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New Probe into Combinatorial Identities

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Abstract

In the first part, we proof some new identities of *Stirling* numbers, *Lah* numbers and Bell numbers using generating functions and inversion formulas. In the second part, we propose a new inversion formula based on inverse functions and discuss some applications of this formula. In the third part, we generalize the definition of binomial coefficients, *Stirling* numbers and *Lah* numbers. Then we discuss some basic properties of the new sequences and show a practical way to obtain new combinatorial identities.

KEYWORDS: Combinatorial identities; *Stirling* numbers; *Lah* numbers; *Bell* numbers; Binomial coefficients ; Generating functions ; Inversion formulas

1 Introduction

1.1 Notations

$[x]_n \quad = x(x-1)\cdots(x-n+1)$ for $n \geq 1$. $[x]_0 = 1$, $[x]_n = 0$ for $n < 0$.

$[x]^n \quad = x(x+1)\cdots(x+n-1)$ for $n \geq 1$. $[x]^0 = 1$, $[x]^n = 0$ for $n < 0$.

$C_n^k, \binom{n}{k} \quad = \frac{[n]_k}{k!}$, where $n \in C$, $k \in N$.

$s(n,k)$ *Stirling* number of the first kind, see Definition 1.

$S(n,k)$ *Stirling* number of the second kind, see Definition 2.

$L(n,k)$ *Lah* number, see Definition 3.

B_n *Bell* number, see Definition 4.

δ_{ij} *Kronecker* symbol. $\delta_{ij} = 0$, if $i \neq j$; $\delta_{ij} = 1$, if $i = j$.

i, j, k, l, n are the indexes of a sequence, they are integers without specification.

1.2 Definitions[1]

Definition 1. *Stirling* number of the first kind $s(n,k)$ is the coefficient of x^k in the expansion of $[x]_n$, or $[x]_n = \sum_{k=0}^{n} s(n,k)x^k$. $s(n,k)$ is defined as 0 when $k > n \geq 0$ or $k < 0$.

Since $[x]_0, [x]_1, [x]_2, \cdots, [x]_n, \cdots$ are linearly independent, they can be the basis of polynomial ring $\boldsymbol{R}[x]$, thus we have the following

Definition 2. *Stirling* number of the second kind $S(n,k)$ is the coefficient of $[x]_k$ in the expansion of x^n, or $x^n = \sum_{k=0}^{n} S(n,k)[x]_k$. $S(n,k)$ is defined as 0 when $k > n \geq 0$ or $k < 0$.

Definition 3. *Lah* number is the coefficient of $[x]_k$ in the expansion of $[-x]_n$, or $[-x]_n = \sum_{k=0}^{n} L(n,k)[x]_k$. $L(n,k)$ is defined as 0 when $k > n \geq 0$ or $k < 0$.

Definition 4. *Bell* number $B_n = \sum_{k=0}^{n} S(n,k)$, where $n \in \boldsymbol{N}$.

1.3 Genarating Functions [1] [2]

Definition 5[1]**.** The ordinary generating function of $\{a_n\}_{n\geq 0}$ is

$$f(x)=\sum_{n=0}^{\infty}a_n x^n \qquad (1.1)$$

Definition 6[1]. The exponential generating function of $\{a_n\}_{n\geq 0}$ is

$$g(x)=\sum_{n=0}^{\infty}a_n \frac{x^n}{n!} \qquad (1.2)$$

(1.1) and (1.2) are called formal power series. In combinatorics we seldom deal with their convergence or other properties that are often involved in mathematical analysis. For rigorous theory of generating functions, please refer to [4][5]. We will list some basic properties[1] of generating functions without proof. Denote the ordinary generating functions of $\{a_n\}$ and $\{b_n\}$ as $A_1(x)$ and $B_1(x)$ respectively, and the exponential generating functions as $A_2(x)$ and $B_2(x)$ respectively. We have the following

Property 1. $\{a_n\}=\{b_n\} \Leftrightarrow A_1(x)=B_1(x) \Leftrightarrow A_2(x)=B_2(x)$

Property 2. The ordinary generating function and exponential generating function of $\{\alpha a_n\}$ are $\alpha A_1(x)$ and $\alpha A_2(x)$ respectively, where α is a complex constant.

Property 3. The ordinary generating function and exponential generating function of $\{a_n+b_n\}$ are $A_1(x)+B_1(x)$ and $A_2(x)+B_2(x)$ respectively.

Property 4. $A_1(x)B_1(x)=\sum_{n=0}^{\infty}c_n x^n$, where $c_n=\sum_{k=0}^{n}a_k b_{n-k}$,

$$A_2(x)B_2(x)=\sum_{n=0}^{\infty}d_n\frac{x^n}{n!}, \text{ where } d_n=\sum_{k=0}^{n}C_n^k a_k b_{n-k}.$$

Definition 7[1]. If $A(x)=\sum_{n=0}^{\infty}a_n x^n$, $B(x)=\sum_{n=0}^{\infty}b_n x^n$, $C(x)=\sum_{n=0}^{\infty}c_n x^n$ satisfy that $B(x)C(x)=A(x)$, then $C(x)$ is called the quotient of $A(x)$ divided by $B(x)$, and denoted as $C(x)=\dfrac{A(x)}{B(x)}$.

Definition 8[1]. The formal derivative of $A(x)=\sum_{n=0}^{\infty}a_n x^n$ is denoted as

$DA(x)=\sum_{n=1}^{\infty} na_n x^{n-1}$.

Definition 9[1]**.** The higher derivative of $A(x)$ is

$$D^n A(x)=\begin{cases} A(x) & n=0 \\ D(D^{n-1}A(x)) & n\geq 1 \end{cases} \qquad (1.3)$$

Definition 10[1]**.** If $B(x)$ satisfy that $A(x)=DB(x)$, then $B(x)$ is called the formal primitive function of $A(x)$.

2 Identities of *Stirling* numbers, *Lah* numbers and *Bell* Numbers

2.1 Basic properties

Theorem 1[1]**.** (1) $s(n+1,k)=s(n,k-1)-ns(n,k)$ (2.1)

(2) $S(n+1,k)=S(n,k-1)+kS(n,k)$ (2.2)

(3) $L(n+1,k)=-(n+k)L(n,k)-L(n,k-1)$ (2.3)

(4) $B_{n+1}=\sum_{k=0}^{n} C_n^k B_k$ (2.4)

Proof. For a more generalized proof of (2.1)(2.2)(2.3), see Theorem 20. We only present a new proof of (2.4). First we proof

$$\sum_{k=0}^{n} S(n,k)x^k = e^{-x}\sum_{k=0}^{\infty} k^n \frac{x^k}{k!} \qquad (2.5)$$

by induction. Denote $f_n(x)=\sum_{k=0}^{n} S(n,k)x^k$.

If $n=0$, we have $f_n(x)=1$, and $e^{-x}\sum_{k=0}^{\infty} k^n \frac{x^k}{k!}=e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}=e^{-x}\cdot e^{x}=1$.

(2.5) holds for $n=0$.

