On generalizations of an inequality

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Abstract

The following inequality is proved in reference[1]:

If $a+b+c+d=4,\ a,b,c,d\in \mathrm{R}_{+}$, then: $a^{-2}+b^{-2}+c^{-2}+d^{-2}\geq a^2+b^2+c^2+d^2$

This dissertation has made an in-depth and comprehensive discussion on the above inequality. On one hand, we've extended the four-variables to multi-variables; on the other hand, we've extended the quadratic inequality to high-power ones. We have proved whether the extended proposition is true or not and came up with three theorems and one truth table. Meanwhile, we've brought forward a worth-thinking problem at the end of the dissertation.

Keyword: inequality, multi-variables, high-power, and convexity of function , substitution based on classification.

Part 1 Generalized proposition to multi-variable cases

Consider the following:

Proposition 1:

Assume that $n \in \mathrm{N}_+$, $x_i \in \mathrm{R}_+$ $(i = 1, 2 \ldots n)$, and $\sum_{i=1}^{n} x_i = n$, then

$$\sum_{i=1}^{n} x_i^{-2} \geq \sum_{i=1}^{n} x_i^2 \qquad (1)$$

Now let $P(n)$ denote proposition 1 for convenience. Define $P(n)$ to be 1 if $P(n)$ holds, otherwise define $P(n)$ to be 0 .

According to the results of reference[1] , we can conclude that $P(4)=1$. Specially, if we take several $x_i = 1$, then we can conclude that when $n \leq 4$, $P(n) = 1$. So now we can assume $n \geq 5$. Actually, if $P(m)=1$ for some $m \in \mathrm{N}_+$,then $P(n) = 1$ for any $n<m$. Meanwhile, if $P(m)=0$ for some $m \in \mathrm{N}_+$, then $P(n) = 0$ for any $n>m$. Now we define $f(x) = \frac{1}{x^2} - x^2 (x \in \mathrm{R}_+)$, then (1) is equivalent to:

$$\sum_{i=1}^{n} f(x_i) \geq 0 \quad (2)$$

Now we have:

Theorem 1: $P(n) = 1$ holds iff $n \leq 10$.

Theorem 1 is equivalent to P(10)=1 and P(11)=0. Now we give the proving:

First we will explore the properties of $f(x)$. Its derivative $f'(x) = -\frac{2}{x^3} - 2x$ is negative when $x \in (0, +\infty)$,indicating that $f(x)$ is strictly decreasing on $x \in (0, +\infty)$. Its 2nd derivative $f''(x) = 6x^{-4} - 2$ is a convex function when $x \in \left(0, 3^{\frac{1}{4}}\right)$, indicating that $f(x)$ is a convex function when $x \in (0,1]$.

Now we give the following lemmas:

Lemma 1: $P(11) = 0$

Proof: take $x_1 = x_2 = ... = x_{10} = \frac{3}{5}$, $x_{11} = 5$, then $\sum_{i=1}^{11} f(x_i) < 0$, the proof is finished.

Lemma 2: Assume $0 < x \le 1, y \ge 1, 9x + y = 10$, then $9f(x) + f(y) \ge 0$

Proof:

Take $\frac{9x}{y} = t$, then we have $t \in (0,9], x = \frac{10t}{9(t+1)}, y = \frac{10}{t+1}$

Then $9f(x) + f(y) \ge 0$

$\Leftrightarrow 900t^2(t+1)^2[9f(x) + f(y)] \ge 0$

$\Leftrightarrow (9t+9)^4 - (10t)^4 + 9t^2[(t+1)^4 - 10^4] \ge 0$

Define $g(t)$ to be the above LHS, then

$g(t) = (9-t)^2(9t^4 + 198t^3 - 550t^2 + 342t + 81)$

The same time, $9t^4 + 198t^3 - 550t^2 + 342t + 81$

$\ge (2\sqrt{9 \times 81} + 2\sqrt{198 \times 342} - 550)t^2$

$\ge (54 + 520 - 550)t^2 > 0$

So $g(t) \ge 0$. The proof is finished.

