le \frac{2k_\alpha \pi}{(\alpha + 1)}, \quad k_\alpha \in Z^+$$

$$\frac{2(k_\beta - 1)\pi}{(\beta + 1)} < t \le \frac{2k_\beta \pi}{(\beta + 1)}, \quad k_\beta \in Z^+$$

Establish a mapping φ_p :

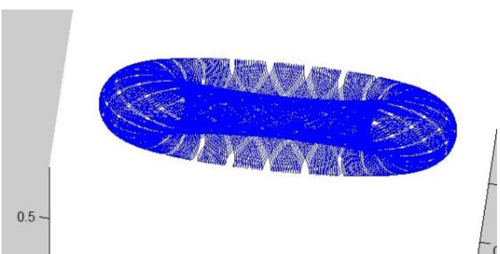
$$P(x_2(t), y_2(t)) \to \varphi_p(k_\alpha, k_\beta)$$

Once again quasicycloid is divided into curves, mapping to $\varphi_p(k_\alpha, k_\beta)$. $k_\alpha, k_\beta \in Z^+$ a $\bullet$ $a = a$, thus the cardinal number of $\varphi_p(k_\alpha, k_\beta)$ is denumerable. Every curve is bounded open set thus Quasicycloid is measurable.

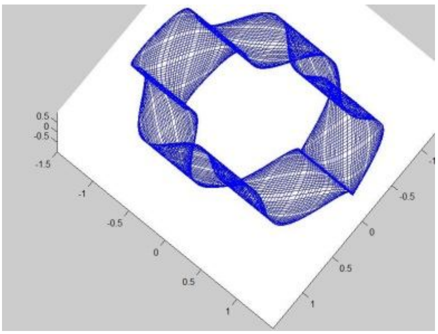
5. Conjectures and Prospects.

Conjecture 1: Conclusions still applies to the three-dimensional hypocycloid and quasicycloid, but not for coefficient of complex number.
Illustration: Three-dimensional quasicycloid (plus a z-axis, z (t)= sint),

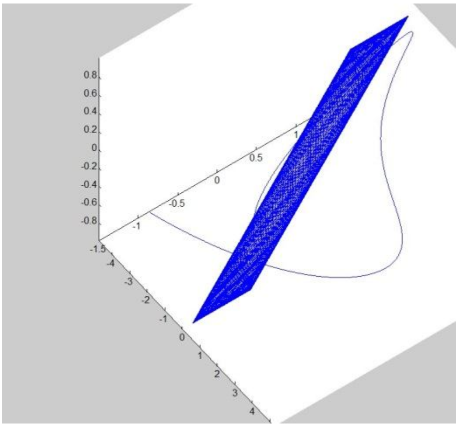
Hypocycloids: Three-dimensional hypocycloid (coefficients are two π, Z-axis: z=sint).



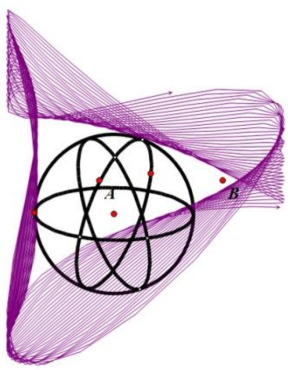
Quasicycloids: Coefficients are 6 and e, respectively:



Complex number i as the coefficient. From this figure area that is not dense could be easily observed.

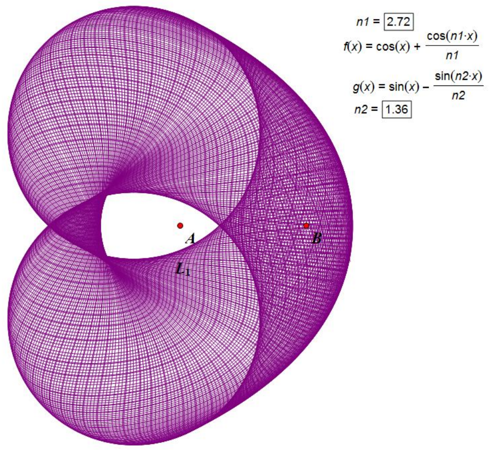


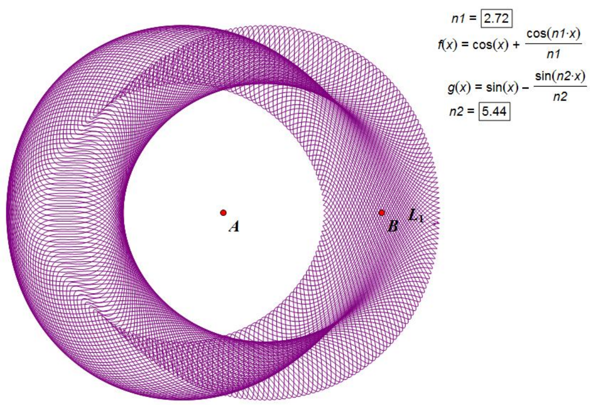
Rationale: Two-dimensions hypocycloidal itself is very similar to a three dimensional projection. So I guess dimensional projection of a hypocycloid is itself under the three-dimensional motion. If it is restored to three-dimensional, two-dimensional conclusions can be similarly applied in three-dimensions.