Suppose (2.5) holds for n, with (2.2) we have

$$f_{n+1}(x)=\sum_{k=0}^{n+1} S(n+1,k)x^k=\sum_{k=0}^{n+1}(S(n,k-1)x^k+kS(n,k)x^k)$$

$$=\sum_{k=0}^{n} S(n,k)x^{k+1}+\sum_{k=0}^{n} kS(n,k)x^k = x(f_n(x)+Df_n(x))$$

By induction hypothesis,

$$Df_n(x) = e^{-x}\sum_{k=1}^{\infty} k^n \frac{x^{k-1}}{(k-1)!} - e^{-x}\sum_{k=0}^{\infty} k^n \frac{x^k}{k!}$$

$$= e^{-x}\sum_{k=0}^{\infty}(k+1)^n \frac{x^k}{k!} - f_n(x)$$

Thus,

$$f_{n+1}(x) = x(f_n(x) + Df_n(x)) = e^{-x}\sum_{k=0}^{\infty}(k+1)^n \frac{x^{k+1}}{k!}$$

$$= e^{-x}\sum_{k=0}^{\infty}(k+1)^{n+1}\frac{x^{k+1}}{(k+1)!} = e^{-x}\sum_{k=0}^{\infty}k^{n+1}\frac{x^k}{k!}$$

According to mathematical induction, (2.5) holds for arbitrary non-negative integer n.
Set x=1, we obtain

$$B_n = f_n(1) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} \tag{2.6}$$

Hence,

$$\sum_{k=0}^{n} C_n^k B_k = \frac{1}{e}\sum_{i=0}^{n} C_n^i \sum_{k=0}^{\infty}\frac{k^i}{k!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i=0}^{n} C_n^i k^i = \frac{1}{e}\sum_{k=0}^{\infty}\frac{(k+1)^n}{k!}$$

$$= \frac{1}{e}\sum_{k=0}^{\infty}\frac{(k+1)^{n+1}}{(k+1)!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^{n+1}}{k!} = B_{n+1}$$

□

Theorem 2[1]**.** (1) $\sum_{n=0}^{\infty} s(n,k)\frac{x^n}{n!} = \frac{1}{k!}(\ln(1+x))^k$ (2.7)

$$(2)\ \sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k \tag{2.8}$$

$$(3)\ \sum_{n=0}^{\infty} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \tag{2.9}$$

$$(4)\ \sum_{n=0}^{\infty} L(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{-x}{1+x}\right)^k \tag{2.10}$$

$$(5)\ \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1} \tag{2.11}$$

Proof is omitted, We remark that (2.11) implies (2.6). In fact,

$$\sum_{n=0}^{\infty} B_n\frac{x^n}{n!} = e^{e^x-1} = \frac{1}{e}e^{e^x} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{(e^x)^k}{k!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=0}^{\infty}\frac{(kx)^n}{n!}\right) = \frac{1}{e}\sum_{n=0}^{\infty}\frac{x^n}{n!}\left(\sum_{k=0}^{\infty}\frac{k^n}{k!}\right)$$

Compare the coefficient of $\frac{x^n}{n!}$, we obtain (2.6).

In **Theorem 7** there is a new proof for (2.7) .

Theorem 3[1]**.** (1) $s(n,k)=(-1)^{n+k}\sum_{0<\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_{n-k}<n}\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-k} \quad (n\geq k)$ (2.12)

(2) $S(n,k)=\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_k=n-k\\ \varepsilon_1,\varepsilon_2,\cdots\varepsilon_k\geq 0}}1^{\varepsilon_1}2^{\varepsilon_2}\cdots k^{\varepsilon_k} \quad (n\geq k)$ (2.13)

(3) $S(n,k)=\frac{1}{k!}\sum_{i=1}^{k}(-1)^{k-i}C_k^i i^n \quad (n\geq k)$ (2.14)

(4) $L(n,k)=(-1)^n\frac{n!}{k!}C_{n-1}^{k-1} \quad (n\geq k)$ (2.15)

(5) $B_n=\sum_{k=0}^{n}\frac{1}{k!}\sum_{i=1}^{k}(-1)^{k-i}C_k^i i^n \quad (n\geq k)$ (2.16)

Proof is omitted. In **Theorem 22** there is a generalized proof for (2.12) and (2.13).

2.2 Identities of *Stirling* numbers

The following identities[2] are famous :

$$\mathrm{C}_{n+m}^k=\sum_{i=0}^{k}\mathrm{C}_n^i\mathrm{C}_m^{k-i} \quad \text{(Vandermonde's identity)} \tag{2.17}$$

$$\sum_{k=n}^{m}\mathrm{C}_k^n=\mathrm{C}_{n+1}^{m+1} \tag{2.18}$$

The following two theorems proof similar identities of *Stirling* number.

Theorem 4. $s(n+m,k)=\sum_{i=0}^{k}\sum_{j=i}^{m}(-n)^{j-i}s(n,k-i)s(m,j)C_j^i$

$$=\sum_{i=0}^{k}\sum_{j=i}^{n}(-m)^{j-i}s(m,k-i)s(n,j)C_j^i \tag{2.19}$$

Proof. Compare the two sides of $[x]_{n+m}=[x]_n[x-n]_m$,

$$[x]_{n+m}=\sum_{k=0}^{n+m}s(n+m,k)x^k$$

$$
\begin{aligned}
[x]_n[x-n]_m &= \left(\sum_{i=0}^{n} s(n,i)x^i\right)\left(\sum_{j=0}^{m} s(m,j)(x-n)^j\right)\\
&= \left(\sum_{i=0}^{n} s(n,i)x^i\right)\left(\sum_{j=0}^{m} s(m,j)\sum_{l=0}^{j}(-n)^{j-l}C_j^l x^l\right)\\
&= \left(\sum_{i=0}^{n} s(n,i)x^i\right)\left(\sum_{l=0}^{m} x^l\sum_{j=l}^{m}(-n)^{j-l}C_j^l s(m,j)\right)\\
&= \sum_{k=0}^{n+m} x^k\sum_{i=0}^{k}\sum_{j=i}^{m}(-n)^{j-i}s(n,k-i)s(m,j)C_j^i
\end{aligned}
$$

Compare the coefficient of x^k , we obtain

$$
s(n+m,k)=\sum_{i=0}^{k}\sum_{j=i}^{m}(-n)^{j-i}s(n,k-i)s(m,j)C_j^i\,,
$$

Exchange m,n , we have

$$
s(n+m,k)=\sum_{i=0}^{k}\sum_{j=i}^{n}(-m)^{j-i}s(m,k-i)s(n,j)C_j^i \qquad \square
$$

We don't have similar identities for *Stirling* numbers of the second kind, for the expansion of $[x]_i[x]_j$ using basis $\{[x]_n\}$ contains $s(n,k)$ and $S(n,k)$ (see **Theorem 8**), which leads to a trivial identity.

