New aspect of chocolate games II

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1 New aspect of chocolate games

In this paper we study Chocolate Bar game. The definition of Chocolate Bar game is very simple, but this game is very rich in its contents. The most well known example of mathematical structure that has a simple definition and very rich contents is the theory of integers. The definition of integers is very simple, but its theory is very beautiful and full of unsolved problems.

The world of Chocolate Bar game may be smaller than that of integers in the scale and richness of its contents, but it is also a world that is worth exploring.
The team of students and a teacher in Kwansei Gakuin Highs school has been doing mathematics research for 20 years, and have published papers on probability theory and discrete mathematics. They began to study combinatorial game theory about 10 years ago. Since then they have studied Chocolate Bar game. As for the paper published by this team of students and the teacher see [7], [8],[10],[11], [12],[13], [14], [15], [16], [18], [9] and [17]. They discovered that certain types of Chocolate Bar games have Grundy numbers that are expressed by Nim-sum, and after that they have been trying hard to find all Chocolate Bar games that have Grundy numbers expressed by Nim-sum, i.e.,the necessary and sufficient conditions for Chocolate Bar to have Grundy numbers expressed by nim-sum.
In this paper the authors present a very general condition for Chocolate Bar to have Grundy numbers expressed by Nim-sum. Unfortunately these are still sufficient conditions for Chocolate Bars to have such a Grundy number, but this condition is general enough to include the resulf ot the research on chocolates games that are in previous research of the team (Yau Award paper 2012,[10] and [17]) and completely new chocolate games that have not been discovered before. The member of Yau Award 2012 submitted their paper to a math journal. Please see the comment of the referee in Section 6, since this shows the importance of this year's research.

In this paper the authors also present formulas for Grundy number of the Chocolate Bar that satisfies the inequality $y \leq z$. The members of the research team have been studying this type of Chocolate Bar for more than eight years, and at last they discovered these formulas and could prove them.
The author have also discovered that these formulas can be applied to many types of Chocolate Bar, and to find all the chocolate that have these kind of formulas became a new topic of research.
In this way a new discovery leads to new unsolved problems, and this shows the richness of the world of Chocolate Bar problem.
There is another similarity between Chocolate Bar game and the theory of integers. Chocolate Bar game is a natural generalization of the game of Nim, and hence it can be a corner stone of the combinatorial game theory in general. A very simple but deep theory can be a corner stone of a big branch of mathematics.
In this way the authors are not only studying interesting problems called Chocolate Bar game, but also creating a new field of research called Chocolate Bar game.

The original Chocolate Bar game, see Robin [2], had a rectangular bar of chocolate with one corner poisoned. Each player in turn breaks the bar in a straight line along the grooves and eats the piece he breaks off. The player who eats the poisoned square loses. Since the horizontal and vertical grooves are independent, a $m \times n$ bar (of squares) is equivalent to a game of NIM with up to four heaps of sizes equal to the number of grooves above, below, to the left and to the right of the poisoned square. For example, please try playing with the bar in Figure 1.1. In this paper we consider other shaped bars as in Figures 1.2 through 1.4 where the gray blocks are sweet chocolate that can be eaten, and the black block is the poisoned square. In these cases, a vertical break can reduce the number of horizontal breaks. We can still think of the game as being played with heaps but now a move may change more than one heap.
There are other types of chocolate games, and one of the most well known is CHOMP. Although many people have studied CHOMP, the winning strategy has not been discovered. As to a good result of the research of CHOMP, see Zeilberger [3].

Chocolate Bar game is more promising theme of research than CHOMP, since there are a lot of games that have formulas for Grundy numbers and there are a lot of unsolved problems.

Example 1.1. *Examples of Chocolate Bar games.*

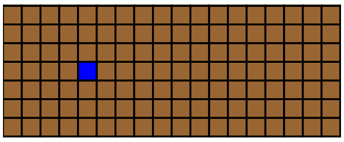


Figure 1.1.

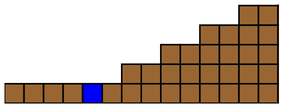


Figure 1.2.

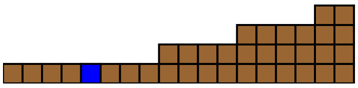


Figure 1.3.

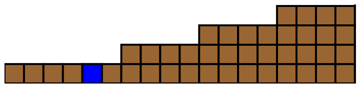


Figure 1.4.

For completeness, we give a quick review of the necessary game theory concepts, see [4] for more details.

Definition 1.1. *We define the nim-sum $x \oplus y$ by*

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i, \tag{1.1}$$

where $w_i = x_i + y_i \pmod 2$.

Since Chocolate Bar games are impartial games without draws there will only be two outcome classes.

Definition 1.2. (*a*) *$\mathcal{N}$-positions, from which the next player can force a win, as long as he plays correctly at every stage.*
(*b*) *$\mathcal{P}$-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.*

Definition 1.3. *The disjunctive sum of two games, denoted $\mathbf{G} + \mathbf{H}$ is a super-game where a player may move either in $\mathbf{G}$ or in $\mathbf{H}$ but not both.*

In Figures 1.2 through 1.4, each game is the disjunctive sum of the single strip of chocolate to the left and the Chocolate Bar to the right of the poisoned square.
Here the height of each column of the Chocolate Bar to the left is 1.

Definition 1.4. *For any position $\mathbf{p}$ of a game $\mathbf{G}$, there is a set of positions that can be reached by making precisely one move in $\mathbf{G}$, which we will denote by $move(\mathbf{p})$.*

Remark 1.1. *As to the examples of move please see Example 1.5.*

Definition 1.5. (1) *The minimum excluded value (mex) of a set, S, of non-negative integers is the least non-negative integer which is not in S.*
(2) *Each position $\mathbf{p}$ of a impartial game has an associated Grundy number, and we denoted it by $G(\mathbf{p})$. Grundy number is found recursively: $G(\mathbf{p}) = mex\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.*

The power of the Sprague-Grundy theory for impartial games is contained in the next result.

Theorem 1.1. *Let $\mathbf{G}$ and $\mathbf{H}$ be impartial games.*
[1] *Then for any position $\mathbf{g}$ we have $G(\mathbf{g}) = 0$ if and only if $\mathbf{g}$ is a P-position.*
[2] *The Grundy number of a position $\{\mathbf{g}, \mathbf{h}\}$ in the game $\mathbf{G} + \mathbf{H}$ is $G(\mathbf{G}) \oplus G(\mathbf{H})$.*

For a proof of this theorem see [4].

Definition 1.6. *Let x be a non-negative integer, and write it in base 2, so $x = \sum_{i=0}^{n} x_i 2^i$ with $x_i \in \{0, 1\}$.*
We denote $\sum_{i=n-k}^{n} x_i 2^{i-n+k}$ by $\{x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}\}$.

Note that the expression used in Definition 1.6 is new, and this is introduced in this paper by the authors to make proofs more simple. Although proofs in this paper are still complicated, they become a lot more complicated if we do not use these expressions.
When we use this expression $\{x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}\}$, we assume that $x = \sum_{i=0}^{n} x_i 2^i$.

Example 1.2. *Let $x = 45$,and write it in base 2, so $45 = x = 1\times 2^5+0\times 2^4+1\times 2^3+1\times 2^2+0\times 2^1+1\times 2^0$.*
Then by Definition 1.6 $\{x_5, x_4, x_3, x_2, x_1, x_0\} = \{1, 0, 1, 1, 0, 1\} = x = 45$.
$\{x_5\} = \{1\} = x_5 \times 2^{5-5} = 1 \times 1 = 1$.
$\{x_5, x_4, x_3\} = \{1, 0, 1\} = x_5 \times 2^{5-3} + x_4 \times 2^{4-3} + x_3 \times 5^{3-3} = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 5$.
Similarly Let $y = 26$,and write it in base 2, so $y = 26 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$.
Then by Definition 1.6 $\{y_4, y_3, y_2\} = \{1, 1, 0\} = y_4\times 2^{4-2}+y_3\times 2^{3-2}+y_2\times 2^{2-2} = 1\times 2^2+1\times 2^1+0\times 2^0 = 6$.
$\{y_4, y_3\} = \{1, 1\} = y_4 \times 2^{4-3} + y_3 \times 2^{3-3} = 1 \times 2^1 + 1 \times 2^0 = 3$.

In this paper the authors present Grundy number of Chocolate Bar. For a general bar, the strategies seem complicated. We focus on bars that grow regularly in height.

Definition 1.7. *Let f be a function that satisfies the following conditions.*
(1) $f(0) = f(1) = 0$ and $f(t) \in Z_{\geq 0}$ for $t \in Z_{\geq 0}$.
(2) f is monotonically increasing,i.e., we have $f(u) \leq f(v)$ for $u, v \in Z_{\geq 0}$ with $u \leq v$.
(3) If $s = f(t)$, then we have (3.1) or (3.2).
(3.1) $2s = f(2t) = f(2t + 1)$.
(3.2) $2s + 1 = f(2t) = f(2t + 1)$.

Remark 1.2. *Throughout this paper we assume that the function f satisfies the conditions in Definition 1.7.*
The condition (3) of Definition 1.7 is the same as the following condition (a).
Note that $2s = \{y_n, y_{n-1}, ..., y_{n-k}, 0\}$ and $2s + 1 = \{y_n, y_{n-1}, ..., y_{n-k}, 1\}$ for $s = \{y_n, y_{n-1}, ..., y_{n-k}\}$, and $2t = \{z_n, z_{n-1}, ..., z_{n-k}, 0\}$ and $2t + 1 = \{z_n, z_{n-1}, ..., z_{n-k}, 1\}$ for $t = \{z_n, z_{n-1}, ..., z_{n-k}\}$.
(a) If $\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-k}\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}\})$, then we have (a.1) or (a.2)
(a.1) $\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-k}, 0\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, 0\}) = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, 1\})$
(a.2) $\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-k}, 1\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, 0\}) = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, 1\})$

In Lemma 1.1, Lemma 1.2 and Lemma 1.3 we present examples of functions that satisfy (1),(2) and (3) of Definition 1.7.
Note that these are only examples, and there are a lot of functions that satisfy the conditions of Definition 1.7.

Lemma 1.1. *Let $f(t) = \lfloor \frac{t}{k} \rfloor$ for an even number k. Then $f(t)$ satisfies the conditions (1), (2) and (3) of Definition 1.7.*

Proof. It is clear that $f(t)$ satisfies (1) and (2).
We prove (3). If $s = f(t) = \lfloor \frac{t}{k} \rfloor$, then $sk \leq t < (s + 1)k$, and hence we have the following (3.1) and (3.2).
(3.1) If $sk \leq t < (s + \frac{1}{2})k$, then $2sk \leq 2t < (2s + 1)k$. Since k is even, we also have $2sk \leq 2t + 1 < (2s + 1)k$.

Therefore we have $2s \leq \frac{2t}{k}, \frac{2t+1}{k} < 2s+1$, which implies that $f(2t) = \lfloor \frac{2t}{k} \rfloor = 2s$ and $f(2t+1) = \lfloor \frac{2t+1}{k} \rfloor = 2s$.
(3.2) If $(s + \frac{1}{2})k \leq t < (s+1)k$, then $(2s+1)k \leq 2t < (2s+2)k$. Since $(2s+2)k$ is even, we also have $(2s+1)k \leq 2t+1 < (2s+2)k$. Therefore we have $2s+1 \leq \frac{2t}{k}, \frac{2t+1}{k} < 2s+2$ which implies that $f(2t) = \lfloor \frac{2t}{k} \rfloor = 2s+1$ and $f(2t+1) = \lfloor \frac{2t+1}{k} \rfloor = 2s+1$.

□

Lemma 1.2. *Let n be a natural number and Let $f(0) = f(1) = ... = f(2^n - 1) = 0$ and $f(t) = 2^{\lfloor log_2 t \rfloor - n}$ for $t \geq 2^n$. Then $f(t)$ satisfies the conditions (1), (2) and (3) of Definition 1.7.*

Proof. It is clear that $f(t)$ satisfies (1) and (2).
Let $s = 0 = f(t)$. Then we have two cases (a) and (b).
(a) If $t < 2^{n-1}$, then $2t, 2t+1 < 2^n$ and $f(2t) = f(2t+1) = 0 = 2s$.
(b) If $2^{n-1} \leq t < 2^n$, then $2^n \leq 2t, 2t+1 < 2^{n+1}$ and $f(2t) = 2^{\lfloor log_2 2t \rfloor - n} = 2^0 = 1 = 2s+1$. Similarly $f(2t+1) = 2^{\lfloor log_2 (2t+1) \rfloor - n} = 2^0 = 1 = 2s+1$.
(c) Next we suppose that $t \geq 2^n$ and $s = f(t) = 2^{\lfloor log_2 t \rfloor - n}$. Then $f(2t) = 2^{\lfloor log_2 2t \rfloor - n} = 2^{\lfloor log_2 t \rfloor - n+1} = 2s$. Let $a = \lfloor log_2 2t \rfloor$, then $2^a \leq 2t < 2^{a+1}$ which also implies that $2^a \leq 2t+1 < 2^{a+1}$. Therefore $a = \lfloor log_2 (2t+1) \rfloor$ and $f(2t+1) = 2^{\lfloor log_2 (2t+1) \rfloor - n} = 2s$. □

Lemma 1.3. *We define a function f by the following way.
Let $f(0) = 0$ and for a positive integer t let $f(t)$*

$$= \begin{cases} \frac{2^{\lfloor \log_2 t \rfloor + 1} - 1}{3} & \text{when } 2^n \leq t \leq 2^{n+1} \text{ for an odd number } n. \quad (1.2) \\ \frac{2^{\lfloor \log_2 t \rfloor + 1} - 2}{3} & \text{when when } 2^n \leq t \leq 2^{n+1} \text{ for an even number } n. \quad (1.3) \end{cases}$$

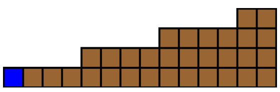
Then $f(t)$ satisfies the conditions (1), (2) and (3) of Definition 1.7.

Proof. First we assume that $2^n \leq t < 2^{n+1}$ and n is odd. Then by (1.2) we have $f(t) = \frac{2^{n+1}-1}{3}$.
Then $2^{n+1} \leq 2t, 2t+1 < 2^{n+2}$ and $n+1$ is even, and hence $f(2t) = f(2t+1) = \frac{2^{n+2}-2}{3} = 2 \times \frac{2^{n+1}-1}{3} = 2f(t)$.
Next we assume that $2^n \leq t < 2^{n+1}$ and n is even. Then by (1.3) we have $f(t) = \frac{2^{n+1}-2}{3}$.
Then $2^{n+1} \leq 2t, 2t+1 < 2^{n+2}$ and $n+1$ is odd, and hence $f(2t) = f(2t+1) = \frac{2^{n+2}-1}{3} = 2 \times \frac{2^{n+1}-2}{3} + 1 = 2f(t)+1$. □

Definition 1.8. *Let f be the function that satisfies the conditions in Definition 1.7.
For $y, z \in Z_{\geq 0}$ the Chocolate Bar will consist of $z+1$ columns where the 0th column is the poison square and the height of the i-th column is $t(i) = \min(f(i), y) + 1$ for i = 0,1,...,z. We will denote this by $CB(f, y, z)$. Thus the width of the chocolate is determined by the value of z, and the height of the i-th column is determined by the value of $\min(f(i), y) + 1$ that is determined by f and y.*

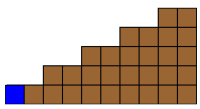
Remark 1.3. *In Reference [1] chocolate bars that satisfy the inequality $y \leq \lfloor \frac{z}{k} \rfloor$ is treated, and by Lemma 1.1 it is a special case of the bar defined in Definition 1.8.*

Example 1.3. *Examples of Chocolate Bar games $CB(f, y, z)$.*



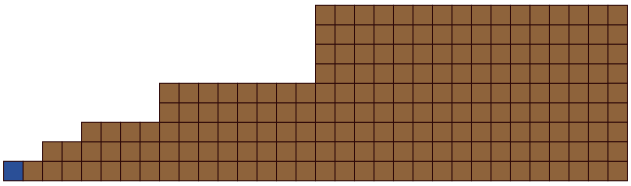
$CB(f, 3, 13)$ $f(t) = \lfloor \frac{t}{4} \rfloor$

Figure 1.5.



$CB(f, 4, 9)$ $f(t) = \lfloor \frac{t}{2} \rfloor$

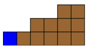
Figure 1.6.



$CB(f, 8, 31)$ $f(0) = f(1) = 0$ *and* $f(t) = 2^{\lfloor log_2 t \rfloor - 1}$ *for* $t > 1$.

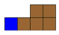
Figure 1.7.

Example 1.4. *Here we have four examples of positions of chocolates that appear when we play the chocolate game of Fig. 1.8. These chocolates are* $CB(f, y, z)$*, where* $f(t) = \lfloor \frac{t}{2} \rfloor$*.*



$CB(f, 2, 5)$

Figure 1.8.

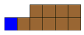


$CB(f, 1, 3)$

Figure 1.9.

$CB(f, 0, 5)$

Figure 1.10.



$CB(f, 1, 5)$

Figure 1.11.

Example 1.5. *Here we fix a function* f *that satisfies the conditions of Definition 1.7, and denote* $CB(f, y, z)$ *by* $\{y, z\}$*.*
We study the function $move(\{y, z\})$ *using examples of positions in Example 1.4. If we start with the position* $\{y, z\} = \{2, 5\}$ *and reduce* $z = 5$ *to* $w = 3$*, then the* y*-coordinate (the first coordinate) will be* $\min(2, \lfloor 3/2 \rfloor) = \min(2, 1) = 1$.
Therefore we have $\{1, 3\} \in move(\{2, 5\})$*. It is easy to see that* $\{0, 5\} \in move(\{2, 5\})$*.*

In our proofs, it will be useful to have the disjunctive sum of a Chocolate bar to the right of the poisoned square and a single strip of chocolate to the left, as in Figures 1.2, 1.3 and 1.4. We will denote such a position by $\{x, y, z\}$ where x is the maximum number moves in the strip on the left, y is the maximum number of vertical moves in the bar and z the maximum number of horizontal moves. See Figures 1.2, 1.3 and 1.4 for examples.
As for the disjunctive sum of the chocolate game with $CB(f, y, z)$ to the right of the poisoned square and a single strip of chocolate to the left, we will show that the $\mathcal{P}$-positions are when $x \oplus y \oplus z = 0$ so that Grundy number of the Chocolate Bar $CB(f, y, z)$ to the right is $x = y \oplus z$.