Lemma 3:

Assume $0 < x \le 1, y_1 \ge 1, y_2 \ge 1, 8x + y_1 + y_2 = 10$, then $8f(x) + f(y_1) + f(y_2) \ge 0$

Proof: First we prove:

Assume $y_1, y_2 \ge 1, s = y_1 + y_2$, then $f(y_1) + f(y_2) \ge \frac{(s-2)(s^2 - s + 2)}{1-s}$ (3)

According our assumption, we have $(y_i - 1)(y_j - 1) \ge 0$, $(1 \le i < j \le 2)$.

Now we take $t_{12} = y_1 y_2$, then $t_{12} \ge s - 1 \ge 1$,

$$f(y_1)+f(y_2)=\frac{1}{y_1^2}+\frac{1}{y_2^2}-y_1^2-y_2^2\ge\frac{2}{y_1y_2}-[(y_1+y_2)^2-2y_1y_2]$$

$$=2(\frac{1}{t_{12}}+t_{12})-s^2$$

When $t\ge 1$, $t+\frac{1}{t}$ is strictly increasing. So $t_{12}+\frac{1}{t_{12}}\ge\frac{1}{(s-1)}+(s-1)$.

Then $f(y_1)+f(y_2)\ge 2(\frac{1}{s-1}+s-1)-s^2=\frac{(s-2)(s^2-s+2)}{1-s}$, indicating that (3) is true.

Now if we use (3) to prove $8f(x)+f(y_1)+f(y_2)\ge 0$, we only have to prove:

$8f(x)+\frac{(s-2)(s^2-s+2)}{1-s}\ge 0$, and $s=10-8x$。

Simplifying the above inequality, we have:

$-\frac{8(x-1)^2(72x^3-81x^2+10x+9)}{x^2(8x-9)}\ge 0$, which is:

$(8x-9)(72x^3-81x^2+10x+9)\le 0$,

$\Leftrightarrow 72x^3-81x^2+10x+9\ge 0\ (\because x\in(0,1]\ \therefore 8x-9<0)$

Now define $g(x)$ as the above LHS, then $g'(x)=216x^2-162x+10$

$g'(x)$ has two roots when $x\in(0,1]$: $x_1=\frac{27-\sqrt{489}}{72}, x_2=\frac{27+\sqrt{489}}{72}$

Then we can conclude that $g(x)$ is strictly increasing when $x\in(0,x_1]$ or $(x_2,1]$, and is decreasing when $x\in(x_1,x_2]$.

So we only need to prove that $g(x_2)\ge 0, g(0)\ge 0$

Obviously, $g(0)=9>0$ and $g(x_2)=\frac{4455-163\sqrt{489}}{864}>0$. The proof is finished.

Lemma 4:

Assume that $0 < x \le 1, y_1, y_2, y_3 \ge 1, 7x + y_1 + y_2 + y_3 = 10$

Then $7f(x) + f(y_1) + f(y_2) + f(y_3) \ge 0$

Proof: First prove:

Assume $y_1, y_2, y_3 \ge 1, s = y_1 + y_2 + y_3$, then

$$2(y_1y_2 + y_2y_3 + y_1y_3) + y_1^{-2} + y_2^{-2} + y_3^{-2} \ge 2(2s-3) + \frac{9}{2s-3} \qquad (4)$$

According to our assumption, we can have $(y_i - 1)(y_j - 1) \ge 0$,($1 \le i < j \le 3$). Now we take $t_{ij} = y_i y_j$, then $t_{ij} \ge y_i + y_j - 1 \ge 1$. Let $g(t) = 2t + \frac{1}{t}(t \ge 1)$, then obviously its derivative $g'(t) = 2 - \frac{1}{t^2}$ is greater than 0 when $x \in (1, +\infty)$. Moreover, its 2nd derivative $g''(x) = \frac{2}{t^3}$ is greater than 0 when $x \in (1, +\infty)$. So $g(t)$ is a strictly increasing convex function when $x \in (1, +\infty)$. Also,

$$y_1y_2 + y_1y_3 + y_2y_3 \ge 2y_1 - 1 + 2y_2 - 1 + 2y_3 - 1 = 2s - 3$$

Plus $\because$ $y_1^{-2} + y_2^{-2} + y_3^{-2} \ge \frac{1}{y_1y_2} + \frac{1}{y_1y_3} + \frac{1}{y_2y_3}$, then

$$2(y_1y_2 + y_1y_3 + y_2y_3) + y_1^{-2} + y_2^{-2} + y_3^{-2} \ge \sum_{1 \le i < j \le 3} (2t_{ij} + \frac{1}{t_{ij}}) = \sum_{1 \le i < j \le 3} g(t_{ij}) \ge$$

$$3g(\frac{1}{3}\sum_{1 \le i < j \le 3} t_{ij}) \ge 3g(\frac{2s-3}{3}) = 3[\frac{2}{3}(2s-3) + \frac{3}{2s-3}] = 2(2s-3) + \frac{9}{2s-3}$$

Then we've proved (4) is true.