Theorem 5. $\displaystyle\sum_{k=n}^{m}\frac{(-1)^k}{k!}s(k,n)=\frac{(-1)^m}{m!}s(m+1,n+1)$ (2.20)

$$
\sum_{k=n}^{m}\frac{S(k,n)}{(n+1)^k}=\frac{S(m+1,n+1)}{(n+1)^m} \tag{2.21}
$$

Proof. According to **Theorem 1**,

$$
\begin{aligned}
\sum_{k=n}^{m}\frac{(-1)^k}{k!}s(k,n) &= \sum_{k=n}^{m}\frac{(-1)^k}{k!}\left(s(k+1,n+1)+ks(k,n+1)\right)\\
&= \sum_{k=n}^{m}\left(\frac{(-1)^k}{k!}s(k+1,n+1)-\frac{(-1)^{k-1}}{(k-1)!}s(k,n+1)\right)\\
&= \frac{(-1)^m}{m!}s(m+1,n+1)-\frac{(-1)^{n-1}}{(n-1)!}s(n,n+1)=\frac{(-1)^m}{m!}s(m+1,n+1)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=n}^{m}\frac{S(k,n)}{(n+1)^k} &= \sum_{k=n}^{m}\frac{1}{(n+1)^k}\left(S(k+1,n+1)-(n+1)S(k,n+1)\right)\\
&= \sum_{k=n}^{m}\left(\frac{S(k+1,n+1)}{(n+1)^k}-\frac{S(k,n+1)}{(n+1)^{k-1}}\right)
\end{aligned}
$$

$$= \frac{S(m+1,n+1)}{(n+1)^m} - \frac{S(n,n+1)}{(n+1)^{n-1}} = \frac{S(m+1,n+1)}{(n+1)^m} \qquad \square$$

The following theorem present an expansion of $[x+m]_n$ using basis $\{[x]_n\}$.

Theorem 6. $[x+m]_n = \sum_{0\le i\le j\le k\le n} C_k^j m^{k-j} s(n,k) S(j,i)[x]_i$ (2.22)

Proof. $[x+m]_n = \sum_{k=0}^{n} s(n,k)(x+m)^k = \sum_{k=0}^{n}\sum_{j=0}^{k} s(n,k) C_k^j m^{k-j} x^j$

$$= \sum_{k=0}^{n}\sum_{j=0}^{k} s(n,k) C_k^j m^{k-j} \sum_{i=0}^{j} S(j,i)[x]_i = \sum_{0\le i\le j\le k\le n} C_k^j m^{k-j} s(n,k) S(j,i)[x]_i \quad \square$$

Set $x = n$, we obtain

Corollary 6.1. $C_{n+m}^m = \frac{1}{n!}\sum_{0\le i\le j\le k\le n} C_k^j C_n^i m^{k-j} s(n,k) S(j,i) i!$ (2.23)

Theorem 7.

$$\frac{(ln(x+m))^k}{k!} = \begin{cases} \sum_{n=0}^{\infty} s(n,k)\frac{x^n}{n!} & \text{m=1[1]} \quad (2.24) \\ \sum_{n=0}^{\infty} \frac{x^n}{m^n n!}\sum_{i=0}^{k} \frac{s(n,i)(\ln m)^{k-i}}{(k-i)!} & m \neq 1 \quad (2.25) \end{cases}$$

Proof. $(x+m)^t = \sum_{n=0}^{\infty}\frac{[t]_n}{n!} m^{t-n} x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!} m^{t-n} x^n s(n,k) t^k = \sum_{k=0}^{n} t^k \sum_{n=0}^{\infty}\frac{1}{n!} m^{t-n} s(n,k) x^n$

$$(x+m)^t = e^{t\ln(x+m)} = \sum_{k=0}^{\infty}\frac{t^k}{k!}(\ln(x+m))^k$$

If $m = 1$, then compare the coefficient of t^k we have

$$\frac{(ln(x+1))^k}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{x^n}{n!} \quad [1]$$

which proof (2.7) in **Theorem 2.**

If $m \neq 1$, then m^t depends on t,

$$m^t = e^{t\ln m} = \sum_{k=0}^{\infty}\frac{t^k}{k!}(\ln m)^k$$

Henceforth,

$$(x+m)^t = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{s(k,j)}{m^k k!}x^k \frac{t^{i+j}}{i!}(\ln m)^i$$
$$= \sum_{i=0}^{\infty} t^i \sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{s(k,j)}{m^k k!}x^k \frac{(\ln m)^{i-j}}{(i-j)!}$$
$$= \sum_{k=0}^{\infty} t^k \sum_{i=0}^{k}\sum_{n=0}^{\infty}\frac{s(n,i)}{m^n n!}x^n \frac{(\ln m)^{k-i}}{(k-i)!}$$

Compare the coefficient of t^k, we obtain

$$\frac{(\ln(x+m))^k}{k!} = \sum_{n=0}^{\infty}\frac{x^n}{m^n n!}\sum_{i=0}^{k}\frac{s(n,i)(\ln m)^{k-i}}{(k-i)!} \quad \square$$

Set $m=e$ in (2.25), we have

Corollary 7.1. $$\frac{(\ln(x+e))^k}{k!} = \sum_{n=0}^{\infty}\frac{x^n}{e^n n!}\sum_{i=0}^{k}\frac{s(n,i)}{(k-i)!} \tag{2.26}$$

Set $k=1$ in (2.25), we have

Corollary 7.2. $$\ln(\frac{x}{m}+1) = \ln(x+m) - \ln m = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{m^n n} \tag{2.27}$$

Which is the same as Taylor expansion of $\ln(x+1)$. Thus (2.25) is a generalization of (2.27).

The following theorem present an expansion of $[x]_m[x]_n$ using basis $\{[x]_n\}$.

Theorem 8. $$[x]_m[x]_n = \sum_{i=0}^{m+n}[x]_i \sum_{k=i}^{m+n}\sum_{j=0}^{k} S(k,i)s(n,j)s(m,k-j) \tag{2.28}$$

Proof. $$[x]_m[x]_n = (\sum_{i=0}^{n} s(n,i)x^i)(\sum_{j=0}^{m} s(m,j)x^j)$$

$$= \sum_{k=0}^{m+n} x^k \sum_{j=0}^{k} s(n,j)s(m,k-j)$$

$$= \sum_{k=0}^{m+n}(\sum_{i=0}^{k} S(k,i)[x]_i)(\sum_{j=0}^{k} s(n,j)s(m,k-j))$$

$$= \sum_{i=0}^{m+n}[x]_i \sum_{k=i}^{m+n}\sum_{j=0}^{k} S(k,i)s(n,j)s(m,k-j) \quad \square$$

Set $x=m+n$, we have

Corollary 8.1. $$m!n!(C_{m+n}^n)^2 = \sum_{i=0}^{m+n} i!C_{m+n}^i \sum_{k=i}^{m+n}\sum_{j=0}^{k} S(k,i)s(n,j)s(m,k-j) \tag{2.29}$$

2.3 Identities of *Lah* numbers

Theorem 9 [1] **.** $$\frac{[x]^n}{n!}=\sum_{k=1}^{n}C_{n-1}^{k-1}\frac{[x]_k}{k!} \tag{2.30}$$

Proof. $[x]^n=(-1)^n[-x]_n=(-1)^n\sum_{k=0}^{n}L(n,k)[x]_k$

By **Theorem 3**, we have

$$L(n,k)=(-1)^n\frac{n!}{k!}C_{n-1}^{k-1}$$

$$[x]^n=(-1)^n\sum_{k=0}^{n}(-1)^n\frac{n!}{k!}C_{n-1}^{k-1}[x]_k=n!\sum_{k=0}^{n}C_{n-1}^{k-1}\frac{[x]_k}{k!}$$

$$\frac{[x]^n}{n!}=\sum_{k=1}^{n}C_{n-1}^{k-1}\frac{[x]_k}{k!}$$ □

(2.30) can be written as

$$\binom{x+n-1}{\text{n}}=\sum_{k=1}^{n}\binom{n-1}{n-k}\binom{x}{k}$$

Which is the same as Vandermonde's identity when $x\in \boldsymbol{N}_+$.

