An inequality chain of Seiffert mean, NS mean and power mean

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directory

Abstract 3
I. Introduction 5
II. Lemmas..7
III. Proof of Theorems 1.1-1.4..8
IV. The application of various binary means 11
V. Project prospect 13
Reference..14
Appendix: Elementary Proof of Theorem1.2........................15

摘要

本文研究了幂平均与Seiffert平均、NS平均、对数平均、指数平均等的不等关系，特别是，通过证明$M_{2/3} < I < M_{\ln 2}$，$Z < M_{3/2} < P^{II}$，$M_{1/2} < P^{I}$，$M_{1/3} < \bar{h} < M_{1/2}$，从而加长了已有的二元平均不等式链,并且用给出的四个引理衍生出了上百个 Agarwal 型不等式.同时通过给出反例指出了一些论文中关于指数平均与对数平均不等式的错误结论.

关键词：Seiffert 平均、NS 平均、幂平均、平均不等式、Agarwal 型不等式

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Abstract

In this paper, we study the inequality relationship between power mean with Seiffert mean, NS mean, logarithmic mean and exponential mean. Particularly, we extend a pre-existing inequality chain in two variables by proving $M_{2/3} < I < M_{\ln 2}$, $M_{1/3} < \bar{h} < M_{1/2}$, $M_{1/2} < P^{I}$, $Z < M_{3/2} < P^{II}$.Additionally, hundreds of Agarwal inequalities are obtained by using the given lemmas. At the same time, we point out some wrong conclusions in some papers about exponential mean and logarithmic mean through some counter examples.

Key words: Seiffert mean, NS mean, power mean, mean inequality, Agarwal inequality.

I. Introduction

During the process of studying Maths in high school, we continually meet a series of the final questions about power mean in the simulation test of the College Entrance Examination, which involves the relationship between power mean, exponential mean and logarithmic mean.

Currently, the exploration of the relationship of these means is a very hot topic, particularly, power mean's inequality, which often appears in the College Entrance Examination and mathematical competitions.

Mean of two positive number is a simple but important mean. Therefore, various mathematical theory workers and lovers are attempting to do researches on various kinds of means. Up till now, all have been defined for the means as follow:

Definition[1] ：Logarithmic mean：$L(a,b)=\begin{cases}\dfrac{a-b}{\ln a-\ln b}, a\neq b,\\ a, \quad a=b.\end{cases}$

Exponential mean： $I(a,b)$， $E(a,b)$

$$I(a,b)=\begin{cases}\dfrac{1}{e}(\dfrac{a^a}{b^b})^{\frac{1}{a-b}}, a\neq b,\\ a, \quad a=b.\end{cases}$$

$$E(a,b)=\begin{cases}e(\dfrac{a^b}{b^a})^{\frac{1}{b-a}}, a\neq b,\\ a, \quad a=b.\end{cases}$$

Power-exponential mean： $B(a,b)=a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$,

Inverse Power-exponential mean： $D(a,b)=a^{\frac{b}{a+b}}b^{\frac{a}{a+b}}$,

Symmetric means： $Q_p(a,b)=\frac{1}{2}(a^r b^s+a^s b^r)$ ；

$r=\frac{1}{2}(1+\sqrt{p}), s=\frac{1}{2}(1-\sqrt{p})$,

Hölder mean： $M_p(a,b)=\begin{cases}\left(\dfrac{a^p+b^p}{2}\right)^{1/p}, p\neq 0\\ \sqrt{ab}, \quad p=0\end{cases}$

specially， $p=3,2,1,0,-1,-2$ obtained respectively

cube root mean M_3、square root mean M_2、arithmetic mean A、geometric mean G、harmonic mean H、harmonic square root mean M_{-2} .

Lemhmer mean： $L_p(a,b)=\dfrac{a^p+b^p}{a^{p-1}+b^{p-1}}$,

Stolarsky mean： $S_p(a,b)=\begin{cases}(\dfrac{b^p-a^p}{p(b-a)})^{\frac{1}{p-1}}, a\neq b, p\neq 0,1\\ b, \quad a=b\end{cases}$

Toader mean: $T(a,b)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta=\begin{cases}\frac{2a}{\pi}\varepsilon(\sqrt{1-(\frac{b}{a})^2}),a>b,\\ \frac{2b}{\pi}\varepsilon(\sqrt{1-(\frac{a}{b})^2}),a<b,\\ a, \qquad a=b.\end{cases}$

in here, $\varepsilon=\varepsilon(r)=\int_0^{\frac{\pi}{2}}\sqrt{1-r^2\sin^2\theta}\,d\theta, r\in[0,1]$.