As a consequence of

$$-\sum_{i=1}^{3} f(y_i) = \sum_{i=1}^{3}(y_i^2 - \frac{1}{y_i^2}) = s^2 - (2\sum_{1 \le i < j \le 3} y_i y_j + \sum_{i=1}^{3}\frac{1}{y_i^2}),$$

and by using (4) at the same time, we have $-\sum_{i=1}^{3} f(y_i) \le S^2 - 4S + 6 - \frac{9}{2S-3}$.

So in order to prove our lemma, we only need to prove

$$7f(x) \ge S^2 - 4S + 6 - \frac{9}{2S-3} \qquad (5).$$

When $x = 1$, we can prove that (5) is true easily, so we assume $0 < x < 1$

Also $S - 3 = 7(1-x)$, so

$$(5) \Leftrightarrow \frac{1+x+x^2+x^3}{x^2} \ge \frac{2S^2-5S+9}{2S-3} = S - 1 + \frac{6}{2S-3}$$
$$\Leftrightarrow (1+x-8x^2+8x^3)(17-14x) \ge 6x^2$$
$$\Leftrightarrow (x-1)(-112x^3+136x^2-20x-17) \ge 0$$

$$\Leftrightarrow 112x^3 - 136x^2 + 20x + 17 \ge 0 \qquad (6)$$

Take $h(x) = 112x^3 - 136x^2 + 20x + 17$

Then we only need to prove that $h(x) \ge 0$ when $x \in [0,1]$.

As a consequence of $h'(x) = 4(84x^2 - 68x + 5) = 0$, we have

$$x_1 = \frac{17-2\sqrt{46}}{42}, x_2 = \frac{17+2\sqrt{46}}{42}$$

Then we can conclude that $h(x)$ is strictly increasing when $x \in (0, x_1]$ or $(x_2, 1]$, and is decreasing when $x \in (x_1, x_2]$.

So we only need to prove that $h(x_2) \ge 0, h(0) \ge 0$

Obviously, $h(0) = 17 > 0, h(x_2) = \frac{13549 - 1472\sqrt{46}}{1323} > 0$, then the proof is finished.

Lemma 5:

Assume that $0 < x \le 1, y_1, y_2, y_3, y_4 \ge 1, 6x + y_1 + y_2 + y_3 + y_4 = 10$, then

$$6f(x) + f(y_1) + f(y_2) + f(y_3) + f(y_4) \ge 0$$

Proof: First prove:

Assume that $y_1, y_2, y_3, y_4 \ge 1, s = y_1 + y_2 + y_3 + y_4$, then

$$2\sum_{1 \le i < j \le 4} y_i y_j + \sum_{i=1}^{4} y_i^{-2} \ge 6(s-2) + \frac{8}{s-2} \qquad (7)$$

According to our assumption, we have $(y_i-1)(y_j-1)\ge 0$, ($1\le i<j\le 4$).

Take $t_{ij}=y_iy_j$, then $t_{ij}\ge y_i+y_j-1\ge 1$.

Let $g(t)=2t+\frac{2}{3t}(t\ge 1)$, then we find $g'(t)$ and $g''(t)$ are all greater than 0 when $x\in(1,+\infty)$, so $g(t)$ is a strictly increasing convex function when $x\in(1,+\infty)$.

Plus $\sum_{1\le i<j\le 4} t_{ij}\ge 3y_1+3y_2+3y_3+3y_4-6=3s-6$ and

$y_1^{-2}+y_2^{-2}+y_3^{-2}+y_4^{-2}\ge\frac{2}{3}\sum_{1\le i<j\le 4}\frac{1}{t_{ij}}$。

$$\therefore 2\sum_{1\le i<j\le 4} t_{ij}+\sum_{i=1}^{4} y_i^{-2}\ge\sum_{1\le i<j\le 4}(2t_{ij}+\frac{2}{3t_{ij}})=\sum_{1\le i<j\le 4} g(t_{ij})\ge 6g(\frac{1}{6}\sum_{1\le i<j\le 4} t_{ij})$$

$$\ge 6g(\frac{3s-6}{6})=6g(\frac{s-2}{2})=6(s-2+\frac{4}{3(s-2)})=6(s-2)+\frac{8}{s-2},$$

Then we can say (7) is true.