By (2.12), we know that $|s(n,i)|=(-1)^{n+i}s(n,i)$, compare the coefficient of x^i in (2.30), we obtain

Corollary 9.1. $$\frac{1}{n!}\left|s(n,i)\right|=\sum_{k=1}^{n}\frac{1}{k!}C_{n-1}^{k-1}s(k,i)\qquad(0\le i\le n) \tag{2.31}$$

With a more generalized form of Vandermonde's identity [2]

$$\binom{x+y}{k}=\sum_{i=0}^{k}\binom{x}{i}\binom{y}{k-i} \tag{2.32}$$

where x,y are arbitrary complex numbers, we can proof a new identity.

Theorem 10. $$2^j s(k,j)=\sum_{i=0}^{k}\sum_{l=0}^{j}s(i,j)s(k-i,j-l)C_k^i \tag{2.33}$$

Proof. Set $x=y$ in (2.32), we get

$$\binom{2x}{k}=\sum_{i=0}^{k}\binom{x}{i}\binom{x}{k-i}$$

Compare the expansion of both sides,

$$\binom{2x}{k}=\frac{[2x]_k}{k!}=\frac{1}{k!}\sum_{j=0}^{k}2^j s(k,j)x^j$$

$$\sum_{i=0}^{k}\binom{x}{i}\binom{x}{k-i}=\frac{1}{k!}\sum_{i=0}^{k}C_k^i[x]_i[x]_{k-i}=\frac{1}{k!}\sum_{i=0}^{k}C_k^i\sum_{j=0}^{k}x^j\sum_{l=0}^{j}s(i,l)s(k-i,j-l)$$

$$=\frac{1}{k!}\sum_{j=0}^{k}x^j\sum_{i=0}^{k}\sum_{l=0}^{j}s(i,l)s(k-i,j-l)C_k^i$$

We obtain

$$2^j s(k,j)=\sum_{i=0}^{k}\sum_{l=0}^{j}s(i,l)s(k-i,j-l)C_k^i \qquad \square$$

In (2.33), set $j=k$, the left hand side is 2^k, the right hand side is

$$\sum_{i=0}^{k}C_k^i\sum_{l=0}^{k}s(i,l)s(k-i,k-l)=\sum_{i=0}^{k}C_k^i$$

And we obtain

Corollary 10.1. $2^k=\sum_{i=0}^{k}C_k^i$ (2.34)

which is a well-known identity. (2.34) is a special case of (2.33).

Theorem 11. $\sum_{k=0}^{n}L(n,k)=n!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^iC_{n+i-1}^n$ (2.35)

Proof. By (2.10), we have

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}(\sum_{k=0}^{n}L(n,k))=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}L(n,k)\frac{x^n}{n!}=\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{-x}{1+x})^k$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{1}{1+x}-1)^k=\sum_{k=0}^{\infty}\frac{1}{k!}(\sum_{i=0}^{\infty}C_k^i(-1)^{k-i}(\frac{1}{1+x})^i)$$

$$=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{k-i}\frac{1}{k!}C_k^i(\sum_{j=0}^{\infty}(-1)^jx^jC_{i+j-1}^j)$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}}{k!}C_k^iC_{i+j-1}^jx^j$$

$$=\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^iC_{n+i-1}^nx^n$$

Compare the coefficient of x^n, we obtain

$$\sum_{k=0}^{n}L(n,k)=n!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^iC_{n+i-1}^n \qquad \square$$

Together with (2.15), we have

Corollary 11.1. $$\sum_{k=0}^{n}\frac{C_{n-1}^{k-1}}{k!}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}C_j^iC_{n+i-1}^n \qquad (2.36)$$

By the relation between *Lah* numbers and *Stirling* numbers [1],

$$L(n,k)=\sum_{j=0}^{n}(-1)^j s(n,j)S(j,k)$$

We obtain

Corollary 11.2. $$\sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^l s(n,l)S(l,k)=n!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^iC_{n+i-1}^n \qquad (2.37)$$

Theorem 12. $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\frac{k!}{j!}C_j^iC_{k+i-1}^kL(n,k)=1 \qquad (2.38)$$

Proof. Apply the inversion formula of *Lah* number (**Theorem 14.**(3))

$$a_n=\sum_{k=0}^{n}L(n,k)b_k \Leftrightarrow b_n=\sum_{k=0}^{n}L(n,k)a_k$$

to **Theorem 11**, where $b_n=1$ and $a_n=n!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^iC_{n+i-1}^n$,

we obtain

$$1=\sum_{k=0}^{n}L(n,k)k!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}}{j!}C_j^iC_{k+i-1}^k$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\frac{k!}{j!}C_j^iC_{k+i-1}^kL(n,k) \qquad \square$$

Together with (2.15), we have

$$1=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k+n}\frac{n!}{j!}C_j^iC_{k+i-1}^kC_{n-1}^{k-1}$$

$$=\sum_{k=0}^{n}C_{n-1}^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+k+n}\frac{n!}{j!}C_j^iC_{k+i-1}^k$$

$$=\sum_{k=0}^{n-1}C_{n-1}^{k}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+k+1+n}\frac{n!}{j!}C_j^iC_{k+i}^{k+1}$$

Change n into $n+1$,

$$1=\sum_{k=0}^{\infty}C_n^k\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+k+n}\frac{(n+1)!}{j!}C_j^iC_{k+i}^{k+1}$$

$$=\frac{(n+1)!}{(-1)^n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}}{j!}C_n^k C_j^i C_{k+i}^{k+1}$$

Corollary 12.1. $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}}{j!}C_n^k C_j^i C_{k+i}^{k+1}=\frac{(-1)^n}{(n+1)!} \quad (2.39)$$

Calculate the sum of (2.39) with variable n, the right hand side is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}=1-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}=1-\frac{1}{e}$$

which implies

Corollary 12.2. $$\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}}{j!}C_n^k C_j^i C_{k+i}^{k+1}=1-\frac{1}{e} \quad (2.40)$$

(2.39) can be written as

$$\sum_{k=0}^{n}C_n^k\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}}{j!}C_j^i C_{k+i}^{k+1}=\frac{(-1)^n}{(n+1)!} \quad (2.41)$$

Apply the inversion formula of binomial coefficients (**Theorem 14.**(1))

$$a_n=\sum_{k=0}^{n}C_n^k b_k \Leftrightarrow b_n=\sum_{k=0}^{n}(-1)^{n-k}C_n^k a_k$$

to (2.41), where $a_n=\frac{(-1)^n}{(n+1)!}$ and $b_n=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^i C_{n+i}^{n+1}$

we obtain $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+n}}{j!}C_j^i C_{n+i}^{n+1}=\sum_{k=0}^{n}(-1)^{n-k}C_n^k\frac{(-1)^k}{(k+1)!}$