Lemma 1.4. *Let* $w \in Z_{\geq 0}$*, and write it in base 2, so* $w = \sum_{i=0}^{n} w_i 2^i$ *with* $w_i \in \{0, 1\}$*. Then there exists* $v \in Z_{\geq 0}$ *such that* $v = \sum_{i=0}^{n} v_i 2^i$ *with* $v_i \in \{0, 1\}$ *and for* $i = n, n-1, n-2, ..., 2, 1, 0$
$\{v_n, v_{n-1}, v_{n-2}, ..., v_i\}$
$= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_i\})$.
In particular $v = f(w)$*.*

Proof. We construct v recursively. By Definition 1.7 $f(0) = f(1) = 0$, and hence we let $v_n = 0 = f(w_n) = f(\{w_n\})$, where $w_n = 0$ or 1.
Suppose that there exist $v_{n-1}, v_{n-2}, ..., v_{n-k}$ such that for $i = n, n-1, ..., n-k$ $\{v_n, v_{n-1}, v_{n-2}, ..., v_i\}$

$= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_i\})$.
By Definition 1.7 and Remark 1.2
we have two equations (1.4) and (1.5) or two equations (1.6) and (1.7).

$$\begin{aligned} &\{v_n, v_{n-1}, v_{n-2}, ..., v_{n-k}, 0\} \\ &= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, 0\}) \quad (1.4) \\ &= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, 1\}). \quad (1.5) \end{aligned}$$

$$\begin{aligned} &\{v_n, v_{n-1}, v_{n-2}, ..., v_{n-k}, 1\} \\ &= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, 0\}) \quad (1.6) \\ &= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, 1\}). \quad (1.7) \end{aligned}$$

Since $w_{n-k-1} = 0$ or 1, by equations (1.4), (1.5), (1.6) and (1.7) we have the following (1.8) or (1.7).

$$\begin{aligned} &\{v_n, v_{n-1}, v_{n-2}, ..., v_{n-k}, 0\} \\ &= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, w_{n-k-1}\}). \quad (1.8) \end{aligned}$$

$$\begin{aligned} &\{v_n, v_{n-1}, v_{n-2}, ..., v_{n-k}, 1\} \\ &= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, w_{n-k-1}\}). \quad (1.9) \end{aligned}$$

If we have equations (1.8), then we let $v_{n-k-1} = 0$.
If we have equations (1.7), then we let $v_{n-k-1} = 1$.
Therefore we have

$$\{v_n, v_{n-1}, v_{n-2}, ..., v_{n-k}, v_{n-k-1}\} = f(\{w_n, w_{n-1}, w_{n-2}, ..., w_{n-k}, w_{n-k-1}\}).$$

In this way we define v_i such that $\{v_n, v_{n-1}, v_{n-2}, ..., v_i\}$
$= f(\{w_n, w_{n-1}, w_{n-2}, ..., w_i\})$ for $i = n, n-1, n-2, ..., 2, 1, 0$.
When $i = 0$, we have $v = f(w)$. □

Lemma 1.5. *(a)*

$$y = f(z) \quad (1.10)$$

if and only if

$$\{y_n, y_{n-1}, ..., y_i\} = f(\{z_n, z_{n-1}, ..., z_i\}) \text{ for } i = n, n-1, ..., 0. \quad (1.11)$$

(b)

$$y > f(z) \quad (1.12)$$

if and only if there exists k such that

$$\{y_n, y_{n-1}, ..., y_i\} = f(\{z_n, z_{n-1}, ..., z_i\}) \text{ for } i = n, n-1, ..., n-k \quad (1.13)$$

and

$$\{y_n, y_{n-1}, ..., y_{n-k}, 0\} = f(\{z_n, z_{n-1}, ..., z_{n-k}, z_{n-k-1}\}) \text{ and } y_{n-k-1} = 1. \quad (1.14)$$

(c)

$$y < f(z) \quad (1.15)$$

if and only if there exists k such that

$$\{y_n, y_{n-1}, ..., y_i\} = f(\{z_n, z_{n-1}, ..., z_i\}) \text{ for } i = n, n-1, ..., n-k \quad (1.16)$$

and

$$\{y_n, y_{n-1}, ..., y_{n-k}, 1\} = f(\{z_n, z_{n-1}, ..., z_{n-k}, z_{n-k-1}\}) \text{ and } y_{n-k-1} = 0. \quad (1.17)$$

Proof. $[I.a]$ We prove that (1.11) implies (1.10).
We suppose (1.11). Let $i = 0$, then $y = \{y_n, y_{n-1}, y_{n-2}, ..., y_0\}$
$= f(\{z_n, z_{n-1}, z_{n-2}, ..., z_0\}) = f(z)$.
$[I.b]$ We prove that (1.13) and (1.14) imply (1.12).
We suppose (1.13) and (1.14). Let $v = f(z)$. Then by Lemma 1.4 we have

$$\begin{aligned} &\{v_n, v_{n-1}, v_{n-2}, ..., v_i\} \\ &= f(\{z_n, z_{n-1}, z_{n-2}, ..., z_i\}) \end{aligned} \quad (1.18)$$

for $i = n, n-1, ..., 1, 0$.
By (1.13), (1.14) and (1.18) $y_i = v_i$ for $i = n, n-1, ..., n-k$ and $v_{n-k-1} = 0 < 1 = y_{n-k-1}$.
Clearly we have $y > v = f(z)$.
$[I.c]$ We prove that (1.16) and (1.17) imply (1.15).
We suppose (1.16) and (1.17).
Let $v = f(z)$. Then by Lemma 1.4 we have

$$\begin{aligned} &\{v_n, v_{n-1}, v_{n-2}, ..., v_i\} \\ &= f(\{z_n, z_{n-1}, z_{n-2}, ..., z_i\}) \end{aligned} \quad (1.19)$$

for $i = n, n-1, ..., 1, 0$.
By (1.16), (1.17) and (1.19) $y_i = v_i$ for $i = n, n-1, ..., n-k$ and $v_{n-k-1} = 1 > 0 = y_{n-k-1}$, and hence we have $y < v = f(z)$.
$[II]$ Next we prove three propositions.
The first one is that (1.10) implies (1.11) . The second one is that (1.12) implies (1.13) and (1.14).
The third one is that (1.15) implies (1.16) and (1.17).
If (1.11) is not true, then there exists k such that $\{y_n, y_{n-1}, y_{n-2}, ..., y_i\}$
$= f(\{z_n, z_{n-1}, z_{n-2}, ..., z_i\})$ for $i = n, n-1, ..., n-k$ and

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-k}, y_{n-k-1}\} \neq f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, z_{n-k-1}\}). \quad (1.20)$$

By Definition 1.7 and Remark 1.2 we have

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-k}, 0\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, z_{n-k-1}\}) \quad (1.21)$$

or

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-k}, 1\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-k}, z_{n-k-1}\}). \quad (1.22)$$

If we have (1.21), then by (1.20) we have $y_{n-k-1} = 1$. This is the case of (1.13) and (1.14).
If we have (1.22), then by (1.20) we have $y_{n-k-1} = 0$. This is the case of (1.16) and (1.17).
Therefore we have proved that there are mutually exclusive three cases, and at least one of them is true in any case.
The first case is (1.11), and by $[1.a]$ we have $y = f(z)$.
The second case is (1.13) and (1.14), and by $[1.b]$ we have $y > f(z)$.
The third case is (1.16) and (1.17), and by $[1.c]$ we have $y < f(z)$.
If (1.11) is not true, then we have the second or the third case, and hence we have $y \neq f(z)$.

Therefore (1.10) implies (1.11).
If the condition "(1.13) and (1.14)" is not true, we have the first or the third case, and hence we have $y \leq f(z)$.
Therefore (1.12) implies (1.13) and (1.14).
If the condition "(1.16) and (1.17)" is not true, then we have the first or the second case, and hence we have $y \geq f(z)$.
Therefore (1.15) implies (1.16) and (1.17). □

Lemma 1.6. [*I*] *Suppose that*

$$x \oplus y \oplus z = 0 \tag{1.23}$$

for $x, y, z \in Z_{\geq 0}$ *and*

$$y < f(z), \tag{1.24}$$

then there exists $j \leq n$ *such that for* $i = n, n-1, ..., n-j$

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_i\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_i\}) \tag{1.25}$$

and

$$\{y_n, y_{n-1}, ..., y_{n-j-1}\} < \{y_n, y_{n-1}, ..., y_{n-j}, 1\} = f(\{z_n, z_{n-1}, ..., z_{n-j-1}\}), \tag{1.26}$$

where $y_{n-j-1} = 0$.
Here for $i = n, n-1, ..., n-j$ y_i *and* z_i *are uniquely determined by* f *and* $x_n, x_{n-1}, ..., x_{n-j}$.
[*II*] *Suppose that*

$$x \oplus v \oplus w = 0 \tag{1.27}$$

for $x, v, w \in Z_{\geq 0}$ *and*

$$v = f(w), \tag{1.28}$$

then we have

$$\{v_n, v_{n-1}, v_{n-2}, ..., v_i\} = f(\{w_n, w_{n-1}, w_{n-2}, ..., w_i\}) \tag{1.29}$$

for $i = n, n-1, ..., 0$. *Here for* $i = n, n-1, ..., 0$ v_i *and* w_i *are uniquely determined by* f *and* $x_n, x_{n-1}, ..., x_1, x_0$.
[*III*] *We assume (1.23),(1.24),(1.28) and (1.27), then we have* $v > y$.

Proof. Let $x = \sum_{i=0}^{n} x_i 2^i$, $y = \sum_{i=0}^{n} y_i 2^i$ and $z = \sum_{i=0}^{n} z_i 2^i$, where $x_i, y_i, z_i \in \{0, 1\}$ and $x_n = 1$.
[*I*] If $y < f(z)$, then Equation (1.25) and Inequality (1.26) are direct from Lemma 1.5. By Definition 1.7 $0 = f(1)$, and hence by Inequality (1.24) and Equation (1.23) we have $y_n = 0$ and $z_n = 1$. Here y_n, z_n are uniquely determined by x_n and f.
By (3) of Definition 1.7 we have

$$\{y_n, 0\} = f(\{z_n, 0\}) = f(\{z_n, 1\}) \tag{1.30}$$

or

$$\{y_n, 1\} = f(\{z_n, 0\}) = f(\{z_n, 1\}). \tag{1.31}$$

Since $x_{n-1} + y_{n-1} + z_{n-1} = 0(mod\ 2)$, y_{n-1}, z_{n-1} are determined by the following ways.
If the function f satisfies Equation (1.30), then we let $y_{n-1} = 0$ and $z_{n-1} = x_{n-1} + y_{n-1} = x_{n-1}$.
If the function f satisfies Equation (1.31), then we let $y_{n-1} = 1$ and $z_{n-1} = x_{n-1} + y_{n-1} = x_{n-1} + 1 = 1 - x_{n-1}$ $(mod\ 2)$. In both cases y_{n-1}, z_{n-1} are uniquely determined by x_{n-1} and f.
Therefore we have $\{y_n, y_{n-1}\} = f(\{z_n, z_{n-1}\})$.
Note that by using (3) of Definition 1.7 we determine the value of y_{n-1}, and after that we determine the value of z_{n-1} by the value of x_{n-1} and the fact that $x_{n-1} + y_{n-1} + z_{n-1} = 0$ $(mod\ 2)$.

Suppose that $y_n, y_{n-1}, ..., y_{n-i}, z_n, z_{n-1}, ..., z_{n-i}$ are uniquely determined by $x_n, z_{n-1}, ..., x_{n-i}$ and we have $\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-i}\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-i}\})$, then by (3) of Definition 1.7 we have

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-i}, 0\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-i}, 0\}) = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-i}, 1\}) \tag{1.32}$$

or

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-i}, 1\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-i}, 0\}) = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-i}, 1\}). \tag{1.33}$$

Since $x_{n-i-1} + y_{n-i-1} + z_{n-i-1} = 0 \ (mod\ 2)$ y_{n-i-1}, z_{n-i-1} are determined by the following ways.
If the function f satisfies Equation (1.32), then $y_{n-j-1} = 0$ and $z_{n-j-1} = x_{n-j-1} + y_{n-j-1} = x_{n-j-1}$.
If the function f satisfies Equation (1.33), then $y_{n-j-1} = 1$ and $z_{n-j-1} = x_{n-j-1} + y_{n-j-1} = x_{n-j-1} + 1 = 1 - x_{n-j-1} \ (mod\ 2)$.
Therefore we have $\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-i-1}\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-i-1}\})$.
Note that by using (3) of Definition 1.7 we determine the value of y_{n-i-1}, and after that we determine the value of z_{n-i-1} by the value of x_{n-i-1} and the fact that $x_{n-i-1} + y_{n-i-1} + z_{n-i-1} = 0$.
In this way y_i, z_i are uniquely determined for $i = n, n-1, ..., n-j$ by $x_n, x_{n-1}, ..., x_{n-j}$.
$[II]$ If $v = f(w)$, by the method similar to the one used in $[I]$ we prove that we have Equation (1.29) for $i = n, n-1, ..., 0$. Here for $i = n, n-1, ..., 0$ v_i and w_i are uniquely determined by f and $x_n, x_{n-1}, ..., x_1, x_0$.
$[III]$ We assume (1.23),(1.24), (1.27), (1.28). Then there exists $j \leq n$ such that for $i = n, n-1, ..., n-j$ we have (1.25), (1.26) and $y_{n-j-1} = 0$.
Here for $i = n, n-1, ..., n-j$ y_i and z_i are uniquely determined by f and $x_n, x_{n-1}, ..., x_{n-j}$.
We have (1.29) for $i = n, n-1, ..., 0$. Here for $i = n, n-1, ..., 0$ v_i and w_i are uniquely determined by $x_n, x_{n-1}, ..., x_1, x_0$.
, and hence for $i = n, n-1, ..., n-j$.

$$v_i = y_i \text{ and } w_i = z_i \tag{1.34}$$

By (1.26), Definition 1.7 and Remark 1.2

$$\begin{aligned} \{y_n, ..., y_{n-j-1}\} < \{y_n, ..., y_{n-j}, 1\} &= f(\{z_n, ..., z_{n-j}, z_{n-j-1}\}) \\ &= f(\{z_n, ..., z_{n-j}, 0\}) = f(\{z_n, ..., , z_{n-j}, 1\}), \end{aligned} \tag{1.35}$$

where $y_{n-j-1} = 0$.

By using (1.29) for $i = n-j-1$ we have

$$\{v_n, ..., v_{n-j}, v_{n-j-1}\} = f(\{w_n, ..., w_{n-j}, w_{n-j-1}\}). \tag{1.36}$$

By (1.34), (1.35), (1.36) and the fact that $w_{n-j-1} = 0$ or 1 we have

$$\begin{aligned} \{v_n, ..., v_{n-j}, v_{n-j-1}\} &= f(\{w_n, ..., w_{n-j}, w_{n-j-1}\}) \\ &= f(\{z_n, ..., z_{n-j}, w_{n-j-1}\}) \\ &= \{y_n, ..., y_{n-j}, 1\} = \{v_n, ..., v_{n-j}, 1\}, \end{aligned} \tag{1.37}$$

and hence we have $y_{n-j-1} = 0 < 1 = v_{n-j-1}$. Therefore by (1.34) we have $v > y$. □

Lemma 1.7. *Suppose that $x \oplus y \oplus z \neq 0$ and $y \leq f(z)$.*
Then at least one of the following (1), (2), (3) and (4) is true.
(1) *$u \oplus y \oplus z = 0$ for some $u \in Z_{\geq 0}$ such that $u < x$.*
(2) *$x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$.*
(3) *$x \oplus y \oplus w = 0$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq f(w)$.*
(4) *$x \oplus v \oplus w' = 0$ for some $v, w' \in Z_{\geq 0}$ such that $v < y, w' < z$ and $v = f(w')$.*

Proof. Suppose that $x_i + y_i + z_i = 0$ $(mod\ 2)$ for $i = n, n-1, ..., n-k$ and $x_{n-k-1} + y_{n-k-1} + z_{n-k-1} \neq 0$ $(mod\ 2)$.
(i) If $x_{n-k-1} = 1$, we define $u = \sum_{i=1}^{n} u_i 2^i$ by $u_i = x_i$ for $i = n, n-1, ..., n-k$, $u_{n-k-1} = 0 < x_{n-k-1}$ and $u_i = y_i + z_i$ for $i = n-k-2, n-k-3, ..., 0$. Then we have $u \oplus y \oplus z = 0$ and $u < x$. Therefore we have (1) of this lemma.
(ii) If $y_{n-k-1} = 1$, then by the method that is similar to the one used in (i) we prove that $x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$. Therefore we have (2) of this lemma.
(iii). We suppose that

$$x_{n-k-1} = y_{n-k-1} = 0 \text{ and } z_{n-k-1} = 1. \tag{1.38}$$

For $i = n, n-1, ..., n-k$ let

$$w_i = z_i \tag{1.39}$$

and $w_i = x_i + y_i$ $(mod\ 2)$ for $i = n-k-1, ..., 0$. Note that

$$w_{n-k-1} = 0 < 1 = z_{n-k-1}, \tag{1.40}$$

since $w_{n-k-1} = x_{n-k-1} + y_{n-k-1} = 0$ $(mod\ 2)$.
$(iii.1)$ If $y \leq f(w)$, then we have (3) of this lemma..
$(iii.2)$ If $y > f(w)$, then there exists s such that for $i = n, n-1, ..., n-s$

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_i\} = f(\{w_n, w_{n-1}, w_{n-2}, ..., w_i\}) \tag{1.41}$$

and

$$\{y_n, y_{n-1}, ..., y_{n-s-1}\} > \{y_n, y_{n-1}, ..., y_{n-s}, 0\} = f(\{w_n, w_{n-1}, ..., w_{n-s-1}\}), \tag{1.42}$$

where $y_{n-s-1} = 1$.
If $n-k \leq n-s-1$, then by (1.39) and (1.42)

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_{n-s-1}\} > f(\{z_n, z_{n-1}, z_{n-2}, ..., z_{n-s}, z_{n-s-1}\}). \tag{1.43}$$

By (1.39) and (1.41) we have for $i = n, n-1, ..., n-s$

$$\{y_n, y_{n-1}, y_{n-2}, ..., y_i\} = f(\{z_n, z_{n-1}, z_{n-2}, ..., z_i\}). \tag{1.44}$$

By (1.43), (1.44) and (b) of Lemma 1.5 we have $y > f(z)$. This contradicts the fact that $y \leq f(z)$.
Therefore we assume that $n-k > n-s-1$.
For $i = n, n-1, ..., n-s$ let

$$v_i = y_i \text{ and } w'_i = w_i \tag{1.45}$$

Let $v_{n-s-1} = 0 < 1 = y_{n-s-1}$, then by (1.42) and (1.45) we have

$$v < y \tag{1.46}$$

and

$$\begin{aligned}\{v_n, ..., v_{n-s}, v_{n-s-1}\} &= \{y_n, ..., y_{n-s}, 0\} = f(\{w_n, w_{n-1}, ..., w_{n-s-1}\}) \\ &= f(\{w_n, ..., w_{n-s}, 0\}) = f(\{w_n, ..., , w_{n-s}, 1\}).\end{aligned} \tag{1.47}$$

Let

$$w'_{n-s-1} = x_{n-s-1} + v_{n-s-1} = x_{n-s-1}. \tag{1.48}$$

Then by (1.45), (1.47) and the fact that $w'_{n-s-1} = 0$ or 1 we have we have

$$\begin{aligned}\{v_n, ..., v_{n-s-1}\} &= f(\{w_n, ..., w_{n-s}, w'_{n-s-1}\}) \\ &= f(\{w'_n, ..., w'_{n-s}, w'_{n-s-1}\}).\end{aligned} \tag{1.49}$$

Next by a method that is similar to the one used in [1] of Lemma 1.6 we define v_i and w'_i for $i = n-s-2, ..., 0$ recursively.
We suppose that $x_i + v_i + w'_i = 0 \ (mod\ 2)$ and $\{v_n, ..., v_i\} = f(\{w'_n, ..., w'_i\})$ for $i = n, n-1, ..., n-j$. Then by Definition 1.7 and Remark 1.2 we have

$$\begin{aligned}\{v_n, ..., v_{n-j}, 0\} &= f(\{w'_n, ..., w'_{n-j}, 0\}) \\ &= f(\{w'_n, ..., w'_{n-j}, 1\}) \end{aligned} \tag{1.50}$$

or

$$\begin{aligned}\{v_n, ..., v_{n-j}, 1\} &= f(\{w'_n, ..., w'_{n-j}, 0\}) \\ &= f(\{w'_n, ..., w'_{n-j}, 1\}). \end{aligned} \tag{1.51}$$

If we have (1.50), then we let $v_{n-j-1} = 0$ and let $w'_{n-j-1} = x_{n-j-1} + v_{n-j-1} = x_{n-j-1} (mod\ 2)$. If we have (1.51), then we let $v_{n-j-1} = 1$ and let $w'_{n-j-1} = x_{n-j-1} + 1 = 1 - x_{n-j-1} (mod\ 2)$.
In this way we define v_i and w'_i for $i = n-s-2, ..., 0$ recursively, and we let $v = \{v_n, ..., v_0\}$ and $w' = \{w'_n, ..., w'_0\}$. Then

$$v = f(w'). \tag{1.52}$$

Since $n-k > n-s-1$, we have $n-k-1 \geq n-s-1$, and hence we have $n-k-1 \geq n-s$ or $n-k-1 = n-s-1$.
If $n-k-1 \geq n-s$, then by (1.39), (1.40) and (1.45) we have
$w'_i = w_i = z_i$ for $i = n, ..., n-k$ and $w'_{n-k-1} = w_{n-k-1} = 0 < 1 = z_{n-k-1}$. Therefore $w' < z$.
If $n-k-1 = n-s-1$, then by (1.38) and (1.48)

$$w'_{n-k-1} = x_{n-k-1} = 0 \ (mod\ 2) \text{ and } z_{n-k-1} = 1. \tag{1.53}$$

By (1.39), (1.40) and (1.45) we have $w'_i = w_i = z_i$ for $i = n, n-1, ..., n-k$, and hence by (1.53) $w' < z$.
By (1.46) $v < y$.
Therefore by (1.52) we have $v < y, w' < z$ and $v = f(w')$, and we have (4) of this lemma. □

Lemma 1.8. *If $x \oplus y \oplus z = 0$ and $y \leq f(z)$, then*
(1) $u \oplus y \oplus z \neq 0$ for any $u \in Z_{\geq 0}$ such that $u < x$.
(2) $x \oplus v \oplus z \neq 0$ for any $v \in Z_{\geq 0}$ such that $v < y$.
(3) $x \oplus y \oplus w \neq 0$ for any $w \in Z_{\geq 0}$ such that $w < z$.
(4) $x \oplus v \oplus w \neq 0$ for any $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = f(w)$.