According to (7), in order to prove:

$$6f(x)+f(y_1)+f(y_2)+f(y_3)+f(y_4)\ge 0$$

we only need to prove that:

$$6f(x)+6(s-2)+\frac{8}{s-2}-s^2\ge 0,$$

Simplifying the above inequality, we have:

$$-\frac{6(x-1)^2(21x^3-28x^2+5x+4)}{x^2(3x-4)}\ge 0,$$ which is :

$$(3x-4)(21x^3-28x^2+5x+4)\le 0,\quad \because x\in(0,1]\therefore 3x-4<0$$

$$\Leftrightarrow 21x^3-28x^2+5x+4\ge 0$$

Let the above LHS=$h(x)$, then its derivative $h'(x)=63x^2-56x+5$ has two roots when $x\in(0,1]$: $x_1=\frac{28-\sqrt{469}}{63}$, $x_2=\frac{28+\sqrt{469}}{63}$

Then we can conclude that $h(x)$ is strictly increasing when $x\in(0,x_1]$ or $(x_2,1]$, and is decreasing when $x \in (x_1, x_2]$.

So we only need to prove that $h(x_2) \geq 0, h(0) \geq 0$

Obviously, $h(0) = 4 > 0$ and $h(x_2) = \frac{4312 - 134\sqrt{469}}{1701} > 0$. The proof is finished.

Lemma 6: $P(10) = 1$

Proof:

First prove that if $a + b \leq 2$, then $f(a) + f(b) \geq 2f(\frac{a+b}{2})$ (8)

Because $\sqrt{ab} \leq \frac{a+b}{2} \leq 1$ and $f(x)$ is strictly decreasing, then we have

$f(\sqrt{ab}) \geq f(\frac{a+b}{2})$

So we only need to prove that $f(a) + f(b) \geq 2f(\sqrt{ab})$ (9)

Besides (9) $\Leftrightarrow (\frac{1-a^2b^2}{a^2b^2})(a^2 + b^2) \geq 2(\frac{1}{ab} - ab)$

$\Leftrightarrow (\frac{1-a^2b^2}{a^2b^2})(\frac{a^2+b^2}{ab} - 2) \geq 0$

So (9) is true, which means (8) is true as well.

Now according to our assumption: $\sum_{i=1}^{10} x_i = 10$, let $\sigma = \sum_{i=1}^{10} f(x_i)$

Divide $x_1, x_2, \ldots x_{10}$ into 2 parts, without losses of generality .We assume that $x_1, x_2, \ldots x_k$ is no more than 1, and we denote the other elements, which are greater than 1, to be $y_1, y_2, \ldots y_l (k + l = 10)$. If $l = 0$, then it's easy to conclude that $x_1 = x_2 = \ldots x_{10} = 1$, indicating that the above conclusion is true. Also, since $k \geq 1$, so if $l > k$, let m denote $l - k, m \in \mathrm{N}_+$.

Since $10 = \sum_{i=1}^{k}(x_i + y_i) + \sum_{j=1}^{m} y_{k+j} > \sum_{i=1}^{k}(x_i + y_i) + m$, $\sum_{i=1}^{k}(x_i + y_i) < 10 - m = 2k$, there exists an integer t in the range [1,k], so that $x_t + y_t \le 2$. In this case, we replace x_t, y_t with $\frac{x_t + y_t}{2} (\le 1)$. From (8) we can infer that σ is not increasing. After this substitution, the sum of the 10 figures is still 10, but there's one more figure which is less than 1. After several arrangements like this, σ is non-decreasing, and the sum of the 10 figures is still 10, but $k > l > 0$. Now we denote $x = \frac{1}{k}\sum_{i=1}^{k} x_i$, and substitute x with $x_1, x_2, \ldots x_k$. According to the convexity of $f(x)$, σ is non-decreasing, so we only need to prove $kf(x) + \sum_{i=1}^{l} f(y_j) \ge 0$. Also, since $k + l = 10, 0 < l < k$, so the only possible value for l is 1,2,3,4.