Corollary 12.3. $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}C_j^i C_{n+i}^{n+1}=\sum_{k=0}^{n}\frac{C_n^k}{(k+1)!} \quad (2.42)$$

2.4 Identities of *Bell* numbers

Theorem 13. $$\sum_{k=1}^{n}\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_k=n\\ \varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k\geq 1}}\frac{(-1)^{k-1}}{k(\varepsilon_1!\cdots\varepsilon_k!)}B_{\varepsilon_1}\cdots B_{\varepsilon_k}=\frac{1}{n!} \quad (2.43)$$

Proof. According to (2.11), we have

$$ln(1+\sum_{n=1}^{\infty}B_n\frac{x^n}{n!})=e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}.$$

By Taylor expansion $ln(1+x)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$, we have

$$\sum_{n=1}^{\infty}\frac{x^n}{n!}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\sum_{n=1}^{\infty}B_n\frac{x^n}{n!})^k$$

By multinomial theorem

$$(x_1+x_2+\cdots+x_k)^n=\sum_{\substack{c_1+\cdots+c_k=n\\ c_1,c_2,\cdots,c_k\ge 0}}\frac{n!}{c_1!\cdots c_k!}x_1^{c_1}x_2^{c_2}\cdots x_k^{c_k}$$

and comparing the coefficient of x^n, we have

$$\frac{1}{n!}=\sum_{k=1}^{n}\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_k=n\\ \varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k\ge 1}}\frac{(-1)^{k-1}}{k(\varepsilon_1!\cdots\varepsilon_k!)}B_{\varepsilon_1}\cdots B_{\varepsilon_k}$$ □

3 On inversion formulas

3.1 Remarks on some well-known inversion formulas

Consider two series $\{a_n\}$ and $\{b_n\}$, we can set up two groups of equations, in each group we use one of the series to represent the element of another. The two groups of equations are equivalence, that is if one group of the equations holds, the other holds too. These equations are called inversion formulas. Due to the equivalence of the equations, if we proof one group of them we will obtain the other immediately, which is convenience for proving some combinatorial identities. Here are some frequently used inversion formulas.

Theorem 14[1] **.**

（1） $$\begin{cases} a_n=\sum_{k=0}^{n}C_n^k b_k & \forall n\in \boldsymbol{N} \\ b_n=\sum_{k=0}^{n}(-1)^{n-k}C_n^k a_k & \forall n\in \boldsymbol{N}\end{cases}$$

（2） $$\begin{cases} a_n=\sum_{k=0}^{n}s(n,k)b_k & \forall n\in \boldsymbol{N} \\ b_n=\sum_{k=0}^{n}S(n,k)a_k & \forall n\in \boldsymbol{N}\end{cases}$$

（3） $$\begin{cases} a_n=\sum_{k=0}^{n}L(n,k)b_k & \forall n\in \boldsymbol{N} \\ b_n=\sum_{k=0}^{n}L(n,k)a_k & \forall n\in \boldsymbol{N}\end{cases}$$

The generalization of **Theorem 14** is the following.

Theorem 15 [3] **.** $\{f_n(x)\}_{n\ge 0}$ and $\{g_n(x)\}_{n\ge 0}$ are two groups of linearly independent functions respectively (i.e. $f_n(x)$ and $g_n(x)$ are both polynomials of degree n), functions of two variables $\alpha(n,k)$, $\beta(n,k)$ satisfy that

$$f_n(x)=\sum_{k=0}^{n}\alpha(n,k)g_k(x) \quad \forall n\in \boldsymbol{N}$$
$$g_n(x)=\sum_{k=0}^{n}\beta(n,k)f_k(x) \quad \forall n\in \boldsymbol{N}$$

then we have an inversion formula

$$\begin{cases} a_n=\sum_{k=0}^{n}\alpha(n,k)b_k & \forall n\in \boldsymbol{N} \quad (3.1)\\ b_n=\sum_{k=0}^{n}\beta(n,k)a_k & \forall n\in \boldsymbol{N} \quad (3.2)\end{cases}$$

Proof. $f_n(x)=\sum_{k=0}^{n}\alpha(n,k)g_k(x)=\sum_{k=0}^{n}\alpha(n,k)\sum_{i=0}^{k}\beta(k,i)f_i(x)=\sum_{i=o}^{n}f_i(x)\sum_{k=i}^{n}\alpha(n,k)\beta(k,i)$

$$g_n(x)=\sum_{k=0}^{n}\beta(n,k)f_k(x)=\sum_{k=0}^{n}\beta(n,k)\sum_{i=0}^{k}\alpha(k,i)g_i(x)=\sum_{i=o}^{n}g_i(x)\sum_{k=i}^{n}\beta(n,k)\alpha(k,i)$$

Since $\{f_n(x)\}_{n\geq 0}$ and $\{g_n(x)\}_{n\geq 0}$ are linearly independent respectively, we obtain

$$\sum_{k=i}^{n}\alpha(n,k)\beta(k,i)=\sum_{k=i}^{n}\beta(n,k)\alpha(k,i)=\delta_{ni}$$

Thus, if (3.1) holds for arbitrary non-negative integer n, then

$$\sum_{k=0}^{n}\beta(n,k)a_k=\sum_{k=0}^{n}\beta(n,k)\sum_{i=0}^{k}\alpha(k,i)b_i=\sum_{i=0}^{n}b_i\sum_{k=i}^{n}\beta(n,k)\alpha(k,i)=\sum_{i=0}^{n}b_i\delta_{ni}=b_n$$

This implies (3.1) $\Rightarrow$ (3.2). (3.2) $\Rightarrow$ (3.1) can be proved in the same way. □

The connection between **Theorem 14** and **Theorem 15** is shown by the following table.

Theorem 15 / **Theorem 14**	$f_n(x)$	$g_n(x)$	$\alpha(n,k)$	$\beta(n,k)$
(1)	x^n	$(x+1)^n$	$(-1)^{n-k}C_n^k$	C_n^k
(2)	$[x]_n$	x^n	$s(n,k)$	$S(n,k)$
(3)	$[x]_n$	$(-1)^n[x]^n$	$L(n,k)$	$L(n,k)$

The inversion technique is powerful because it can transform an identity into another easier identity equivalently, as we can see in **Theorem 12**.

In the following section, we will use generating functions to find inversion formulas.

3.2 A new inversion formula

We should investigate the following theorem first.

Theorem 16[3]**.** (1) Suppose the ordinary generating functions of $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ are $f(x)$ and $g(x)$ respectively. If $f(x)g(x)=1$, then

$$a_n=\sum_{k=0}^{n}f_kb_{n-k} \quad (\forall n\in \boldsymbol{N}) \quad \Leftrightarrow \quad b_n=\sum_{k=0}^{n}g_ka_{n-k} \quad (\forall n\in \boldsymbol{N}) \qquad (3.3)$$

(2) Suppose the exponential generating functions of $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ are $f(x)$ and $g(x)$ respectively. If $f(x)g(x)=1$, then

$$a_n=\sum_{k=0}^{n}C_n^k f_k b_{n-k}\quad (\forall n\in \boldsymbol{N})\quad \Leftrightarrow\quad b_n=\sum_{k=0}^{n}C_n^k g_k a_{n-k}\quad (\forall n\in \boldsymbol{N}) \tag{3.4}$$

Proof is omitted. Suppose the ordinary generating functions (or exponential generating functions) of $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are $A(x)$ and $B(x)$ respectively. Since $f(x)g(x)=1$, we have

$$A(x)=B(x)f(x)\quad \Leftrightarrow\quad B(x)=A(x)g(x) \tag{3.5}$$

Compare the coefficients, we obtain (3.3) and (3.4) immediately. With two equivalence equations of generating functions (i.e. (3.5)), we have an inversion formula correspondingly. Suppose $g(x)$ is the inverse function of $f(x)$, we have

$$A(x)=f(B(x))\quad \Leftrightarrow\quad B(x)=g(A(x)) \tag{3.6}$$

We obtain a new inversion formula.