Seiffert first mean: $P^I(a,b)=\begin{cases}\frac{a-b}{4\arctan\sqrt{a/b}-\pi}, a\neq b\\ a, \qquad a=b\end{cases}$

or $P^I(a,b)=\begin{cases}\frac{a-b}{2\arcsin[(a-b)/(a+b)]}, a\neq b\\ a, \qquad a=b\end{cases}$

Seiffert second mean: $P^{II}(a,b)=\begin{cases}\frac{a-b}{2\arctan[(a-b)/(a+b)]}, a\neq b\\ a, \qquad a=b\end{cases}$

NS mean: $Z(a,b)=\begin{cases}\frac{a-b}{2\operatorname{ar}\sin h[(a-b)/(a+b)]}, a\neq b\\ a, \qquad a=b\end{cases}$

Heron mean: $h(a,b)=\frac{a+\sqrt{ab}+b}{3}$,

Conjugate Heron mean: $\overline{h}(a,b)=\frac{a+4\sqrt{ab}+b}{6}$,

Centroid mean: $g(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)}$,

Inverse harmonic mean: $C(a,b)=\frac{a^2+b^2}{a+b}$.

and we know

Theorem 0 For $0<a<b$, then

$$a<M_{-2}<H<G<Q_{1/3}<L<M_{1/3}<M_{1/2}<h<M_{2/3}<I<A<g<M_2<M_3<L_2<b. \quad (1.1)$$

(see [2]) obtained

$$D<M_{-2}<H<G<L<\overline{h}<P^I<h<I<A<Z<P^{II}<g<M_2<B<M_3<C. \quad (1.2)$$

(see[3]) obtained: $M_{\ln 2/\ln\pi}<P^I<M_{2/3}$. (1.3)

In1998, (see[4]), the author Vuorinen gave the following conjecture:

$$M_{3/2}(a,b)<T(a,b). \quad (1.4)$$

This conjecture is proved in the (see[5]) and (see[6]) . In 2004,(see[7]), the author proved: $T(a,b)<M_{(\ln 2)/(\ln\pi/2)}(a,b)$. (1.5)

Main results in this paper are:

Theorem 1.1 For $0<a<b$, then

$$M_{2/3} < I < M_{\ln 2} \Leftrightarrow (\frac{a^{2/3}+b^{2/3}}{2})^{3/2} < \frac{1}{e}(\frac{a^a}{b^b})^{\frac{1}{a-b}} < (\frac{a^{\ln 2}+b^{\ln 2}}{2})^{1/\ln 2} . \quad (1.6)$$

Theorem 1.2 For $0<a<b$, then $Z < M_{3/2} < P^{II}$.

That is

$$\frac{a-b}{2ar\sinh[(a-b)/(a+b)]} < (\frac{a^{3/2}+b^{3/2}}{2})^{2/3} < \frac{a-b}{2\arctan[(a-b)/(a+b)]} . \quad (1.7)$$

Theorem 1.3 For $0<a<b$, then $M_{1/2} < P^I$.

That is $(\frac{a^{1/2}+b^{1/2}}{2})^2 < \frac{a-b}{2\arcsin[(a-b)/(a+b)]}$. (1.8)

Theorem 1.4 For $0<a<b$, then $M_{1/3} < \overline{h} < M_{1/2}$

That is $(\frac{a^{1/3}+b^{1/3}}{2})^3 \le \frac{a+4\sqrt{ab}+b}{6} \le (\frac{a^{1/2}+b^{1/2}}{2})^2$. (1.9)

From these we get a lengthened chain of inequalities:

Theorem 1.5 For $0<a<b$, then

$$a < D < M_{-2} < H < G < Q_{1/3} < L < M_{1/3} < \overline{h} < M_{1/2} < M_{\ln 2/\ln\pi} < P^I < h < M_{2/3} < I < M_{\ln 2}$$
$$< M_{7/10} < A < Z < M_{3/2} < P^{II} < g < T < M_{(\ln 2)/(\ln\pi/2)} < M_2 < B < M_3 < L_2 = C < b$$

(1.10)

II. Lemmas

Lemma 2.1: If $0 < H \le \beta \le \alpha \le A$, then

$$0 \le \alpha - \beta \le \alpha[(a-b)/(a+b)]^2 . \quad (2.1)$$

Proof: from $0 < H \le \beta, 0 < \alpha \le A$, we get $0 < \alpha H \le \beta A$, that is $\alpha \cdot \frac{H}{A} - \beta \le 0$,

that is

$\alpha \cdot 4ab/(a+b)^2 - \beta \le 0$, $\alpha \cdot [1-[(a-b)/(a+b)]^2] - \beta \le 0$, that is

$0 \le \alpha - \beta \le \alpha \cdot [(a-b)/(a+b)]^2$.