According to lemma 2 and 5, we can conclude that lemma 6 is true. According to lemma 1 and 6, we can conclude that proposition 1 is true. The proof is finished.

Now we can extend proposition 1:

Proposition 2: Assume that $x_i \in \mathrm{R}_+ (i = 1, 2, \ldots, n)$, $\sum_{x=1}^{n} x_i = a$,(a is a positive constant), then:

1).when $a \le n \le 10$, $\sum_{i=1}^{n} f(x_i) \ge 0$;

2).when $a \ge n \ge 11$, $\sum_{i=1}^{n} f(x_i) \ge 0$ is not always true.

Now we give the proof:

1).We only need to discuss such circumstances that $n = 10$ and $a < 10$.

Let $b = \frac{10 - a}{a}$, then $b \in (0,1)$. Take $y_i = x_i + b, (i = 1,2,\ldots 10)$. Obviously, $y_i \in R_+$ and $\sum_{i=1}^{10} y_i = 10b + \sum_{i=1}^{10} x_i = 10$. According to proposition 1, we have : $\sum_{i=1}^{n} f(y_i) \ge 0$. Also, since $y_i = x_i + b > x_i$, and $f(x)$ is strictly decreasing, so

$f(y_i) < f(x_i)$, then $\sum_{i=1}^{n} f(x_i) \geq 0$. The proof is finished.

2).We only need to give proof when $n = 11$

Take those $x_1, x_2, ..., x_{11}$ from Lemma 1, then $\sum_{i=1}^{11} x_i = 11$ and $\sum_{i=1}^{11} f(x_i) < 0$.

Take $y_i = \frac{a}{11} x_i$, then $y_i \geq x_i > 0$ and $\sum_{i=1}^{11} y_i = a$. Also, since $f(y_i) \leq f(x_i)$, so $\sum_{i=1}^{11} f(y_i) < 0$. As a consequence, when $a \geq n \geq 11$, $\sum_{i=1}^{n} f(x_i) \geq 0$ is not always true.

When $n = 11$ and $a = 10$, there must be one figure among $x_1, x_2 ... x_{11}$, which is less than 1. We can call it x_{11}, then $f(x_{11}) \geq 0$. Since $f(x)$ is strictly decreasing, we have $f(x_{10}) \geq f(x_{10} + x_{11})$, then $\sum_{i=1}^{11} f(x_i) \geq \sum_{i=1}^{9} f(x_i) + f(x_{10} + x_{11}) \geq 0$.

Therefore: $b \in [10,11)$ such that :

$$\sum_{i=1}^{11} f(x_i) \geq \sum_{i=1}^{9} f(x_i) + f(x_{10} + x_{11}) \geq 0.$$

From the above proof，there exists $b \in [10,11)$, such that

$$\forall x_i \in \mathrm{R}_+ \ (i = 1, 2, ..., 11), \sum_{i=1}^{11} x_i = b, \text{ and} \sum_{i=1}^{11} f(x_i) \geq 0.$$

Part 2 Generalized proposition to high-degree cases

Consider the more general problem and denote the following proposition as $P(n,k)$.

Assume that $x_i \in \mathrm{R}_+$, $n,k \in \mathrm{N}_+$, $\sum_{i=1}^{n} x_i = n$, $f_k(x) = x^{-k} - x^k$, then $\sum_{i=1}^{n} f_k(x_i) \geq 0$.

Similarly, we use $P(n,k) = 1$ to denote that for (n,k), $P(n,k)$ is true. Otherwise, $P(n,k)=0$

According to part one, we can conclude that when $P(n,k)=1$, for $\forall m \in \mathrm{N}_+, m \le n$, we have $P(m,k)=1$, and when $P(n,k)=0$, for $\forall m \in \mathrm{N}_+, m \ge n$, we also have $P(m,k)=0$. Furthermore, we can infer from the first and second derivatives of $f(x)$, $f_k^{'}(x)=-k(x^{-k-1}+x^{k-1})$ and $f_k^{''}(x)=\frac{k(k+1)}{x^{k+2}}-k(k-1)x^{k-2}$, that $f_k(x)$ is a decreasing function on R_+, and is a convex function on $(0,1]$.