Theorem 17. Suppose the exponential generating functions of $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ are $f(x)$ and $g(x)$ respectively. If $f(x)$ is the inverse function of $g(x)$, then we have the following inversion formula

$$\begin{cases} a_n=\sum\limits_{i=1}^{n} f_i \sum\limits_{\varepsilon_1+\cdots+\varepsilon_i=n} b_{\varepsilon_1}b_{\varepsilon_2}\cdots b_{\varepsilon_i}\cdot\dfrac{1}{i!} & \forall n\in \boldsymbol{N}_+ \quad (3.7)\\ b_n=\sum\limits_{i=1}^{n} g_i \sum\limits_{\varepsilon_1+\cdots+\varepsilon_i=n} a_{\varepsilon_1}a_{\varepsilon_2}\cdots a_{\varepsilon_i}\cdot\dfrac{1}{i!} & \forall n\in \boldsymbol{N}_+ \quad (3.8)\end{cases}$$

Proof. Suppose the ordinary generating functions of $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are $A(x)$ and $B(x)$ respectively. If (3.7) holds for arbitrary positive integer n, then

$$f(B(x))=\sum_{i=1}^{\infty}\frac{f_i}{i!}\Big(\sum_{j=1}^{\infty}b_j x^j\Big)^i=\sum_{n=1}^{\infty}x^n\cdot\sum_{i=1}^{n}\frac{f_i}{i!}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}b_{\varepsilon_1}b_{\varepsilon_2}\cdots b_{\varepsilon_i}=\sum_{n=1}^{\infty}a_n x^n=A(x)$$

Thus $B(x)=g(A(x))$, which is equivalence to

$$b_n=\sum_{i=1}^{n}\frac{g_i}{i!}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}a_{\varepsilon_1}a_{\varepsilon_2}\cdots a_{\varepsilon_i}\qquad \forall n\in \boldsymbol{N}_+$$

So (3.7) $\Rightarrow$ (3.8). Similarly, (3.8) $\Rightarrow$ (3.7). □

In order to apply **Theorem 17**, we give tow pairs of reciprocal functions first.

(1) $f(x)=e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}$

and $g(x)=ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\frac{x^n}{n!}$

(2) $f(x)=\frac{1}{\sqrt{1-4x}}-1=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-4x)^k-1=\sum_{k=1}^{\infty}k!C_{2k}^{k}\frac{x^k}{k!}$

and $g(x)=\frac{1}{4}\left(1-\frac{1}{(x+1)^2}\right)=\frac{1}{4}\left(1-\sum_{k=0}^{\infty}\binom{-2}{k}x^k\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4}(k+1)!\frac{x^k}{k!}$

By **Theorem 17**, we have

Corollary 17.1. $a_n=\sum_{i=1}^{n}\frac{1}{i!}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}b_{\varepsilon_1}b_{\varepsilon_2}\cdots b_{\varepsilon_i}$ $(\forall n\in \boldsymbol{N}_+)$

$$\Leftrightarrow \quad b_n=\sum_{i=1}^{n}\frac{(-1)^{i+1}}{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}a_{\varepsilon_1}a_{\varepsilon_2}\cdots a_{\varepsilon_i} \quad (\forall n\in \boldsymbol{N}_+)$$

Corollary 17.2. $a_n=\sum_{i=1}^{n}C_{2i}^{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}b_{\varepsilon_1}b_{\varepsilon_2}\cdots b_{\varepsilon_i}$ $(\forall n\in \boldsymbol{N}_+)$

$$\Leftrightarrow \quad b_n=\sum_{i=1}^{n}\frac{(-1)^{i-1}(i+1)}{4}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}a_{\varepsilon_1}a_{\varepsilon_2}\cdots a_{\varepsilon_i} \quad (\forall n\in \boldsymbol{N}_+)$$

Theorem 18. $x=\sum_{i=1}^{n}\frac{1}{i!}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}\frac{(1-(1-x)^{\varepsilon_1})\cdots(1-(1-x)^{\varepsilon_i})}{\varepsilon_1\cdots\varepsilon_i}$ (3.9)

Proof. By **Corollary 17.1**, it is sufficient to proof that

$$\frac{1-(1-x)^n}{n}=\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}x^i$$

Since the equation $\varepsilon_1+\cdots+\varepsilon_i=n$ has C_{n-1}^{i-1} different positive integer solutions, thus it is sufficient to proof that

$$\frac{1-(1-x)^n}{n}=\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}C_{n-1}^{i-1}x^i$$

In fact,

$$\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}C_{n-1}^{i-1}x^i=\sum_{i=1}^{n}\frac{(-1)^{i-1}}{n}C_n^i x^i=\frac{1}{n}-\frac{1}{n}\sum_{i=0}^{n}C_n^i(-x)^i=\frac{1-(1-x)^n}{n}$$ □

Set $x=1$ and $x=2$ in **Theorem 18** respectively, we have

Corollary 18.1. $$1=\sum_{i=1}^{n}\frac{1}{i!}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}\frac{1}{\varepsilon_1\cdots\varepsilon_i} \tag{3.10}$$

Corollary 18.2. $$2=\sum_{i=1}^{n}\frac{1}{i!}\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_i=n\\ 2\nmid\varepsilon_1,\cdots,\varepsilon_i}}\frac{2^i}{\varepsilon_1\cdots\varepsilon_i} \tag{3.11}$$

Theorem 19. $$x=\sum_{i=1}^{n}C_{2i}^{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}\frac{1}{4^i}x^i(1-x)^{n-2i}(2-x-\varepsilon_1 x)\cdots(2-x-\varepsilon_i x) \tag{3.12}$$

Proof. According to **Corollary 17.2**, it is sufficient to proof that

$$\frac{1}{4}\sum_{i=1}^{n}(-1)^{i+1}(i+1)C_{n-1}^{i-1}x^i=\frac{1}{4}x(1-x)^{n-2}(2-x-nx)$$

Denote the left hand side as $f(x)$, then $f(x)=Dg(x)$, where

$$g(x)=\frac{1}{4}\sum_{i=1}^{n}C_{n-1}^{i-1}(-x)^{i+1}=\frac{x^2}{4}\sum_{i=1}^{n}C_{n-1}^{i-1}(-x)^{i-1}=\frac{x^2(1-x)^{n-1}}{4}$$

Thus,

$$f(x)=Dg(x)=\frac{2x(1-x)^{n-1}}{4}-\frac{(n-1)x^2(1-x)^{n-2}}{4}=\frac{1}{4}x(1-x)^{n-2}(2-x-nx)$$ □

Set $x=2$, -1, $\frac{1}{2}$, in **Theorem 19** respectively, we have

Corollary 19.1. $$(-1)^n 2=\sum_{i=1}^{n}(-1)^i C_{2i}^{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}\varepsilon_1\cdots\varepsilon_i \tag{3.13}$$

Corollary 19.2. $$-\frac{1}{2^n}=\sum_{i=1}^{n}(-\frac{1}{16})^i C_{2i}^{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}(\varepsilon_1+3)\cdots(\varepsilon_i+3) \tag{3.14}$$

Corollary 19.3. $$2^{n-1}=\sum_{i=1}^{n}(\frac{1}{4})^i C_{2i}^{i}\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}(3-\varepsilon_1)\cdots(3-\varepsilon_i) \tag{3.15}$$

We can see that **Theorem 17** can be used to discover and prove a new kind of combinatorial identities which contain symbol $\sum_{\varepsilon_1+\cdots+\varepsilon_i=n}$. This new inversion formula is a useful new method.