Lemma 2.2: If $0 < G \le \beta \le \alpha \le A$, then

$$0 \le \alpha - \beta \le \alpha \cdot (\sqrt{a}-\sqrt{b})^2/(a+b) . \quad (2.2)$$

Proof: from $0 < G \le \beta, 0 < \alpha \le A$, we get $0 < \alpha G \le \beta A$, that is $\alpha \cdot \frac{G}{A} - \beta \le 0$,

that is

$\alpha \cdot 2\sqrt{ab}/(a+b) - \beta \le 0$, $\alpha \cdot [1-(\sqrt{a}-\sqrt{b})^2/(a+b)] - \beta \le 0$, that is

$0 \le \alpha - \beta \le \alpha \cdot (\sqrt{a}-\sqrt{b})^2/(a+b)$.

Lemma 2.3: If $0 < H \le \beta \le \alpha \le C$, then

$$0 \le \alpha - \beta \le \alpha \cdot (a-b)^2/(a^2+b^2) . \quad (2.3)$$

Proof: from $0 < H \le \beta, 0 < \alpha \le C$, we get $0 < \alpha H \le \beta C$, that is $\alpha \cdot H/C - \beta \le 0$, that is

$\alpha \cdot \frac{2ab}{a^2+b^2} - \beta \le 0$, $\alpha \cdot [1-\frac{(a-b)^2}{a^2+b^2}] - \beta \le 0$, that is $0 \le \alpha - \beta \le \alpha \cdot \frac{(a-b)^2}{a^2+b^2}$.

Lemma 2.4: If $0 < H \le \beta \le \alpha \le g$, then

$$0 \le \alpha - \beta \le \alpha \cdot (a-b)^2/(a^2+ab+b^2) . \quad (2.4)$$

Proof: from $0 < H \le \beta, 0 < \alpha \le g$,we get $0 < \alpha H \le \beta g$, that is $\alpha \cdot H / g - \beta \le 0$, that is $\alpha \cdot 3ab/(a^2+ab+b^2) - \beta \le 0$, $\alpha \cdot [1-(a-b)^2/(a^2+ab+b^2)] - \beta \le 0$, that is $0 \le \alpha - \beta \le \alpha \cdot (a-b)^2/(a^2+ab+b^2)$.

III. Proof of Theorems 1.1-1.4

Proof of Theorem1.1: observe and study

$$M_p = (\frac{a^p+b^p}{2})^{\frac{1}{p}} \text{ and } I(a,b) = \begin{cases} \frac{1}{e}(\frac{a^a}{b^b})^{\frac{1}{a-b}}, a \ne b, \\ a, \quad a = b. \end{cases}$$

Let $x = a/b > 1$, then $a = bx$.

Take the natural logarithm, $\ln M_p = \ln(\frac{a^p+b^p}{2})^{\frac{1}{p}} = -\frac{1}{p}\ln\frac{2}{1+x^{-p}} + \ln a$,

$$\ln I(a,b) = \frac{1}{a-b}\ln\frac{a^a}{b^b} - 1 = \frac{\ln x}{x-1} + \ln a - 1 .$$

Let $f(x) = \frac{\ln x}{x-1} + \frac{1}{p}\ln\frac{2}{1+x^{-p}} - 1, x > 1$, then

$$f'(x) = \frac{1}{(x-1)^2}[1 - \frac{1}{x} - \ln x + \frac{(x-1)^2}{x(x^p+1)}] .$$

Let $g(x) = 1 - \frac{1}{x} - \ln x + \frac{(x-1)^2}{x(x^p+1)} = \frac{x^2-2x+1}{x^{p+1}+x} - \frac{1}{x} - \ln x + 1, x > 1$, $g(1) = 0$,

then $g'(x) = \dfrac{(1-p)x^{p+2} - x^{2p+1} + (2p-2)x^{p+1} + x^{2p} + (1-p)x^p + x^2 - x}{x^2(x^p+1)^2}$.

Let $h(x) = (1-p)x^{p+2} - x^{2p+1} + (2p-2)x^{p+1} + x^{2p} + (1-p)x^p + x^2 - x, x > 1$, $h(1) = 0$, then $h'(x) = (2-p-p^2)x^{p+1} - (2p+1)x^{2p} + (2p^2-2)x^p + 2px^{2p-1} + p(1-p)x^{p-1} + 2x - 1$, $h'(1) = 0$,

$$h''(x) = (2+p-2p^2-p^3)x^p - 2p(2p+1)x^{2p-1} + p(2p^2-2)x^{p-1} + (4p^2-2p)x^{2p-2}$$
$$+(-p^3+2p^2-p)x^{p-2} + 2 , \quad h''(1) = -6p + 4 .$$

Let $h''(1) \ge 0$, then $p \le \frac{2}{3}$. If $h''(1) < 0$, then there exists $h(x) < 0$, $f(x)$ will first increased and then decreased.