Now we give such following:

Lemma 7. $P(5,3)=P(4,4)=P(4,5)=P(3,6)=0$

Proof:

Take $(x,y)=(\frac{7}{10},\frac{11}{5})$, then $4x+y=5$, and by directly calculating, we can see that $4f_3(x)+f_3(y)<0$, so we've proved $P(5,3)=0$.

Take $(x,y)=(\frac{3}{5},\frac{11}{5})$, then $3x+y=4$, and we've found that $3f_4(x)+f_4(y)<0$, $3f_5(x)+f_5(y)<0$, so we've proved $P(4,4)=P(4,5)=0$.

Take $(x,y)=(\frac{3}{4},\frac{3}{2})$, then $2x+y=3$, and we've found that $2f_6(x)+f_6(y)<0$, so we've proved $P(3,6)=0$.

Lemma 8:Assume that $a,b \in \mathrm{R}_+$, $a+b \le 2$, then $f_k(a)+f_k(b) \ge 2f_k(\frac{a+b}{2}) \ge 0$.

Proof: We can easily prove by the assumption that $\sqrt{ab} \le \frac{a+b}{2} \le 1$.

So $f_k(\sqrt{ab}) \ge f_k(\frac{a+b}{2}) \ge f_k(1)=0$.

$f_k(a)+f_k(b) \ge 2f_k(\sqrt{ab})$

$\Leftrightarrow (a^{-k}+b^{-k})-(a^k+b^k) \ge 2(ab)^{-\frac{k}{2}}-2(ab)^{\frac{k}{2}}$

$\Leftrightarrow [\frac{1}{(\sqrt{a})^k}-\frac{1}{(\sqrt{b})^k}]^2 \ge [(\sqrt{a})^k-(\sqrt{b})^k]^2$

$\Leftrightarrow (a^{\frac{k}{2}}-b^{\frac{k}{2}})^2(\frac{1}{a^k b^k}-1) \ge 0$

From $\sqrt{ab} \le 1$ we can know that $a^k b^k \le 1$, so Lemma 8 is proved.

From the convexity of $f_k(x)$, which is mentioned above, and Lemma 8, quite similar to the proof in lemma 6 in part one, we need to prove the proposition $P(n,k)=1$ only in the following case:

$k \ge l$ and those x_i which are not greater than 1 can be replaced by their average value.

Lemma 9. $P(4,3)=1$

Proof:

We only need to prove that $3f_3(x)+f_3(y) \ge 0$, when $0 < x \le 1 < y, 3x + y = 4$.

Assume that $t=\frac{y}{x}(t \ge 1)$, then we can conclude that

$$\{\begin{matrix} 3x+y=4 \\ y=tx \end{matrix} \Rightarrow \{\begin{matrix} x=\frac{4}{3+t} \\ y=\frac{4t}{3+t} \end{matrix}$$

$$3x^{-3}-3x^3+y^{-3}-y^3$$

$$=\frac{3(t-1)^2}{64t^3(t+3)^3}(g(t)\cdot t+243),$$

in which $g(t)=972+2106t+53t^2-497t^3+174t^4+20t^5+t^6$

$$\ge (2\sqrt{972}+2\sqrt{2106\times 20}+2\sqrt{53\times 174}-497)t^3$$

$$\ge 167t^3$$

$$\ge 0$$

So $3f_3(x)+f_3(y) \ge 0$. The proof is finished.

Lemma 10: $P(3,5)=1$

Proof:

We only need to prove that $2f_5(x)+f_5(y) \ge 0$, when $0 < x \le 1 < y, 2x + y = 3$.

Assume that $t = \frac{y}{x} (t \ge 1)$, then we can conclude that

$$\begin{cases} 2x + y = 3 \\ y = tx \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2+t} \\ y = \frac{3t}{2+t} \end{cases}$$

$$2x^{-5} - 2x^5 + y^{-5} - y^5$$

$$= \frac{2(t-1)^2}{3^5 t^5 (t+2)^5} (g(t) \cdot t + 2^9)$$

In which $g(t) =$

$$3584 + 12416t + 28928t^2 + 52160t^3 + 21399t^4 - 2562t^5 - 14523t^6 - 11034t^7 + 5905t^8 + 1384t^9 + 223t^{10} + 22t^{11} + t^{12}$$