Proof. (1),(2) and (3) are direct from the definition of nim-sum.
We prove (4). We suppose that $x \oplus v \oplus w = 0$ and $v = f(w)$ for some $w \in Z_{\geq 0}$ such that $v < y, w < z$. If $y < f(z)$, then by Lemma 1.5 we have $y < v$, and this contradicts the fact $v < y$. If $y = f(z)$, then by Lemma 1.5 we have $y = v$, and this contradicts the fact $v < y$. Therefore $x \oplus v \oplus w \neq 0$. □

Definition 1.9. *Let $A_k = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, y \leq f(z)$ and $x \oplus y \oplus z = 0\}$, $B_k = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, y \leq f(z)$ and $x \oplus y \oplus z \neq 0\}$.*

Theorem 1.2. *Let A_k and B_k be the sets defined in Definition 1.9. A_k is the set of P-positions and B_k is the set of N-positions of the disjunctive sum of the chocolate game with $CB(f, y, z)$ to the right of the poisoned square and a single strip of chocolate to the left.*

Proof. If we start the game with a position $\{x, y, z\} \in A_k$, then by Lemma 1.8 any option by us leads to a position $\{p, q, r\}$ in B_k. From this position $\{p, q, r\}$ by Lemma 1.7 our opponent can choose a proper option that leads to a position in A_k. Note that any option reduces some of the numbers in the coordinates. In this way our opponent can always reach a position in A_k, and finally he wins by reaching $\{0, 0, 0\} \in A_k$. Note

that position $\{0, 0, 0\}$ represent the poisoned square itself. Therefore A_k is the set of $\mathcal{P}$-positions.
If we start the game with a position $\{x, y, z\} \in B_k$, then by Lemma 1.7 we can choose a proper option leads to a position $\{p, q, r\}$ in A_k. From $\{p, q, r\}$ any option by our opponent leads to a position in B_k. In this way we win the game by reaching $\{0, 0, 0\}$. Therefore B_k is the set of $\mathcal{N}$-positions. □

Theorem 1.3. *Let f be the function defined in Definition 1.7. Then the Grundy number of $CB(f, y, z)$ is $y \oplus z$.*

Proof. By Theorem 1.2 a position $\{x, y, z\}$ of the sum of the chocolate is a $\mathcal{P}$-position when $x \oplus y \oplus z = 0$ so that the Grundy number of the Chocolate bar to the right is $x = y \oplus z$. □

2 New conditions for functions

In Definition 2.1 we define a function g that is slightly different from the function f of Definition 1.7.

Definition 2.1. *Let f be a function that is defined in Definition 1.7.*
Let g be a function that satisfies the following condition.
There exist $s_1, s_2, ..., s_n$ and $t_1, t_2, ..., t_n$ such that

$$s_i = f(t_i) = g(t_i), \tag{2.1}$$
$$2s_i + 1 = f(2t_i) = f(2t_i + 1) \tag{2.2}$$
$$\textit{and } 2s_i = g(2t_i) = g(2t_i + 1). \tag{2.3}$$

Clearly $g(t) \leq f(t)$ for $t \in Z_{\geq 0}$.

Lemma 2.1. *Let g be a function defined in Definition 2.1.*
Suppose that $x \oplus y \oplus z \neq 0$ and $y \leq g(z)$.
Then at least one of the following (1), (2), (3) and (4) is true.
(1) $u \oplus y \oplus z = 0$ for some $u \in Z_{\geq 0}$ such that $u < x$.
(2) $x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$.
(3) $x \oplus y \oplus w = 0$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq g(w)$.
(4) $x \oplus v \oplus w' = 0$ for some $v, w' \in Z_{\geq 0}$ such that $v < y, w' < z$ and $v = g(w')$.

Proof. Since $y \leq g(z) \leq f(z)$, by using Lemma 1.7 we have the following (i),(ii),(iii) or (iv).
(i) $u \oplus y \oplus z = 0$ for some $u \in Z_{\geq 0}$ such that $u < x$, then we have (1) of this lemma.
(ii) $x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$, then we have (2) of this lemma.
(iii) $x \oplus y \oplus w = 0$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq f(w)$. If $y \leq g(w)$, then we have (3) of this lemma.
If $y > g(w)$, then $f(w) \geq y > g(w)$. Then by Definition 2.1 $f(w) = y = g(w) + 1$, and by using (2.1), (2.2) and (2.3) we have the following (2.4), (2.5) and (2.6) respectively.

$$\{y_n, ..., y_1\} = f(\{w_n, ..., w_1\}), \tag{2.4}$$
$$y = \{y_n, ..., y_1, 1\} = f(\{w_n, ..., w_1, 0\}) = f(\{w_n, ..., w_1, 1\}) \tag{2.5}$$
$$\text{and } \{y_n, ..., y_1, 0\} = g(\{w_n, ..., w_1, 0\}) = g(\{w_n, ..., w_1, 1\}). \tag{2.6}$$

Note that $w = \{w_n, ..., w_1, 1\}$ or $= \{w_n, ..., w_1, 0\}$.
Let $v = \{y_n, ..., y_1, 0\}$ and $w' = \{w_n, w_{n-1}, ..., w_1, x_0\}$. Then by (2.5) $v < y$, $v = g(w')$ and $x \oplus v \oplus w' = 0$.
If $w' < z$, then we have (4) of this lemma. If $w' = z$, then we have (2) of this lemma.
(iv) $x \oplus v \oplus w = 0$ for some $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = f(w)$.

If $f(w) = g(w)$, then we have (4) of this lemma.
Next we suppose that $g(w) < f(w)$. Then by Definition 2.1 $f(w) = v = g(w) + 1$, and by using (2.1), (2.2) and (2.3) we have the following (2.7), (2.8) and (2.9) respectively

$$\{v_n, ..., v_1\} = f(\{w_n, ..., w_1\}), \tag{2.7}$$
$$v = \{v_n, ..., v_1, 1\} = f(\{w_n, ..., w_1, 0\}) = f(\{w_n, ..., w_1, 1\}) \tag{2.8}$$
$$\text{and } \{v_n, ..., v_1, 0\} = g(\{w_n, ..., w_1, 0\}) = g(\{w_n, ..., w_1, 1\}). \tag{2.9}$$

Note that $w = \{w_n, ..., w_1, 1\}$ or $= \{w_n, ..., w_1, 0\}$.
Let $v' = \{v_n, ..., v_1, 0\}$ and $w' = \{w_n, w_{n-1}, ..., w_1, x_0\}$. Then $v' < v < y$. Since $w < z$, $w' \leq z$. If $w' < z$, then we have (4) of this lemma.
If $w' = z$, then we have (2) of this lemma. □

Lemma 2.2. *If $x \oplus y \oplus z = 0$ and $y \leq g(z)$, then*
(1) $u \oplus y \oplus z \neq 0$ for any $u \in Z_{\geq 0}$ such that $u < x$.
(2) $x \oplus v \oplus z \neq 0$ for any $v \in Z_{\geq 0}$ such that $v < y$.
(3) $x \oplus y \oplus w \neq 0$ for any $w \in Z_{\geq 0}$ such that $w < z$.
(4) $x \oplus v \oplus w \neq 0$ for any $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = g(w)$.

Proof. (1),(2) and (3) are direct from Definition 1.1 (the definition of nim-sum).
We prove (4). We suppose that $x \oplus v \oplus w = 0$, $v = g(w)$,$v < y$ and $w < z$ for some $v, w \in Z_{\geq 0}$.
If $f(w) = g(w)$, then this contradicts (4) of Lemma 1.8. If $f(w) > g(w)$, then by Definition 2.1 $f(w) = g(w) + 1$, and by using (2.1), (2.2) and (2.3) we have the following (2.10), (2.11) and (2.12) respectively

$$\{v_n, ..., v_1\} = f(\{w_n, ..., w_1\}), \tag{2.10}$$
$$\{v_n, ..., v_1, 1\} = f(\{w_n, ..., w_1, 0\}) = f(\{w_n, ..., w_1, 1\}) \tag{2.11}$$
$$\text{and } v = \{v_n, ..., v_1, 0\} = g(\{w_n, ..., w_1, 0\}) = g(\{w_n, ..., w_1, 1\}). \tag{2.12}$$

Let $v' = \{v_n, v_{n-1}, ..., v_1, 1\}$ and $w' = \{w_n, ..., w_1, 1 - x_0\}$, then we have $v' = f(w')$ and $x \oplus v' \oplus w' = 0$.
Since $v < y, w < z$, we have $v' \leq y$,$w' \leq z$. If $w' = z$, then by (2.12) we have $v = g(z)$. By the assumption of this lemma $g(z) \geq y$, and hence we have $v \geq y$ which contradicts the fact that $v < y$.
Therefore we have $w' < z$. If $v' = y$, then this contradict (3) of Lemma 1.8. If $v' < y$, then this contradict (4) of Lemma 1.8. □

Definition 2.2. *Let $A'_k = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, y \leq g(z)$ and $x \oplus y \oplus z = 0\}$, $B'_k = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, y \leq g(z)$ and $x \oplus y \oplus z \neq 0\}$.*

Theorem 2.1. *Let A'_k and B'_k be the sets defined in Definition 2.2. A'_k is the set of P-positions and B'_k is the set of N-positions of the disjunctive sum of the chocolate game with $CB(g, y, z)$ to the right of the poisoned square and a single strip of chocolate to the left.*

Proof. We can prove this theorem by the same method used in Theorem 1.2 using Lemma 2.1 and Lemma 2.2. □

Theorem 2.2. *The Grundy number of $CB(g, y, z)$ is $y \oplus z$.*

Proof. We can prove this theorem by the same method used in Theorem 1.3 using Theorem 2.1. □

By Definition 2.1, Definition 1.7 and Theorem 2.2 we know that there are quite a lot of functions that produce Chocolate bars with Grundy number expressed by nim-sum.

3 Grundy number of $CB(f,y,z)$, where $f(t)=t$

In this section we study Grundy number of $CB(f,y,z)$ with $f(t)=t$, and present a proof for closed formulas of Grundy number of $CB(f,y,z)$. We prove by mathematical induction, and the only tool for the proof is calculation. It seems that this is the only tool available now for this problem.
Let with $Z_{\geq 0}$ be the set of non-negative integers.
This Chocolate Bar was studied in the paper presented for Yau Award of 2013, but this is the first time that we discovered closed formulas of Grundy number of this Chocolate Bar. A study on the Grundy number of this Chocolate Bar was presented in [18] without any closed formulas.

Figure 3.1 is $CB(f,7,7)$, where $f(t)=t$.

We present the definition of *move* here again. This was the same definition in Definition 1.4.

For any position G, there is a set of positions (games) that can be reached by making precisely one move in G, which we will denote by $move(G)$.

Since $move(\{y,z\})$ is the set of positions (games) that can be reached by making precisely one move in $\{y,z\}$, $move(\{y,z\}) = \{\{y-1,z\},\{y-2,z\},\{y-3,z\},...,\{0,z\}\} \cup \{\min(y,z-1),z-1\},...,\{\min(y,0),0\}\}$.

Example 3.1.

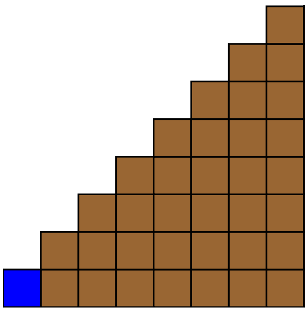


Figure 3.1. $CB(f,7,7)$ $f(t)=t$

Figure 3.2 is the table of Grundy number $G(\{y,z\})$ of $CB(f,y,z)$ with $f(t)=t$ for $z=0,1,2,3,...,17, 0 \leq y \leq z$, $y \in Z_{\geq 0}$.
We present an example of the method used in the proof of Theorem 3.1.

Z\Y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0																	
1	1	2																
2	2	1	3															
3	3	4	1	5														
4	4	3	5	1	6													
5	5	6	4	7	1	8												
6	6	5	7	4	8	1	9											
7	7	8	6	9	4	10	1	11										
8	8	7	9	6	10	4	11	1	12									
9	9	10	8	11	7	12	4	13	1	14								
10	10	9	11	8	12	7	13	4	14	1	15							
11	11	12	10	13	9	14	7	15	4	16	1	17						
12	12	11	13	10	14	9	15	7	16	4	17	1	18					
13	13	14	12	15	11	16	10	17	7	18	4	19	1	20				
14	14	13	15	12	16	11	17	10	18	7	19	4	20	1	21			
15	15	16	14	17	13	18	12	19	10	20	7	21	4	22	1	23		
16	16	15	17	14	18	13	19	12	20	10	21	7	22	4	23	1	24	
17	17	18	16	19	15	20	14	21	13	22	10	23	7	24	4	25	1	26

Figure 3.2.

To prove Theorem 3.1 we use mathematical induction. [1] *We assume that we know the values of Grundy numbers;* $\{G(\{y,z\});\{y,z\} \in move(\{6,16\})\}$

$$= \{G(\{5,16\}), G(\{4,16\}), ..., G(\{0,16\})\} \tag{3.1}$$
$$\cup\{G(\{6,15\}), G(\{6,14\}), ..., G(\{6,7\})\} \tag{3.2}$$
$$\cup\{G(\{6,6\}), G(\{5,5\}), ..., G(\{0,0\})\}. \tag{3.3}$$

, and we calculate the value of $G(\{6,16\})$.
By the definition of Grundy number $G(\{6,16\})$ *is the smallest non-negative integer not contained in the sets of Grundy numbers (3.1), (3.2) and (3.3).*
In Figure 3.2 (3.1), (3.2) and (3.3) are in red, yellow and blue squares. As it is easily seen (3.1), (3.2) and (3.3) are $\{13,18,14,17,15,16\}$, $\{12,17,10,15,7,13,4,11,1\}$ *and* $\{9,8,6,5,3,2,0\}$ *respectively, and 19 is the smallest non-negative integer not contained in these sets of numbers. Therefore we have* $G(\{6,16\}) = 19$.

Z\Y	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0													
1	1	2												
2	2	1	3											
3	3	4	1	5										
4	4	3	5	1	6									
5	5	6	4	7	1	8								
6	6	5	7	4	8	1	9							
7	7	8	6	9	4	10	1	11						
8	8	7	9	6	10	4	11	1	12					
9	9	10	8	11	7	12	4	13	1	14				
10	10	9	11	8	12	7	13	4	14	1	15			
11	11	12	10	13	9	14	7	15	4	16	1	17		
12	12	11	13	10	14	9	15	7	16	4	17	1	18	
13	13	14	12	15	11	16	10	17	7	18	4	19	1	20

Figure 3.3.

Figure 3.3 is the table of Grundy number $G(\{y,z\})$ *of* $CB(f,y,z)$ *with* $f(t)=t$ *for* $z=0,1,2,3,...,17, 0 \leq y \leq z$, $y \in Z_{\geq 0}$ *and* $z=0,1,2,3,...,13, 0 \leq y \leq z$, $y \in Z_{\geq 0}$
We present another example of the method used in the proof of Theorem 3.1.
[2] *We assume that we know the values of Grundy numbers;* $\{G(\{y,z\}); \{y,z\} \in move(\{5,12\})$

$$= \{G(\{4,12\}), G(\{3,12\}), ..., G(\{0,12\})\} \tag{3.4}$$

$$\cup \{G(\{5,11\}), G(\{5,10\}), ..., G(\{5,6\})\} \tag{3.5}$$

$$\cup \{G(\{5,5\}), G(\{4,4\}), ..., G(\{0,0\})\}. \tag{3.6}$$

, and we calculate the value of $G(\{5,12\})$.
By the definition of Grundy number $G(\{5,12\})$ *is the smallest non-negative integer not contained in the sets of Grundy numbers (3.4), (3.5) and (3.6).*
In Figure 3.3 (3.4), (3.5) and (3.6) are in red, yellow and blue squares. As it is easily seen (3.4), (3.5) and (3.6) are $\{14,10,13,11,12\}$, $\{14,7,12,4,10,1\}$ *and* $\{8,6,5,3,2,0\}$ *respectively, and* 9 *is the smallest non-negative integer not contained in these sets of numbers. Therefore we have* $G(\{5,12\}) = 9$.