(i) If $p \ge \ln 2$, because $M_{\ln 2} > I(a,b)$, we know when $p \ge \ln 2$, $f(x)$ is constant negative.

(ii) If $2/3 < p < \ln 2$, when $x \to 1$, there exists $f(x) < 0$; when $x \to \infty$, there exists $f(x) > 0$.

(ii)If $p \le 2/3$, we unable to determine the monotony. Because $M_{2/3} < I(a,b)$, we know when $p \le 2/3$, then $f(x)$ is positive forever.

From(i)、(ii)、(iii),we know when $2/3 < p < \ln 2$, there does not exist $I(a,b)$'s indeed circles . So the upper bound for the $I(a,b)$ is $M_{\ln 2}$; and because $M_{2/3} < I(a,b)$, so the maximum infimum of $I(a,b)$ is $M_{2/3}$.

Via this exploration, we obtain $M_{\ln 2} > I(a,b) > M_{2/3}$ and prove that the range of p is the most precise.

Ratiocination: Because we can not infinitely narrow $I(a,b)$ indeed the world, it is speculated that the real p does not exist to make $M_p = I(a,b)$, whose proof is so easy that is omitted.

Proof of Theorem1.2: As for $Z < M_{3/2}$, we need to prove

$$\frac{a-b}{2ar\sinh[(a-b)/(a+b)]} < (\frac{a^{3/2}+b^{3/2}}{2})^{2/3}. \quad (3.1)$$

Let $t=\sqrt{\frac{a}{b}}>1$, then $\frac{a-b}{a+b}=\frac{t^2-1}{t^2+1}$,

then $\frac{t^2-1}{2ar\sinh[(t^2-1)/(t^2+1)]} < (\frac{t^3+1}{2})^{2/3}$ (3.2)

is equivalent to

$$ar\sinh[(t^2-1)/(t^2+1)] > \frac{(t^2-1)}{2^{1/3}(t^3+1)^{2/3}}, \quad (3.3)$$

let $f(t)=ar\sinh[(t^2-1)/(t^2+1)]-\frac{(t^2-1)}{2^{1/3}(t^3+1)^{2/3}}, t>1$,

then $f'(t)=\frac{2^{3/2}t}{(t^2+1)\sqrt{t^4+1}}-\frac{2^{2/3}t(t+1)}{(t^3+1)^{5/3}}, t>1$,

we just need to prove:

$$\frac{2^{3/2}t}{(t^2+1)\sqrt{t^4+1}} > \frac{2^{2/3}t(t+1)}{(t^3+1)^{5/3}}, \quad (3.4)$$

that is: $\frac{2^5}{(t^2+1)^6(t^4+1)^3} > \frac{(t+1)^6}{(t^3+1)^{10}}$,

is equivalent to:

$$2^5(t^3+1)^{10} > (t+1)^6(t^2+1)^6(t^4+1)^3. \quad (3.5)$$

Let $g(t)=10\ln(t^3+1)-3\ln(t^4+1)-6\ln(t^2+1)-6\ln(t+1)+5\ln 2$,

then
$$g'(t)=\frac{30t^2}{t^3+1}-\frac{12t^3}{t^4+1}-\frac{12t}{t^2+1}-\frac{6}{t+1}$$
$$=\frac{6(t^7-t^6-3t^5+t^4-t^3+3t^2-t-1)}{(t^4+1)(t^3+1)(t^2+1)}$$
$$=\frac{6(t-1)(t^6+2t^5-t^4-t^2+2t+1)}{(t^4+1)(t^3+1)(t^2+1)}$$
$$=\frac{6(t-1)[t^6+t^4(t-1)+t^2(t^3-1)+2t+1]}{(t^4+1)(t^3+1)(t^2+1)}>0,$$

so $g(t) > g(1) = 5\ln 2+10\ln 2-6\ln 2-6\ln 2-3\ln 2=0$.

Then prove: $M_{3/2} < P^{II}$,

We need to prove:

$$(\frac{a^{3/2}+b^{3/2}}{2})^{2/3} < \frac{a-b}{2\arctan[(a-b)/(a+b)]}. \quad (3.6)$$

Let $t=\sqrt{\frac{a}{b}}>1$， then $\frac{a-b}{a+b}=\frac{t^2-1}{t^2+1}$,

$$\frac{t^2-1}{2\arctan[(t^2-1)/(t^2+1)]}>(\frac{t^3+1}{2})^{2/3} \qquad (3.7)$$

is equivalent to

$$\arctan[(t^2-1)/(t^2+1)]<\frac{t^2-1}{2^{1/3}(t^3+1)^{2/3}}. \qquad (3.8)$$

Let $f(t)=\arctan[(t^2-1)/(t^2+1)]-\frac{t^2-1}{2^{1/3}(t^3+1)^{2/3}}, t>1$.