4 Generalization and simplification – a promising new way

In the research of combinatorial identities, we often face a paradox: on the one hand, we expect the identities to be concise and symmetric; on the other hand, many concise identities are based on some specific properties of several sequences such as binomial coefficients, which makes them hard to be generalized. In view of such a paradox, we propose that we should start from generalizing the definitions, and deduce the common properties of the generalized sequence. Then by adding some specific properties to the sequence, we obtain identities which are original and simple as well.

4.1 Generalization of definitions

Definition 11. Given a sequence $\{a_n\}_{n\geq 1}$, let $f_n(x)=\prod_{i=1}^{n}(x-a_i)$ $(n\in \boldsymbol{N}_+)$, and $f_0(x)=1$.

Definition 12. Sequence $\{b_1(n,k)\}$ satisfy that $f_n(x)=\sum_{k=0}^{n} b_1(n,k)X^k$, and define $b_1(n,k)=0$ when $k>n\geq 0$ or $k<0$.

Since $\{f_n(x)\}$ are lineary independent, we can introduce

Definition 13. Sequence $\{b_2(n,k)\}$ satisfy that $X^n=\sum_{k=0}^{n} b_2(n,k)f_k(x)$, and define $b_2(n,k)=0$ when $k>n\geq 0$ or $k<0$.

Definition 14. Sequence $\{d(n,k)\}$ satisfy that $f_n(-x)=\sum_{k=0}^{n} d(n,k)f_k(x)$ and define $d(n,k)=0$ when $k>n\geq 0$ or $k<0$.

Set $a_n=n-1$, then $b_1(n,k)=s(n,k)$, $b_2(n,k)=S(n,k)$, $d(n,k)=L(n,k)$.

Set $a_n=-1$, then $b_1(n,k)=C_n^k$, $b_2(n,k)=(-1)^{n-k}C_n^k$, $d(n,k)=(-1)^k 2^{n-k}C_n^k$.

Henceforth, the definitions above are generalizations of binomial coefficients, *Stirling* numbers and *Lah* numbers.

4.2 Generalization of basic properties

Theorem 20. (1) $b_1(n+1,k)=b_1(n,k-1)-a_{n+1}b_1(n,k)$ (4. 1)

(2) $b_2(n+1,k)=b_2(n,k-1)+a_{k+1}b_2(n,k)$ (4. 2)

(3) $d(n+1,k)=-(a_{n+1}+a_{k+1})d(n,k)-d(n,k-1)$ (4. 3)

Proof. (1) $\sum_{k=0}^{n+1} b_1(n+1,k)x^k = f_{n+1}(x) = f_n(x)(x-a_{n+1}) = \sum_{k=0}^{n}(b_1(n,k)x^{k+1}-a_{n+1}b_1(n,k)x^k)$

Compare the coefficient of x^k, we have

$$b_1(n+1,k)=b_1(n,k-1)-a_{n+1}b_1(n,k)$$

(2) $\sum_{k=0}^{n+1} b_2(n+1,k)f_k(x)=x^{n+1}=x\sum_{k=0}^{n+1} b_2(n,k)f_k(x)=\sum_{k=0}^{n} b_2(n,k)(a_{k+1}f_k(x)+f_{k+1}(x))$

Compare the coefficient of $f_k(x)$, we have

$$b_2(n+1,k)=b_2(n,k-1)+a_{k+1}b_2(n,k)$$

(3)Since $f_{n+1}(-x)=-f_n(-x)(x+a_{n+1})$,

$$\sum_{k=0}^{n+1} d(n+1,k)f_k(x)=-(x+a_{n+1})\sum_{k=0}^{n} d(n,k)f_k(x)$$

$$=-\sum_{k=0}^{n} d(n,k)(a_{n+1}f_k(x)+a_{k+1}f_k(x)+f_{k+1}(x))$$

$$=-\sum_{k=0}^{n}((a_{n+1}+a_{k+1})d(n,k)+d(n,k-1))f_k(x)$$

Compare the coefficient of $f_k(x)$, we have

$$d(n+1,k)=-(a_{n+1}+a_{k+1})d(n,k)-d(n,k-1)$$ □

Theorem 21. $$\sum_{n=0}^{\infty} b_2(n,k)x^n=\frac{x^k}{\prod_{i=1}^{k+1}(1-a_i x)} \qquad (4.4)$$

Proof. Denote $G_k(x)=\sum_{n=0}^{\infty} b_2(n,k)x^n$. When $k>0$,

$$\frac{G_k(x)}{x}=\sum_{n=0}^{\infty} b_2(n+1,k)x^n=\sum_{n=0}^{\infty} b_2(n,k-1)x^n+a_{k+1}\sum_{n=0}^{\infty} b_2(n,k)x^n=G_{k-1}(x)+a_{k+1}G_k(x)$$

Thus,

$$G_k(x)=\frac{x}{1-a_{k+1}x}G_{k-1}(x)$$

Since $a_1^n=\sum_{k=0}^{n} b_2(n,k)f_k(a_1)=b_2(n,0)$, we have

$$G_0(x)=\sum_{n=0}^{\infty} b_2(n,0)x^n=\sum_{n=0}^{\infty}(a_1x)^n=\frac{1}{1-a_1x}$$

$$G_k(x)=\sum_{n=0}^{\infty} b_2(n,k)x^n=\frac{x^k}{\prod_{i=1}^{k+1}(1-a_i x)}$$ □

Theorem 22. (1) $$b_1(n+1,k)=(-1)^{n+k}\sum_{1\le\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_{n-k}\le n} a_{\varepsilon_1}a_{\varepsilon_2}\cdots a_{\varepsilon_{n-k}} \qquad (n\ge k) \qquad (4.5)$$

(2) $$b_2(n+1,k)=\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_{k+1}=n-k\\ \varepsilon_1,\,\cdots,\,\varepsilon_{k+1}\ge 0}} a_1^{\varepsilon_1}a_2^{\varepsilon_2}\cdots a_{k+1}^{\varepsilon_{k+1}} \qquad (n\ge k) \qquad (4.6)$$

Proof. (1) $f_n(x)=\sum_{k=0}^{n} b_1(n,k)x^k = \prod_{i=1}^{n}(x-a_i)$

Compare the coefficient of x^k, we obtain (4.5).