Theorem 3.1. *Let* $f(t)=t$. *Then Grundy number* $G(\{y,z\})$ *of* $CB(f,y,z)$ *satisfies the following equations for* $p,r \in Z_{\geq 0}$.

$$G(\{2p, 2(p+r)\}) = 3p+2r. \tag{3.7}$$

$$G(\{2p+1, 2(p+r)\}) = p+2r-1-\max(0, p-r+1). \tag{3.8}$$

$$G(\{2p, 2(p+r)+1\}) = p+2r+1-\max(0, p-r). \tag{3.9}$$

$$G(\{2p+1, 2(p+r)+1\}) = 3p+2r+2. \tag{3.10}$$

Proof. *We prove by mathematical induction. We assume that equations (3.7),(3.8),(3.9),(3.10) are valid when* $p \leq m$ *and* $p+r < m+s$ *or* $p < m$ *and* $p+r \leq m+s$, *where* $m,s,p,r \in Z_{\geq 0}$.
[1] *First we prove (3.9). Therefore under the assumption of mathematical induction we prove that*

$$G(\{2m, 2(m+s)+1\}) = m+2s+1-\max(0, m-s). \tag{3.11}$$

Since $move(2m, 2(m+s)+1)$

$$= \{\{2m-1, 2(m+s)+1\}, \{2m-2, 2(m+s)+1\}, ...\{1, 2(m+s)+1\}, \{0, 2(m+s)+1\}\} \tag{3.12}$$

$$\cup \{\{2m, 2(m+s)\}, \{2m, 2(m+s)-1\}, ...\{2m, 2m+1\}, \{2m, 2m\}\} \tag{3.13}$$

$$\cup \{\{2m-1, 2m-1\}, \{2m-2, 2m-2\}, ...\{1,1\}, \{0,0\}\}, \tag{3.14}$$

by the definition of Grundy number we have to prove that $m+2s+1-\max(0,m-s)$ *is the smallest non-negative integer not included in sets of Grundy numbers of positions in (3.12),(3.13),(3.14), if we are to prove (3.11).*

By the hypothesis of mathematical induction the Grundy numbers of the positions in (3.12) are $G(\{2m-1, 2(m+s)+1\}) = 3m+2s+1$, $G(\{2m-2, 2(m+s)+1\}) = m+2s+2-\max(0, m-s-2)$, $G(\{2m-3, 2(m+s)+1\}) = 3m+2s$, $G(\{2m-4, 2(m+s)+1\}) = m+2s+3-\max(0, m-s-4)$,...,$G(\{2, 2(m+s)+1\}) = 2m+2s-\max(0, 2-m-s)$,$G(\{1, 2(m+s)+1\}) = 2m+2s+2$ *and* $G(\{0, 2(m+s)+1\}) = 2m+2s+1$. *These Grundy numbers are bigger than* $m+2s+1-\max(0,m-s)$, *and hence these are irrelevant to the proof of (3.11).*

By the hypothesis of mathematical induction we calculate Grundy numbers of positions in (3.13).
We divide these into two groups. The first group consists of Grundy numbers of positions in (3.13) whose z-coordinate (the second coordinate) are even numbers;

$$G(\{2m, 2(m+s)\}) = 3m+2s, G(\{2m, 2(m+s-1)\}) = 3m+2s-2, ..., G(\{2m, 2s+2\}) = m+2s+2, \tag{3.15}$$

$$G(\{2m,2s\}) = 2s+m, G(\{2m,2s-2\}) = 2s+m-2, ..., G(\{2m,2m+2\}) = 3m+2, G(\{2m,2m\}) = 3m. \tag{3.16}$$

The second group consists of Grundy numbers of positions in (3.13) whose z-coordinate (the second coordinate) are odd numbers;

$$G(\{2m,2(m+s-1)+1\}) = m+2s-1-\max(0,m-s+1), ...,$$
$$G(\{2m,2(m+s-k)+1\}) = m+2s-(2k-1)-\max(0,m-s+k), ..., \tag{3.17}$$
$$G(\{2m,4m+1\}) = 3m+1-\max(0,2m-2m) = 3m+1, \tag{3.18}$$
$$G(\{2m,4m-1\}) = 3m-1-\max(0,2m-2m+1) = 3m-2,$$
$$G(\{2m,4m-3\}) = 3m-5, ..., G(\{2m,2m+3\}) = 4, G(\{2m,2m+1\}) = 1. \tag{3.19}$$

By the hypothesis of mathematical induction we calculate Grundy numbers of positions in (3.14);

$$G(\{2m-1,2m-1\}) = 3m-1, G(\{2m-2,2m-2\}) = 3m-3, G(\{2m-3,2m-3\}) = 3m-4, ...$$
$$G(\{2m-(2k-1),2m-(2k-1)\}) = 3m-3k+2, G(\{2m-2k,2m-2k\}) = 3m-3k, ...,$$
$$G(\{1,1\}) = 2, G(\{0,0\}) = 0. \tag{3.20}$$

Note that the set of these Grundy numbers does not include numbers of the type $3h+1$, where h is a non-negative integer, since $G(\{2m-(2k-1),2m-(2k-1)\}) = 3m-3k+2$ and $G(\{2m-2k,2m-2k\}) = 3m-3k$ for $k \in Z_{\geq 0}$ with $1 \leq k \leq m$. In the proof of (3.11) we have three cases. The first case is when $m-s+1 \leq 0$, the second case is when $m-s > 0$ and the third case is when $m-s = 0$.
[1.1] *Suppose that $m-s+1 \leq 0$, then $m+2s+1-\max(0,m-s) = m+2s+1$, which is the right term of (3.11). Grundy numbers of positions in (3.15) are bigger than $m+2s+1$, and hence these Grundy numbers are irrelevant to the proof of (3.11).*
Since $m-s+1 \leq 0$, Grundy numbers in (3.18) and (3.19) are

$$\{m+2s-1, m+2s-3, ..., 3m+1\} \tag{3.21}$$
$$\text{and } \{3m-2, ..., 4, 1\}. \tag{3.22}$$

Note that (3.21) and (3.22) are arithmetic progressions with common difference of -2 and -3 respectively. Grundy numbers of the positions in (3.16), (3.20) are

$$\{m+2s, m+2s-2, ..., 3m+2, 3m\} \tag{3.23}$$
$$\text{and } \{3m-1, 3m-3, 3m-4..., 3m-3k, 3m-3k-1, ..., ..., 3, 2, 0\}, \tag{3.24}$$

where (3.23) is an arithmetic progression with common difference of -2 and (3.24) does not contains the number of the type $3h+1$ with $h \in Z_{\geq 0}$.
The union of the sets of Grundy numbers in (3.21) and (3.23) is the set

$$\{m+2s, m+2s-1, ..., 3m\}, \tag{3.25}$$

and this is an arithmetic progression with common difference of -1.

The union of the sets of Grundy numbers in (3.22) and (3.24) is the set

$$\{3m-1, 3m-2, ..., 1, 0\}, \tag{3.26}$$

since (3.22) consists of the number of the type $3h+1$ with $h \in Z_{\geq 0}$ and (3.24) does not contains the number of the type $3h+1$ with $h \in Z_{\geq 0}$.

(3.26) is an arithmetic progression with common difference of -1.
Therefore by (3.25) and (3.26) the Grundy numbers of the positions in $move(\{2m, 2(m+s)+1\})$ contains $\{m+2s, m+2s-1, ..., 1, 0\}$ and other Grundy numbers in $move(\{2m, 2(m+s)+1\})$ are bigger than $m+2s+1$.
Therefore by the definition of Grundy number we have $G(\{2m, 2(m+s)+1\}) = m+2s+1$. Therefore we proved (3.11).
[1.2] Next we suppose that $m-s > 0$. Since the right term of (3.11) is $m+2s+1-\max(0, m-s) = 3s+1 \leq 3m-2$, Grundy numbers in (3.15),(3.16) are bigger than $3s+1$, and hence they are irrelevant to the proof of (3.11).
Since $m-s > 0$, Grandy numbers in (3.17) and (3.19) are

$$G(\{2m, 2(m+s-1)+1\}) = m+2s-1-\max(0, m-s+1) = 3s-2,$$
$$G(\{2m, 2(m+s-2)+1\}) = m+2s-3-\max(0, m-s+2) = 3s-5, ...$$
$$G(\{2m, 2(m+s-k)+1\}) = m+2s-(2k-1)-\max(0, m-s+k) = 3s-3k+1, ...,$$
$$G(\{2m, 2m+3\}) = 4, G(\{2m, 2m+1\}) = 1. \tag{3.27}$$

Note that we do not have Grundy numbers in (3.18), since $3m+1 > 3s-2$. Clearly (3.27) is an arithmetic progression with common difference of -3.
We divide Grundy number of (3.20) into two groups (3.28) and (3.29).

$$G(\{2m-1, 2m-1\}) = 3m-1, ..., G(\{2s+1, 2s+1\}) = 3s+2, \tag{3.28}$$
$$G(\{2s, 2s\}) = 3s, ..., G(\{1, 1\}) = 2, G(\{0, 0\}) = 0. \tag{3.29}$$

Note that (3.28) and (3.29) do not contain numbers of the type $3h+1$, where $h \in Z_{\geq 0}$. Therefore they do not contain $3s+1$.
Grundy numbers in (3.28) are bigger than $3s+1$, and these Grundy numbers are irrelevant to the proof of (3.11).
Since $m > s$, the union of the set of Grundy numbers (3.27) and (3.29) contains the set $\{0, 1, ..., 3s\}$, and hence Grundy numbers of the positions in $move(\{2m, 2(m+s)+1\})$ contains $\{0, 1, ..., 3s\}$ and other Grundy numbers in $move(\{2m, 2(m+s)+1\})$ are bigger than $3s+1$.
Therefore by the definition of Grundy number we have $G(\{2m, 2(m+s)+1\}) = 3s+1$.
[1.3] Suppose that $m-s = 0$. Then Grundy numbers in (3.15),(3.16) except $3m$ are bigger than $3s+1$ that is the right term of (3.11). Since $s = m$, Grundy numbers in (3.28) do not exist. Since $m = s$, the union of the set of Grundy numbers (3.27) and (3.29) is the set $\{0, 1, ..., 3s-1\}$, which is an arithmetic progression with common difference of 1.
Since $3s = 3m$ is in (3.16), by the definition of Grundy number we have $G(\{2m, 2(m+s)+1\}) = 3s+1$.
[2] Next we prove (3.7). Therefore under the assumption of mathematical induction we prove that

$$G(\{2m, 2(m+s)\}) = 3m+2s. \tag{3.30}$$

Since $move(\{2m, 2(m+s)\})$

$$= \{\{2m-1, 2(m+s)\}, \{2m-2, 2(m+s)\}, ..., \{1, 2(m+s)\}, \{0, 2(m+s)\}\} \tag{3.31}$$
$$\cup\{\{2m, 2(m+s)-1\}, \{2m, 2(m+s)-2\}, ..., \{2m, 2m\}\} \tag{3.32}$$
$$\cup\{\{2m-1, 2m-1\}, \{2m-2, 2m-2\}, ...\{1, 1\}, \{0, 0\}\}, \tag{3.33}$$

by the definition of Grundy number we have to show that $3m+2s$ is the smallest non-negative integer that is not included in the sets of Grundy numbers of positions in (3.31),(3.32),(3.33), if we are to prove (3.30).
We calculate Grundy numbers of positions in (3.31) using the assumption of mathematical induction, and divide them into two groups.

The first group is the Grundy number of the positions in (3.31) whose y-coordinate (the first coordinate) is even;

$$G(\{2m-2,2(m+s)\}) = 3m+2s-1, G(\{2m-4,2(m+s)\}) = 3m+2s-2, ..., \\ G(\{2m-2k,2(m+s)\}) = 3m+2s-k, ..., G(\{0,2(m+s)\}) = 2m+2s. \tag{3.34}$$

The second group is the Grundy number of the positions in (3.31) whose y-coordinate (the first coordinate) is odd;

$$G(\{2m-1,2(m+s)\}) = m+2s-\max(0,m-s-1), \\ G(\{2m-3,2(m+s)\}) = m+2s+1-\max(0,m-s-3), \\ ...G(\{2m-(2k+1),2(m+s)\}) = m+2s+k-\max(0,m-s-2k-1), ..., \\ G(\{1,2(m+s)\}) = 2m+2s-1-\max(0,-m-s+1). \tag{3.35}$$

We calculate Grundy numbers of positions in (3.32) using the assumption of mathematical induction, and divide them into two groups.
The first group is the Grundy number of the positions in (3.32) whose y-coordinate is even;

$$G(\{2m,2(m+s)-2\}) = 3m+2s-2, ..., G(\{2m,2s\}) = 2s+m, \\ G(\{2m,2s-2\}) = 2s+m-2, ..., G(\{2m,2m+2\}) = 3m+2, G(\{2m,2m\}) = 3m. \tag{3.36}$$

The second group is the Grundy number of the positions in (3.32) whose y-coordinate is odd;

$$G(\{2m,2(m+s-1)+1\}) = m+2s-1-\max(0,m-s+1), ... \\ G(\{2m,2(m+s-k)+1\}) = m+2s-(2k-1)-\max(0,m-s+k), ... \tag{3.37}$$

$$G(\{2m,4m+1\}) = 3m+1-\max(0,2m-2m) = 3m+1, \tag{3.38}$$

$$G(\{2m,4m-1\}) = 3m-1-\max(0,2m-2m+1) = 3m-2, \\ G(\{2m,4m-3\}) = 3m-5..., G(\{2m,2m+3\}) = 4, G(\{2m,2m+1\}) = 1. \tag{3.39}$$

We calculate Grundy numbers of positions in (3.33) using the assumption of mathematical induction;

$$G(\{2m-1,2m-1\}) = 3m-1, G(\{2m-2,2m-2\}) = 3m-3, G(\{2m-3,2m-3\}) = 3m-4, ..., \\ G(\{2m-2k,2m-2k\}) = 3m-3k, G(\{2m-(2k+1),2m-(2k+1)\}) = 3m-3k-1, ... \\ G(\{1,1\}) = 2, G(\{0,0\}) = 0. \tag{3.40}$$

In the proof of (3.30) we have three cases. The first case is when $m-s+1 \leq 0$, the second case is when $m-s-1 \geq 0$ and the third case is when $m-s=0$.
Note that the negation of $m-s+1 \leq 0$ is $m-s-1 \geq 0$ or $m-s=0$.
[2.1] *We suppose that $m-s+1 \leq 0$. Then the Grundy numbers of the postions in (3.34) and (3.35) are $\{3m+2s-1, 3m+2s-2..., 2m+2s\}$ and $\{m+2s, m+2s+1, ..., 2m+2s-1\}$ respectively, and the union of these sets is the set*

$$\{3m+2s-1, 3m+2s-2, ..., m+2s\}, \tag{3.41}$$

and this is an arithmetic progression with common difference of -1.
Grundy numbers in (3.36) is the set

$$\{3m+2s-2, 3m+2s-4, ..., 3m\}, \tag{3.42}$$

and this is an arithmetic progression with common difference of -2.
Since $m - s + 1 \leq 0$, *Grundy numbers in (3.37) and (3.38) is the set*

$$\{m + 2s - 1, m + 2s - 3, ..., 3m + 1\}. \tag{3.43}$$

This is an arithmetic progression with common difference of -2.
The union of sets (3.42) and (3.43) contains the set

$$\{m + 2s - 1, m + 2s - 2, ..., 3m\}, \tag{3.44}$$

and this is an arithmetic progression with common difference of -1.
Grundy numbers in the union of sets (3.42) and (3.43) that do not belong to (3.44) is bigger than $m+2s-1$ *and smaller than* $3m + 2s$, *and hence they belong to (3.41). Therefore they are irrelevant to the proof of (3.30).*
Grundy numbers of the positions in (3.39) is the set

$$\{3m - 2, 3m - 5, ..., 1\}, \tag{3.45}$$

and this is an arithmetic progression with common difference of -3. *Grundy numbers of the positions in (3.40) is the set*

$$\{3m - 1, 3m - 3, ..., 3, 2, 0\}, \tag{3.46}$$

and this set does not contain numbers of the type $3h + 1$ *with* $h \in Z_{\geq 0}$. *The union of sets (3.45) and (3.46) is the set*

$$\{3m - 1, 3m - 2, ...0\}, \tag{3.47}$$

and this is an arithmetic progression with common difference of -1.
By (3.41),(3.44),(3.47) the Grundy numbers of the positions in $move(\{2m, 2n\})$ *is the set* $\{0, 1, 2, 3..., 3m + 2s - 1\}$, *and hence we have* $G(\{2m, 2(m + s)\}) = 3m + 2s$.
[2.2] *Next we suppose that* $m-s-1 \geq 0$. *Let* $k_- = \lfloor \frac{m-s-1}{2} \rfloor$ *and* $k_+ = \lceil \frac{m-s-1}{2} \rceil$. *Note that* $k_- = \max(k, k \in Z_{\geq 0}$ *and* $m - s - 2k - 1 \geq 0)$ *and* $k_+ = \min(k, k \in Z_{\geq 0}$ *and* $m - s - 2k - 1 \leq 0)$.
Grundy numbers of the positions in (3.35) are

$$G(\{2m - 1, 2(m + s)\}) = m + 2s - \max(0, m - s - 1) = 3s + 1,$$
$$G(\{2m - 3, 2(m + s)\}) = m + 2s + 1 - \max(0, m - s - 3) = 3s + 4, ...,$$
$$G(\{2m - (2k_- + 1), 2(m + s)\}) = m + 2s + k_- - \max(0, m - s - (2k_- + 1)) = 3s + 3k_- + 1, \tag{3.48}$$
$$G(\{2m - (2k_+ + 1), 2(m + s)\}) = m + 2s + k_+ - \max(0, m - s - (2k_+ + 1)) = m + 2s + k_+,$$
$$G(\{2m - (2k_+ + 3), 2(m + s)\}) = m + 2s + k_+ + 1 - \max(0, m - s - (2k_+ + 3)) = m + 2s + k_+ + 1,$$
$$..., G(\{1, 2(m + s)\}) = 2m + 2s - 1. \tag{3.49}$$

Since $m - s - 1 \geq 0$,

$$\begin{aligned} & 3m - 1 - (3s + 3k_- + 2) \\ & \geq 3m - 3s - 3 - 3 \times \frac{m - s - 1}{2} \\ & = \frac{3m - 3s - 3}{2} \\ & = \frac{3(m - s - 1)}{2} \geq 0. \end{aligned} \tag{3.50}$$

If $m - s - 1$ *is even, then* $k_+ = k_-$ *and*

$$m + 2s + k_+ - (3s + 3k_- + 1) = m - s - 2k_1 - 1 = m - s - (m - s - 1) - 1 = 0. \tag{3.51}$$

If $m-s-1$ is odd, then

$$\begin{aligned} m+2s+k_+-(3s+3k_-+1) &= m-s+k_+-3k_--1 \\ &= m-s+\frac{m-s}{2}-3\times\frac{m-s-2}{2}-1=2. \end{aligned} \tag{3.52}$$