So $f'(t)=\frac{2t}{t^4+1}-\frac{2^{2/3}t(t+1)}{(t^3+1)^{5/3}}, t>1$.

We just need to prove：

$$\frac{2}{t^4+1}<\frac{2^{2/3}(t+1)}{(t^3+1)^{5/3}}, \qquad (3.9)$$

that is：

$$2(t^3+1)^{5/3}<2^{2/3}(t+1)(t^4+1) \qquad (3.10)$$

is equivalent to

$$2(t^3+1)^5<(t+1)^3(t^4+1)^3. \qquad (3.11)$$

Let $g(t)=3\ln(t^4+1)+3\ln(t+1)-5\ln(t^3+1)-\ln 2$,

then $g'(t)=\frac{2t^3}{t^4+1}+\frac{3}{t+1}-\frac{15t^2}{t^3+1}$

$$=-\frac{3(t-1)[(t^2-1)^2-2t^2]}{(t^4+1)(t^3+1)}$$

$$=-\frac{3(t-1)(t^2+\sqrt{2}t-1)(t^2-\sqrt{2}t-1)}{(t^4+1)(t^3+1)}$$

$$=-\frac{3(t-1)(t^2+\sqrt{2}t-1)(t-\frac{\sqrt{6}+\sqrt{2}}{2})(t+\frac{\sqrt{6}-\sqrt{2}}{2})}{(t^4+1)(t^3+1)}.$$

When $1<t<\frac{\sqrt{6}+\sqrt{2}}{2}$， then $g'(t)>0$,. $g(t)>g(1)=3\ln 2+3\ln 2-5\ln 2-\ln 2=0$;

When $t>\frac{\sqrt{6}+\sqrt{2}}{2}$， then $g'(t)<0$， $g(t)>\lim_{t\to\infty}g(t)>0$

We conclude that $g(t)>0$.

Proof of Theorem1.3: $M_{1/2}<P^I \Leftrightarrow (\frac{a^{1/2}+b^{1/2}}{2})^2<\frac{a-b}{2\arcsin[(a-b)/(a+b)]}$

$$\Leftrightarrow [\frac{(a/b)^{1/2}+1}{2}]^2<\frac{a/b-1}{2\arcsin[(a-b)/(a+b)]}. \qquad (3.12)$$

Let a and b be two positives, and $a/b>1$， let $t=\frac{a-b}{a+b}=\frac{a/b-1}{a/b+1}\in(0,1)$,

and $\frac{a}{b}=\frac{1+t}{1-t}$， then

$$[\frac{(a/b)^{1/2}+1}{2}]^2<\frac{a/b-1}{2\arcsin[(a-b)/(a+b)]}$$ is equivalent to

$$[\frac{[(1+t)/(1-t)]^{1/2}+1}{2}]^2<\frac{(1+t)/(1-t)-1}{2\arcsin t}\Leftrightarrow\frac{t}{\arcsin t}>\frac{1+\sqrt{1-t^2}}{2}. \quad (3.13)$$

Let $u=\arcsin t\in(0,\frac{\pi}{2})$, then $\frac{t}{\arcsin t}>\frac{1+\sqrt{1-t^2}}{2}$ is equivalent to

$$\frac{\sin u}{u}>\frac{1+\cos u}{2}\Leftrightarrow 2\sin u>u+u\cos u. \quad (3.14)$$

Let $f(u)=2\sin u-u-u\cos u$, $u\in(0,\frac{\pi}{2})$, then

$f^{'}(u)=2\cos u-1-\cos u+u\sin u=\cos u-1+u\sin u$,

$f^{''}(u)=-\sin u+\sin u+ucosu=ucosu>0$,

We conclude that $f^{'}(u)$ is monotone increasing in $(0,\frac{\pi}{2})$, so

$f^{'}(u)>f^{'}(0)=0$, so $f(u)$ is monotone increasing in $(0,\frac{\pi}{2})$, so

$f(u)>f(0)=0$, that is $2\sin u>u+u\cos u$.