$$\ge 21399t^4 - 2562t^5 + (2\sqrt{3584} + 2\sqrt{12416 \times 22} + 2\sqrt{28928 \times 223} + 2\sqrt{52160 \times 1384} - 14523)t^6 - 11034t^7 + 5905t^8$$

$$\ge 21399t^4 - 2562t^5 + (32\sqrt{14} + 32\sqrt{1067} + 32\sqrt{25199} + 32\sqrt{281990} - 14523)t^6 - 11034t^7 + 5905t^8$$

$$\ge 21399t^4 - 2562t^5 + 8712t^6 - 11034t^7 + 5905t^8$$

$$= (21399t^4 - 2562t^5 + 3556t^6) + (5156t^6 - 11034t^7 + 5905t^8)$$

$$\ge 0$$

So $2f_5(x) + f_5(y) \ge 0$. The proof is finished.

Lemma 11:

Assume that $0 < a < b$, $2a + b = 3$, and $2f_k(a) + f_k(b) < 0$, then $\exists\, a_1, b_1$, so that $0 < a_1 < b_1$, $2a_1 + b_1 = 3$, and $2f_{k+1}(a_1) + f_{k+1}(b_1) < 0$ 。

Proof:

We can easily tell from the assumption that

$$0 < a < 1 < b, \quad \frac{2}{a^k} + \frac{1}{b^k} < 2a^k + b^k. \qquad (10)$$

Take $r=\frac{k}{k+1}$, then $0<r<1$ and $0<a^r<1<b^r<b$. Using power-mean inequality we can conclude that:

$$(\frac{2a^r+b^r}{3})^{\frac{1}{r}}\le(\frac{2a^\alpha+b^\alpha}{3})^{\frac{1}{\alpha}},(r<\alpha)$$

Take $\alpha=1$, and consider $a<b$, we have $(\frac{2a^r+b^r}{3})^{\frac{1}{r}}<1$.

So $2a^r+b^r<3$ (11)

Since $(a^r)^{k+1}=a^k,(b^r)^{k+1}=b^k$, (10) is equivalent to

$$\frac{2}{(a^r)^{k+1}}+\frac{1}{(b^r)^{k+1}}<2(a^r)^{k+1}+(b^r)^{k+1} \qquad (12)$$

Also, from inequality (11) we can conclude that

$$\delta=1-\frac{2a^r+b^r}{3}>0.$$

Now we take $a_1=a^r+\delta,b_1=b^r+\delta.$ From $0<a^r<b^r$ we have $0<a_1<b_1$, and $2a_1+b_1=3$.

Noticing that $a_1^{k+1}=(a^r+\delta)^{k+1}>(a^r)^{k+1},b_1^{k+1}=(b^r+\delta)^{k+1}>(b^r)^{k+1}$, we can conclude that

$$2a_1^{k+1}+b_1^{k+1}>2(a^r)^{k+1}+(b^r)^{k+1},\frac{2}{a_1^{k+1}}+\frac{1}{b_1^{k+1}}<\frac{2}{(a^r)^{k+1}}+\frac{1}{(b^r)^{k+1}}$$

Also, considering (12) ,we can know that $\frac{2}{a_1^{k+1}}+\frac{1}{b_1^{k+1}}<2a_1^{k+1}+b_1^{k+1}$,

which is exactly $2f_{k+1}(a_1)+f_{k+1}(b_1)<0$,

The proof of Lemma 11 is finished.

At the end, we give two following corollaries:

Corollary 1:

Assume that $P(3,k)=0,(k\in\mathrm{N}_+,k>2)$, then we have $a,b\in\mathrm{R}_+$.

$0<a<b,2a+b=3,$ and $2f_k(a)+f_k(b)<0$

Proof:

From the assumption we can know that $x_1, x_2, x_3 \in \mathrm{R}_+$,

$x_1 + x_2 + x_3 = 3$ and $\sum_{i=1}^{3} f_k(x_i) < 0$.

So we can conclude that at least one of x_1, x_2, x_3 is greater than 1.

And let's just take $x_3 > 1$, then $x_1 + x_2 < 2$.

From Lemma 8 we can conclude that $f_k(x_1) + f_k(x_2) \geq 2 f_k(\frac{x_1 + x_2}{2})$.