Grundy numbers of the positions in (3.48) is the set

$$\{3s+1, 3s+4, ..., 3s+3k_-+1\}, \tag{3.53}$$

and this is an arithmetic progression with common difference of 3.
Since $m-s-1 \geq 0$, the union of the set of Grundy numbers of the positions in (3.37) and (3.39) is the set

$$\{m+2s-1-(m-s+1)=3s-2, m+2s-3-(m-s+2)=3s-5, ..., 4, 1\}, \tag{3.54}$$

and this is an arithmetic progression with common difference of -3. *By the fact that $m-s>0$ we have $3s-2<3m+1$, and hence the Grundy number of the position in (3.38) does not exist.*
The union of the sets of Grundy numbers of the positions in (3.53) and (3.54) is the set

$$\{3s+3k_-+1, ..., 4, 1\}, \tag{3.55}$$

and this is an arithmetic progression with common difference of -3. *Grundy numbers in (3.40) is the set*

$$\{3m-1, 3m-3, 3m-4, ...2, 0\}, \tag{3.56}$$

and this set does not contain numbers of the type $3h+1$ with $h \in Z_{\geq 0}$.
By (3.50) the Grundy numbers in (3.55) and (3.56) is the set

$$\{3s+3k_-+2, ..., 3, 2, 1, 0\}, \tag{3.57}$$

and this is an arithmetic progression with common difference of -1.
The union of the sets of Grundy numbers in (3.34) and (3.49) is the set

$$\{m+2s+k_+, ...3m+2s-2, 3m+2s-1\}, \tag{3.58}$$

and this is an arithmetic progression with common difference of 1.
By (3.34), (3.51), (3.52), (3.57) and (3.58) the set of Grundy numbers of the positions in move($\{2m, 2(m+s)\}$) is the set

$$\{0, 1, ...3m+2s-1\}, \tag{3.59}$$

and by the definition of Grundy number $G(\{2m, 2(m+s)\} = G(\{2m, 2(m+s)\} = 3m+2s$.
[2.3] *We suppose that $m-s=0$. Then the situation is slightly different from the situation in* [2.1].
The Grundy numbers of the postions in (3.34) and (3.35) are $\{3m+2s-1, 3m+2s-2..., 2m+2s\}$ and $\{m+2s, m+2s+1, ..., 2m+2s-1\}$ respectively, and the union of these sets contatins the set

$$\{3m+2s-1, 3m+2s-2, ..., m+2s\}, \tag{3.60}$$

and this is the same as (3.41) in [2.1].
Grundy numbers in (3.37) and (3.38) is the set $\{m+2s-2, m+2s-3, m+2s-5, .., 3m+1\}$ instead of $\{m+2s-1, m+2s-3, m+2s-5, ..., 3m+1\}$ that is (3.43) in [2.1]. *Therefore we need to find $m+2s-1$. Since we have $m+2s-1=3m-1$ in (3.47), we prove $G(\{2m, 2(m+s)\})=3m+2s$ by modifying the proof in* [2.1].
[3] *Next we prove (3.8). Therefore under the assumption of mathematical induction we prove that*

$$G(\{2m+1, 2(m+s)\}) = m+2s-1-\max(0, m-s+1). \tag{3.61}$$

Since $move(\{2m+1, 2m+2s\})$

$$= \{\{2m, 2(m+s)\}, \{2m-1, 2(m+s)\}, ...\{1, 2(m+s)\}, \{0, 2(m+s)\}\} \tag{3.62}$$

$$\cup\{\{2m+1, 2(m+s)-1\}, \{2m+1, 2(m+s)-2\}, ..., \{2m+1, 2m+1\}\} \tag{3.63}$$

$$\cup\{\{2m, 2m\}, \{2m-1, 2m-1\}, ...\{1, 1\}, \{0, 0\}\}, \tag{3.64}$$

by the definition of Grundy number we have to prove that $m + 2s - 1 - \max(0, m - s + 1)$ *is the smallest non-negative integer not included in sets of Grundy numbers of positions in (3.64),(3.62),(3.63) if we are to prove (3.61).*

We calculate Grundy numbers of the positions in (3.62) using the assumption of mathematical induction; $G(\{2m, 2(m+s)\}) = 3m+2s$, $G(\{2m-1, 2(m+s)\}) = m+2s-\max(0, m-s-1)$, $G(\{2m-2, 2(m+s)\}) = 3m+2s-1$, $G(\{2m-3, 2(m+s)\}) = m+2s+1-\max(0, m-s-3)$,...,$G(\{2, 2(m+s)\}) = 2m+2s+1$, $G(\{1, 2(m+s)\}) = 2m+2s-1-\max(0, -m-s+1)$, $G(\{0, 2(m+s)\} = 2m+2s$. *These Grundy number are bigger than* $m+2s-1-\max(0, m-s+1)$ *that is the right term of (3.61), and hence they are irrelevant to the proof of (3.61).*

We calculate Grundy numbers of the positions in (3.63), and divide then into two groups. The first group is the Grundy number of the positions in (3.63) whose z*-coordinate is odd.*

$$G(\{2m+1, 2(m+s)-1\}) = 3m+2s, G(\{2m+1, 2(m+s)-3\}) = 3m+2s-2, \\ ..., G(\{2m+1, 2s-1\}) = m+2s \tag{3.65}$$

$$G(\{2m+1, 2s-3\}) = m+2s-2, G(\{2m+1, 2s-5\}) = m+2s-4, \\ ..., G(\{2m+1, 2m+3\}) = 3m+4, G(\{2m+1, 2m+1\}) = 3m+2. \tag{3.66}$$

Grundy numbers in (3.65) are bigger than the right term of (3.61), so they are irrelevant to the proof of (3.61).

The second group is the Grundy number of the positions in (3.63) whose z*-coordinate is even.*

$$G(\{2m+1, 2(m+s)-2\}) = m+2s-3-\max(0, m-s+2), \\ G(\{2m+1, 2(m+s)-4\}) = m+2s-5-\max(0, m-s+3), ..., \\ G(\{2m+1, 2(m+s-k)\}) = m+2s-2k-1-\max(0, k+m-s+1), \tag{3.67}$$

$$..., G(\{2m+1, 4m+2\}) = 3m+1-\max(0, 2m-2m) = 3m+1, \tag{3.68}$$

$$G(\{2m+1, 4m\}) = 3m-2, ..., G(\{2m+1, 2m+4\}) = 4, G(\{2m+1, 2m+2\}) = 1. \tag{3.69}$$

We calculate Grundy numbers of the positions in (3.64);

$$G(\{2m, 2m\}) = 3m, G(\{2m-1, 2m-1\}) = 3m-1, G(\{2m-2, 2m-2\}) = 3m-3, ..., \\ G(\{2m-2k, 2m-2k\}) = 3m-3k, G(\{2m-(2k+1), 2m-(2k+1)\}) = 3m-3k-1, ..., \\ G(\{1, 1\}) = 2, G(\{0, 0\}) = 0. \tag{3.70}$$

In the proof of (3.61) we have two cases. The first case is when $m - s + 2 \leq 0$ *and the second case is when* $m - s + 2 > 0$.

[3.1] *We suppose that* $m-s+2 \leq 0$, *then the right part of (3.61) is* $m+2s-1$. *Grundy numbers of positions in (3.66) is the set*

$$\{m+2s-2, m+2s-4, ..., 3m+4, 3m+2\}, \tag{3.71}$$

and this is an arithmetic progression with common difference of -2. *Since* $m - s + 2 \leq 0$, *Grundy numbers of positions in (3.67) and (3.68) is the set*

$$\{m+2s-3, m+2s-5, ..., 3m+1\}, \tag{3.72}$$

and this is an arithmetic progression with common difference of -2. *Grundy numbers of positions in (3.69) is the set*

$$\{3m-2, ..., 4, 1\}, \tag{3.73}$$

and this is an arithmetic progression with common difference of -3.
Grundy numbers of positions in (3.70) is the set

$$\{3m, 3m-1, 3m-3, ..., 3m-3k, 3m-3k-1, ...2, 0\}, \tag{3.74}$$

and this set does not contains the number of the type $3h+1$ *with* $h \in Z_{\geq 0}$.
The union of two sets (3.71) and (3.72) is the set

$$\{m+2s-2, m+2s-3, m+2s-4, ..., 3m+1\}, \tag{3.75}$$

and this is an arithmetic progression with common difference of -1.
The union of two sets (3.73) and (3.74) is the set

$$\{3m, ..., 0\}. \tag{3.76}$$

The union of two sets (3.75) and (3.76) is $\{0, 1, ...m+2s-2\}$, *and by the definition of Grundy number we have* $G(\{2m+1, 2(m+s)\}) = m+2s-1$.
[3.2] *We suppose that* $m-s+2 > 0$. *Then we have* $m-s+1 \geq 0$, *and the right part of (3.61) is* $3s-2$. *By* $m-s+1 \geq 0$ *we have* $3s-2 < 3m+2$, *and hence Grundy numbers in (3.65),(3.66) are bigger than* $3s-2$, *and hence they are irrelevant to the proof of (3.61). Since* $m-s+2 > 0$, *Grundy numbers in (3.69) and (3.67) are*

$$\begin{aligned} G(\{2m+1, 2(m+s)-2\}) &= m+2s-3-\max(0, m-s+2) = 3s-5, \\ G(\{2m+1, 2(m+s)-4\}) &= m+2s-5-\max(0, m-s+3) = 3s-8, ..., \\ G(\{2m+1, 2(m+s-k)\}) &= m+2s-2k-1-\max(0, k+m-s+1) = 3s-3k-2, \\ ..., G(\{2m+1, 4m+2\}) &= 3m+1, G(\{2m+1, 4m\}) = 3m-2, ... \\ , G(\{2m+1, 2m+4\}) &= 4, G(\{2m+1, 2m+2\}) = 1. \end{aligned} \tag{3.77}$$

Grundy numbers in (3.77) is an arithmetic progression with common difference of -3.
Note that Grundy numbers in (3.68) do not exist, since $3s-2 < 3m+2$ *implies* $3s-5 < 3m+1$.
We divide Grundy numbers in (3.70) into two groups.

$$G(\{2m, 2m\}) = 3m, ..., G(\{2s-1, 2s-1\}) = 3s-1, \tag{3.78}$$

$$G(\{2s-2, 2s-2\}) = 3s-3, ..., G(\{1, 1\}) = 2, G(\{0, 0\}) = 0. \tag{3.79}$$

Grundy numbers in (3.78) is bigger than $3s-2$, *and the set of Grundy numbers in (3.79) does not contains the number of the type* $3h+1$ *with* $h \in Z_{\geq 0}$.
The union of two sets (3.77) and (3.79) is $\{3s-3, ..., 1, 0\}$. *By definition of Grundy number we have* $G(\{2m+1, 2(m+s)\}) = 3s-2$.
[4] *Next we prove (3.10). Therefore under the assumption of mathematical induction we prove that*

$$G(\{2m+1, 2(m+s)+1\}) = 3m+2s+2. \tag{3.80}$$

Since move$\{2m+1, 2(m+s)+1\}$

$$= \{\{2m, 2(m+s)+1\}, \{2m-1, 2(m+s)+1\}, ..., \{1, 2(m+s)+1\}, \{0, 2(m+s)+1\}\} \tag{3.81}$$

$$\cup \{\{2m+1, 2(m+s)\}, \{2m+1, 2(m+s)-1\}, ..., \{2m+1, 2m+2\}, \{2m+1, 2m+1\}\} \tag{3.82}$$

$$\cup \{\{2m, 2m\}, \{2m-1, 2m-1\}, ..., \{1, 1\}, \{0, 0\}\}, \tag{3.83}$$

by the definition of Grundy number we have to prove that $3m + 2s + 2$ *is the smallest non-negative integer not included in sets of Grundy numbers of positions in (3.82),(3.82),(3.83), if we are to prove (3.80). We calculate Grundy numbers of the positions in (3.81), and divide them into two groups. The first group is the Grundy numbers of the positions in (3.81) whose y-coordinate is odd;*

$$G(\{2m-1, 2(m+s)+1\}) = 3m+2s+1, G(\{2m-3, 2(m+s)+1\}) = 3m+2s, ...,$$
$$G(\{2m-(2k+1), 2(m+s)+1\}) = 3m+2s+1-k, ..., G(\{1, 2(m+s)+1\}) = 2m+2s+2. \quad (3.84)$$

The second group is the Grundy numbers of the positions in (3.81), whose y-coordinate is even;

$$G(\{2m, 2(m+s)+1\}) = m+2s+1-\max(0, m-s),$$
$$G(\{2m-2, 2(m+s)+1\}) = m+2s+2-\max(0, m-s-2), ...,$$
$$G(\{2m-2k, 2(m+s)+1\}) = m+2s+1+k-\max(0, m-s-2k), ...,$$
$$G(\{0, 2(m+s)+1\}) = 2m+2s+1. \quad (3.85)$$

We calculate Grundy numbers of the positions in (3.82), and divide them into two groups. The first group is the Grundy number of the positions in (3.82) whose z-coordinate is odd;

$$G(\{2m+1, 2(m+s)-1\}) = 3m+2s, G(\{2m+1, 2(m+s)-3\} = 3m+2s-2, ...,$$
$$G(\{2m+1, 2s-1\}) = m+2s, ..., G(\{2m+1, 2m+1\}) = 3m+2. \quad (3.86)$$

The first group is the Grundy number of the positions in (3.82) whose z-coordinate is even;

$$G(\{2m+1, 2(m+s)\}) = m+2s-1-\max(0, m-s+1), ...,$$
$$G(\{2m+1, 2(m+s)-2\}) = m+2s-3-\max(0, m-s+2), ...,$$
$$G(\{2m+1, 2(m+s-k)\}) = m+2s-1-2k-\max(0, m-s+1+k) \quad (3.87)$$
$$, ..., G(\{2m+1, 4m+2\}) = 3m+1-\max(0, 2m-2m) = 3m+1, \quad (3.88)$$
$$G(\{2m+1, 4m\}) = 3m-1-\max(0, 2m-2m+1) = 3m-2,$$
$$G(\{2m+1, 4m-2\}) = 3m-5, ..., G(\{2m+1, 2m+4\}) = 4, G(\{2m+1, 2m+2\}) = 1. \quad (3.89)$$

We calculate Grundy numbers of the positions in (3.83);

$$G(\{2m, 2m\}) = 3m, G(\{2m-1, 2m-1\}) = 3m-1, G(\{2m-2, 2m-2\}) = 3m-3, ...,$$
$$G(\{2m-2k, 2m-2k\}) = 3m-3k, G(\{2m-(2k+1), 2m-(2k+1)\}) = 3m-3k-1, ...,$$
$$G(\{1,1\}) = 2, G(\{0,0\}) = 0. \quad (3.90)$$

In the proof of (3.80) we have two cases. The first case is when $m-s+1 \leq 0$ *and the second case is when* $m-s+1 > 0$.
[4.1] *We suppose that* $m-s+1 \leq 0$.
Grundy numbers of the positions in (3.84) and(3.85) are $\{3m+2s+1, 3m+2s, ..., 2m+2s+2\}$ *and* $\{m+2s+1, m+2s+2, ..., 2m+2s+1\}$, *and the union of these two sets is the set*

$$\{3m+2s+1, 3m+2s, ..., m+2s+1\}, \quad (3.91)$$

and this is an arithmetic progression with common difference of -1.
Grundy numbers of the positions in (3.86) is the set

$$\{3m+2s, 3m+2s-2, ..., 3m+2\}, \quad (3.92)$$

which is an arithmetic progression with common difference of -2.
Grundy numbers of the positions in (3.87) and (3.88) is the set

$$\{m + 2s - 1, m + 2s - 3, ..., 3m + 1\}, \tag{3.93}$$

which is an arithmetic progression with common difference of -2.
The union of sets (3.92) and (3.93) contains the set

$$\{m + 2s, m + 2s - 1, ..., 3m + 1\}, \tag{3.94}$$

and this is an arithmetic progression with common difference of -1.
Grundy numbers that belong to the union of sets (3.92) and (3.93) and do not belong to the set (3.94) are contained in (3.91).
Grundy numbers of the positions in (3.89) is the set

$$\{3m - 2, 3m - 5, ...1\}, \tag{3.95}$$

which is an arithmetic progression with common difference of -3. *Grundy numbers of the positions in (3.90) is the set*

$$\{3m, 3m - 1, ..., 2, 0\}, \tag{3.96}$$

and this set does not contains the number of the type $3h + 1$ *with* $h \in Z_{\geq 0}$.
The union of sets (3.95) and (3.96) is the set

$$\{3m, 3m - 1, ..., 1, 0\}, \tag{3.97}$$

and this is an arithmetic progression with common difference of -1.
By (3.91),(3.94),(3.97) the set of Grundy numbers of the positions in $move(\{2m + 1, 2m + 2s + 1\})$ *is the set* $\{0, 1, ...3m + 2s, 3m + 2s + 1\}$, *and hence* $G(\{2m + 1, 2m + 2s + 1\}) = 3m + 2s + 2$.
[4.2] *We assume that* $m - s + 1 > 0$. *Let* $k_- = \lfloor \frac{m-s}{2} \rfloor$ *and* $k_+ = \lceil \frac{m-s}{2} \rceil$. *Note that* $k_- = \max(k, k \in Z_{\geq 0}$ *and* $m - s - 2k \geq 0)$ *and* $k_+ = \min(k, k \in Z_{\geq 0}$ *and* $m - s - 2k \leq 0)$.
Since $m - s \geq 0$, *Grundy numbers of the positions in (3.85) are*

$$\begin{aligned} G(\{2m, 2(m+s)+1\}) &= m + 2s + 1 - \max(0, m - s) = 3s + 1, \\ G(\{2m - 2, 2(m+s)+1\}) &= m + 2s + 2 - \max(0, m - s - 2) = 3s + 4, ..., \\ G(\{2m - 2k_-, 2(m+s)+1\}) &= m + 2s + k_- + 1 - \max(0, m - s - 2k_-) = 3s + 3k_- + 1, \quad (3.98) \\ G(\{2m - 2k_+, 2(m+s)+1\}) &= m + 2s + k_+ + 1 - \max(0, m - s - 2k_+) = m + 2s + k_+ + 1, \\ &..., G(\{0, 2(m+s)+1\}) = 2m + 2s + 1. \quad (3.99) \end{aligned}$$

If $m - s$ *is even, then* $k_+ = k_-$ *and*

$$m + 2s + k_+ + 1 - (3s + 3k_- + 1) = m - s - 2k_1 = m - s - (m - s) = 0. \tag{3.100}$$

If $m - s$ *is odd, then*

$$\begin{aligned} m + 2s + k_+ + 1 - (3s + 3k_- + 1) &= m - s + k_+ - 3k_- \\ &= m - s + \frac{m - s + 1}{2} - 3 \times \frac{m - s - 1}{2} = 2. \end{aligned} \tag{3.101}$$

Grundy numbers of the positions in (3.98) are

$$\{3s + 1, 3s + 4, ..., 3s + 3k_- + 1\}, \tag{3.102}$$

and this is an arithmetic progression with common difference of 3.
Since $m - s + 1 > 0$, *the union of Grundy numbers of the positions in (3.87) and (3.89) is the set*

$$\{3s - 2, 3s - 5, ..., 4, 1\}, \tag{3.103}$$

and this is an arithmetic progression with common difference of -3.
Note that by $m - s + 1 > 0$ *we have* $3m + 1 > 3s - 2$, *and hence we do not have Grundy numbers in (3.88).*
The union of Grundy numbers of the positions in (3.102) and (3.103) is the set

$$\{3s + 3k_- + 1, ...4, 1\}, \tag{3.104}$$

and this is an arithmetic progression with common difference of -3. *The set of Grundy numbers in (3.90) is the set*

$$\{3m, 3m - 1, 3m - 3, 3m - 4, ..., 2, 0\}, \tag{3.105}$$

and this set does not contains the number of the type $3h + 1$ *with* $h \in Z_{\geq 0}$.
Since $3m + 2s + 1 \geq m + 2s + k_+ + 1$, *The union of the sets of Grundy numbers of the positions in (3.84) and(3.99) contains the set*

$$\{m + 2s + k_+ + 1, ..., 3m + 2s, 3m + 2s + 1\}, \tag{3.106}$$

which is an arithmetic progression with common difference of 1.
Since $m - s + 1 > 0$, *we have* $m - s = 0$ *or* $m - s \geq 1$.
[4.2.1] *Suppose that* $m - s = 0$. *Then* $3s + 3k_- = 3m$, *and hence by (3.104) and (3.105) we have*

$$\{3s + 3k_- + 1, ..., 3, 2, 1, 0\}. \tag{3.107}$$

By (3.100) the union of the sets of Grundy numbers of (3.107) and (3.106) is the set

$$\{0, 1, ..., 3m + 2s + 1\}, \tag{3.108}$$

and this is an arithmetic progression with common difference of 1.
Grundy numbers in (3.86) are contained in (3.108).
Therefore the set of Grundy numbers in $move(\{2m + 1, 2(m + s) + 1\})$ *is (3.108), and we have* $G(\{2m + 1, 2(m + s) + 1\}) = 3m + 2s + 2$.
[4.2.2] *Suppose that* $m - s \geq 1$. *Then* $3m - (3s + 3k_- + 2) \geq 3m - 3s - 3 \times \frac{m-s}{2} - 2 = \frac{3(m-s)-4}{2} \geq \frac{-1}{2}$.
Therefore we have $3m \geq 3s + 3k_- + 2$. *Then by (3.104) and (3.105) we have*

$$\{3s + 3k_- + 2, ..., 3, 2, 1, 0\}, \tag{3.109}$$

and this is an arithmetic progression with common difference of -1.
By (3.101) the union of the sets of Grundy numbers in (3.106) and (3.109) is the set

$$\{0, 1, ..., 3m + 2s + 1\}, \tag{3.110}$$

and this is an arithmetic progression with common difference of 1.
Grundy numbers in (3.86) are contained in (3.110), and hence the set of Grundy numbers of positions in $move(\{2m+1, 2(m+s)+1\})$ *is the set of Grundy numbers in (3.110). Therefore we have* $G(\{2m+1, 2(m+s)+1\}) = 3m + 2s + 2$.