Proof of Theorem1.4: we first prove the left part of the inequality

$$M_{1/3}<\bar{h}\Leftrightarrow(\frac{a^{1/3}+b^{1/3}}{2})^3\le\frac{a+4\sqrt{ab}+b}{6}.$$

Let $a=x^6,b=y^6$, the original inequality can be equivalent to

$$(\frac{x^2+y^2}{2})^3\le\frac{x^6+4x^3y^3+y^6}{6} \quad (3.15)$$

$$\Leftrightarrow x^6-9x^4y^2+16x^3y^3-9x^2y^4+y^6\ge 0$$

$$\Leftrightarrow(x^2+4xy+y^2)(x-y)^4\ge 0.$$

Then we prove the right part of inequality:

$$\bar{h}<M_{1/2}\Leftrightarrow\frac{a+4\sqrt{ab}+b}{6}\le(\frac{a^{1/2}+b^{1/2}}{2})^2.$$

Let $a=x^6,b=y^6$, the original inequality can be equivalent to

$$\frac{x^6+4x^3y^3+y^6}{6}\le(\frac{x^3+y^3}{2})^2 \quad (3.16)$$

$$\Leftrightarrow x^6-2x^3y^3+y^6\ge 0\Leftrightarrow(x^3-y^3)^2\ge 0.$$

IV. The application of various binary means

(i) some new Agarwal type inequalities

During the study, we also found a kind of interesting inequality--Agarwal inequality. In 1996, Agarwal put forward this interesting inequality $0\le L-H\le L[(a-b)/(a+b)]^2$, which is called Agarwal inequality.

In 2011, Yan-Li Bi obtained three similar Agarwal inequalities:

（1） $0\le E-H\le E[(a-b)/(a+b)]^2$,

（2） $0 \le E^{-1} - L \le E^{-1}[(a-b)/(a+b)]^2$,

（3） $0 \le L - G \le L[(a-b)/(a+b)]^2$.

Among them, (2) is proved incorrect while only (1) ,(3) are correct. In fact, with the support of following simple lemmas and obtained conclusions, we can quickly prove Agarwal inequality.

（1） $0 \le I - L \le I[(a-b)/(a+b)]^2$.

（2） $0 \le I - M_{2/3} \le I[(a-b)/(a+b)]^2$.

（3） $0 \le M_{2/3} - h \le M_{2/3}[(a-b)/(a+b)]^2$.

（4） $0 < M_{7/10} - M_{\ln 2} < M_{7/10}[(a-b)/(a+b)]^2$.

（5） $0 \le I - L \le I(\sqrt{a} - \sqrt{b})^2/(a+b)$.

（6） $0 \le L - H \le L(a-b)^2/(a^2+b^2)$.

（7） $0 \le M_3 - M_2 \le M_3(a-b)^2/(a^2+b^2)$.

（8） $0 \le L - H \le L(a-b)^2/(a^2+ab+b^2)$.

Proof：according to lemma2.1-2.4and obtained conclusions ：

$a < D < M_{-2} < H < G < Q_{1/3} < L < M_{1/3} < \bar{h} < M_{1/2} < P^I < h < M_{2/3} < I < M_{\ln 2} < M_{7/10}$ $< A < Z < P^{II} < g < M_2 < B < M_3 < L_2 = C < b$,then the above chain of inequality is established. Some interested colleagues can continue to explore.

(ii) find some mistakes in the former paper

Meanwhile, we found some mistakes in the former paper.

obtained this conclusion(see [1]):

$$H < E < L^{-1} < G < Q_{1/3} < L < E^{-1} < A .$$

In fact , this chain of inequality is incorrect

$E^{-1} < A$ is incorrect, such as, $a = 1, b = 2$, $E^{-1} = 2/e < 3/2 = A$; but $a = 0.1, b = 0.2$,

$E^{-1} = 20/e > 3/20 = A$.

$L < E^{-1}$ is also incorrect , that means $E < L^{-1}$ is incorrect, such as, $a = 1, b = 2$,

$L = 1/\ln 2 \approx 1.44 > 2/e = E^{-1}$; but $a = 0.1, b = 0.2$,

$L = 0.1/\ln 2 \approx 0.144 < 20/e = E^{-1}$. $L < E^{-1}$ is incorrect，which is equivalent to $L^{-1} < E$ is incorrect. But this chain of inequality can be correct to

Theorem 1.6 $H < E < G < Q_{1/3} < L < A$.

Because $H < E < L^{-1} < G < Q_{1/3} < L < E^{-1} < A$ is incorrect , paper[11]'s Agarwal inequality is also incorrect, which means Agarwal inequality: $0 \le E^{-1} - L \le E^{-1}[(a-b)/(a+b)]^2$ is incorrect. We also can find something is incorrect in the process of proof. When (see[11]) proved $0 \le D^{-1} - L \le D^{-1}[(a-b)/(a+b)]^2$, let $x = b/a > 1$, it tried to transform incorrectly to $[(x+1)^2(x-1)]/(x^{x/(x-1)}\ln x) > 4/e$ .In fact: $0 \le D^{-1} - L \le D^{-1}[(a-b)/(a+b)]^2$ can't be equivalent to $[(x+1)^2(x-1)]/(x^{x/(x-1)}\ln x) > 4/e$,but should be equivalent to $[a^2(x+1)^2(x-1)]/(x^{x/(x-1)}\ln x) > 4/e$. Therefore, Bi proved an incorrect conclusion : $0 \le D^{-1} - L \le D^{-1}[(a-b)/(a+b)]^2$.