Two - dimensional hypocycloidal on the right figure，coefficients are 2 and e^3, respectively

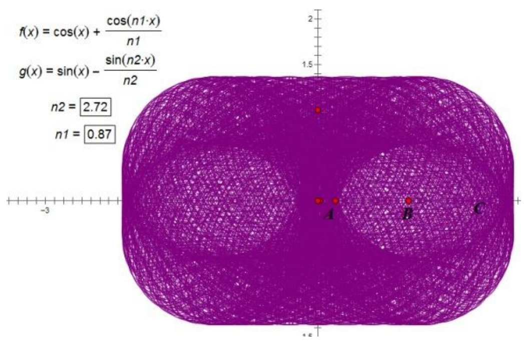
Illustration: quasicycloid (coefficients are e and e/2, e and 2e respectively)

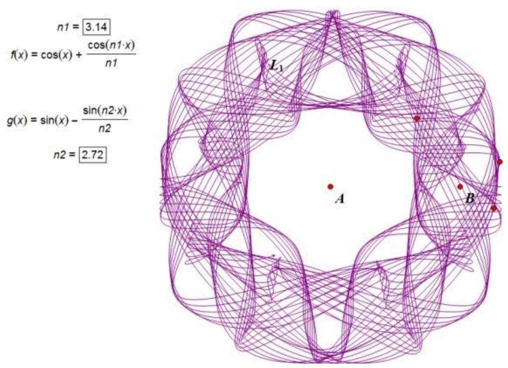


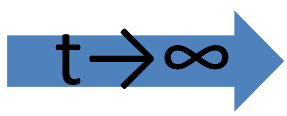


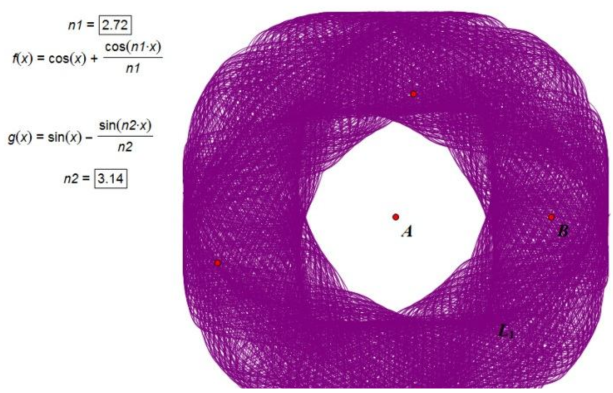
Conjecture 2: Quasicycloid dense in a certain area.

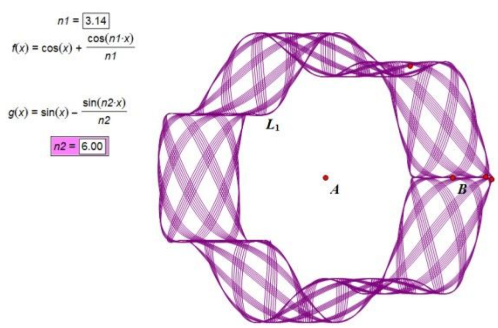
Computer simulation: Due to the limited memory, t cannot be taken to infinity, but big enough to coincidence me denseness idea.

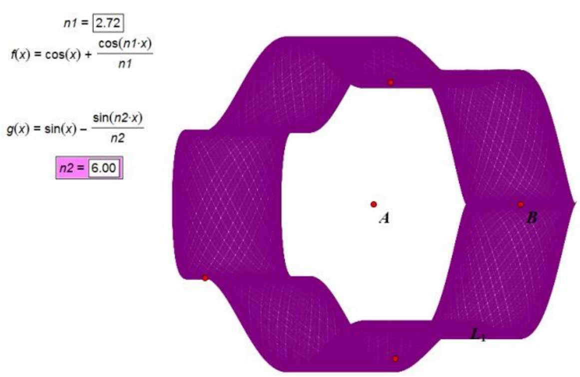


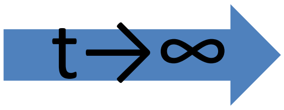












Conjecture 3:

Given any open set $F \subset \overline{H}$,

if $\varphi^{-1}(\mathrm{t}) = \{\mathrm{t} \in R | \varphi(\mathrm{t}) \in F\}$ $E(\tau) = [-\tau, \tau] \cap \varphi^{-1}(\mathrm{t})$, Then

$$\lambda \frac{mF}{mE} \le \underline{\lim}_{\tau \to \infty} \frac{mE(\tau)}{2\tau} \le \lim_{\tau \to \infty} \frac{mE(\tau)}{2\tau} \le \overline{\lim}_{\tau \to \infty} \frac{mE(\tau)}{2\tau} \le \Lambda \frac{mF}{mE}$$

Supplementary of Conjecture 3: It is observed that the mapping $\varphi(\mathrm{t})$ is surjection, but not one to one mapping, which means a point on the image may correspond to a plurality of t. As a result, it might add complexity when doing the reverse mapping. To be more precise and rigorous, the $\varphi^{-1}(\mathrm{t})$ is additionally defined that all values of t are taken at the intersected point $\varphi(\mathrm{t})$. It is easy to prove that the number of intersections is countable.