4 The case of $f(t) = \lfloor \frac{t}{k} \rfloor$ for an odd number k.

The authors discovered the following Prediction 4.1, but they have not find a way to prove this. The authors present a part of a proof for the case of $k = 3$.

Prediction 4.1. *Let $f(t) = \lfloor \frac{t}{k} \rfloor$ for an odd number k. Then Grundy number $G(\{y, z\})$ of $CB(f, y, z)$ satisfies the following equations for $p, q \in Z_{\geq 0}$.*

$$G(\{2p, 2q\}) = p + 2q \tag{4.1}$$

$$G(\{2p + 1, 2q + 1\}) = p + 2q + 2 \tag{4.2}$$

$G(\{2p, 2q + 1\})$

$$= \begin{cases} -p + 2q + 1 & \text{when } 2p \times (k + 1) \leq 2q \quad (4.3) \\ -2p + 2q + 1 + \lfloor \frac{-2p+2q+1}{2k} \rfloor & \text{when } 2p \times (k + 1) > 2q \quad (4.4) \end{cases}$$

$G(\{2p + 1, 2q\})$

$$= \begin{cases} -p + 2q - 1 & \text{when } (2p + 1) \times (k + 1) \leq 2q \quad (4.5) \\ -2p + 2q - 1 + \lfloor \frac{-2p+2q-1}{2k} \rfloor & \text{when } (2p + 1) \times (k + 1) > 2q \quad (4.6) \end{cases}$$

It is clear that Theorem 3.1 is a case for $k = 1$. Note that in Theorem 3.1 the authors used p and r, but in Prediction 4.1 the authors used p, q. Prediction 4.1 is the same as Theorem 3.1 if we let $k = 1$ and $q = p + r$.
The following is the theorem for $k = 3$ and a part of the proof.

Prediction 4.2. *Let $f(t) = \lfloor \frac{t}{3} \rfloor$. Then Grundy number $G(\{y, z\})$ of $CB(f, y, z)$ satisfies the following equations for $p, q \in Z_{\geq 0}$.*

$$G(2p, 2q) = p + 2q. \tag{4.7}$$

$$G(2p + 1, 2q + 1) = p + 2q + 2. \tag{4.8}$$

$G(\{2p, 2q + 1\})$

$$= \begin{cases} -p + 2q + 1 & \text{when } 2p \times 4 \leq 2q \quad (4.9) \\ -2p + 2q + 1 + \lfloor \frac{-2p+2q+1}{6} \rfloor & \text{when } 2p \times 4 > 2q \quad (4.10) \end{cases}$$

$G(\{2p + 1, 2q\})$

$$= \begin{cases} -p + 2q - 1 & \text{when } (2p + 1) \times 4 \leq 2q \quad (4.11) \\ -2p + 2q - 1 + \lfloor \frac{-2p+2q-1}{6} \rfloor & \text{when } (2p + 1) \times 4 > 2q \quad (4.12) \end{cases}$$

Proof. [1]
We prove (4.9) by mathematical induction.
[1.1]
We suppose that $\{2p, 2q+1\} = \{6m, 18m+6s+1\}$ for $m, s \in Z_{\geq 0}$ under the condtion $2p \times 4 \leq 2q$, and prove

$$G(\{6m, 18m + 6s + 1\}) = 15m + 6s + 1. \tag{4.13}$$

In this case the condition $2p \times 4 \leq 2q$ *is mathematically equivalence to the condition* $m \leq s$.
$move(\{6m, 18m + 6s + 1\})$

$$= \{\{6m - 1, 18m + 6s + 1\}, \{6m - 2, 18m + 6s + 1\}, ..., \{1, 18m + 6s + 1\}, \{0, 18m + 6s + 1\}\} \quad (4.14)$$
$$\cup\{\{6m, 18m + 6s\}, \{6m, 18m + 6s - 1\}, ..., \{6m, 18m + 1\}, \{6m, 18m\}\} \quad (4.15)$$
$$\cup\{\{6m - 1, 18m - 1\}, \{6m - 1, 18m - 2\}, ..., \{1, 1\}, \{0, 0\}\}. \quad (4.16)$$

If $m = 0$, *then* $move(\{6m, 18m + 6s + 1\}) = move(\{0, 6s + 1\})$

$$= \{\{0, 6s\}, \{0, 6s - 1\}, ..., \{0, 1\}, \{0, 0\}\}. \quad (4.17)$$

By using (4.7) and (4.9) we calculate Grundy numbers in (4.17), then we have $\{6s, 6s-1, 6s-2, ..., 3, 2, 1, 0\}$, *and hence by definition of Grundy number we have* $(\{0, 6s+1\}) = 6s+1$. *Therefore (4.13) is true for* $m = 0$. *Now we suppose that* $m > 0$.
We calculate Grundy numbers in (4.14). We use (4.8) when the y-coordinate (the first coordinate) is even. When the y-coordinate (the first coordinate) is odd, we use (4.9), since the y-coordinate is smaller than $6m$ *and satisfies the condition for (4.9).*
We have $G(\{6m - 1, 18m + 6s + 1\}) = 21m + 6s + 1, G(\{6m - 2, 18m + 6s + 1\}) = 15m + 6s + 2, G(\{6m - 3, 18m + 6s + 1\}) = 21m + 6s, ..., G(\{1, 18m + 6s + 1\}) = 18m + 6s + 2, G(\{0, 18m + 6s + 1\}) = 18m + 6s + 1$.
These Grundy numbers are bigger than $15m + 6s + 1$ *that is the right term of (4.13), and hence they are irrelevant to the proof of (4.13).*

We calculate Grundy numbers in (4.15), and divide them into two groups. The group of Grundy numbers in positions whose z-coordinate (the second coordinate) is even;

$$G(\{6m, 18m + 6s\}) = 21m + 6s, G(\{6m, 18m + 6s - 2\}) = 21m + 6s - 2, ...$$
$$, G(\{6m, 12m + 6s + 2\} = 15m + 6s + 2, \quad (4.18)$$
$$G(\{6m, 12m + 6s\} = 15m + 6s, G(\{6m, 12m + 6s - 2\} = 15m + 6s - 2, ...,$$
$$G(\{6m, 18m + 2\} = 21m + 2, G(\{6m, 18m\} = 21m. \quad (4.19)$$

These Grundy numbers in (4.18) are bigger than $15m + 6s + 1$ *that is the right term of (4.13), and hence they are irrelevant to the proof of (4.13).*
The group of Grundy numbers in positions whose z-coordinate (the second coordinate) is odd;

$$G(\{6m, 18m + 6s - 1\}) = 15m + 6s - 1, G(\{6m, 18m + 6s - 3\}) = 15m + 6s - 3, ...$$
$$, G(\{6m, 24m + 1\} = 21m + 1 \quad (4.20)$$
$$G(\{6m, 24m - 1\} = 21m - 2, G(\{6m, 24m - 3\} = 21m - 4,$$
$$G(\{6m, 24m - 5\} = 21m - 6, G(\{6m, 24m - 7\} = 21m - 9...,$$
$$G(\{6m, 18m + 3\} = 14m + 3, G(\{6m, 18m + 1\} = 14m + 1 \quad (4.21)$$

Grundy numbers in (4.21) are numbers of the type $7t + 5$ *or* $7t + 3$ *or* $7t + 1$ *for* $t \in Z_{\geq 0}$. *Here we used (4.9) for the calculation of (4.20), and we used (4.10) for the calculation of (4.21).*

We calculate Grundy numbers in (4.16), and divide them into three groups.
In the first group we calculate Grundy numbers of the position $\{(6m - k, 18m - (3k - 2))\}$ *with* $k \in Z_{\geq 0}$. *Here we use (4.7) and (4.8).*

$$G(\{6m - 1, 18m - 1\}) = 21m - 1, G(\{6m - 2, 18m - 4\}) = 21m - 5, G(\{6m - 3, 18m - 7\}) = 21m - 8,$$
$$.., G(\{3, 11\}) = 13, G(\{2, 8\}) = 9, G(\{1, 5\}) = 6, G(\{0, 2\}) = 2 \quad (4.22)$$

These numbers are of the type $7t+6$ *or* $7t+2$ *for* $t \in Z_{\geq 0}$.
In the second group we calculate Grundy numbers of the position $\{(6m-k, 18m-3k)\}$ *with* $k \in Z_{\geq 0}$. *Here we use (4.7) and (4.8).*

$$G(\{6m-1, 18m-3\}) = 21m-3, G(\{6m-2, 18m-6\}) = 21m-7, G(\{6m-3, 18m-9\}) = 21m-10, \\ .., G(\{3,9\}) = 11, G(\{2,6\}) = 7, G(\{1,3\}) = 4, G(\{0,0\}) = 0 \quad (4.23)$$

These numbers are of the type $7t+4$ *or* $7t$ *for* $t \in Z_{\geq 0}$.
In the third group we calculate Grundy numbers of the position $\{(6m-k, 18m-(3k-1))\}$ *with* $k \in Z_{\geq 0}$. *We use (4.9) for* $\{0,1\}$, *and we use (4.10) and (4.12) for other Grundy numbers, since* $m > 0$.

$$G(\{6m-1, 18m-2\}) = 14m-2, G(\{6m-2, 18m-5\}) = 14m-4, \\ G(\{6m-3, 18m-8\}) = 14m-6, G(\{6m-4, 18m-11\}) = 14m-9 \\ .., G(\{3,10\}) = 8, G(\{2,7\}) = 5, G(\{1,4\}) = 3, G(\{0,1\}) = 1 \quad (4.24)$$

These numbers are of the type $7t+5$ *or* $7t+1$ *for* $t \in Z_{\geq 0}$.
The union of the sets of Grundy numbers (4.19) and (4.20) is the set

$$\{15m+6s, 15m+6s-1, ..., 21m+1, 21m\}. \quad (4.25)$$

We divide Grundy numbers in (4.22) into two groups.

$$\{21m-1, 21m-5, 21m-8, ..., 14m+6, 14m+2\}. \quad (4.26)$$

$$\{14m-1, 14m-5, 14m-8.., 11, 7, 4, 0\}. \quad (4.27)$$

We divide Grundy numbers in (4.23) into two groups.

$$\{21m-3, 21m-7, 21m-10, ..., 14m+7, 14m+4\}. \quad (4.28)$$

$$\{14m, 14m-3, 14m-7.., 11, 7, 4, 0\}. \quad (4.29)$$

Therefore the union of the sets (4.21), (4.26) and (4.28) is the set

$$\{21m-1, 21m-2, 21m-3, ..., 14m+2, 14m+1\}. \quad (4.30)$$

The union of the sets (4.24),(4.27) and (4.29) is the set

$$\{14m, 14m-1, 14m-2, ..., 2, 1, 0\}. \quad (4.31)$$

The union of the sets (4.21),(4.30) and (4.31) is

$$\{15m+6s, 15m+6s-1, 15m+6s-2, ..., 2, 1, 0\}, \quad (4.32)$$

which is the set of Grundy numbers in $move(\{6m, 18m+6s+1\})$. *Therefore we have* $G(\{6m, 18m+6s+1\}) = 15m+6s+1$.

Clearly a small part of the proof for $k = 3$ is quite lengthy, so we have to find some ingenious way to prove this kind of theorem for an odd number k in general.

5 Unsolved Problems

There are many unsolved problems, and the authors present some examples hare to show the prospect of the research of chocolate games. The chocolate in Example 5.1 has two functions, and these two functions satisfy the condition of Definition 1.7. It is natural to study this kind of chocolate.
In other examples functions do not satisfy the condition of Definition 1.7, but Grundy numbers can be expressed with nim-sum. These example shows the difficulty of research, since each chocolate has its own unique features.

Example 5.1. *Chocolate Bar in Figure 5.1 have four directions to cut, and this can be expressed with 3 coordinates* $\{x, y, z\}$. *The authors discovered that the Grundy number of this is* $x \oplus y \oplus z$ *by calculation of computer, but they have not proved the fact.*

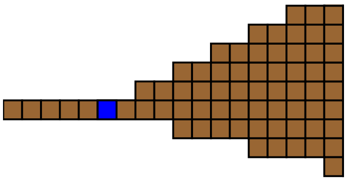


Figure 5.1.

Example 5.2. *The authors discovered that the Grundy number of Chocolate Bar in Figure 5.2 is* $y \oplus z$ *by calculation of computer, but they have not proved the fact.*

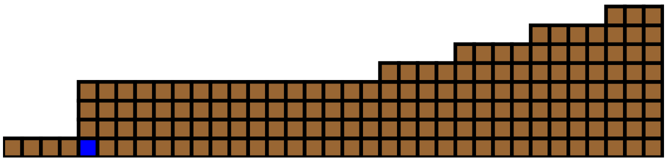


Figure 5.2.
$CB(f, y, z)$, *where* $f(t) = 3$ *for* $t = 0, 1, 2, ..., 11$ *and* $f(t) = \lfloor \frac{t}{4} \rfloor$.

Example 5.3. *The authors discovered that the Grundy number of Chocolate Bar in Figure 5.3 is* $y \oplus z$ *by calculation of computer, but they have not proved the fact.*

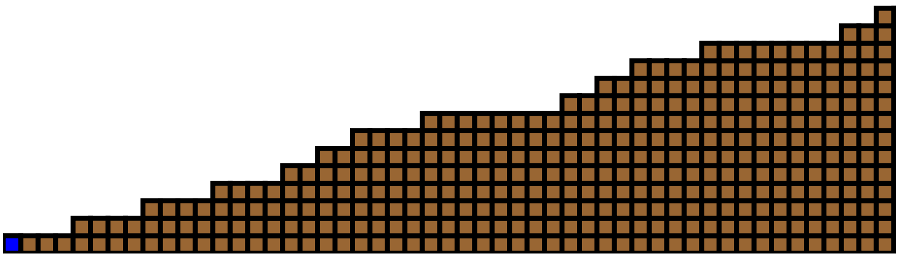


Figure 5.3.

For each $t \in Z_{\geq 0}$ *let* $m = t \pmod{16}$ *and let* $f(t)$

$$= \begin{cases} \lfloor \frac{t}{4} \rfloor & \text{when } t < 16 & (5.1) \\ \lceil \frac{\lfloor \frac{t}{2} \rfloor}{2} \rceil & \text{when } 0 \leq m \leq 3 & (5.2) \\ \lfloor \frac{t}{4} \rfloor + 1 & \text{when } 4 \leq m \leq 7 & (5.3) \\ 2(\lfloor \frac{t}{8} \rfloor) + 1 & \text{when } 8 \leq m \leq 15 & (5.4) \end{cases}$$

Example 5.4. *The authors discovered that the Grundy number of Chocolate Bar in Figure 5.4 is $y \oplus z$ by calculation of computer, but they have not proved the fact.*

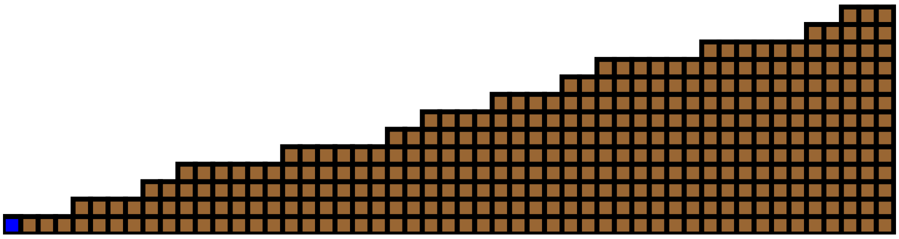


Figure 5.4.

For each $t \in Z_{\geq 0}$ let $v = t \pmod{24}$ and let $f(t)$

$$= \begin{cases} \lfloor \frac{t}{4} \rfloor & \text{when } v < 10 & (5.5) \\ \lfloor \frac{t}{4} \rfloor + 1 & \text{when } v < 12 & (5.6) \\ \lfloor \frac{t}{4} \rfloor & \text{when } v < 20 & (5.7) \\ \lfloor \frac{t}{4} \rfloor - 1 & \text{when } v < 22 & (5.8) \\ \lfloor \frac{t}{4} \rfloor & \text{when } v < 24 & (5.9) \end{cases}$$

Example 5.5. *The authors discovered that the Grundy number of Chocolate Bar in Figure 5.5 is $y \oplus z$ by calculation of computer, but they have not proved the fact.*

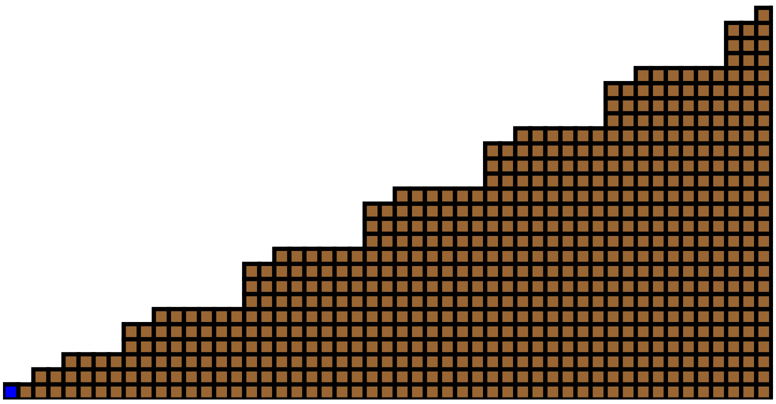


Figure 5.5.