(see[12]) put forward $M_{3/10} < L$, when $M_{3/10} < L$ is incorrect. That means $(\frac{a^{3/10}+b^{3/10}}{2})^{10/3} < \frac{a-b}{\ln a - \ln b}$ is incorrect, and $(\frac{x^{3/10}+1}{2})^{10/3} < \frac{x-1}{\ln x}$ ($x = \frac{a}{b} > 1$) is incorrect. With the support of maple, we find when $a = 600, b = 1$, $(\frac{a^{3/10}+b^{3/10}}{2})^{10/3} > \frac{a-b}{\ln a - \ln b}$.

(iii) The application of mean inequality in the College Entrance Examination

Recently, questions about two positives' mean inequality especially about the arithmetic geometric mean inequality and logarithmic mean inequality are common in the College Entrance Examination.

1. The application of the arithmetic geometric mean inequality in the College Entrance Examination

The inequality between arithmetic mean and geometric mean is applied in function, derivative, trigonometric function, sequence , inequality, solid geometry as well as analytic geometry and so on, such as the sixteenth question in 2014 Zhejiang version, the sixteenth question in 2014 national version, the ninth question in 2014 Hubei version and the sixteenth question in 2014 Hubei version etc.

2. The application of the logarithmic mean inequality in the College Entrance Examination

Logarithmic mean inequality, in the College Entrance Examination, often appears in function inequality proving, permanent inequality establishment and functional solution problem.

The above inequalities are common in the College Entrance Examination, such as 2014 national version , 2014 Shandong version and 2013 Shanxi version.

For $0 < a < b$, then

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2} \Leftrightarrow \sqrt{x} < \frac{x-1}{\ln x} < \frac{x+1}{2}.$$

Specially, when $x > 1$, the above one can be transformed into

$$\frac{2(x-1)}{x+1} < \ln x < \frac{x-1}{\sqrt{x}}.$$

Because, $\sqrt{x} > 1$, so when $x > 1$, $\ln x < x - 1$.

Let $x = x - 1$, then

when $x > -1$, $\ln(x+1) \le x$.

The above examples all are based on the inequality of logarithmic mean, which are easily solved by using the inequality of logarithmic mean.

V. Project prospect

Different kinds of means between two positives are the most basic but important among all means. The means human studied date back to BC 500 years of Pythagoras era, which had the concepts of two positives' arithmetic mean and geometric mean. In the twentieth century, many branches of mathematics appeared gradually, leading to the discovery of different kinds of means, and were widely applied in practice. Hence, studying binary means or even multiple means has broad prospects, especially the research of the relationship between various kinds of the means as well as the practical application, including the College Entrance Examination and the maths competition.

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Appendix: Elementary Proof of Theorem1.2

As for $Z < M_{3/2}$, we need to prove

$$\frac{a-b}{2ar\sinh[(a-b)/(a+b)]} < (\frac{a^{3/2}+b^{3/2}}{2})^{2/3}.$$

Let $a > b$, then $\frac{a}{b} > 1$, let $t = \frac{a-b}{a+b} = \frac{a/b-1}{a/b+1}$, then we easily get $0 < t < 1$, and $\frac{a}{b} = \frac{1+t}{1-t}$, then original inequality can be equivalent to

$$\frac{t}{ar\sinh t} < [\frac{(1+t)^{3/2}+(1-t)^{3/2}}{2}]^{2/3} \quad (1)$$

Let $u = ar\sinh t$, then $t = \sinh u$,

then the above can be equivalent to

$$\frac{\sinh u}{u} < [\frac{(1+\sinh u)^{3/2}+(1-\sinh u)^{3/2}}{2}]^{2/3}. \quad (2)$$

We recognize $\sinh u = \frac{e^u - e^{-u}}{2}$, so let $v = e^u \in (1, \sqrt{2}+1)$, $u = \ln v$,

then $\sinh u = \frac{v - 1/v}{2}$, the above can be equivalent to

$$\frac{v^2-1}{2\ln v} < [\frac{(2v+v^2-1)^{3/2}+(2v-v^2+1)^{3/2}}{2}]^{2/3}. \quad (3)$$