(2)According to **Theorem 21**,

$$\sum_{n=0}^{\infty} b_2(n,k)x^{n-k} = \prod_{i=1}^{k+1}\frac{1}{1-a_i x} = \prod_{i=1}^{k+1}(\sum_{j=0}^{\infty}(a_i x)^j)$$

Compare the coefficient of x^{n-k}, we obtain (4.6) □

4.3 Some common properties

Theorem 23. (1) $\sum_{k=n}^{m}\frac{(-1)^k b_1(k,n)}{a_{k+1}\cdots a_1} = \frac{(-1)^m b_1(m+1,n+1)}{a_{m+1}\cdots a_1} \quad (a_i \neq 0)$ (4.7)

(2) $\sum_{k=n}^{m}\frac{b_2(k,n)}{a_{n+2}^k} = \frac{b_2(m+1,n+1)}{a_{n+2}^m} \quad (a_{n+2} \neq 0)$ (4.8)

Theorem 24. (1) $\sum_{k=m}^{n} b_1(n,k)b_2(k,m) = \sum_{k=m}^{n} b_2(n,k)b_1(k,m) = \delta_{mn}$ (4.9)

(2) $\sum_{k=m}^{n} d(n,k)d(k,m) = \delta_{mn}$ (4.10)

(3) $\alpha_n = \sum_{k=0}^{n} b_1(n,k)\beta_k \ (\forall n \in \boldsymbol{N}_+) \iff \beta_n = \sum_{k=0}^{n} b_2(n,k)\alpha_k (\forall n \in \boldsymbol{N}_+)$ (4.11)

(4) $\alpha_n = \sum_{k=0}^{n} d(n,k)\beta_k \ (\forall n \in \boldsymbol{N}_+) \iff \beta_n = \sum_{k=0}^{n} d(n,k)\alpha_k (\forall n \in \boldsymbol{N}_+)$ (4.12)

Theorem 25. $f_n(x+m) = \sum_{0\le i\le j\le k\le n} m^{k-j} C_k^j b_1(n,k) b_2(j,i) f_i(x)$ (4.13)

Theorem 26. $d(n,k) = \sum_{j=k}^{n}(-1)^j b_1(n,j) b_2(j,k)$ (4.14)

Proof. $\sum_{k=0}^{n} d(n,k) f_k(x) = f_n(-x) = \sum_{j=0}^{n}(-1)^j b_1(n,j)x^j$

$$= \sum_{j=0}^{n}(-1)^j b_1(n,j)\sum_{k=0}^{j} b_2(j,k) f_k(x) = \sum_{k=0}^{n} f_k(x)\sum_{j=k}^{n}(-1)^j b_1(n,j) b_2(j,k)$$

Compare the coefficient of $f_k(x)$, we obtain (4.14) □

4.4 Other properties based on special $\{a_n\}$

Theorem 27. If $a_n = \alpha n + \beta$ where α and β are constants, then

$$b_1(n+m,k) = \sum_{i=0}^{k}\sum_{j=i}^{m}(-\alpha n)^{j-i} b_1(n,k-i) b_1(m,j) C_j^i$$

$$= \sum_{i=0}^{k}\sum_{j=i}^{n}(-\alpha m)^{j-i} b_1(m,k-i) b_1(n,j) C_j^i \qquad (4.15)$$

The proof is similar to **Theorem 4**.

In the following discussion, $a_n = q^{n-1}$ where q is a constant.

Theorem 28. (1) $b_2(n,k) = \dfrac{\prod_{i=1}^{n}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{n-k}(q^i-1)}$ (4.16)

(2) $b_2(n,k) = b_2(n,n-k)$ (4.17)

Proof. (1) We use mathematical induction on n.

$n=0$ is trivial. Suppose (4.16) holds for n, then for $1 \le k \le n+1$

$$b_2(n+1,k) = b_2(n,k-1) + a_{k+1} b_2(n,k)$$

$$= \frac{\prod_{i=1}^{n}(q^i-1)}{\prod_{i=1}^{k-1}(q^i-1)\prod_{i=1}^{n-k+1}(q^i-1)} + q^k \frac{\prod_{i=1}^{n}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{n-k}(q^i-1)}$$

$$= \frac{\prod_{i=1}^{n+1}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{n-k+1}(q^i-1)}\left(\frac{q^k-1}{q^{n+1}-1} + q^k \bullet \frac{q^{n-k+1}-1}{q^{n+1}-1}\right) = \frac{\prod_{i=1}^{n+1}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{n-k+1}(q^i-1)}$$

$$1^{n+1} = \sum_{k=0}^{n+1} b_2(n+1,k) f_k(1) = b_2(n+1,0),$$

Thus (4.16) holds for $n+1$.

According to mathematical induction, (4.16) holds for arbitrary non-negative integer n.

(2) (4.17) follows immediately from (4.16) □

If we set $q \to 1$, according to L'Hospital's rule,

$$\lim_{q\to 1}\frac{\prod_{i=1}^{n}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{n-k}(q^i-1)}=\lim_{q\to 1}\frac{\prod_{i=1}^{n}iq^{i-1}}{\prod_{i=1}^{k}iq^{i-1}\prod_{i=1}^{n-k}iq^{i-1}}=\frac{n!}{k!(n-k)!}=C_n^k .$$

And (4.17) becomes the well-known $C_n^k = C_n^{n-k}$.

If $q^m = 1$ for some integer m, then the denominator and numerator of (4.16) are both zero. Suppose q is a unit root with order d where d is a positive integer, then the number of zeroes among $\prod_{i=1}^{n}(q^i-1)$, $\prod_{i=1}^{k}(q^i-1)$ and $\prod_{i=1}^{n-k}(q^i-1)$ are $\left[\frac{n}{d}\right]$, $\left[\frac{k}{d}\right]$ and $\left[\frac{n-k}{d}\right]$ respectively. Since $\left[\frac{n}{d}\right]\geq\left[\frac{k}{d}\right]+\left[\frac{n-k}{d}\right]$ for all $0\leq k\leq n+1$, we have the following

$$b_2(n,k)=\frac{\prod_{\substack{1\leq i\leq n\\ d\nmid i}}(q^i-1)}{\prod_{\substack{1\leq i\leq k\\ d\nmid i}}(q^i-1)\prod_{\substack{1\leq i\leq n-k\\ d\nmid i}}(q^i-1)} \tag{4.18}$$

With (4.18), it is easy to deduce that

Theorem 29. Suppose q is a unit root with order d, then

$$b_2(n,k)=b_2(r_2,r_1) \tag{4.19}$$

where r_1 and r_2 are the least non-negative residue of k and n modulo d respectively.

Corollary 29.1 If $r_2 < r_1$, then $b_2(n,k)=0$.

5 Concluding remarks

In this article, we obtain many new identities of *Stirling* numbers, *Lah* numbers and *Bell* numbers. We proof a new inversion formula and provide a new method to proof a specific kind of new identities. We also generalize the definition of binomial coefficient, *Stirling* number and *Lah* number, and proof the generalized form of some identities as well as some other new identities. By setting some special $\{a_n\}$, we can obtain some new identities in relatively simple form. This could be a new way to research combinatorial identities.

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