For each $t \in Z_{\geq 0}$ let $m = t \pmod{8}$ and let $f(t)$

$$= \begin{cases} \lfloor \frac{t}{2} \rfloor & \text{when } t \leq 5 & (5.10) \\ \lfloor \frac{t}{2} \rfloor - 1 & \text{when } 6 \leq t \leq 7 & (5.11) \\ \lfloor \frac{t}{2} \rfloor & \text{when } 0 \leq m \leq 1 & (5.12) \\ 4(\lfloor \frac{t}{8} \rfloor) + 1 & \text{when } 2 \leq m \leq 7 & (5.13) \end{cases}$$

Example 5.6. *The authors discovered that the Grundy number of Chocolate Bar in Figure 5.6 is $y \oplus z$ by calculation of computer, but they have not proved the fact.*

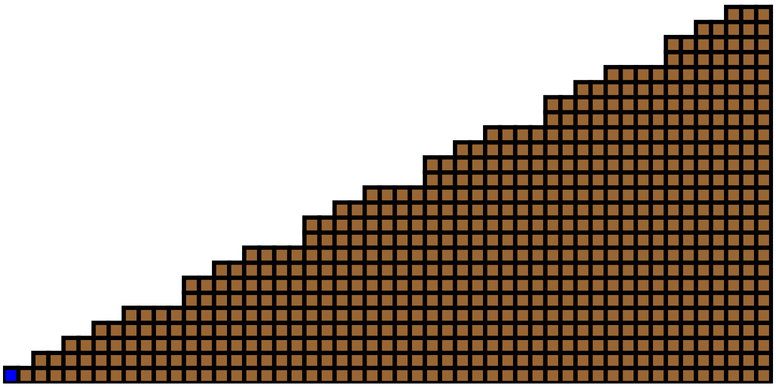


Figure 5.6.

For each $t \in Z_{\geq 0}$ let $m = t \pmod 8$ and let
$f(t)$

$$= \begin{cases} \lfloor \frac{t}{2} \rfloor & \textit{when } n < 4 & (5.14) \\ \lfloor \frac{t}{2} \rfloor & \textit{when } 0 \leq m \leq 1 & (5.15) \\ \lfloor \frac{t}{2} \rfloor - 1 & \textit{when } 2 \leq m \leq 3 & (5.16) \\ \lfloor \frac{t}{2} \rfloor & \textit{when } 4 \leq m \leq 7 & (5.17) \end{cases}$$

Example 5.7. *The authors discovered that the Grundy number of Chocolate Bar in Figure 5.7 is $y \oplus z$ by calculation of computer, but they have not proved the fact.*

Out[365]=

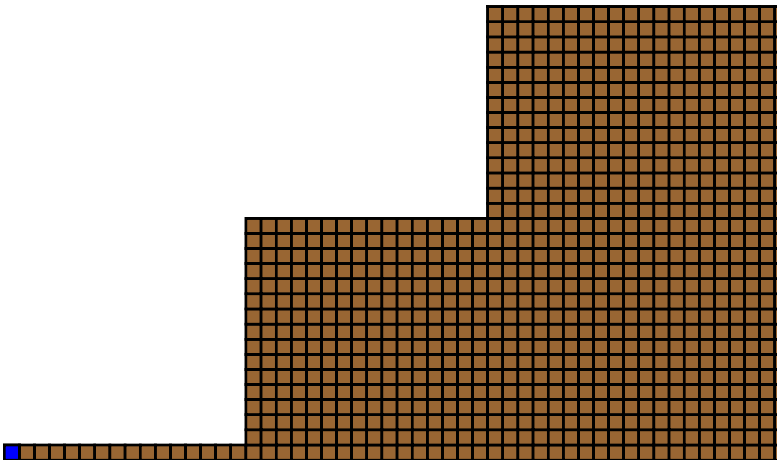


Figure 5.7.

$f(t)$

$$= \begin{cases} \lfloor \frac{t}{16} \rfloor & \textit{when } t \leq 15 & (5.18) \\ \lfloor \frac{t}{32} \rfloor + 15 & \textit{when } 16 \leq t \leq 31 & (5.19) \\ 2^{\lfloor \log_2 \frac{t}{16} + 4 \rfloor} - 1 & \textit{when } 32 \leq t & (5.20) \end{cases}$$

6 Quotation of the comment of the referee of a mathematics journal

The member of the team participated in Yau award 2012 submitted the paper to the mathematics journal "Integer" 2 years ago. The authors chose this journal, since they have special section for combinatorial games.

They received referee's comment twice, and revised their paper twice. They are waiting for the reply of the referee. Please permit us of quoting the following comment of the last referee's report. I regret not being clear enough in my last report. The paper has good results that deserved to be published. Unfortunately, the authors have hidden the results (nim-values of certain Chocolate Bar games) through overly emphasizing discussions and duplication of results. This makes the paper very hard to read and even understand what the authors have proven. In this report, I make detailed suggestions as how to improve the presentation from both mathematical and style viewpoints.

Clearly the referee of the paper think that the result of the research is worth publishing, but the authors of this paper (Yau award 2014) present a result that include the result of the above paper submitted to the journal Integer as a special example. This shows the importance of this year's Yau Award paper.

7 New result of our research

Throughout our research in Yau Award 2014 we have studied the condition for a function f to have Grundy number $G(\{y,z\}) = y \oplus z$.
After we submitted our research, we discovered a necessary and sufficient condition for a function f to have Grundy number $G(\{y,z\}) = y \oplus z$.
We are going to present this new result in the following sections. Definition 7.1 is the same as the definition of a chocolate bar treated in the paper for Yau Award 2014.

Definition 7.1. *Let f be a monotonically increasing function of $Z_{\geq 0}$ into $Z_{\geq 0}$.*
For $y, z \in Z_{\geq 0}$ the Chocolate Bar will consist of $z+1$ columns where the 0th column is the poison square and the height of the i-th column is $t(i) = \min(f(i), y) + 1$ for i = 0,1,...,z. We will denote this by $CB(f, y, z)$. Thus the width of the chocolate is determined by the value of z, and the height of the i-th column is determined by the value of $\min(f(i), y) + 1$ that is determined by h and y.

Let x be an arbitrary non-negative integer, then we write it in base 2, so $x = \sum\limits_{i=0}^{n} x_i 2^i$ with $x_i \in \{0,1\}$. We do the same for any non-negative integers.

8 A proof for a sufficient condition

Let $Z_{\geq 0}$ be the set of non-negative integers. The condition presented in Definition 8.1 is the necessary and sufficient condition for a function f to have Grundy number $G(\{y,z\}) = y \oplus z$.
In this section we prove that this is a sufficient condition.

Definition 8.1. *Let h be a function of $Z_{\geq 0}$ into $Z_{\geq 0}$ that satisfies the following condition* [1].
[1] *Suppose that*

$$h(\sum_{k=0}^{n} z_k 2^k) = \sum_{k=0}^{n} y_k 2^k.$$

Then for any z'_k for $k = 0, ..., i-1$ there exist y'_k for $k = 0, ..., i-2$ such that

$$h(\sum_{k=i}^{n} z_k 2^k + \sum_{k=0}^{i-1} z'_k 2^k) = \sum_{k=i-1}^{n} y_k 2^k + \sum_{k=0}^{i-2} y'_k 2^k.$$

Remark 8.1. *Suppose that $y = h(z)$. Then by Definition 8.1 the coefficients of $2^n, 2^{n-1}, ..., 2^i, 2^{i-1}$ of $h(z)$ are determined by the coefficients of $2^n, 2^{n-1}, ..., 2^i$ of z.*

Throughout this section we assume that the function h satisfies the condition of Definition 8.1.

Definition 8.2. *For $y, z \in Z_{\geq 0}$ we define*
$move(\{y, z\}) = \{\{\{v, z\}; v < y\} \cup \{\{x, \min(y, h(w)), w\}; w < z\}$, where $v, w \in Z_{\geq 0}$.

Lemma 8.1. *Suppose that*

$$h(\sum_{k=0}^{n} z_k 2^k) = \sum_{k=0}^{n} y_k 2^k. \tag{8.1}$$

Then

$$h(\sum_{k=i}^{n} z_k 2^k) \geq \sum_{k=i-1}^{n} y_k 2^k.$$

Proof. *Let $z'_k = 0$ for $k = 0, 1, 2, ..., i-1$.*
By (8.1) and Definition 8.1 there exist y'_k for $k = 0, ..., i-2$ such that

$$h(\sum_{k=i}^{n} z_k 2^k) = h(\sum_{k=i}^{n} z_k 2^k + \sum_{k=0}^{i-1} z'_k 2^k) = \sum_{k=i-1}^{n} y_k 2^k + \sum_{k=0}^{i-2} y'_k 2^k \geq \sum_{k=i-1}^{n} y_k 2^k.$$

Lemma 8.2. *[1] Suppose that*

$$h(\sum_{k=i}^{n} p_k 2^k) < \sum_{k=i-1}^{n} q_k 2^k. \tag{8.2}$$

Then for any p_k for $k = 0, 1, ..., i-1$

$$h(\sum_{k=0}^{n} p_k 2^k) < \sum_{k=i-1}^{n} q_k 2^k. \tag{8.3}$$

[2] Suppose that

$$h(\sum_{k=0}^{n} p_k 2^k) \geq \sum_{k=i-1}^{n} q_k 2^k. \tag{8.4}$$

Then

$$h(\sum_{k=i}^{n} p_k 2^k) \geq \sum_{k=i-1}^{n} q_k 2^k.$$

[3] Suppose that

$$\sum_{k=0}^{n} q_k 2^k \leq h(\sum_{k=0}^{n} p_k 2^k). \tag{8.5}$$

Then

$$h(\sum_{k=i}^{n} p_k 2^k) \geq \sum_{k=i-1}^{n} q_k 2^k. \tag{8.6}$$

[4] Suppose that

$$h(\sum_{k=i}^{n} p_k 2^k) < \sum_{k=j}^{n} q_k 2^k, \tag{8.7}$$

where $j \geq i-1$. Then for any t with $0 \leq t \leq i$ and any p_k for $k = i-t, .., i-1$ we have

$$h(\sum_{k=i-t}^{n} p_k 2^k) < \sum_{k=j}^{n} q_k 2^k.$$

Proof. [1] *Let $h(\sum_{k=0}^{n} p_k 2^k) = \sum_{k=0}^{n} r_k 2^k$. Then there exist r'_k for $k = 0, 1, 2, ..., i-2$ such that $h(\sum_{k=i}^{n} p_k 2^k) = \sum_{k=i-1}^{n} r_k 2^k + \sum_{k=0}^{i-2} r'_k 2^k$. Then by (8.2) we have*

$$\sum_{k=i-1}^{n} q_k 2^k > \sum_{k=i-1}^{n} r_k 2^k + \sum_{k=0}^{i-2} r'_k 2^k,$$

and hence we have

$$\sum_{k=i-1}^{n} q_k 2^k > \sum_{k=i-1}^{n} r_k 2^k + \sum_{k=0}^{i-2} r_k 2^k = \sum_{k=0}^{n} r_k 2^k = h(\sum_{k=0}^{n} p_k 2^k).$$

[2] *This is the contraposition of the proposition in [1].*
[3] *By (8.5) we have*

$$\sum_{k=i-1}^{n} q_k 2^k \leq h(\sum_{k=0}^{n} p_k 2^k), \tag{8.8}$$

and hence by [2] *of this lemma we finish the proof of* [3].
[4] *Let $q_k = 0$ for $k = i-1, ..., j-1$, then by (8.7) we have*

$$h(\sum_{k=i}^{n} p_k 2^k) < \sum_{k=i-1}^{n} q_k 2^k, \tag{8.9}$$

and hence by [1] *of this lemma*

$$h(\sum_{k=i-t}^{n} p_k 2^k) \leq h(\sum_{k=0}^{n} p_k 2^k) < \sum_{k=i-1}^{n} q_k 2^k = \sum_{k=j}^{n} q_k 2^k \tag{8.10}$$

Lemma 8.3. *Suppose that $x \oplus y \oplus z \neq 0$ and*

$$y \leq h(z). \tag{8.11}$$

Then at least one of the following [1], [2], [3] *and* [4] *is true.*
[1] *$u \oplus y \oplus z = 0$ for some $u \in Z_{\geq 0}$ such that $u < x$.*
[2] *$x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$.*
[3] *$x \oplus y \oplus w = 0$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq h(w)$.*
[4] *$x \oplus v \oplus w' = 0$ for some $v, w' \in Z_{\geq 0}$ such that $v < y, w' < z$ and $v = h(w')$.*

Proof. Suppose that there exists a natural number s such that $x_i+y_i+z_i = 0 \ (mod\ 2)$ for $i = n, n-1, ..., n-s$ and

$$x_{n-s-1} + y_{n-s-1} + z_{n-s-1} \neq 0 \ (mod\ 2). \tag{8.12}$$

[*i*] If $x_{n-s-1} = 1$, we define $u = \sum_{i=1}^{n} u_i 2^i$ by $u_i = x_i$ for $i = n, n-1, ..., n-s$, $u_{n-s-1} = 0 < x_{n-s-1}$ and $u_i = y_i + z_i$ for $i = n-s-2, n-s-3, ..., 0$. Then we have $u \oplus y \oplus z = 0$ and $u < x$. Therefore we have [1] of this lemma.
[*ii*] If $y_{n-s-1} = 1$, then by the method that is similar to the one used in [*i*] we prove that $x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$. Therefore we have [2] of this lemma.
[*iii*]. We suppose that

$$z_{n-s-1} = 1. \tag{8.13}$$

For $i = n, n-1, ..., n-s$ let

$$w_i = z_i \tag{8.14}$$

and $w_i = x_i+y_i \ (mod\ 2)$ for $i = n-s-1, ..., 0$. By (8.12) and (8.13) we have $w_{n-s-1} = x_{n-s-1}+y_{n-s-1} = 0 \ (mod\ 2)$, and hence

$$w_{n-s-1} = 0 < 1 = z_{n-s-1}. \tag{8.15}$$

[*iii*.1] If $y \leq h(w)$, then we have [3] of this lemma.
[*iii*.2] Next we suppose that

$$y > h(w). \tag{8.16}$$

By (8.11) we have $\sum_{k=0}^{n} y_k 2^k \leq h(\sum_{k=0}^{n} z_k 2^k)$, and hence by [3] of Lemma 8.2 and Equation (8.14)

$$\sum_{k=n-s-1}^{n} y_k 2^k \leq h(\sum_{k=n-s}^{n} z_k 2^k) = h(\sum_{k=n-s}^{n} w_k 2^k) \leq h(w). \tag{8.17}$$

By inequalities (8.16) and (8.17) there exists a natural number j such that

$$\sum_{k=n-j}^{n} y_k 2^k \leq h(\sum_{k=0}^{n} w_k 2^k) \tag{8.18}$$

$$\text{and} \sum_{k=n-j-1}^{n} y_k 2^k > h(\sum_{k=0}^{n} w_k 2^k). \tag{8.19}$$

By Inequality (8.18) and [2] of Lemma 8.2

$$\sum_{k=n-j}^{n} y_k 2^k \leq h(\sum_{k=n-j+1}^{n} w_k 2^k) \leq h(\sum_{k=n-j}^{n} w_k 2^k). \tag{8.20}$$

By Inequality (8.19)

$$\sum_{k=n-j-1}^{n} y_k 2^k > h(\sum_{k=n-j}^{n} w_k 2^k). \tag{8.21}$$

By (8.20) and (8.21) we have

$$\sum_{k=n-j}^{n} y_k 2^k \leq h(\sum_{k=n-j}^{n} w_k 2^k) \leq \sum_{k=n-j-1}^{n} y_k 2^k. \tag{8.22}$$

We construct v and w' by assigning values to v_i and w'_i for $i = n, n-1, n-2, ...$
First for $i = n, n-1, ..., n-j$ let

$$w'_i = w_i \text{ and } v_i = y_i, \tag{8.23}$$

and let $v_{n-j-1} = 0 < 1 = y_{n-j-1}$ and $w'_{n-j-1} = x_{n-j-1} + v_{n-j-1}$.
Since $v_{n-j-1} = 0$ and $y_{n-j-1} = 1$, by (8.22) and (8.23)

$$\sum_{k=n-j-1}^{n} v_k 2^k \leq h(\sum_{k=n-j}^{n} w'_k 2^k) < \sum_{k=n-j-1}^{n} v_k 2^k + 2^{n-j-1}. \tag{8.24}$$

By (8.24) and [4] of Lemma 8.2 we have

$$\sum_{k=n-j-1}^{n} v_k 2^k \leq h(\sum_{k=n-j-1}^{n} w'_k 2^k) < \sum_{k=n-j-1}^{n} v_k 2^k + 2^{n-j-1}. \tag{8.25}$$

Suppose that we define v_k and w'_k for $n, n-1, ..., t$ such that

$$\sum_{k=t}^{n} v_k 2^k \leq h(\sum_{k=t}^{n} w'_k 2^k) < \sum_{k=t}^{n} v_k 2^k + 2^t. \tag{8.26}$$

If

$$\sum_{k=t}^{n} v_k 2^k + 2^{t-1} \leq h(\sum_{k=t}^{n} w'_k 2^k) < \sum_{k=t}^{n} v_k 2^k + 2^t, \tag{8.27}$$

then let $v_{t-1} = 1$ and $w'_{t-1} = x_{t-1} + v_{t-1}$ $(mod\ 2)$.
Then by (8.27) we have

$$\sum_{k=t-1}^{n} v_k 2^k \leq h(\sum_{k=t}^{n} w'_k 2^k) < \sum_{k=t-1}^{n} v_k 2^k + 2^{t-1}, \tag{8.28}$$

and by [4] of Lemma 8.2

$$\sum_{k=t-1}^{n} v_k 2^k \leq h(\sum_{k=t-1}^{n} w'_k 2^k) < \sum_{k=t-1}^{n} v_k 2^k + 2^{t-1}. \tag{8.29}$$

If

$$\sum_{k=t}^{n} v_k 2^k + 2^{t-1} > h(\sum_{k=t}^{n} w'_k 2^k), \tag{8.30}$$

then let $v_{t-1} = 0$ and $w'_{t-1} = x_{t-1} + v_{t-1}$ $(mod\ 2)$.
Then by (8.26) and (8.30) we have

$$\sum_{k=t-1}^{n} v_k 2^k \leq h(\sum_{k=t}^{n} w'_k 2^k) < \sum_{k=t-1}^{n} v_k 2^k + 2^{t-1}. \tag{8.31}$$

Then by (8.31) and [4] of Lemma 8.2 we have

$$\sum_{k=t-1}^{n} v_k 2^k \leq h(\sum_{k=t-1}^{n} w'_k 2^k) < \sum_{k=t-1}^{n} v_k 2^k + 2^{t-1}. \tag{8.32}$$

We continue to construct v_k and w'_k for $k = n, n-1, ..., 1, 0$, and we have

$$\sum_{k=0}^{n} v_k 2^k \leq h(\sum_{k=0}^{n} w'_k 2^k) < \sum_{k=0}^{n} v_k 2^k + 2^0. \tag{8.33}$$

Therefore we have $\sum_{k=0}^{n} v_k 2^k = h(\sum_{k=0}^{n} w'_k 2^k)$.
Then we have [4] of this lemma. □

Lemma 8.4. *If $x \oplus y \oplus z = 0$ and $y \leq h(z)$, then*
[1] $u \oplus y \oplus z \neq 0$ for any $u \in Z_{\geq 0}$ such that $u < x$.
[2] $x \oplus v \oplus z \neq 0$ for any $v \in Z_{\geq 0}$ such that $v < y$.
[3] $x \oplus y \oplus w \neq 0$ for any $w \in Z_{\geq 0}$ such that $w < z$.
[4] $x \oplus v \oplus w \neq 0$ for any $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = h(w)$.