Let $p = 2v+v^2-1$, $q = 2v-v^2+1$,

from the logarithmic mean we know: $\frac{v-1}{\ln v} < \frac{v+1}{2} (v>1)$, so for proving the above , we need to prove:

$$(\frac{v+1}{2})^2 < (\frac{p^{3/2}+q^{3/2}}{2})^{2/3} \Leftrightarrow 4(p^{3/2}+q^{3/2}) > (v+1)^3. \quad (4)$$

Let $f(v) = 4(p^{3/2}+q^{3/2}) - (v+1)^3, v \in (1, 1+\sqrt{2})$,

then $f'(v) = 6[p^{1/2}(2+2v)+q^{1/2}(2-2v)] - 3(v+1)^2$

$$= 12 \cdot \frac{p(v+1)^2 - q(v-1)^2}{p^{1/2}(v+1)+q^{1/2}(v-1)} - 3(v+1)^2 \geq \frac{24(v^4+4v^2-1)}{\sqrt{(p+q)[(v+1)^2+(v-1)^2]}} - 3(v+1)^2$$

$$\geq 3[\frac{8(v^4+4v^2-1)}{2v+(v^2+1)} - (v+1)^2] = 3 \cdot \frac{7v^4-4v^3+26v^2-4v-9}{2v+(v+1)^2}$$

$$> 3 \cdot \frac{4v^3(v-1)+4v(v-1)+9(v+1)(v-1)}{2v+(v+1)^2} > 0,$$

We conclude that $f(v)$ is monotone increasing in $(1, 1+\sqrt{2})$,

so $f(v) \geq f(1) = 0$.

Then prove: $M_{3/2} < P^{II}$,

We need to prove: $(\frac{a^{3/2}+b^{3/2}}{2})^{2/3} < \frac{a-b}{2\arctan[(a-b)/(a+b)]}$.

The original inequality can be equivalent to

$$[\frac{(1+t)^{3/2}+(1-t)^{3/2}}{2}]^{2/3} < \frac{t}{2\arctan t}. \tag{5}$$

Let $u=\arctan t$， then $t=\tan u$. $u\in(0,\frac{\pi}{4})$,

the above can be equivalent to $[\frac{(1+\tan u)^{3/2}+(1-\tan u)^{3/2}}{2}]^{2/3} < \frac{\tan u}{u}$

$$\Leftrightarrow [\frac{(\cos u+\sin u)^{3/2}+(\cos u-\sin u)^{3/2}}{2}]^{2/3} < \frac{\sin u}{u}. \tag{6}$$

According to paper[10]， $\frac{\sin u}{u} > \frac{1+2\cos u}{2+\cos u}$， we just need to prove

$$[\frac{(\cos u+\sin u)^{3/2}+(\cos u-\sin u)^{3/2}}{2}]^{2/3} < \frac{1+2\cos u}{2+\cos u}$$

$$\Leftrightarrow (\cos u+\sin u)^{3/2}+(\cos u-\sin u)^{3/2} < 2(\frac{1+2\cos u}{2+\cos u})^{3/2}. \tag{7}$$

Let $x=(\cos u+\sin u)^{1/2}, y=(\cos u-\sin u)^{1/2}$， then $\cos u=\frac{x^2+y^2}{2}$,

$x^4+y^4=2, 0<y<1<x<\sqrt[4]{2}$.

Then $x^3+y^3 < 2(\frac{2+2x^2+2y^2}{4+x^2+y^2})^{3/2}$

$$\Leftrightarrow (x^3+y^3)^2(4+x^2+y^2)^3 < 32(1+x^2+y^2)^3. \tag{8}$$

According to cauchy inequality, $(x^3+y^3)^2 \le (x^2+y^2)(x^4+y^4)$,

that is $(x^2+y^2)(4+x^2+y^2)^3 < 16(1+x^2+y^2)^3$. (9)

Let $m=(x^2+y^2)$,for $(x^2+y^2)=2\cos u$, then $m\in(\sqrt{2},2)$.

The old one is equivalent to

$$(4+m)^3 < 16(1+m)^3, \quad \Leftrightarrow m^4-4m^3+16m-16<0 \tag{10}$$

Let $f(m)=m^4-4m^3+16m-16, m\in(\sqrt{2},2)$， $f(2)=0$，

then $f'(m)=4m^3-12m^2+16$， $f'(2)=0$,

$f''(m)=12m^2-24m$， $f''(2)=0$， $f'''(m)=24m-24>0$,

so $f''(m)<f''(2)=0$; $f'(m)>f'(2)=0$; $f(m)<f(2)=0$.

We conclude that when $m\in(\sqrt{2},2)$, $f(m)<0$.

The original one obtains a proof.