Proof: Using above the mapping φ_p :

$$P(x_2(t), y_2(t)) \to \varphi_p(k_\alpha, k_\beta)$$

Thus, all the curves of quasicycloid are mapped to the $\varphi_p(k_\alpha, k_\beta)$, that is, the number of curves is countable. What's more, the number of intersections of a presence on the curves cannot be larger than $N = a \bullet a = a$ (here a is the cardinal number). Thus the set of intersections is denumerable. It would have no impact on conjecture 3.

Conjecture 4:

If $\forall M(a, b) \in \overline{H_2}$, There are $\{x_2(t_n), y_2(t_n)\}$ in H_2 and its limit is M

Definition 13:

If $P(x, y)$, meet the condition of $\forall \varepsilon > 0$ when $t_n \to \infty$

$$\sqrt{|x_2(t_n) - \mathrm{x}|^2 + |y_2(t_n) - \mathrm{y}|^2} < \varepsilon$$

In which, $\{t_n\} \subset \{\mathrm{t} : t \in |R\}$

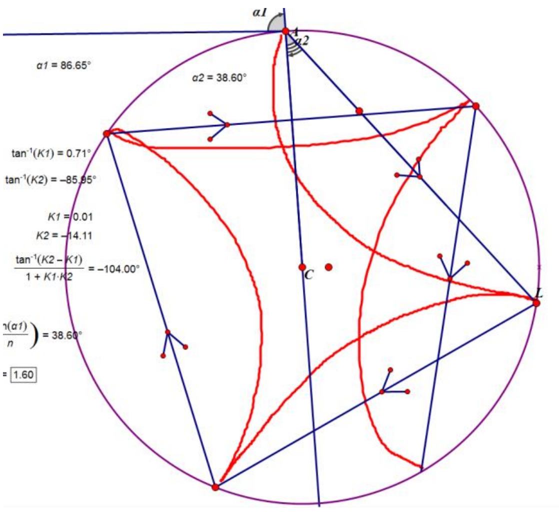
Then define the limits set L_2: $L_2 = \{(x_2, y_2)\}$

Conjecture 5:

$$L_2 = \overline{H_2}$$

Prospect 1: Traditional hypocycloid internal-combustion engines only use the hypocycloid with coefficient 2. In order to optimize the internal-combustion engines maybe the hypocycloid with coefficient as irrational numbers(or rational numbers close to irrational numbers) can do a better job. Because there is only 3 singularities for hypocycloid with coefficient 2,which means the same place on the gear ring, a internal part of the engine gets abraded by the same part of the gear, a thousand times over. However the hypocycloid discussed in this paper might solve this problem perfectly.

Prospect 2: My study on total reflection.



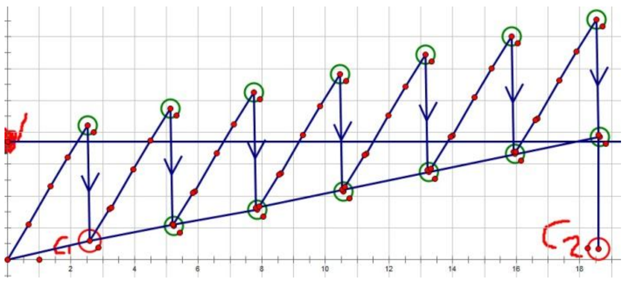
One year ago in my research on physics, I had a theory that under specific conditions, a jet of light shoot into a crystal ball may achieve eternal total reflection without escaping. I wanted to prove that there is no strict period of the motion of the light but failed. However, view this in a hypocycloid way, the light can be mapped to the curve and the curve doesn't have a period. The remained problem, troubling me for a whole year was solved easily by the conclusion of my research later.

Prospect 3: A Fractal Study relatively periodic function of stage N

$$f_N(x)=a_x,\{\mathrm{x}|\ \ x=kS_N\}$$

It's a not-so-standard fractal, with properties similar to self-similarity

Illustration:



Prospect 4: The application of cryptography

Two irrational coefficients, which are randomly selected, can generate a random quasicycloid. Even with the same quasicycloid, with every different value of t, the image can be quite different. Thus if given some information, say, a location, just take a certain part of the quasicycloid and encode it with a certain algorism, it is almost impossible crack it. To decode, the coefficient, the value of t and the algorism are necessary at the same time. This can be a very powerful encryption technique and also a convenient one, as long as you have the three things all together.

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