Proof. [1],[2] and [3] are direct from the definition of nim-sum.
We prove [4]. We suppose that $v = h(w)$ for some $w \in Z_{\geq 0}$ such that $v < y, w < z$.
Suppose that

$$w_i = z_i \tag{8.34}$$

for $i = n, n-1, n-2, ..., j$ and

$$w_{j-1} < z_{j-1}. \tag{8.35}$$

By $y \leq h(z)$ we have $h(\sum_{k=0}^{n} z_k 2^k) \geq \sum_{k=0}^{n} y_k 2^k$, and hence by [3] of Lemma 8.2 we have

$$h(\sum_{k=j}^{n} z_k 2^k) \geq \sum_{k=j-1}^{n} y_k 2^k. \tag{8.36}$$

By $v = h(w)$ we have

$$h(\sum_{k=j}^{n} z_k 2^k) = h(\sum_{k=j}^{n} w_k 2^k) \leq \sum_{k=0}^{n} v_k 2^k. \tag{8.37}$$

Since $v < y$, by (8.36), (8.37) and (8.34) we have $v_{j-1} = y_{j-1}$, and hence by (8.35) and $x_{j-1} + y_{j-1} + z_{j-1} = 0 \ (mod\ 2)$ we have $x \oplus v \oplus w \neq 0$. □

Theorem 8.1. *Let $A_k = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, y \leq h(z)$ and $x \oplus y \oplus z = 0\}$, $B_k = \{\{x, y, z\}; x, y, z \in Z_{\geq 0}, y \leq h(z)$ and $x \oplus y \oplus z \neq 0\}$.*
A_k is the set of P-positions and B_k is the set of N-positions of the disjunctive sum of the chocolate game with $CB(h, y, z)$ to the right of the poisoned square and a single strip of chocolate to the left.

Proof. By using Lemma 8.3, Lemma 8.4 and the method we used in Theorem 1.2 we prove this theorem. □

The following theorem is a generalized version of Theorem 1.3.

Theorem 8.2. *The Grundy number of $CB(h, y, z)$ is $y \oplus z$.*

Proof. By Theorem 8.1 a position $\{x, y, z\}$ of the sum of the chocolate is a $\mathcal{P}$-position when $x \oplus y \oplus z = 0$ so that the Grundy number of the Chocolate bar to the right is $x = y \oplus z$. □

9 A proof for a necessary condition

The condition presented in Definition 8.1 is the necessary and sufficient condition for a function f to have Grundy number $G(\{y, z\}) = y \oplus z$.
In this section we prove that this is a necessary condition.

Definition 9.1. *Let f be a monotonically increasing function of $Z_{\geq 0}$ into $Z_{\geq 0}$ such that $CB(f, y, z)$ has the Grundy number*

$$G(\{y, z\}) = y \oplus z. \tag{9.1}$$

Throughout this section we assume that the function f satisfies the condition in Definition 9.1.

Lemma 9.1. *Let $y, z \in Z_{\geq 0}$ such that $y < y'$, $y = f(z)$ and $y' \leq f(z+1)$. Then $G(\{y, z+1\}) < G(\{y', z+1\})$*

Proof. By $y = f(z)$ we have $f(w) \leq y < y'$ for $w < z+1$, and hence $move(\{y', z+1\}) = \{\{\{v, z+1\}; v < y'\} \cup \{\{\min(y', f(w)), w\}; w < z+1\} = \{\{\{v, z+1\}; v < y'\} \cup \{\{f(w), w\}; w < z+1\} = \{\{y, z+1\}, \{y+1, z+1\}, ..., \{y'-1, z+1\}\} \cup \{\{\{v, z+1\}; v < y\} \cup \{\{f(w), w\}; w < z+1\}$
$= \{\{y, z+1\}, \{y+1, z+1\}, ..., \{y'-1, z+1\}\} \cup \{\{\{v, z+1\}; v < y\} \cup \{\{\min(y, f(w)), w\}; w < z+1\}$
$= \{\{y, z+1\}, \{y+1, z+1\}, ..., \{y'-1, z+1\}\} \cup move(\{y, z+1\})$, where $v, w \in Z_{\geq 0}$. Therefore $G(\{y', z+1\}) = Mex(\{G(\{y, z+1\}), G(\{y+1, z+1\}), ..., G(\{y'-1, z+1\})\} \cup \{G(\{a, b\}); \{a, b\} \in move(\{y, z+1\})\} \geq G(\{y, z+1\})$ □

Lemma 9.2. *For any $y, z \in Z_{\geq 0}$ with $y \leq f(z)$ we have $\{G(\{min(y, f(w)), w\}); w < z\} = \{y \oplus w; w < z\}$.*

Proof. Let $w \in Z_{\geq 0}$ with $w < z$. Let

$$n > \lfloor log_2 max(y, z) \rfloor. \tag{9.2}$$

Then by Inequality (9.2) we have $y \oplus w < y \oplus (z + 2^n) = G(\{y, z + 2^n\})$. By the definition of Grundy number there exist $a, b \in Z_{\geq 0}$ such that $\{a, b\} \in move(\{y, z + 2^n\})$ and $G(\{a, b\}) = y \oplus w$.
By the definition of *move* and Grundy number we have (9.3) or (9.4).

$$G(\{min(y, f(w'), w'\}) = y \oplus w \tag{9.3}$$

for $w' \in Z_{\geq 0}$ with $w' < z + 2^n$.

$$G(\{y', z + 2^n\}) = y \oplus w \tag{9.4}$$

for $y' \in Z_{\geq 0}$ with $y' < y$. (9.4) contradicts to (9.2), since $G(\{y', z + 2^n\}) = y' \oplus (z + 2^n)$. Therefore we have (9.3). If $w' \geq z$, then $f(w') \geq f(z) \geq y$ and $G(\{y, w'\}) = y \oplus w' = y \oplus w$, which contradicts $w < z \leq w'$. Therefore we have $w' < z$, and hence we have $\{G(\{min(y, f(w)), w\}); w < z\} \supset \{y \oplus w; w < z\}$. Since these two sets have the same number of elements, we have $\{G(\{min(y, f(w)), w\}); w < z\} = \{y \oplus w; w < z\}$. □

Lemma 9.3. *For any $a \in Z_{\geq 0}$ $f(2a) = f(2a + 1)$.*

Proof. [1] Suppose that there exists $b \in Z_{\geq 0}$ such that $f(2a) = 2b$ and $f(2a + 1) \geq 2b + 1$.
Then $G(\{2b, 2a + 1\}) = 2b \oplus (2a + 1) > (2b + 1) \oplus (2a + 1) = G(\{2b + 1, 2a + 1\})$, but this contradicts Lemma 9.1.
Note that here we use Lemma 9.1 by letting $y = 2b, y' = 2b + 1, z = 2a$ and $z + 1 = 2a + 1$.
[2] Suppose that there exists $b \in Z_{\geq 0}$ such that $f(2a) = 2b + 1$ and $f(2a + 1) \geq 2b + 2$.
For $2b$ there exists a natural number i such that $2b = c \times 2^i + \sum_{k=1}^{i-2} 2^k$. Then $2b + 2 = c \times 2^i + 2^{i-1}$.
Let $2a = d \times 2^i + \sum_{k=1}^{i-1} d_k 2^k$.
[2.1] Suppose that $d_{i-1} = 1$. Then
$G(\{2b + 1, 2a + 1\}) = (2b + 1) \oplus (2a + 1) = (c \oplus d)2^i + d_{i-1}2^{i-1} + ... > (c \oplus d)2^i + (d_{i-1} \oplus 1)2^{i-1} + ...$
$= (2b + 2) \oplus (2a + 1) = G(\{2b + 2, 2a + 1\})$. Note that $d_{i-1} \oplus 1 = 0$.
[2.2] Suppose that $d_{i-1} = 0$. Then
$G(\{2b + 2, 2a + 1\}) = Mex(\{G(\{y, z\}); \{y, z\} \in move(\{2b + 2, 2a + 1\})$

$$= Mex(\{G(\{min(2b+2, f(k)), k\}); k = 0, 1, 2, ..., 2a\} \cup \{G(\{k, 2a+1\}); k = 0, 1, 2, ..., 2b+1\}) \tag{9.5}$$

Since $\min(2b+2, f(k)) = \min(2b+1, f(k))$ for $k \leq 2a$, by Lemma 9.2 and Definition 9.1 we have (9.5) $= Mex(\{2b+1 \oplus k; k = 0, 1, 2, ..., 2a\} \cup \{k \oplus 2a+1; k = 0, 1, 2, ..., 2b+1\})$.

$$\{2b+1 \oplus k; k = 0, 1, 2, ..., 2a\} = \{(c \times 2^i + \sum_{k=1}^{i-2} 2^k + 1) \oplus k; k = 0, 1, 2, ..., d \times 2^i + \sum_{k=1}^{i-2} d_k 2^k\}$$

$$= \{(c \times 2^i + \sum_{k=1}^{i-2} 2^k + 1) \oplus k; k = 0, 1, 2, ..., d \times 2^i - 1\} \tag{9.6}$$

$$\cup \{(c \times 2^i + \sum_{k=1}^{i-2} 2^k + 1) \oplus k; k = d \times 2^i, d \times 2^i + 1, ..., d \times 2^i + \sum_{k=1}^{i-2} d_k 2^k\}. \tag{9.7}$$

$$\{k \oplus 2a+1; k = 0, 1, 2, ..., 2b+1\}$$

$$= \{k \oplus (d \times 2^i + \sum_{k=1}^{i-1} d_k 2^k + 1); k = 0, 1, 2, ..., c \times 2^i - 1\} \tag{9.8}$$

$$\cup \{k \oplus (d \times 2^i + \sum_{k=1}^{i-1} d_k 2^k + 1); k = c \times 2^i, c \times 2^i + 1, ..., c \times 2^i + \sum_{k=1}^{i-2} 2^k + 1\} \tag{9.9}$$

Since $G(\{2b+2, 2a+1\}) = (c \oplus d)2^i + 2^{i-1} + \sum_{k=1}^{i-2} d_k 2^k + 1 > (c \oplus d)2^i + 2^{i-1}$, $(c \oplus d)2^i + 2^{i-1}$ should be in (9.6) or (9.7) or (9.8) or (9.9). Tt is clear that it is not the case, since $d_{i-1} = 0$. □

Lemma 9.4. *Let*

$$a + 1 = d \times 2^{i+1} + d_i 2^i + e, \tag{9.10}$$

where $e < 2^i$ *and* $0 < d_i 2^i + e$.
If $c \times 2^{i+1} \leq f(a) < c \times 2^{i+1} + 2^i$, *then* $f(a+1) < c \times 2^{i+1} + 2^i$.

Proof. Let

$$f(a) = c \times 2^{i+1} + t, \tag{9.11}$$

where $0 \leq t < 2^i$. We suppose that

$$f(a+1) \geq c \times 2^{i+1} + 2^i, \tag{9.12}$$

and we show that this leads to a contradiction.
[1] If $d_i = 1$, then
$G(\{c \times 2^{i+1} + 2^i, a+1\}) = (c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + d_i 2^i + e)$
$= (c \oplus d)2^{i+1} + e < (c \oplus d)2^{i+1} + d_i 2^i + (t \oplus e) = G(\{c \times 2^{i+1} + t, a+1\})$, which contradicts Lemma 9.1.
[2] If $d_i = 0$, then $G(\{c \times 2^{i+1} + 2^i, a+1\}) = (c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + e)$
$= (c \oplus d)2^{i+1} + 2^i + e > (c \oplus d)2^{i+1} + 2^i$.
Therefore by the definition of Grundy number we have

$$(c \oplus d)2^{i+1} + 2^i \in \{G(\{p, q\}); \{p, q\} \in move(\{c \times 2^{i+1} + 2^i, a+1\})\}. \tag{9.13}$$

$$\{G(\{p,q\});\{p,q\}\in move(\{c\times 2^{i+1}+2^i,a+1\})\}$$

$$=\{G(\{v,d\times 2^{i+1}+e\}),v=0,1,2,...,c\times 2^{i+1}+2^i-1\}$$
$$\cup\{G(\{min(c\times 2^{i+1}+2^i,f(w)),w\}),w=0,1,2,...,d\times 2^{i+1}+e-1\}. \tag{9.14}$$

Here note that $a=d\times 2^{i+1}+e-1$.
For $w\leq a$ we have $f(w)\leq c\times 2^{i+1}+t$,
(9.14) =

$$=\{G(\{v,d\times 2^{i+1}+e\}),v=0,1,2,...,c\times 2^{i+1}+2^i-1\}$$
$$\cup\{G(\{min(c\times 2^{i+1}+t,f(w)),w\}),w=0,1,2,...,d\times 2^{i+1}+e-1\} \tag{9.15}$$

By Lemma 9.2 and Definition 9.1
(9.15)

$$=\{v\oplus(d\times 2^{i+1}+e),v=0,1,2,...,c\times 2^{i+1}+2^i-1\}$$
$$\cup\{(c\times 2^{i+1}+t)\oplus w,w=0,1,2,...,d\times 2^{i+1}+e-1\}$$

$$=\{(c\times 2^{i+1}+k)\oplus(d\times 2^{i+1}+e),k=0,1,2,...,2^i-1\} \tag{9.16}$$
$$\cup\{k\oplus(d\times 2^{i+1}+e),k=0,1,2,...,c\times 2^{i+1}-1\} \tag{9.17}$$
$$\cup\{(c\times 2^{i+1}+t)\oplus(d\times 2^{i+1}+k),k=0,1,2,...,e-1\} \tag{9.18}$$
$$\cup\{(c\times 2^{i+1}+t)\oplus k),k=0,1,2,...,d\times 2^{i+1}-1\}. \tag{9.19}$$

All the numbers in (9.16) are of the type $(c\oplus d)2^{i+1}+(k\oplus e)$, and hence this set does not contains $(c\oplus d)2^{i+1}+2^i$. Note that $k,e<2^i$.
The coefficient of 2^{i+1} of the numbers in (9.17) are not $c\oplus d$, and hence this set does not contains $(c\oplus d)2^{i+1}+2^i$.
All the numbers in (9.18) are of the type $(c\oplus d)2^{i+1}+(t\oplus k)$, and hence this set does not contains $(c\oplus d)2^{i+1}+2^i$.
Note that $k\leq e-1<2^i$ and $t<2^i$.
The coefficient of 2^{i+1} of the numbers in (9.19) are not $c\oplus d$, and hence this set does not contains $(c\oplus d)2^{i+1}+2^i$.
These facts contradict (9.13). Therefore we conclude that (9.12) is false. □

Theorem 9.1. *Suppose that the function f satisfies the condition in Definition 9.1. The function f satisfies the following condition* [1].
[1] *If*

$$f(\sum_{k=0}^{n}z_k 2^k)=\sum_{k=0}^{n}y_k 2^k,$$

then for any z'_k for $k=0,...,j-1$ there exist y'_k for $k=0,...,j-2$ such that

$$f(\sum_{k=j}^{n}z_k 2^k+\sum_{k=0}^{j-1}z'_k 2^k)=\sum_{k=j-1}^{n}y_k 2^k+\sum_{k=0}^{j-2}y'_k 2^k.$$

Proof. If f does not satisfies the condition [1], then there exists i such that $i\geq j-1$,

$$f(\sum_{k=0}^{n}z_k 2^k)=\sum_{k=i+1}^{n}y_k 2^k+0\times 2^i+\sum_{k=0}^{i-1}y_k 2^k$$

and

$$f(\sum_{k=j}^{n} z_k 2^k + \sum_{k=0}^{j-1} z'_k 2^k) = \sum_{k=i+1}^{n} y_k 2^k + 2^i + \sum_{k=0}^{i-1} y'_k 2^k.$$

We denote $\sum_{k=i+1}^{n} y_k 2^k$ by $c \times 2^{i+1}$, then we have

$$f(\sum_{k=0}^{n} z_k 2^k) = c \times 2^{i+1} + 0 \times 2^i + \sum_{k=0}^{i-1} y_k 2^k$$

and

$$f(\sum_{k=j}^{n} z_k 2^k + \sum_{k=0}^{j-1} z'_k 2^k) = c \times 2^{i+1} + 2^i + \sum_{k=0}^{i-1} y'_k 2^k.$$

Let $a = max(\{z; f(z) < c \times 2^{i+1} + 2^i\})$ and $b = min(\{z; f(z) \geq c \times 2^{i+1} + 2^i\})$. Then clearly $b = a + 1$. Since $i + 1 \geq j$, by denoting $\sum_{k=i+1}^{n} z_k 2^k$ by $d \times 2^{i+1}$ we have $a \geq d \times 2^{i+1}$ and $(d+1) \times 2^{i+1} > b = a + 1 > d \times 2^{i+1}$. Therefore there exist d_i and e such that $a + 1 = d \times 2^{i+1} + d_i 2^i + e$, $e < 2^i$ and $0 < d_i 2^i + e$. Clearly this contradicts Lemma 9.4. □

10 Prospect for the future research

We presented chocolate games whose coordinates satisfy the inequality $f(t) = \lfloor \frac{t}{k} \rfloor$ for an even number k in Yau Award 2012, and the chocolate bar $CB(f, y, z)$ has the Grundy number $G(\{y, z\}) = y \oplus z$.
The result of the Yau Award 2012 was accepted by "Integers Electronic Journal of Combinatorial Number theory" for publication.
In Yau Award 2014 we introduced a function f and g for which chocolate bars $CB(f, y, z)$ and $CB(g, y, z)$ have the Grundy number $G(\{y, z\}) = y \oplus z$.
Therefore we discovered a sufficient condition (a condition that is more general than that of Yau Award 2012).
After we submitted our research in 2014, we discovered and proved the necessary and sufficient condition for a chocolate bar $CB(h, y, z)$ to have the Grundy number $G(\{y, z\}) = y \oplus z$.
We are very happy to be able to present the result that is far better than the result that is accepted by a well known mathematics journal.
Next step of our research will be to study the function h, and find properties of it. The function $f(t) = \lfloor \frac{t}{k} \rfloor$ is very concrete, and the functions f, g have a clear method to be constructed, but the newest function h is very abstract, and we still do not have a good method to construct this function.
We are going to discover a method to create concrete functions from this abstract function h if we could move to the final of Yau Award.
Chocolate games whose coordinates satisfy the inequality $f(t) = \lfloor \frac{t}{k} \rfloor$ for an odd number k is still a difficult problem to study.

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