Take $a = \frac{x_1 + x_2}{2}, b = x_3$, then

$0 < a < b$, $2a + b = 3$, and $2 f_k(a) + f_k(b) \leq \sum_{i=1}^{3} f_k(x_i) < 0$.

Corollary 2:

Assume that $P(3,k) = 0$, then for $\forall k_1 \geq k, k_1 \in \mathrm{N}_+$, we have $P(3,k_1) = 0$. Assume that $P(3,k) = 1$, then for $\forall k_1 \leq k, k_1 \in \mathrm{N}_+$, we have $P(3,k_1) = 1$.

Proof:

Assume that $P(3,k) = 0$, from the results of Corollary 1 and Lemma 11 we can conclude that $P(3,k+1) = 0$, and by using this result repeatedly we can conclude that $P(3,k_1) = 0$. And if $P(3,k) = 1$, using proof of contradiction and the former result we can know that $P(3,k-1) = 1$. Using this result repeatedly, we can conclude that $P(3,k_1) = 1$.

Now by summarizing the whole dissertation, we can conclude that:

Theorem 3:

1).$P(1,k) = P(n,1) = P(2,k) = 1 (\forall n,k \in \mathrm{N}_+)$

2).$P(n,2) = 1 \Leftrightarrow n \leq 10$

3).$P(n,3) = 1 \Leftrightarrow n \leq 4$

4).$P(n,4) = P(n,5) = 1 \Leftrightarrow n \leq 3$

5).$P(n,k) = 1, (k \geq 6) \Leftrightarrow n \leq 2$

Proof:

1)We can easily prove that $P(1,k)=P(n,1)=1$.Also,from Lemma 8, we have $P(2,k)=1$.
2)From Theorem 1, we can conclude that $P(n,2)=1 \Leftrightarrow n \le 10$

3)From Lemma 9 and the properties of $P(n,k)$, we have $P(n,3)=0(n \ge 5)$,
and from the convexity of $f_k(x)$ on $(0,1]$ and Lemma 8, we know that
Lemma 9 proved $P(4,3)=1$, so $P(n,3)=1 \Leftrightarrow n \le 4$.

4)From Lemma 7 we can know that $P(4,4)=0$, so $P(n,4)=0(n \ge 4)$,
and Lemma 10 proved that $P(3,5)=1$. From Corollary 2, we can conclude that
$P(3,4)=1$, so $P(n,4)=1 \Leftrightarrow n \le 3$.
Similarly, from $P(3,5)=1$, we can conclude that $P(n,5)=1 \Leftrightarrow n \le 3$.
From Lemma 7, we have $P(4,5)=0$, and then $P(n,5)=0(n \ge 4)$, so $P(n,5)=1 \Leftrightarrow n \le 3$

5)When $n \le 2, P(n,k)=1$, assume that $k \ge 6, n \ge 3$. From Lemma 7 $P(3,6)=0$
and Corollary 2, we can conclude that $P(3,k)=0(k \ge 6)$.
Also, from the properties of the proposition, we have $P(n,k)=0(n \ge 3)$.
So when $k \ge 6$, $P(n,k)=1 \Leftrightarrow n \le 2$.

In summary, we've finished the proof.

According to Theorem 3, we can get the truth table of $P(n,k)$ as following:

k \ n	1	2	3	4	5	6	7	8	9	10	11	…
1	1	1	1	1	1	1	1	1	1	1	1	…
2	1	1	1	1	1	1	1	1	1	1	0	…
3	1	1	1	1	0	0	0	0	0	0	0	…
4	1	1	1	0	0	0	0	0	0	0	0	…
5	1	1	1	0	0	0	0	0	0	0	0	…
6	1	1	0	0	0	0	0	0	0	0	0	…
7	1	1	0	0	0	0	0	0	0	0	0	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

At the end, we indicate that, we may extend Theorem 3 the way we've extended Theorem 1 to Theorem 2.

Summary:

During our research, we are deeply amazed by the beauty and complexity of algebra, as we can produce a series of worth-thinking and in-depth extensions from a simple inequality. What we've done is extending the original inequality to higher-order ones, but we also think it's possible to extend it to real inequalities. As math enthusiasts in the new century, we hope that one day, this extension will be proved.

Reference:

[1] High School Mathematics, 2010 Supplement, Page 22, Tianjin Normal University Press, Editor: LiHua Tang